PHASE STABILITY

by

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Bibliography : Old Books

M. Stanley Livingston **High Energy Accelerators**
(Interscience Publishers, 1954)

J.J. Livingood **Principles of cyclic Particle Accelerators**
(D. Van Nostrand Co Ltd , 1961)

M. Stanley Livingston and J. B. Blewett **Particle Accelerators**
(Mc Graw Hill Book Company, Inc 1962)

K.G. Steffen **High Energy optics**
(Interscience Publisher, J. Wiley & sons, 1965)

H. Bruck **Accelerateurs circulaires de particules**
(PUF, Paris 1966)

M. Stanley Livingston (editor) The development of High Energy Accelerators
(Dover Publications, Inc, N. Y. 1966)

A.A. Kolomensky & A.W. Lebedev **Theory of cyclic Accelerators**
(North Holland Publihers Company, Amst. 1966)

E. Persico, E. Ferrari, S.E. Segre **Principles of Particles Accelerators**
(W.A. Benjamin, Inc. 1968)

P.M. Lapostolle & A.L. Septier **Linear Accelerators**
(North Holland Publihers Company, Amst. 1970)

A.D. Vlasov **Theory of Linear Accelerators**
(Programm for scientific translations, Jerusalem 1968)

Bibliography : New Books

M. Conte, W.W. Mac Kay **An Introduction to the Physics of particle Accelerators** (World Scientific, 1991)

P. J. Bryant and K. Johnsen **The Principles of Circular Accelerators and Storage Rings** (Cambridge University Press, 1993)

D. A. Edwards, M. J. Syphers **An Introduction to the Physics of High Energy Accelerators** (J. Wiley & sons, Inc, 1993)

H. Wiedemann **Particle Accelerator Physics** (Springer-Verlag, Berlin, 1993)

M. Reiser **Theory and Design of Charged Particles Beams** (J. Wiley & sons, 1994)

A. Chao, M. Tigner **Handbook of Accelerator Physics and Engineering** (World Scientific 1998)

K. Wille **The Physics of Particle Accelerators: An Introduction** (Oxford University Press, 2000)

E.J.N. Wilson **An introduction to Particle Accelerators** (Oxford University Press, 2001)

And CERN Accelerator Schools (CAS) Proceedings

Main Characteristics of an Accelerator

ACCELERATION is the main job of an accelerator.

- The accelerator provides **kinetic energy** to charged particles, hence increasing their **momentum**.
- In order to do so, it is necessary to have an electric field E, preferably along the direction of the initial momentum.

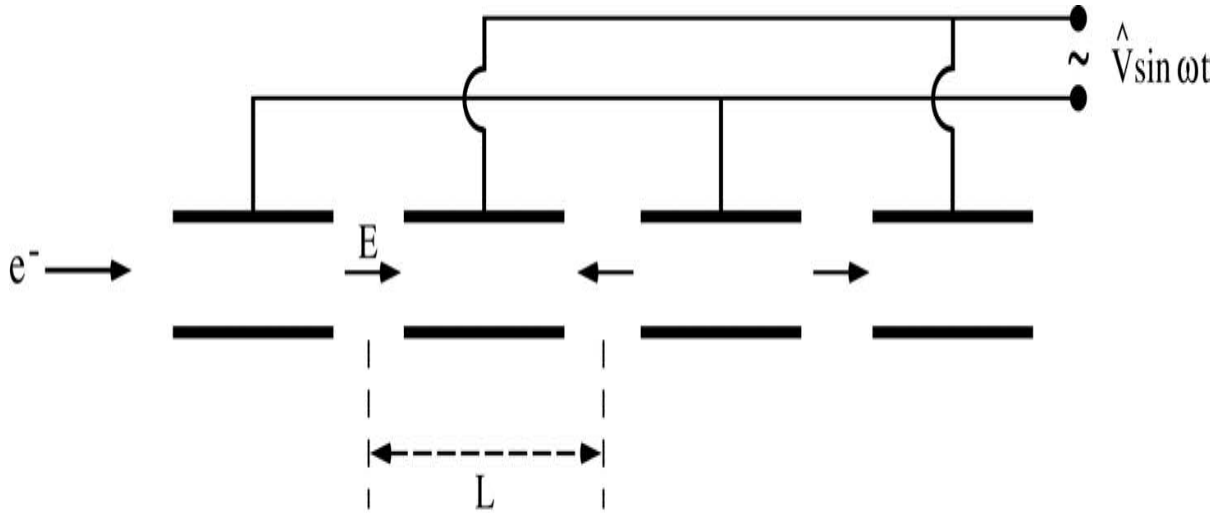
$$\boxed{\frac{dp}{dt} = eE}$$

BENDING is generated by a magnetic field perpendicular to the plane of the particle trajectory. The bending radius ρ obeys to the relation :

$$\boxed{\frac{p}{e} = B\rho}$$

FOCUSING is a second way of using a magnetic field, in which the bending effect is used to bring the particles trajectory closer to the axis, hence to increase the beam density.

Radio-Frequency Acceleration

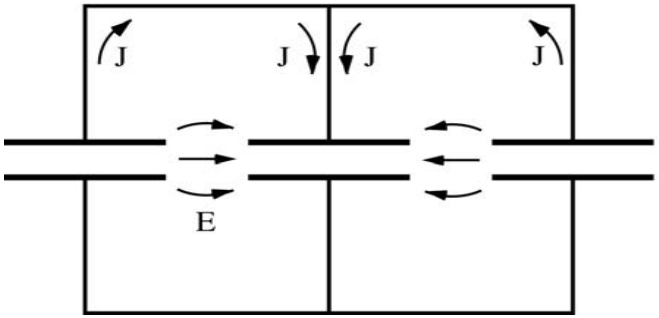


Cylindrical electrodes separated by gaps and fed by a RF generator, as shown on the Figure, lead to an alternating electric field polarity

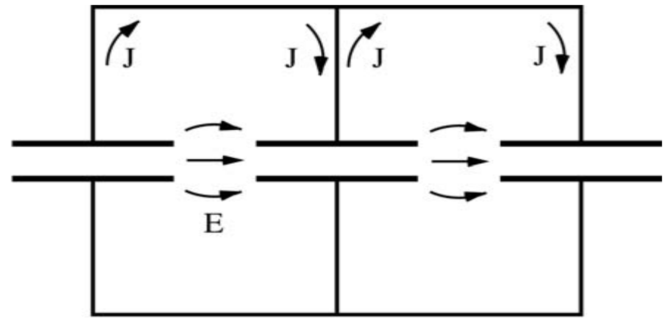
Synchronism condition $\longrightarrow$ $L = v\,T/2$

Radio-Frequency Acceleration (2)

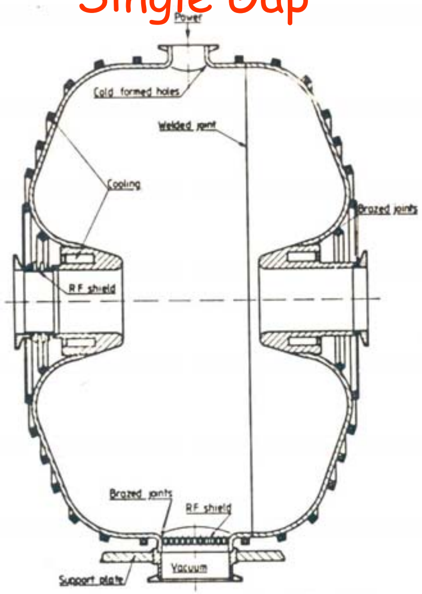
L = vT/2 (π mode)



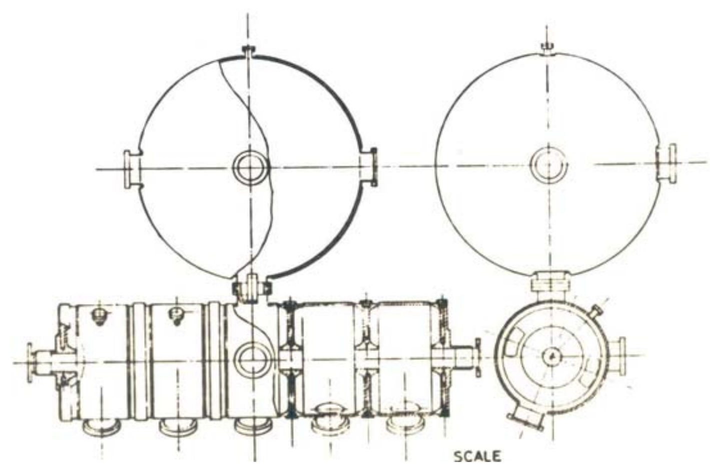
L = vT (2π mode)



Single Gap



Multi-Gap



Energy Gain

Newton-Lorentz Force

$$\frac{d\vec{p}}{dt} = e\,\vec{E}$$

Relativistics Dynamics

$$E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad dE = v\,dp$$

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

$$dE = dW = eE_z\,dz \quad \Rightarrow \quad W = \int E_z\,dz$$

RF Acceleration

$$E_z = \hat{E}_z \cos\omega_{RF}t = \hat{E}_z \cos\Phi(t)$$

$$\int \hat{E}_z\,dz = \hat{V}$$

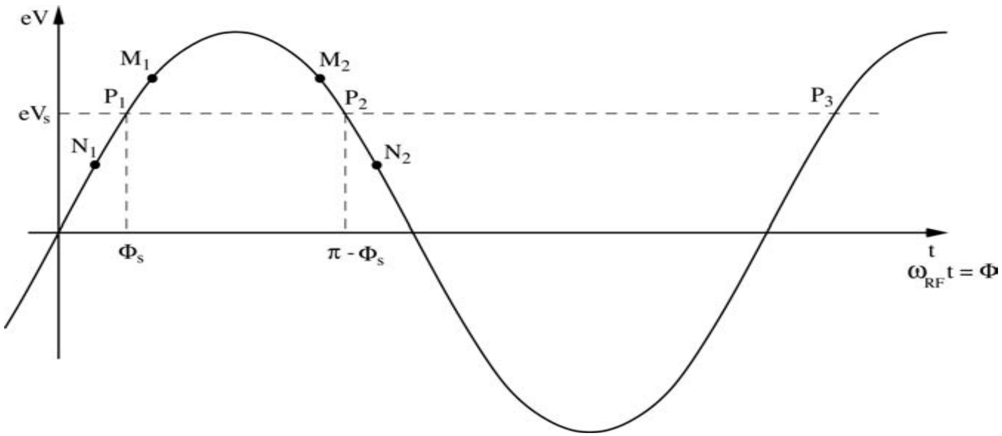
$$W = e\hat{V}\cos\Phi$$

(neglecting transit time factor)

Principle of Phase Stability

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

For a 2π mode, the electric field is the same in all gaps at any given time.



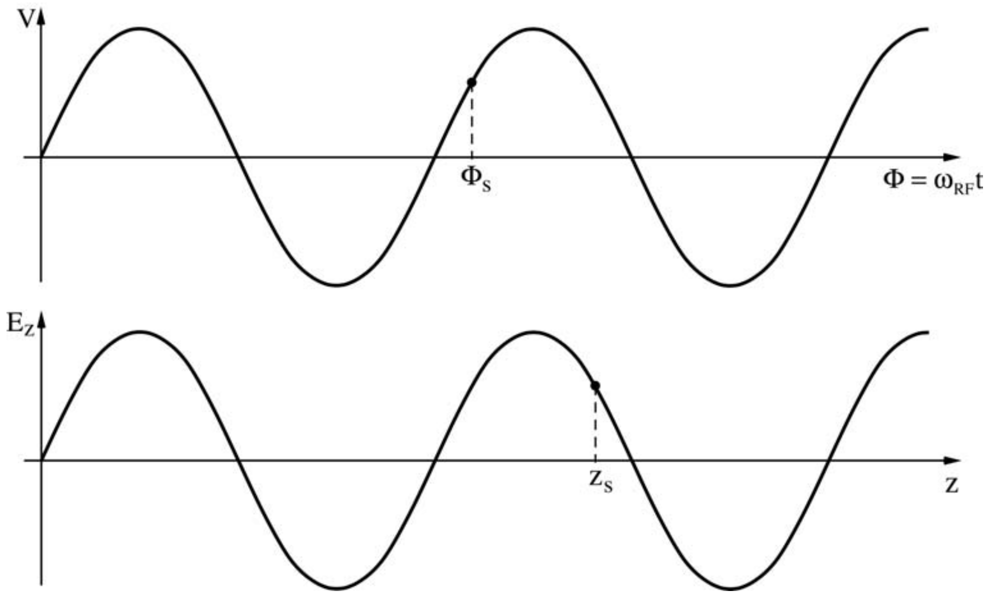
$eV_S = e\hat{V}\sin\Phi_S$ **is the energy gain in one gap for the particle to reach the next gap with the same RF phase : P_1 ,P_2, …… are fixed points.**

If an increase in energy is transferred into an increase in velocity, M_1 & N_1 will move towards P_1(stable), while M_2 & N_2 will go away from P_2 (unstable).

A Consequence of Phase Stability

Transverse Instability

Longitudinal phase stability means : $$\frac{\partial V}{\partial t} > 0 \Rightarrow \frac{\partial E_z}{\partial z} < 0$$



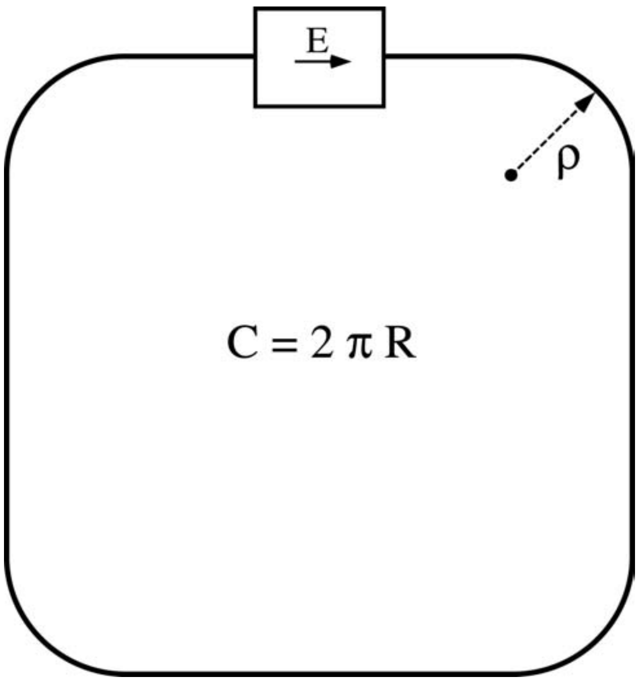
Defocusing RF force

The divergence of the field is zero according to Maxwell : $$\nabla.\vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} > 0$$

External focusing (solenoid, quadrupole) is then necessary

The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$e\hat{V}\sin\Phi \longrightarrow$ Energy gain per turn

$\Phi = \Phi_s = cte \longrightarrow$ Synchronous particle

$\omega_{RF} = h\omega_r \longrightarrow$ RF synchronism

$\rho = cte \quad R = cte \longrightarrow$ Constant orbit

$B\rho = P/e \Rightarrow B \longrightarrow$ Variable magnetic field

If v = c, ω_r hence ω_{RF} remain constant (ultra-relativistic e^-)

The Synchrotron (2)

Energy ramping is simply obtained by varying the B field:

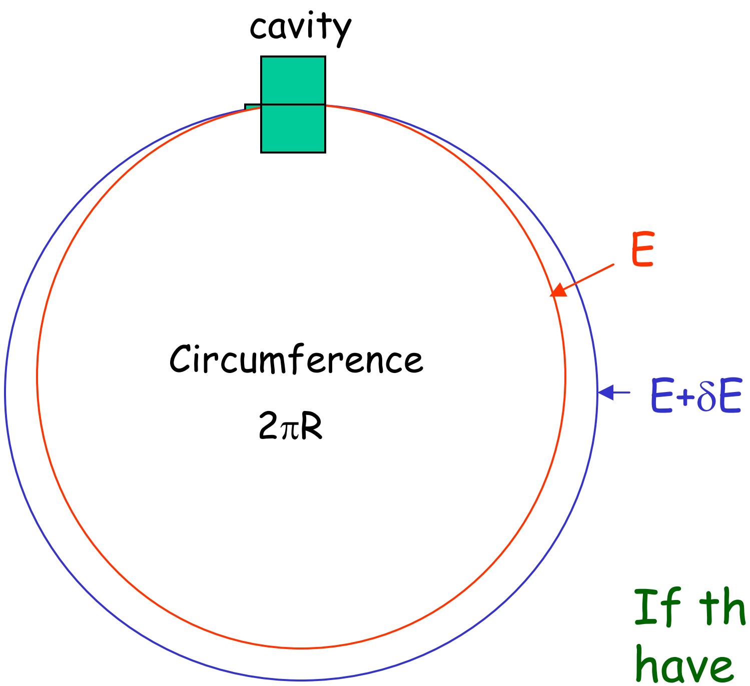
$$p = eB\rho \quad \Rightarrow \quad \frac{dp}{dt} = e\rho\, B' \quad \Rightarrow (\Delta p)_{turn} = e\rho\, B'T_r = \frac{2\pi\, e\rho\, RB'}{v}$$

Since: $$E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad \Delta E = v\Delta p$$

$$(\Delta E)_{turn} = (\Delta W)_{turn} = 2\pi\rho RB' = eV\,\sin\Phi_s$$

- The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.
- Each synchronous particle satifies the relation p=eBρ. They have the nominal energy and follow the nominal trajectory.

Dispersion Effects in a Synchrotron



If a particle is slightly shifted in momentum it will have a different orbit:

$$\alpha = \frac{p}{R}\frac{dR}{dp}$$

This is the "momentum compaction" generated by the bending field.

If the particle is shifted in momentum it will have also a different velocity. As a result of both effects the revolution frequency changes:

$$\eta = \frac{p}{f_r}\frac{df_r}{dp}$$

p=particle momentum

R=synchrotron physical radius

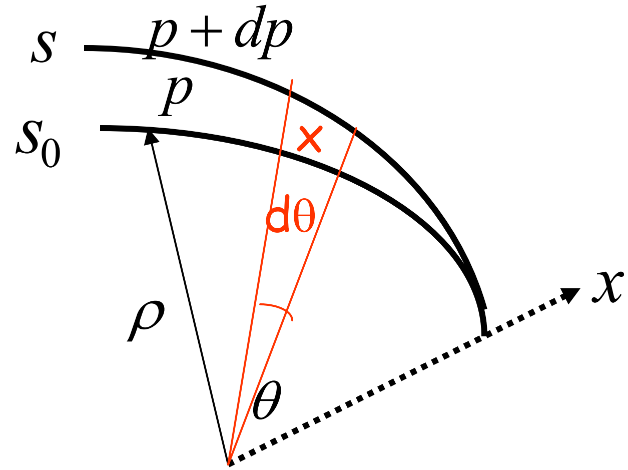
f_r=revolution frequency

Dispersion Effects in a Synchrotron (2)

$$\alpha = \frac{p}{R}\frac{dR}{dp}$$

$$ds_0 = \rho d\theta$$

$$ds = (\rho + x)d\theta$$



The elementary path difference from the two orbits is:

$$\frac{ds - ds_0}{ds_0} = \frac{dl}{ds_0} = \frac{x}{\rho}$$

leading to the total change in the circumference:

$$\int dl = 2\pi dR = \int \frac{x}{\rho} ds_0 = \frac{1}{\rho}\int_m x ds_0 \quad \Rightarrow \quad dR = \langle x \rangle_m$$

$\langle\ \rangle_m$ means that the average is considered over the bending magnet only

Since: $x = D_x \dfrac{dp}{p}$ **we get:** $\alpha = \dfrac{\langle D_x \rangle_m}{R}$

Dispersion Effects in a Synchrotron (3)

$$\boxed{\eta = \frac{p}{f_r}\frac{df_r}{dp}}$$

$$f_r = \frac{\beta c}{2\pi R} \Rightarrow \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R}$$

$$p = mv = \beta\gamma\frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-\frac{1}{2}}}{\left(1-\beta^2\right)^{-\frac{1}{2}}} = \left(1-\beta^2\right)^{-1}\frac{d\beta}{\beta}$$

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right)\frac{dp}{p} \longrightarrow \boxed{\eta = \frac{1}{\gamma^2} - \alpha}$$

η=0 at the transition energy

$$\boxed{\gamma_{tr} = \frac{1}{\sqrt{\alpha}}}$$

Phase Stability in an Electron Synchrotron

In an electron synchrotron γ is generally very large and:

$$\eta = \frac{1}{\gamma^2} - \alpha \cong -\alpha < 0$$

(since α>0 in most cases)

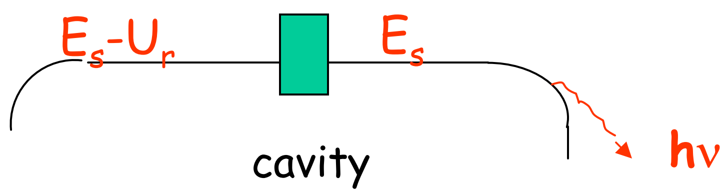
Hence dp>0 translates into df_r<0, which is a consequense of a longer orbit while the velocity remains constant=c.

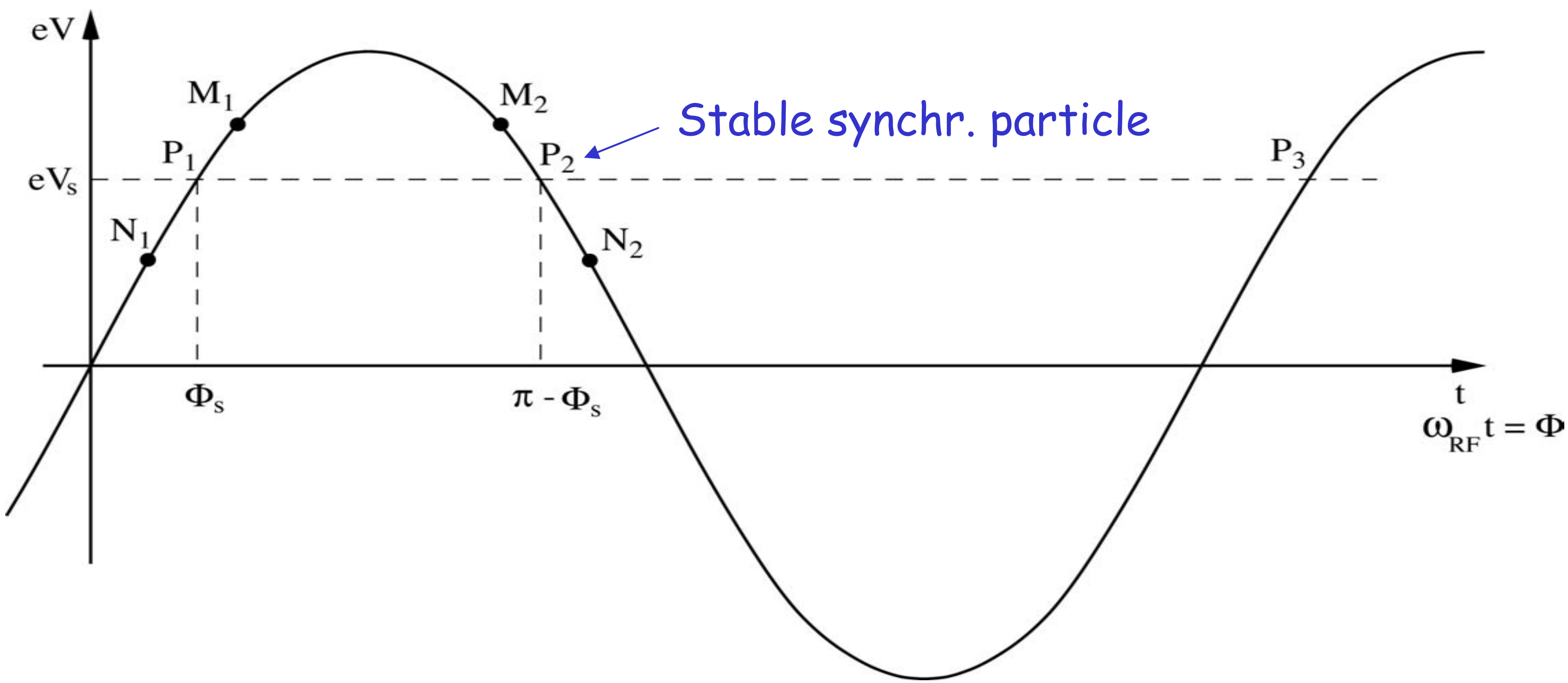
A delayed particle with respect to the synchronous one will get closer to it if it gets a smaller energy increase when going through the cavity, since then it will go faster around the machine because of a smaller path.

Consequently the stable synchronous phase has to sit on the negative slope of the RF voltage ($\pi - \phi_s$)

Phase Stability in an Electron Synchrotron (2)

Radiation loss: $U_r \propto \dfrac{E_s^{\,4}}{\rho}$





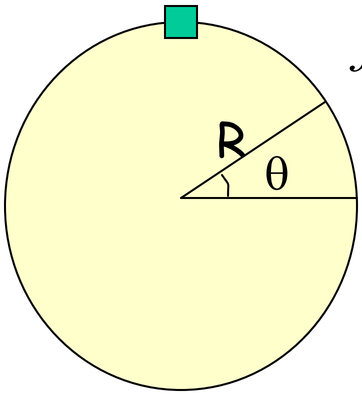
Longitudinal Dynamics

It is also often called " synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle. Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle. So let's introduce the following reduced variables:

revolution frequency	:	$\Delta f_r = f_r - f_{rs}$
particle RF phase	:	$\Delta\phi = \phi - \phi_s$
particle momentum	:	$\Delta p = p - p_s$
particle energy	:	$\Delta E = E - E_s$
azimuth angle	:	$\Delta\theta = \theta - \theta_s$

First Energy-Phase Equation



$$f_{RF} = h f_r \quad \Rightarrow \quad \Delta\phi = -h\Delta\theta \quad with \quad \theta = \int \omega_r dt$$

For a given particle with respect to the reference one:

$$\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $$\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s = \frac{E_s}{\omega_{rs}}\left(\frac{d\omega_r}{dE}\right)_s \cong -\alpha \quad (electron\ case)$$

one gets: $$\frac{\Delta E}{E_s} = \frac{1}{\omega_{rs}\alpha h}\frac{d\phi}{dt} = \frac{R}{c\alpha h}\dot{\phi}$$

Second Energy-Phase Equation

The energy gained by a particle at each turn is: $e\hat{V}\sin\phi$

and when compare to the reference one: $(\Delta E)_{turn} = e\hat{V}(\sin\phi - \sin\phi_s)$

The rate of relative energy gain can be approximated to first order:

$$\frac{d(\Delta E)}{dt} \cong (\Delta E)_{turn} f_{rs} = \frac{c}{2\pi R} e\hat{V}(\sin\phi - \sin\phi_s)$$

leading to the second energy-phase equation:

$$\frac{d}{dt}\left(\frac{\Delta E}{E_s}\right) = \frac{ce\hat{V}}{2\pi R E_s}(\sin\phi - \sin\phi_s)$$

Small Amplitude Oscillations

$$\frac{d}{dt}\left(\frac{\Delta E}{E_s}\right) = \frac{ce\hat{V}}{2\pi R E_s}\left(\sin\phi - \sin\phi_s\right) \cong \frac{ce\hat{V}\cos\phi_s}{2\pi R E_s}\Delta\phi$$

(for small $\Delta\phi$)

$$\frac{\Delta E}{E_s} = \frac{1}{\omega_{rs}\alpha h}\frac{d\phi}{dt} = \frac{R}{c\alpha h}\dot{\phi} \quad \Rightarrow \quad \ddot{\phi} = \frac{c\alpha h}{R}\frac{d}{dt}\left(\frac{\Delta E}{E_s}\right)$$

By combining the two equations one gets a second order linear differential equation:

$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0$$

$$\Omega_s^2 = -\left(\frac{c}{R}\right)^2 \frac{\alpha h e\hat{V}\cos\phi_s}{2\pi E_s}$$

Since $\alpha > 0$, stable oscillations mean $\cos\phi_s < 0$, which correspond to the negative slope of the RF as mentioned already ($\pi/2 < \phi_s < \pi$)

Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

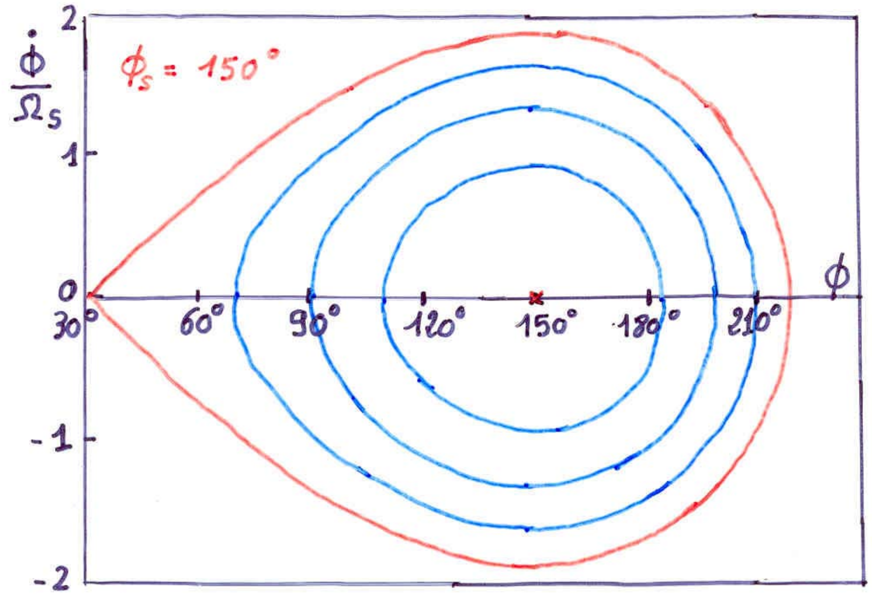
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{(\Delta\phi)^2}{2} = I$$

(the variable is $\Delta\phi$ and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches π-ϕ_s the force goes to zero and beyond it becomes non restoring. Hence π-ϕ_s is an extreme amplitude for a stable motion which in the phase space($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extremum when $\ddot{\phi} = 0$, hence corresponding to $\phi = \phi_s$.

Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{\max} = 2\Omega_s^2\left\{2 + \left(2\phi_s - \pi\right)\tan\phi_s\right\}$$

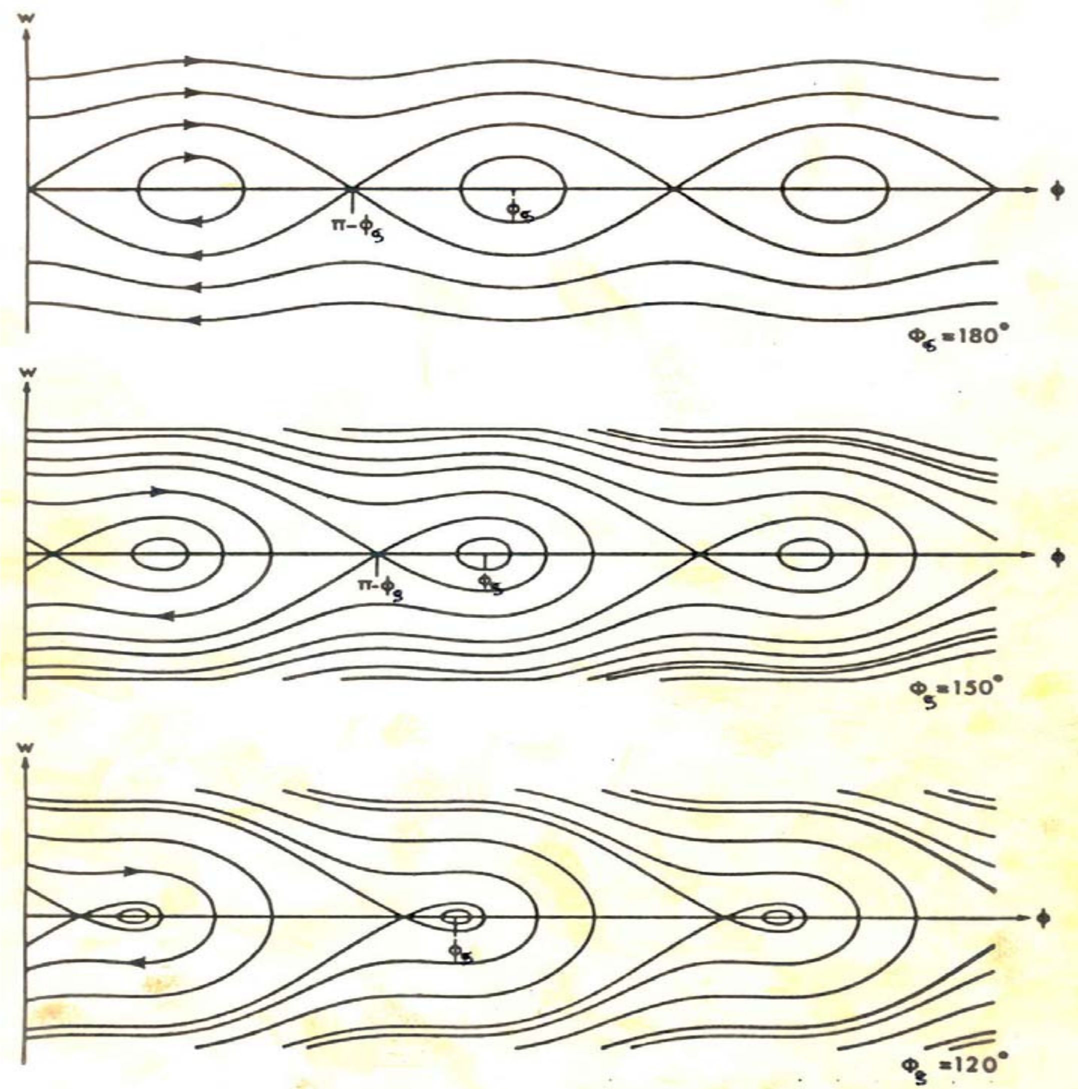
That translates into an acceptance in energy:

$$\left(\frac{\Delta E}{E_s}\right)_{\max} = \mp\left\{-\frac{e\hat{V}}{\pi\alpha h E_s}G(\phi_s)\right\}^{\frac{1}{2}}$$

$$G(\phi_s) = \left[2\cos\phi_s + \left(2\phi_s - \pi\right)\sin\phi_s\right]$$

This "RF acceptance" depends strongly on ϕ_s and plays an important role for the electron capture at injection, and the stored beam lifetime.

RF Acceptance versus Synchronous Phase



As the synchronous phase gets closer to 90° the area of stable motion (closed trajectories) gets smaller. These areas are often called "BUCKET".

The number of circulating buckets is equal to "h".

The phase extension is maximum close to 180° but the synchronous phase will be slightly different to compensate for radiation losses. The RF acceptance will increase with the RF voltage.

From Synchrotron to Linac

In the linac there is no bending magnets, hence there is no dispersion effects on the orbit and α=0.

Provided the cavities are periodically spaced to fulfill the synchronism condition, the longitudinal dynamics treatment remains valid and one ends up with a phase oscillation frequency Ω_s=0. In other words the distance in phase between particles is frozen while energies change which is a consequence of constant velocities c.

Hence in an ultra-relativistic electron linac it is important to inject a short bunch on the crest of the RF voltage such that the bunch length will stay short and the relative energy spread small.

However when the particle energy is too low (v«c), the factor $1/\gamma^2$ in η is no more negligible and energy and phase are coupled together again leading to a longitudinal oscillation with $\Omega_s \propto \gamma^{-3}$. Note that in the case of a linac, the term c/R which appears in the writing of Ω_s is directly related to the RF angular frequency. Moreover in a linac the equivallent h will be equal to 1.

Adiabatic Damping

Though there are many physical processes that can damp the longitudinal oscillation amplitudes, one is directly generated by the acceleration process itself. It will happen in the electron synchrotron when ramping the energy but not in the ultra-relativistic electron linac which does not show any oscillation.

As a matter of fact, when E_s varies with time, one needs to be more careful in combining the two first order energy-phase equations in one second order equation:

$$\frac{d}{dt}\left(E_s\dot{\phi}\right) = -\Omega_s^2 E_s \Delta\phi$$

$$E_s\ddot{\phi} + \dot{E}_s\dot{\phi} + \Omega_s^2 E_s \Delta\phi = 0$$

$$\ddot{\phi} + \frac{\dot{E}_s}{E_s}\dot{\phi} + \Omega_s^2\left(E_s\right)\Delta\phi = 0$$

The damping coefficient is proportional to the rate of energy variation and from the definition of Ω_s one has:

$$\frac{\dot{E}_s}{E_s} = -2\frac{\dot{\Omega}_s}{\Omega_s}$$