Beam instabilities (II)

Giovanni Rumolo
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Acknowledgments: H. Bartosik, H. Damerau, G. Iadarola, K. Li, E. Métral, N. Mounet, G. Papotti, B. Salvant, C. Zannini

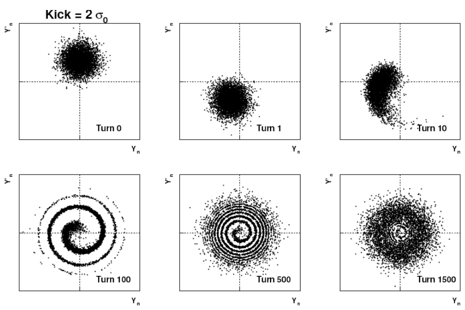
CAS

Summary of the first part

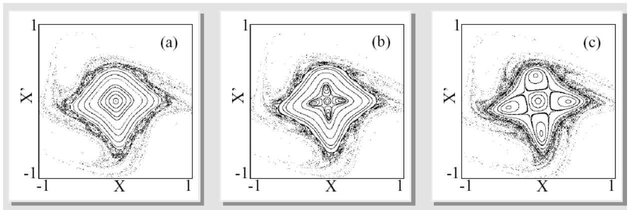
- What is a beam instability?
  - A beam becomes unstable when a moment of its distribution exhibits an exponential growth (e.g. mean positions <x>, <y>, <z>, standard deviations σ_x, σ_y, σ_z, etc.) – resulting into beam loss or emittance growth!
- Instabilities are caused by the electro-magnetic fields trailing behind charged particles moving at the speed of light
  - Origin: discontinuities, finite conductivity
  - Described through wake functions and impedances

⇒ Longitudinal plane
  - Energy loss and potential well distortion
    - → Synchronous phase shift
    - → Bunch lengthening/shortening, synchrotron tune shift
  - Instabilities
    - Robinson instability (dipole mode)
    - Coupled bunch instabilities
    - Single bunch instabilities

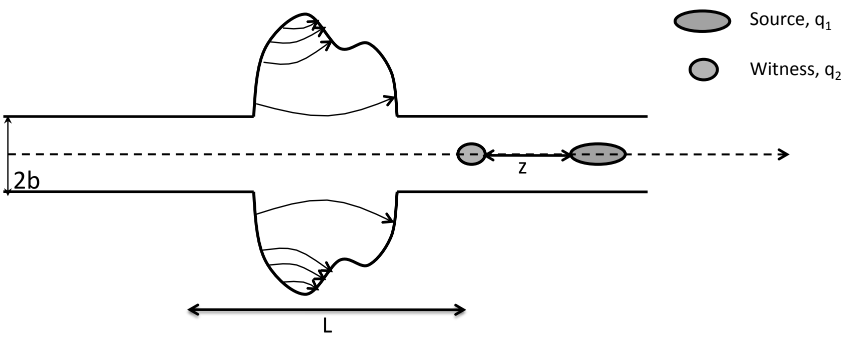
2



2. The transverse plane



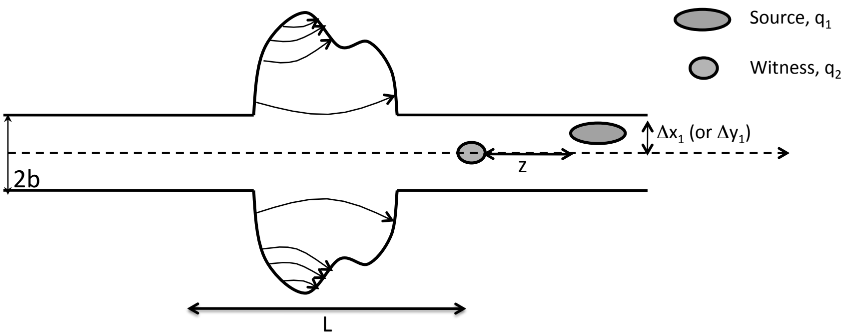
Transverse wake function: definition



- In an axisymmetric structure (or simply with a top-bottom and left-right symmetry) a source particle traveling on axis cannot induce net transverse forces on a witness particle also following on axis
- At the zero-th order, there is no transverse effect
- We need to introduce a breaking of the symmetry to drive transverse effect, but at the first order there are two possibilities, i.e. offset the source or the witness

4

Transverse **dipolar** wake function: definition



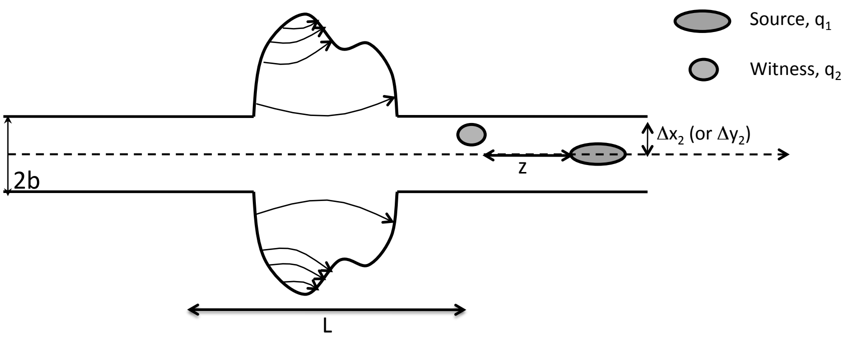
$$\int_0^L F_x(s,z)ds = -q_1 q_2 W_x(z)\Delta x_1$$
$$\int_0^L F_y(s,z)ds = -q_1 q_2 W_y(z)\Delta y_1$$

$$F_{x,y}(s,z) = q_2\left(\vec{E} + \vec{v}\times\vec{B}\right)_{x,y}$$

$$\Delta E_{2x,y} \quad \frac{\Delta E_{2x,y}}{E_0} = \Delta x_2',\ \Delta y_2'$$

5

Transverse **quadrupolar** wake function: definition



$$\int_0^L F_x(s,z)ds = -q_1 q_2 W_{Qx}(z)\Delta x_2$$
$$\int_0^L F_y(s,z)ds = -q_1 q_2 W_{Qy}(z)\Delta y_2$$

$$F_{x,y}(s,z) = q_2\left(\vec{E} + \vec{v}\times\vec{B}\right)_{x,y}$$

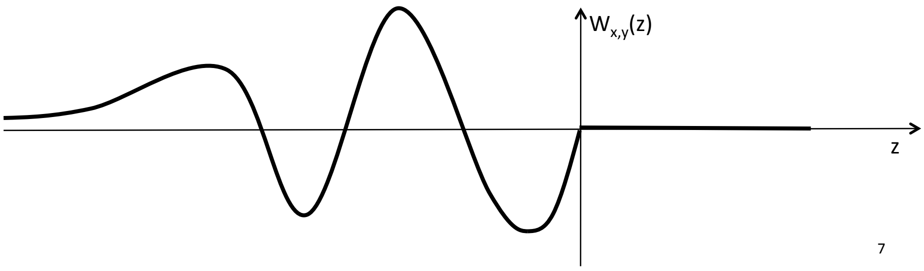
$$\Delta E_{2x,y} \quad \frac{\Delta E_{2x,y}}{E_0} = \Delta x_2',\ \Delta y_2'$$

6

Transverse dipolar wake function

$$W_x(z) = -\frac{E_0}{q_1 q_2}\frac{\Delta x_2'}{\Delta x_1} \qquad \begin{matrix} z \to 0 \\ q_2 \to q_1 \end{matrix} \qquad W_x(0) = 0$$

- The value of the transverse wake functions in 0, $\boldsymbol{W_{x,y}(0)}$, must vanish because the source has only longitudinal momentum and thus cannot lose energy in the transverse plane
- $\boldsymbol{W_{x,y}(0^-)<0}$ since trailing particles are deflected toward the source particle (Δx_1 and $\Delta x'_2$ have the same sign)
- $\boldsymbol{W_{x,y}(z)}$ has a discontinuous derivative in z=0 and it vanishes for all z>0 because of the ultra-relativistic approximation

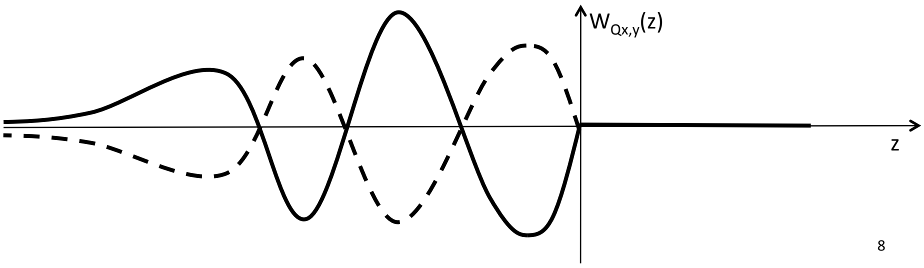


7

Transverse quadrupolar wake function

$$W_{Qx}(z) = -\frac{E_0}{q_1 q_2}\frac{\Delta x_2'}{\Delta x_2} \qquad \begin{matrix} z \to 0 \\ q_2 \to q_1 \end{matrix} \qquad W_{Qx}(0) = 0$$

- The value of the transverse quadrupolar wake functions in 0, $\boldsymbol{W_{Qx,y}(0)}$, must vanish because dipolar and quadrupolar wake functions should exhibit the same behavior when the witness tends to source
- $\boldsymbol{W_{Qx,y}(0^-)}$ can be of either sign since trailing particles can be either attracted or deflected even more off axis (depends on geometry and boundary conditions)
- $\boldsymbol{W_{x,y}(z)}$ has a discontinuous derivative in z=0 and it vanishes for all z>0 because of the ultra-relativistic approximation



8

Transverse impedance

- The transverse wake function of an accelerator component is basically its Green function in time domain (i.e., its response to a pulse excitation)
  - ⇒ Very useful for macroparticle models and simulations, because it relates source perturbations to the associated kicks on trailing particles!
- We can also describe it as a transfer function in frequency domain
- This is the definition of **transverse beam coupling impedance** of the element under study

$$Z_{\perp}^{\mathrm{dip}}(\omega) = i \int_{-\infty}^{\infty} W_{\perp}(z) \exp\left(-\frac{i\omega z}{c}\right) \frac{dz}{c}$$

$$Z_{\perp}^{\mathrm{qua}}(\omega) = i \int_{-\infty}^{\infty} W_{Q\perp}(z) \exp\left(-\frac{i\omega z}{c}\right) \frac{dz}{c}$$

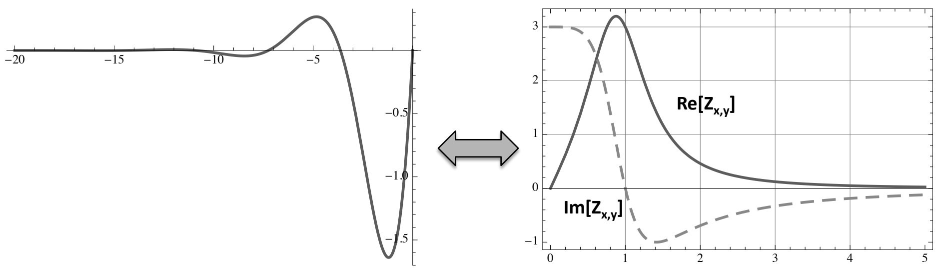
$Z_{\perp}^{\mathrm{dip}}(\omega)$, $Z_{\perp}^{\mathrm{qua}}(\omega)$ → [Ω/m]

$W_{\perp}(z)$, $W_{Q\perp}(z)$ → [Ω/m/s]

** m^{-1} refers then to a transverse offset and does not represent a normalization per unit length of the structure

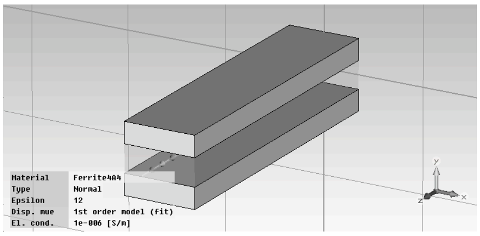
Transverse impedance

$$Z_{\perp}^{\mathrm{dip}}(\omega) = i \int_{-\infty}^{\infty} W_{\perp}(z) \exp\left(-\frac{i\omega z}{c}\right) \frac{dz}{c}$$

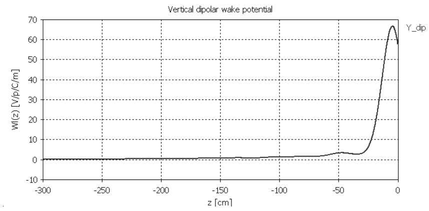


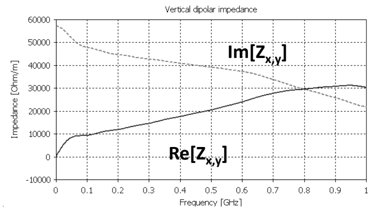
- Shape of wake function can be similar to that in longitudinal plane, determined by the oscillations of the trailing electromagnetic fields
- Contrary to longitudinal impedances, Re[$Z_{x,y}$] is an odd function of frequency, while Im[$Z_{x,y}$] is an even function

Transverse impedance



- An example: magnetic kickers are usually large contributors to the transverse impedance of a machine
- It is a broad band contribution
  - No trapped modes
  - Losses both in vacuum chamber and ferrite (kicker heating and outgassing)

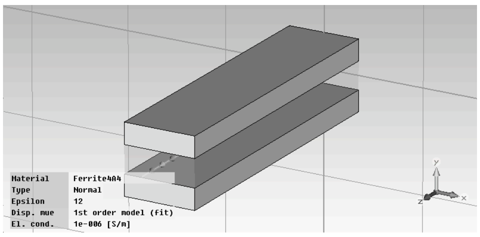




Dipolar

11

Transverse impedance



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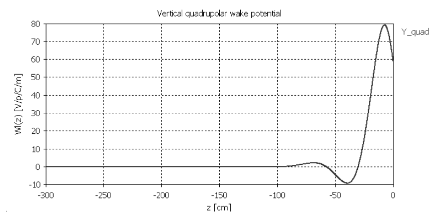
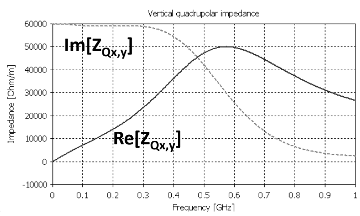
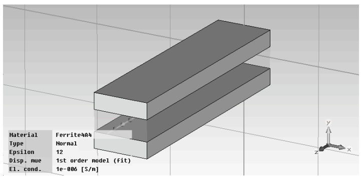




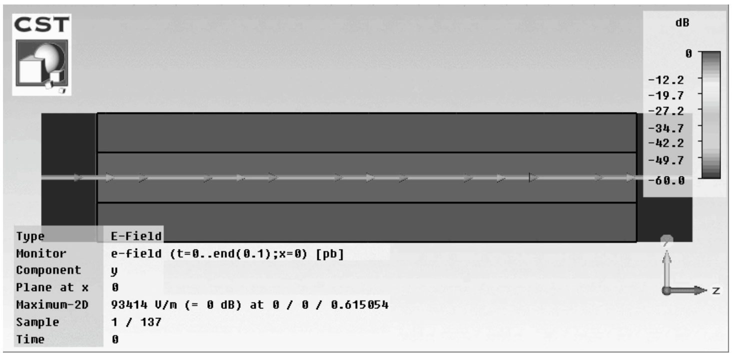
Quadrupolar

12

Transverse impedance

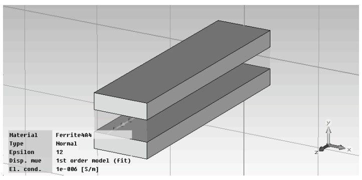


– Evolution of the electromagnetic fields (E_y) in the kicker while and after the beam has passed

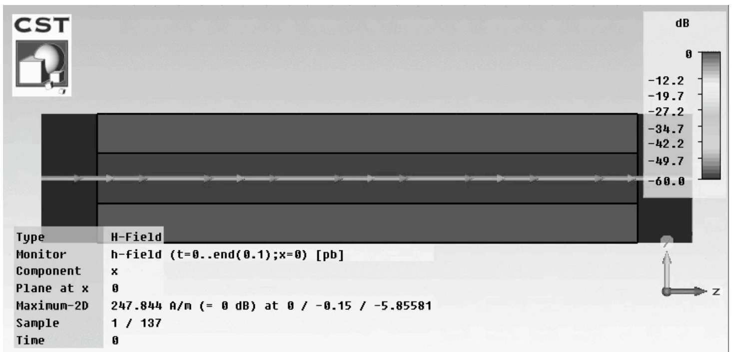


13

Transverse impedance

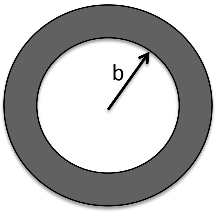


– Evolution of the electromagnetic fields (H_x) in the kicker while and after the beam has passed

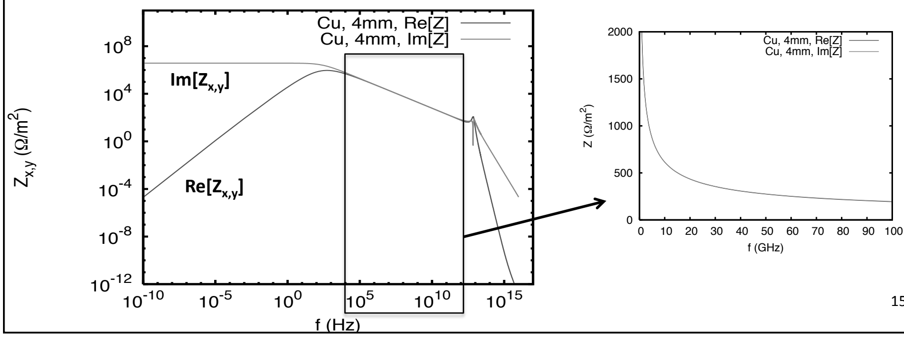


14

Transverse impedance

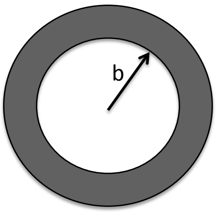


- Another interesting example: a conductive pipe (e.g. Cu, b=4mm)
- Its impedance extends over a very wide range of frequencies
  - At low frequencies there is an inductive by-pass effect
  - At high frequencies there is a small peak due to the ac conductivity
  - At intermediate frequencies there is a "classical" behavior

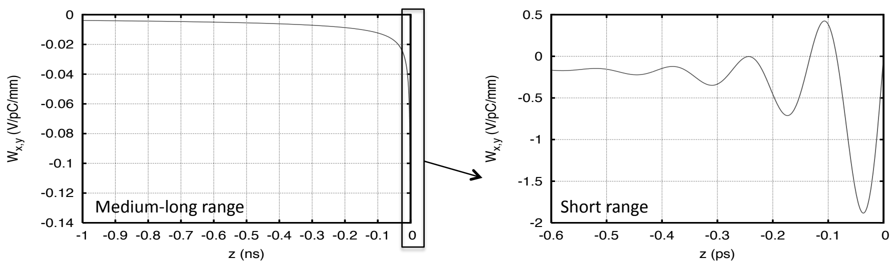


15

Transverse impedance



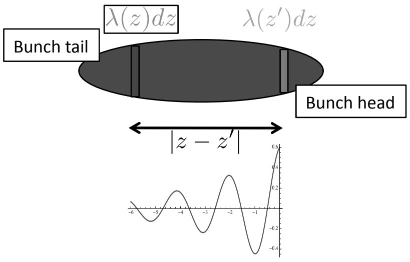
- Another interesting example: a conductive pipe (e.g. Cu, b=4mm)
- Corresponding to the different frequency ranges, the wake field has
  - A medium range behavior (coupled bunch) following the classical shape
  - A long range behavior (multi-turn) dominated by the inductive by-pass
  - A short range behavior (single bunch) dominated by the ac conductivity peak



16

Single particle equations of the transverse motion in presence of dipolar wake fields

- The single particle in the witness slice λ(z)dz will feel the external focusing forces and that associated to the wake in s_0
- Space charge here neglected
- The wake contribution can extend to several turns



$$\frac{d^2x}{ds^2} + K_x(s)x = -\left(\frac{e^2}{m_0c^2}\right)\sum_{k=-\infty}^{\infty}\frac{N}{\gamma C}\int_{-\infty}^{\infty}\lambda(z'+kC)\langle x\rangle(s_0,z'+kC)W_x(s_0,z-z'-kC)dz'$$

$$\frac{d^2y}{ds^2} + K_y(s)y = -\left(\frac{e^2}{m_0c^2}\right)\sum_{k=-\infty}^{\infty}\frac{N}{\gamma C}\int_{-\infty}^{\infty}\lambda(z'+kC)\langle y\rangle(s_0,z'+kC)W_y(s_0,z-z'-kC)dz'$$

External Focusing

Wake fields

17

The Rigid Bunch Instability

- To illustrate the rigid bunch instability we will use some simplifications:
  - ⇒ The bunch is point-like and feels an external linear force (i.e. it would execute linear betatron oscillations in absence of the wake forces)
  - ⇒ Longitudinal motion is neglected
  - ⇒ Smooth approximation → constant focusing + distributed wake

- In a similar fashion as was done for the Robinson instability in the longitudinal plane we want to
  - ⇒ Calculate the betatron tune shift due to the wake
  - ⇒ Derive possible conditions for the excitation of an unstable motion

18

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$$\frac{d^2y}{ds^2} + \left(\frac{\omega_\beta}{c}\right)^2 y = -\left(\frac{e^2}{m_0c^2}\right)\frac{N}{\gamma C}\sum_{k=-\infty}^{\infty} y(s-kC)W_y(kC)$$

$$y \propto \exp\left(\frac{-i\Omega s}{c}\right) \quad \Rightarrow \quad \Omega^2 - \omega_\beta^2 = \frac{Ne^2}{m_0\gamma C}\sum_{k=-\infty}^{\infty}\exp\left(ik\Omega T_0\right)W_y(kC)$$

$$= -i\frac{Ne^2}{m_0\gamma CT_0}\sum_{p=-\infty}^{\infty} Z_y(p\omega_0+\Omega)$$

Comes from the definition of Z_y

19

The Rigid Bunch Instability

- ⇒ We assume a small deviation from the betatron tune
- ⇒ Re(Ω - ω_β) → Betatron tune shift
- ⇒ Im(Ω - ω_β) → Growth/damping rate, if it is positive there is an instability!

$$\Omega^2 - \omega_\beta^2 \approx 2\omega_\beta\cdot(\Omega-\omega_\beta)$$

$$\frac{1}{4\pi}\left[\beta_y\frac{eI_b\mathrm{Im}(Z_y^{\mathrm{eff}})}{E}\right] = \frac{1}{4\pi}\oint\beta_y(s)\Delta k(s)ds$$

$$\frac{\mathrm{Re}\left(\Omega-\omega_\beta\right)}{\omega_0} = \Delta\nu_y \approx \frac{Ne^2\beta_y}{4\pi m_0\gamma cC}\sum_{p=-\infty}^{\infty}\mathrm{Im}\left[Z_y(p\omega_0+\omega_\beta)\right]$$

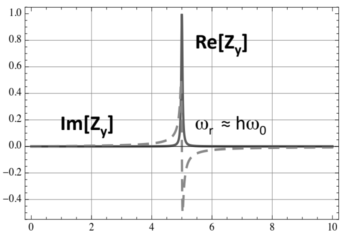
$$\mathrm{Im}\left(\Omega-\omega_\beta\right) = \tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2}\sum_{p=-\infty}^{\infty}\mathrm{Re}\left[Z_y(p\omega_0+\omega_\beta)\right]$$

20

The Rigid Bunch Instability

$$\mathrm{Im}\,(\Omega - \omega_\beta) = \tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2}\sum_{p=-\infty}^{\infty}\mathrm{Re}\left[Z_y(p\omega_0+\omega_\beta)\right]$$

- ⇒ We assume the impedance to be peaked at a frequency ω_r close to $h\omega_0$ (e.g. RF cavity fundamental mode or HOM)
- ⇒ Defining the tune $\nu_y = n_y + \Delta_{\beta y}$ with $-0.5<\Delta_{\beta y}<0.5$, we can easily express the only two leading terms left in the summation at the RHS of the equation for the growth rate

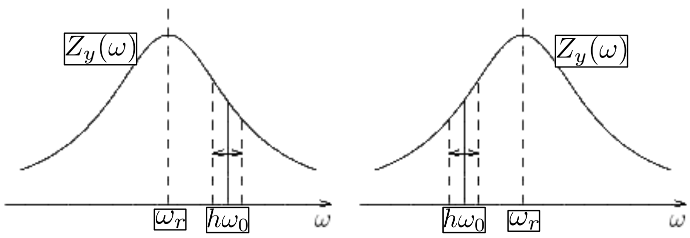


$$\tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2}\left(\mathrm{Re}\left[Z_y(h\omega_0+\Delta_{\beta y}\omega_0)\right]-\mathrm{Re}\left[Z_y(h\omega_0-\Delta_{\beta y}\omega_0)\right]\right)$$

21

The Rigid Bunch Instability

$$\tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2}\left(\mathrm{Re}\left[Z_y(h\omega_0+\Delta_{\beta y}\omega_0)\right]-\mathrm{Re}\left[Z_y(h\omega_0-\Delta_{\beta y}\omega_0)\right]\right)$$



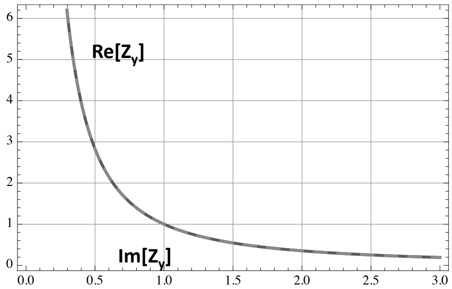
	$\omega_r < h\omega_0$	$\omega_r > h\omega_0$
Tune above integer ($\Delta_{\beta y}>0$)	unstable	stable
Tune below integer ($\Delta_{\beta y}<0$)	stable	unstable

22

The Rigid Bunch Instability

$$\mathrm{Im}\left(\Omega-\omega_\beta\right)=\tau_y^{-1}\approx-\frac{Ne^2\beta_y}{2m_0\gamma C^2}\sum_{p=-\infty}^{\infty}\mathrm{Re}\left[Z_y(p\omega_0+\omega_\beta)\right]$$

⇒ We assume the impedance to be of resistive wall type, i.e. strongly peaked in the very low frequency range (→ 0)
⇒ Using the same definitions for the tune as before, we can easily express the only two leading terms left in the summation at the RHS of the equation for the growth rate

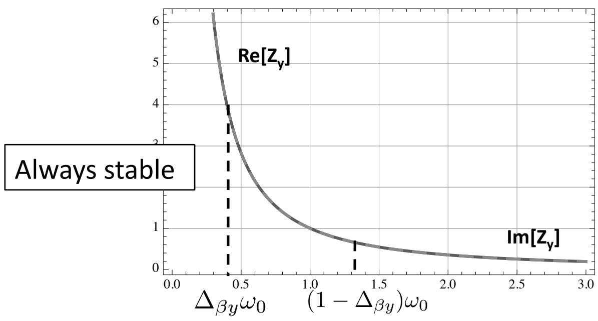


23

The Rigid Bunch Instability

⇒ Using the same definitions for the tune as before, we can easily express the only two leading terms left in the summation at the RHS of the equation for the growth rate

$$\begin{cases}\Delta_{\beta y}>0\\ \tau_y^{-1}\approx-\frac{Ne^2\beta_y}{2m_0\gamma C^2}\left(\mathrm{Re}\left[Z_y(\Delta_{\beta y}\omega_0)\right]-\mathrm{Re}\left[Z_y((1-\Delta_{\beta y})\omega_0)\right]\right)<0\end{cases}$$



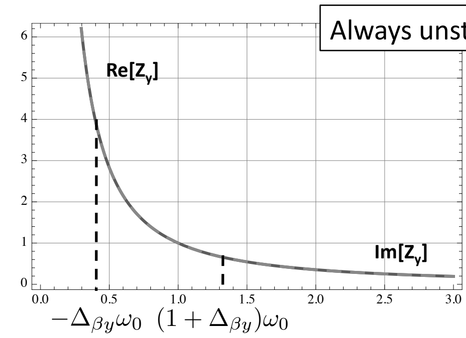
24

The Rigid Bunch Instability

⇒ Using the same definitions for the tune as before, we can easily express the only two leading terms left in the summation at the RHS of the equation for the growth rate

$$\begin{cases} \Delta_{\beta y} < 0 \\ \tau_y^{-1} \approx -\dfrac{Ne^2\beta_y}{2m_0\gamma C^2}\left(\mathrm{Re}\left[Z_y((1+\Delta_{\beta y})\omega_0)\right] - \mathrm{Re}\left[Z_y(-\Delta_{\beta y}\omega_0)\right]\right) > 0 \end{cases}$$

Always unstable



And this is the reason why most of the running machines are usually made to operate with a fractional part of the tune below 0.5! However, tunes above the half integer can be used, if the resistive wall instability is Landau damped or efficiently suppressed with a feedback system

25

The Strong Head Tail Instability (aka Transverse Mode Coupling Instability)

– To illustrate TMCI we will need to make use of some simplifications:
  - ⇒ The bunch is represented through two particles carrying half the total bunch charge and placed in opposite phase in the longitudinal phase space
  - ⇒ They both feel external linear focusing in all three directions (i.e. linear betatron focusing + linear synchrotron focusing).
  - ⇒ Zero chromaticity ($Q'_{x,y}$=0)
  - ⇒ Constant transverse wake left behind by the leading particle
  - ⇒ Smooth approximation → constant focusing + distributed wake

– We will
  - ⇒ Calculate a stability condition (threshold) for the transverse motion
  - ⇒ Have a look at the excited oscillation modes of the centroid

26

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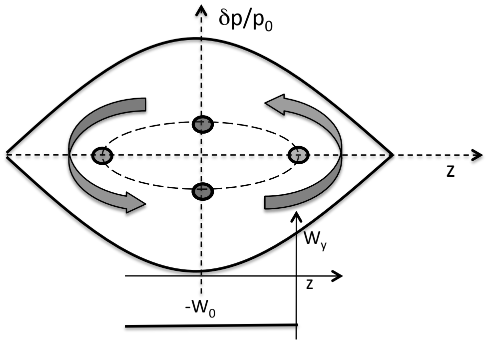


27

The Strong Head Tail Instability Equations of motion

⇒ During the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1

$$\begin{cases} \dfrac{d^2y_1}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_1 = 0 \\ \dfrac{d^2y_2}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_2 = \left(\dfrac{e^2}{m_0c^2}\right)\dfrac{NW_0}{2\gamma C} y_1(s) \end{cases} \qquad 0 < s < \frac{\pi c}{\omega_s}$$

28

The Strong Head Tail Instability
Equations of motion

- ⇒ During the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1
- ⇒ During the second half of the synchrotron period, the situation is reversed

$$\begin{cases} \dfrac{d^2y_1}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_1 = \left(\dfrac{e^2}{m_0c^2}\right)\dfrac{NW_0}{2\gamma C}y_2(s) \\ \dfrac{d^2y_2}{ds^2} + \left(\dfrac{\omega_\beta}{c}\right)^2 y_2 = 0 \end{cases} \qquad \frac{\pi c}{\omega_s} < s < \frac{2\pi c}{\omega_s}$$

29

The Strong Head Tail Instability
Equations of motion

- ⇒ We solve with respect to the complex variables defined below (k=1,2) during the first half of synchrotron period
- ⇒ $y_1(s)$ will be trivially a free betatron oscillation
- ⇒ $y_2(s)$ will be the sum of a free betatron oscillation plus a driven oscillation with $y_1(s)$ being its driving term

$$\tilde{y_k}(s) = y_k(s) + i\frac{c}{\omega_\beta}y_k'$$

$$\tilde{y_1}(s) = \tilde{y}_1(0)\exp\left(\frac{-i\omega_\beta s}{c}\right)$$

$$\tilde{y_2}(s) = \underbrace{\tilde{y}_2(0)\exp\left(-\frac{i\omega_\beta s}{c}\right)}_{\text{Free oscillation term}} + \underbrace{i\frac{Ne^2W_0}{4m_0\gamma cC\omega_\beta}\left[\frac{c}{\omega_\beta}\tilde{y}_1^*(0)\sin\left(\frac{\omega_\beta s}{c}\right) + \tilde{y}_1(0)s\exp\left(-\frac{i\omega_\beta s}{c}\right)\right]}_{\text{Driven oscillation term}}$$

30

The Strong Head Tail Instability Transfer map

$$\tilde{y_1}\left(\frac{\pi c}{\omega_s}\right) = \tilde{y_1}(0)\exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right)$$

$$\tilde{y_2}\left(\frac{\pi c}{\omega_s}\right) = \tilde{y_2}(0)\exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right) +$$
$$+i\frac{Ne^2W_0}{4m_0\gamma cC\omega_\beta}\left[\frac{c}{\omega_\beta}\tilde{y}_1^*(0)\sin\left(\frac{\pi\omega_\beta}{\omega_s}\right) + \tilde{y}_1(0)\left(\frac{\pi c}{\omega_s}\right)\exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right)\right]$$

⇒ Second term in RHS equation for $y_2(s)$ negligible if $\omega_s \ll \omega_\beta$
⇒ We can now transform these equations into linear mapping across half synchrotron period

$$\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=\pi c/\omega_s} = \exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right)\cdot\begin{pmatrix}1 & 0\\ i\Upsilon & 1\end{pmatrix}\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=0}$$

$$\Upsilon = \frac{\pi Ne^2W_0}{4m_0\gamma C\omega_\beta\omega_s}$$

31

The Strong Head Tail Instability Transfer map

⇒ In the second half of synchrotron period, particles 1 and 2 exchange their roles
⇒ We can therefore find the transfer matrix over the full synchrotron period for both particles
⇒ Stability will require that its eigenvalues be smaller than 1

$$\Upsilon = \frac{\pi Ne^2W_0}{4m_0\gamma C\omega_\beta\omega_s}$$

$$\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=2\pi c/\omega_s} = \exp\left(-\frac{i2\pi\omega_\beta}{\omega_s}\right)\cdot\begin{pmatrix}1 & i\Upsilon\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ i\Upsilon & 1\end{pmatrix}\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=0}$$

$$\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=2\pi c/\omega_s} = \exp\left(-\frac{i2\pi\omega_\beta}{\omega_s}\right)\cdot\begin{pmatrix}1-\Upsilon^2 & i\Upsilon\\ i\Upsilon & 1\end{pmatrix}\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=0}$$

32

The Strong Head Tail Instability
Stability condition

- ⇒ Since the product of the eigenvalues is 1, the only condition for stability is that they both be purely imaginary
- ⇒ From the second equation for the eigenvalues, it is clear that this is true only when sin(ϕ/2)<1
- ⇒ This translates into a condition on the beam/wake parameters

$$\lambda_1 \cdot \lambda_2 = 1 \quad \Rightarrow \quad \lambda_{1,2} = \exp(\pm i\phi)$$

$$\lambda_1 + \lambda_2 = 2 - \Upsilon^2 \quad \Rightarrow \quad \sin\left(\frac{\phi}{2}\right) = \frac{\Upsilon}{2}$$

$$\Upsilon = \frac{\pi N e^2 W_0}{4 m_0 \gamma C \omega_\beta \omega_s} \leq 2$$

33

The Strong Head Tail Instability
Stability condition

$$N_{\text{thr}} \leq \frac{8}{\pi e^2} \frac{p_0 \omega_s}{\beta_y} \left(\frac{C}{W_0}\right)$$

- ⇒ Proportional to p_0 → obviously bunches with higher energy tend to be more stable
- ⇒ Proportional to ω_s → the quicker is the longitudinal motion within the bunch, the more stable is the bunch
- ⇒ Inversely proportional to β_y → the effect of the impedance is enhanced if the kick is given at a location with large beta function

- ⇒ Inversely proportional to the wake per unit length along the ring, W_0/C → obviously a large integrated wake (impedance) lowers the instability threshold

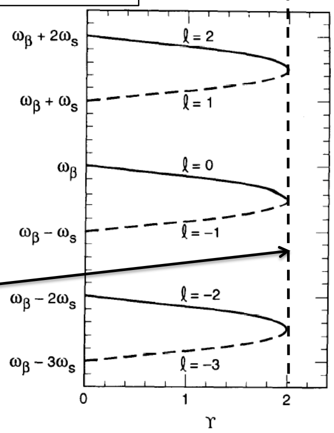
34

The Strong Head Tail Instability
Why TMCI?

- ⇒ A careful analysis of the centre of charge (centroid) motion, shows that the $(y_1+y_2)(s)$ signal is the combination of an infinite number of modes
- ⇒ The frequency of each of these modes is intensity dependent and can be expressed as function of ϕ

$$+\ \text{mode:}\qquad \omega_\beta + l\omega_s - \frac{\phi}{2\pi}\omega_s,\qquad l\ \text{even}$$

$$-\ \text{mode:}\qquad \omega_\beta + l\omega_s + \frac{\phi}{2\pi}\omega_s,\qquad l\ \text{odd}.$$



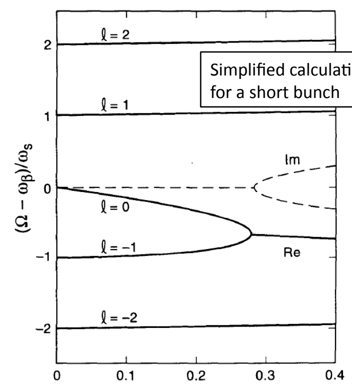
When $\phi=\pi$, all odd and even modes merge by pairs, and above this threshold the system is unstable

That's the reason why this type of instability is called Transverse Mode Coupling Instability!

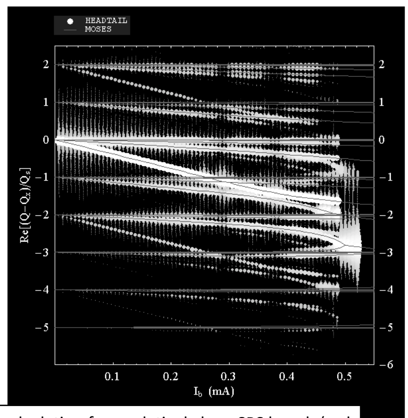
35

The Strong Head Tail Instability
Why TMCI?

- ⇒ For a real bunch, modes exhibit a more complicated shift pattern
- ⇒ The shift of the modes can be calculated via Vlasov equation or can be found through macroparticle simulations



Simplified calculation for a short bunch



Full calculation for a relatively long SPS bunch (red lines) + macroparticle simulation (white traces)

36

The Head Tail Instability

- To illustrate the head-tail instability we will need to make use of some simplifications:
  - ⇒ The bunch is represented through two particles carrying half the total bunch charge and placed in opposite phase in the longitudinal phase space
  - ⇒ They both feel external linear focusing in all three directions (i.e. linear betatron focusing + linear synchrotron focusing).
  - ⇒ **Chromaticity is different from zero ($Q'_{x,y}\neq 0$)**
  - ⇒ Constant transverse wake left behind by the leading particle
  - ⇒ Smooth approximation → constant focusing + distributed wake

- We will
  - ⇒ Show that this system is intrinsically unstable
  - ⇒ Calculate the growth time of the excited oscillation modes

37

The Head Tail Instability
Equations of motion

⇒ As for the TMCI, during the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1

⇒ During the second half of the synchrotron period, the situation is reversed

$$\begin{cases} \dfrac{d^2 y_1}{ds^2} + \left[\dfrac{\omega_\beta(1+\xi_y\delta(s))}{c}\right]^2 y_1 = 0 \\ \dfrac{d^2 y_2}{ds^2} + \left[\dfrac{\omega_\beta(1+\xi_y\delta(s))}{c}\right]^2 y_2 = \left(\dfrac{e^2}{m_0c^2}\right)\dfrac{NW_0}{2\gamma C}y_1(s) \end{cases} \qquad 0 < s < \frac{\pi c}{\omega_s}$$

Difference! → now the frequency of free oscillation is modulated by the momentum spread, δ(s)

38

The Head Tail Instability
Equations of motion

⇒ Let's first write the solution without wake field assuming a linear synchrotron motion and particles in opposite phase (z_1=-z_2)

⇒ It is already clear that head and tail of the bunch exhibit a phase difference given by the chromatic term

$$\tilde{y}_1(0)\exp\left[-i\omega_\beta\frac{s}{c}+i\frac{\xi_y\omega_\beta}{c\eta}\hat{z}\sin\left(\frac{\omega_s s}{c}\right)\right]$$

$$\tilde{y}_2(0)\exp\left[-i\omega_\beta\frac{s}{c}-i\frac{\xi_y\omega_\beta}{c\eta}\hat{z}\sin\left(\frac{\omega_s s}{c}\right)\right]$$

$\dfrac{\xi_y\omega_\beta\hat{z}}{c\eta}$ is the head-tail phase shift

39

The Head Tail Instability
Equations of motion

⇒ The free oscillation is the correct solution for $y_1(s)$ in the first half synchrotron period

⇒ For $y_2(s)$ we assume a similar type of solution, allowing for a slowly time varying coefficient

⇒ Substituting into the equation of motion this yields

$$\tilde{y}_1(0)\exp\left[-i\omega_\beta\frac{s}{c}+i\frac{\xi_y\omega_\beta}{c\eta}\hat{z}\sin\left(\frac{\omega_s s}{c}\right)\right]$$

$$\tilde{y}_2(s)\exp\left[-i\omega_\beta\frac{s}{c}+i\frac{\xi_y\omega_\beta}{c\eta}\hat{z}\sin\left(\frac{\omega_s s}{c}\right)\right]$$

$$\tilde{y}_2'(s)\approx\left(\frac{e^2}{m_0c}\right)\frac{NW_0}{4\gamma C\omega_\beta}\tilde{y}_1(0)\exp\left[2i\frac{\xi_y\omega_\beta}{c\eta}\hat{z}\sin\left(\frac{\omega_s s}{c}\right)\right]$$

40

The Head Tail Instability
Transfer map

⇒ For small head-tail shifts, we can expand the exponential in Taylor series and find an expression for $y_2(s)$

⇒ We can write a transfer map over the first half of synchrotron period in the same form as was done for the study of the TMCI

⇒ This time ϒ is a complex parameter!

$$\tilde{y}_2(s) \approx \tilde{y}_2(0) + \left(\frac{e^2}{m_0 c}\right)\frac{NW_0}{4\gamma C\omega_\beta}\tilde{y}_1(0)\left[s + i\frac{2\xi_y\omega_\beta\hat{z}}{\eta\omega_s}\left(1-\cos\frac{\omega_s s}{c}\right)\right]$$

$$\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=\pi c/\omega_s} = \begin{pmatrix}1 & 0\\ i\Upsilon & 1\end{pmatrix}\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=0}$$

$$\Upsilon = \frac{\pi N e^2 W_0}{4 m_0\gamma C\omega_\beta\omega_s}\left(1 + i\frac{4\xi_y\omega_\beta\hat{z}}{\pi c\eta}\right)$$

41

The Head Tail Instability
Oscillation modes

⇒ For weak beam intensities (|ϒ|<<1), we have simple expressions for the eigenvalues of the transfer matrix

⇒ With ϒ complex, even for low bunch intensities there is no possible stable solution

$$\Upsilon = \frac{\pi N e^2 W_0}{4 m_0\gamma C\omega_\beta\omega_s}\left(1 + i\frac{4\xi_y\omega_\beta\hat{z}}{\pi c\eta}\right)$$

$$\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=2\pi c/\omega_s} = \begin{pmatrix}i\Upsilon & 1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=\pi c/\omega_s} = \begin{pmatrix}1-\Upsilon^2 & i\Upsilon\\ i\Upsilon & 1\end{pmatrix}\cdot\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix}_{s=0}$$

$|\Upsilon| \ll 1$ ⇒

$$\lambda_\pm \approx \exp(\pm i\Upsilon)$$

+ mode is "in-phase" mode → the two particles oscillate in phase (ω_β)

- mode is "out-phase" mode → the two particles oscillate in opposition of phase ($\omega_\beta \pm \omega_s$)

42

The Head Tail Instability
Growth/damping time

$$\tau^{-1} = \mathrm{Im}\left(\pm\Upsilon \cdot \frac{\omega_s}{2\pi}\right) = \mp\frac{e^2}{2\pi} \cdot \frac{N\xi_y\hat{z}}{p_0\eta}\left(\frac{W_0}{C}\right)$$

- ⇒ Inversely proportional to p_0 → obviously bunches with higher energy tend to be less affected by impedances
- ⇒ Proportional to N → the more intense is the bunch, the more sensitive it is
- ⇒ Proportional to bunch length → this depends on the chosen shape of the wake
- ⇒ Proportional to ξ_y → higher chromaticity enhances the head-tail effect
- ⇒ Inversely proportional to η → the head-tail exchange through synchrotron motion is an essential mechanism

⇒ Proportional to the wake per unit length along the ring, W_0/C → obviously a large integrated wake (impedance) gives a stronger effect

43

The Head Tail Instability
Growth/damping time

$$\tau^{-1} = \mathrm{Im}\left(\pm\Upsilon \cdot \frac{\omega_s}{2\pi}\right) = \mp\frac{e^2}{2\pi} \cdot \frac{N\xi_y\hat{z}}{p_0\eta}\left(\frac{W_0}{C}\right)$$

Mode 0 (+)

	$\xi_y>0$	$\xi_y<0$
Above transition (η>0)	damped	unstable
Below transition (η<0)	unstable	damped

Mode 1 (−)

	$\xi_y>0$	$\xi_y<0$
Above transition (η>0)	unstable	damped
Below transition (η<0)	damped	unstable

44

The Head Tail Instability

- The head-tail instability is unavoidable in the two-particle model
  - Either mode 0 or mode 1 is unstable
  - Growth/damping times are in all cases identical
- Fortunately, the situation is less dramatic in reality
  - The number of modes increases with the number of particles we consider in the model (and becomes infinite in the limit of a continuous bunch)
  - The instability conditions for mode 0 remain unchanged, but all the other modes become unstable with much longer rise times when mode 0 is stable

Mode 0

	$\xi_y>0$	$\xi_y<0$
Above transition ($\eta>0$)	damped	unstable
Below transition ($\eta<0$)	unstable	damped

$$\sum_{l=-\infty}^{\infty} \frac{1}{\tau_l} = 0$$

All modes >0

	$\xi_y>0$	$\xi_y<0$
Above transition ($\eta>0$)	unstable	damped
Below transition ($\eta<0$)	damped	unstable

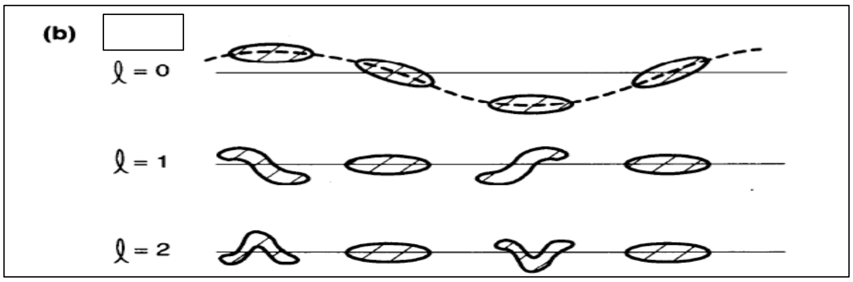
The Head Tail Instability

- The head-tail instability is unavoidable in the two-particle model
  - Either mode 0 or mode 1 is unstable
  - Growth/damping times are in all cases identical
- Fortunately, the situation is less dramatic in reality
  - The number of modes increases with the number of particles we consider in the model (and becomes infinite in the limit of a continuous bunch)
  - The instability conditions for mode 0 remain unchanged, but all the other modes become unstable with much longer rise times when mode 0 is stable

  - Therefore, the bunch can be in practice stabilized by using the settings that make mode 0 stable ($\xi<0$ below transition and $\xi>0$ above transition) and relying on feedback or Landau damping for the other modes
- To be able to study these effects we would need to resort to a more detailed description of the bunch
  - Vlasov equation (kinetic model)
  - Macroparticle simulations

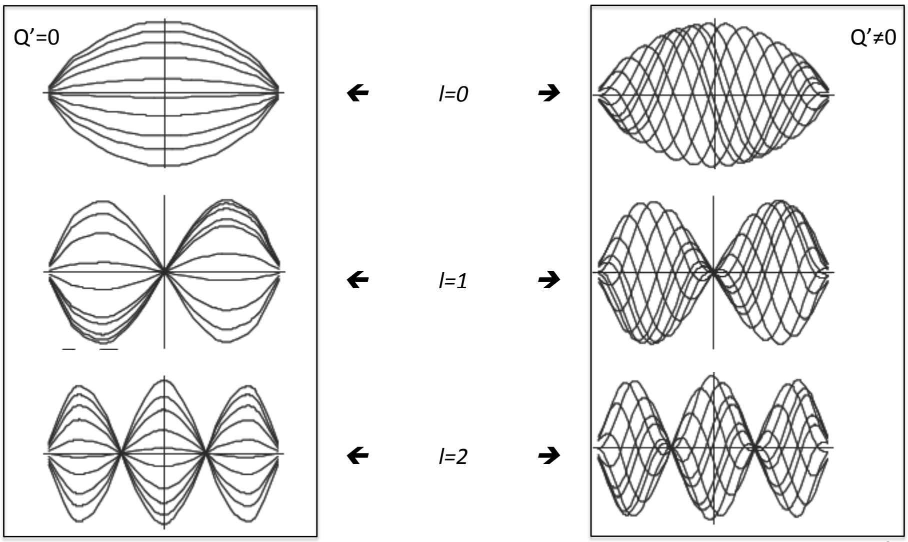
A glance into the head-tail modes

- Different transverse head-tail modes correspond to different parts of the bunch oscillating with relative phase differences. E.g.
  - Mode 0 is a rigid bunch mode
  - Mode 1 has head and tail oscillating in counter-phase
  - Mode 2 has head and tail oscillating in phase and the bunch center in opposition



47

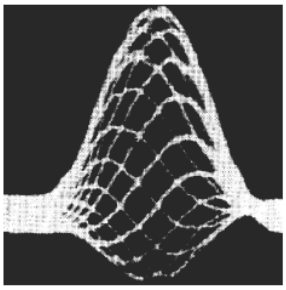
A glance into the head-tail modes (as seen at a wide-band BPM)



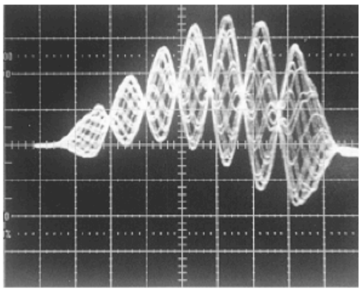
48

A glance into the head-tail modes (experimental observations)

Observation in the CERN PSB in ~1974
(J. Gareyte and F. Sacherer)



Observation in the CERN PS in 1999

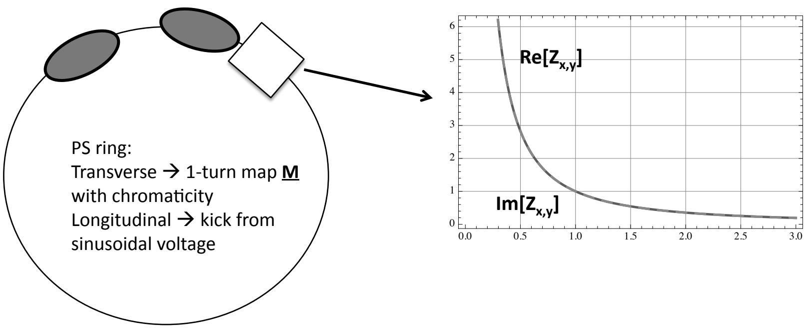


- The mode that gets first excited in the machine depends on
  - The spectrum of the exciting impedance
  - The chromaticity setting
- Head-tail instabilities are a good diagnostics tool to identify and quantify the main impedance sources in a machine

49

Macroparticle simulation

- We have simulated the evolution of a long PS bunch under the effect of a transverse resistive wall impedance lumped in one point of the ring
- We have used parameters at injection (below transition!) and three different chromaticity values: $\xi_{x,y}$= ±0.15, -0.3



50

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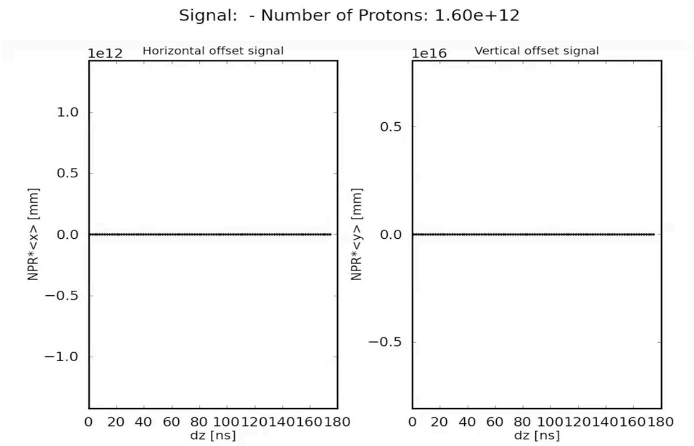

51

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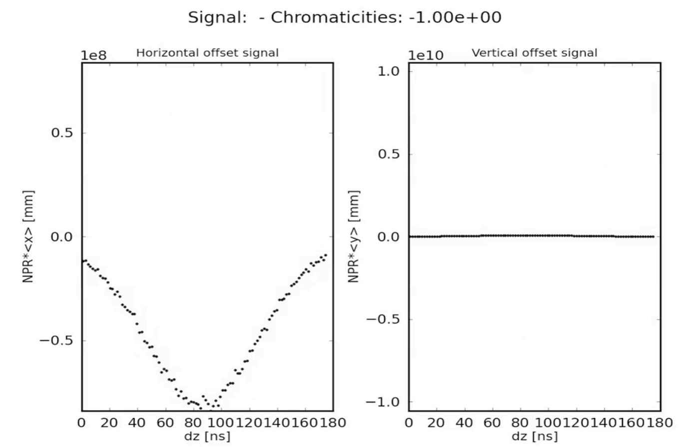

52

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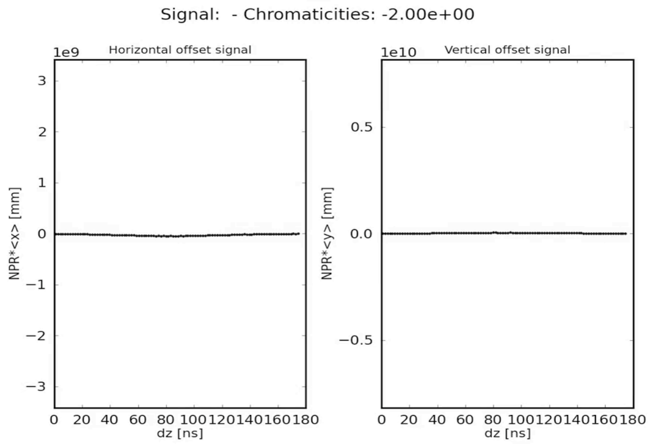

53