Short Introduction to (Classical) Electromagnetic Theory

(.. and applications to accelerators)

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(http://cern.ch/Werner.Herr/CAS2014_Chavannes/em.pdf)

Why electrodynamics ?

- **Accelerator physics relies on electromagnetic concepts:**
  - **Beam dynamics**
  - **Magnets, cavities**
  - **Beam instrumentation**
  - **Powering**
  - **...**

OUTLINE

- **Some mathematics (intuitive, mostly illustrations), see also lecture R. Steerenberg**
- **Basic electromagnetic phenomena**
- **Maxwell's equations**
- **Lorentz force**
- **Motion of particles in electromagnetic fields**
- **Electromagnetic waves in vacuum**
- **Electromagnetic waves in conducting media**
  - **Waves in RF cavities**
  - **Waves in wave guides**

Reading Material

- **J.D. Jackson,** *Classical Electrodynamics* **(Wiley, 1998 ..)**
- **L. Landau, E. Lifschitz,** *KlassischeFeldtheorie*, **Vol2. (Harri Deutsch, 1997)**
- **W. Greiner,** *Classical Electrodynamics*, **(Springer, February, 22nd, 2009)**
- **J. Slater, N. Frank,** *Electromagnetism*, **(McGraw-Hill, 1947, and Dover Books, 1970)**
- **R.P. Feynman,** *Feynman lectures on Physics*, **Vol2.**

First some mathematics (vectors, potential, calculus ....)

Reminder: mathematics used here

- **Addition to previous lecture (R.S.)**
- **Not all details are strictly needed to understand, but required for calculations**
- **I shall introduce:**
  - **Scalar and vector fields**
  - **Calculation on fields (vector calculus)**
  - **Illustrations and examples ...**

    Remark: **many illustrations only in 2 dimensions**

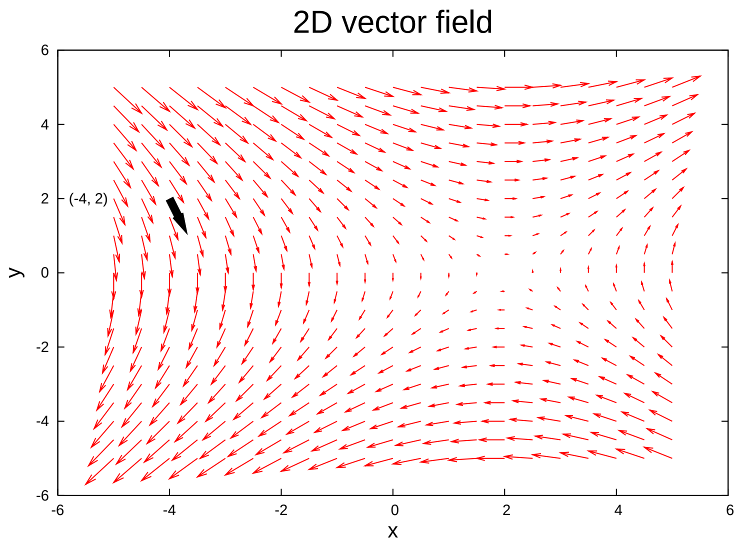
A bit on scalar fields (potentials)

- **At each point in space has assigned a quantity with a value (real or complex)**
- **Described by a scalar** $\phi(x, y, z)$ **(a number)**

  Example: $\phi(x, y, z) \; = \; 0.1x^2 - 0.2 \cdot x \cdot y + z^2$

➡ **We get for** $(x = 4, y = 2, z = 1)$**:** $\phi(-4, 2, 1) \; = \; 4.2$

A bit on vector fields ...



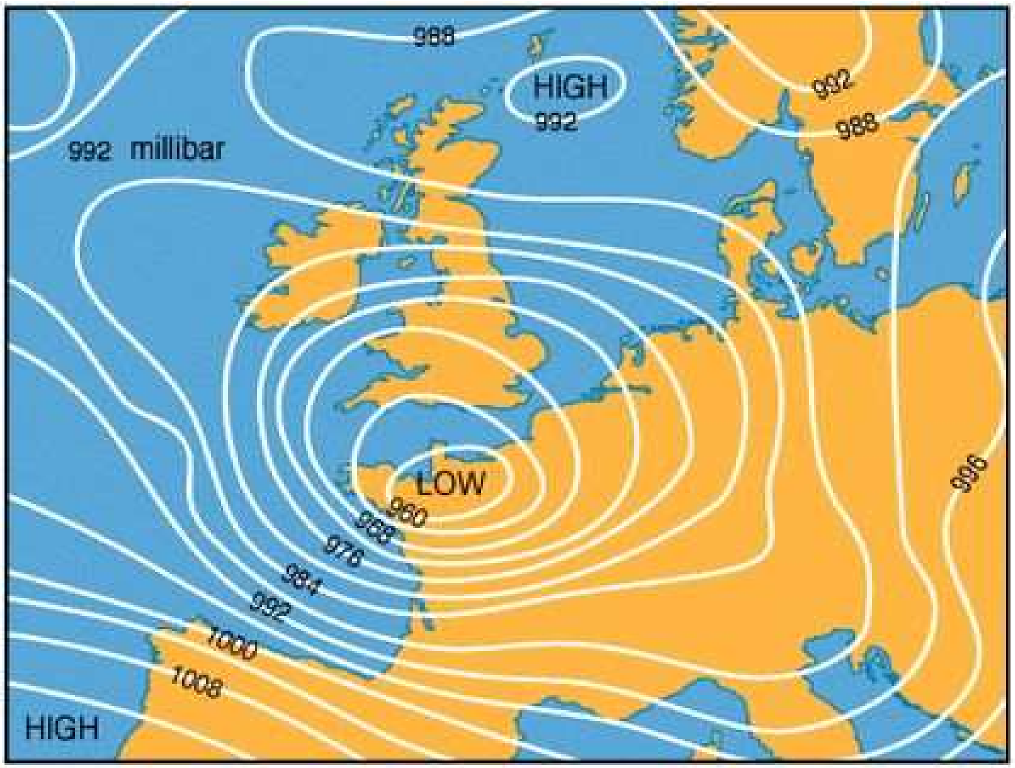
- At each point in space (or plane): a quantity with a length and direction, (typically 2, 3, 4, 6 components)
- A vector with 3 components: $\vec{F}(x, y, z) \;=\; (F_x, F_y, F_z)$
- Example (in 2D): $\vec{F}(x, y) \;=\; (0.1y, 0.1x - 0.2)$

➡ We get: $\vec{F}(-4, 2) \;=\; (0.2, -0.6)$

Examples:

- **Scalar fields:**
  - **Atmospheric pressure**
  - **Temperature in a room**
  - **Density of molecules in a gas**
- **Vector fields:**
  - **Speed and direction of wind ..**
  - **Heat flow**
  - **Velocity and direction of moving molecules in a gas**

Example: scalar field/potential ...



Lines of pressure (isobars)

Function of longitude, latitude and altitude (x, y, z)

Example: vector field ...

Example for an extreme vector field ..

What we shall talk about

Maxwell's equations relate Electric and Magnetic fields from charge and current distributions (SI units).

$\vec{E}$	=	**electric field [V/m]**
$\vec{H}$	=	**magnetic field [A/m]**
$\vec{D}$	=	**electric displacement [C/m^2]**
$\vec{B}$	=	**magnetic flux density [T]**
q	=	**electric charge [C]**
ρ	=	**electric charge density [C/m^3]**
$\vec{j}$	=	**current density [A/m^2]**
μ_0	=	**permeability of vacuum, 4** $\pi \cdot 10^{-7}$ **[H/m or N/A^2]**
ϵ_0	=	**permittivity of vacuum, 8.854** $\cdot 10^{-12}$ **[F/m]**
c	=	**speed of light, 2.99792458** $\cdot 10^{8}$ **[m/s]**

Electromagnetic fields

In electrodynamics we talk about vector fields:

Electric phenomena: $\vec{E}$ **and** $\vec{D}$

Magnetic phenomena: $\vec{H}$ **and** $\vec{B}$

➡ **Electrodynamics: need vectors with 3 components**

➡ **Need to know how to calculate with vectors**

- **Scalar and vector products**
- **Vector calculus products**

Scalar products

Define a scalar product for (usual) vectors like: $\vec{a} \cdot \vec{b}$,

$$\vec{a} \; = \; (x_a, y_a, z_a) \qquad \vec{b} \; = \; (x_b, y_b, z_b)$$

$$\vec{a} \cdot \vec{b} \;\; = \;\; (x_a, y_a, z_a) \;\cdot\; (x_b, y_b, z_b) = (x_a \cdot x_b \; + y_a \cdot y_b \; + \; z_a \cdot z_b)$$

This product of two vectors is a <u>scalar</u> (number) not a vector.

(on that account: Scalar Product)

Example:

$$(-2, 2, 1) \;\cdot\; (2, 4, 3) \;=\; -2 \cdot 2 + 2 \cdot 4 + 1 \cdot 3 \;\; = \;\; 7$$

Vector products (sometimes cross product)

Define a vector product for (usual) vectors like: $\vec{a} \times \vec{b}$**,**

$$\vec{a} \;=\; (x_a, y_a, z_a) \qquad\qquad \vec{b} \;=\; (x_b, y_b, z_b)$$

$$\begin{aligned} \vec{a} \times \vec{b} \;&=\; (x_a, y_a, z_a) \;\times\; (x_b, y_b, z_b) \\ &= (\underbrace{y_a \cdot z_b - z_a \cdot y_b}_{x_{ab}}, \;\; \underbrace{z_a \cdot x_b - x_a \cdot z_b}_{y_{ab}}, \;\; \underbrace{x_a \cdot y_b - y_a \cdot x_b}_{z_{ab}}) \end{aligned}$$

This product of two vectors is a <u>vector</u>, not a scalar (number), (on that account: Vector Product)

Example 1:

$$(-2, 2, 1) \;\times\; (2, 4, 3) \;\;=\;\; (2, 8, -12)$$

Example 2 (two components only in the $x - y$ **plane):**

$(-2, 2, 0) \;\times\; (2, 4, 0) \;\;=\;\; (0, 0, -12)$ **(see R. Steerenberg)**

Vector calculus ...

We can define a special vector ∇ (sometimes written as $\vec{\nabla}$):

$$\nabla = (\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z})$$

It is called the "gradient" and invokes "partial derivatives".
It can operate on a scalar function $\phi(x, y, z)$:

$$\nabla\phi = (\frac{\partial\phi}{\partial x},\ \frac{\partial\phi}{\partial y},\ \frac{\partial\phi}{\partial z}) \quad = \quad \vec{G} = (G_x, G_y, G_z)$$

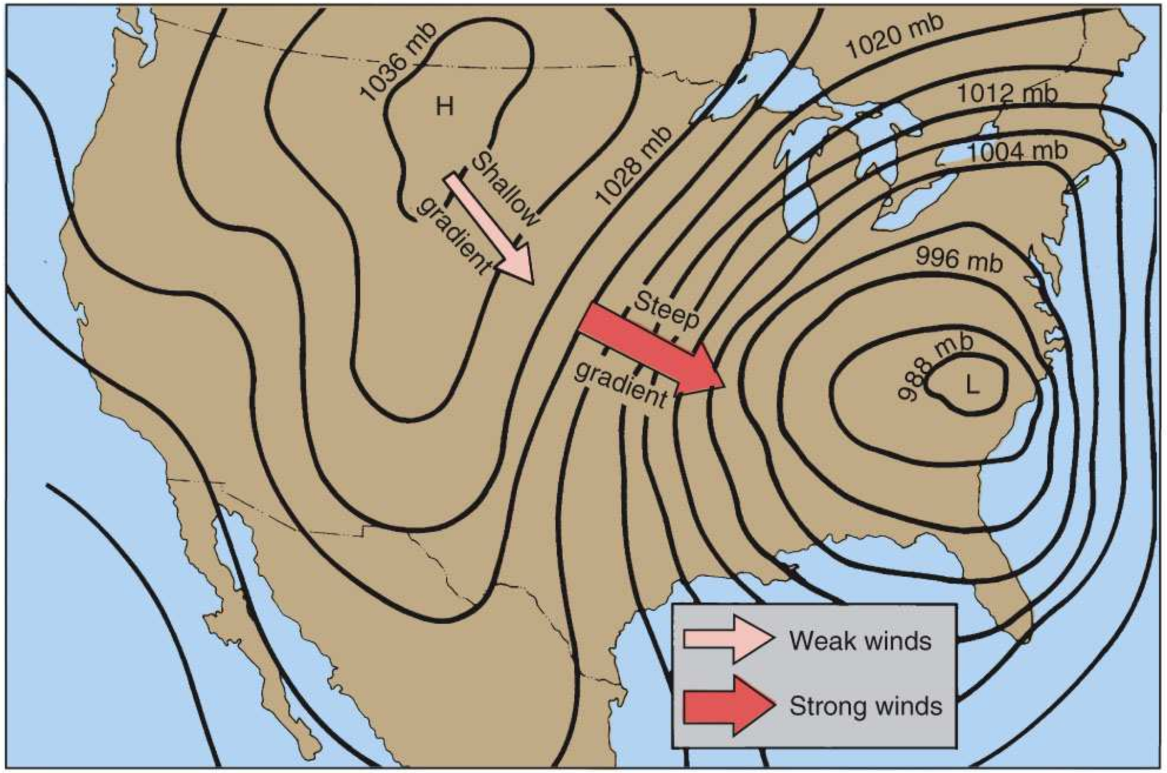
and we get a vector $\vec{G}$. It is a kind of "slope" (steepness ..) in the 3 directions.

Example: $\phi(x, y, z) \ = \ 0.1x^2 - 0.2 \cdot x \cdot y + z^2$

➡

$$\nabla\phi \ = \ \vec{G}(x, y, z) \ = \ (G_x, G_y, G_z) \ = \ (0.2x - 0.2y,\ \ -0.2x,\ 2z)$$

Gradient (slope) of a scalar field



Lines of pressure (isobars)

Gradient is large (steep) where lines are close (fast change of pressure)

Vector calculus ...

The gradient ∇ can be used as scalar or vector product with a vector $\vec{F}$, sometimes written as $\vec{\nabla}$
Used as:

$$\nabla \cdot \vec{F} \qquad \text{or} \qquad \nabla \times \vec{F}$$

Same definition for products as before, ∇ treated like a "normal" vector, but results depends on how they are applied:
$\nabla \cdot \Phi$ is a vector
$\nabla \cdot \vec{F}$ is a scalar
$\nabla \times \vec{F}$ is a vector

Operations on vector fields ...

Two operations of ∇ have special names:

Divergence (scalar product of gradient with a vector):

$$\mathsf{div}(\vec{F}) \quad = \quad \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Physical significance: ”amount of density”, (see later)

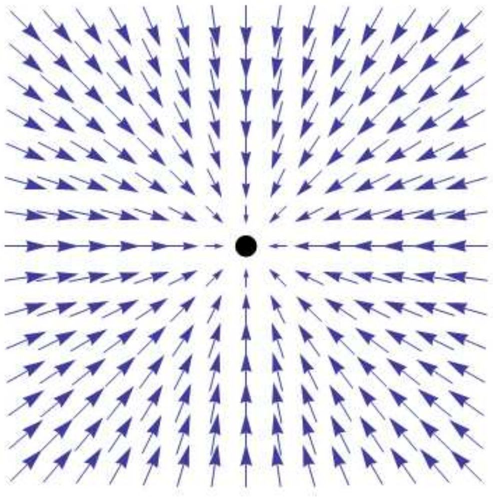
Curl (vector product of gradient with a vector):

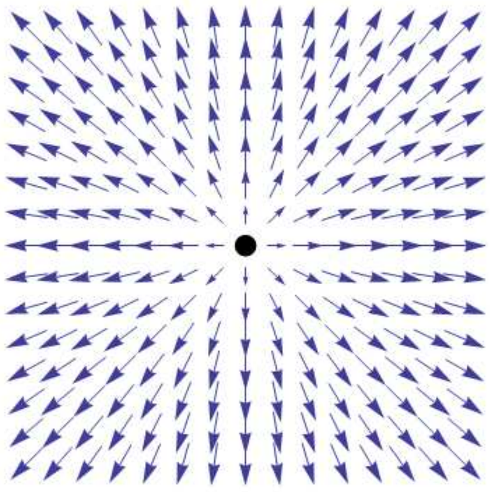
$$\mathsf{curl}(\vec{F}) \quad = \quad \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \; \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \; \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

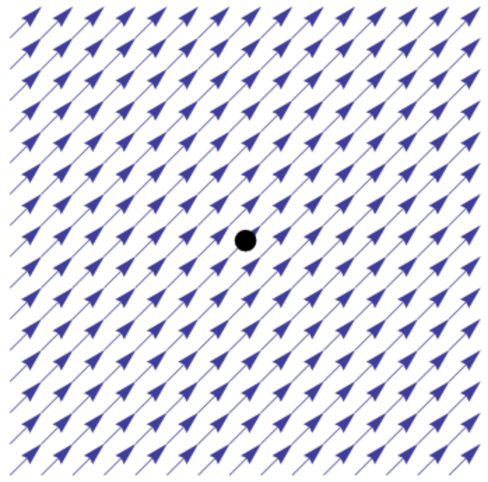
Physical significance: ”amount of rotation”, (see later)

Meaning of Divergence of fields ...

Field lines seen from some origin:







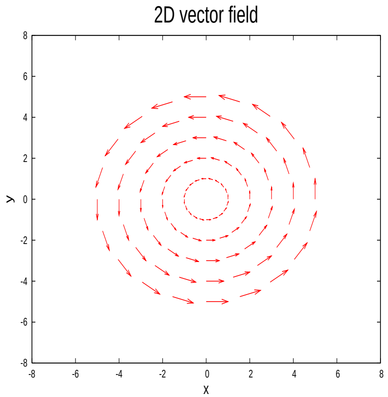
$\nabla \vec{F} < 0$
(sink)

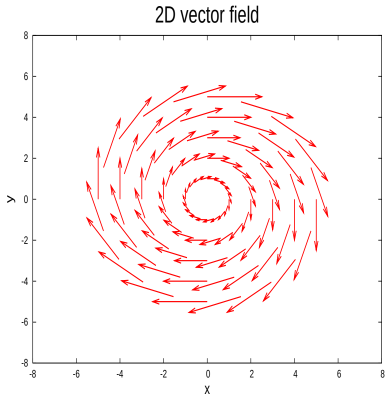
$\nabla \vec{F} > 0$
(source)

$\nabla \vec{F} = 0$
(fluid)

The divergence (scalar, a single number) characterizes what comes from (or goes to) the origin

Meaning of Curl of fields ...





Here we have fields in $x - y$ plane::

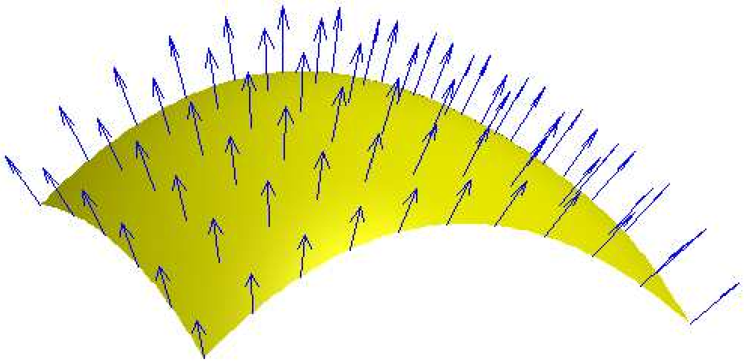
$\vec{F}_1 = (-0.2y, +0.2x, 0)$ $\quad$ $\vec{F}_2 = (+0.5y, -0.5x, 0)$

$\nabla \times \vec{F}_1 = \text{curl}\vec{F}_1 = (0, 0, +0.4)$ $\quad$ $\nabla \times \vec{F}_2 = \text{curl}\vec{F}_2 = (0, 0, -1.0)$

Vectors in z-direction, perpendicular to $x - y$ plane

Values characterize "strength" and "direction" of rotation

Integration of (vector-) fields



Surface integrals: integrate field vectors passing (perpendicular) through a surface S (or area A), we obtain the **Flux**:

$$\rightarrow \quad \int\int_A \vec{F} \cdot d\vec{A}$$

Density of field lines through the surface

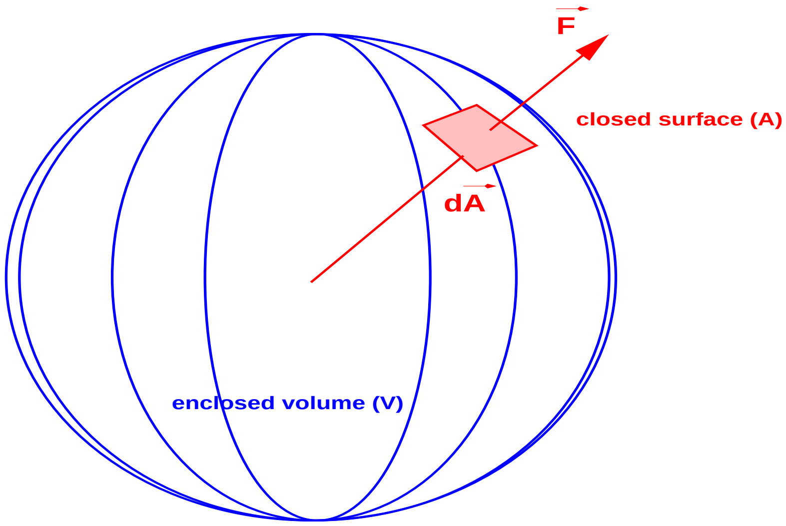
(e.g. amount of heat passing through a surface)

Easier Integration of (vector-) fields

Gauss' Theorem:

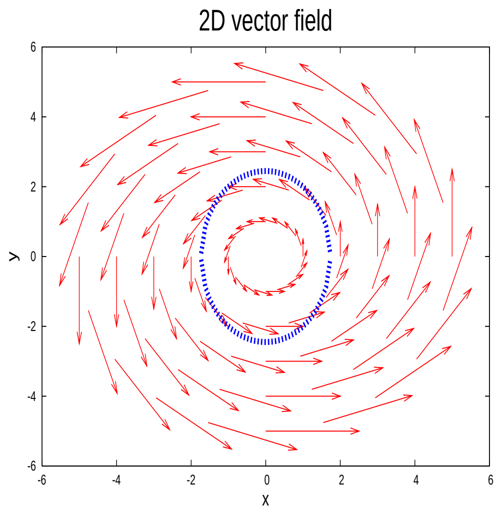
Integral through a closed surface (flux) is integral of divergence in the enclosed volume

$$\int\int_A \vec{F} \cdot d\vec{A} = \int\int\int_V \nabla \cdot \vec{F} \cdot dV$$



Relates surface integral to divergence

Integration of (vector-) fields



Line integrals: integrate field vectors along a line C:

$$\rightarrow \quad \oint_C \vec{F} \cdot d\vec{r}$$

"sum up" vectors (length) in <u>direction</u> of line C
Integral often called Circulation.

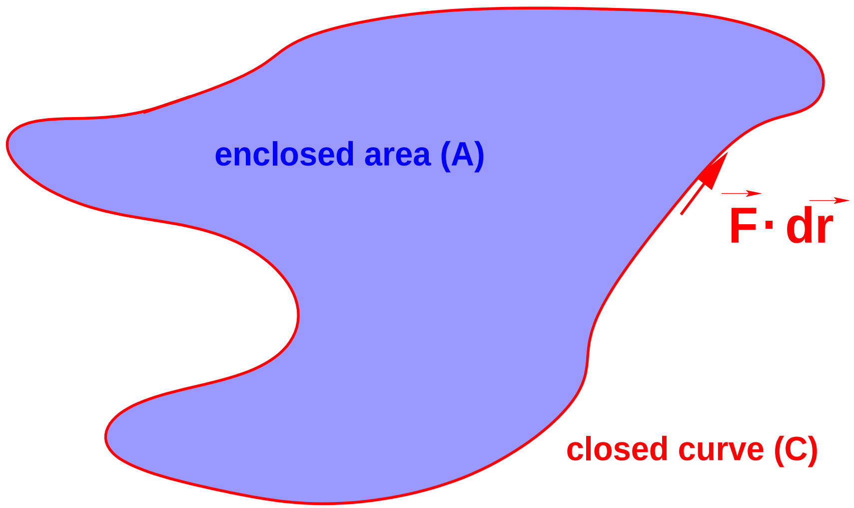
(e.g. work performed along a path ...)

Easier Integration of (vector-) fields

Stokes' Theorem:

Integral along a closed line is integral of curl in the enclosed area

$$\oint_C \vec{F} \cdot d\vec{r} = \int\int_A \nabla \times \vec{F} \cdot d\vec{A}$$



Relates line integral to curl

To remember: ...

Not really rigorous, but:

- DIV **measures what is coming out (or going in), integral is called the FLUX**
- $CURL$ **measures what is circulating, integral is called the CIRCULATION**

In general: a closed surface or closed line "measures" what is happening inside ...

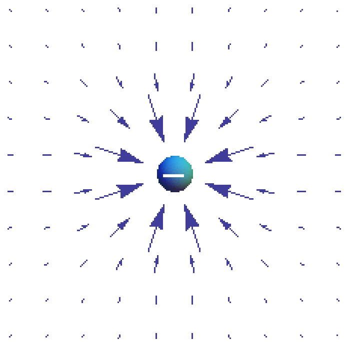
- BACK to ELECTRODYNAMICS -

How do we use all that stuff ?

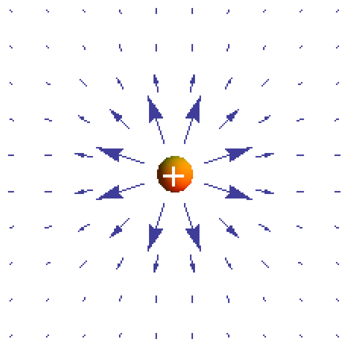
Some generalities

- **Electric fields $\vec{E}$ are generated by charges**
- **Magnetic fields $\vec{B}$ are generated by moving charges**
- **Quantified by strength and density of field vectors**

Electric fields from charges



(negative charges)



(positive charges)

Assume fields from a positive or negative charge q

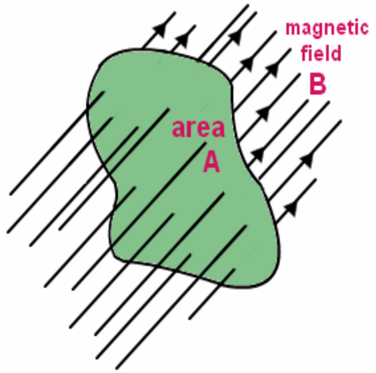
Electric field $\vec{E}$ is written as (Coulomb law):

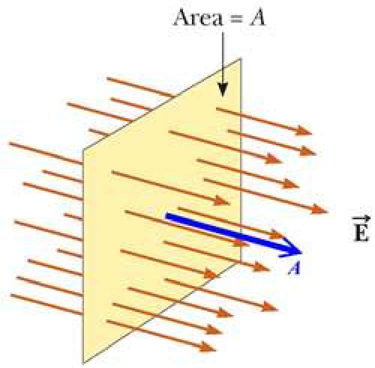
$$\vec{E} \;=\; \frac{\pm q}{4\pi\epsilon_0} \cdot \frac{\vec{r}}{|r|^3}$$

with:

$$\vec{r} \;=\; (x, y, z), \qquad\qquad |r| = \sqrt{x^2 + y^2 + z^2}$$

Electric and Magnetic flux





$$\int\int_A \vec{B} \cdot d\vec{A}$$

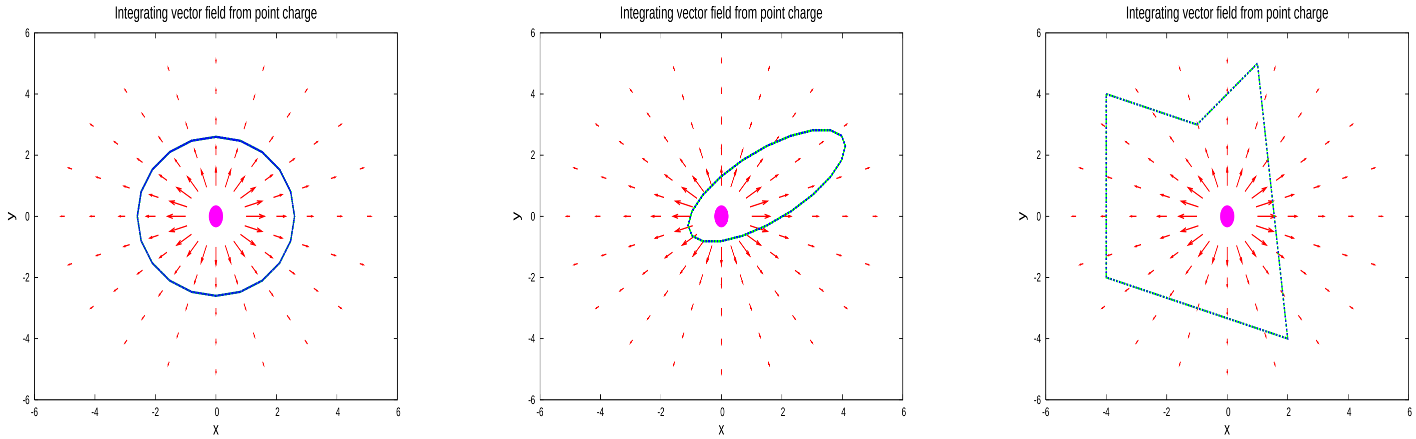
$$\int\int_A \vec{E} \cdot d\vec{A}$$

Integrate (count) field vectors through an area (or surface)

"Measures" the strength of the fields

Gives flux of electric and magnetic fields

Integrating fields from charges (2D !) ..

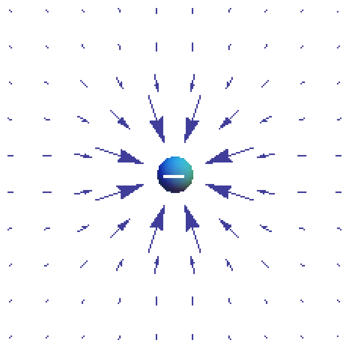


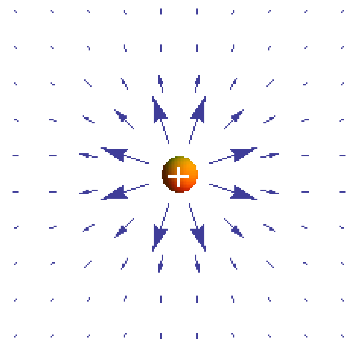
- To compute the flux, add field lines through the surface: $\int\int_A \vec{E} \cdot d\vec{A}$
- Put any <u>closed</u> surface around charges (sphere, box, ...). If all charges are enclosed: independent of shape !

→ If <u>positive</u>: total net charge enclosed <u>positive</u>

→ If <u>negative</u>: total net charge enclosed <u>negative</u>

Applying Divergence and charges ..





We can do the (non-trivial) computation of the divergence:

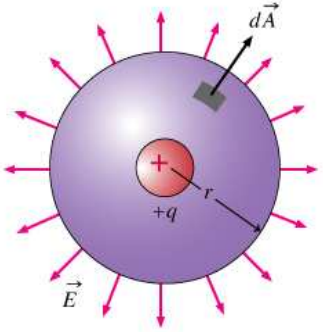
$$\mathsf{div}\vec{E} = \nabla\vec{E} = \frac{dE_x}{dx} + \frac{dE_y}{dy} + \frac{dE_z}{dz} = \frac{\rho}{\epsilon_0}$$

(negative charges) $\nabla \cdot \vec{E} < 0$

(positive charges) $\nabla \cdot \vec{E} > 0$

Divergence related to charge density ρ **generating the field** $\vec{E}$

More formal: Gauss's Theorem
(Maxwell's first equation ...)



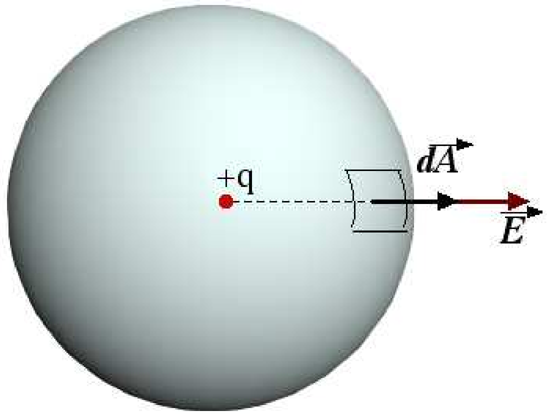
$$\frac{1}{\epsilon_0} \int \int_A \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} \int \int \int_V \nabla \vec{E} \cdot dV = \frac{q}{\epsilon_0}$$

$$\nabla \vec{E} = \frac{\rho}{\epsilon_0}$$

Flux of electric field $\vec{E}$ through a closed surface proportional to net electric charge q enclosed in the region (Gauss's Theorem). Written with charge density ρ we get Maxwell's first equation:

$$\text{div}\vec{E} = \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Example: field from a charge q



A charge q generates a field $\vec{E}$ according to:

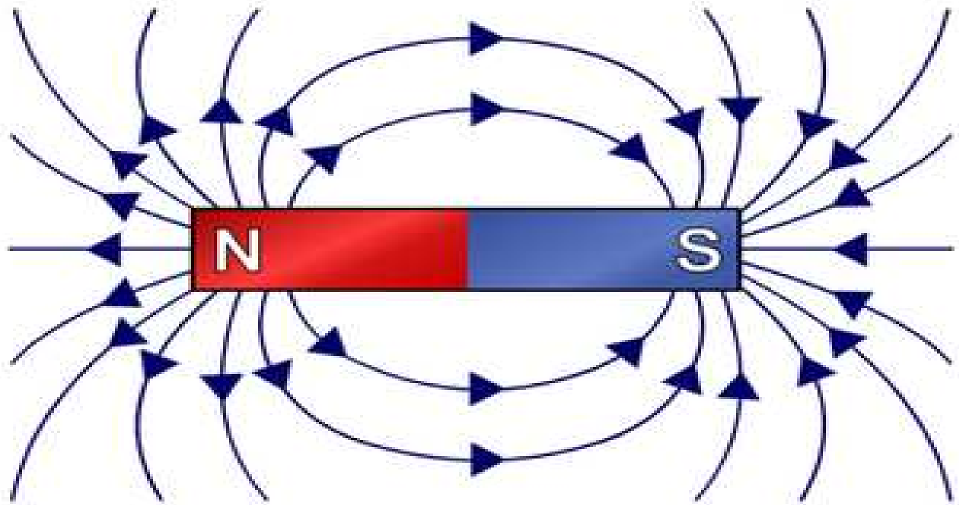
$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

Enclose it by a sphere: $\vec{E} = const.$ on a sphere (area is $4\pi \cdot r^2$):

$$\int\int_{sphere} \vec{E} \cdot d\vec{A} = \frac{q}{4\pi\epsilon_0}\int\int_{sphere}\frac{dA}{r^2} = \frac{q}{\epsilon_0}$$

Surface integral through sphere A is charge inside the sphere

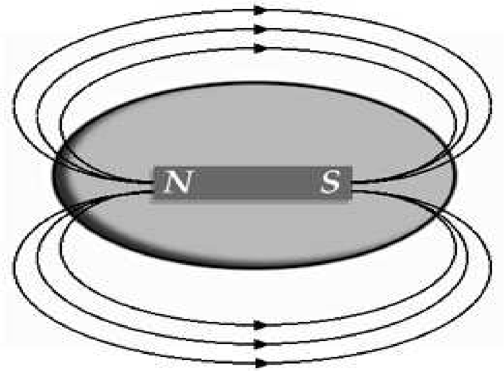
Divergence of magnetic fields



Definitions

- **Magnetic field lines from North to South**
- **Q: which is the direction of the earth magnetic field lines ?**

Maxwell's second equation ...



$$\int\int_A \vec{B} \;\; d\vec{A} = \int\int\int_V \nabla\vec{B} \;\; dV \;\; = \;\; 0$$

$$\nabla\vec{B} = 0$$

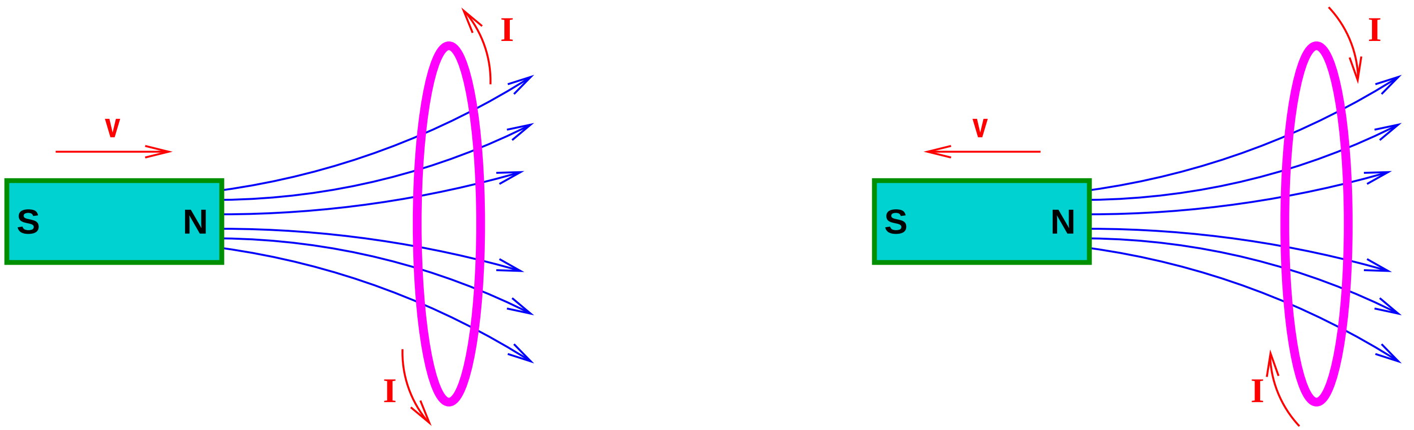
Closed field lines of magnetic flux density ($\vec{B}$): What goes out ANY closed surface also goes in, Maxwell's second equation:

$$\nabla\vec{B} = \mu_0\nabla\vec{H} = 0$$

➡ **Physical significance: no Magnetic Monopoles**

Maxwell's third equation ...

Faradays law:



- **Changing magnetic flux through area of a coil introduces electric current I**
- **Can be changed by moving magnet or coil**

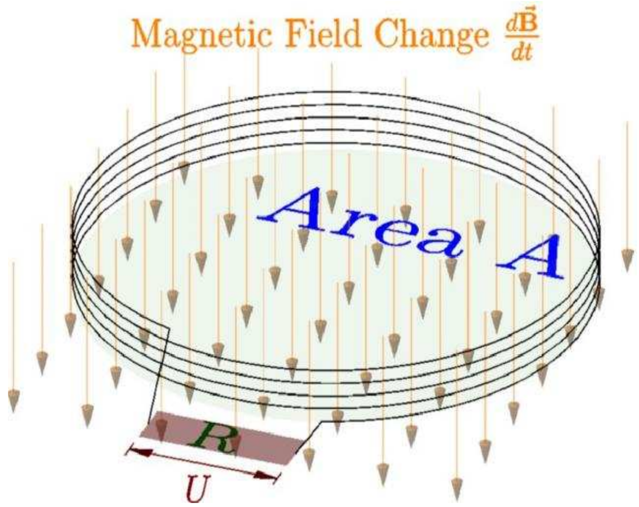
Maxwell's third equation ...

A changing flux Ω through an area A produces circulating electric field $\vec{E}$, i.e. a current I (Faraday)

$$-\frac{\partial\Omega}{\partial t} = \frac{\partial}{\partial t}\underbrace{\int_A \vec{B}d\vec{A}}_{flux\ \Omega} = \oint_C \vec{E}\cdot d\vec{r}$$

- **Flux can be changed by:**
  - **Change of magnetic field $\vec{B}$ with time t (e.g. transformers)**
  - **Change of area A with time t (e.g. dynamos)**

Formally: Maxwell's third equation ...



$$-\int_A \frac{\partial \vec{B}}{\partial t} d\vec{A} = \underbrace{\int_A \nabla \times \vec{E}\, d\vec{A} = \oint_C \vec{E} \cdot d\vec{r}}_{Stoke's formula}$$

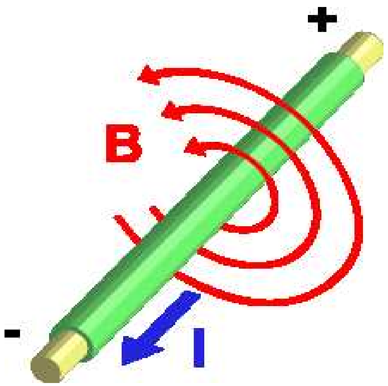
Changing magnetic field through an area induces electric field in coil around the area (Faraday)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Remember: strong $curl$ **= strong circulating field**

Maxwell's fourth equation (part 1) ...

From Ampere's law, for example current density $\vec{j}$:



<u>Static</u> electric current induces circulating magnetic field

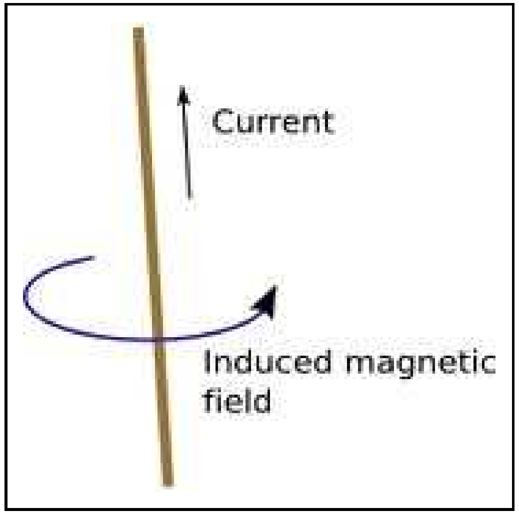
$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

or in integral form the currect density becomes the current I:

$$\int \int_A \nabla \times \vec{B} \;\; d\vec{A} \;\; = \;\; \int \int_A \mu_0 \vec{j} \;\; d\vec{A} \;\; = \;\; \mu_0 \vec{I}$$

Maxwell's fourth equation - application

For a static electric current I in a single wire we get Biot-Savart law (we have used Stoke's theorem and area of a circle $A = r^2 \cdot \pi$):



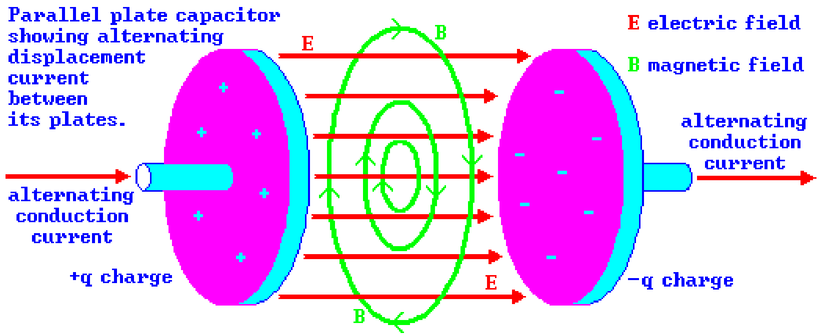
$$\vec{B} = \frac{\mu_0}{4\pi} \oint \vec{I} \cdot \frac{\vec{r} \cdot d\vec{r}}{r^3}$$

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{\vec{I}}{r}$$

For magnetic field calculations in electromagnets

Maxwell's fourth equation (part 2)...

From displacement current, for example charging capacitor $\vec{j}_d$:



- **Defining a Displacement Current $\vec{I}_d$:**

  Not a current from moving charges

  But a current from time varying electric fields

Maxwell's fourth equation (part 2) ...

Displacement current I_d produces magnetic field, just like "actual currents" do ...

→ **Time varying electric field induce magnetic field (using the current density $\vec{j}_d$**

$$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Remember: strong *curl* = strong circulating field

Maxwell's complete fourth equation ...

Magnetic fields $\vec{B}$ can be generated by two ways:

$\nabla \times \vec{B} = \mu_0 \vec{j}$ **(electrical current)**

$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$ **(changing electric field)**

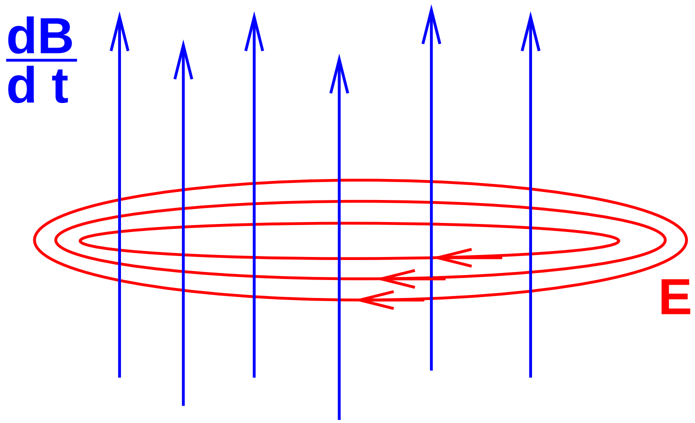
or putting them together:

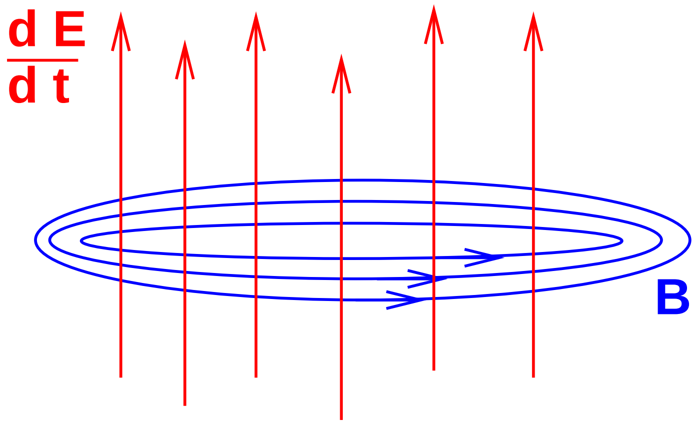
$$\nabla \times \vec{B} = \mu_0 (\vec{j} + \vec{j}_d) = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

or in integral form (using Stoke's formula):

$$\underbrace{\oint_C \vec{B} \cdot d\vec{r} \;=\; \int_A \nabla \times \vec{B} \cdot d\vec{A}}_{Stoke's formula} = \int_A \left(\mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

Summary: Static and Time Varying Fields



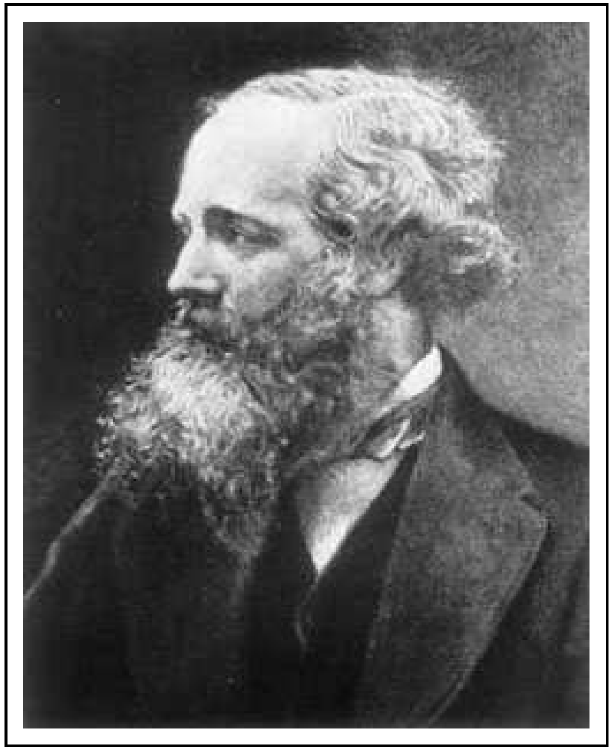


- **Time varying magnetic fields produce circulating electric field:** $\mathrm{curl}(\vec{E}) \ = \ \nabla \times \vec{E} \ = \ -\frac{d\vec{B}}{\partial t}$

- **Time varying electric fields produce circulating magnetic field:** $\mathrm{curl}(\vec{B}) \ = \ \nabla \times \vec{B} \ = \ \mu_0 \epsilon_0 \frac{d\vec{E}}{\partial t}$

because of the $\times$ they are perpendicular: $\vec{E} \perp \vec{B}$

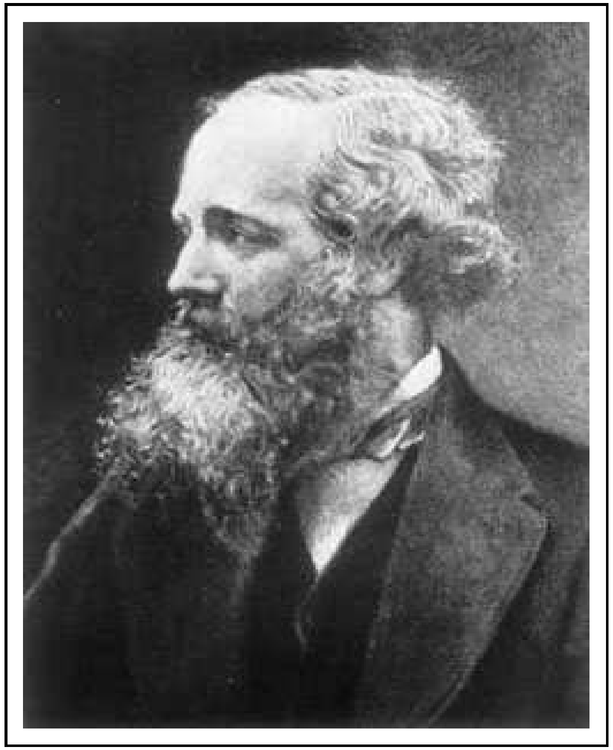
Summary: Maxwell's Equations



$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$
$$\int_A \vec{B} \cdot d\vec{A} = 0$$
$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt}\right) \cdot d\vec{A}$$
$$\oint_C \vec{B} \cdot d\vec{r} = \int_A \left(\mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}\right) \cdot d\vec{A}$$

Written in Integral form

Summary: Maxwell's Equations



$$\nabla \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

Written in Differential form

Summary: Maxwell's Equations

1. **Electric fields $\vec{E}$ are generated by charges and proportional to total charge**

2. **Magnetic monopoles do <u>not</u> exist**

3. **Changing magnetic flux generates circulating electric fields/currents**

4.1 **Changing electric flux generates circulating magnetic fields**

4.2 **Static electric current generates circulating magnetic fields**

Written in Physical terms

Interlude and Warning !!

Maxwell's equation can be written in other forms.
Often used: cgs (Gaussian) units instead of SI units, example:

Starting from (SI):

$$\nabla \cdot \vec{E} \; = \; \frac{\rho}{\epsilon_0}$$

we would use:

$$\vec{E}_{cgs} \quad = \quad \tfrac{1}{c} \quad \cdot \vec{E}_{SI} \qquad \text{and} \qquad \epsilon_0 \;\; = \;\; \tfrac{1}{4\pi \cdot c}$$

and arrive at (cgs):

$$\nabla \cdot \vec{E} \; = \; 4\pi \cdot \rho$$

Beware: there are more different units giving: $\nabla \cdot \vec{E} \; = \; \rho$

Electromagnetic fields in material

In vacuum:

$$\vec{D} = \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_0 \cdot \vec{H}$$

In a material:

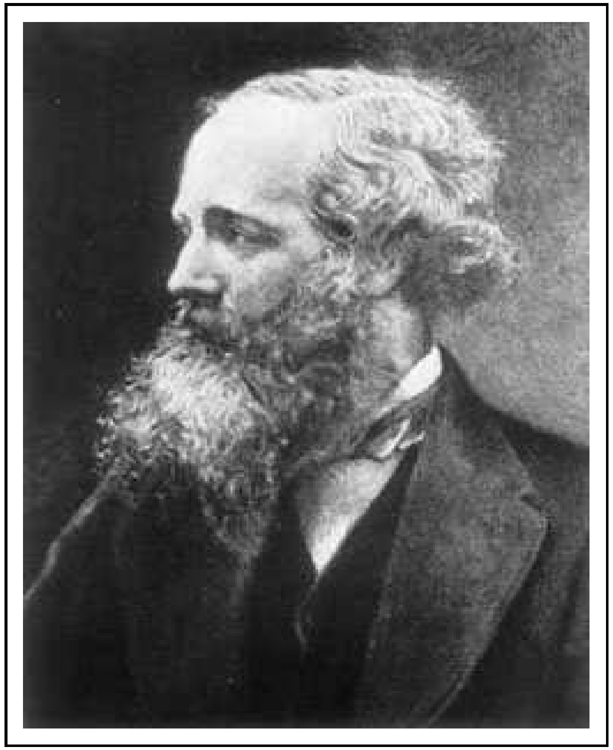
$$\vec{D} = \epsilon_r \cdot \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_r \cdot \mu_0 \cdot \vec{H}$$

ϵ_r **is** **<u>relative</u> permittivity** $\approx \quad [1 - 10^5]$

μ_r **is** **<u>relative</u> permeability** $\approx \quad [0(!) - 10^6]$

Origin: polarization and Magnetization

Once more: Maxwell's Equations



$$\nabla \vec{D} = \rho$$
$$\nabla \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$
$$\nabla \times \vec{H} = \vec{j} + \frac{d\vec{D}}{dt}$$

Re-factored in terms of the free current density $\vec{j}$ and free charge density ρ ($\mu_0 = 1, \epsilon_0 = 1$):

Applications of Maxwell's Equations

- **Lorentz force, motion in EM fields**
  - Motion in electric fields
  - Motion in magnetic fields
- **EM waves (in vacuum and in material)**
- **Boundary conditions**
- **EM waves in cavities and wave guides**

Lorentz force on charged particles

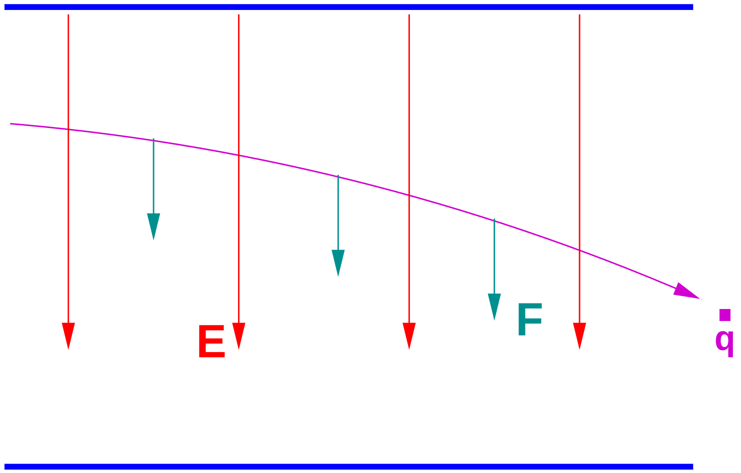
Moving ($\vec{v}$) charged (q) particles in electric ($\vec{E}$) and magnetic ($\vec{B}$) fields experience a force $\vec{f}$ like (Lorentz force):

$$\vec{f} \quad = \quad q \cdot (\vec{E} \; + \; \vec{v} \; \times \; \vec{B})$$

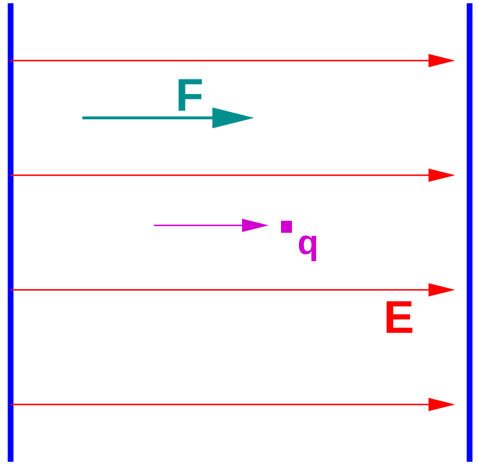
for the equation of motion we get (using Newton's law and relativistic γ);

$$\frac{d}{dt}(m_0 \gamma \vec{v}) \quad = \quad \vec{f} \quad = \quad q \cdot (\vec{E} \; + \; \vec{v} \; \times \; \vec{B})$$

Motion in electric fields



$\vec{v} \perp \vec{E}$



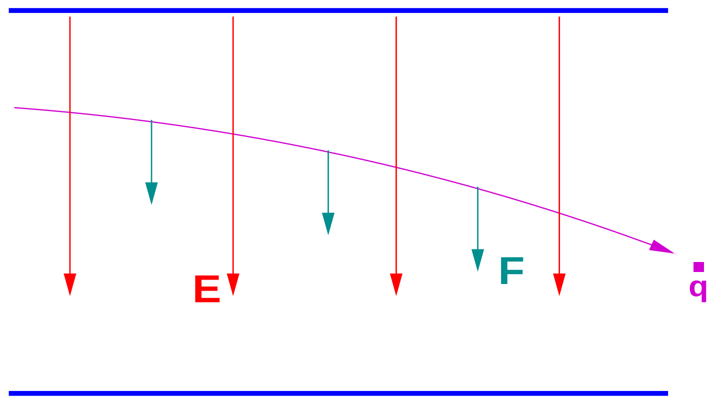
$\vec{v} \parallel \vec{E}$

Assume no magnetic field:

$$\frac{d}{dt}(m_0 \gamma \vec{v}) \quad = \quad \vec{f} \quad = \quad q \cdot \vec{E}$$

Force always in direction of field $\vec{E}$, also for particles at rest.

Motion in electric fields



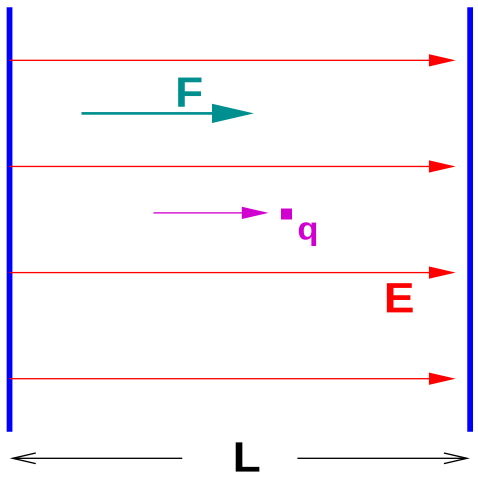
$$\frac{d}{dt}(m_0\gamma\vec{v}) \quad = \quad \vec{f} \quad = \quad q \cdot \vec{E}$$

The solution is:

$$\vec{v} \quad = \quad \frac{q \cdot \vec{E}}{\gamma \cdot m_0} \cdot \; t \qquad \Rightarrow \qquad \vec{x} \quad = \quad \frac{q \cdot \vec{E}}{\gamma \cdot m_0} \cdot \; t^2 \qquad (\textbf{parabola})$$

Constant E-field deflects beams: TV, electrostatic separators (SPS,LEP)

Motion in electric fields



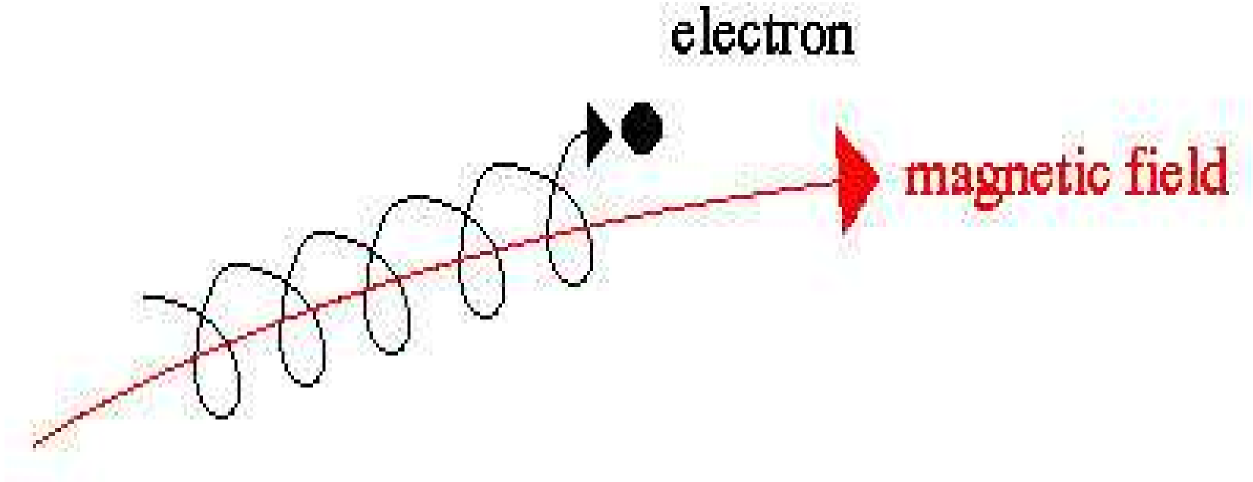
$$\frac{d}{dt}(m_0 \gamma \vec{v}) \quad = \quad \vec{f} \quad = \quad q \cdot \vec{E}$$

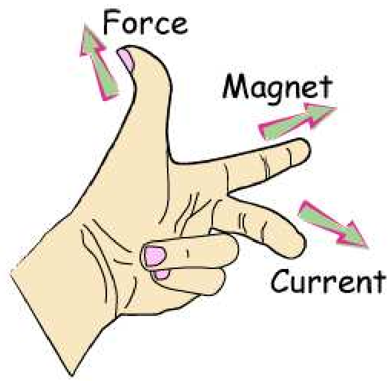
For constant field $\vec{E} = (E, 0, 0)$ in x-direction the energy gain is:

$$m_0 c^2 (\gamma - 1) \quad = \quad qE \cdot L$$

It is a line integral of the force along the path !
Constant E-field gives uniform acceleration over length L

Motion in magnetic fields





Assume first no electric field:

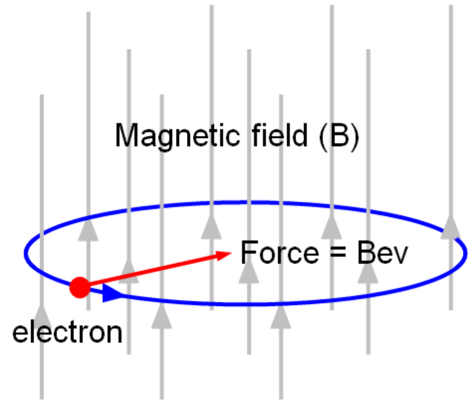
$$\frac{d}{dt}(m_0 \gamma \vec{v}) \quad = \quad \vec{f} \quad = \quad q \cdot \vec{v} \; \times \; \vec{B}$$

Force is perpendicular to both, $\vec{v}$ **and** $\vec{B}$

No forces on particles at rest !

Particles will spiral around the magnetic field lines ...

Motion in magnetic fields



Assuming that $v_\perp$ is perpendicular to $\vec{B}$
We get a circular motion with radius ρ:

$$\rho = \frac{m_0 \gamma v_\perp}{q \cdot B}$$

defines the Magnetic Rigidity: $B \cdot \rho = \frac{m_0 \gamma v}{q} = \frac{p}{q}$

Magnetic fields deflect particles, but no acceleration (synchrotron, ..)

Motion in magnetic fields

Practical units:

$$B[T] \cdot \rho[m] \quad = \quad \frac{p[ev]}{c[m/s]}$$

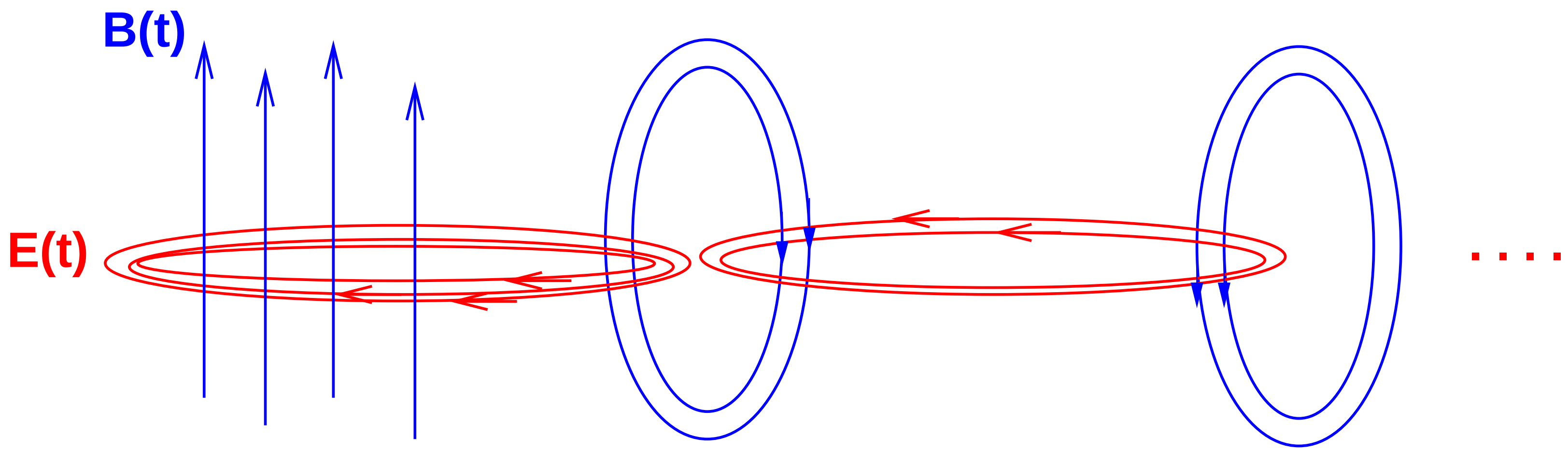
Example LHC:

B = 8.33 T, p = 7000 GeV/c ➡ ρ **= 2804 m**

Use of static fields (some examples, incomplete)

- **Magnetic fields**
  - **Bending magnets**
  - **Focusing magnets (quadrupoles)**
  - **Correction magnets (sextupoles, octupoles, orbit correctors, ..)**
- **Electric fields**
  - **Electrostatic separators (beam separation in particle-antiparticle colliders)**
  - **Very low energy machines**
- **What about non-static, time-varying fields ?**

Time Varying Fields



Time varying magnetic fields produce circulating electric fields

Time varying electric fields produce circulating magnetic fields

➡ Can produce self-sustaining, propagating fields (i.e. waves)

Electromagnetic waves in vacuum

Vacuum: only fields, no charges ($\rho = 0$), no current ($j = 0$) ...

From: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\Longrightarrow \quad \nabla \times (\nabla \times \vec{E}) = -\nabla \times (\frac{\partial \vec{B}}{\partial t})$$

$$\Longrightarrow \quad -(\nabla^2 \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\Longrightarrow \quad -(\nabla^2 \vec{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

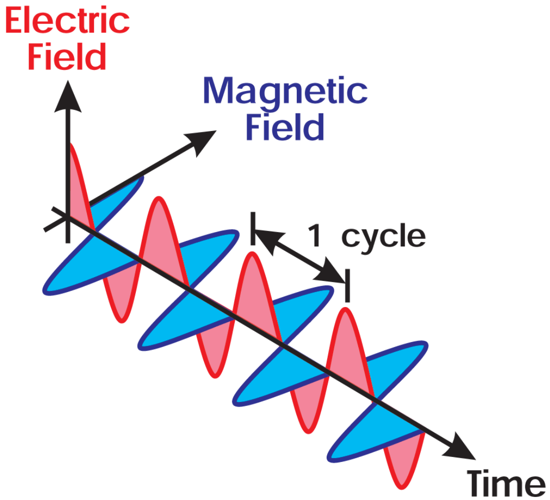
$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{E}}{\partial t^2}$$

Similar expression for the magnetic field:

$$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{B}}{\partial t^2}$$

Equation for a plane wave with velocity: $c = \frac{1}{\sqrt{\mu_0 \cdot \epsilon_0}}$

Electromagnetic waves



$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$

$$\vec{B} = \vec{B}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$

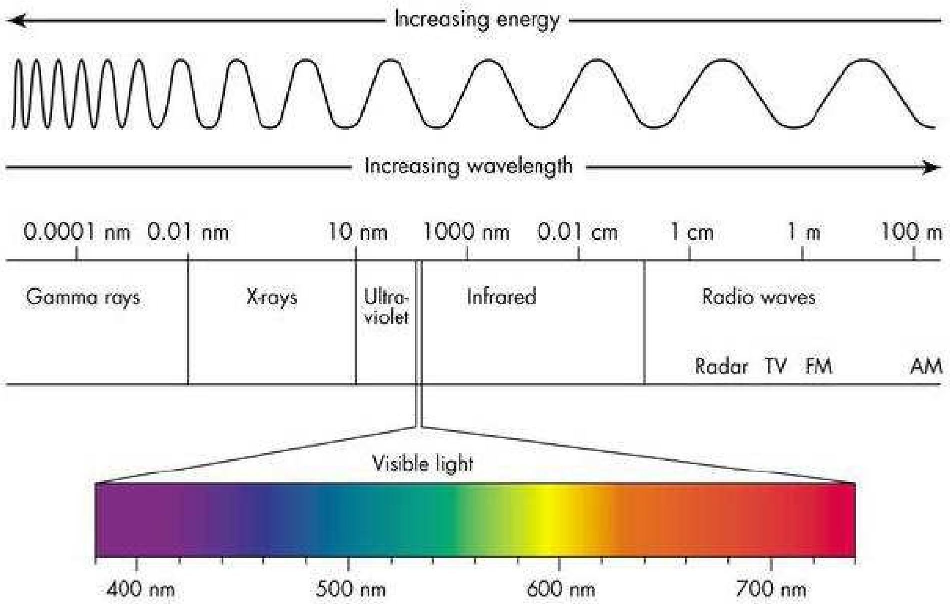
$$|\vec{k}| = \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad (\textbf{propagation vector})$$

$$\lambda = (\textbf{wave length, 1 cycle})$$

$$\omega = (\textbf{frequency} \cdot 2\pi)$$

Magnetic and electric fields are transverse to direction of propagation: $\vec{E} \perp \vec{B} \perp \vec{k}$

Spectrum of Electromagnetic waves



Example: yellow light ➡ $\approx 5 \cdot 10^{14}$ **Hz (i.e.** $\approx$ **2 eV !)**

gamma rays ➡ $\leq 3 \cdot 10^{21}$ **Hz (i.e.** $\leq$ **12 MeV !)**

LEP (SR) ➡ $\leq 2 \cdot 10^{20}$ **Hz (i.e.** $\approx$ **0.8 MeV !)**

Waves hitting material

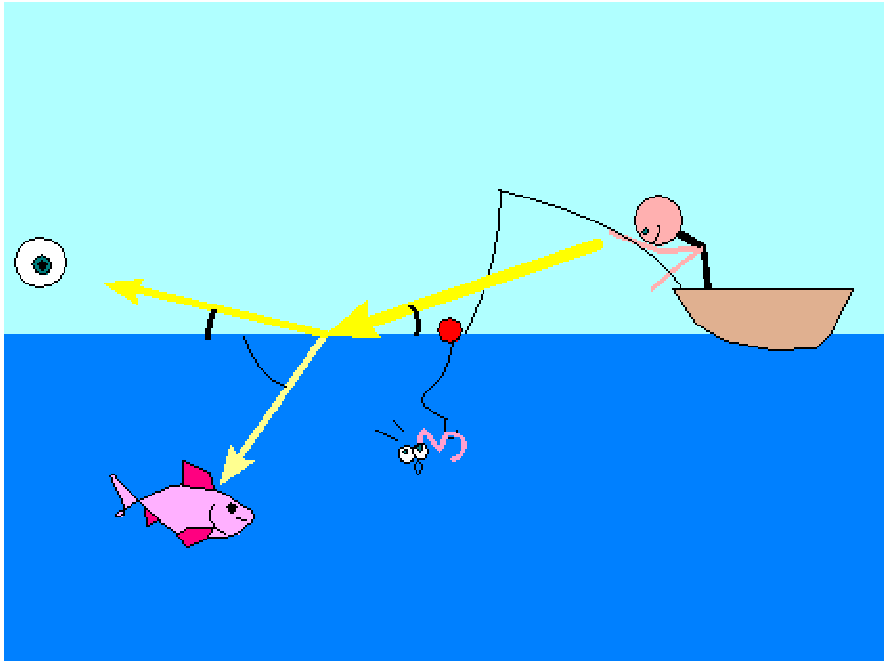
Need to look at the behaviour of electromagnetic fields at <u>boundaries between different materials</u> (air-glass, air-water, vacuum-metal, ...).

Important for highly conductive materials, e.g.:

- **RF systems**
- **Wave guides**
- **Impedance calculations**

Can be derived from Maxwell's equations, here only the results !

Observation: between air and water

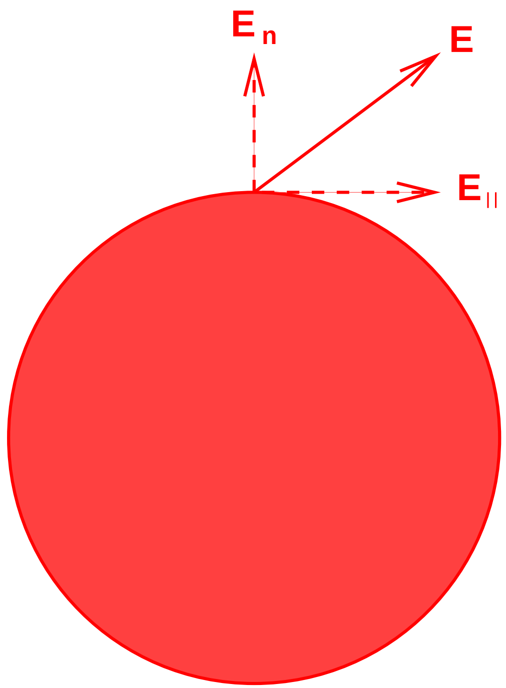


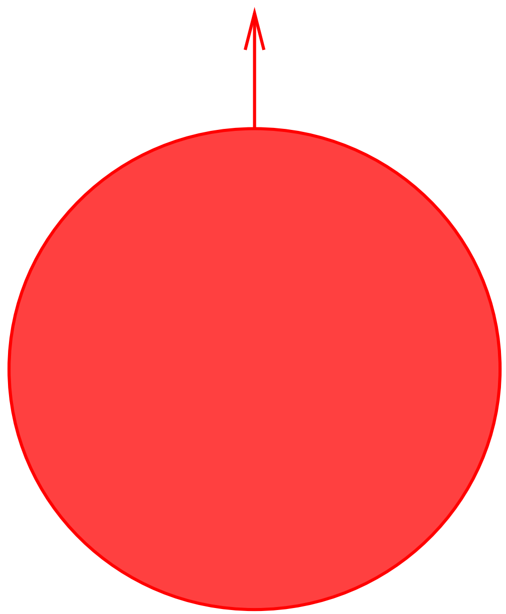
- **Some of the light is reflected**
- **Some of the light is transmitted and refracted**

→ **Reason are boundary conditions for fields**

Boundary conditions: air and conductor

A simple case as demonstration ($\vec{E}$-fields on a conducting sphere):

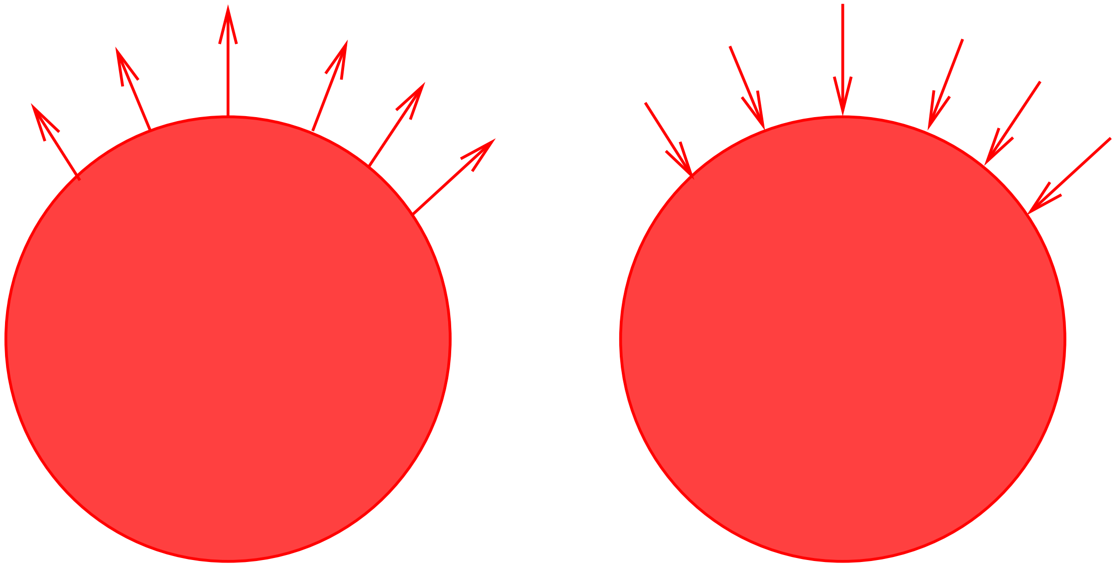




- **Field parallel to surface $E_{||}$ cannot exist (it would move charges and we get a surface current)**
- **Only field normal to surface E_n is possible**

Boundary conditions for fields

All electric field lines must be normal (perpendicular) to surface of a conductor.



- **All conditions for $\vec{E}, \vec{D}, \vec{H}, \vec{B}$ can be derived from Maxwell's equations (see bibliography, e.g. R.P.Feynman or J.D.Jackson)**

Boundary conditions for fields*

Electromagnetic fields at boundaries between different materials with different permittivity and permeability ($\epsilon^a, \epsilon^b, \mu^a, \mu^b$).

The requirements for the components are (summary of the results, not derived here !):

- $(E^a_{\parallel} = E^b_{\parallel}), (E^a_n \neq E^b_n)$
- $(D^a_{\parallel} \neq D^b_{\parallel}), (D^a_n = D^b_n)$
- $(H^a_{\parallel} = H^b_{\parallel}), (H^a_n \neq H^b_n)$
- $(B^a_{\parallel} \neq B^b_{\parallel}), (B^a_n = B^b_n)$

Conditions are used to compute reflection, refraction and refraction index n.

Extreme case: ideal conductor

For an ideal conductor (i.e. no resistance) the tangential electric field must vanish, otherwise a surface current becomes infinite. Similar conditions for magnetic fields. We must have:

$$\vec{E}_{\|} \ = \ 0, \qquad \vec{B}_n \ = \ 0$$

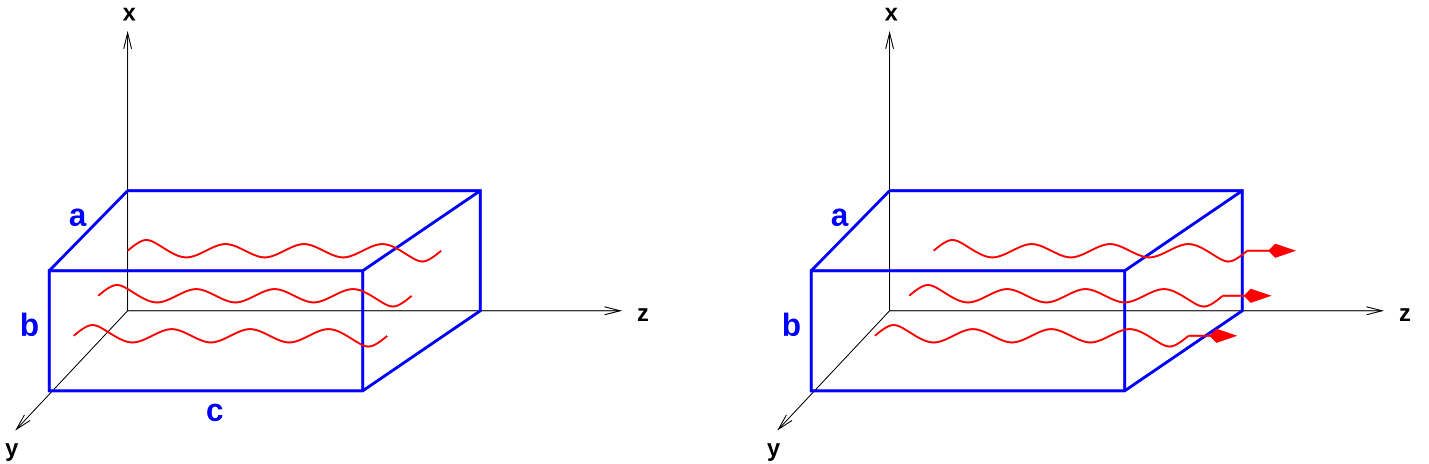
This implies:

- **All energy of an electromagnetic wave is reflected from the surface.**
- **Fields at any point in the conductor are zero.**
- **Only some fieldpatterns are allowed in waveguides and RF cavities**

A very nice lecture in R.P.Feynman, Vol. II

Examples: cavities and wave guides

Rectangular, conducting cavities and wave guides (schematic) with dimensions $a \times b \times c$ and $a \times b$:



- **RF cavity, fields can persist and be stored (reflection !)**
- **Plane waves can propagate along wave guides, here in z-direction**

Fields in RF cavities

Assume a rectangular RF cavity ($a,\ b,\ c$), ideal conductor.

Without derivations, the components of the fields are:

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

$$E_z = E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_x = \frac{i}{\omega}(E_{y0} k_z - E_{z0} k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_y = \frac{i}{\omega}(E_{z0} k_x - E_{x0} k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_z = \frac{i}{\omega}(E_{x0} k_y - E_{y0} k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

Consequences for RF cavities

Field must be zero at conductor boundary, only possible under the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

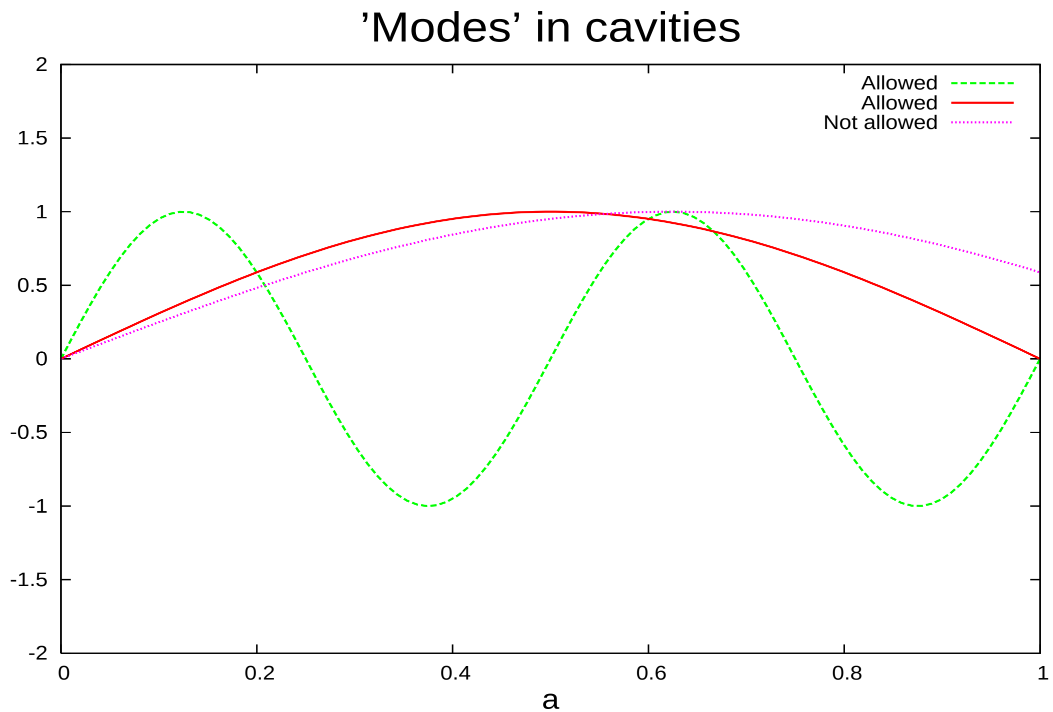
and for k_x, k_y, k_z **we can write:**

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \frac{m_z \pi}{c},$$

The integer numbers m_x, m_y, m_z **are called mode numbers, important for shape of cavity !**

It means that a half wave length $\lambda/2$ **must always fit exactly the size of the cavity.**

Allowed modes



- **Only modes which 'fit' into the cavity are allowed**
- $\frac{\lambda}{2} = \frac{a}{4}$, $\frac{\lambda}{2} = \frac{a}{1}$, $\frac{\lambda}{2} = \frac{a}{0.8}$
- **No electric field at boundaries**

Fields in wave guides

Similar considerations lead to (propagating) solutions in (rectangular) wave guides:

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

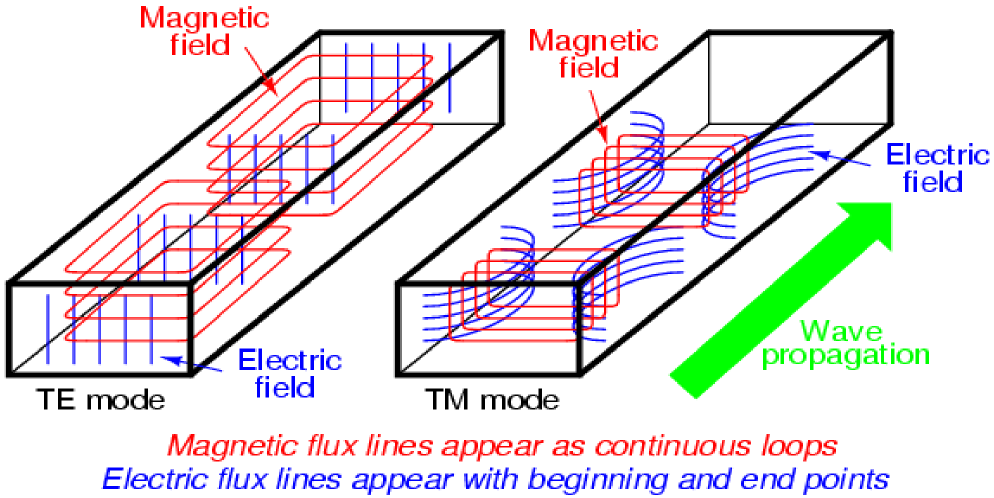
$$E_z = i \cdot E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$B_x = \frac{1}{\omega}(E_{y0}k_z - E_{z0}k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$B_y = \frac{1}{\omega}(E_{z0}k_x - E_{x0}k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$B_z = \frac{1}{i \cdot \omega}(E_{x0}k_y - E_{y0}k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

The fields in wave guides



- **Electric and magnetic fields through a wave guide**
- **Shapes are consequences of boundary conditions !**
- **Can be Transverse Electric (TE, no E-field in z-direction) or Transverse Magnetic (TM, no B-field in z-direction)**

Consequences for wave guides

Similar considerations as for cavities, no field at boundary.

We must satisfy again the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

This leads to modes like:

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b},$$

The numbers m_x, m_y are called mode numbers for planar waves in wave guides !

Consequences for wave guides

Re-writing the condition as:

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2$$

Propagation without losses requires k_z to be real, i.e.:

$$\frac{\omega^2}{c^2} > k_x^2 + k_y^2 = (\frac{m_x \pi}{a})^2 + (\frac{m_y \pi}{b})^2$$

which defines a cut-off frequency ω_c.

- **Above cut-off frequency: propagation without loss**
- **Below cut-off frequency: attenuated wave (means it does not "really fit" and k is complex).**

Done ...

- **Review of basics and Maxwell's equations**
- **Lorentz force**
- **Motion of particles in electromagnetic fields**
- **Electromagnetic waves in vacuum**
- **Electromagnetic waves in conducting media**
  - **Waves in RF cavities**
  - **Waves in wave guides**

- BACKUP SLIDES -

Some popular confusion ..

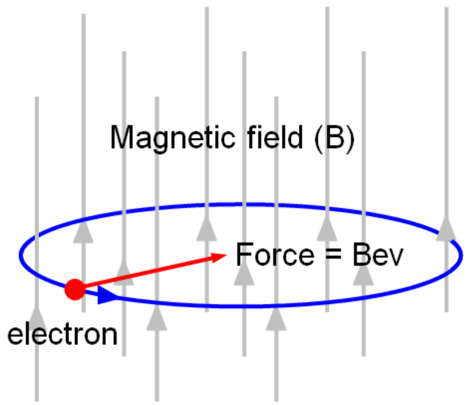
V.F.A.Q: why this strange mixture of $\vec{E}, \vec{D}, \vec{B}, \vec{H}$??

Materials respond to an applied electric E field and an applied magnetic B field by producing their own internal charge and current distributions, contributing to E and B. Therefore H and D fields are used to re-factor Maxwell's equations in terms of the free current density $\vec{j}$ and free charge density ρ:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$
$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$\vec{M}$ **and** $\vec{P}$ **are** *Magnetization* **and** *Polarisation* **in material**

Is that the full truth ?



If we have a circulating E-field along the circle of radius R ?

➡ **should get acceleration !**

Remember Maxwell's third equation:

$$\oint_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt}\int_A \vec{B} \cdot d\vec{A}$$

$$\rightarrow 2\pi R E_\theta = -\frac{d\Phi}{dt}$$

Motion in magnetic fields

- This is the principle of a **Betatron**
  - Time varying magnetic field creates circular electric field !
  - Time varying magnetic field deflects the charge !

For a constant radius we need:

$$-\frac{m \cdot v^2}{R} = e \cdot v \cdot B \quad \Rightarrow \quad B = -\frac{p}{e \cdot R}$$

$$\frac{\partial}{\partial t} B(r,t) = -\frac{1}{e \cdot R}\frac{dp}{dt}$$

$$\Rightarrow \quad B(r,t) = \frac{1}{2}\frac{1}{\pi R^2}\int\int B dS$$

B-field on orbit must be half the average over the circle

➡ **Betatron condition**

Other case: finite conductivity

Assume conductor with finite conductivity ($\sigma_c = \rho_c^{-1}$) , waves will penetrate into surface. Order of the skin depth is:

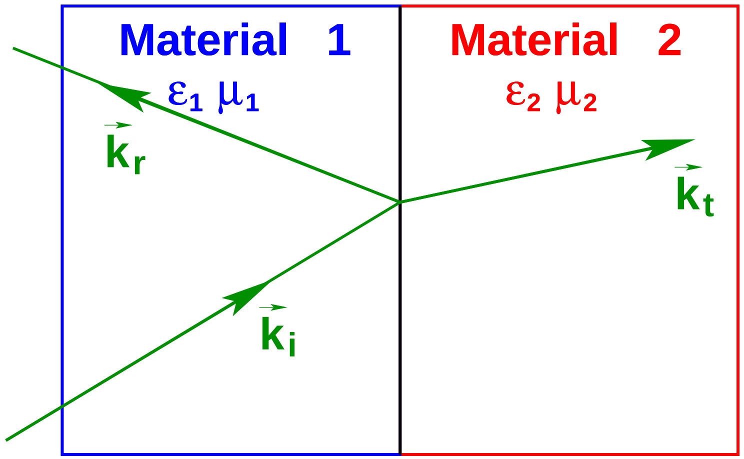
$$\delta_s = \sqrt{\frac{2\rho_c}{\mu\omega}}$$

i.e. depend on resistivity, permeability and frequency of the waves (ω).
We can get the surface impedance as:

$$Z = \sqrt{\frac{\mu}{\epsilon}} = \frac{\mu\omega}{k}$$

the latter follows from our definition of k and speed of light. Since the wave vector k is complex, the impedance is also complex. We get a phase shift between electric and magnetic field.

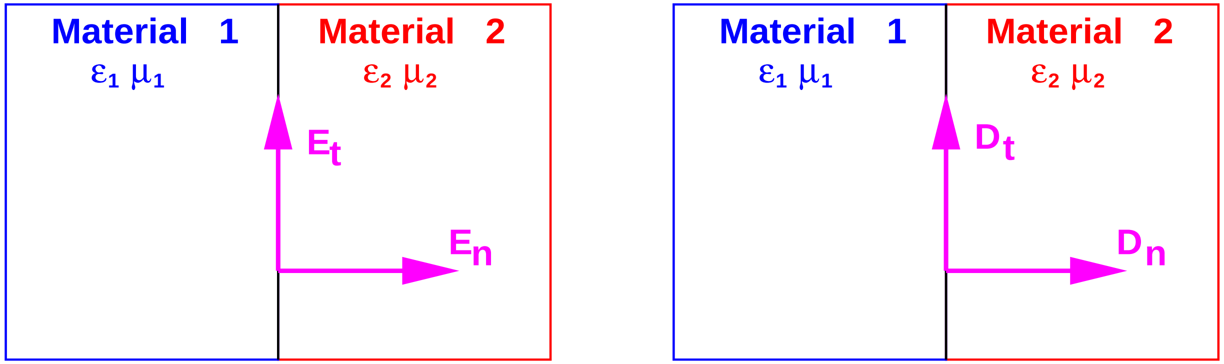
Boundary conditions for fields



What happens when an incident wave ($\vec{K}_i$) encounters a boundary between two different media ?

- **Part of the wave will be reflected ($\vec{K}_r$), part is transmitted ($\vec{K}_t$)**
- **What happens to the electric and magnetic fields ?**

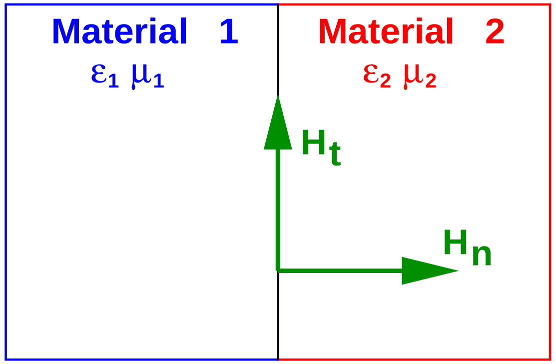
Boundary conditions for fields

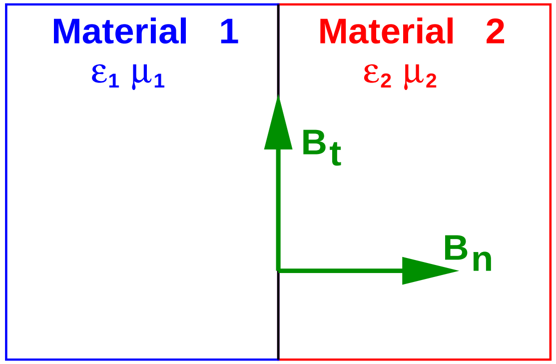


Assuming <u>no</u> surface charges:

- **<u>tangential $\vec{E}$-field</u> constant across boundary** $(E_{1t} \;=\; E_{2t})$
- **<u>normal $\vec{D}$-field</u> constant across boundary** $(D_{1n} \;=\; D_{2n})$

Boundary conditions for fields





Assuming <u>no</u> surface currents:

- **<u>tangential $\vec{H}$-field</u> constant across boundary** $(H_{1t} = H_{2t})$
- **<u>normal $\vec{B}$-field</u> constant across boundary** $(B_{1n} = B_{2n})$