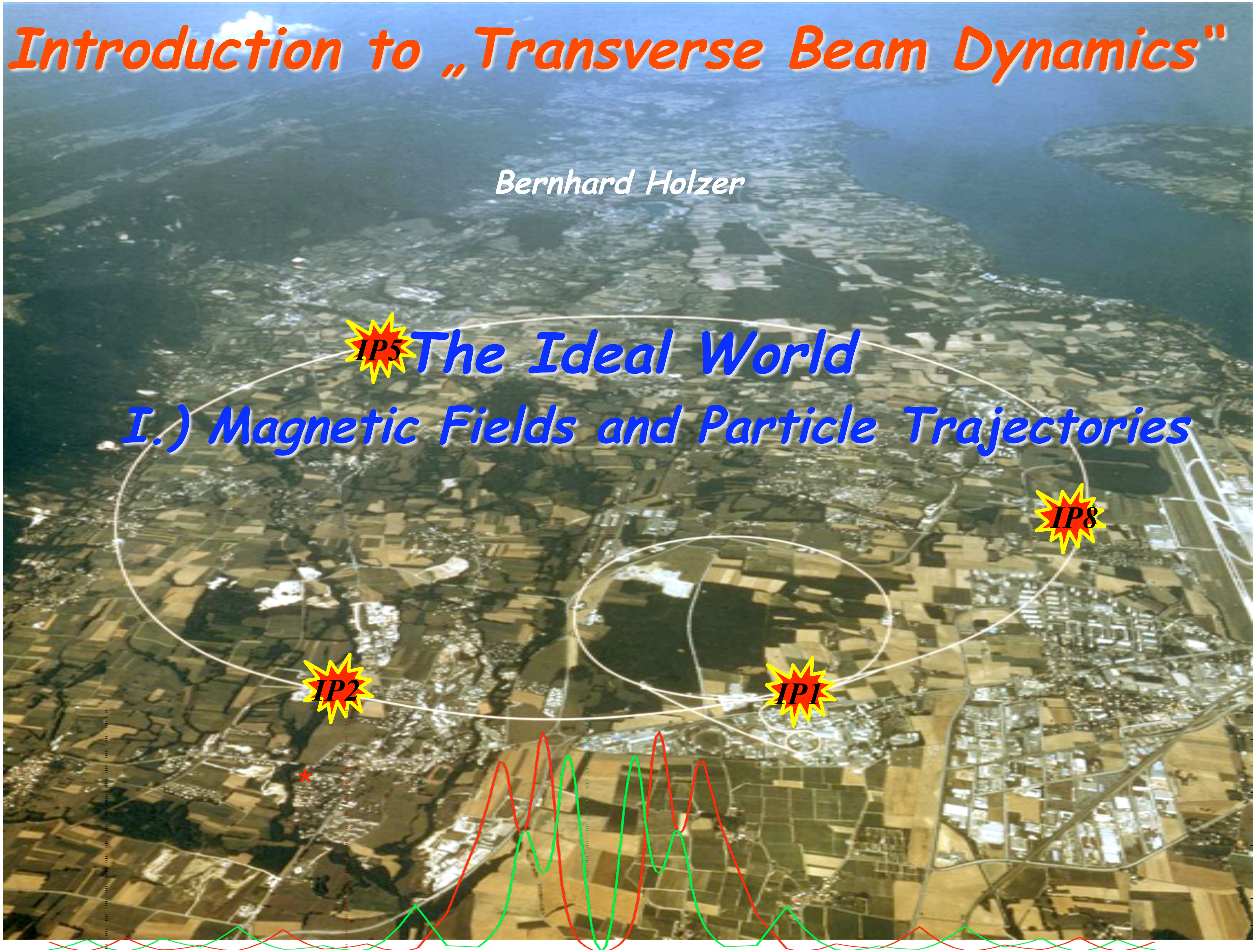
Introduction to „Transverse Beam Dynamics“

Bernhard Holzer

The Ideal World

I.) Magnetic Fields and Particle Trajectories



Transverse Beam Dynamics I

„ ... and so I hope that everybody will find something useful in these lectures ...

... be it physics, entertainement or ... consolation"

I.) Linear Beam Optics

Single Particle Trajectories

Magnets and Focusing Fields

Tune & Orbit

Luminosity Run of a typical storage ring:

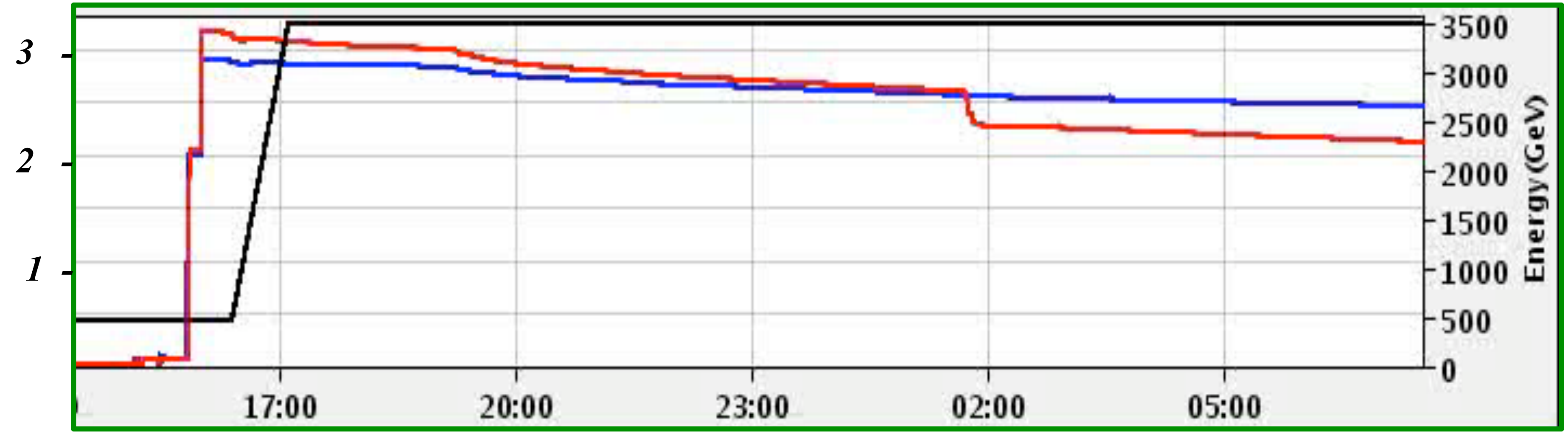
*LHC Storage Ring: Protons **accelerated and** stored for 12 hours*

distance of particles travelling at about v ≈ c

$L = 10^{10}$-10^{11} km

... several times Sun - Pluto and back ♪

intensity (10^{11})



→ *guide the particles **on a well defined orbit** („design orbit“)*

→ *focus the particles **to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.***

1.) Introduction and Basic Ideas

„ … in the end and after all it should be a kind of circular machine"

→ need transverse deflecting force

Lorentz force $\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$

typical velocity in high energy machines: $v \approx c \approx 3 * 10^8 \, m/s$

Example:

$$B = 1\,T \quad \rightarrow \quad F = q * 3 * 10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}$$

$$F = q * \underbrace{300 \frac{MV}{m}}_{E}$$

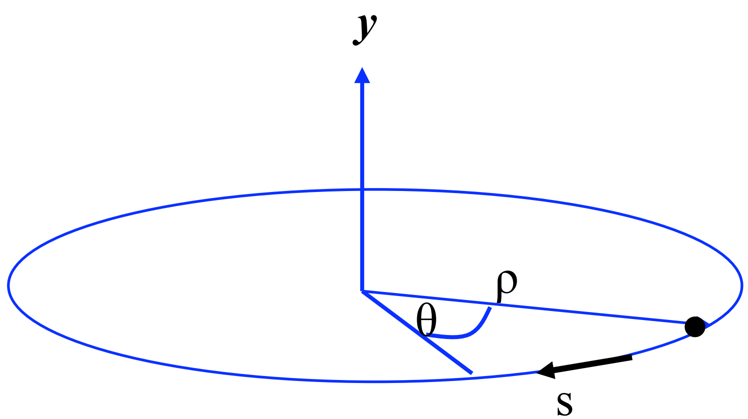
equivalent electrical field E

technical limit for electrical field

$$E \leq 1 \frac{MV}{m}$$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $$F_L = e\,v\,B$$

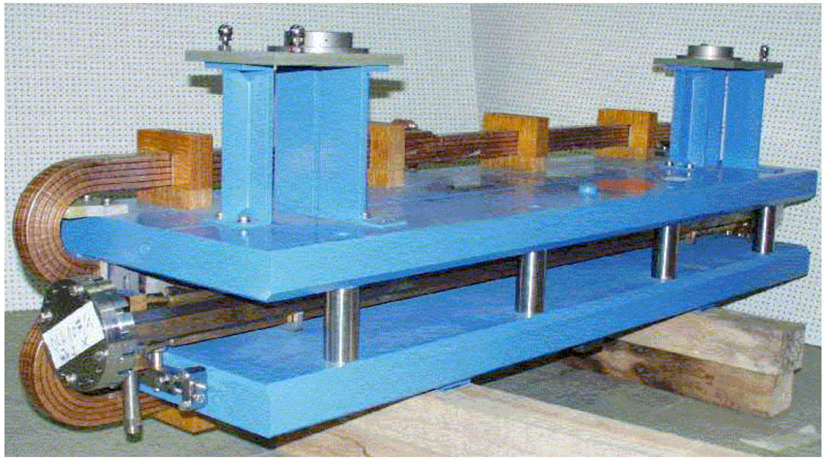
centrifugal force $$F_{centr} = \frac{\gamma\, m_0\, v^2}{\rho}$$

$$\frac{\gamma\, m_0\, v^{\cancel{2}}}{\rho} = e\,\cancel{v}\,B$$

$$\boxed{\frac{p}{e} = B\,\rho}$$

$B\rho$ = "beam rigidity"

2.) The Magnetic Guide Field



Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes

$$B = \frac{\mu_0 \, n \, I}{h}$$

$$\frac{\boldsymbol{p}}{\boldsymbol{e}} = \boldsymbol{B}\,\rho \longrightarrow \rho = \frac{p}{B * e}$$

The bending radius ... and so the size of the machine is determined by the dipole field and the particle momentum

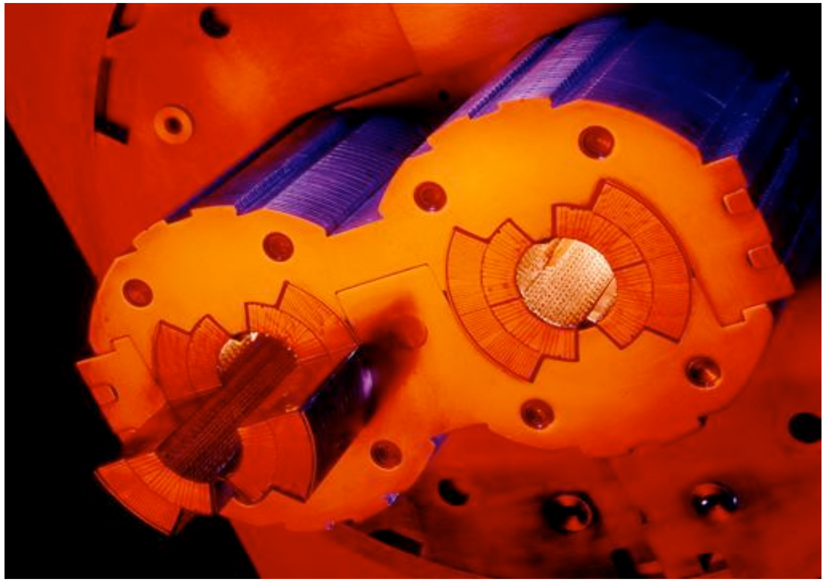
convenient units:

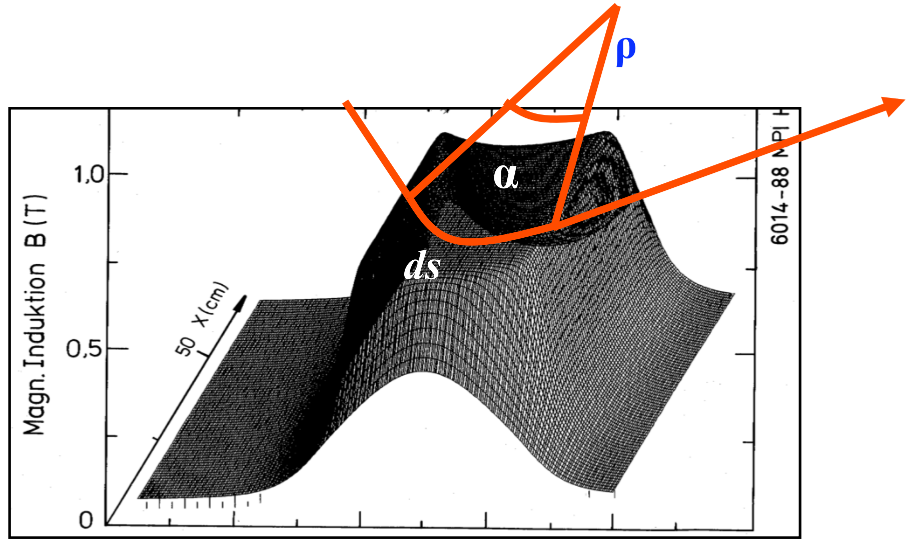
$$\boldsymbol{B} = [\boldsymbol{T}] = \left[\frac{\boldsymbol{Vs}}{\boldsymbol{m}^2}\right] \qquad \boldsymbol{p} = \left[\frac{\boldsymbol{GeV}}{\boldsymbol{c}}\right]$$

Example LHC:

$$\left.\begin{array}{l} \boldsymbol{B} = 8.3\,\boldsymbol{T} \\ \boldsymbol{p} = 7000\,\dfrac{\boldsymbol{GeV}}{\boldsymbol{c}} \end{array}\right\} \qquad \rho = 2.53\,\boldsymbol{km}$$

The Magnetic Guide Field





field map of a storage ring dipole magnet

$$B \approx 1 \ldots 8\ T$$

The dipole magnets of a storage ring (or synchrotron) create a circle (... better polygon) of circumference 2πρ and define the maximum momentum of the particle beam.

Example LHC: → $2\pi\rho = 17.6\ km$
$\approx 66\%$

About 1/3 of the ring size is still needed for straight sections, rf cavities, diagnostics, injection, extraction, high energy physics detectors etc etc

The Problem:

LHC Design Magnet current: I=11850 A

and the machine is 27 km long !!!

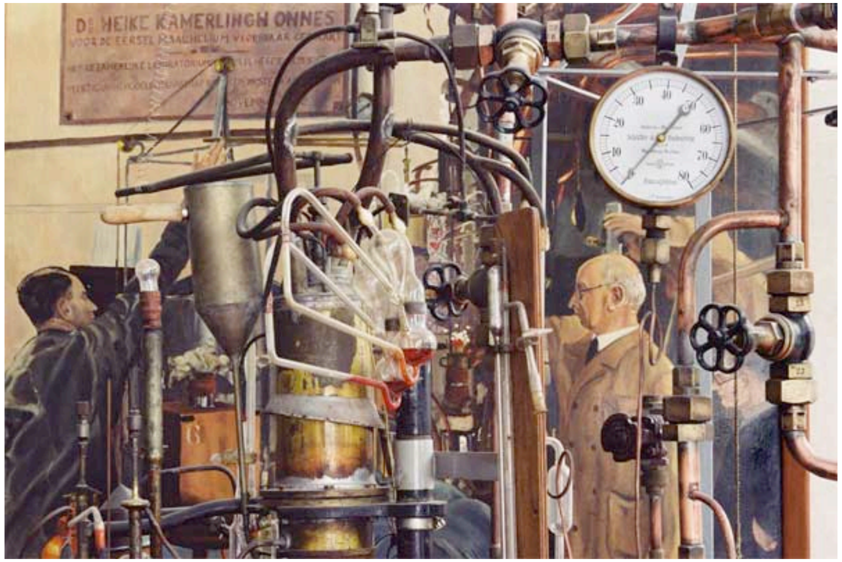
Ohm's law: $U = R * I, \qquad P = R * I^2$

Problem:

reduce ohmic losses to the absolute minimum

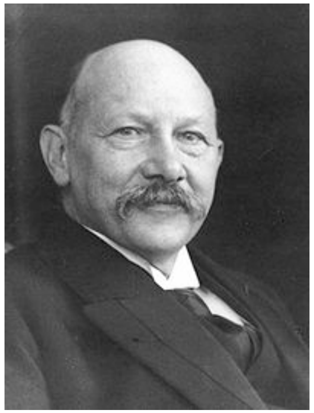
The Solution:

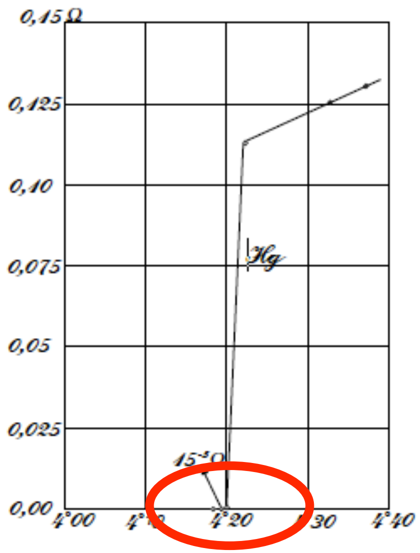
super conductivity

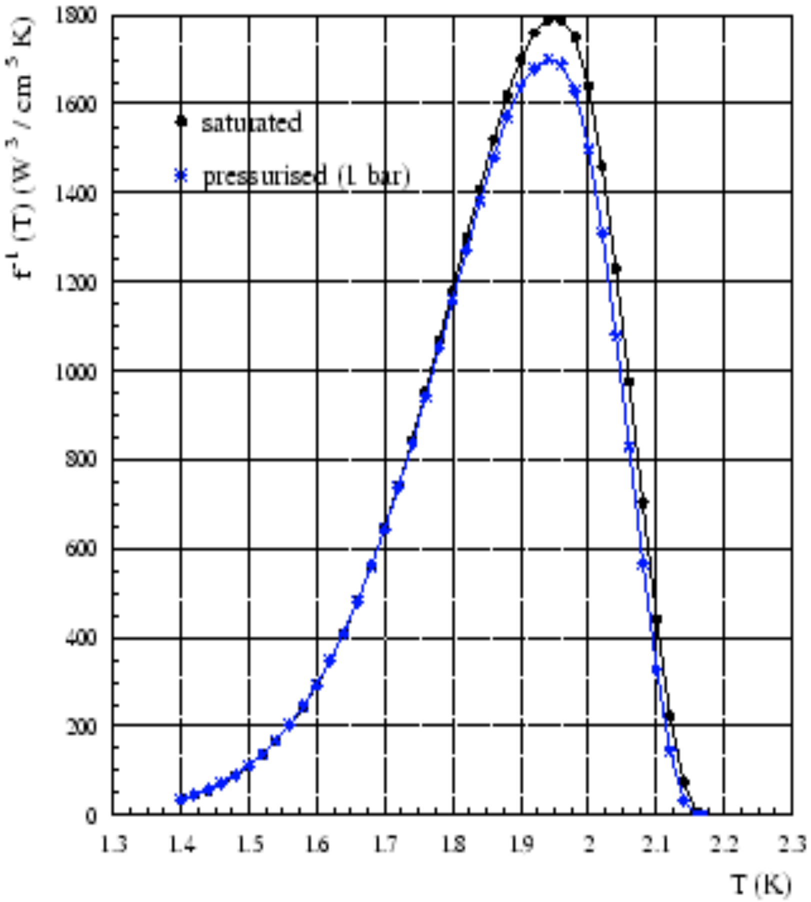


Super Conductivity ...
... and why we run at 1.9 K

discovery of sc. by H. Kammerling Onnes, Leiden 1911

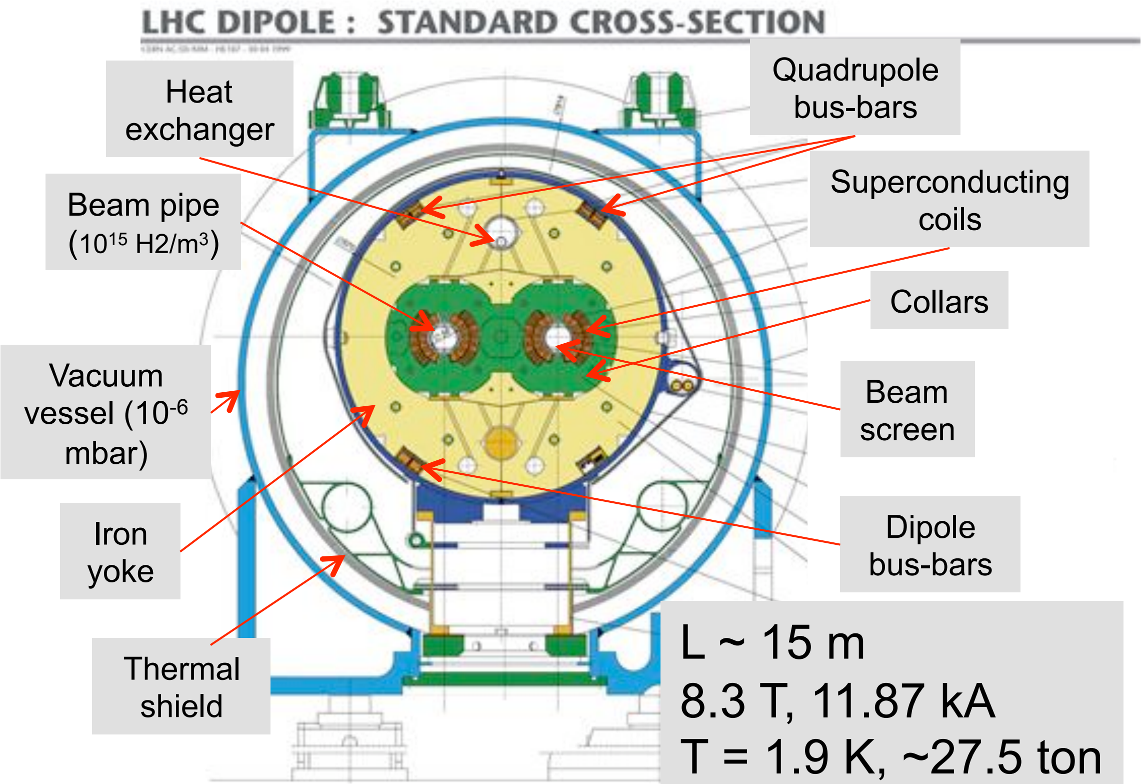




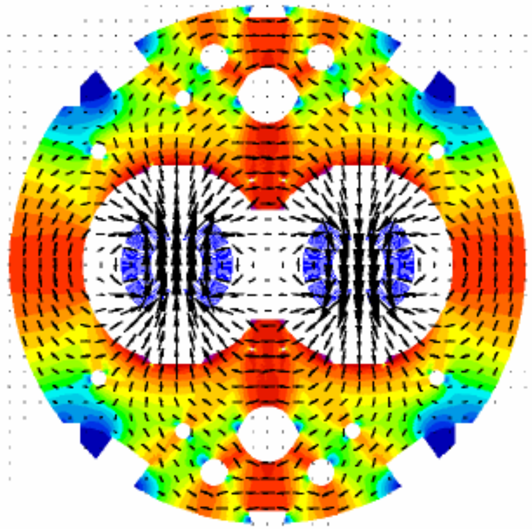


thermal conductivity of fl. Helium in supra fluid state

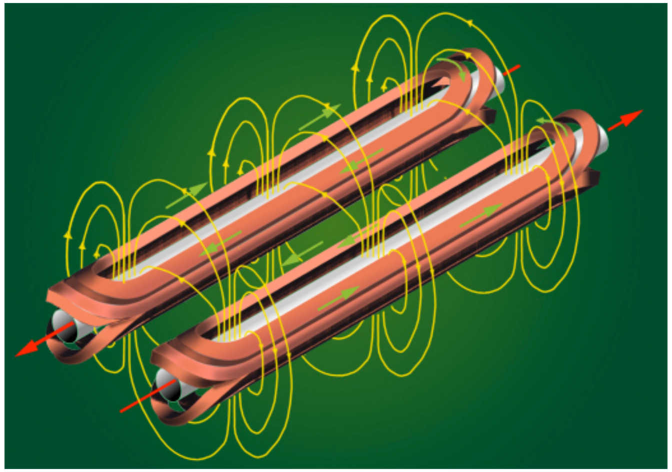
LHC: The -1232- Main Dipole Magnets

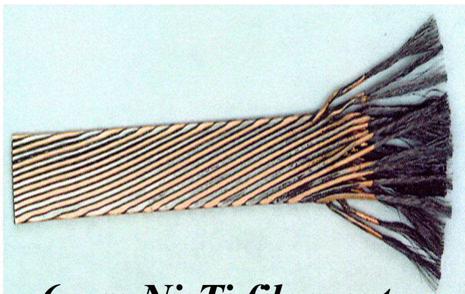


L ~ 15 m
8.3 T, 11.87 kA
T = 1.9 K, ~27.5 ton

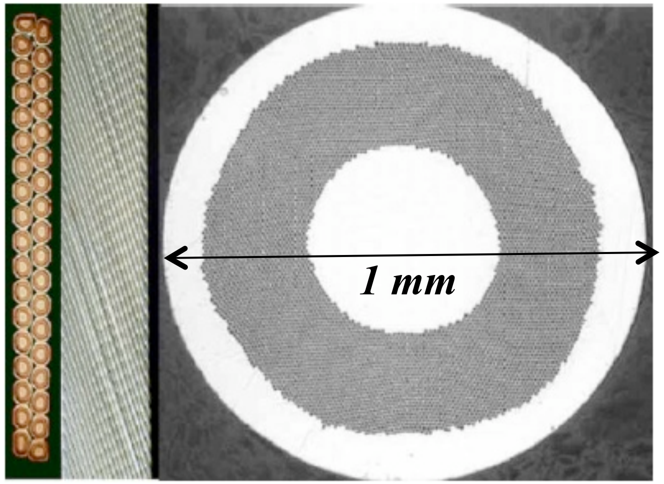


required field quality:
$\Delta B/B=10^{-4}$





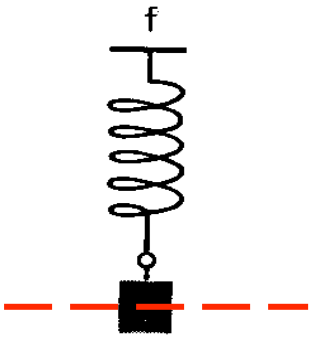
6 μm Ni-Ti filament



3.) Focusing Properties – Transverse Beam Optics

... keeping the flocs together:
In addition to the pure bending of the beam
we have to keep 10^{11} particles close together

classical mechanics:
pendulum



there is a restoring force, proportional to the elongation x:

$$F = m * a$$

$$= m * \frac{d^2 x}{dt^2} = -c * x$$

general solution: free harmonic oszillation of a pendulum

$$x(t) = A * \cos(\omega t + \varphi)$$

Quadrupole Magnets:

In a Storage Ring: ***we need a Lorentz force that rises as a function of the distance to .... ? ...... the design orbit***

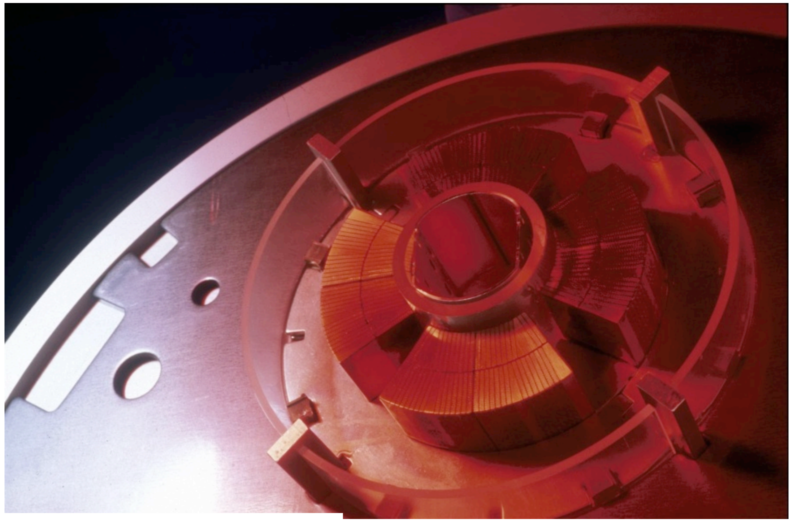
$$F(x) = q * v * B(x)$$

required: ***focusing forces to keep trajectories in vicinity of the ideal orbit***

linear increasing Lorentz force

linear increasing magnetic field

$$B_y = g\,x \qquad B_x = g\,y$$



LHC main quadrupole magnet

Integrated Gradient	690	T
Nominal Temperature	1.9	K
Nominal Gradient	223	T/m
Peak Field in Conductor	6.85	T
Temperature Margin	2.19	K
Working Point on Load Line	80.3	%
Nominal Current	11870	A
Magnetic Length	3.10	M
Beam Separation distance (cold)	194.0	mm

$$g \approx 25 \ldots 220 \;\; T/m$$

Focusing forces and particle trajectories:

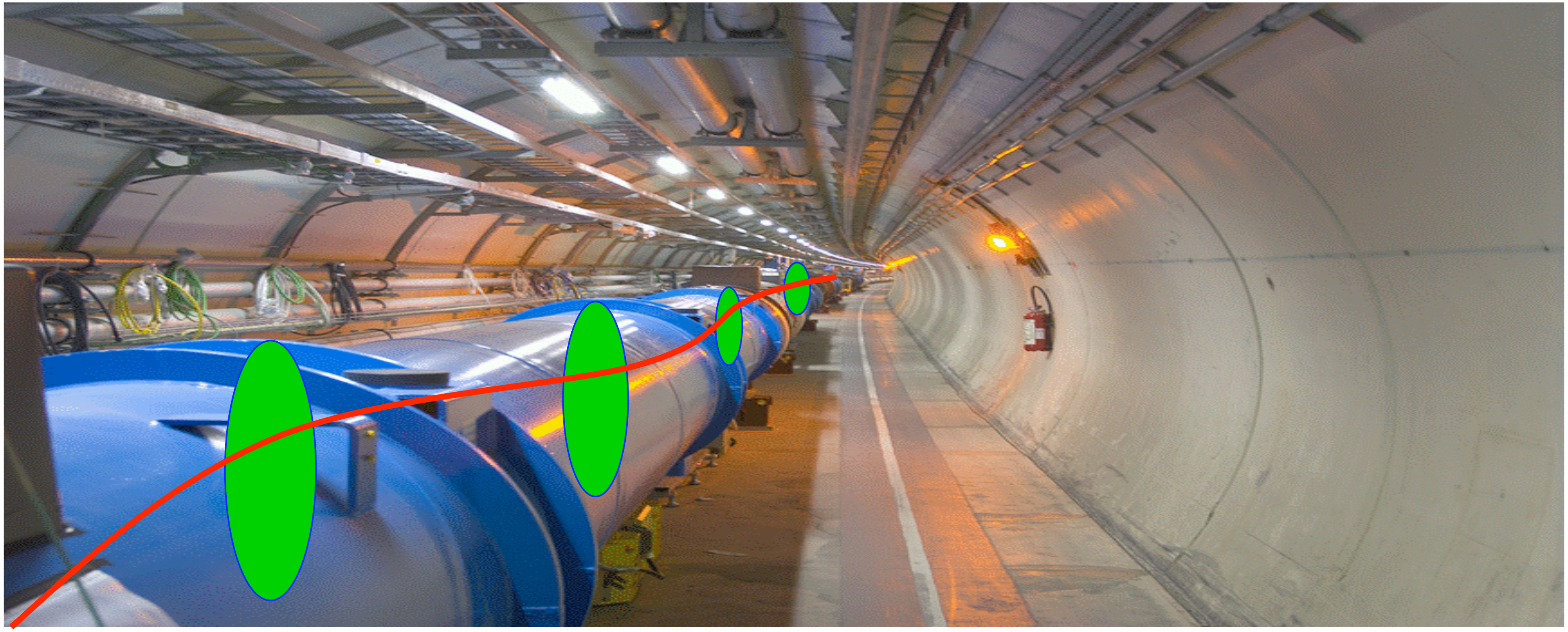
normalise magnet fields to momentum
(remember: ***B*ρ = p / q*** *)*

Dipole Magnet

$$\frac{B}{p/q} = \frac{B}{B\rho} = \frac{1}{\rho}$$

Quadrupole Magnet

$$k := \frac{g}{p/q}$$



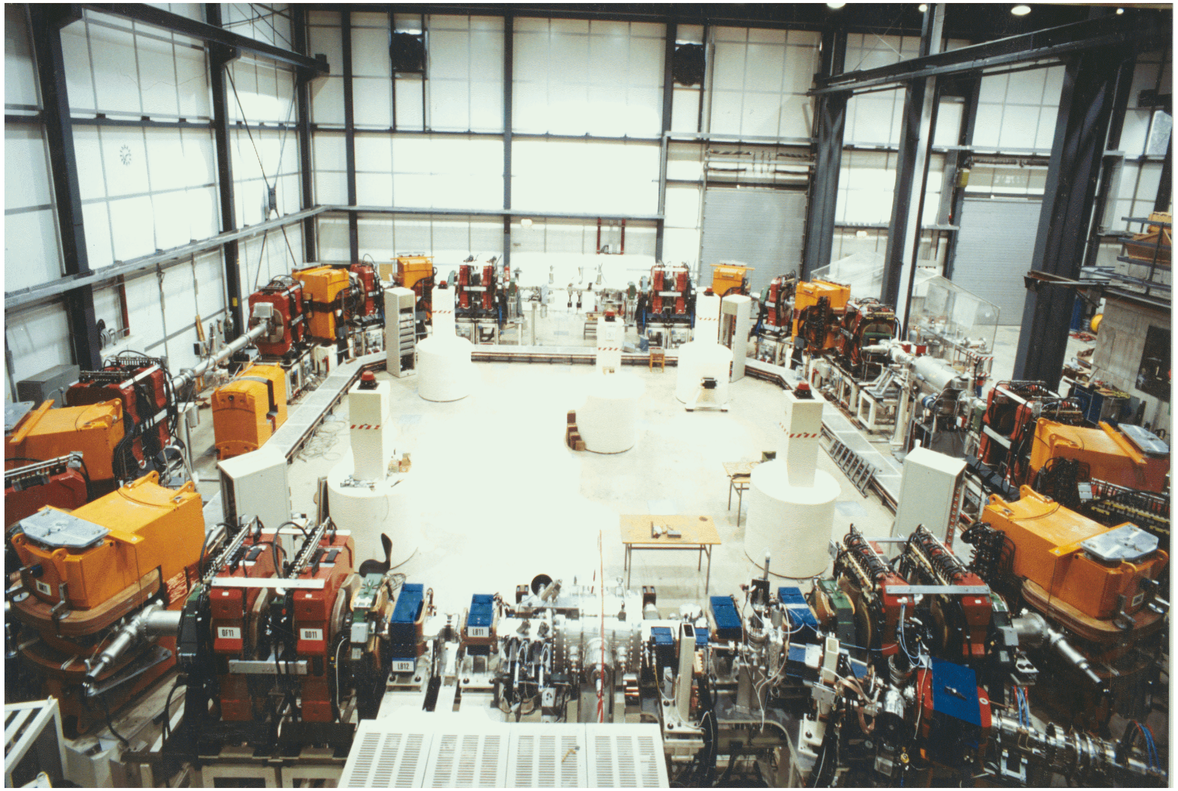
4.) A Bit of Theory

The large Storage Rings and „Synchrotrons“

The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \cancel{\frac{1}{2!}m\,x^2} + \cancel{\frac{1}{3!}n\,x^3} + \ldots$$

only terms linear in x, y taken into account *dipole fields*
quadrupole fields



Separate Function Machines:

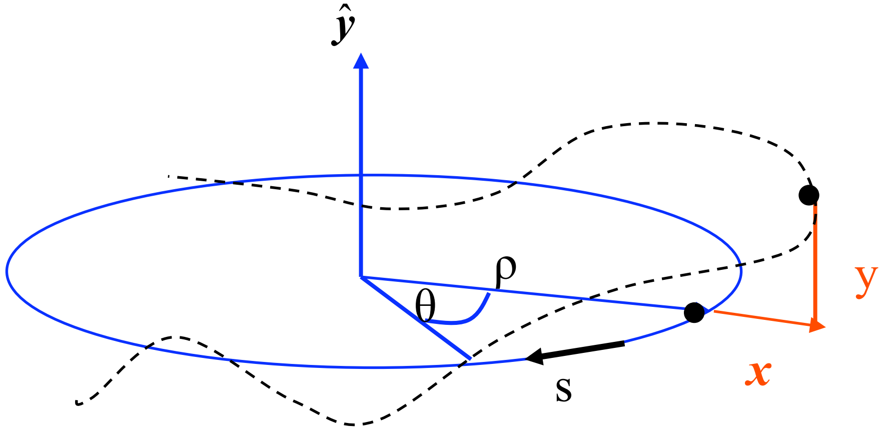
Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

* *man sieht nur dipole und quads → linear*

The Equation of Motion:

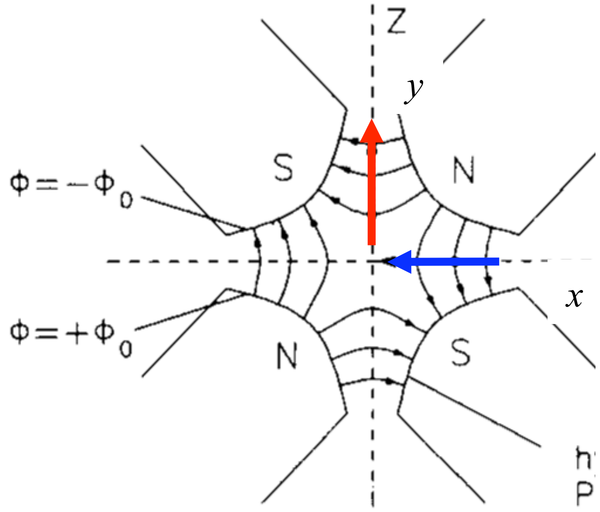


* *Equation for the horizontal motion:*

$$x'' + x\left(\frac{1}{\rho^2} + k\right) = 0$$

***x* = particle amplitude**
***x'* = angle of particle trajectory (wrt ideal path line)**

* *Equation for the vertical motion:*



$\frac{1}{\rho^2} = 0$ ***no dipoles … in general …***

$k \leftrightarrow -k$ ***quadrupole field changes sign***

$$y'' - k\, y = 0$$

5.) Solution of Trajectory Equations

Define … hor. plane: $K = 1/\rho^2 + k$

… vert. Plane: $K = -k$

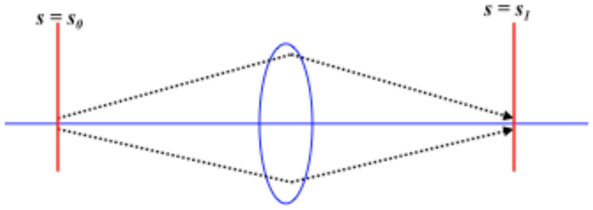
$$\boxed{x'' + K\,x = 0}$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: ***Hor. Focusing Quadrupole K > 0:***

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x_0' \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x_0' \cdot \cos(\sqrt{|K|}s)$$



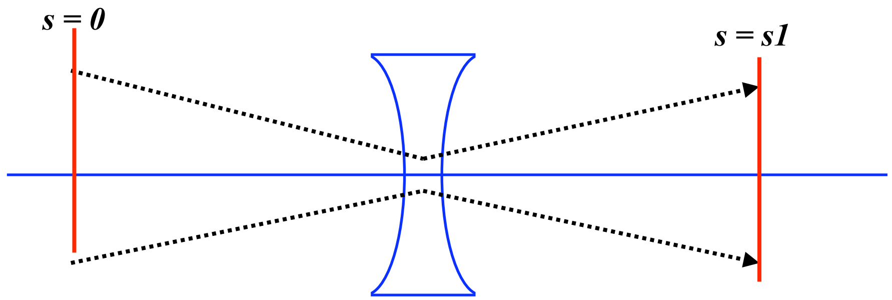
For convenience expressed in matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M_{foc} = \begin{pmatrix} \cos\left(\sqrt{|K|}l\right) & \frac{1}{\sqrt{|K|}} \sin\left(\sqrt{|K|}l\right) \\ -\sqrt{|K|} \sin\left(\sqrt{|K|}l\right) & \cos\left(\sqrt{|K|}l\right) \end{pmatrix}$$

hor. defocusing quadrupole:

$$x'' - K\,x = 0$$



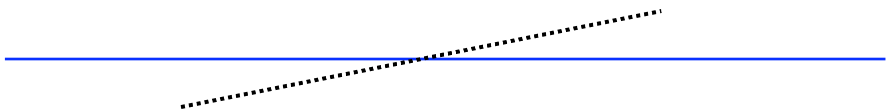
Ansatz: ***Remember from school***

$$x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$$

$$M_{defoc} = \begin{pmatrix} \cosh \sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}} \sinh \sqrt{|K|}\,l \\ \sqrt{|K|} \sinh \sqrt{|K|}\,l & \cosh \sqrt{|K|}\,l \end{pmatrix}$$

drift space:

$$K = 0$$



$$x(s) = x_0' * s$$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“

„veni vidi vici ...“ .... or in english .... „we got it !“

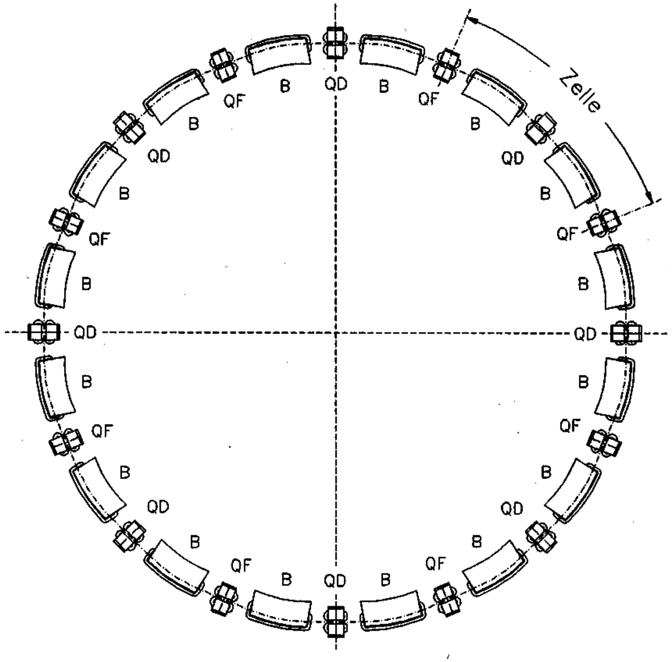
* we can calculate the trajectory of a single particle, inside a storage ring magnet (lattice element)

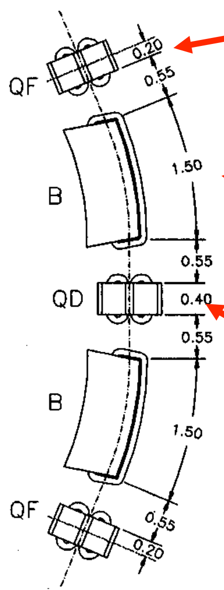
* for arbitrary initial conditions x_0, x'_0

* we can combine these trajectory parts (also mathematically) and so get the complete transverse trajectory around the storage ring

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*} .....$$

Beispiel:
Speichering für Fußgänger
(Wille)





horizontal focussing quadrupole lens

dipole magnet

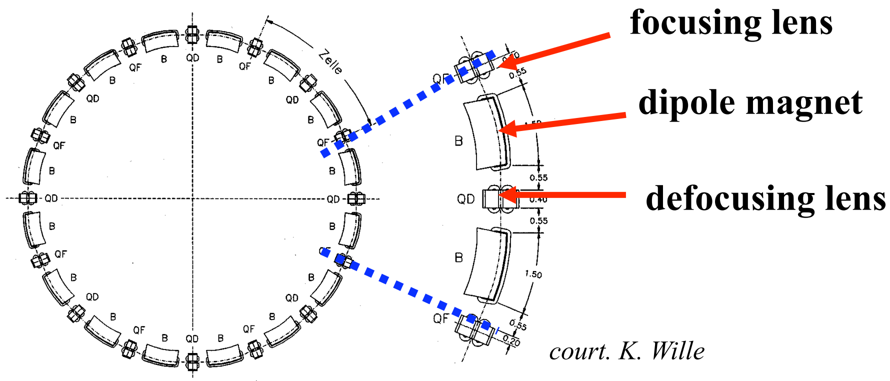
horizontal defokussing quadrupole lens

Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$

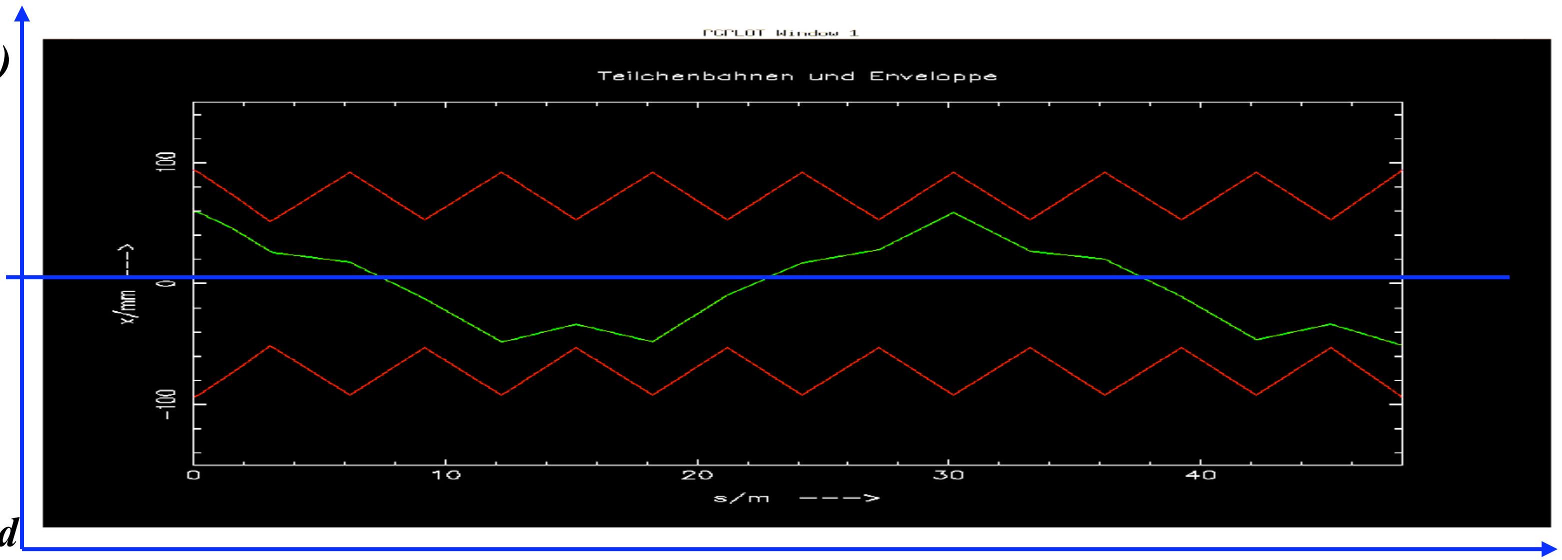


court. K. Wille

in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator ! ! !

x(s)

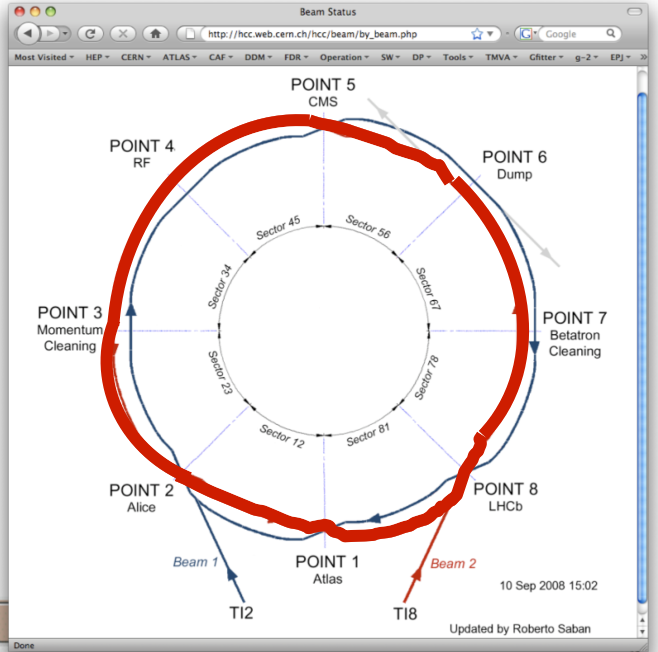
0



typical values in a strong foc. machine:

$x \approx mm,\ x' \leq mrad$

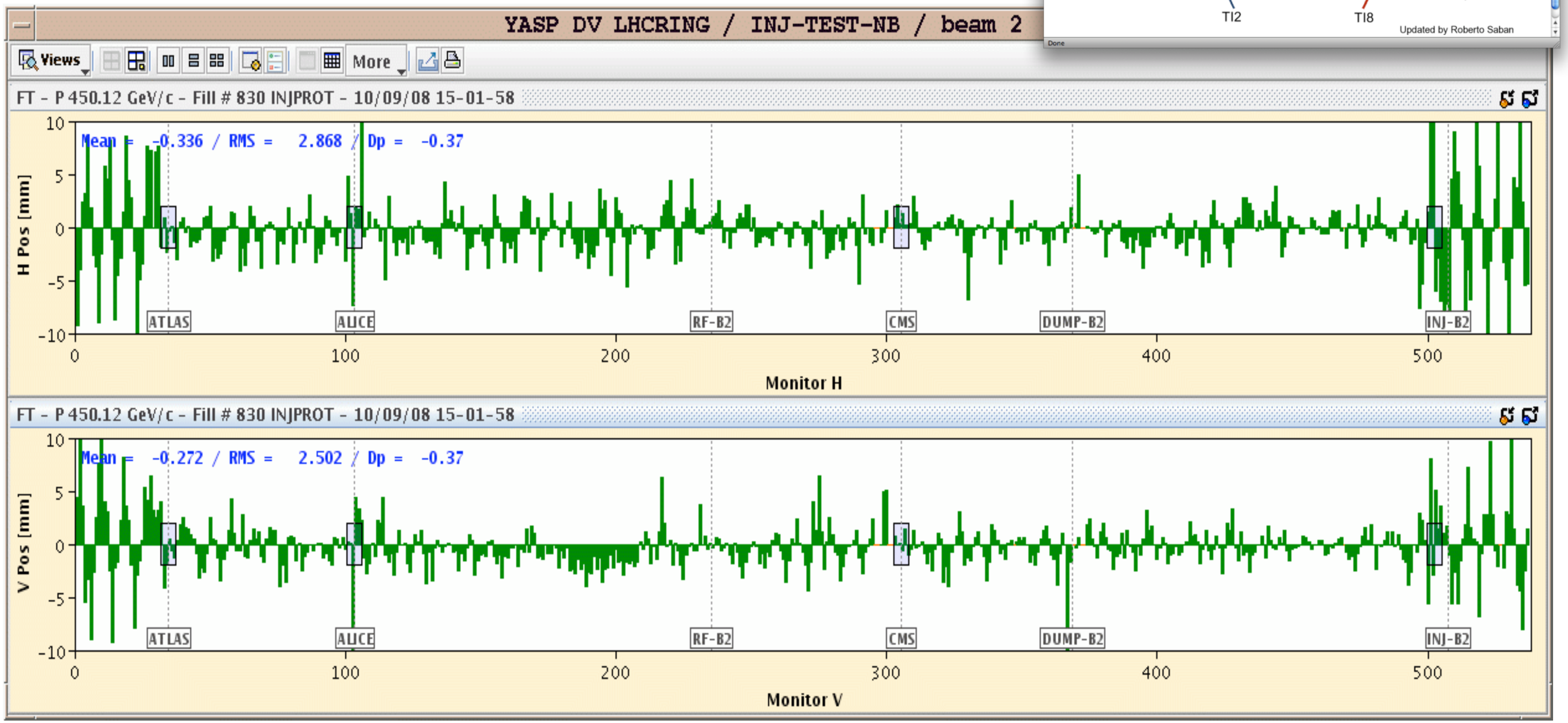
LHC Operation: Beam Commissioning



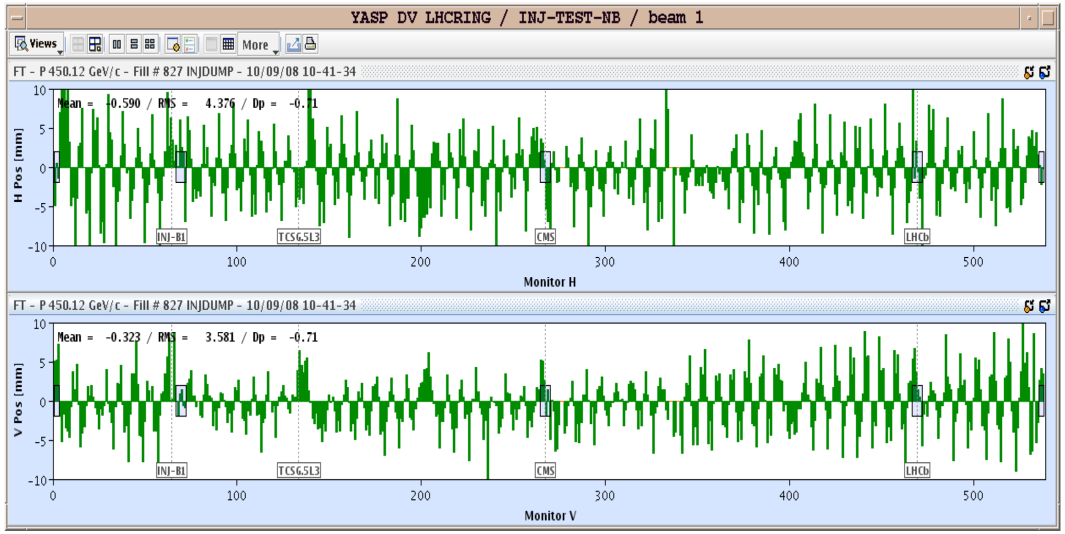
First turn steering "by sector:"

- One beam at the time
- Beam through 1 sector (1/8 ring),

correct trajectory, open collimator and move on.



6.) Orbit & Tune:



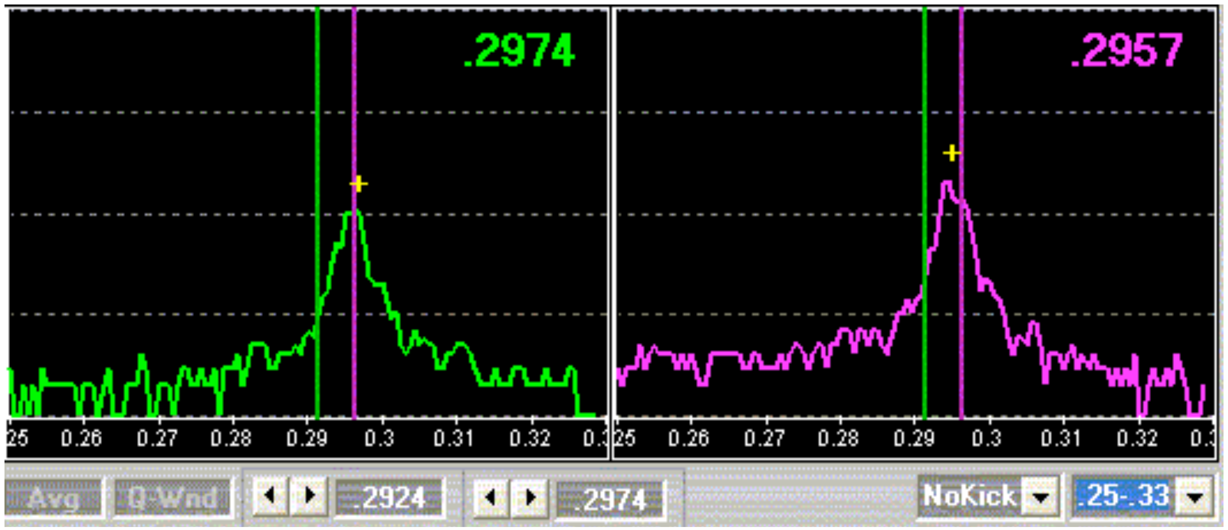
Tune: number of oscillations per turn

64.31
59.32

***Relevant** for beam stability:*
non integer part

LHC revolution frequency:** **11.3 kHz

$$0.31 * 11.3 = 3.5kHz$$



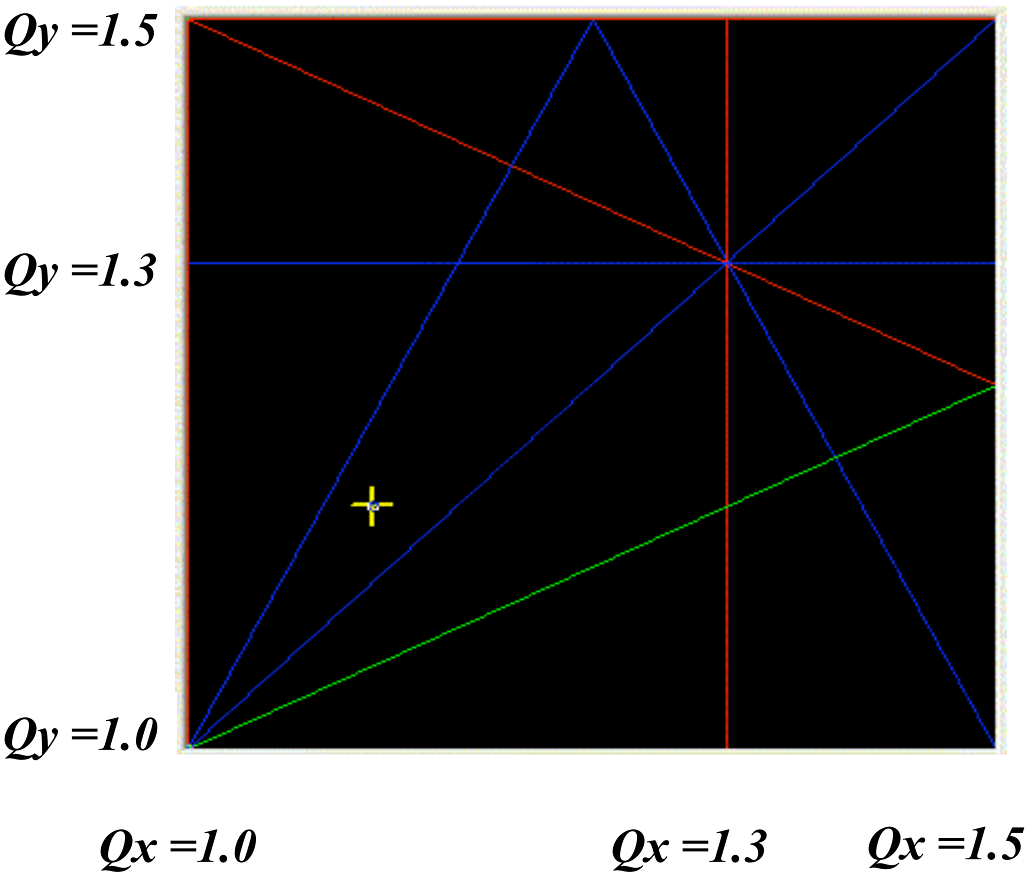
... and the tunes in x and y are different.

i.e. we can apply different focusing forces in the two planes

i.e. we can create different beam sizes in the two planes

Tune and Resonances

To avoid resonance conditions the frequency of the transverse motion must not be equal (or a integer multiple) of the revolution frequency



$1*Q_x = 1 \quad -> \quad Q_x = 1$

$2*Q_x = 1 \quad -> \quad Q_x = 0.5$

in general:

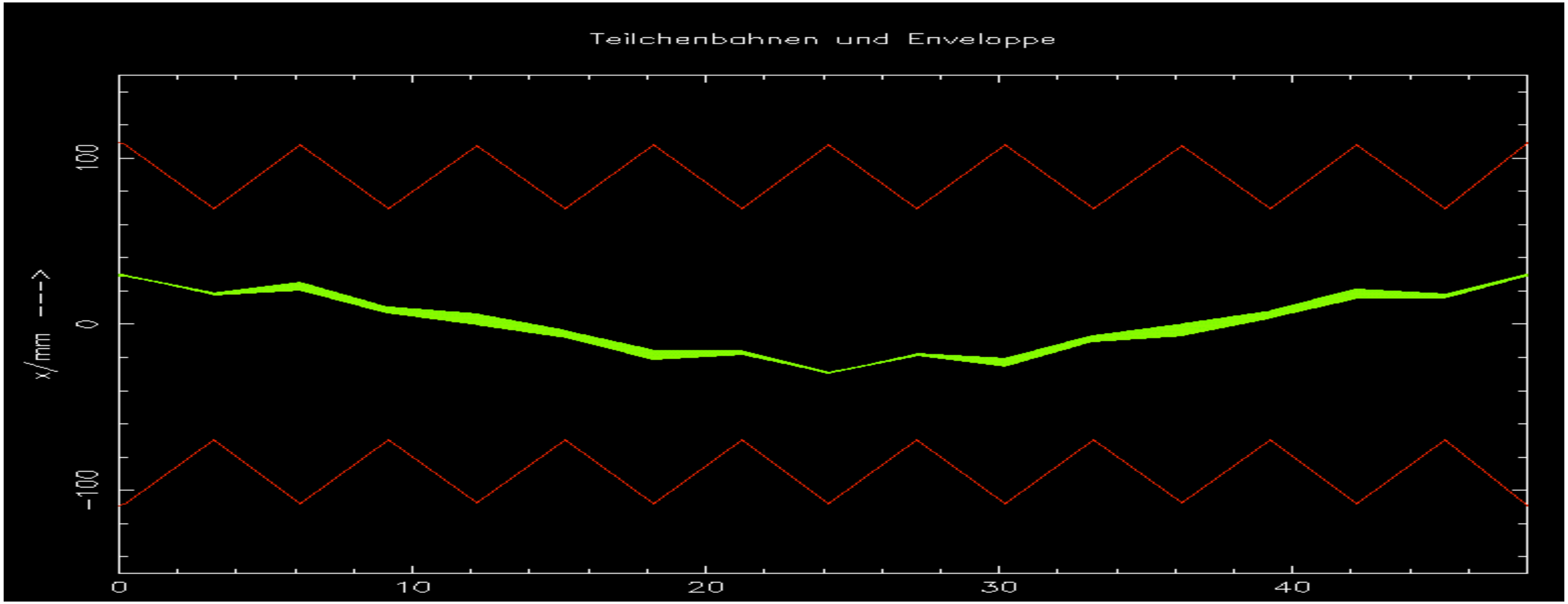
$m*Q_x+n*Q_y+l*Q_s = integer$

Tune diagram up to 3rd order

Resonance Problem:

Why do we have so stupid non-integer tunes ?
"Q = 64.0" sounds much better

Qualitatively spoken: ***Integer tunes lead to a resonant increase of the closed orbit amplitude in presence of the smallest dipole field error.***



Orbit in case of a small dipole error:

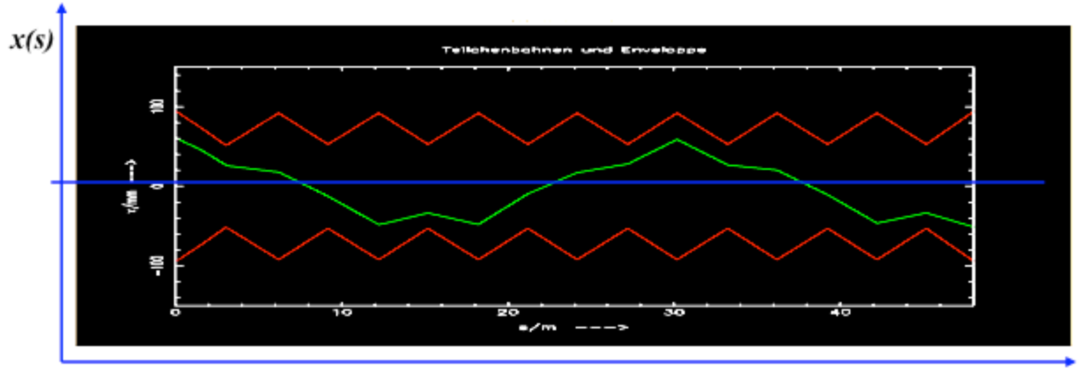
$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(\psi_{s1} - \psi_s - \pi Q)\, ds}{\underbrace{2 \sin \pi Q}}$$

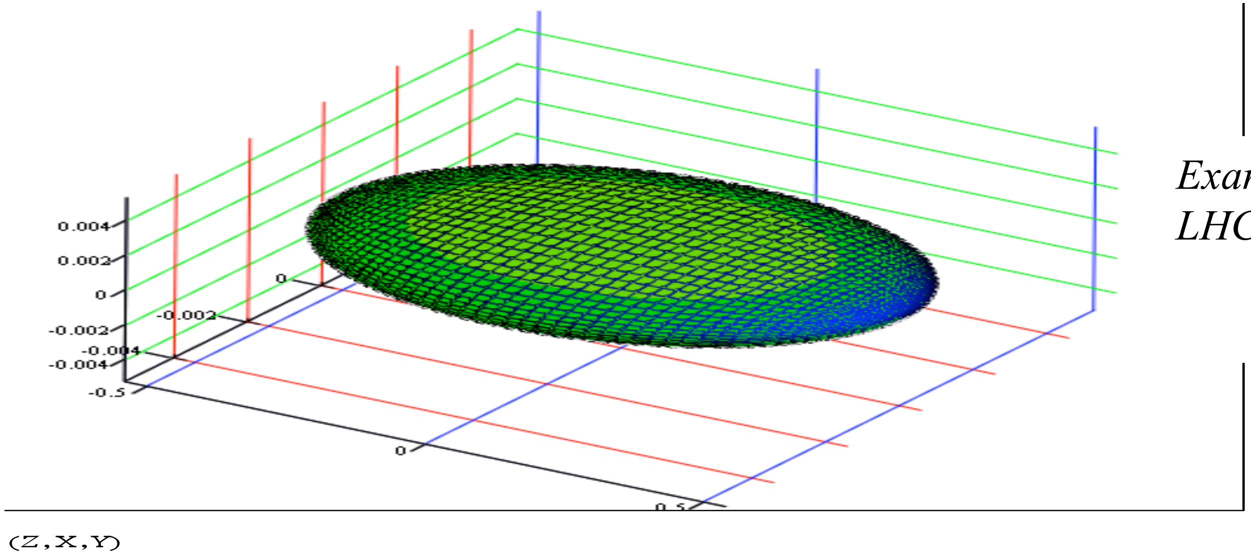
Assume: Tune = integer $Q = 1 \quad \rightarrow \quad 0$

Quadrupole Magnets ... and Beam Size

Quadrupoles ...
... focus every single particle trajectory towards the centre of the vacuum chamber
... define the beam size ... and divergence
... „produce" the tune
... increase the luminosity

Example: particle trajectory defined in amplitude and oscillation frequency by the quadrupole gradients

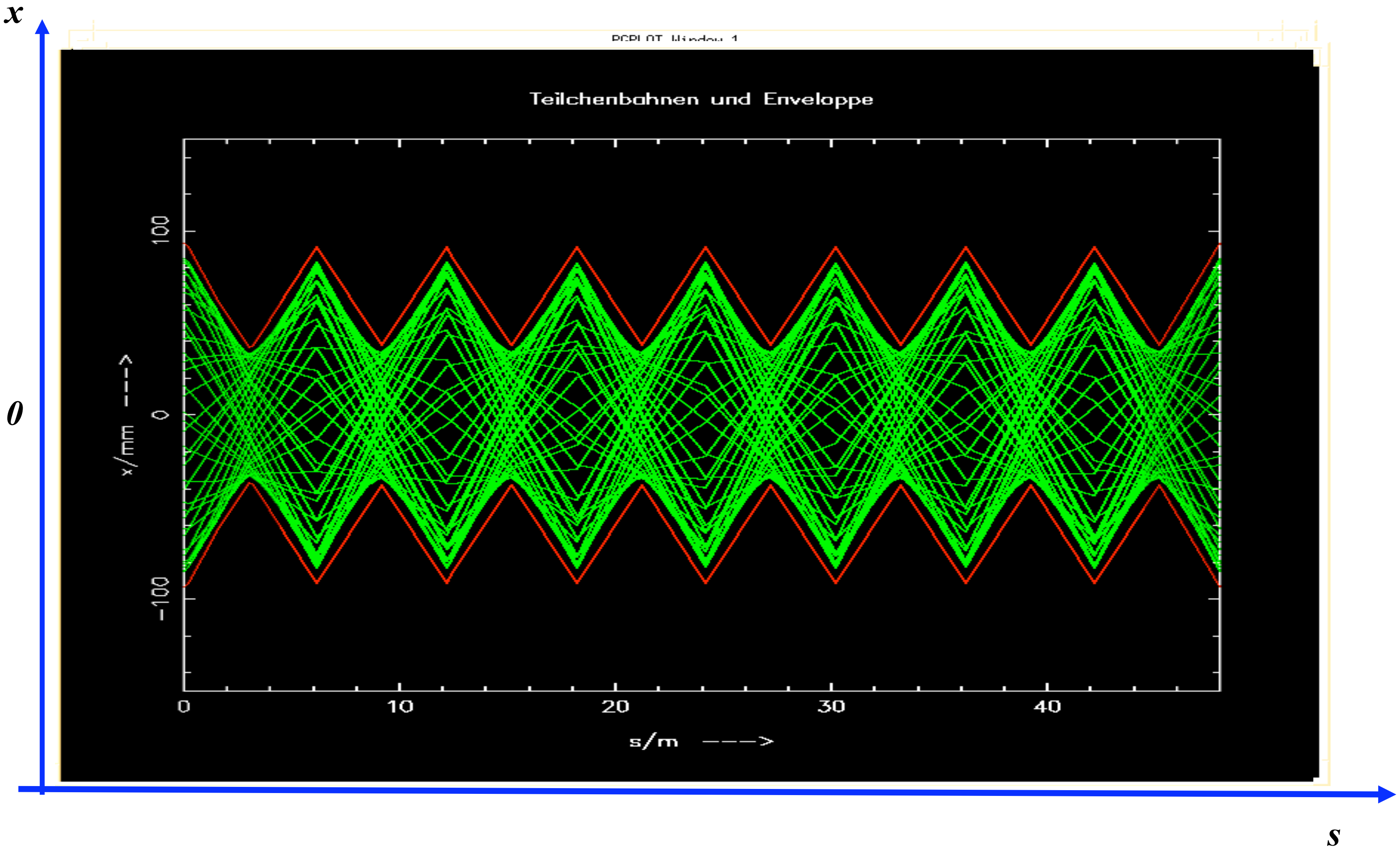




Example:
LHC bunch in the arc of the storage ring
$l \approx 13$ cm,
$x \approx y \approx 0.3$mm

Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



Transverse Beam Dynamics II

I) ***Linear Beam Optics***

Single Particle Trajectories

Magnets and Focusing Fields

Tune & Orbit

II) The State of the Art in High Energy Machines:

The Theory of Synchrotrons:

Linear Beam Optics

The Beam as Particle Ensemble

Emittance and Beta-Function

Colliding Beams & Luminosity

„... how does it work ?"

„...does it ?"

II Storage Rings

Lattice Design and Acceleration

19th century:

Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $x''(s) - k(s)x(s) = 0$

restoring force $\neq$ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

7.) The Beta Function

„it is convenient to see" ... *after some beer* ... *we make two statements:*

1.) There exists a mathematical function, that defines the envelope of all particle trajectories and so can act as measure for the beam size. We call it the β – function.

2.) Whow !!

A particle oscillation can then be written in the form

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

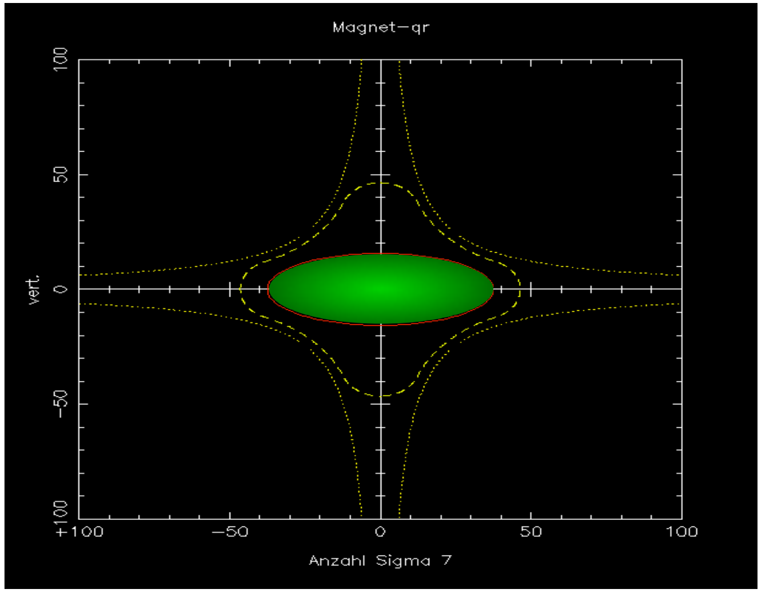
β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.
scientifiquely spoken: area covered in transverse x, x´ phase space

… and it is constant !!!

The Beta Function



If we obtain the x, x' coordinates of a particle trajectory via

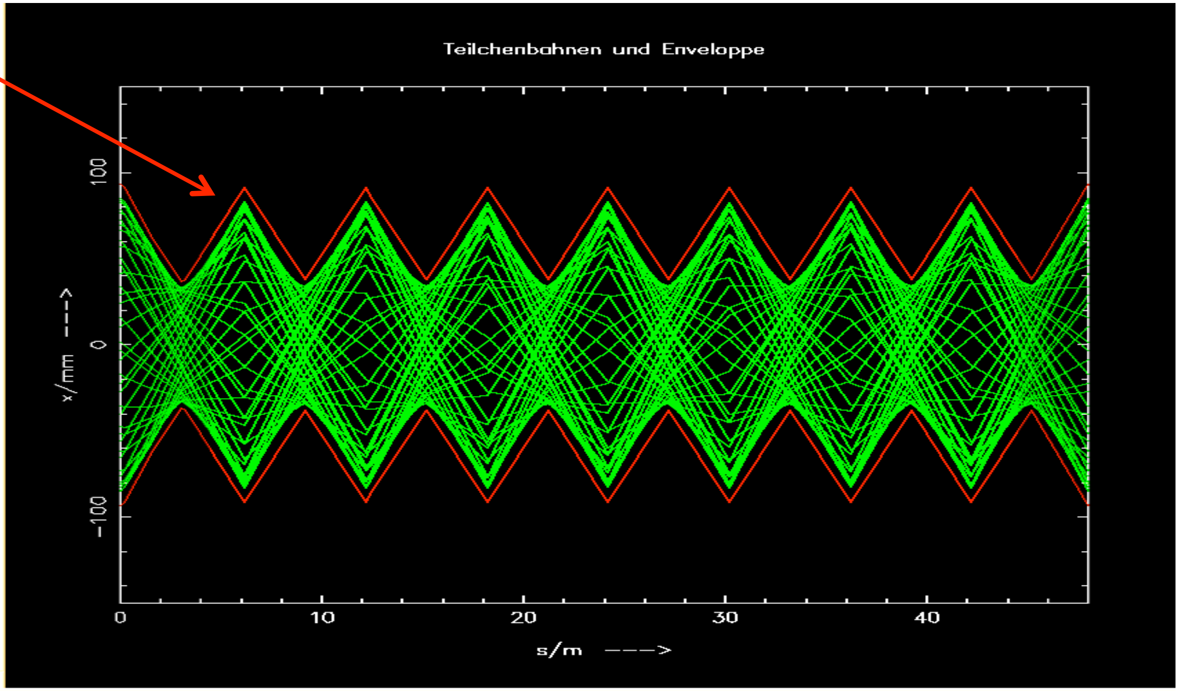
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$

The maximum size of any particle amplitude at a position "s" is given by

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$

β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.



8.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\begin{cases} (1) \quad x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi) \\ (2) \quad x'(s) = -\dfrac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\right\} \end{cases}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

Insert into (2) and solve for ε

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

**** ε is a constant of the motion … it is independent of „s“***
**** parametric representation of an ellipse in the x x‘ space***
**** shape and orientation of ellipse are given by α, β, γ***

Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ $\longrightarrow$ ***x´ at that position …?***

… put $\hat{x}(s)$ ***into*** $\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$ ***and solve for x´***

$$\varepsilon = \gamma\cdot\varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

$$\longrightarrow \quad x' = -\alpha\cdot\sqrt{\varepsilon/\beta}$$

* ***A high β-function means a large beam size and a small beam divergence.*** ***!***
  … et vice versa !!!

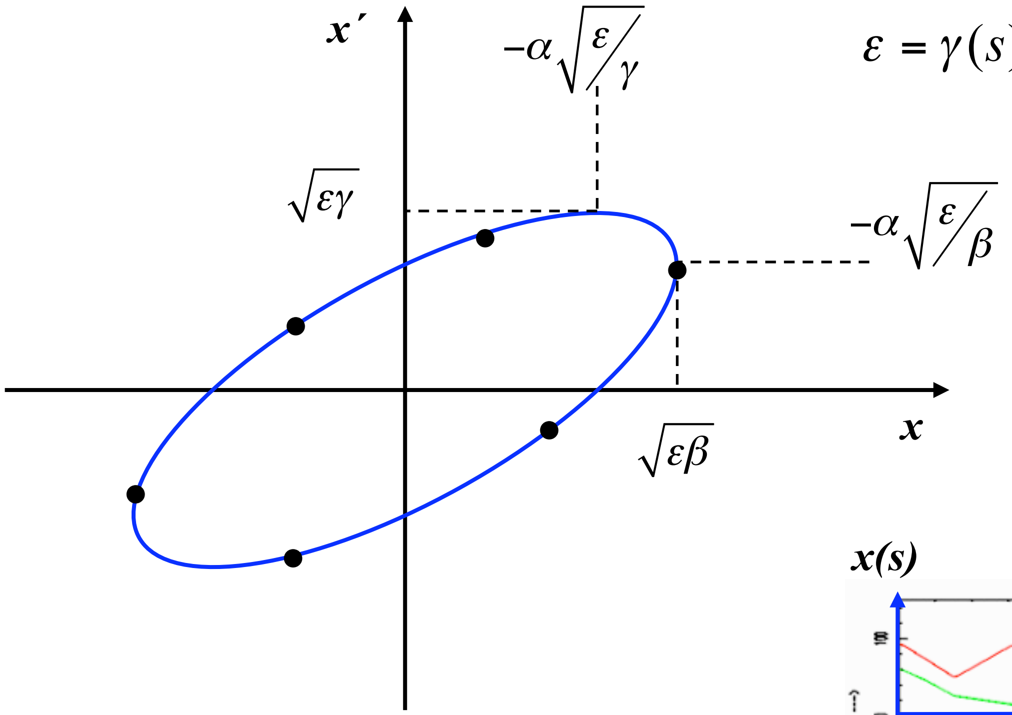
* ***In the middle of a quadrupole β = maximum,***
  α = zero
  $\Big\}\quad x' = 0$

… and the ellipse is flat

Beam Emittance and Phase Space Ellipse

In phase space x, x' a particle oscillation, observed at a given position "s" in the ring is running on an ellipse ... making Q revolutions per turn.

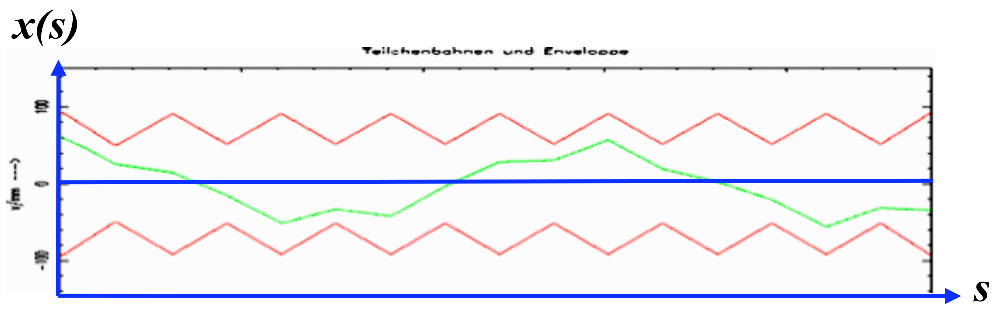
$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$



$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

***Liouville:* in reasonable storage rings**
area in phase space is constant.

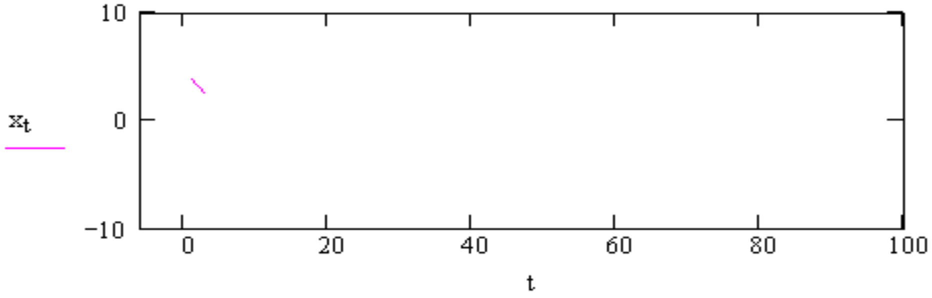
A = π*ε=const

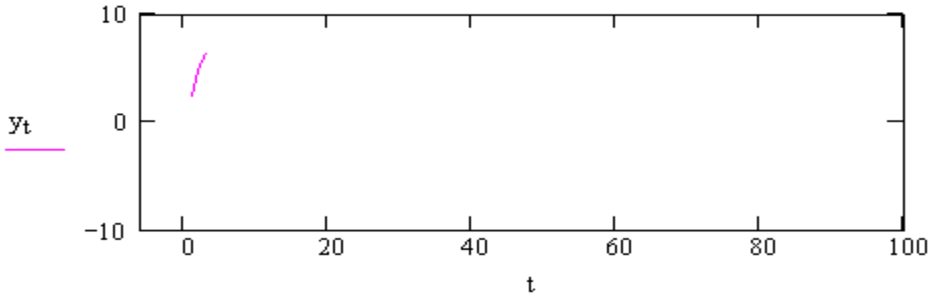


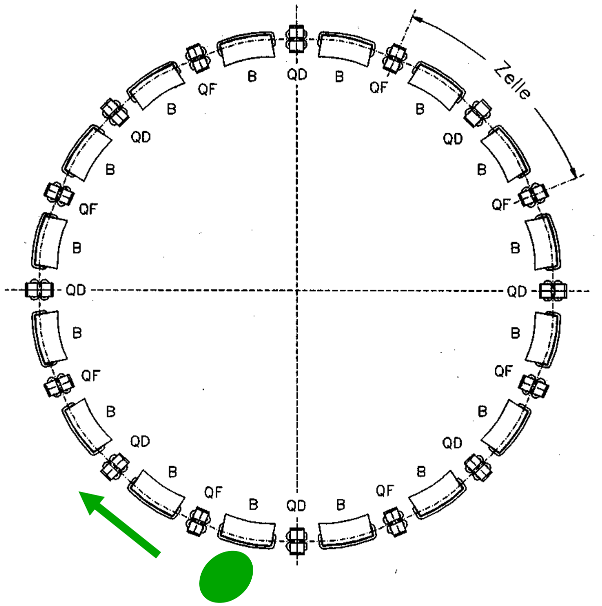
Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

plot x, x´as a function of „s"

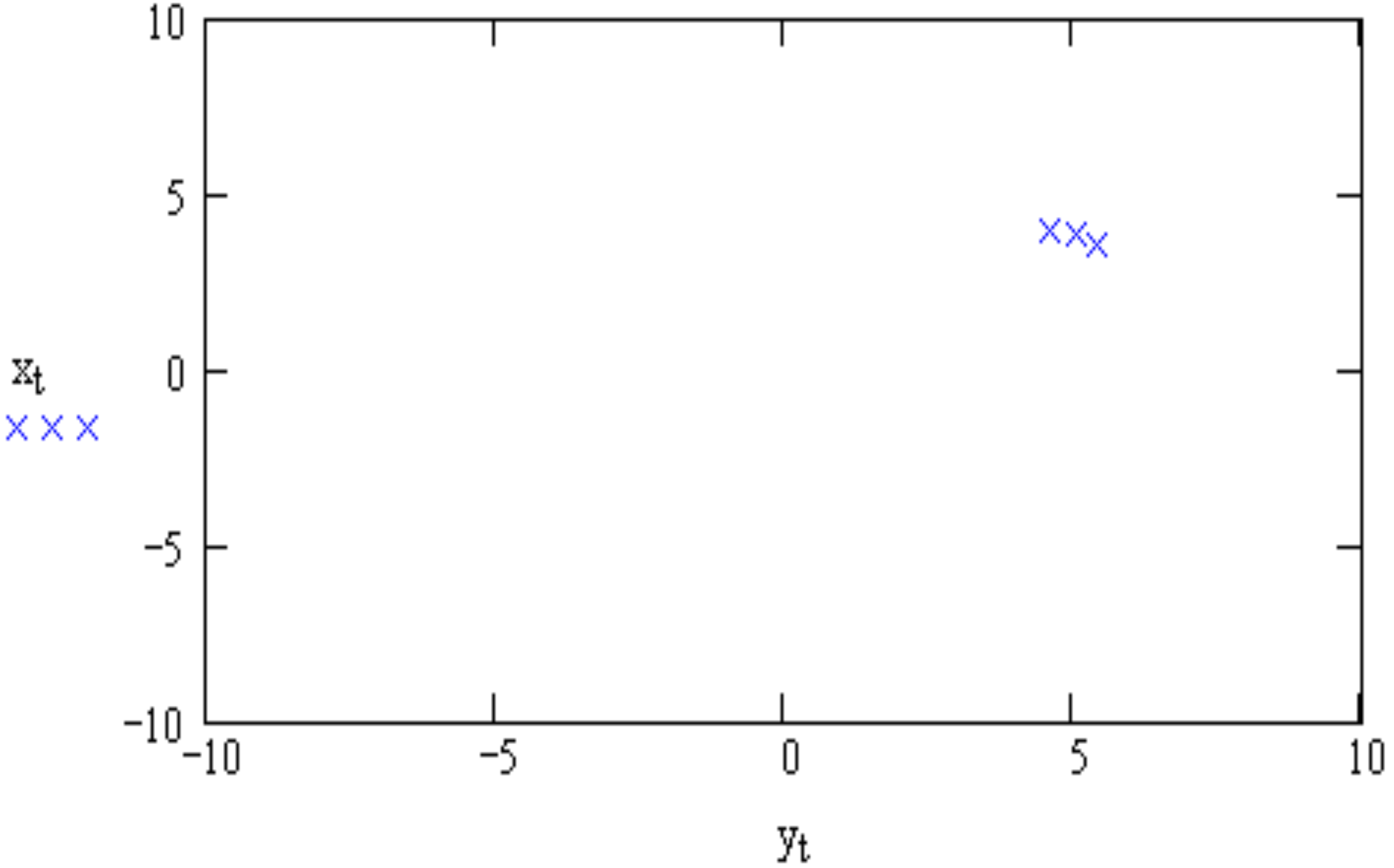




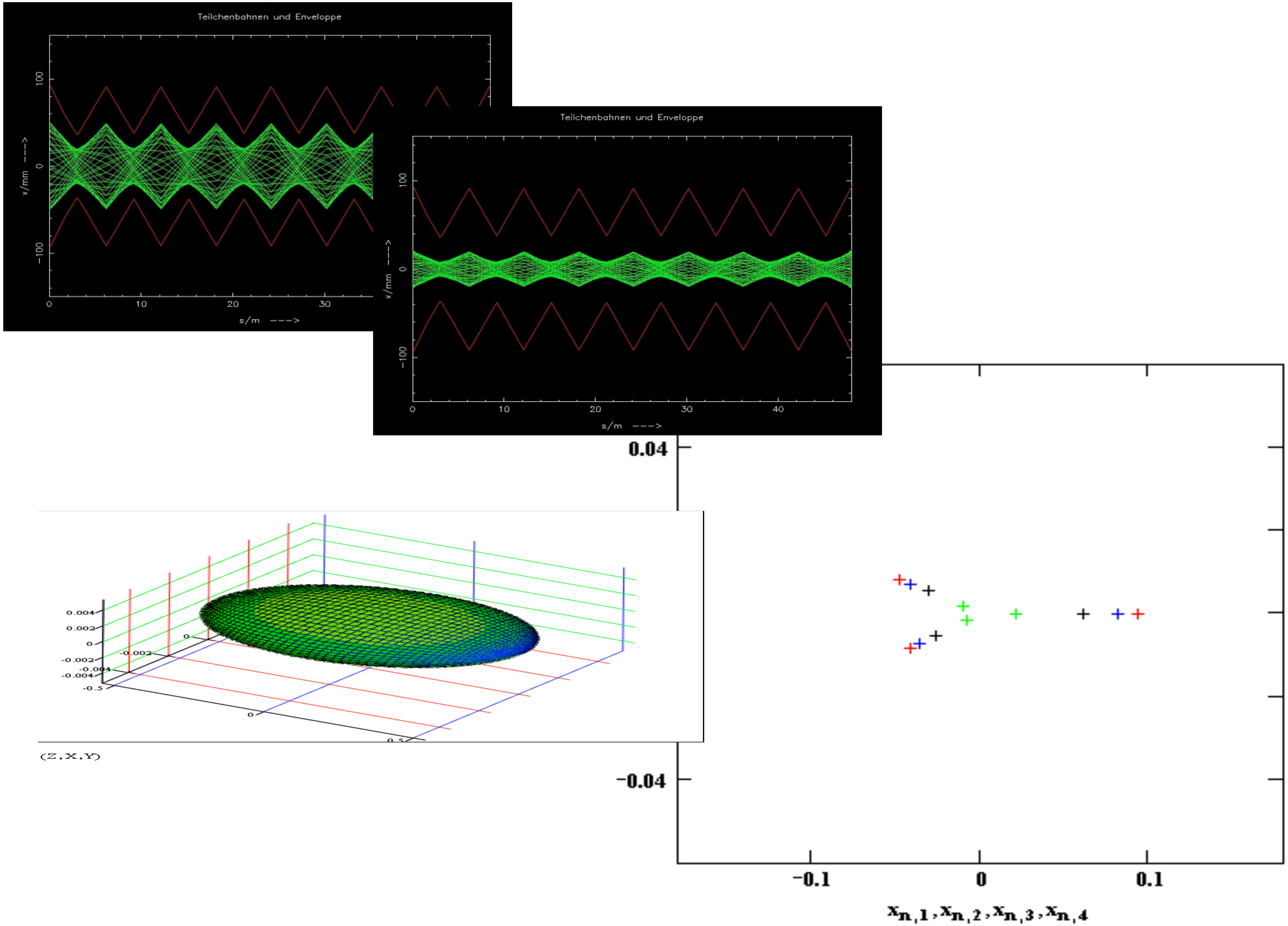


… and now the ellipse:

note for each turn x, x´at a given position „s_1" and plot in the phase space diagram

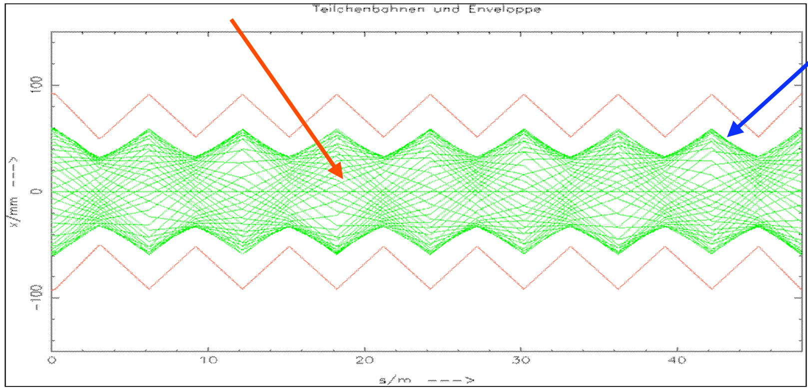


Emittance of the Particle Ensemble:



Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, N ≈ 10 [11] per bunch

Gauß Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x{}^2}}$$

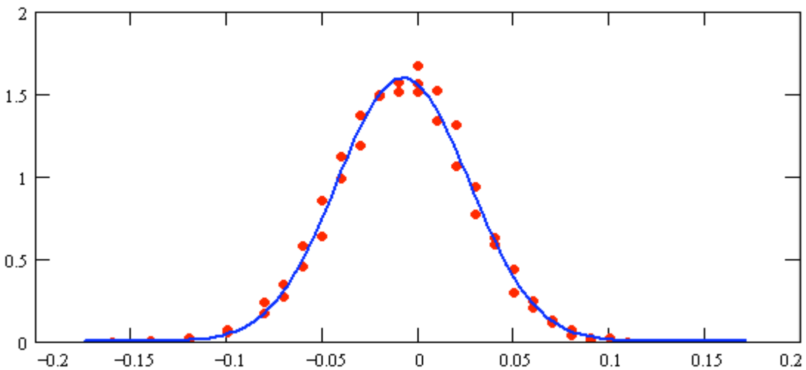
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

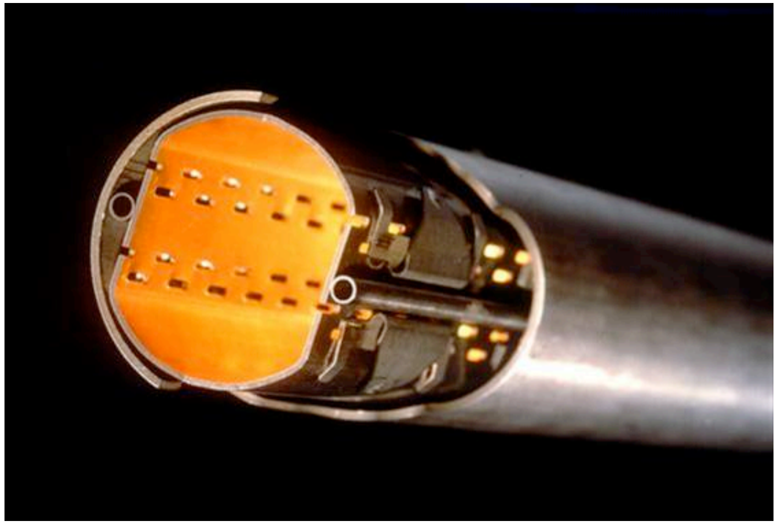
LHC:

$$\beta = 180\,m$$

$$\varepsilon = 5*10^{-10}\,m\,rad$$

$$\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5*10^{-10}\,m*180\,m} = 0.3\,mm$$

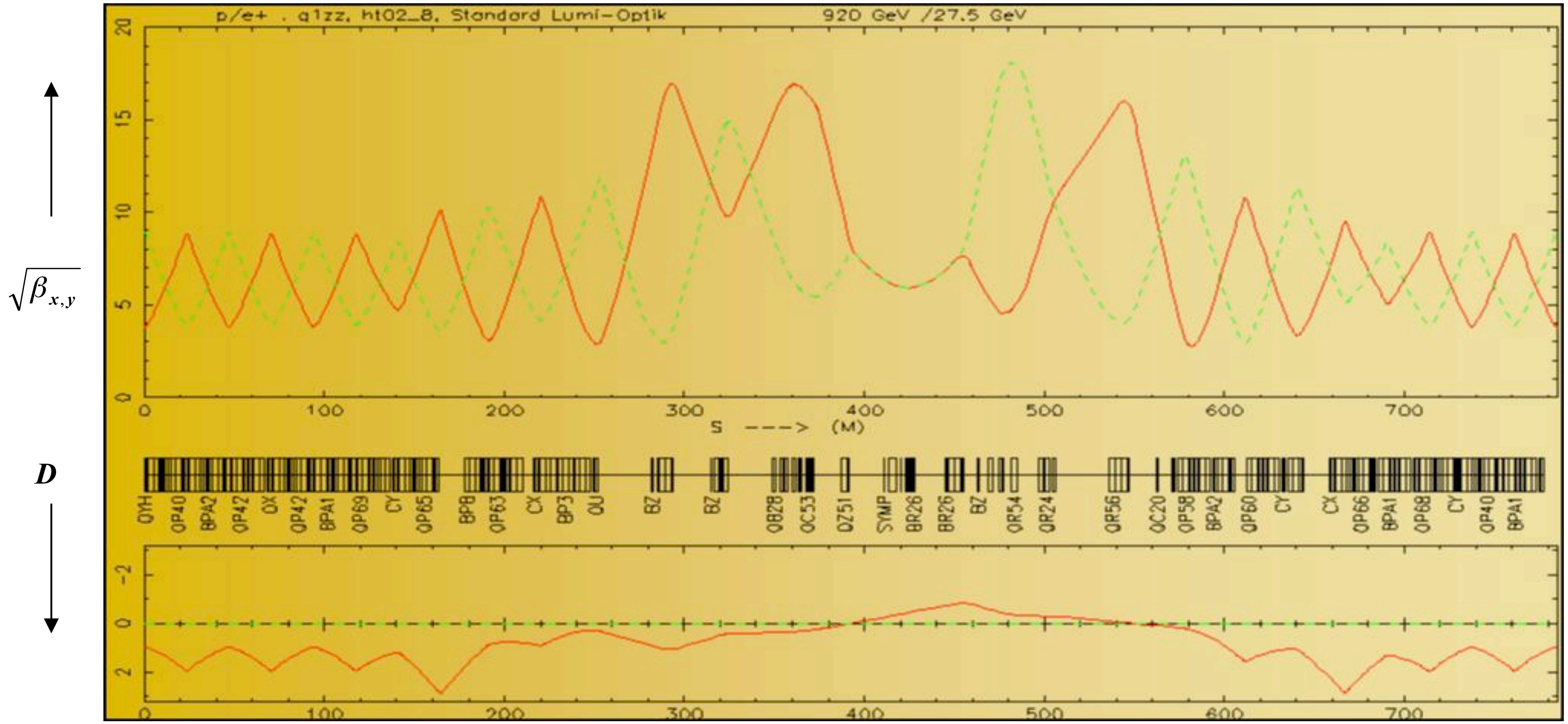




aperture requirements: $r_0 = 12 * \sigma$

The „not so ideal“ World

Lattice Design in Particle Accelerators



1952: Courant, Livingston, Snyder:

Theory of strong focusing in particle beams

Recapitulation: ...the story with the matrices !!!

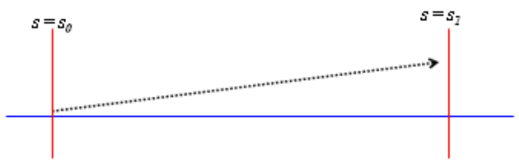
Equation of Motion:

$$x'' + K\,x = 0 \qquad K = 1/\rho^2 - k \;\; \dots \textit{ hor. plane:}$$

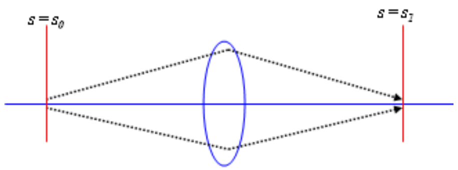
$$K = k \qquad \dots \textit{ vert. Plane:}$$

Solution of Trajectory Equations

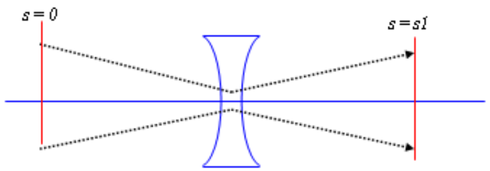
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$



$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}l) \\ \sqrt{|K|}\sinh(\sqrt{|K|}l) & \cosh(\sqrt{|K|}l) \end{pmatrix}$$

$$M_{total} = M_{QF} * M_D * M_B * M_D * M_{QD} * M_D * \quad \dots$$

9.) Lattice Design: *„… how to build a storage ring"*

Geometry of the ring: $B * \rho = p / e$

p = momentum of the particle,
ρ = curvature radius

Bρ= beam rigidity

Circular Orbit: bending angle of one dipole

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$

The angle run out in one revolution must be 2π, so for a full circle

$$\alpha = \frac{\int Bdl}{B\,\rho} = 2\pi$$

$$\int Bdl = 2\pi \frac{p}{q}$$

… defines the integrated dipole field around the machine.

7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\,dl \approx N\,l\,B = 2\pi\;p/e$$

$$B \approx \frac{2\pi\;\;7000\;\;10^{9}\,eV}{1232\;\;15\,m\;\;3\;10^{8}\,\dfrac{m}{s}\;\;\;e} = 8.3\,Tesla$$

10.) Transfer Matrix M

my appologies: two slides for the experts

... and for Verena who will need it afterwards ...

general solution of Hill´s equation

$$\begin{cases} x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \\ x'(s) = \dfrac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right] \end{cases}$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point s(0) = s_0, where we put Ψ(0) = 0

$$\left.\begin{aligned} \cos\phi &= \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \\ \sin\phi &= -\frac{1}{\sqrt{\varepsilon}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{aligned}\right\}$$

inserting above …

$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s + \alpha_0 \sin\psi_s\right\} x_0 + \left\{\sqrt{\beta_s\beta_0}\,\sin\psi_s\right\} x_0'$$

$$x'(s) = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s\right\} x_0 + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s - \alpha_s \sin\psi_s\right\} x_0'$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

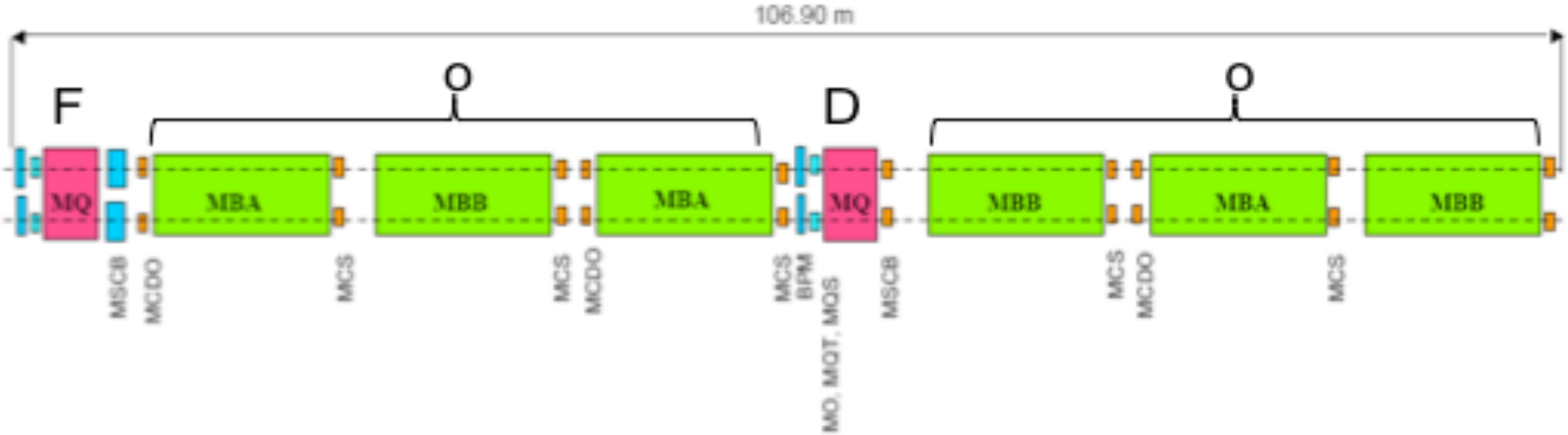
* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***

* ***and nothing but the α β γ at these positions.***

* ... **!**

* *Äquivalenz der Matrizen*

LHC: Lattice Design
the ARC 90° FoDo in both planes

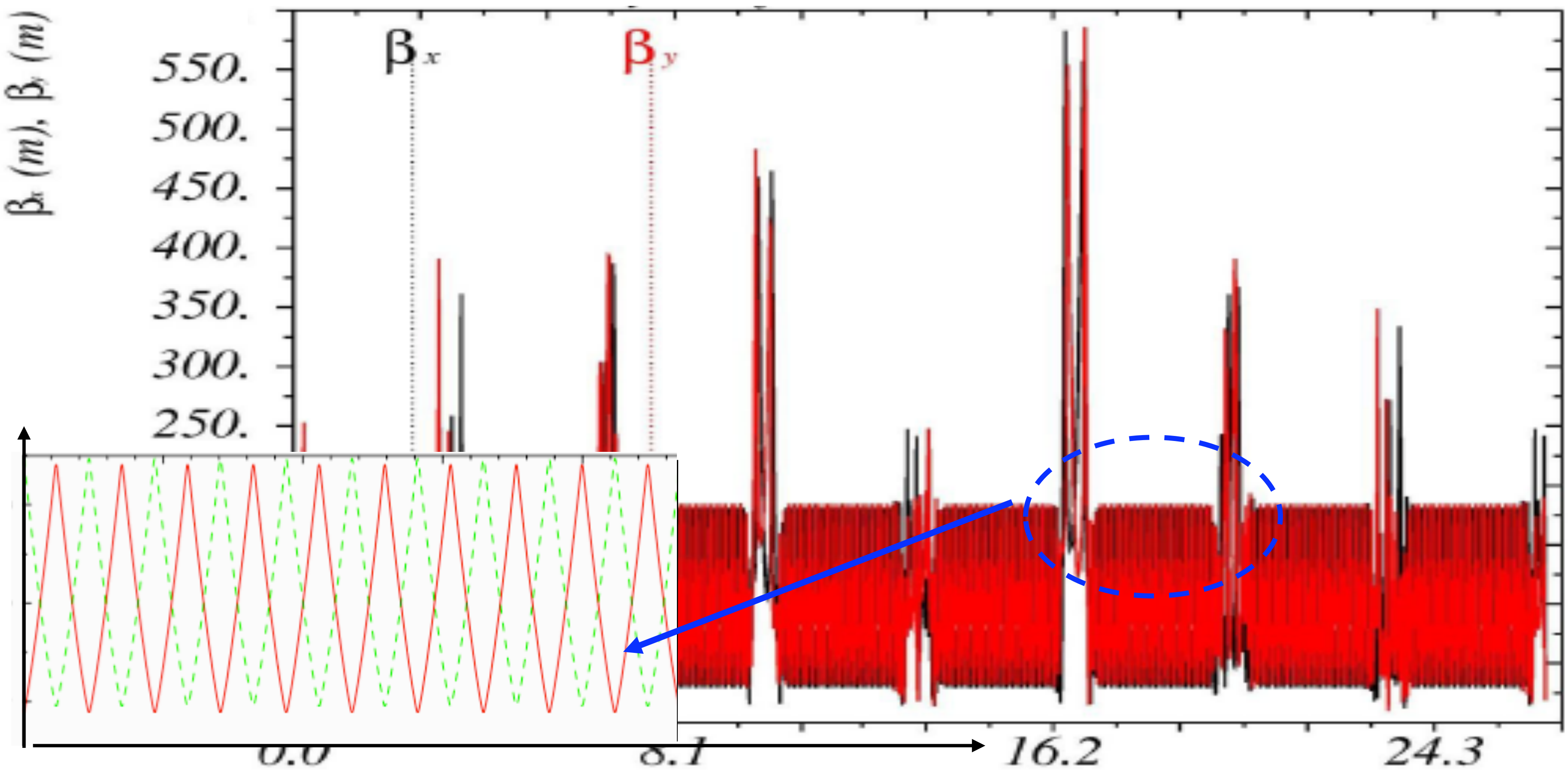


equipped with additional corrector coils

MB: main dipole
MQ: main quadrupole
MQT: Trim quadrupole
MQS: Skew trim quadrupole
MO: Lattice octupole (Landau damping)
MSCB: Skew sextupole
Orbit corrector dipoles
MCS: Spool piece sextupole
MCDO: Spool piece 8 / 10 pole
BPM: Beam position monitor + diagnostics

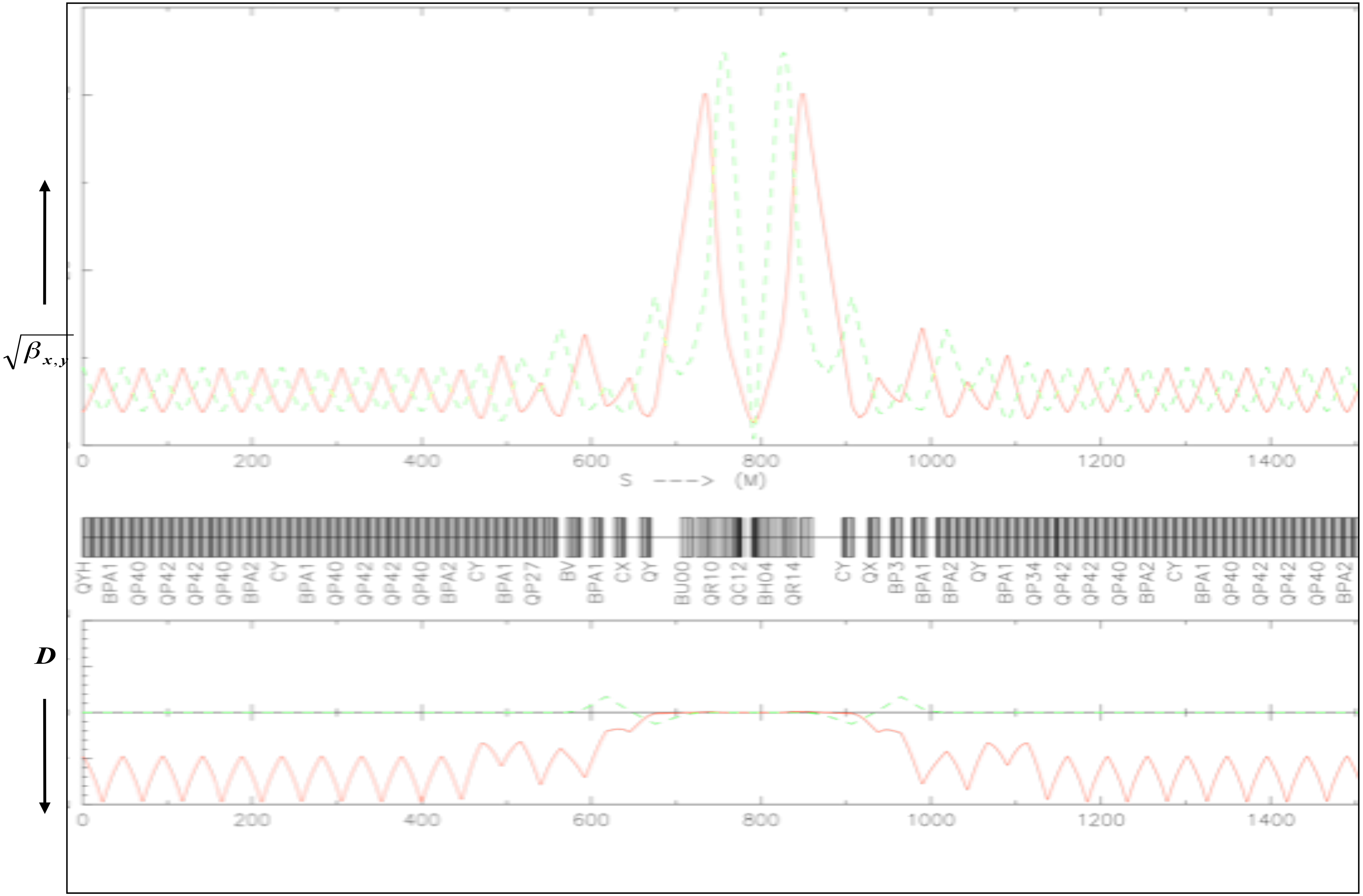
FoDo-Lattice

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in .
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



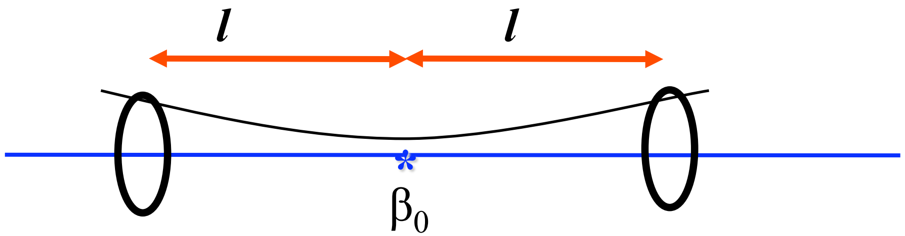
Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell μ = 45°,
→ calculate the twiss parameters for a periodic solution

11.) Insertions



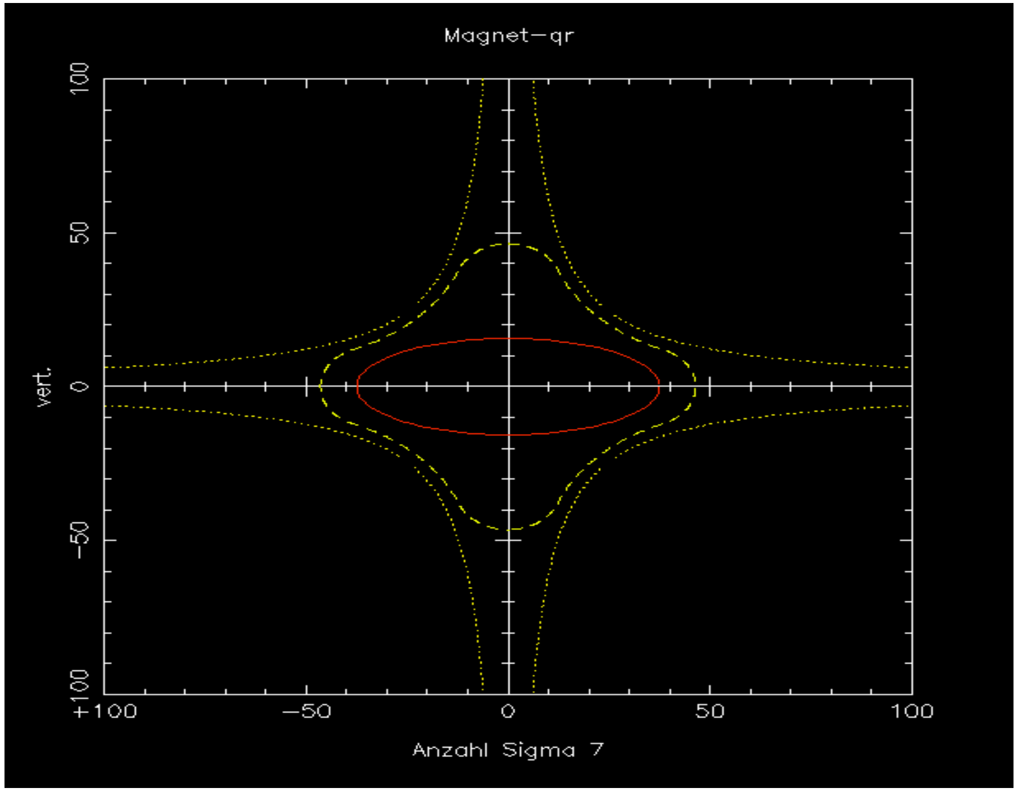
β-Function in a Drift:

$$\beta(\ell) = \beta_0 + \frac{\ell^2}{\beta_0}$$



At the end of a long symmetric drift space the beta function reaches its maximum value in the complete lattice.
-> here we get the largest beam dimension.

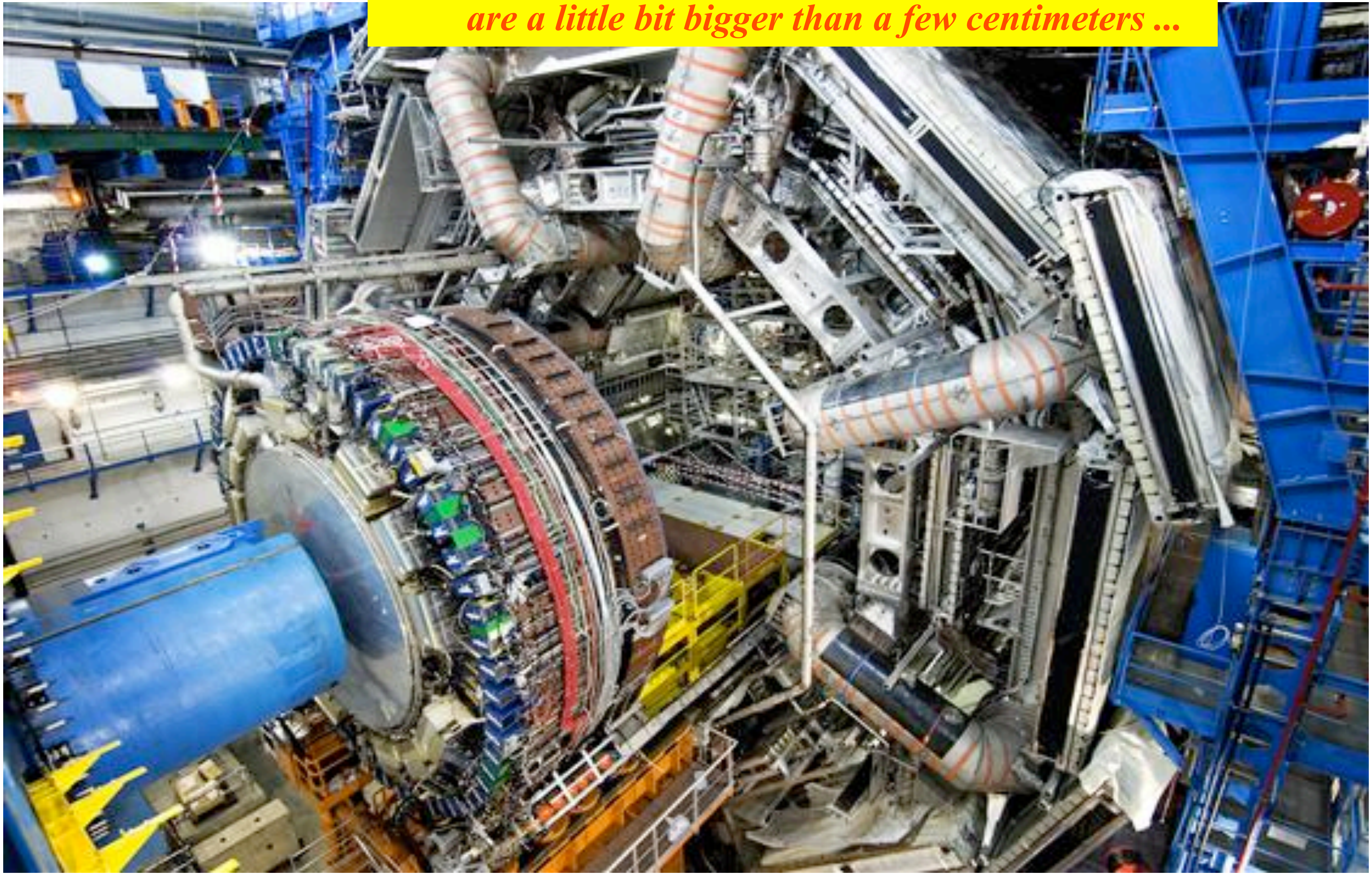
-> keep l as small as possible



7 sigma beam size inside a mini beta quadrupole

... clearly there is an

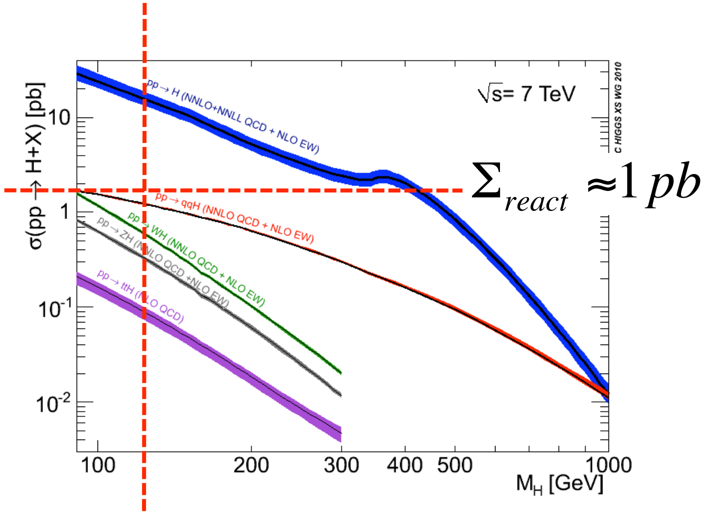
... unfortunately ... in general
high energy detectors that are
installed in that drift spaces
are a little bit bigger than a few centimeters ...

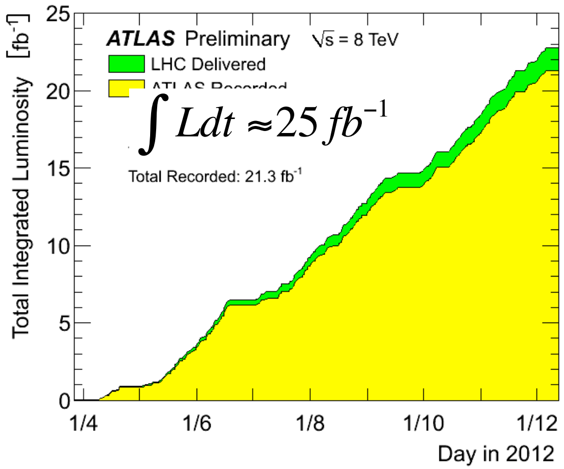


11.) The Mini-β Insertion & Luminosity:

production rate of events is determined by the cross section Σ_{react} and a parameter L that is given by the design of the accelerator:
… the luminosity

$$R = L * \Sigma_{react} \qquad \approx 10^{-12} b \cdot 25 \frac{1}{10^{-15} b} = some\, 1000\, H$$



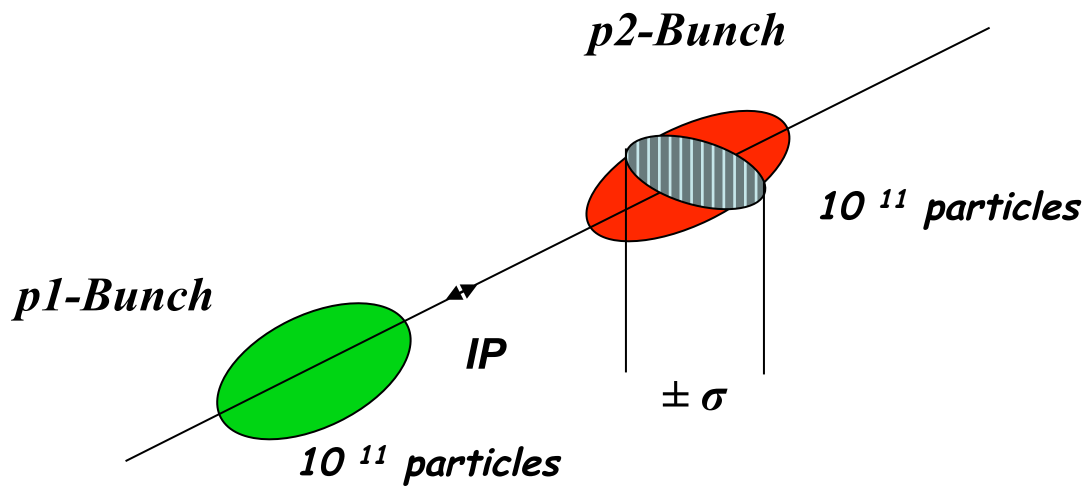


remember:
$1b = 10^{-24}\, cm^2$

The luminosity is a storage ring quality parameter and depends on beam size (β !!) and stored current

$$L = \frac{1}{4\pi e^2 f_0 \, b} * \frac{I_1 * I_2}{\sigma^*_x * \sigma^*_y}$$

Luminosity



Example: Luminosity run at LHC

$$\beta_{x,y} = 0.55\ m \qquad f_0 = 11.245\ kHz$$

$$\varepsilon_{x,y} = 5*10^{-10}\ rad\ m \qquad n_b = 2808$$

$$\sigma_{x,y} = 17\ \mu m$$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

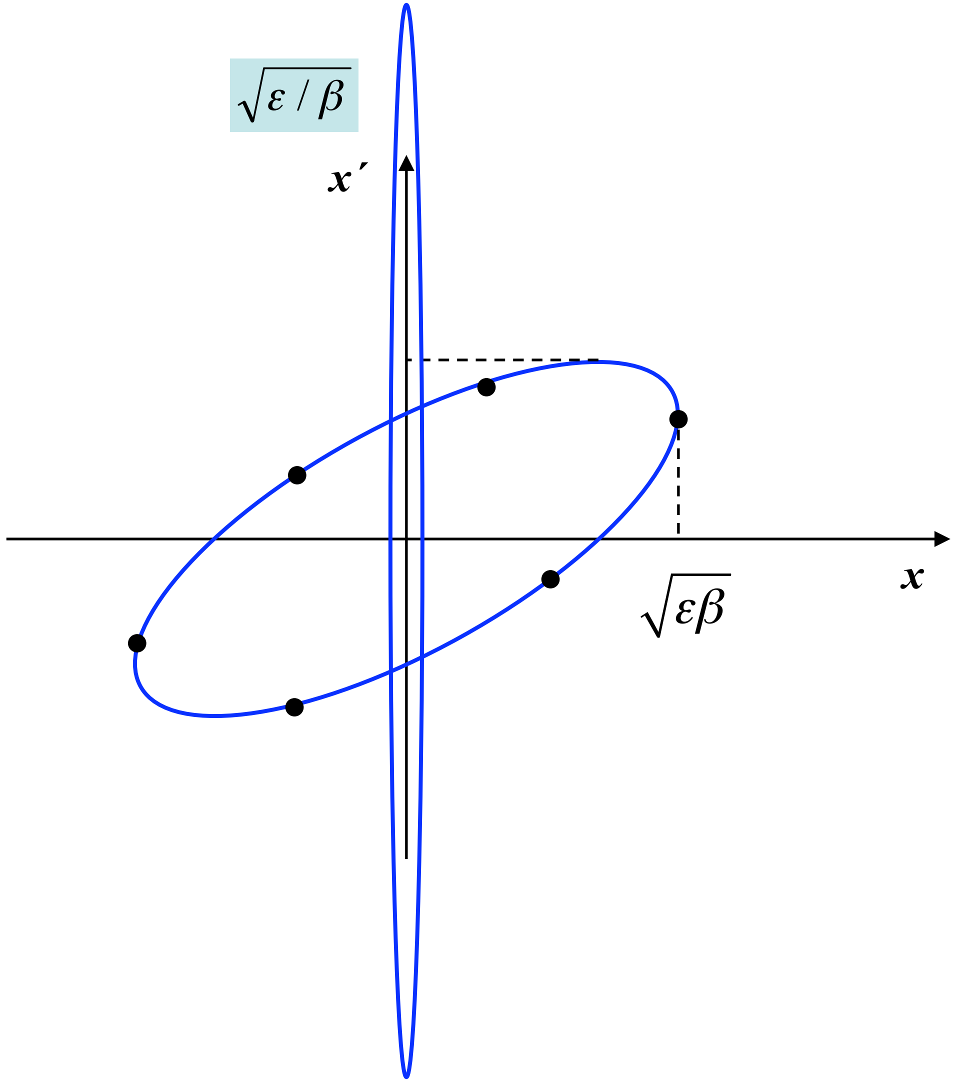
$$I_p = 584\ mA$$

$$L = 1.0*10^{34}\ {}^{1}\!/_{cm^2 s}$$

Mini-β Insertions: Betafunctions

A mini-β insertion is always a kind of special symmetric drift space.
→greetings from Liouville

the smaller the beam size
the larger the bam divergence



Mini-β Insertions: some guide lines

** calculate the periodic solution in the arc*

** introduce the drift space needed for the insertion device (detector ...)*

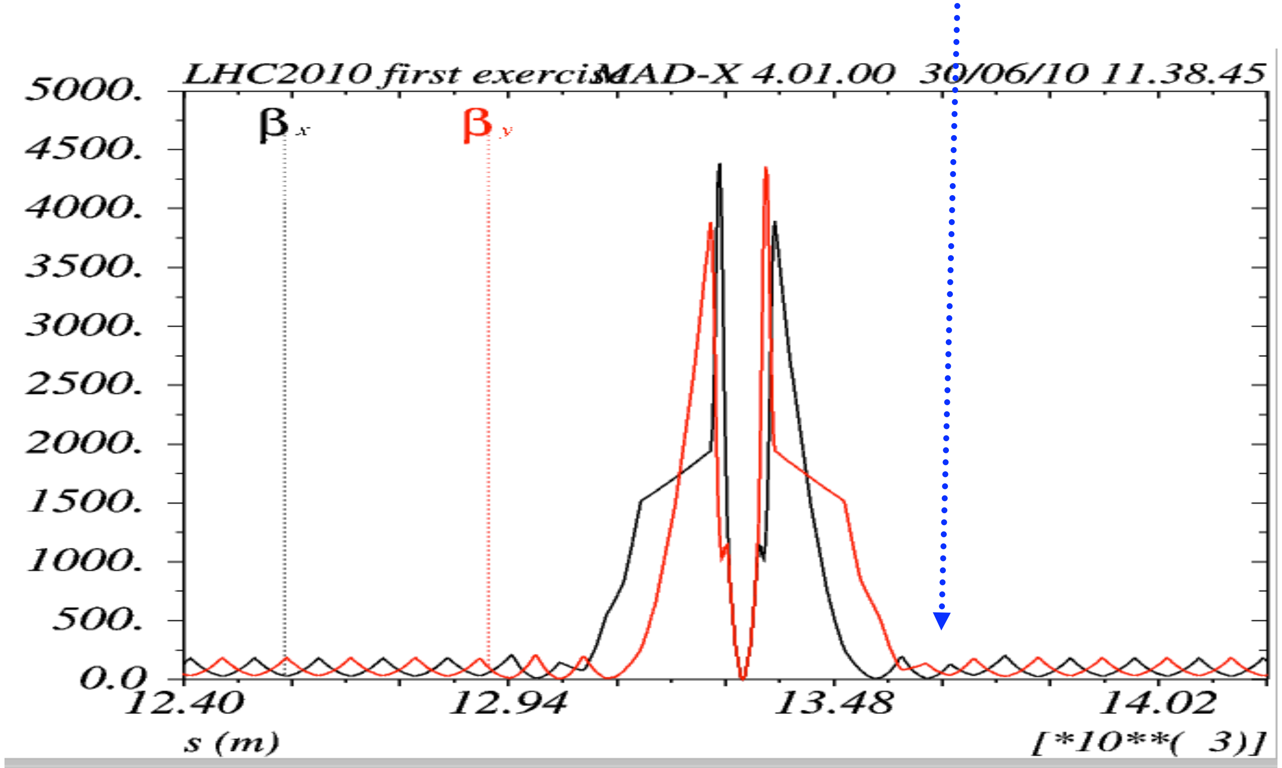
** put a quadrupole doublet (triplet ?) as close as possible*

** introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure*

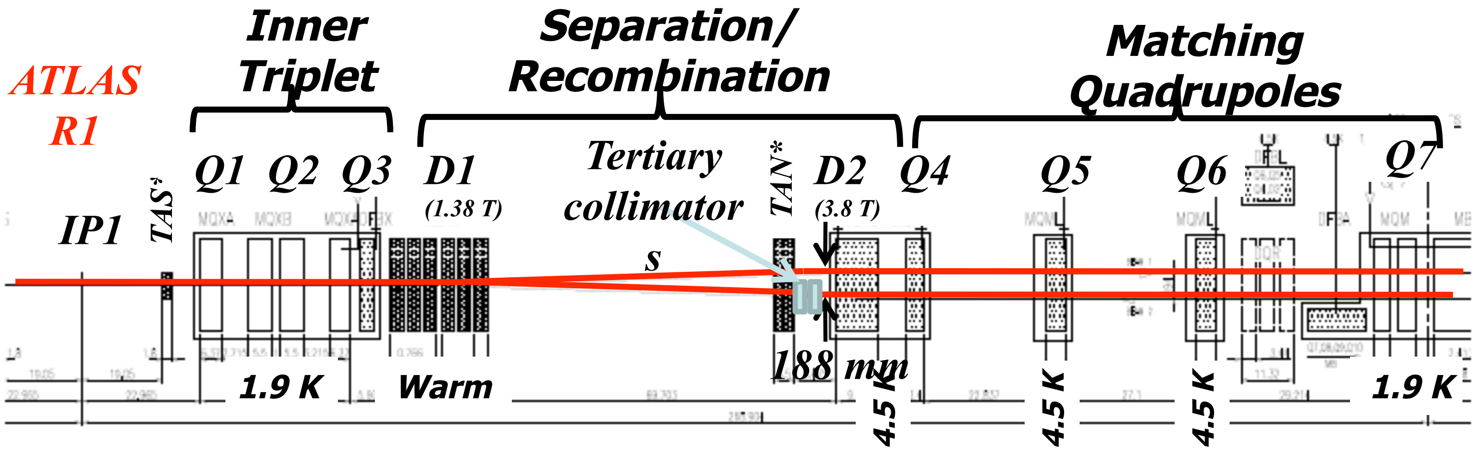
parameters to be optimised & matched to the periodic solution:

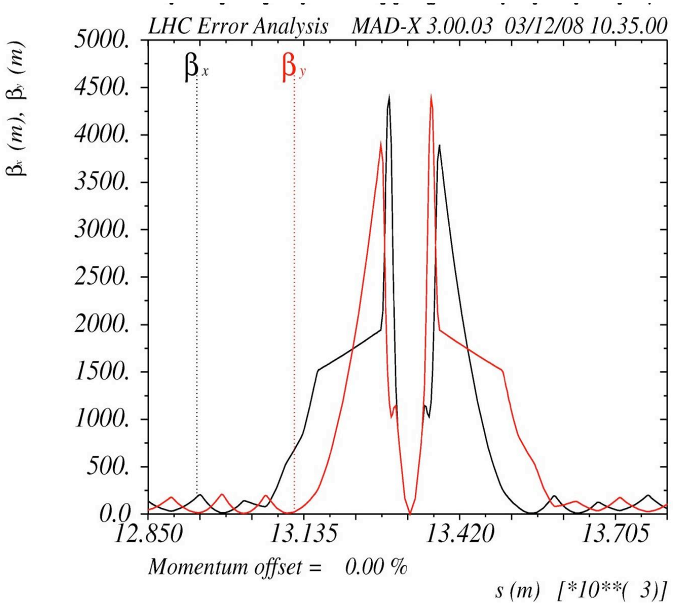
$$\alpha_x,\ \beta_x \qquad D_x,\ D_x'$$
$$\alpha_y,\ \beta_y \qquad Q_x,\ Q_y$$

8 individually powered quad magnets are needed to match the insertion
(... at least)

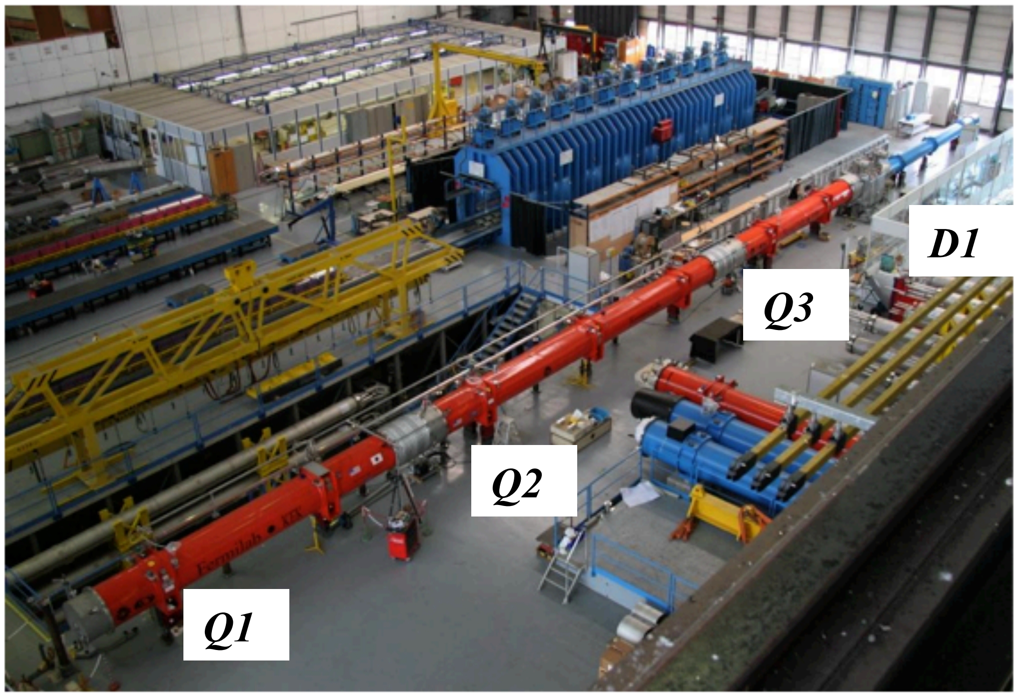


The LHC Insertions





mini β optics



Transverse Beam Dynamics III

I) Linear Beam Optics

Single Particle Trajectories
Magnets and Focusing Fields
Tune & Orbit

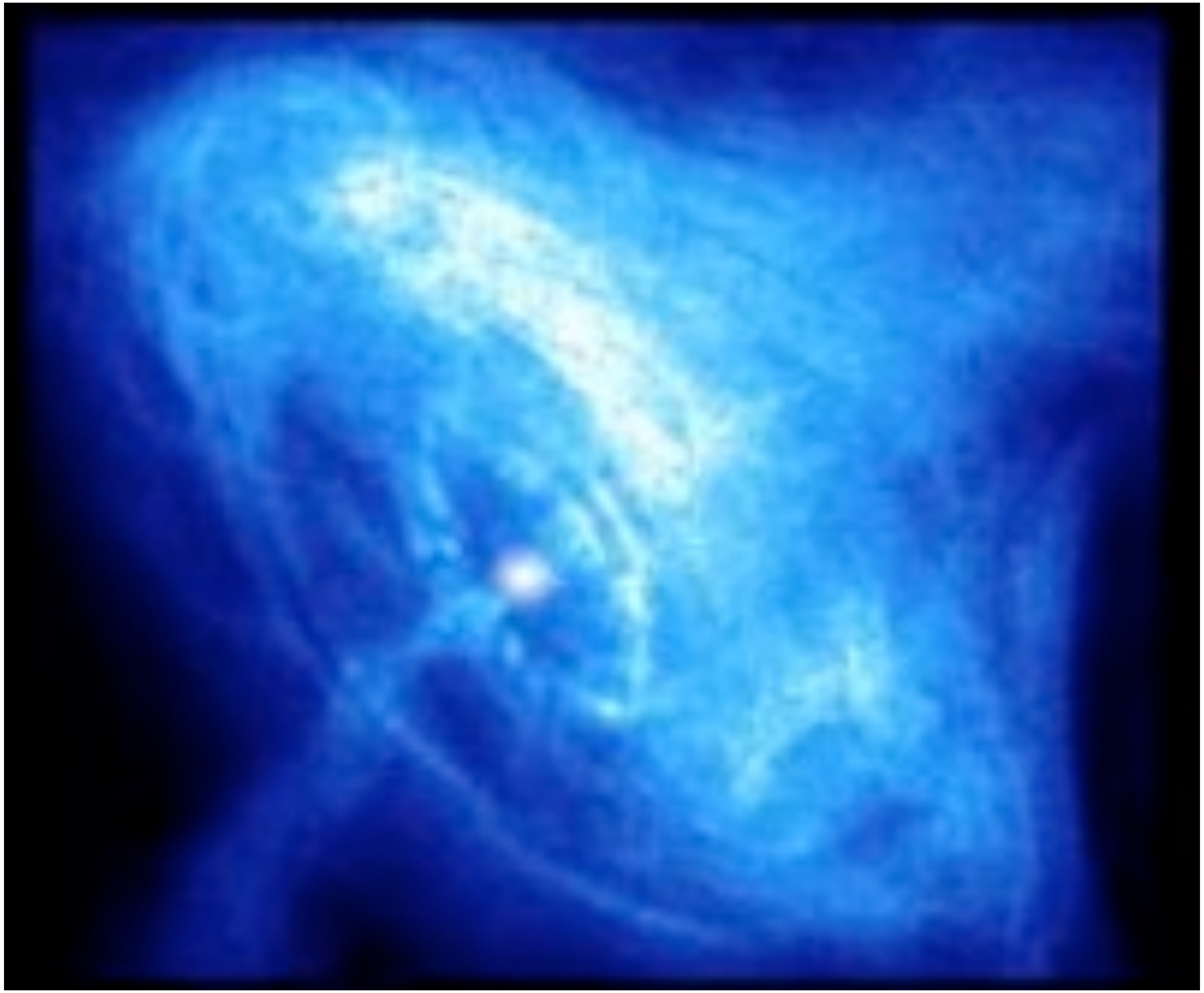
II) The State of the Art in High Energy Machines:

The Beam as Particle Ensemble
Emittance and Beta-Function
Colliding Beams & Luminosity

III) Errors in Field and Gradient:

Liouville during Acceleration
The Δp/p ≠0 problem
Dispersion
Chromaticity

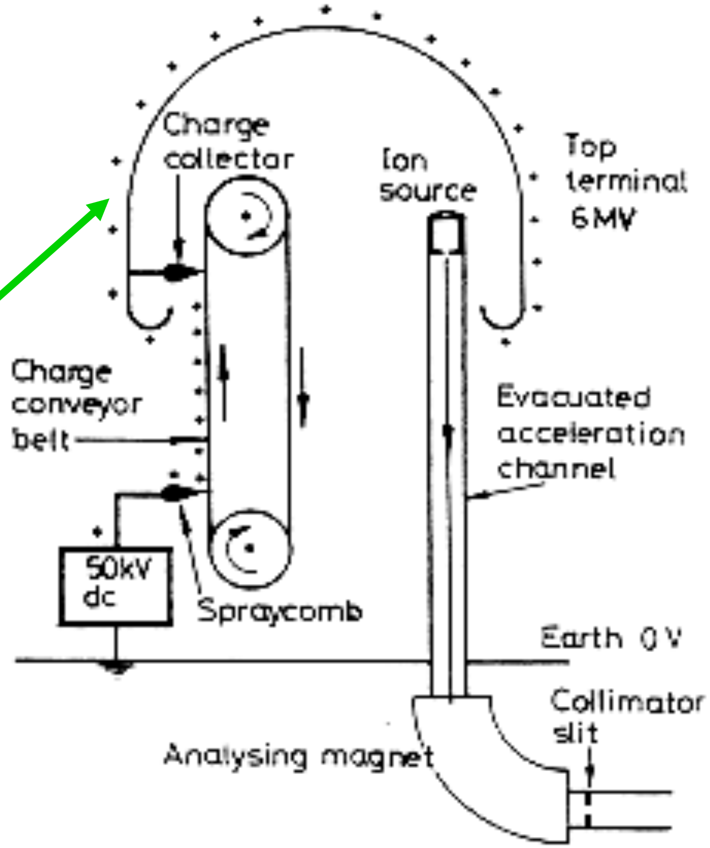
12) … let´s talk about acceleration



Electrostatic Machines

(Tandem -) van de Graaff Accelerator

creating high voltages by mechanical transport of charges



** Terminal Potential: U ≈ 12 ...28 MV*
using high pressure gas to suppress discharge (SF_6)

*Problems: * Particle energy limited by high voltage discharges*
** high voltage can only be applied once per particle ...*
... or twice ?

The „Tandem principle“: Apply the accelerating voltage twice ...
... by working with negative ions (e.g. H^-) and
stripping the electrons in the centre of the
structure

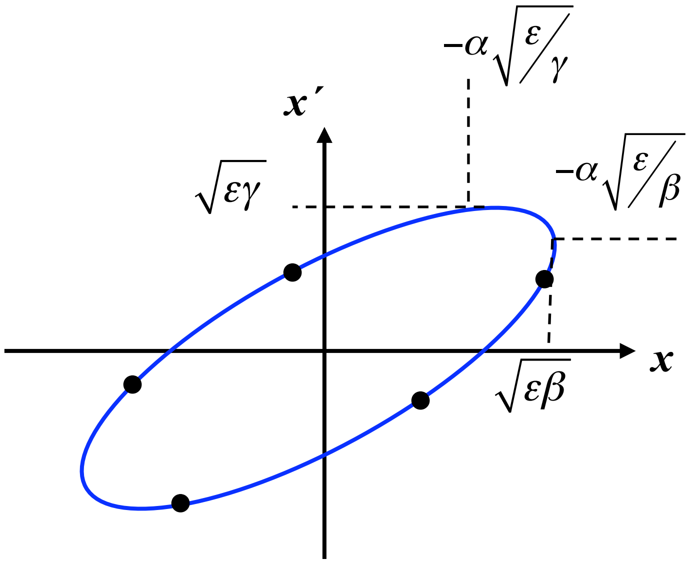
$$dW = dE = eE_z ds \quad \Rightarrow \quad W = e\int E_z ds = eV$$

nota bene: all particles are “synchron” with the acceleration potential

Electro Static Accelerator: 12 MV-Tandem van de Graaff Accelerator at MPI Heidelberg

13.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$



***Beam Emittance* corresponds to the area covered in the x, x' Phase Space Ellipse**

***Liouville:* Area in phase space is constant.**

But so sorry ... $\varepsilon \neq const$!

Classical Mechanics:

***phase space* = diagram of the two canonical variables**
position **&** ***momentum***
x p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

$q = position = x$
$p = momentum = \gamma m v = mc\gamma\beta_x$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $\int p\,dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\beta_x = v_x / c$

$$\int p\,dq = mc\int \gamma\beta_x dx$$

$$\int p\,dq = mc\gamma\beta \underbrace{\int x'dx}_{\varepsilon}$$

$$\Rightarrow \quad \varepsilon = \int x'dx \propto \frac{1}{\beta\gamma}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

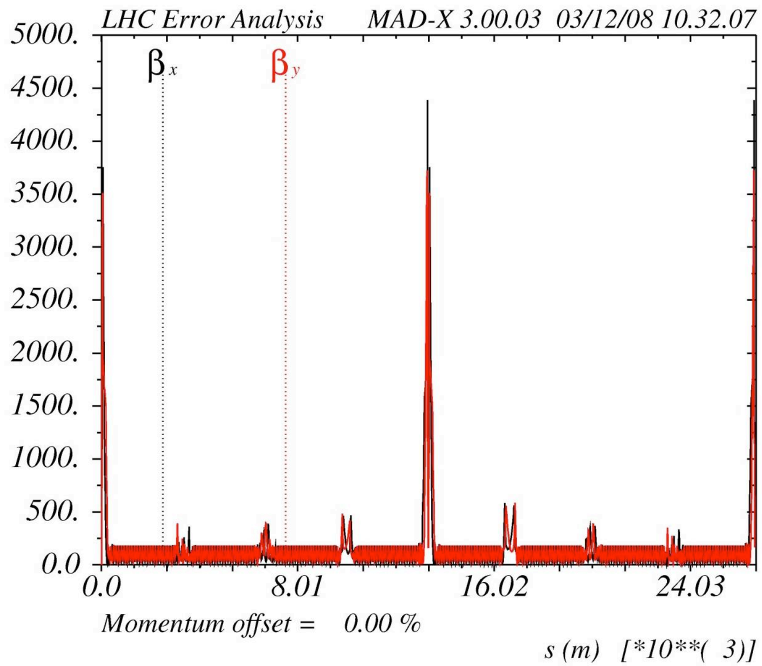
Nota bene:

1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

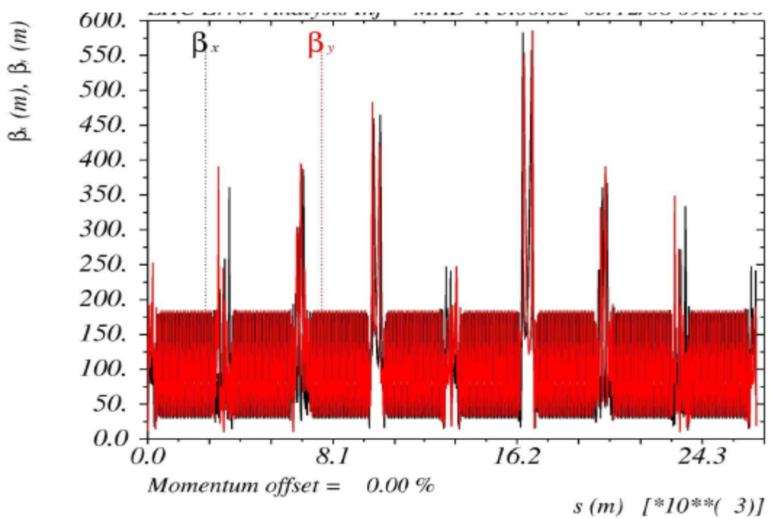
$$\sigma = \sqrt{\varepsilon\beta}$$

2.) At lowest energy the machine will have the major aperture problems,
→ here we have to minimise $\hat{\beta}$

3.) we need different beam optics adopted to the energy:
A Mini Beta concept will only be adequate at flat top.



LHC mini beta optics at 7000 GeV

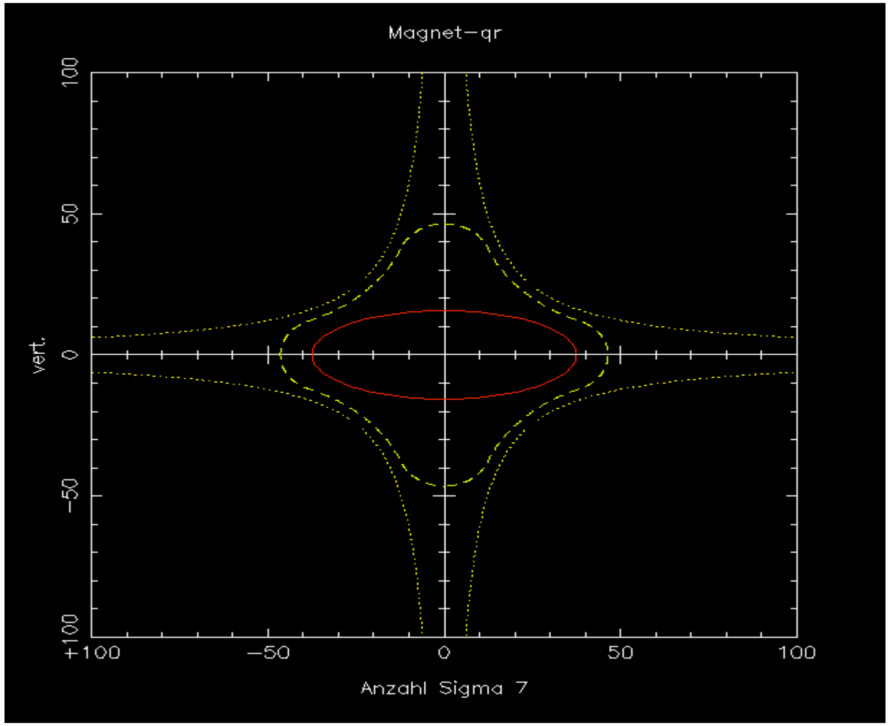


LHC injection optics at 450 GeV

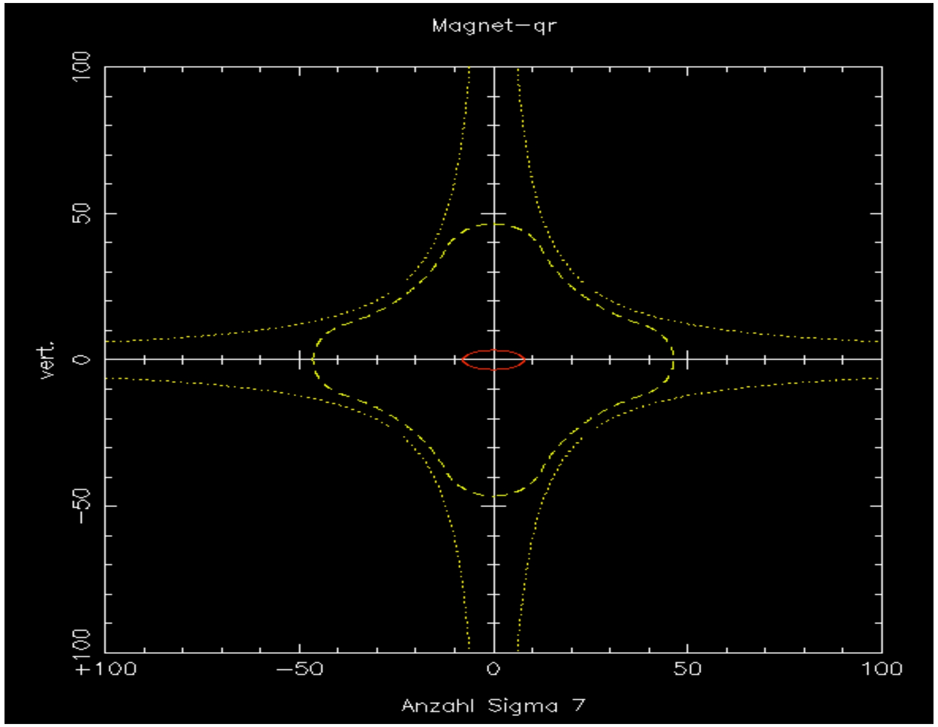
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance $\varepsilon\,(40GeV) = 1.2 * 10^{-7}$
$\varepsilon\,(920GeV) = 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

The „ not so ideal world "

14.) The „ $\Delta p / p \neq 0$" Problem

ideal accelerator: ***all particles will see the same accelerating voltage.***

$\rightarrow \Delta p / p = 0$

„nearly ideal" accelerator: ***Cockroft Walton or van de Graaf***

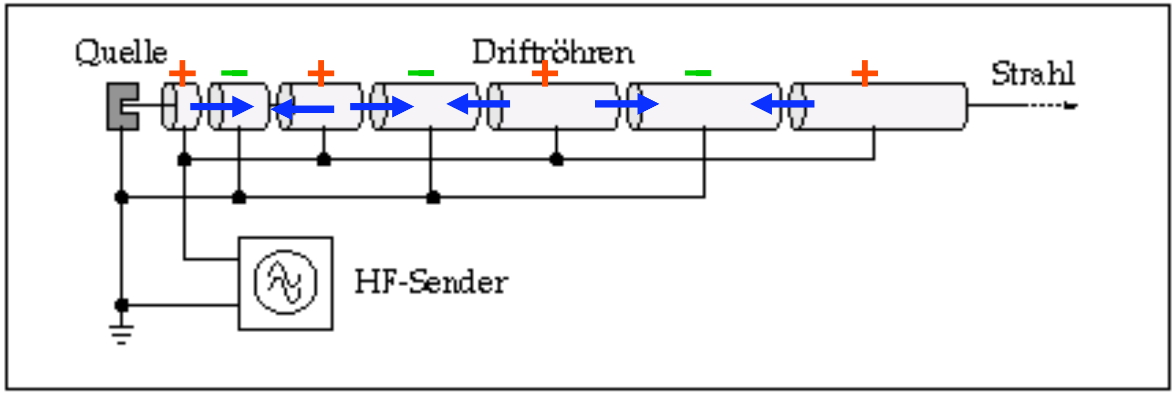
$\Delta p / p \approx 10^{-5}$

Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

RF Acceleration

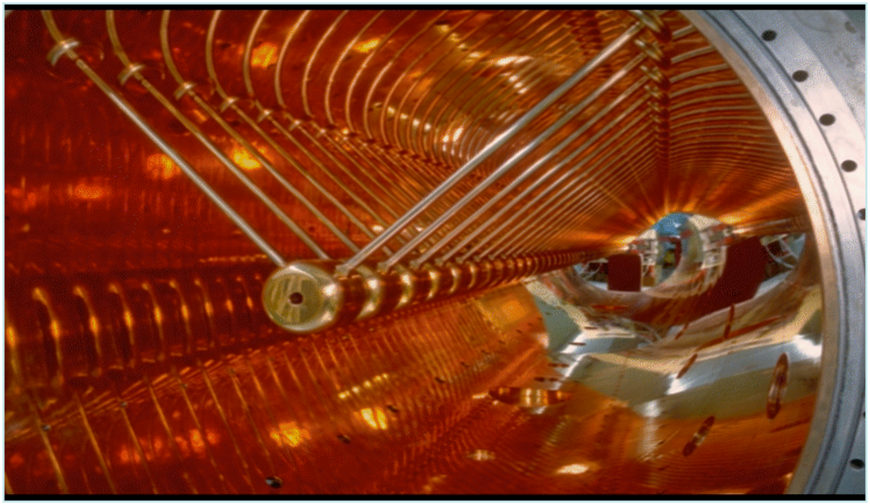
1928, Wideroe



Energy Gain per „Gap":

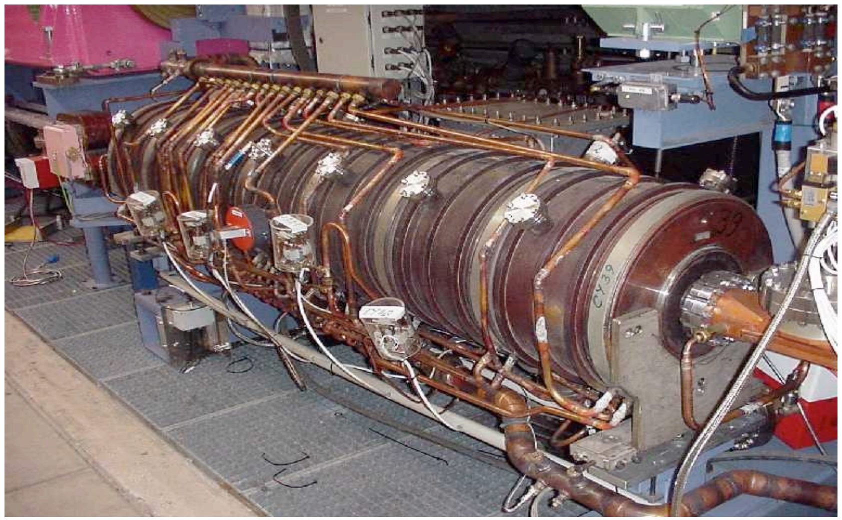
$$W = n * q \, U_0 \sin\omega_{RF} t$$

drift tube structure at a proton linac (GSI Unilac)



n *number of gaps between the drift tubes*
q *charge of the particle*
$\boldsymbol{U_0}$ *Peak voltage of the RF System*
$\boldsymbol{\Psi_S}$ *synchronous phase of the particle*

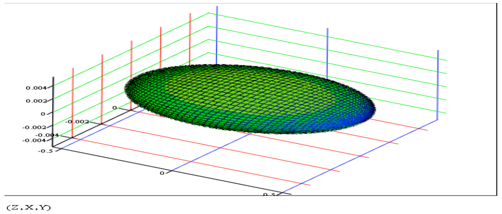
500 MHz cavities in an electron storage ring



* ***RF Acceleration:*** ***multiple application of the same acceleration voltage; brillant idea to gain higher energies***

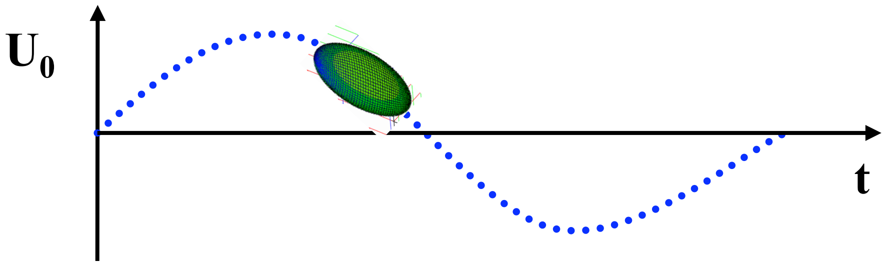
RF Acceleration-Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



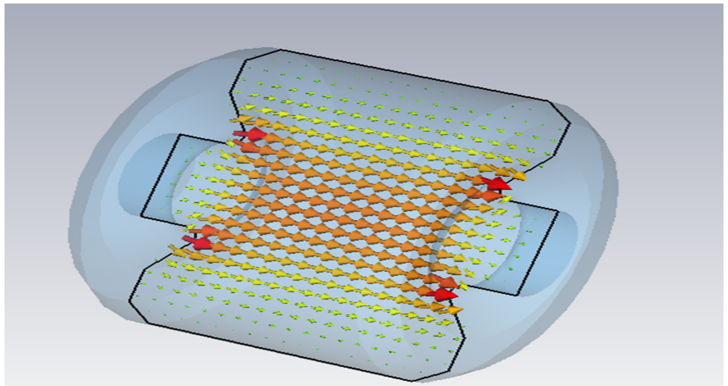
Bunch length of Electrons ≈ 1cm

just a stupid (and nearly wrong) example)



$\lambda = 75\ cm$

$$\nu = 400\,MHz$$
$$c = \lambda\,\nu$$
$$\lambda = 75\ cm$$



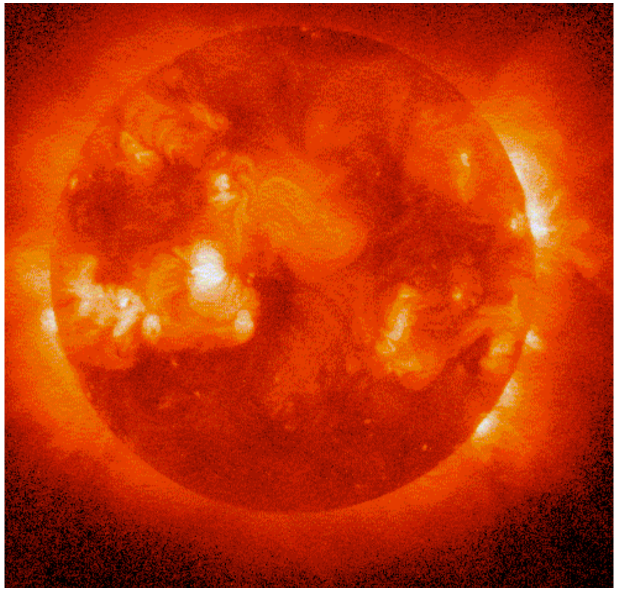
$$\sin(90^o) = 1$$
$$\sin(84^o) = 0.994$$

$$\frac{\Delta U}{U} = 6.0\ \ 10^{-3}$$

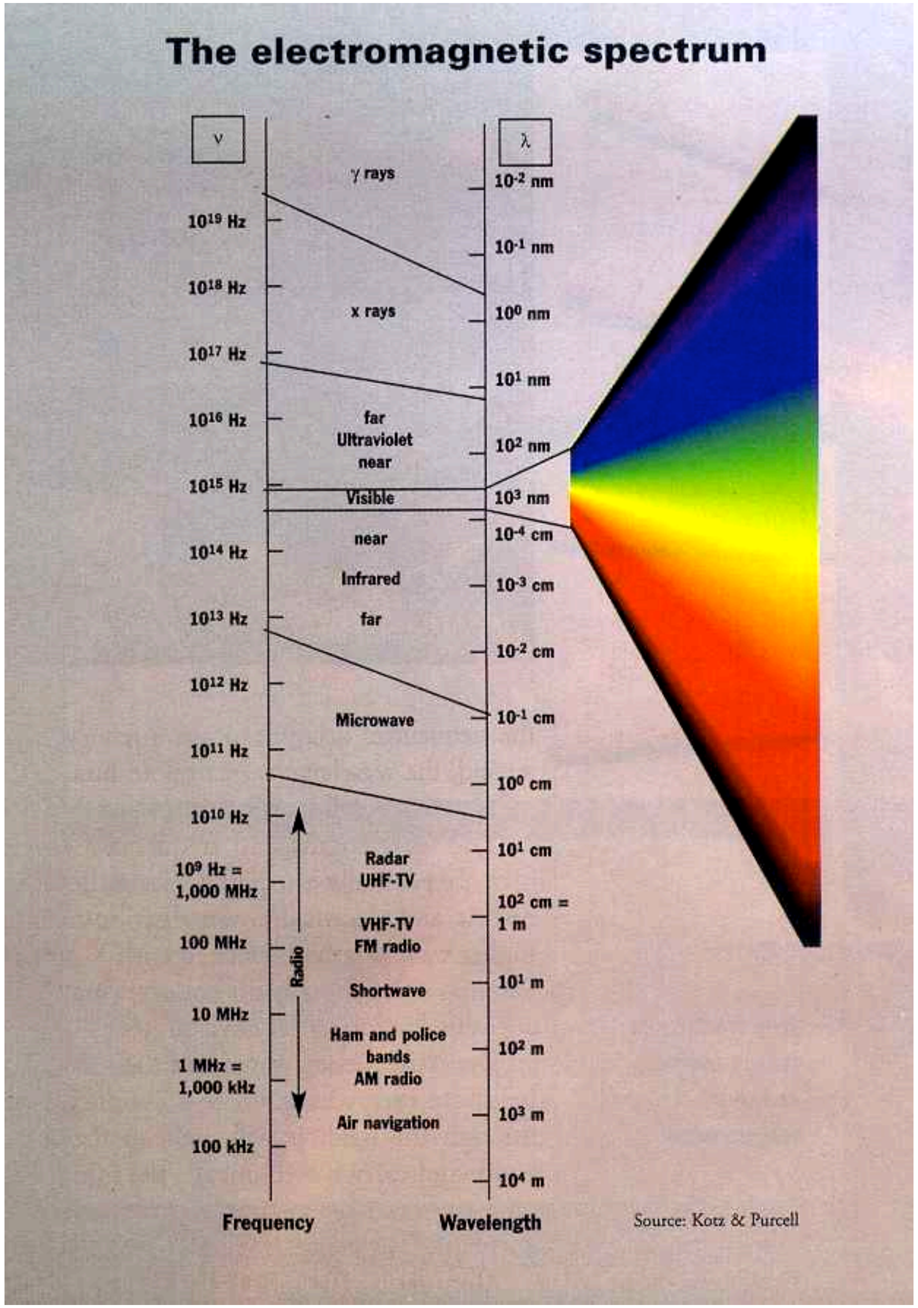
typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0\ \ 10^{-3}$$

Electromagnetic Spectrum:



Sun, looking a bit closer ...



visible light:

$\lambda \approx 400$ nm … 800 nm

1 Oktave

Dispersive and Chromatic Effects: $\Delta p/p \neq 0$

Are there any Problems ???

Sure there are !!!

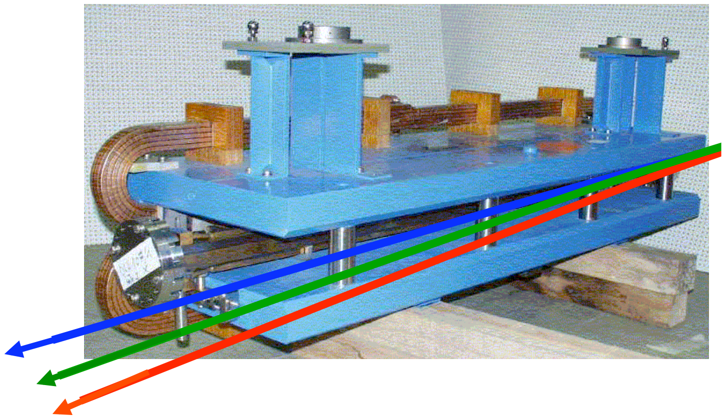
font colors due to pedagogical reasons

15.) Dispersion and Chromaticity: Magnet Errors for $\Delta p/p \neq 0$

Influence of external fields on the beam: ***prop. to magn. field & prop. zu 1/p***

dipole magnet

$$\alpha = \frac{\int B\,dl}{p/e}$$



$$x_D(s) = D(s)\,\frac{\Delta p}{p}$$

focusing lens

$$k = \frac{g}{p/e}$$

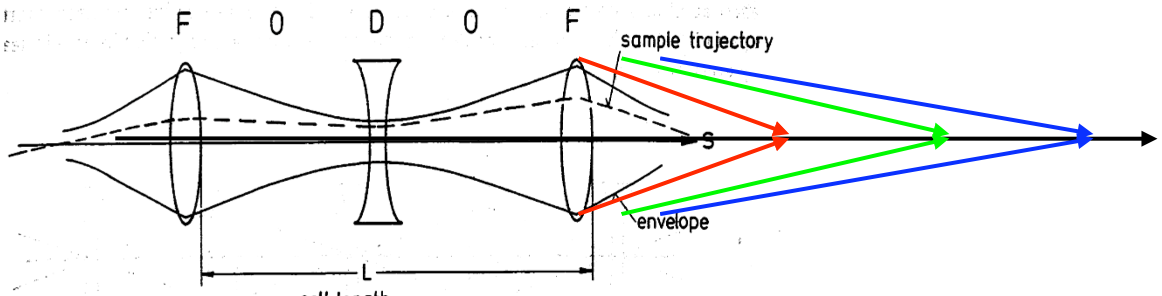


Figure 29: FODO cell

particle having ...

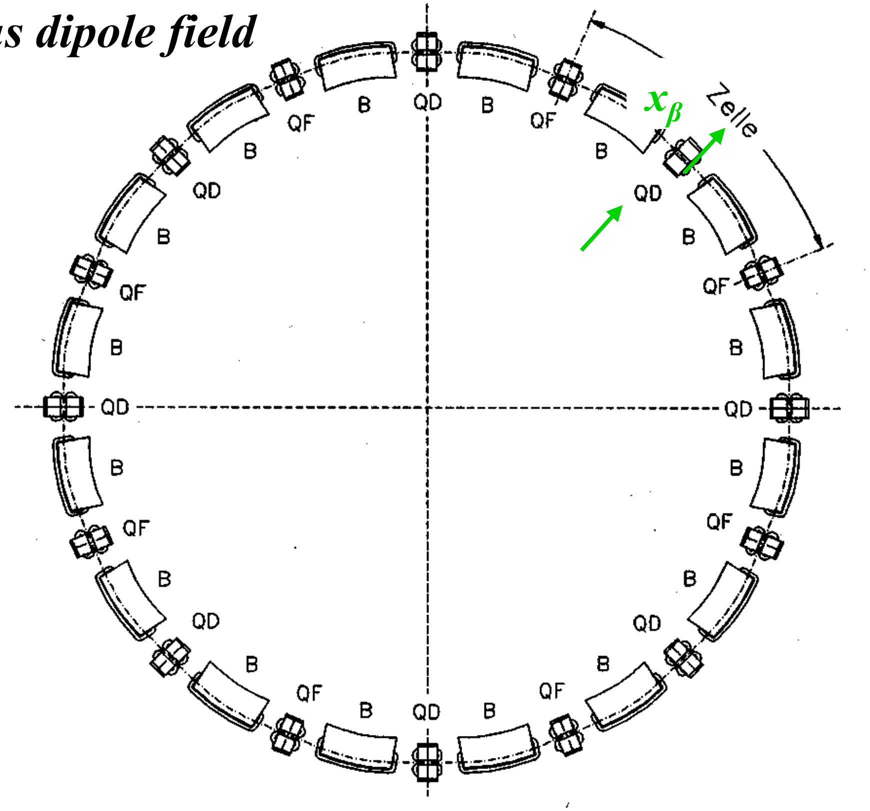
to high energy

to low energy

ideal energy

Dispersion

Example: homogeneous dipole field



ıit for Δp/p > 0

$: D(s) \cdot \frac{\Delta p}{p}$

Matrix formalism:

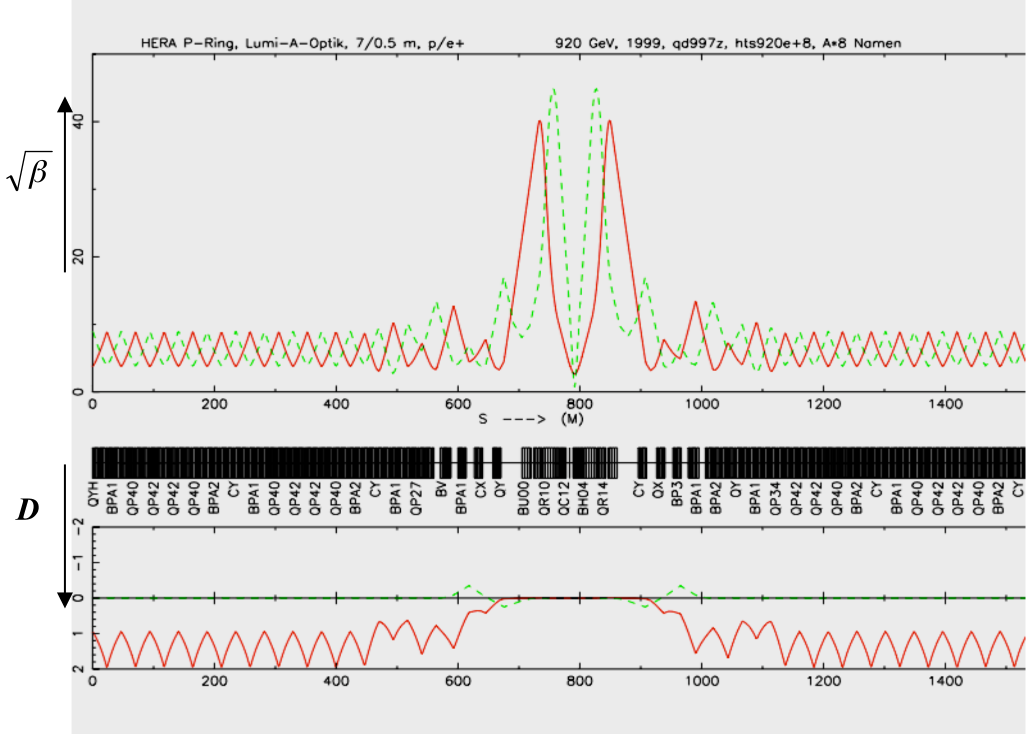
$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example

$$x_\beta = 1 \ldots 2\,mm$$
$$D(s) \approx 1 \ldots 2\,m$$
$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation

contribution due to Dispersion ≈ beam size

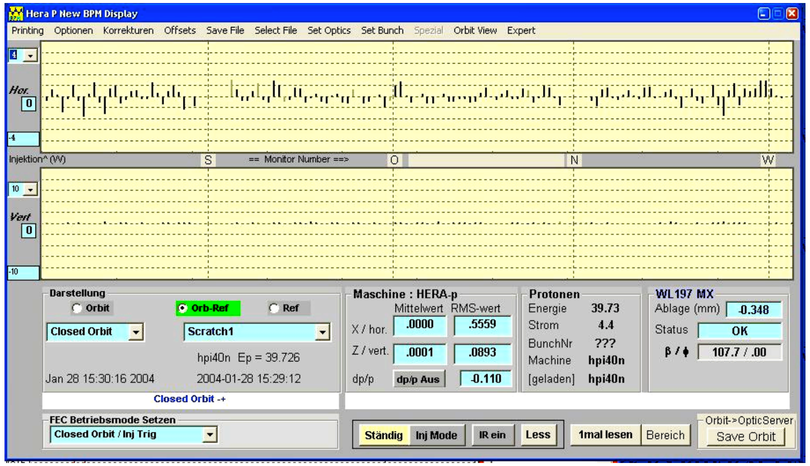
→ Dispersion must vanish at the collision point

!

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\,d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\,d\tilde{s}$$

Dispersion is visible



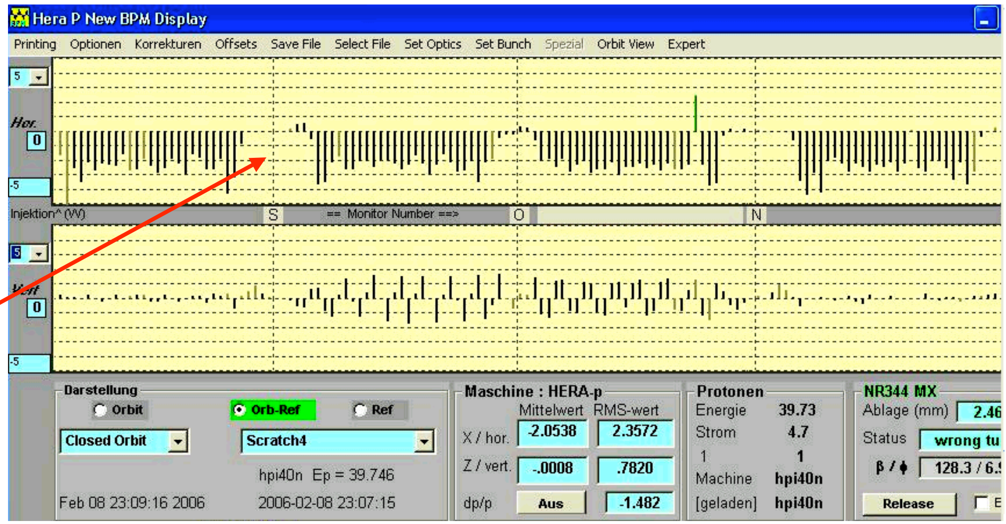
HERA Standard Orbit

HERA Dispersion Orbit

dedicated energy change of the stored beam

→ closed orbit is moved to a dispersions trajectory

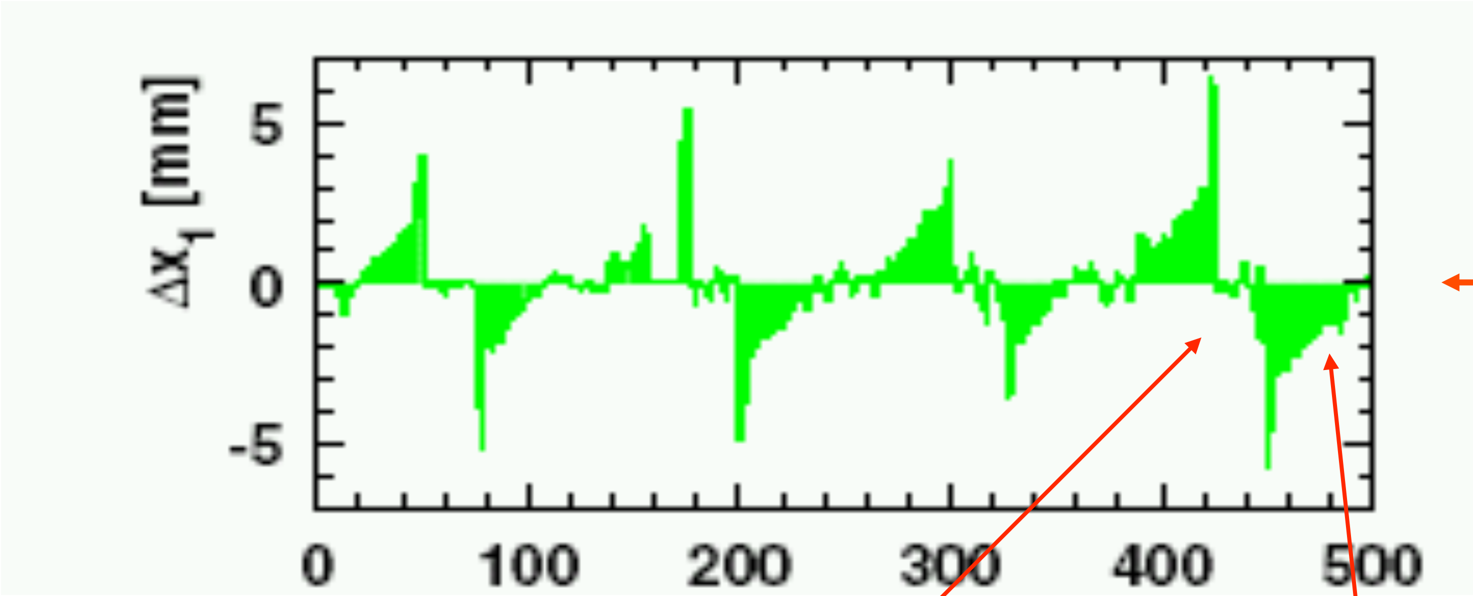
$$x_D = D(s) * \frac{\Delta p}{p}$$



Attention: at the Interaction Points we require D=D´= 0

Periodic Dispersion:

„Sawtooth Effect“ at LEP (CERN)



Electron course

BPM Number

In the straight sections they are accelerated by the rf cavities so much that they „overshoot“ and reach nearly the outer side of the vacuum chamber.

In the arc the electron beam loses so much energy in each octant that the particle are running more and more on a dispersion trajectory.

16.) Chromaticity: A Quadrupole Error for Δp/p ≠ 0

Influence of external fields on the beam: prop. to magn. field & prop. zu 1/p

Remember the normalisation of the external fields:

focusing lens $$k = \frac{g}{p/e}$$

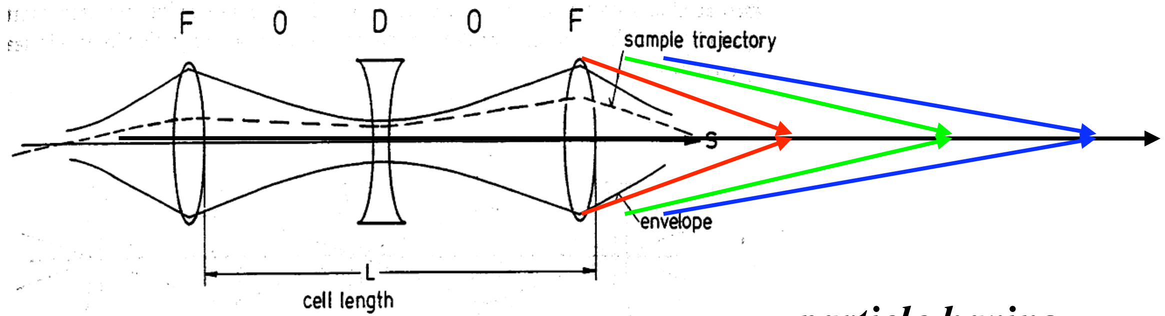


Figure 29: FODO cell

particle having ...
to high energy
to low energy
ideal energy

a particle that has a higher momentum feels a weaker quadrupole gradient and has a lower tune.

$$\Delta Q = -\frac{1}{4\pi}\frac{\Delta p}{p_0} k_0 \beta(s) ds$$

definition of chromaticity:

$$\boxed{\Delta Q = Q' * \frac{\Delta p}{p}}$$

… what is wrong about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

Q' is a number indicating the size of the tune spot in the working diagram,
Q' is always created if the beam is focussed
→ *it is determined by the focusing strength k of all quadrupoles*

$$\boldsymbol{Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds}$$

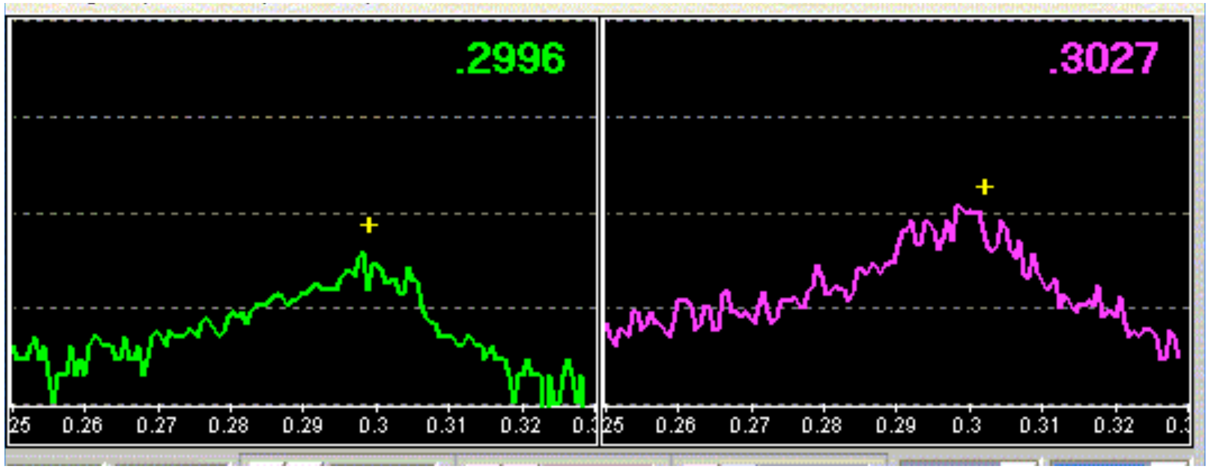
k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: LHC

$$\left.\begin{array}{l} Q' = 250 \\ \Delta p/p = +/- \; 0.2 \cdot 10^{-3} \\ \Delta Q = 0.256 \ldots 0.36 \end{array}\right\}$$

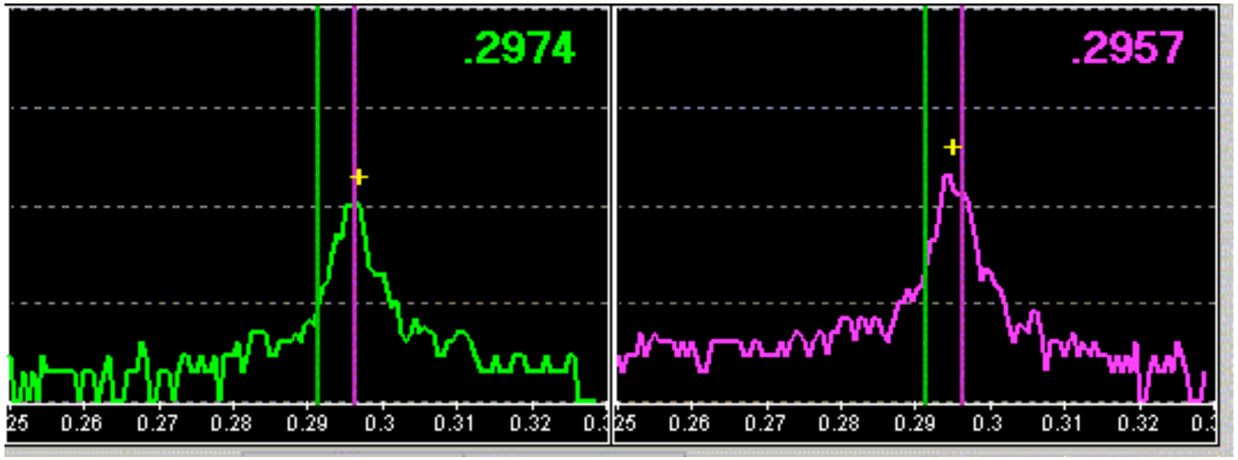
→ *Some particles get very close to resonances and are lost*

in other words: the tune is not a point
it is a pancake



Tune signal for a nearly uncompensated cromaticity (Q' ≈ 20)

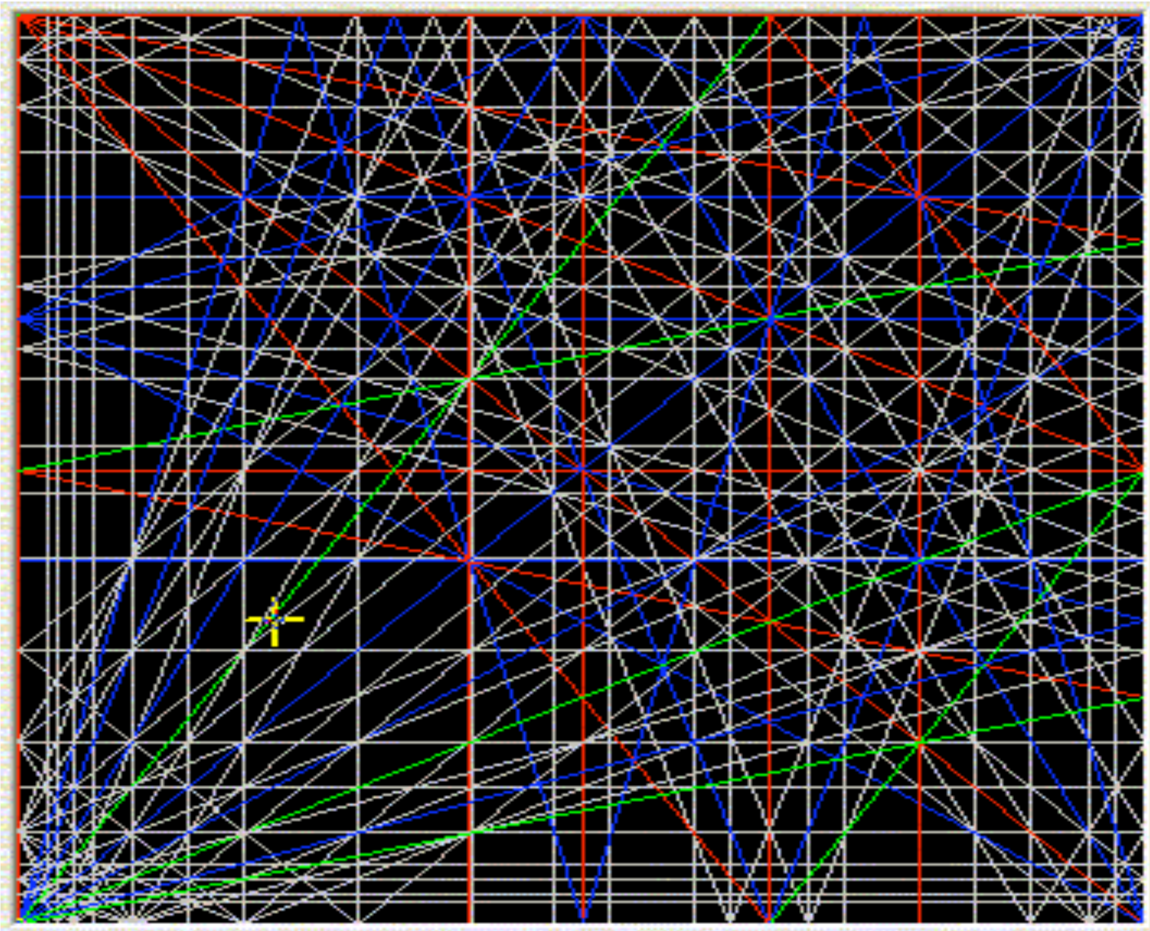
Ideal situation: cromaticity well corrected, (Q' ≈ 1)



Tune and Resonances

$$m*Q_x+n*Q_y+l*Q_s = integer$$

Tune diagram up to 3rd order



… and up to 7th order

Homework for the operateurs: find a nice place for the tune where against all probability the beam will survive

Chromaticity Correction:

We need a magnetic field that focuses stronger those individual particles that have larger momentum and focuses weaker those with lower momentum.

... but that does not exist.

Trick: 1.) sort the particle trajectories according to their energy

2.) introduce magnetic fields that increase stronger than linear with the distance Δx to the centre

3.) calculate these fields (sextupoles) in a way that the lack of focusing strength is exactly compensated.

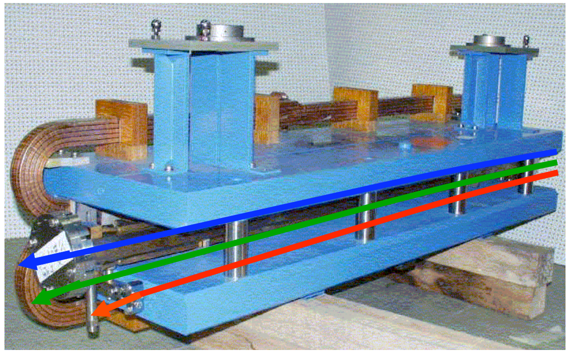
we use the dispersion to do the job

Correction of Q':

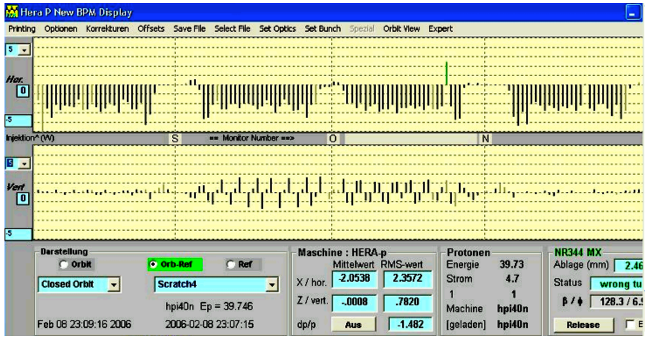
Need: additional quadrupole strength for each momentum deviation Δp/p

1.) sort the particles acording to their momentum

$$x_D(s) = D(s)\frac{\Delta p}{p}$$



… using the dispersion function



2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left.\begin{aligned} B_x &= \tilde{g}xz \\ B_z &= \frac{1}{2}\tilde{g}(x^2 - z^2) \end{aligned}\right\} \qquad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

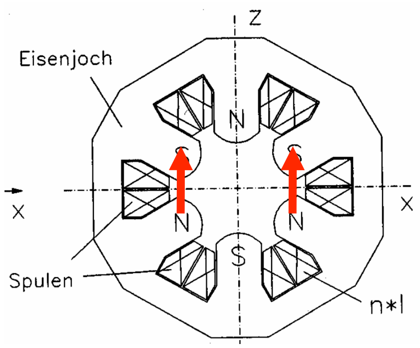
linear rising „gradient“:

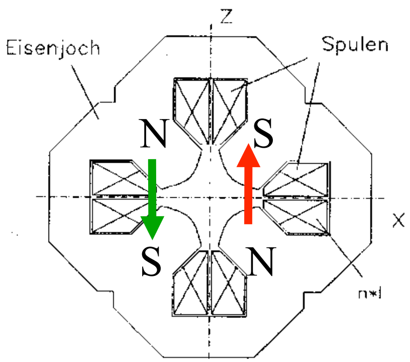
Correction of Q':

k_1 normalised quadrupole strength

k_2 normalised sextupole strength

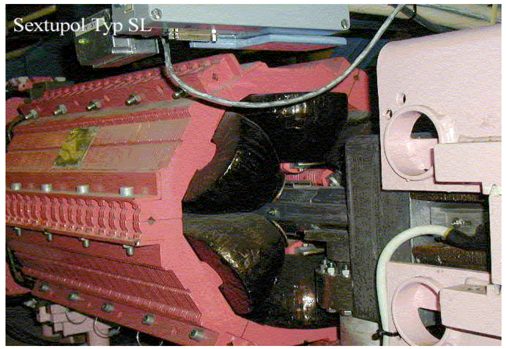
Sextupole Magnets:





$$k_1(sext) = \frac{\tilde{g}\, x}{p/e} = k_2 * x$$

$$k_1(sext) = k_2 * D * \frac{\Delta p}{p}$$



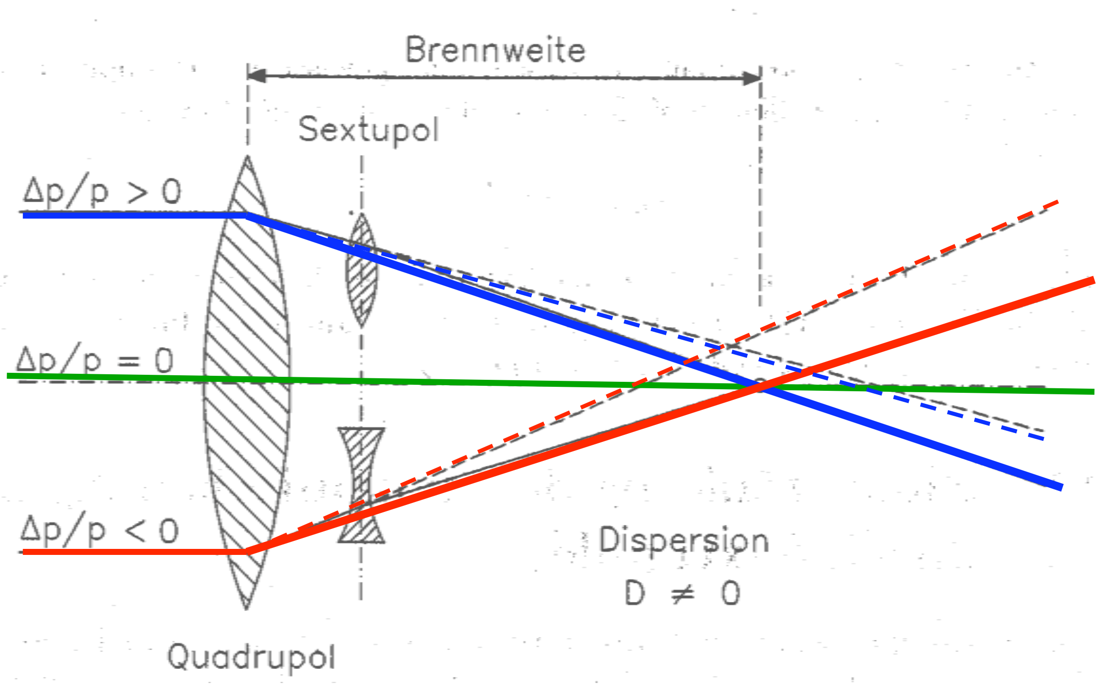
corrected chromaticity

considering a single cell:

$$Q'_{cell\_x} = -\frac{1}{4\pi}\left\{k_{qf}\hat{\beta}_x l_{qf} - k_{qd}\breve{\beta}_x l_{qd}\right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

$$Q'_{cell\_y} = -\frac{1}{4\pi}\left\{-k_{qf}\breve{\beta}_y l_{qf} + k_{qd}\hat{\beta}_y l_{qd}\right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

Chromatizitätskorrektur:



Einstellung am Speicherring:

Sextupolströme so variieren, dass $\xi \approx +1 \ldots +2$

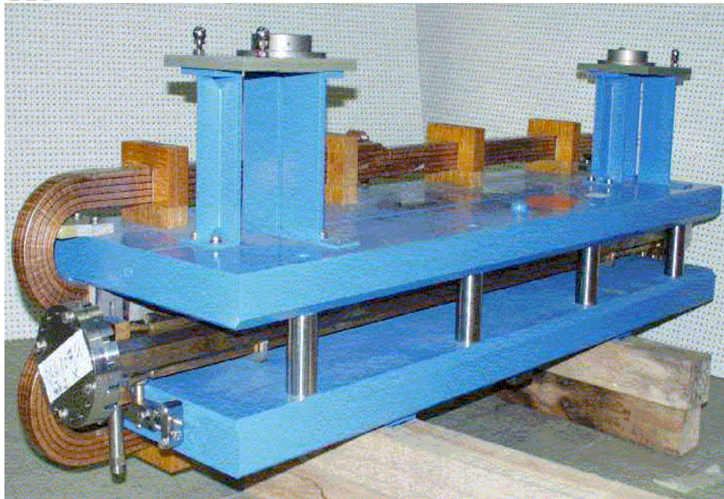
A word of caution: keep non-linear terms in your storage ring low.

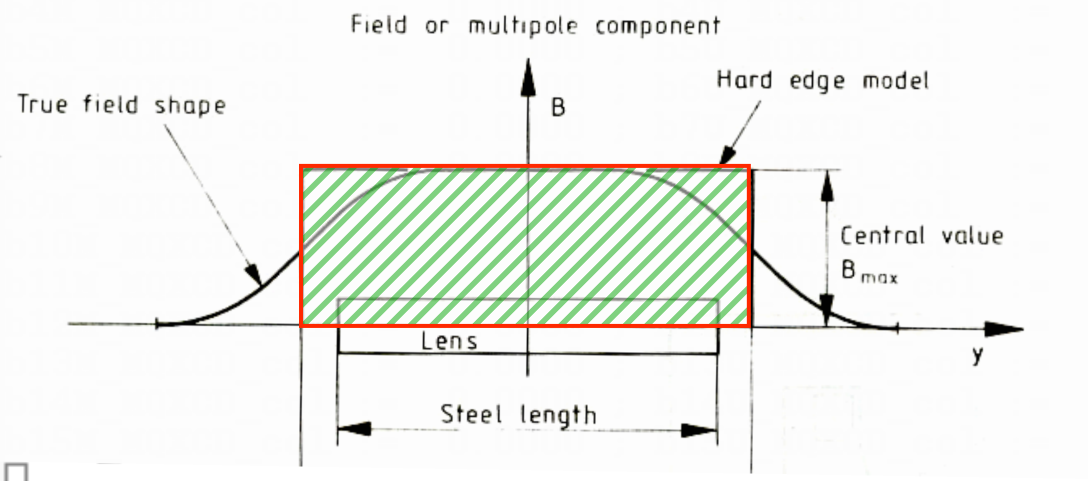
$$B_y + iB_x = B_{ref} * \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x+iy}{r_0} \right)^{n-1}$$

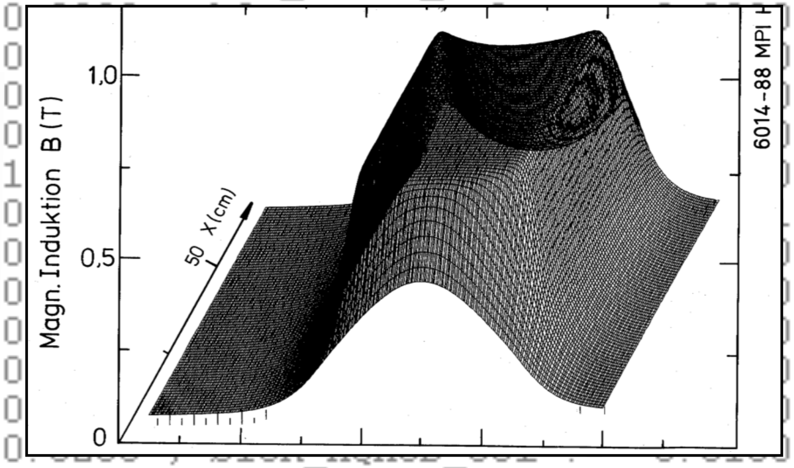
"effective magnetic length"

$$B * l_{eff} = \int_0^{l_{mag}} B\,ds$$

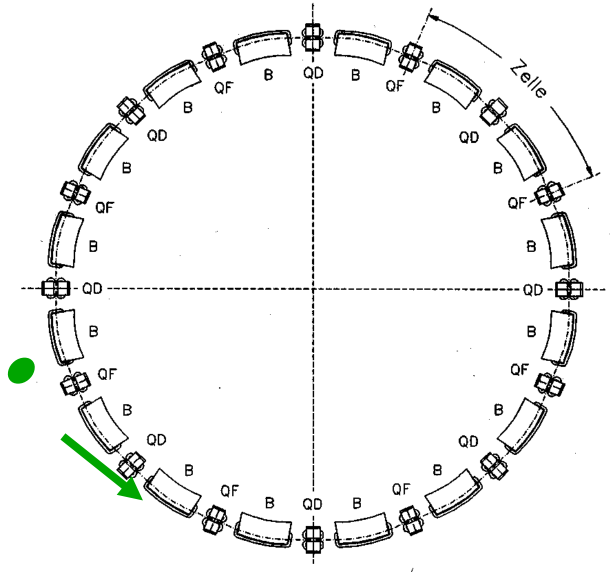
```
bn at injection
b1M_MQXCD_inj  :=  0.0000 ; b1U_MQXCD_inj   :=               0000
b2M_MQXCD_inj  :=  0.0000 ; b2U_MQXCD_inj   :=               0000
                                            :=               8900
                                            :=               6400
                                            :=               4600
                                            :=               2800
                                            :=               2100
                                            :=               1600
b9M_MQXCD_inj  :=  0.0000 ; b9U_MQXCD_inj   :=               0800
b10M_MQXCD_inj :=  0.5000 ; b10U_MQXCD_inj  :=               0600
b11M_MQXCD_inj :=  0.0000 ; b11U_MQXCD_inj  :=               0300
                            12U_MQXCD_inj   :=               0200
                            13U_MQXCD_inj   :=               0100
b14M_MQXCD_inj := -0.2700 ; b14U_MQXCD_inj  :=  0.0300 ; b14R_MQXCD_inj  :=  0.0100
b15M_MQXCD                  b15U_MQXCD_inj  :=  0.0000 ; b15R_MQXCD_inj  :=  0.0000

bn in collision
b1M_MQXCD_c                 b1U_MQXCD_col   :=  0.0000 ; b1R_MQXCD_col   :=  0.0000
b2M_MQXCD_col  :=  0.0000 ; b2U_MQXCD_col   :=
                            0.0400 ; b14R_MQXCD_col  :=  0.0100
                            0.0000 ; b15R_MQXCD_col  :=  0.0000
```







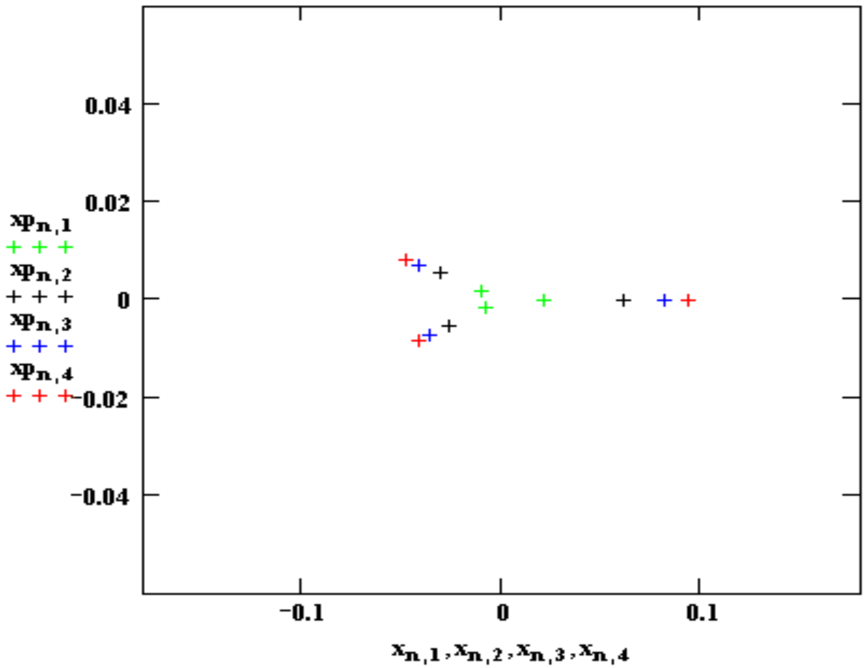
Clearly there is another problem ...
... if it were easy everybody could do it



Again: the phase space ellipse

for each turn write down – at a given position „s" in the ring – the single partilce amplitude x and the angle x´... and plot it.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{turn} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

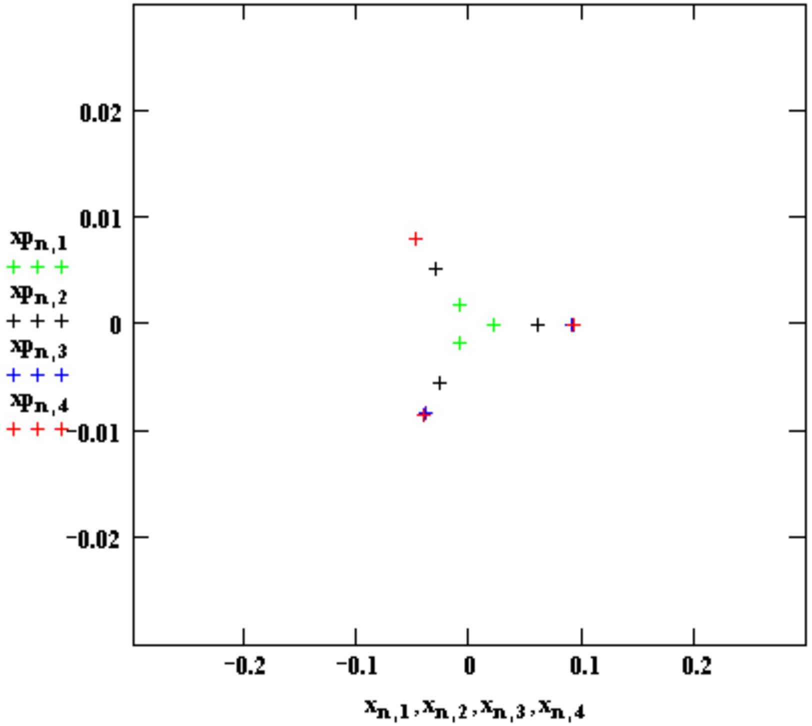


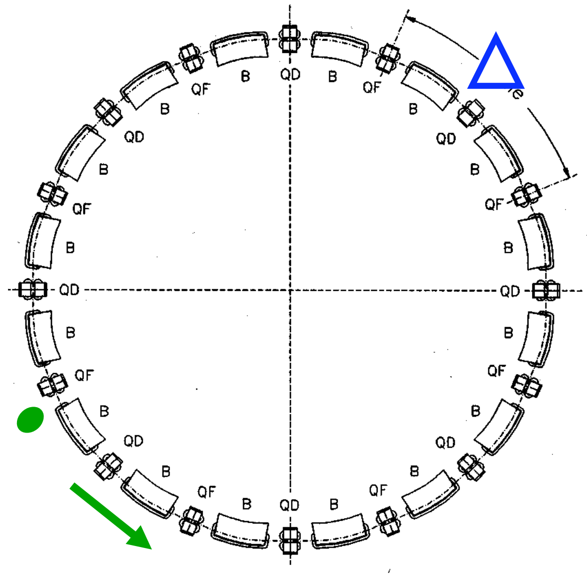
A beam of 4 particles

– each having a slightly different emittance:

Installation of a weak (!!!) sextupole magnet

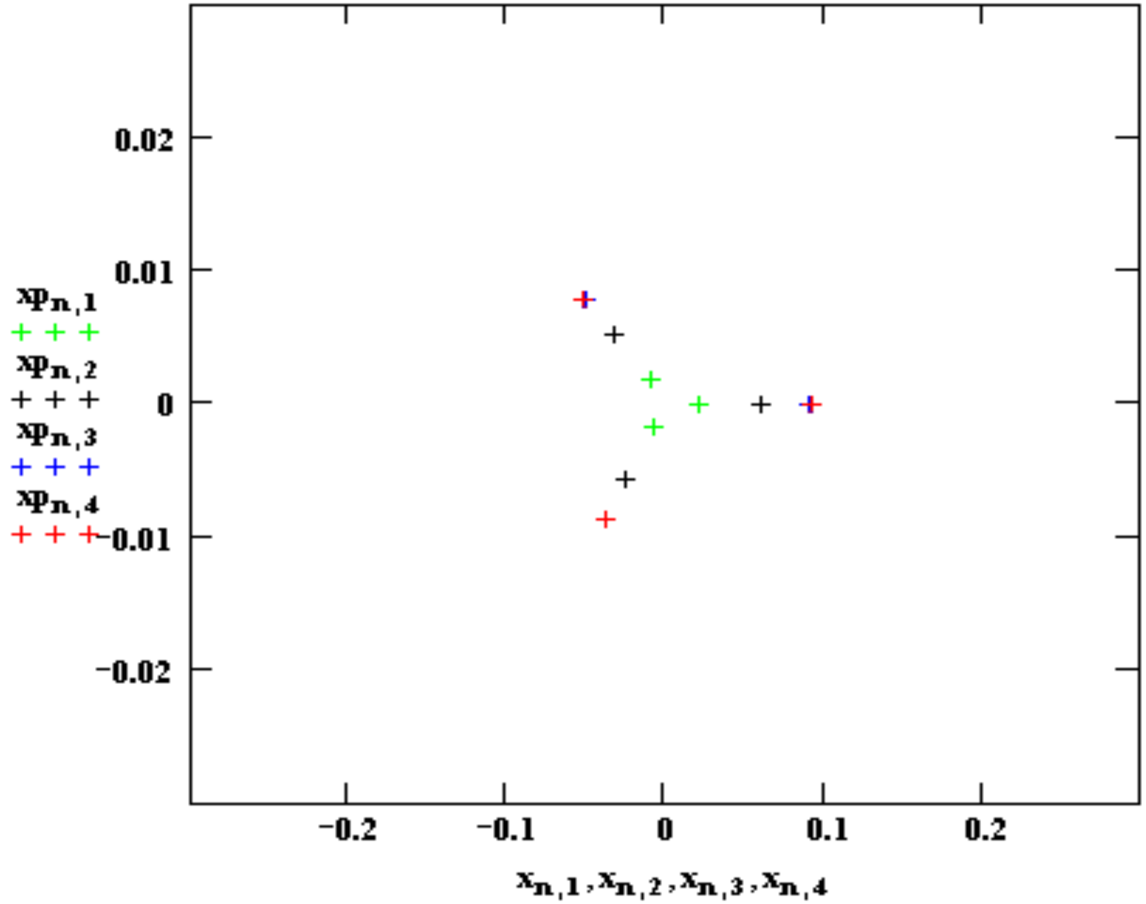
The good news: sextupole fields in accelerators cannot be treated analytically anymore.
→ no equatiuons; instead: Computer simulation
„ particle tracking "

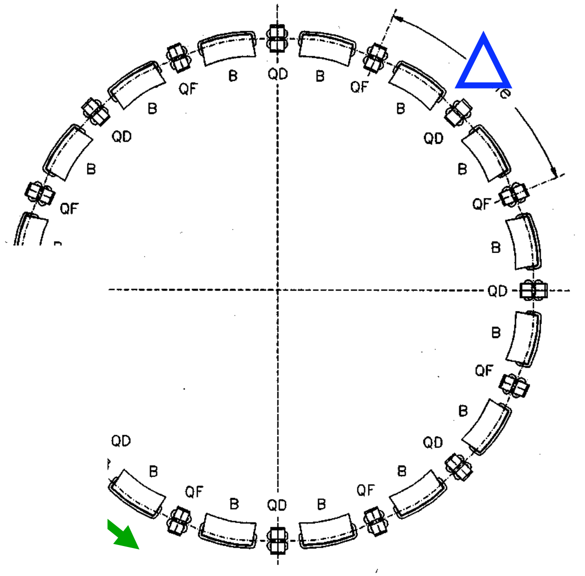




Effect of a strong (!!!) Sextupole …

→ *Catastrophy !*





„dynamic aperture"

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