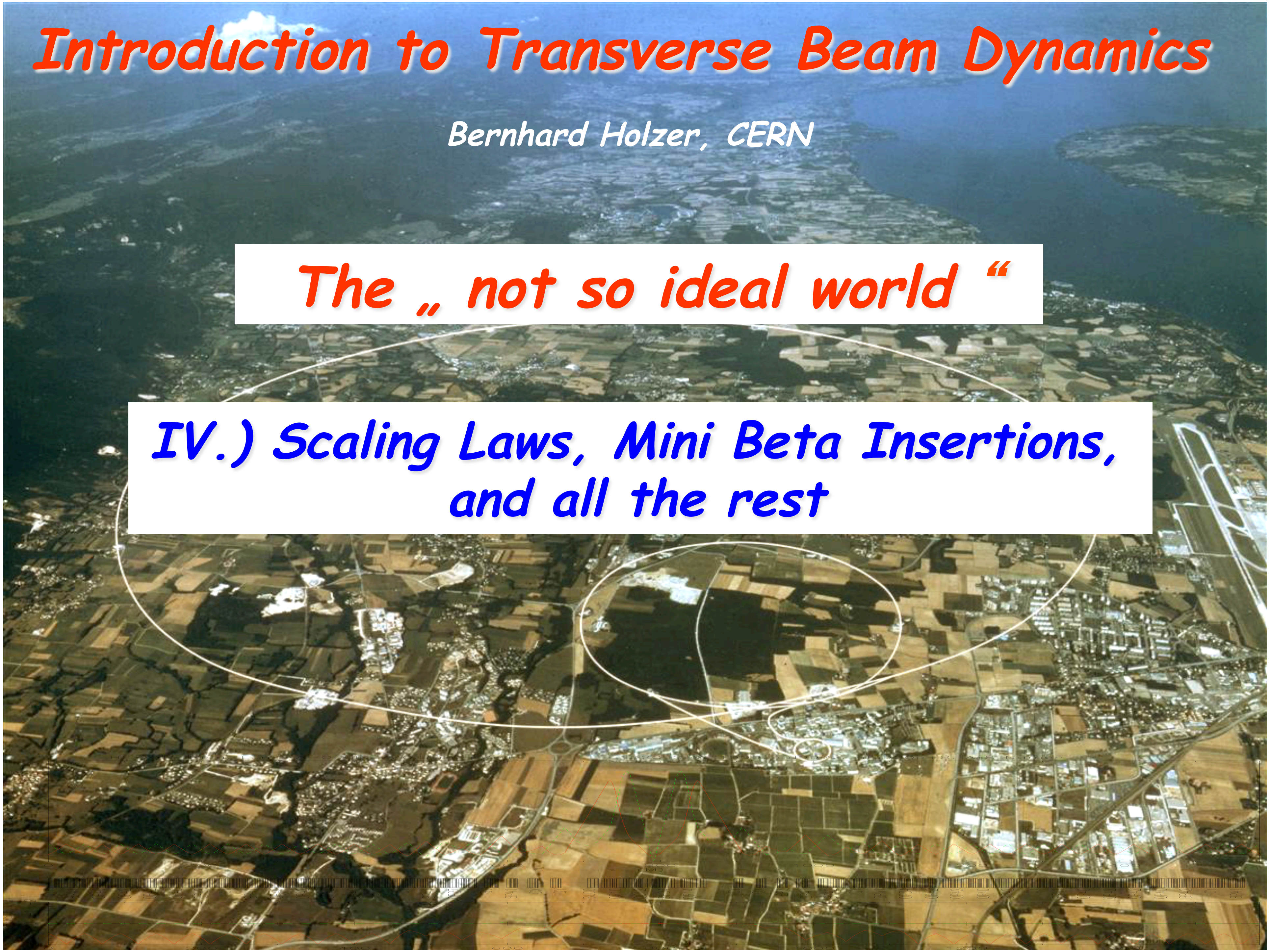
Introduction to Transverse Beam Dynamics

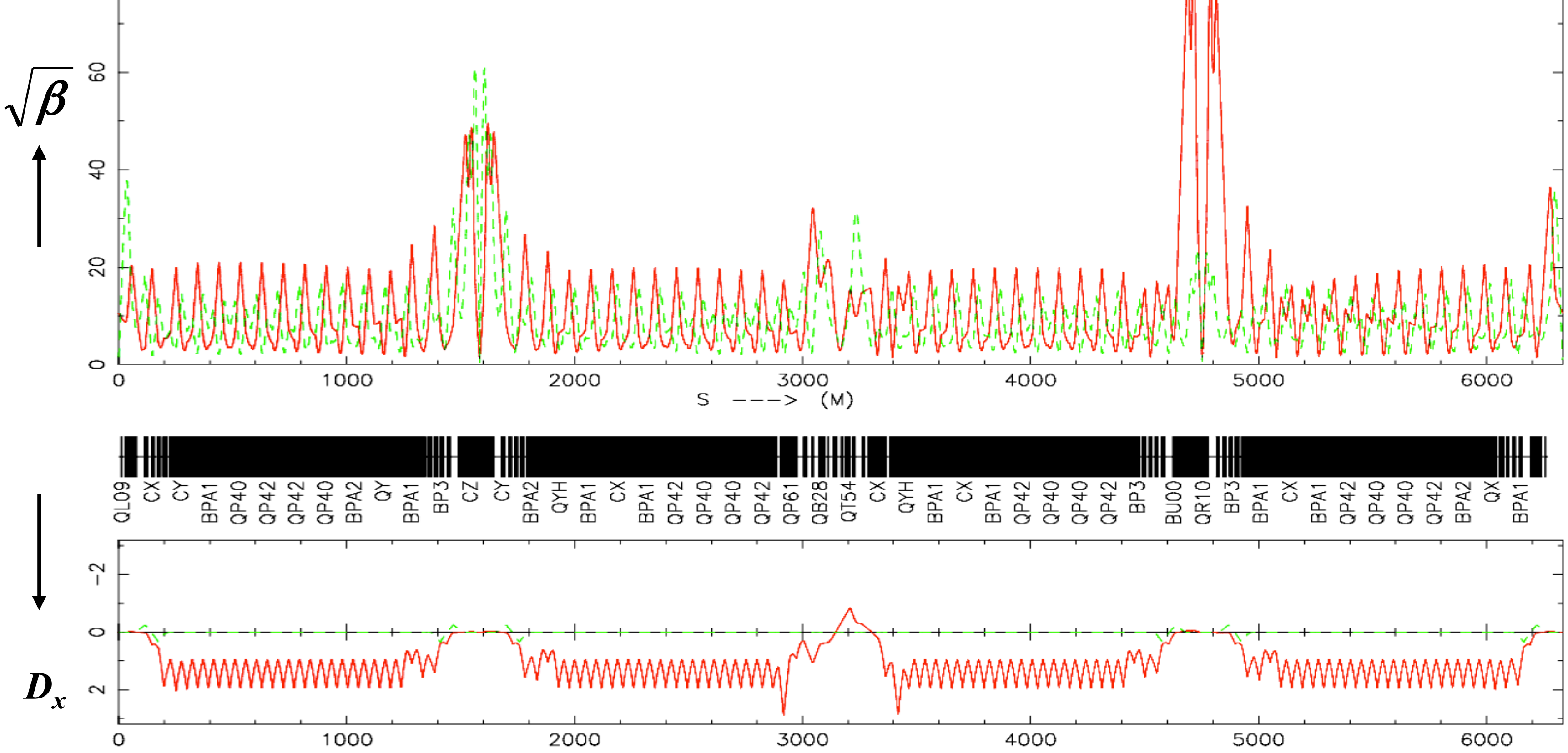
Bernhard Holzer, CERN

The „ not so ideal world “

IV.) Scaling Laws, Mini Beta Insertions, and all the rest



17.) Quadrupole Errors



Quadrupole Errors

go back to Lecture I, page 1

single particle trajectory

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{QF} * \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

Solution of equation of motion

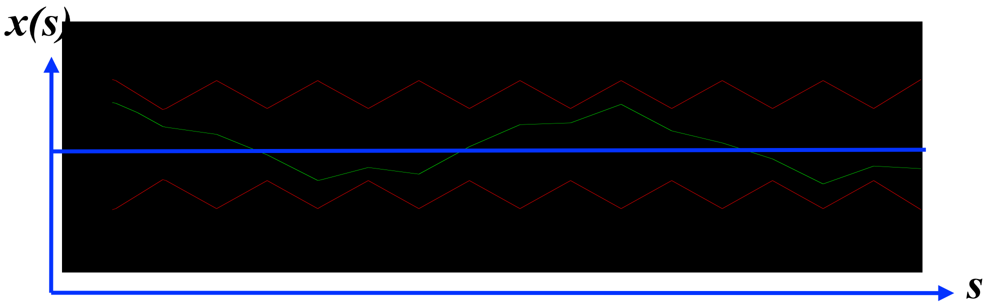
$$x = x_0 \cos(\sqrt{k}\, l_q) + x_0' \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q)$$

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{k}\, l_q) & \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q) \\ -\sqrt{k} \sin(\sqrt{k}\, l_q) & \cos(\sqrt{k}\, l_q) \end{pmatrix} \quad , \quad M_{thinlens} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$$M_{turn} = M_{QF} * M_{D1} * M_{QD} * M_{D2} * M_{QF} \ldots$$

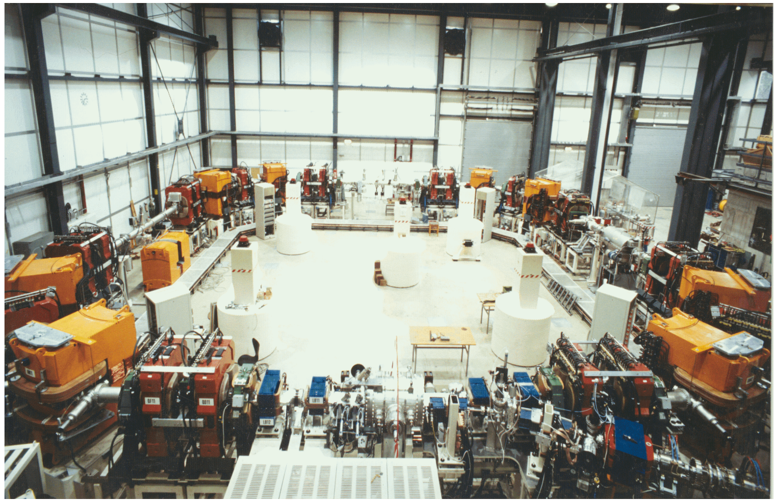
Definition: ***phase advance of the particle oscillation per revolution in units of 2π is called tune***

$$Q = \frac{\psi_{turn}}{2\pi}$$



Matrix in Twiss Form

Transfer Matrix from point „0“ in the lattice to point „s“:



$$M(s)=\begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\psi_s+\alpha_0\sin\psi_s) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos(\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos(\psi_s-\alpha_0\sin\psi_s) \end{pmatrix}$$

For one complete turn the Twiss parameters have to obey periodic bundary conditions:

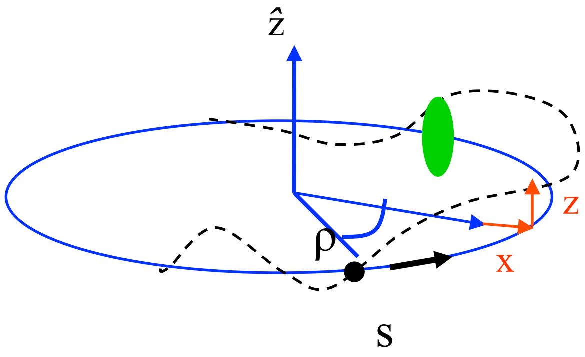
$$\beta(s+L)=\beta(s)$$
$$\alpha(s+L)=\alpha(s)$$
$$\gamma(s+L)=\gamma(s)$$

$$M(s)=\begin{pmatrix} \cos\psi_{turn}+\alpha_s\sin\psi_{turn} & \beta_s\sin\psi_{turn} \\ -\gamma_s\sin\psi_s & \cos\psi_{turn}-\alpha_s\sin\psi_{turn} \end{pmatrix}$$

Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

$$M_{dist} = M_{\Delta k} \cdot M_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta k ds & 1 \end{pmatrix}}_{\text{quad error}} \cdot \underbrace{\begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}}_{\text{ideal storage ring}}$$



$$M_{dist} = \begin{pmatrix} \cos\psi_0 + \alpha \sin\psi_0 & \beta \sin\psi_0 \\ \Delta k ds (\cos\psi_0 + \alpha \sin\psi_0) - \gamma \sin\psi_0 & \Delta k ds\, \beta \sin\psi_0 + \cos\psi_0 - \alpha \sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$Trace(M) = 2\cos\psi = 2\cos\psi_0 + \Delta k ds \beta \sin\psi_0$$

Quadrupole error → Tune Shift

$$\boxed{\psi = \psi_0 + \Delta\psi} \quad \longrightarrow \quad \cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta kds\,\beta\sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos\psi_0\underbrace{\cos\Delta\psi}_{\approx 1} - \sin\psi_0\underbrace{\sin\Delta\psi}_{\approx\Delta\psi} = \cos\psi_0 + \frac{kds\,\beta\sin\psi_0}{2}$$

$$\Delta\psi = \frac{kds\,\beta}{2}$$

and referring to Q instead of ψ:

$$\psi = 2\pi Q$$

$$\boxed{\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k(s)\beta(s)ds}{4\pi}}$$

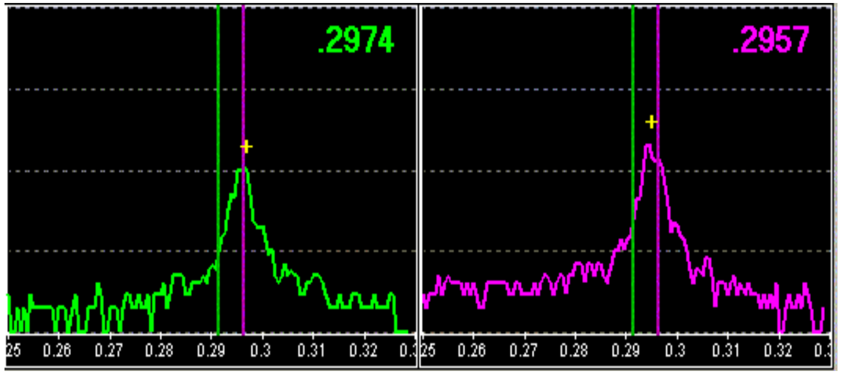
! the tune shift is proportional to the β-function at the quadrupole

!! field quality, power supply tolerances etc are much tighter at places where β is large

!!! mini beta quads: β ≈ 1900 m
arc quads: β ≈ 80 m

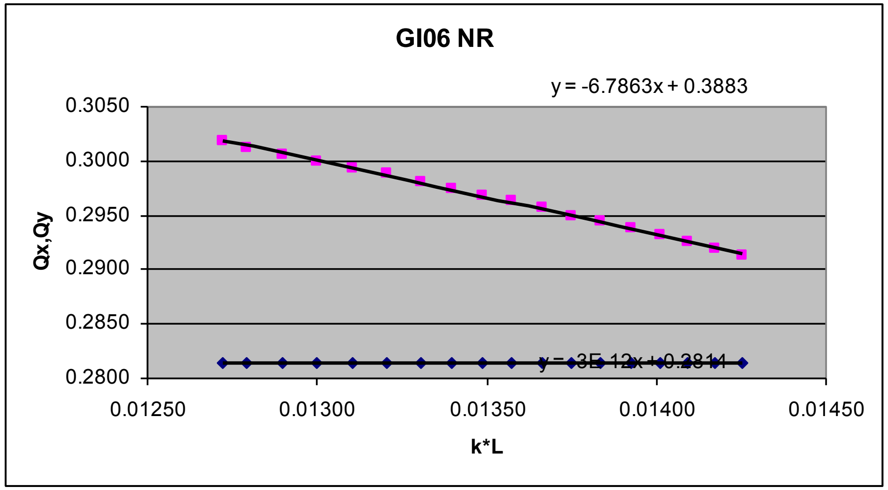
!!!! β is a measure for the sensitivity of the beam

a quadrupol error leads to a shift of the tune:



$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k\, \beta(s)}{4\pi}\, ds \approx \frac{\Delta k l_{quad} \bar{\beta}}{4\pi}$$

Example: measurement of β in a storage ring:
tune spectrum



Quadrupole error: Beta Beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta K\cos\left(2\left|\psi_{s1}-\psi_{s0}\right|-2\pi Q\right)ds$$

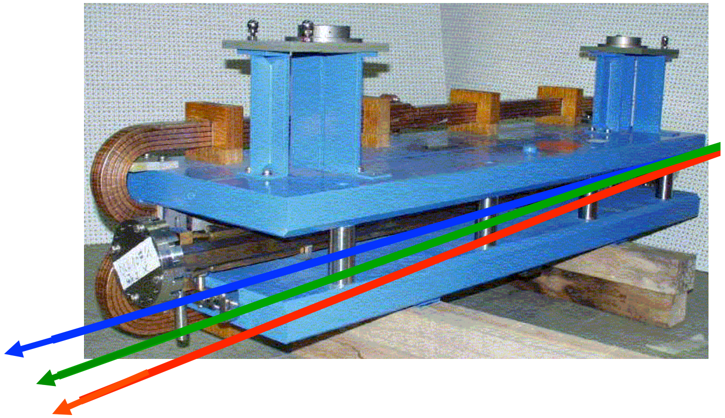
(proof: see appendix)

$\sqrt{\beta}$

18.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

dipole magnet $$\alpha = \frac{\int B\,dl}{p/e}$$



$$x_D(s) = D(s)\,\frac{\Delta p}{p}$$

focusing lens $$k = \frac{g}{p/e}$$

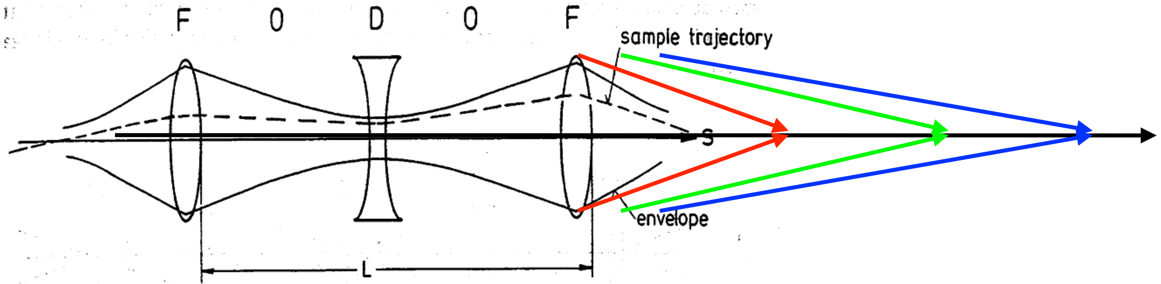


Figure 29: FODO cell

particle having ...

to high energy

to low energy

ideal energy

Chromaticity: Q'

$$k = \frac{g}{p/e} \qquad\qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$\boldsymbol{k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}(1 - \frac{\Delta p}{p_0})\, g = k_0 + \Delta k}$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

… which acts like a quadrupole error in the machine and leads to a tune spread:

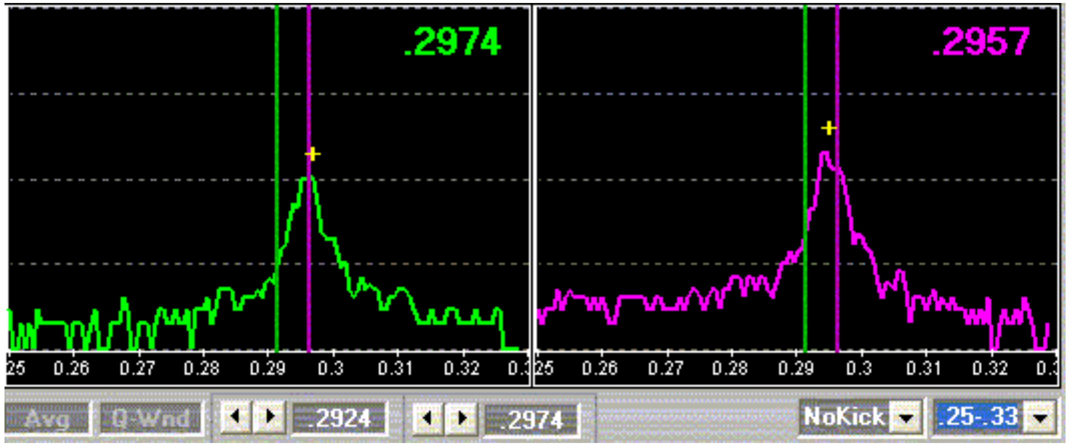
$$\Delta Q = -\frac{1}{4\pi}\frac{\Delta p}{p_0} k_0 \beta(s) ds$$

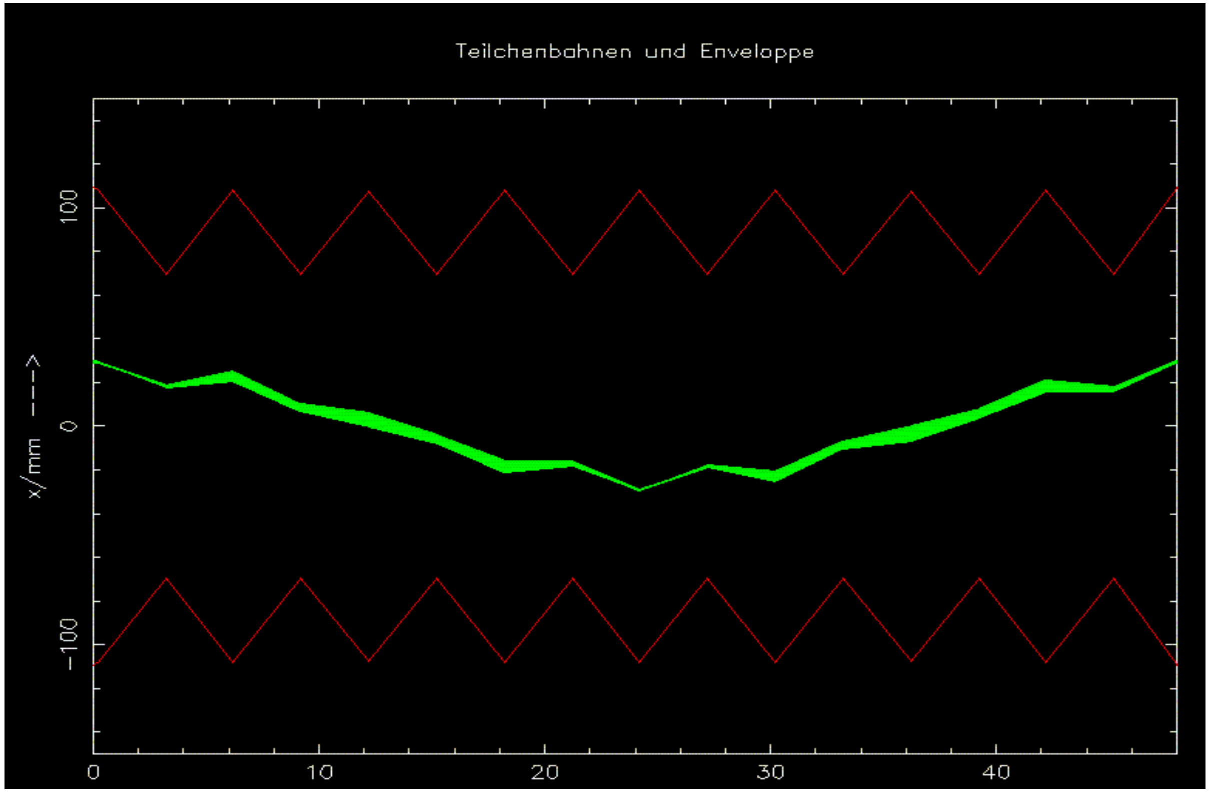
definition of chromaticity:

$$\Delta Q = Q' \frac{\Delta p}{p} \quad ; \qquad Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

Where is the Problem ?

Tunes and Resonances





avoid resonance conditions:

$$m\,Q_x + n\,Q_y + l\,Q_s = integer$$

… for example: $1\,Q_x = 1$

… and now again about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

***Q'** is a **number** indicating the **size of the tune spot** in the working diagram,*
***Q'** is always created if the beam is focussed*
*→ it is determined by the focusing strength **k** of all quadrupoles*

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

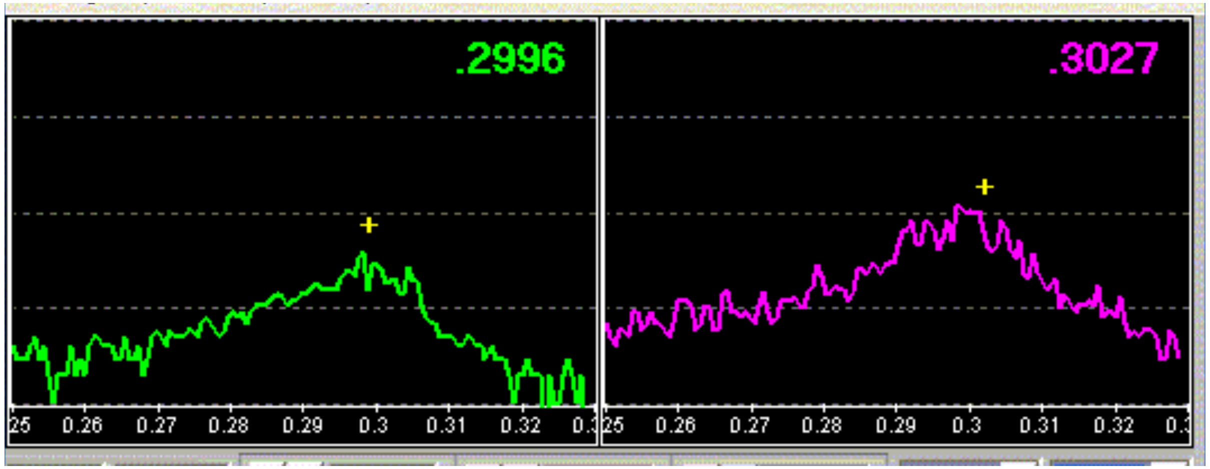
k = quadrupole strength
*β = **betafunction** indicates the beam size … and even more the **sensitivity of the beam to external fields***

Example: LHC

Q' = 250
$\Delta p/p = +/- 0.2 * 10^{-3}$
$\Delta Q = 0.256 \ldots 0.36$

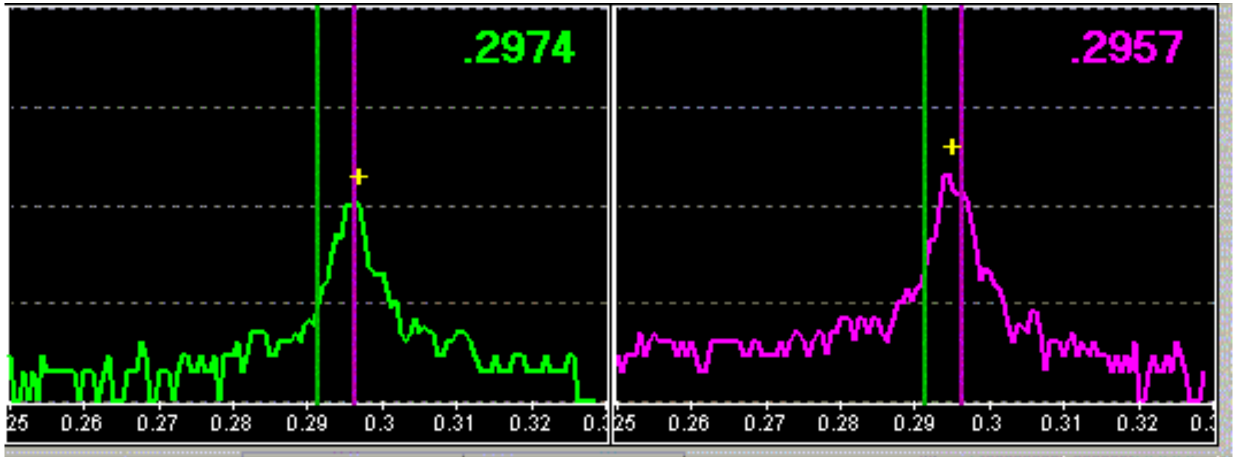
→Some particles get very close to resonances and are lost

in other words: the tune is not a point
it is a pancake



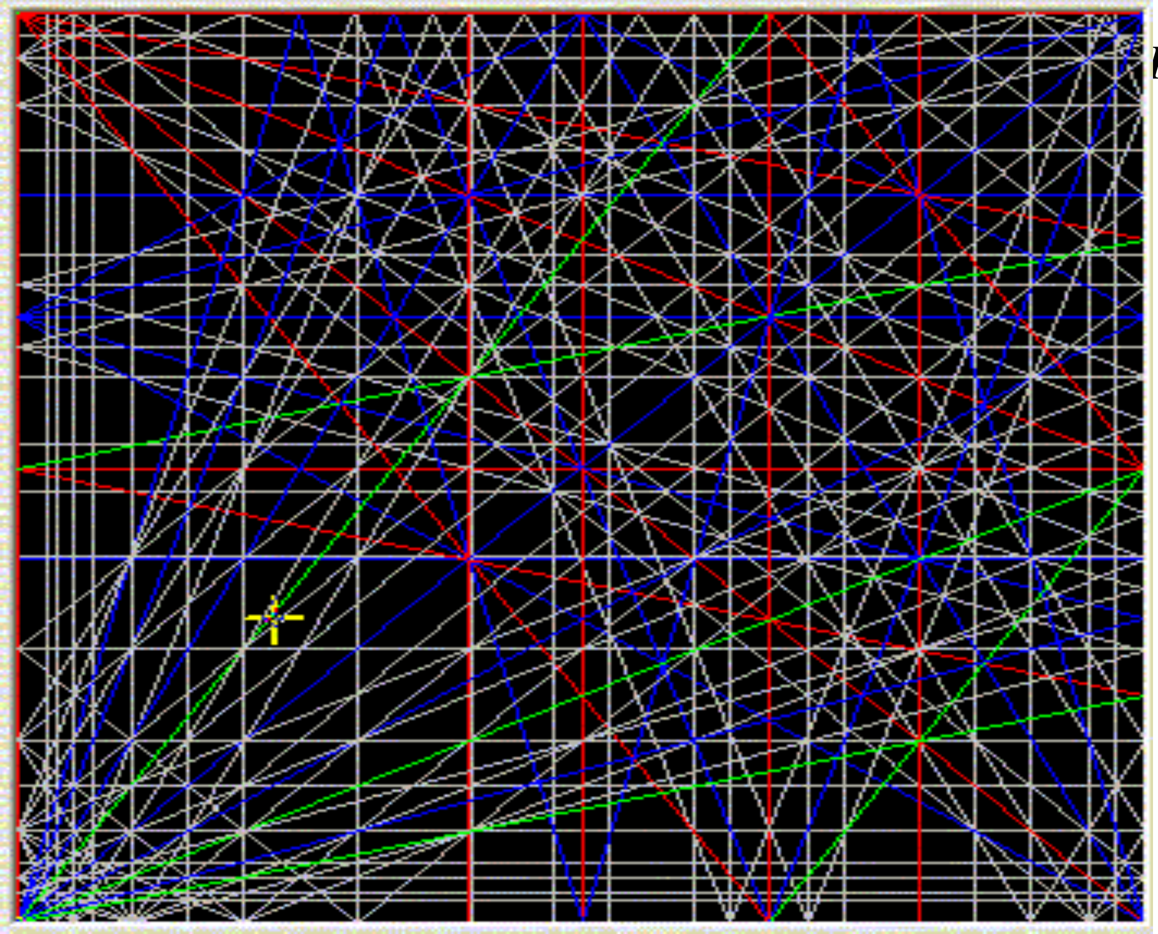
Tune signal for a nearly uncompensated cromaticity (Q' ≈ 20)

Ideal situation: cromaticity well corrected, (Q' ≈ 1)



Tune and Resonances

$$m*Q_x+n*Q_y+l*Q_s = \text{integer}$$



RA e Tune diagram up to 3rd order

… and up to 7th order

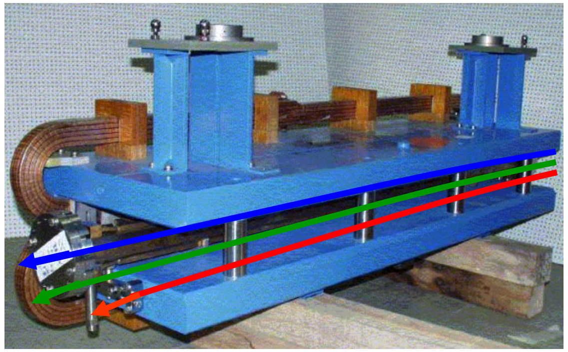
Homework for the operateurs:
find a nice place for the tune
where against all probability
the beam will survive

Correction of Q':

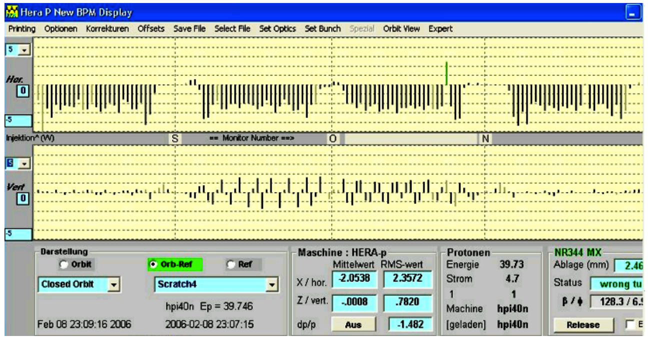
Need: additional quadrupole strength for each momentum deviation Δp/p

1.) sort the particles acording to their momentum

$$x_D(s) = D(s)\frac{\Delta p}{p}$$



… using the dispersion function



Darstellung
○ Orbit ● Orb-Ref ○ Ref
Closed Orbit
Scratch4
hpi40n Ep = 39.746
Feb 08 23:09:16 2006
2006-02-08 23:07:15

Maschine : HERA-p

	Mittelwert	RMS-wert
X / hor.	-2.0538	2.3572
Z / vert.	-.0008	.7820
dp/p	Aus	-1.482

Protonen

Energie	39.73
Strom	4.7
1	1
Machine	hpi40n
[geladen]	hpi40n

NR344 MX
Ablage (mm) 2.46
Status wrong tu
β / ϕ 128.3 / 6.

***2.) apply a** magnetic field that rises quadratically **with x** (sextupole field)*

$$\left.\begin{aligned} B_x &= \tilde{g}xy \\ B_y &= \frac{1}{2}\tilde{g}(x^2 - y^2) \end{aligned}\right\} \qquad \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{g}x$$

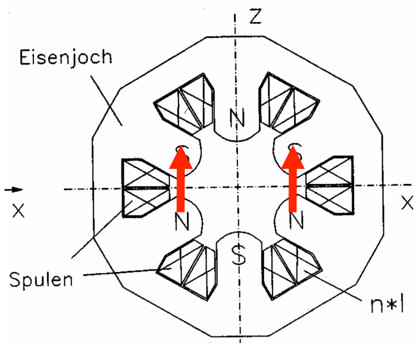
linear amplitude dependent „gradient“:

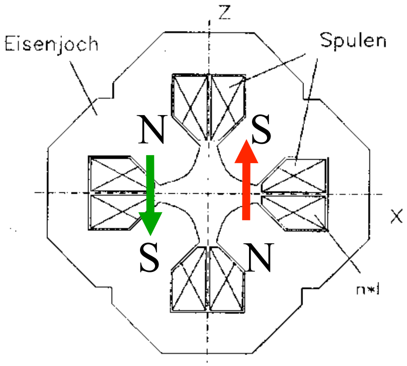
Correction of Q':

k_1 normalised quadrupole strength

k_2 normalised sextupole strength

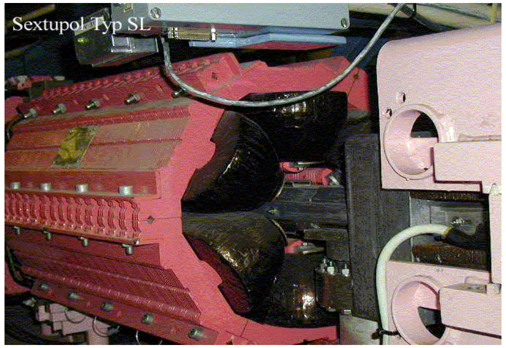
Sextupole Magnets:





$$k_1(sext) = \frac{\tilde{g}x}{p/e} = k_2 * x$$

$$= k_2 * D\,\frac{\Delta p}{p}$$



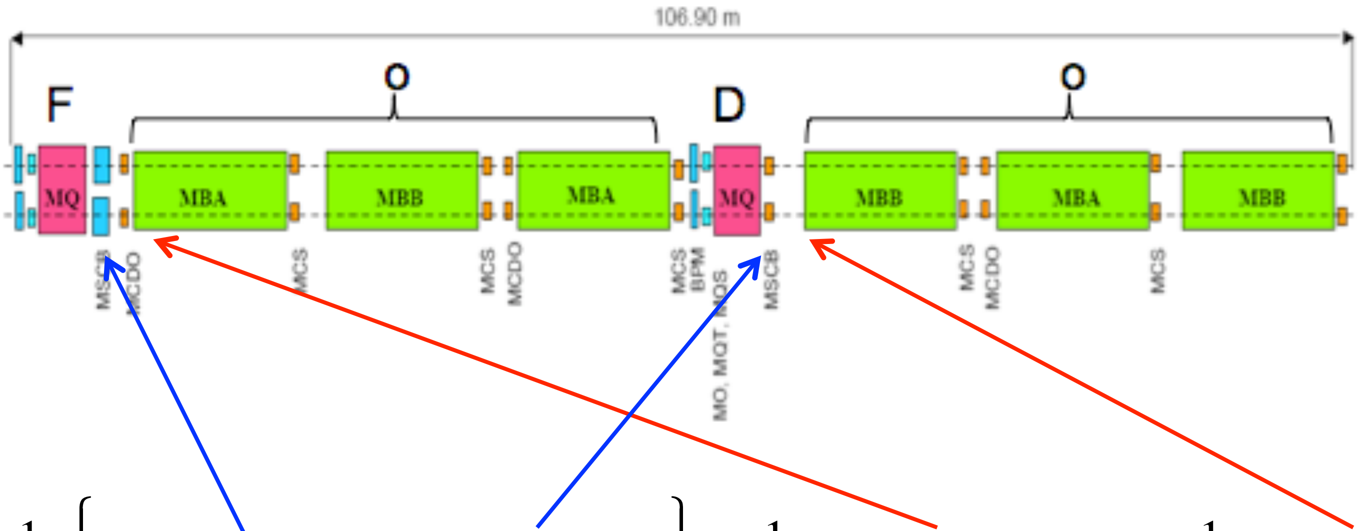
Combined effect of „natural chromaticity“ and Sextupole Magnets:

$$Q' = -\frac{1}{4\pi}\left\{\int k_1(s)\beta(s)ds + \int k_2 * D(s)\beta(s)ds\right\}$$

You only should not forget to correct Q ‘ in both planes ...
and take into account the contribution from quadrupoles of both polarities.

corrected chromaticity

considering an arc built out of single cells:

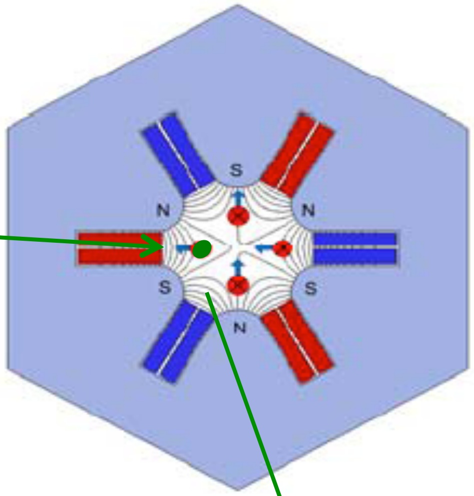


$$Q'_x = -\frac{1}{4\pi}\left\{ \sum_{F\,quad} k_{qf}\hat{\beta}_x l_{qf} - \sum_{D\,quad} k_{qd}\breve{\beta}_x l_{qd} \right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

$$Q'_y = -\frac{1}{4\pi}\left\{ -\sum_{F\,quad} k_{qf}\breve{\beta}_y l_{qf} + \sum_{D\,quad} k_{qd}\hat{\beta}_y l_{qd} \right\} - \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F + \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

25.) Particle Tracking Calculations

particle vector: $\begin{pmatrix} x \\ x' \end{pmatrix}$



$$B = \begin{pmatrix} g'xy \\ \frac{1}{2} g'(x^2 - y^2) \end{pmatrix}$$

Idea: ***calculate the particle coordiantes x, x´ through the linear lattice***
… using the matrix formalism.
if you encounter a nonlinear element (e.g. sextupole): stop
calculate explicitly the magnetic field at the particles coordinate

calculate kick on the particle

$$\Delta x_1' = \frac{B_y l}{p/e} = \frac{1}{2}\frac{g'}{p/e} l(x_1^2 - y_1^2) = \frac{1}{2} m_{sext} l(x_1^2 - y_1^2)$$

$$\Delta y'_1 = \frac{B_x l}{p/e} = \frac{g' x_1 y_1}{p/e} l = m_{sext} l x_1 y_1$$

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_1' + \Delta x' \end{pmatrix}$$

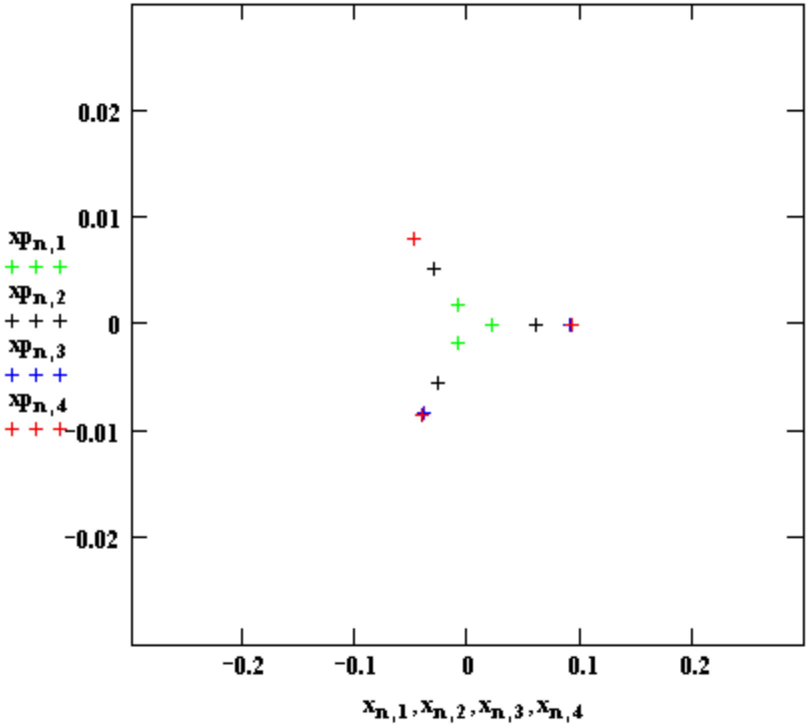
$$\begin{pmatrix} y_1 \\ y'_1 \end{pmatrix} \rightarrow \begin{pmatrix} y_1 \\ y'_1 + \Delta y'_1 \end{pmatrix}$$

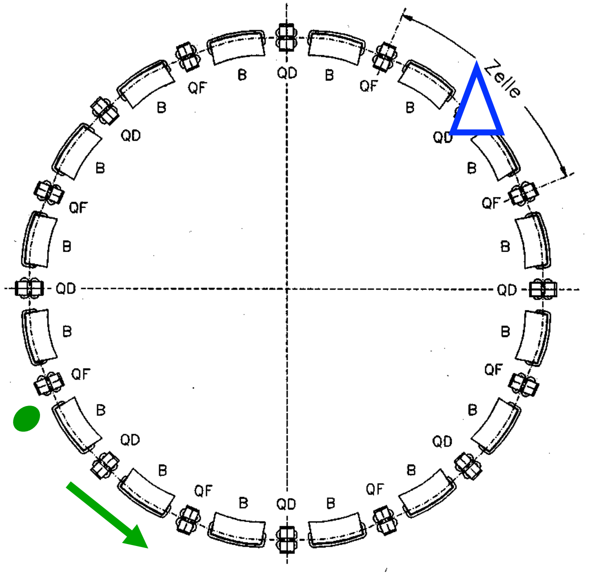
and continue with the linear matrix transformations

Installation of a weak (!!!) sextupole magnet

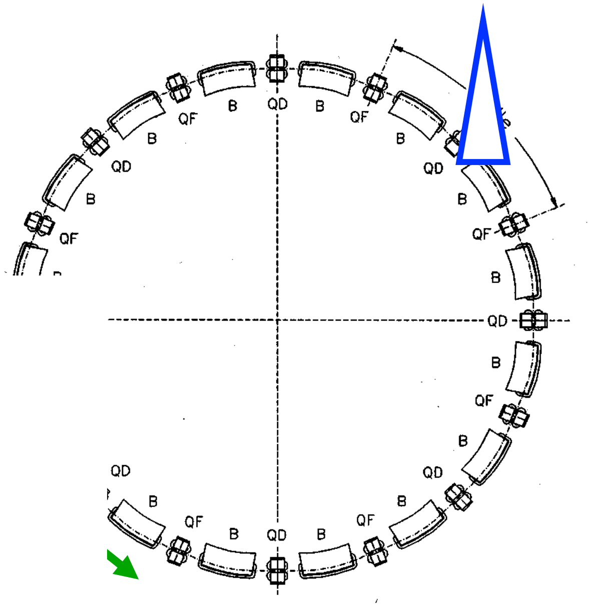
The good news: sextupole fields in accelerators cannot be treated with conventional methods.

→ no equatiuons; instead: Computer simulation „ particle tracking “

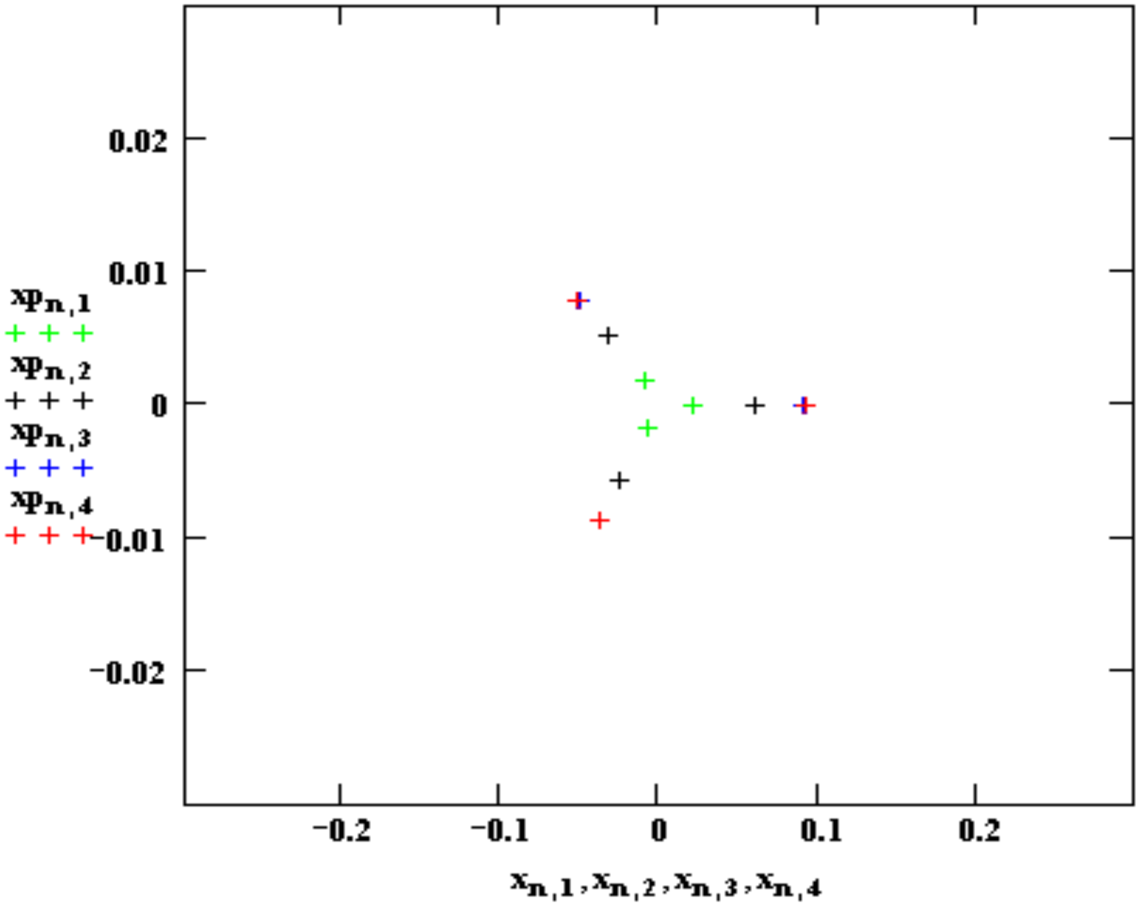




Effect of a strong (!!!) Sextupole …

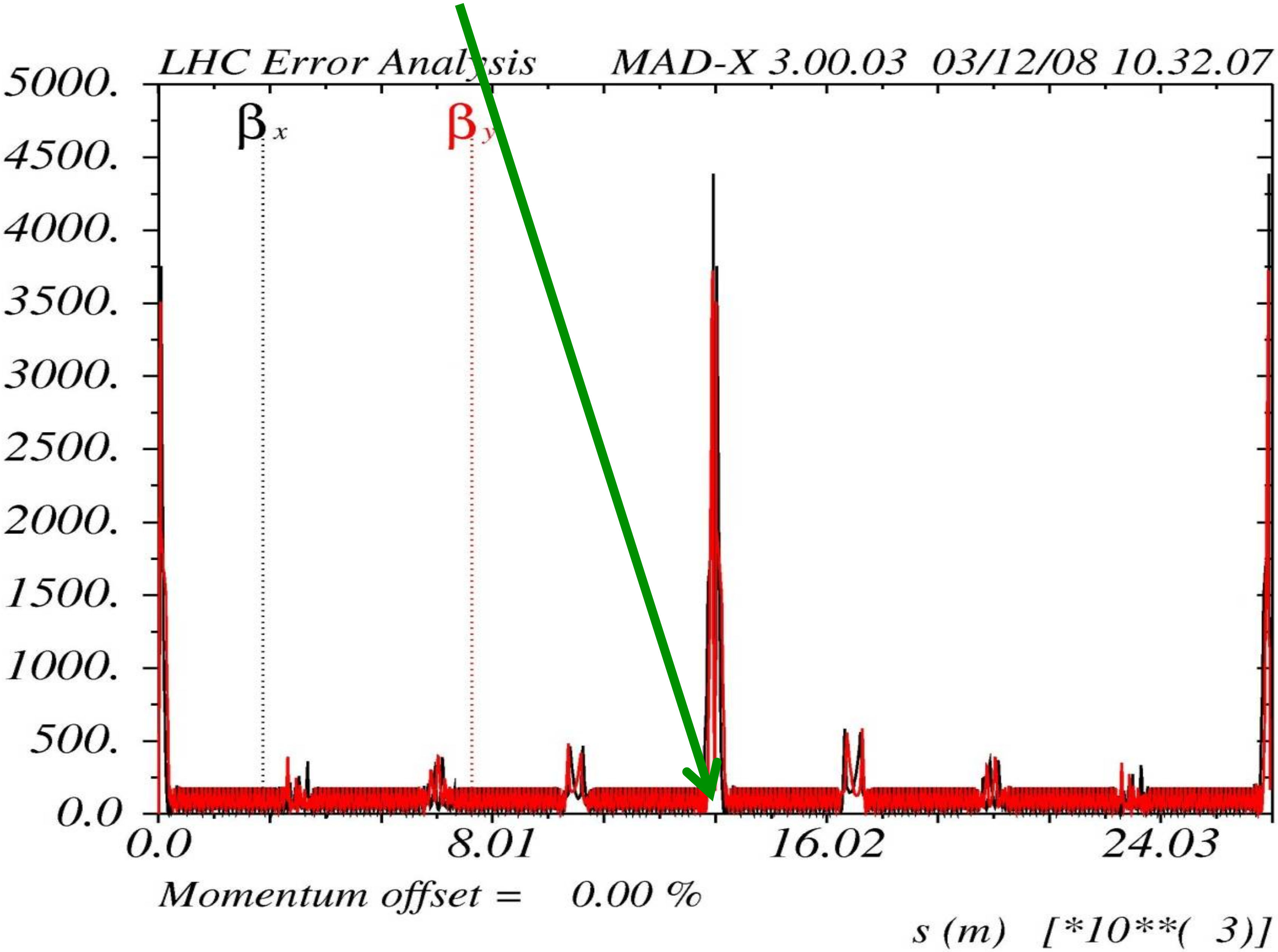


→ Catastrophy !



„dynamic aperture“

20.) Insertions



Insertions

... the most complicated one: the drift space

Question to the audience: what will happen to the beam parameters α, β, γ if we stop focusing for a while ...?

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_S = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

where e.g. for one element

$$M_{QF} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}\, l_q) & \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q) \\ -\sqrt{k} \sin(\sqrt{k}\, l_q) & \cos(\sqrt{k}\, l_q) \end{pmatrix}$$

transfer matrix for a drift:

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \longrightarrow$$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

$$\alpha(s) = \alpha_0 - \gamma_0 s$$

$$\gamma(s) = \gamma_0$$

β-Function in a Drift:

let ‘s assume we are at a symmetry point in the center of a drift.

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

as $\alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^{\ 2}}{\beta_0} = \frac{1}{\beta_0}$

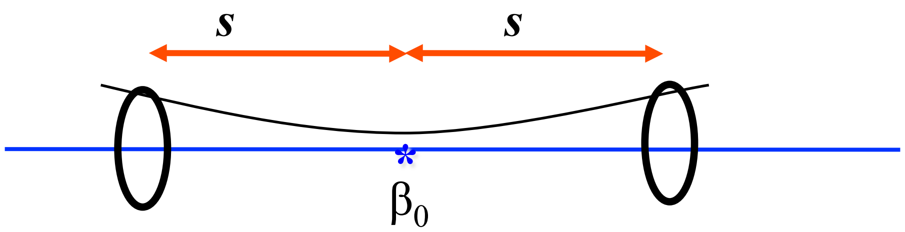
and we get for the β function in the neighborhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

! ! !

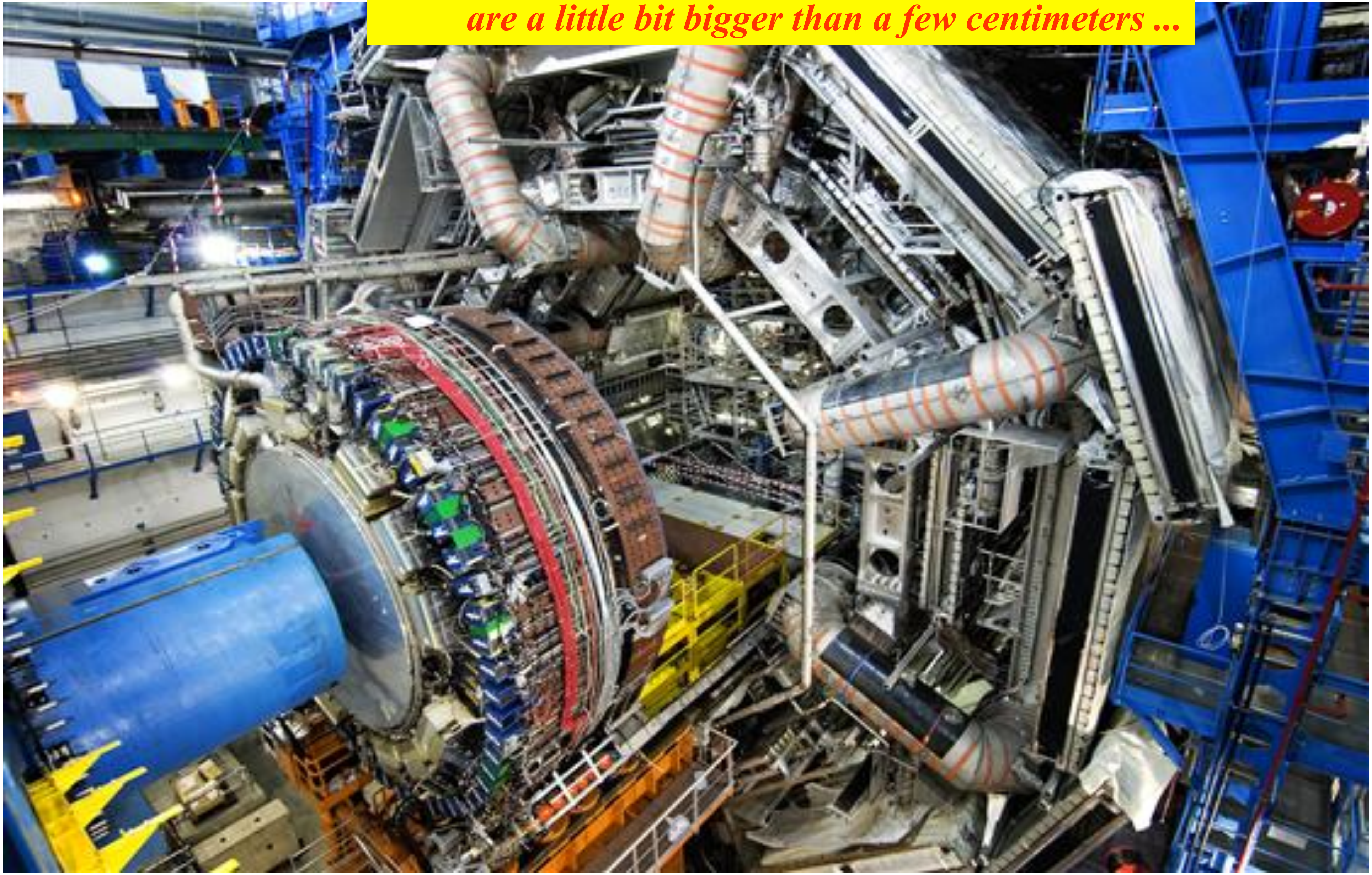
At the end of a long symmetric drift space the beta function reaches its maximum value in the complete lattice.
-> here we get the largest beam dimension.

-> keep l as small as possible

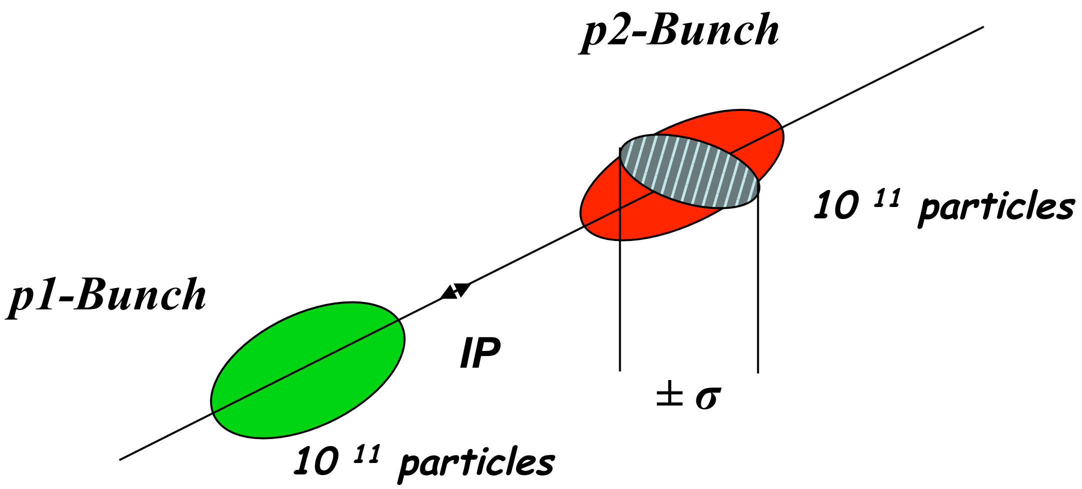


... clearly there is an

***But:* ... unfortunately ... in general**
***high energy detectors* that are**
installed in that drift spaces
are a little bit bigger than a few centimeters ...



21.) Luminosity



Example: Luminosity run at LHC

$\beta_{x,y} = 0.55\ m$

$\varepsilon_{x,y} = 5 * 10^{-10}\ rad\ m$

$\sigma_{x,y} = 17\ \mu m$

$f_0 = 11.245\ kHz$

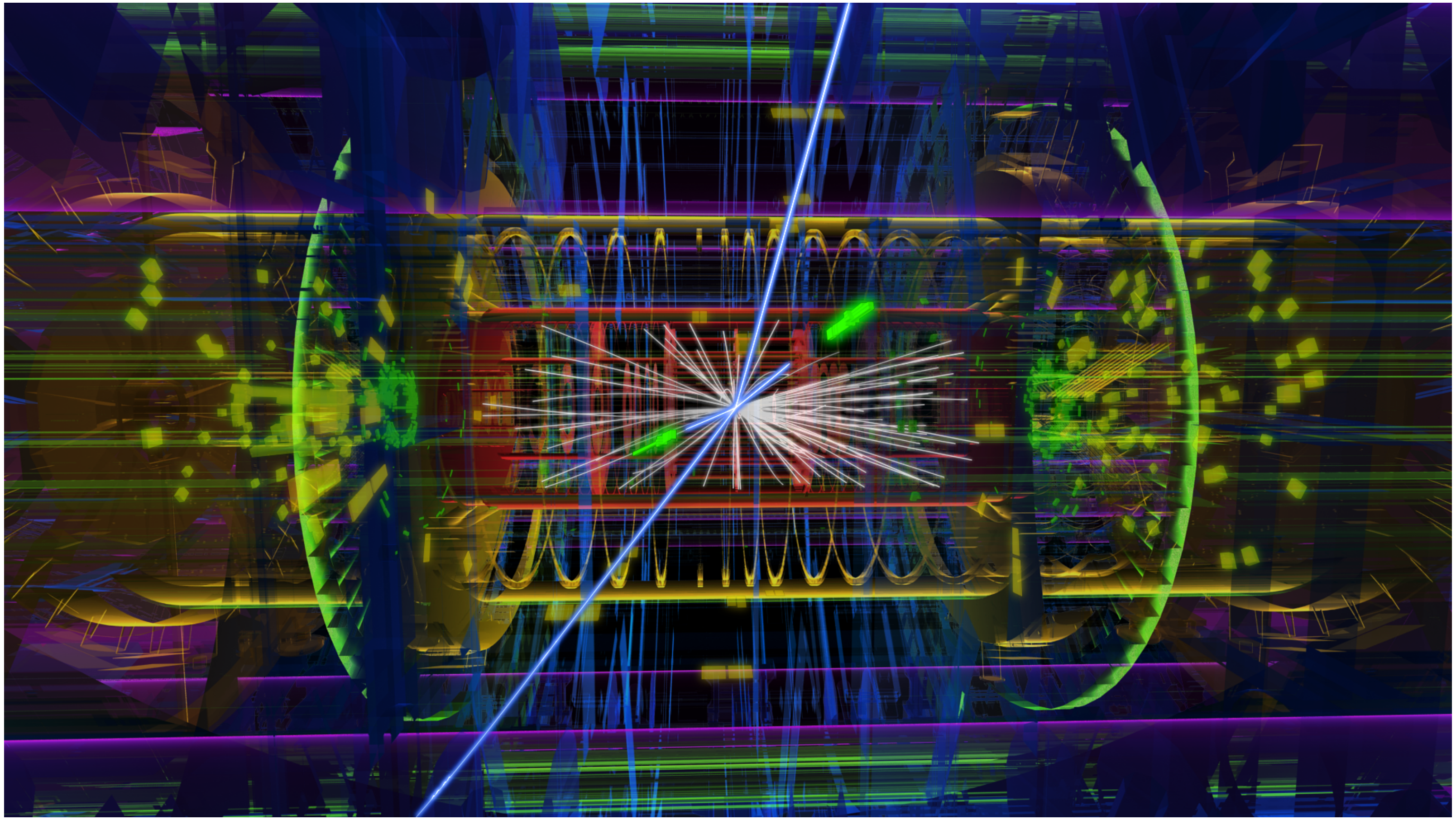
$n_b = 2808$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

$I_p = 584\ mA$

$$L = 1.0 * 10^{34}\ {}^{1}\!/_{cm^2 s}$$

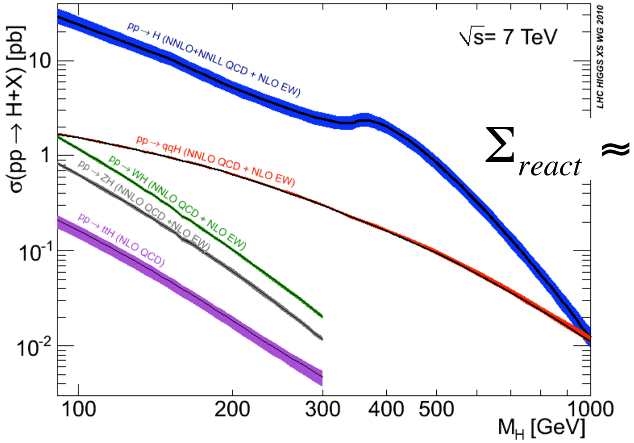
High Light of the HEP-Year 2012 / 13 naturally the HIGGS



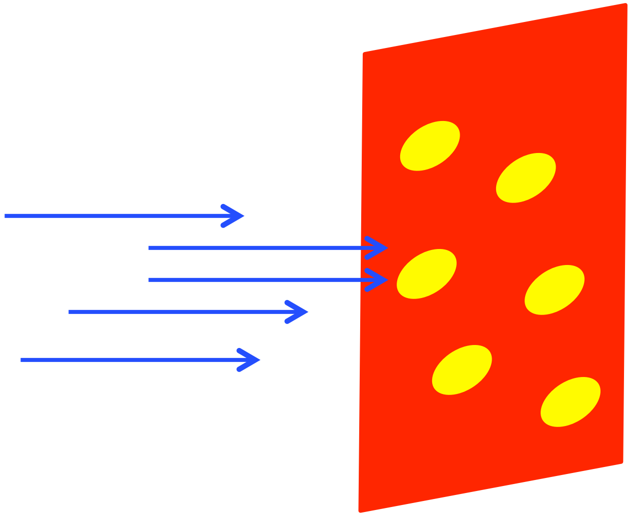
ATLAS event display: Higgs => two electrons & two muons

Problem: Our particles are VERY small !!

Overall cross section of the Higgs:



$$\Sigma_{react} \approx 1\,pb$$

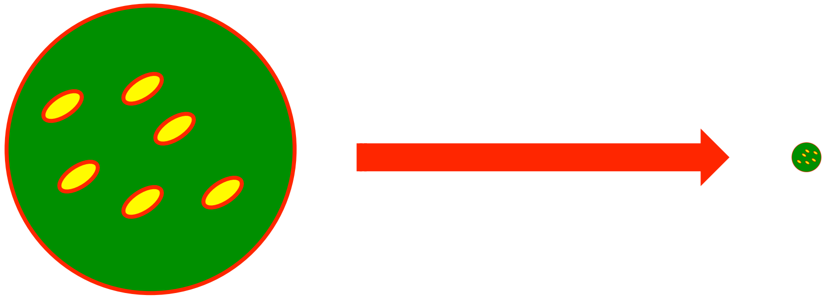


$$1b \;= 10^{-24}\,cm^2$$

$$1pb = \quad 10^{-12} * 10^{-24}\,cm^2 = 1/mio * 1/mio * 1/mio * 1/mio * 1/mio * 1/10000\ mm^2$$

The particles are "very small"

The only chance we have:
compress the transverse beam size … at the IP



LHC typical:

σ = 0.1 mm → 16 μm

Mini-β Insertions: some guide lines♪

** calculate the periodic solution in the arc*

** introduce the drift space needed for the insertion device (detector ...)*

** put a quadrupole doublet (triplet ?) as close as possible*

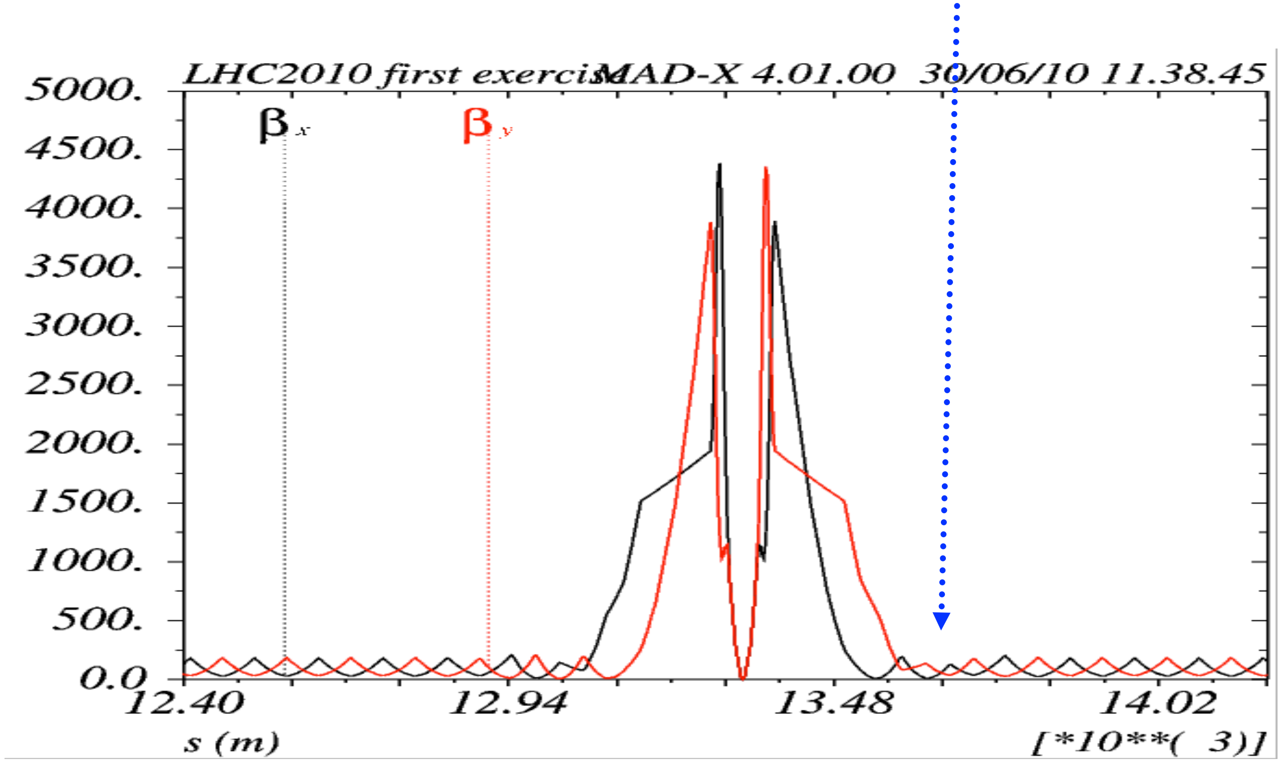
** introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure*

parameters to be optimised & matched to the periodic solution:

$$\alpha_x,\ \beta_x \qquad D_x,\ D_x'$$

$$\alpha_y,\ \beta_y \qquad Q_x,\ Q_y$$

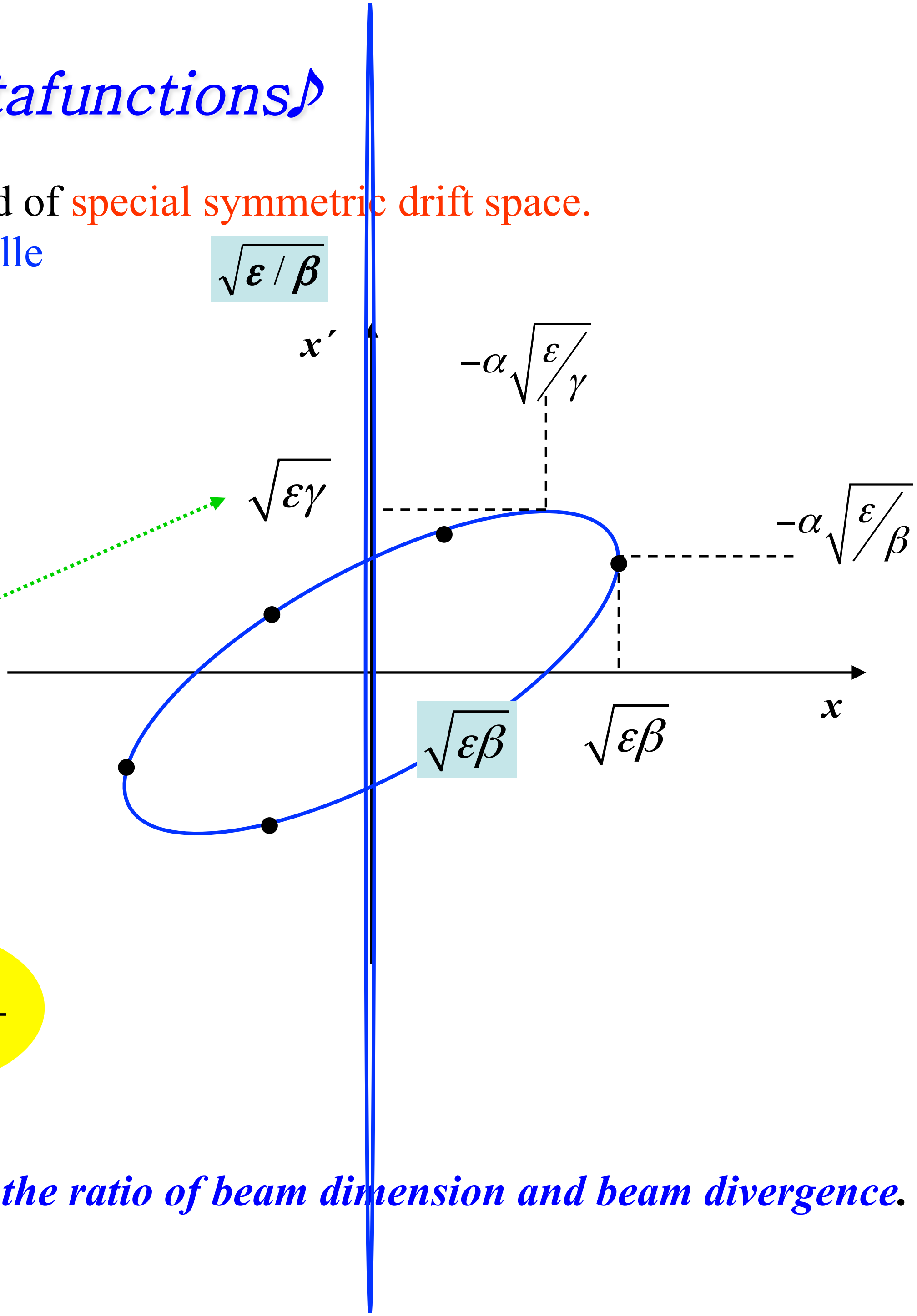
8 individually powered quad magnets are needed to match the insertion
(... at least)



Mini−β Insertions: Betafunctions♪

A mini-β insertion is always a kind of special symmetric drift space.

→greetings from Liouville



$$\alpha^* = 0$$

$$\gamma^* = \frac{1+\alpha^2}{\beta} = \frac{1}{\beta^*}$$

$$\sigma'^* = \sqrt{\frac{\varepsilon}{\beta^*}}$$

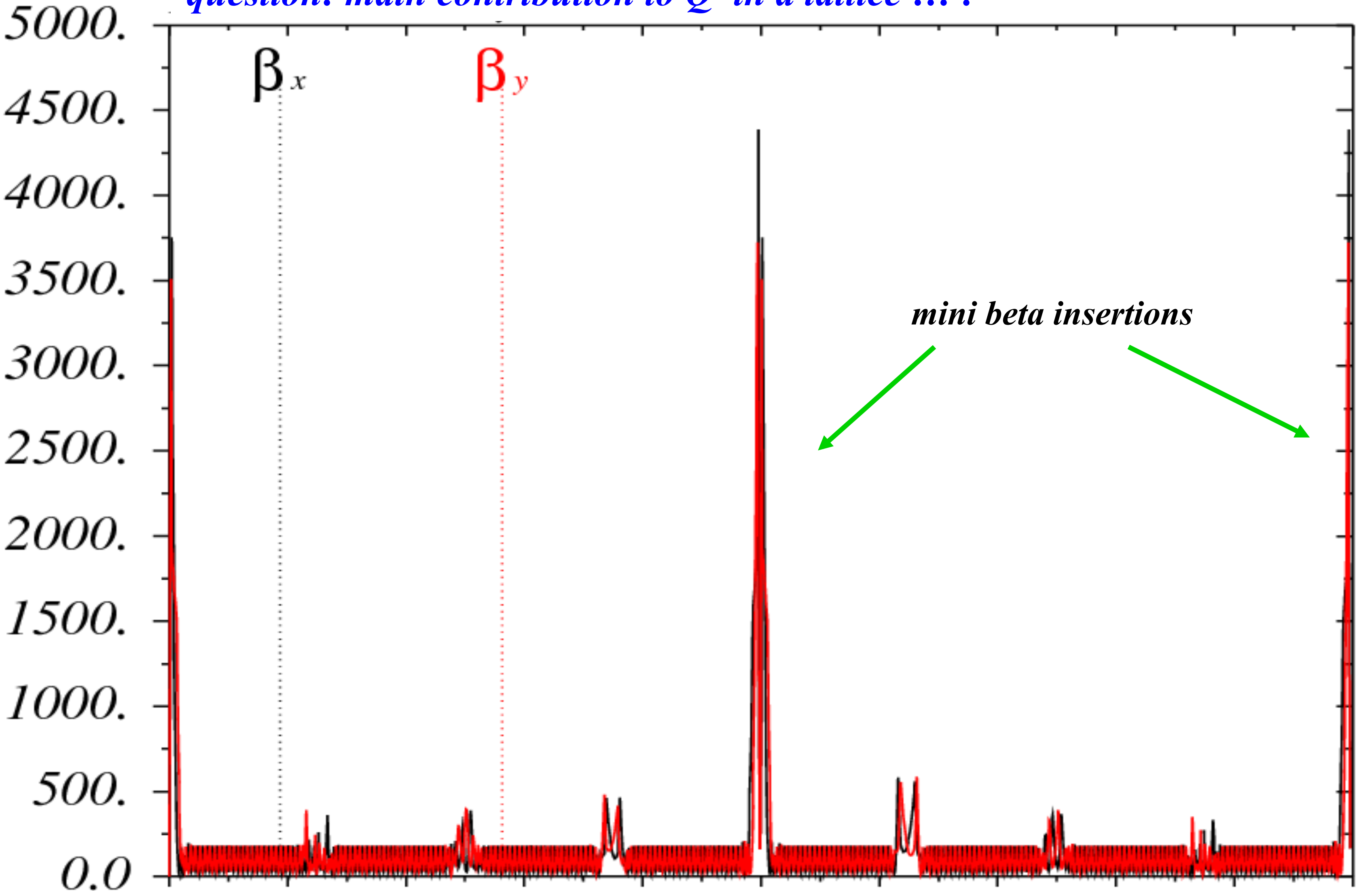
$$\beta^* = \frac{\sigma^*}{\sigma'^*}$$

at a symmetry point β is just the ratio of beam dimension and beam divergence.

… and now back to the Chromaticity

$$Q' = -\frac{1}{4\pi}\oint k(s)\boxed{\beta(s)}ds$$

question: main contribution to Q' in a lattice … ?



Resume´:

quadrupole error: tune shift

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s)\; l_{quad}\; \overline{\beta}}{4\pi}$$

beta beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta k\cos\left(2(\psi_{s1}-\psi_{s0})-2\pi Q\right)ds$$

chromaticity

$$\Delta Q = Q'\,\frac{\Delta p}{p}$$

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

momentum compaction

$$\frac{\delta\, l_{\varepsilon}}{L} = \alpha_p\,\frac{\Delta p}{p}$$

$$\alpha_p \approx \frac{2\pi}{L}\,\langle D\rangle \approx \frac{\langle D\rangle}{R}$$

beta function in a symmateric drift

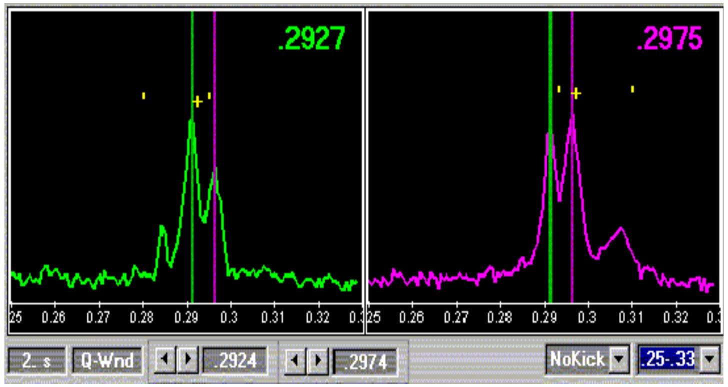
$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

Appendix:

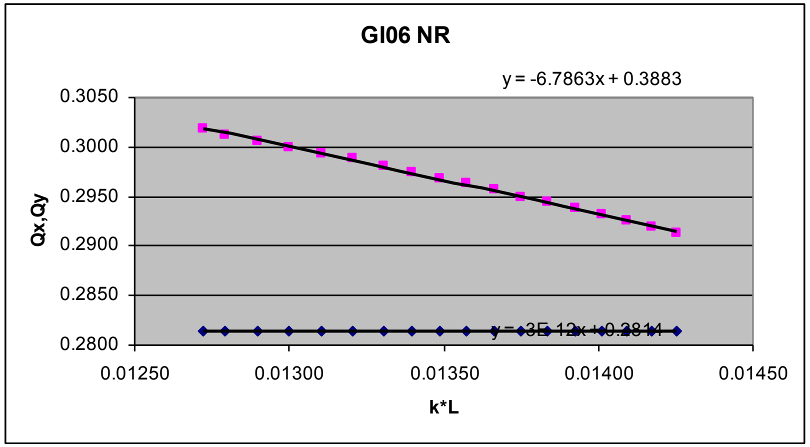
Quadrupole Error and Beta Function

a change of quadrupole strength in a synchrotron leads to tune sift:

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s) * l_{quad} * \overline{\beta}}{4\pi}$$



tune spectrum …



tune shift as a function of a gradient change

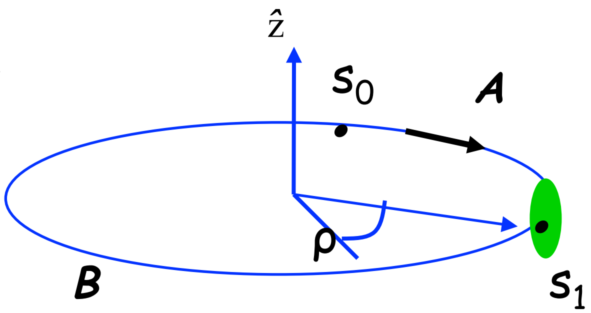
But we should expect an error in the β-function as well …

… shouldn´t we ???

Quadrupole Errors and Beta Function

a quadrupole error will not only influence the oscillation frequency … „tune"
… but also the amplitude … „beta function"

split the ring into 2 parts, described by two matrices A and B



$$M_{turn} = B * A$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

matrix of a quad error between A and B

$$M_{dist} = \begin{pmatrix} m^*_{11} & m^*_{12} \\ m^*_{21} & m^*_{22} \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ -\Delta kds & 1 \end{pmatrix} A$$

$$M_{dist} = B \begin{pmatrix} a_{11} & a_{12} \\ -\Delta kdsa_{11} + a_{12} & -\Delta kdsa_{12} + a_{22} \end{pmatrix}$$

$$M_{dist} = \begin{pmatrix} \sim & b_{11}a_{12} + b_{12}(-\Delta kdsa_{12} + a_{22}) \\ \sim & \sim \end{pmatrix}$$

the beta function is usually obtained via the matrix element „m12“, which is in Twiss form for the undistorted case

$$m_{12} = \beta_0 \sin 2\pi Q$$

and including the error:

$$m_{12}^{*} = \underbrace{b_{11}a_{12} + b_{12}a_{22}}_{m_{12} = \beta_0 \sin 2\pi Q} - b_{12}a_{12}\Delta k ds$$

$$(1) \quad m_{12}^{*} = \beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds$$

As M is still a matrix for one complete turn we still can express the element m_{12} in twiss form:*

$$(2) \quad m_{12}^{*} = (\beta_0 + d\beta) * \sin 2\pi (Q + dQ)$$

Equalising (1) and (2) and assuming a small error

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta) * \sin 2\pi (Q + dQ)$$

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta) * \sin 2\pi Q \underbrace{\cos 2\pi dQ}_{\approx\ 1} + \cos 2\pi Q \underbrace{\sin 2\pi dQ}_{\approx 2\pi\, dQ}$$

$$\cancel{\beta_0 \sin 2\pi Q} - a_{12} b_{12} \Delta k ds = \cancel{\beta_0 \sin 2\pi Q} + \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q + \cancel{d\beta_0 2\pi dQ} \cos 2\pi Q$$

ignoring second order terms

$$-a_{12} b_{12} \Delta k ds = \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

remember: tune shift dQ due to quadrupole error: $dQ = \dfrac{\Delta k \beta_1 ds}{4\pi}$

(index „1“ refers to location of the error)

$$-a_{12} b_{12} \Delta k ds = \frac{\beta_0 \Delta k \beta_1 ds}{2} \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

solve for dβ

$$d\beta_0 = \frac{-1}{2 \sin 2\pi Q} \left\{ 2 a_{12} b_{12} + \beta_0 \beta_1 \cos 2\pi Q \right\} \Delta k ds$$

express the matrix elements a_{12}, b_{12} in Twiss form

$$M = \begin{pmatrix} \sqrt{\dfrac{\beta_s}{\beta_0}} \left(\cos\psi_s + \alpha_0 \sin\psi_s \right) & \sqrt{\beta_s \beta_0} \sin\psi_s \\ \dfrac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\dfrac{\beta_0}{\beta s}} \left(\cos\psi_s - \alpha_s \sin\psi_s \right) \end{pmatrix}$$

$$d\beta_0 = \frac{-1}{2\sin 2\pi Q}\left\{2a_{12}b_{12} + \beta_0\beta_1\cos 2\pi Q\right\}\Delta k ds$$

$$\boldsymbol{a}_{12} = \sqrt{\boldsymbol{\beta}_0\boldsymbol{\beta}_1}\,\sin\Delta\boldsymbol{\psi}_{0\rightarrow 1}$$

$$\boldsymbol{b}_{12} = \sqrt{\boldsymbol{\beta}_1\boldsymbol{\beta}_0}\,\sin(2\pi\boldsymbol{Q} - \Delta\boldsymbol{\psi}_{0\rightarrow 1})$$

$$d\beta_0 = \frac{-\beta_0\beta_1}{2\sin 2\pi Q}\underbrace{\left\{2\sin\Delta\psi_{12}\sin(2\pi Q - \Delta\psi_{12}) + \cos 2\pi Q\right\}}\Delta k ds$$

… after some TLC transformations … $= \cos(2\Delta\psi_{01} - 2\pi Q)$

$$\Delta\beta(s_0) = \frac{-\beta_0}{\boxed{2\sin 2\pi Q}}\int_{s1}^{s1+l}\beta(s_1)\Delta k\cos\left(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\right)ds$$

Nota bene: ! the beta beat is *proportional to the strength of the error Δk*

!! and to the *β function* at the place of the error ,

!!! and to the *β function* at the observation point,
(… remember orbit distortion !!!)

!!!! there is a *resonance denominator*