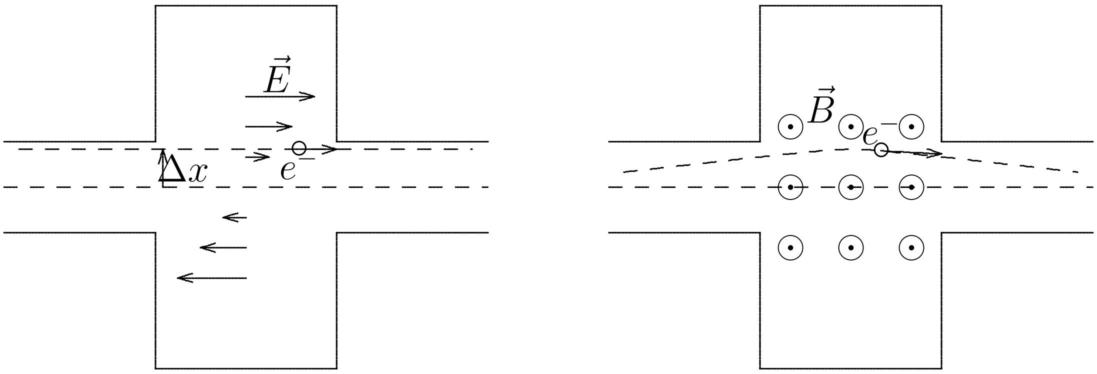
INSTABILITIES II
TRANSVERSE INSTABILITIES

TRANSVERSE IMPEDANCE



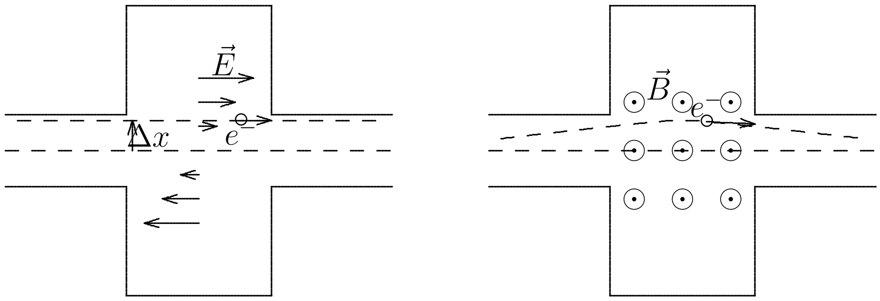
A transverse impedance is excited by the longitudinal bunch motion and produces a deflection field. It is illustrated by a cavity oscillating with ω in a mode (dipole mode) having a longitudinal field E_z with a constant transverse gradient $\partial E_z/\partial x$. E_z vanishes on axis and is only excited by a bunch with a transverse off-set giving a dipole moment $I_b \Delta x$. After 1/4 oscillation the E_z field is converted into B_y field which deflects the beam in the x-direction. Maxwell's equation $\dot{\vec{B}} = -\text{curl}\vec{E}$ in integral form $\int \dot{\vec{B}} d\vec{a} = -\oint \vec{E} d\vec{s}$ gives

$$E = E_z = \frac{\partial \hat{E}}{\partial x} x \cos(\omega t) \ \rightarrow \ B = B_y = \frac{1}{\omega}\frac{\partial \hat{E}}{\partial x} \sin(\omega t)$$

To describe a general deflecting field we introduce a transverse impedance, Z_T or $Z_\perp$ in analogy to the longitudinal one

$$Z_T(\omega) = j\frac{\int \left(\vec{E}(\omega) + [\vec{v} \times \vec{B}(\omega)]\right)_T ds}{I x(\omega)} = -\frac{\omega \int \left(\vec{E}(\omega) + [\vec{v} \times \vec{B}(\omega)]\right)_T ds}{I \dot{x}(\omega)}$$

using $e^{j\omega t}$. The first impedance definition above relates deflecting field to exciting dipole moment. If the two are in phase there is no energy transfer to the transverse motion, therefore the factor 'j' in front. However, if deflecting field and transverse velocity are in phase there is energy transfer as stated in the second definition.



In our cavity mode the dipole moment Ix also induces a longitudinal field giving a longitudinal impedance Z_L. An excitation at distance x_0 gives a gradient of the dE_z/dx related to Ix_0 by a factor k

$$\frac{dE_z}{dx} = kIx_0 \text{ and } E_z(x) = \frac{dE_z}{dx}x = kIx_0x \quad , \quad E_z(x_0) = kIx_0^2$$

The longitudinal impedance of this mode is

$$Z_L(x_0) = -\frac{\int E_z(x_0)dz}{I} = kx_0^2, \ell$$

ℓ is the cavity length. Maxwell's eq. $\int \dot{\vec{B}} d\vec{a} = -\oint \vec{E} d\vec{s}$ gives $\dot{B}_y x\ell = -x\ell dE_z/dx$ and transforms electric field gradient into magnetic field. With $I(t) = \hat{I}\mathrm{e}^{j\omega t}$ we get

$$\dot{B}_y = \hat{B}j\omega\mathrm{e}^{j\omega t} = -\frac{d\hat{E}_z}{dx}\mathrm{e}^{j\omega t} \quad , \quad B_y = j\frac{1}{\omega}\frac{dE_z}{dx} \text{ giving}$$

$$Z_T(\omega) = j\frac{\int \left(\vec{E}(\omega) + [\vec{v}\times\vec{B}(\omega)]\right)_T ds}{Ix(\omega)} = -j\frac{B_y c\ell}{Ix_0} = \frac{c}{\omega}k\ell = \frac{2c}{\omega}\frac{d^2 Z_L}{dx^2}.$$

Our transverse impedance is related to the second derivative of the longitudinal one. From this we get the symmetry relations

$$\begin{array}{rl} \text{longitudinal} & Z_r(-\omega) = Z_r(\omega)\ ,\ Z_i(-\omega) = -Z_i(\omega) \\ \text{transverse} & Z_{Tr}(-\omega) = -Z_{Tr}(\omega)\ ,\ Z_{Ti}(-\omega) = Z_{Ti}(\omega) \end{array}$$

For a ring radius R and vacuum chamber radius b the impedances, averaged over resonances of different modes, have a ratio

$$Z_T(\omega) \approx \frac{2R}{b^2}\frac{Z(\omega)}{(\omega/\omega_0)}$$

TRANSVERSE INSTABILITIES WITH Q'=0

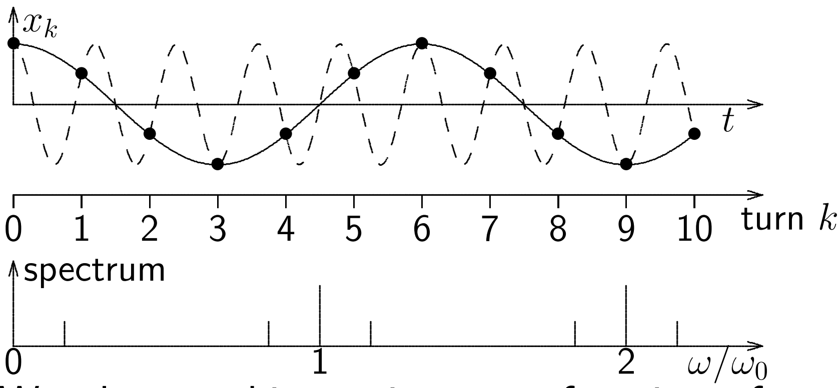
Transverse dynamics summary

The transverse focusing provided by the quadrupoles keeps the beam in the vicinity of the nominal orbit. A particle executes a betatron motion around this orbit. This motion has the form of an oscillation which is not harmonic but has a phase advance per unit length which varies around the ring. Often this is approximated by a smooth focusing given by

$$\ddot{x} + \omega_0^2 Q_x^2 x = 0$$

with ω_0 being the revolution frequency and Q_x the horizontal tune, i.e. the number of betatron oscillation executed per turn.

A stationary observer, or the impedance, sees the particle position x_k only at one location each turn k and has no information what the particle does in the rest of the ring

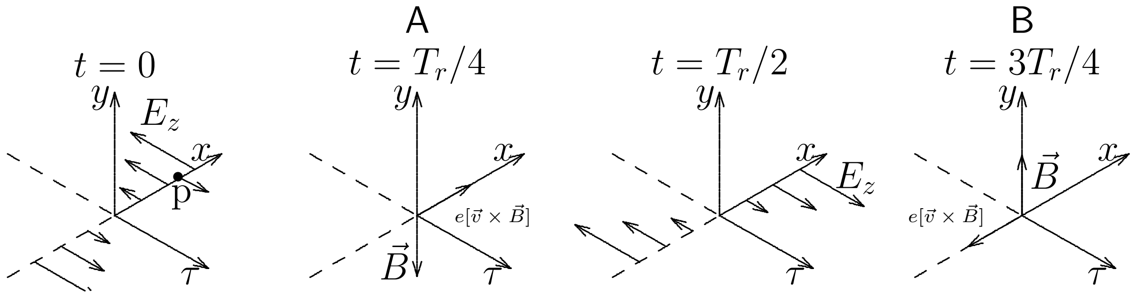


$$x_k = \hat{x}\cos(2\pi q k)$$

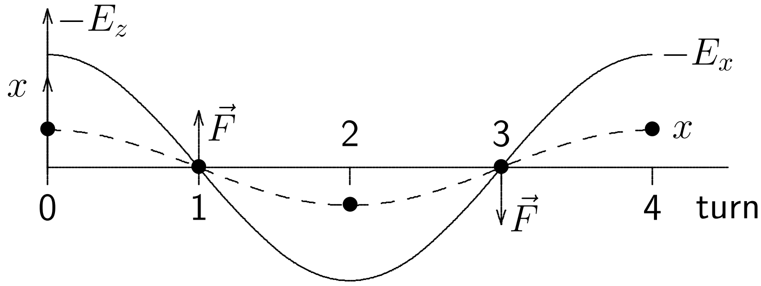
$$x'_k = -\frac{\hat{x}}{\beta_x}\sin(2\pi q k).$$

We observe this motion as a function of turn k. We can make a harmonic fit, i.e. a Fourier analysis. For a single bunch circulating in the machine we find at the revolution harmonic $p\omega_0$ an upper and lower sideband. The distance of the sideband is given by the tune $Q_x =$ integer $+ q$. The fractional part q is the only part which matters since the integer cannot be observed. For a very short bunch these sidebands will appear at very high frequencies, for longer bunches they will get smaller. A transverse impedance (or a position monitor) is sensitive to the dipole moment Ix of the current and does not see the revolution harmonics.

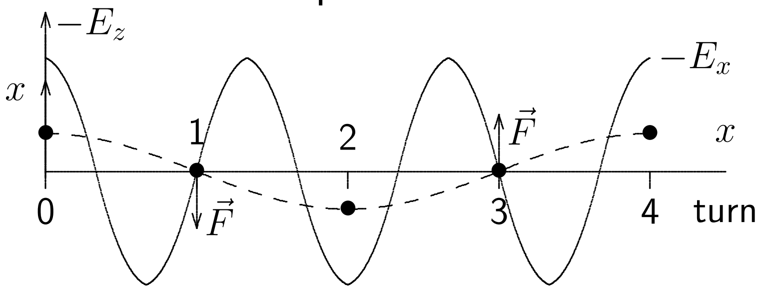
Transverse multi-traversal instability of a single bunch



A bunch p passes with a displacement x through the cavity and excites a fields $\vec{E}$ which converts after $T_r/4$ into a field $-\vec{B}$, then into $-\vec{E}$ and after into $\vec{B}$. The bunch is making a betatron oscillation with frequency $\omega_0 Q$, Q is the tune having a fractional part q. For a stationary observer the oscillating bunch has sidebands at $\omega_0(\text{integer} \pm q)$ (take $q = 1/4$).



A) With the cavity tuned to the upper sideband the bunch will traverse it in the next turn at the situation 'A', $t = T_r(k+1/4)$ with a transverse velocity in the $-x$ direction and receive by the magnetic field a force in the opposite direction which damps the oscillation.



B) With the cavity tuned to lower sideband the bunch traverses it next turn at situation 'B', $t = T_r(k'+3/4) = T_r(k'+1-1/4)$ with negative velocity and receives a force in the same direction. This increases the oscillation and leads to an instability.

The resistive impedance at the upper sideband damps, the one at the lower sideband excites the oscillation. If we have a more general impedance extending over several sidebands $\omega_0(p+q)$ and $\omega_0(p-q)$ we expect that the growth or damping rate of the oscillation is given by an expression of the form

$$\frac{1}{\tau_s} \propto \sum_p \left(I_{p+}^2 Z_{Tr}(\omega_{p+}) - I_{p-}^2 Z_{Tr}(\omega_{p-}) \right) \text{ with } \omega_{p\pm} = \omega_0 \left(p \pm q\right)$$

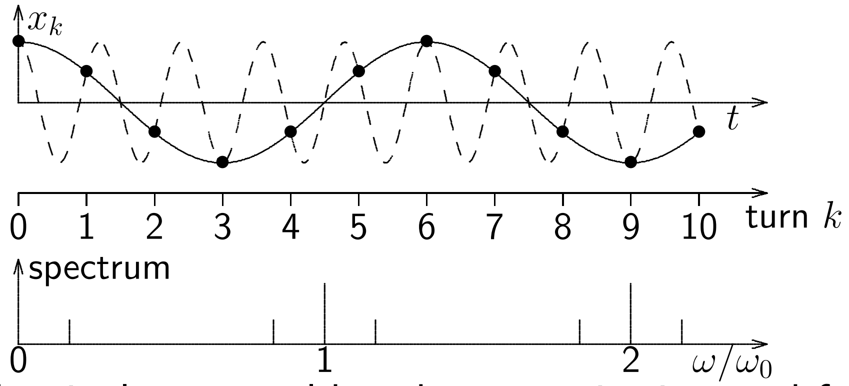
where $I_{p\pm}$ is the Fourier component of the beam current at the upper or lower sidebands. It appears here as the square I_p^2 since the instability is driven by the energy transfer from the longitudinal to the transverse motion.

We can estimate some properties of the proportionality factor missing in the above equation. The product $I_p^2 Z_T = P/y$ represents a power transfer per unit length. To get a growth rate we have to divide this by the energy of the bunch having N_b particles which can be related to the average current of the bunch $I_0 = eN_b\omega_0/2\pi$

$$\frac{1}{\tau} \approx \frac{P}{m_0 c^2 \gamma N_0} = \frac{e\omega_0 P}{2\pi m_0 c^2 \gamma I_0}$$

Calculation of the transverse instability for a single bunch

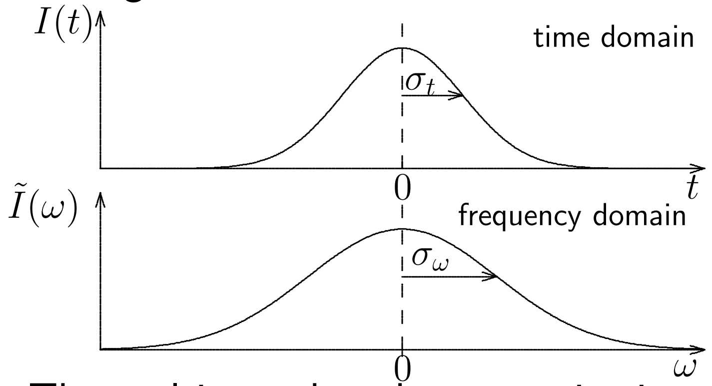
A transverse impedance $Z_T(\omega)$ in a symmetry point $\beta_x' = 0$ interacts with a bunch executing a betatron oscillation with tune $Q_x =$ integer$+q$. Its position and angle at impedance location as a function of turn k are



$$x_k = \hat{x}\cos(2\pi qk)$$

$$x_k' = -\frac{\hat{x}}{\beta_x}\sin(2\pi qk).$$

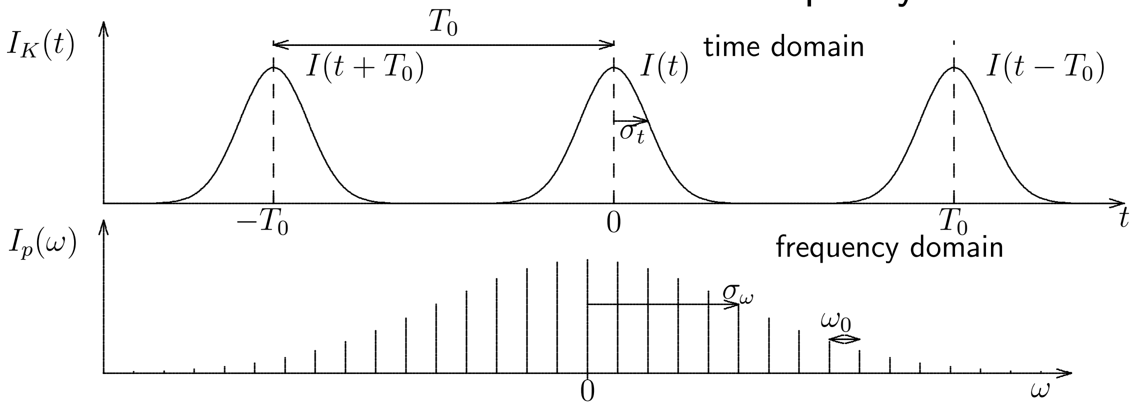
The single traversal bunch current in time and frequency domain is



$$I(t)$$

$$\tilde{I}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} I(t)\mathrm{e}^{-j\omega t}dt.$$

The multi turn bunch current in time and frequency domain is



$I_K(t) = \sum_{-\infty}^{\infty} I(t - kT_0)$ is periodic of period T_0 and Fourier series

$$I_K(t) = \sum_{p=-\infty}^{\infty} I_p \mathrm{e}^{jp\omega_0 t} \;\; , \;\; I_p = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} I(t)\mathrm{e}^{-jp\omega_0 t}dt = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(p\omega_0)$$

$$I_K(t) = \sum_{p=-\infty}^{\infty} I_p \mathrm{e}^{jp\omega_0 t} = I_0 + 2\sum_{p=1}^{\infty} I_p\cos(p\omega_0 t) \;\; , \;\; I_0 = \langle I(t)\rangle.$$

The dipole moment of oscillating bunch at turn k and as function of t is

$$D_k = x_k I_k \ , \ D_K(t) = \hat{x} \sum_{k=-\infty}^{\infty} \cos(2\pi q k) I(t - kT_0)$$

To express this in a series we get **Fourier transform** of $I_K(t)$

$$\begin{aligned}\tilde{I_K}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} I(t - kT_0) \mathrm{e}^{-j\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} \sum \mathrm{e}^{-k\omega T_0} \int_{-\infty}^{\infty} I(t - kT_0) \mathrm{e}^{-j\omega(t-kT_0)} dt = \tilde{I}(\omega) \sum_{k=-\infty}^{\infty} \mathrm{e}^{-jk\omega T_0}\end{aligned}$$

The Fourier transform of the dipole moment is

$$\tilde{D_x}(\omega) = \hat{x} \tilde{I}(\omega) \sum_{-\infty}^{\infty} \cos(2\pi q k) \mathrm{e}^{-jk\omega T_0} = \frac{\hat{x}\tilde{I}(\omega)}{2} \sum_{-\infty}^{\infty} \left[\mathrm{e}^{-jk(\omega T_0 + 2\pi q)} + \mathrm{e}^{-jk(\omega T_0 - 2\pi q)} \right]$$

The sums are ∞ if the exponent is of form $2\pi p$ and vanish otherwise

$$\sum_{k=-\infty}^{\infty} \mathrm{e}^{-jkx} = 2\pi \sum_{p=-\infty}^{\infty} \delta(x - 2\pi p) \ \text{ and } \ \delta(ax) = \frac{1}{a}\delta(x) \text{ gives}$$

$$\tilde{D}_K(\omega) = \hat{x} \frac{\omega_0 \tilde{I}(\omega)}{2} \sum_{-\infty}^{\infty} \left[\delta(\omega - (p-q)\omega_0) + \delta(\omega - (p+q)\omega_0) \right]$$

The inverse Fourier transform gives the oscillating dipole in time domain

$$D_K(t) = \frac{\omega_0 \hat{x}}{2\sqrt{2\pi}} \sum_{-\infty}^{\infty} \left[\tilde{I}((p+q)\omega_0) \mathrm{e}^{j((p+q)\omega_0 t)} + \tilde{I}((p-q)\omega_0) \mathrm{e}^{j((p-q)\omega_0 t)} \right]$$

Using $\ \omega_{p+} = (p+q)\omega_0, \ \omega_{p-} = (p-q)\omega_0, \ I_{p\pm} = \frac{\omega_0}{\sqrt{2\pi}} \tilde{I}(\omega_{p\pm})$ gives

$$D_K k(t) = \frac{\hat{x}}{2} \sum_{p=-\infty}^{\infty} \left[I_{p+} \mathrm{e}^{j(t\omega_{p+})} + I_{p-} \mathrm{e}^{j(t\omega_{p-})} \right].$$

Combining terms $p > 0$ from first, $p < 0$ from second part and vice versa, using $\tilde{I}(\omega) = \tilde{I}(-\omega)$

$$D_K(t) = \hat{x} \sum_{\omega > 0}^{\infty} [I_{p+} \cos(\omega_{p+} t) + I_{p-} \cos(\omega_{p-} t)].$$

A charge e going through the impedance element at turn k feels a transverse force changing its momentum $\Delta p_{ke} = F_T \Delta t \approx F_T \Delta s/c$

$$\Delta p_{ke} = \frac{e}{c}\int \left[\vec{E}(\omega) + [\vec{v}\times\vec{B}(\omega)]\right]_T ds = \frac{-jeD_k(t)Z_T}{c}.$$

We get the momentum change of the **whole** bunch by a convolution between its charge distribution given by the **single traversal current** $dq/dt = I(t)$ with the momentum change $\Delta p_x(t+kT_0)$ in turn k

$$\begin{aligned}\Delta p_k &= -j\frac{1}{c}\int_{-\infty}^{\infty} I(t) D_k(t+kT_0) Z_T dt \\ &= -j\frac{\hat{x}}{2c}\sum_{p=-\infty}^{\infty}\int_{-\infty}^{\infty} I(t)\left[I_{p+}Z_T(\omega_{p+})\mathrm{e}^{j\omega_{p+}(t+kT_0)t} + I_{p-}Z_T(\omega_{p-})\mathrm{e}^{j\omega_{p-}(t+kT_0)}\right] dt\end{aligned}$$

This contains integrals of the form

$$\int_{-\infty}^{\infty} I(t)\mathrm{e}^{-j(t+kT_0)\omega_{p+}} dt = \sqrt{2\pi}\mathrm{e}^{-jT_0 k\omega_{p+}}\tilde{I}(\omega_{p+}) = \frac{2\pi}{\omega_0}\mathrm{e}^{-j2\pi qk} I_{p+}$$

giving

$$\Delta p_k = -j\frac{\hat{c}T_0}{2c}\sum_{-\infty}^{\infty}\left[I_{p+}^2 Z_T(\omega_{p+})\mathrm{e}^{-j2\pi qk} + I_{p-}^2 Z_T(\omega_{p-})\mathrm{e}^{j2\pi qk}\right].$$

Combining terms $p > 0$ from first, $p < 0$ from second part and vice versa, using relations $Z_{Tr}(\omega) = Z_{Tr}(-\omega)$, $Z_{Ti}(\omega) = Z_{Ti}(-\omega)$ gives

$$\begin{aligned}\Delta p_k = -\frac{T_0}{c}\sum_{\omega>0} &\left[\left(I_{p+}^2 Z_{Tr}(\omega_{p+}) - I_{p-}^2 Z_{Tr}(\omega_{p-})\right)\hat{x}\sin(2\pi qk)\right. \\ &- \left.\left(I_{p+}^2 Z_{Ti}(\omega_{p+}) + I_{p-}^2 Z_{Ti}(\omega_{p-})\right)\hat{x}\cos(2\pi qk)\right].\end{aligned}$$

using the form of the betatron oscillation we started from

$$x_k = \hat{x}\cos(2\pi qk) \;\; , \;\; x'_k = -\frac{\hat{x}}{\beta_x}\sin(2\pi qk) \;\; , \;\; \dot{x}_k = cx'_k = -\frac{\hat{x}c}{\beta_x}\sin(2\pi qk)$$

$$\begin{aligned}\Delta p_k = \frac{T_0}{c^2}\sum_{\omega>0} &\left[\left(I_{p+}^2 Z_{Tr}(\omega_{p+}) - I_{p-}^2 Z_{Tr}(\omega_{p-})\right)\beta_x\dot{x}_k\right. \\ &+ \left.\left(I_{p+}^2 Z_{Ti}(\omega_{p+}) + I_{p-}^2 Z_{Ti}(\omega_{p-})\right)cx_k\right].\end{aligned}$$

Transverse velocity and angle change with the transverse momentum

$$\Delta x'_k = \frac{\Delta \dot{x}_k}{c} = \frac{\Delta p_k}{N_0 m_0 \gamma c} = \frac{e \Delta p_k}{m_0 \gamma c I_0 T_0}$$

$$\Delta \dot{x}_k = \frac{e}{m_0 c^2 \gamma I_0} \sum_{\omega>0} \Big[\Big(I_{p+}^2 Z_{Tr}(\omega_{p+}) - I_{p-}^2 Z_{Tr}(\omega_{p-}) \Big) \beta_x \dot{x}_k \\ + \Big(I_{p+}^2 Z_{Ti}(\omega_{p+}) + I_{p-}^2 Z_{Ti}(\omega_{p-}) \Big) c x_k \Big].$$

The velocity change has a component proportional to velocity and resistive impedance and one proportional to displacement and reactive impedance. The first can lead to exponential growth or damping the second to a change of betatron frequency.

The first part alone with a smooth approximation gives an acceleration $\ddot{x} = \Delta \dot{x} \omega_0 / 2\pi$ which we ad to the one due to focusing by beam optics

$\ddot{x} + 2a\dot{x} + Q_x^2 \omega_0^2 = 0,$ solution: $x = x_0 \mathrm{e}^{-at} \cos(Q_x \omega_0 t + \phi)$ if $a \ll Q_x \omega_0$

$$a = \frac{1}{\tau} = \frac{e \omega_0 \beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \Big(I_{p+}^2 Z_{Tr}(\omega_{p+}) - I_{p-}^2 Z_{Tr}(\omega_{p-}) \Big).$$

using $\omega_- = (p-q)\omega_0 = -(-p+q)\omega_0 = -(|p|+q)\omega_0$ for $p < 0$ and $Z_{Tr}(\omega) = -Z_{Tr}(-\omega)$ gives a sum with positive and negative frequencies

$$a = \frac{1}{\tau} = \frac{e \omega_0 \beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{p=-\infty}^{\infty} I_{p+}^2 Z_{Tr}(\omega_{p+}).$$

The reactive impedance alone gives angle change $\Delta x'_k = \Delta \dot{x}_k / c$ proportional to x_k. This represents a focusing element of strength

$$\frac{1}{f} = -\frac{\Delta x'_k}{x_k} = -\frac{e}{m_0 c^2 \gamma I_0} \sum_{\omega_\pm>0} \Big(I_{p+}^2 Z_{Ti}(\omega_{p+}) + I_{p-}^2 Z_{Ti}(\omega_{p-}) \Big) x_k$$

which results in a tune change $\Delta Q_x = \beta_x / (4\pi f)$

$$\Delta \omega_\beta = \omega_0 \Delta Q_x \;=\; -\frac{e \omega_0 \beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega_\pm>0} \Big(I_{p+}^2 Z_{Ti}(\omega_{p+}) + I_{p-}^2 Z_{Ti}(\omega_{p-}) \Big) \\ = \; -\frac{e \omega_0 \beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{p=-\infty}^{\infty} I_{p+}^2 Z_{Ti}(\omega_{p+}).$$

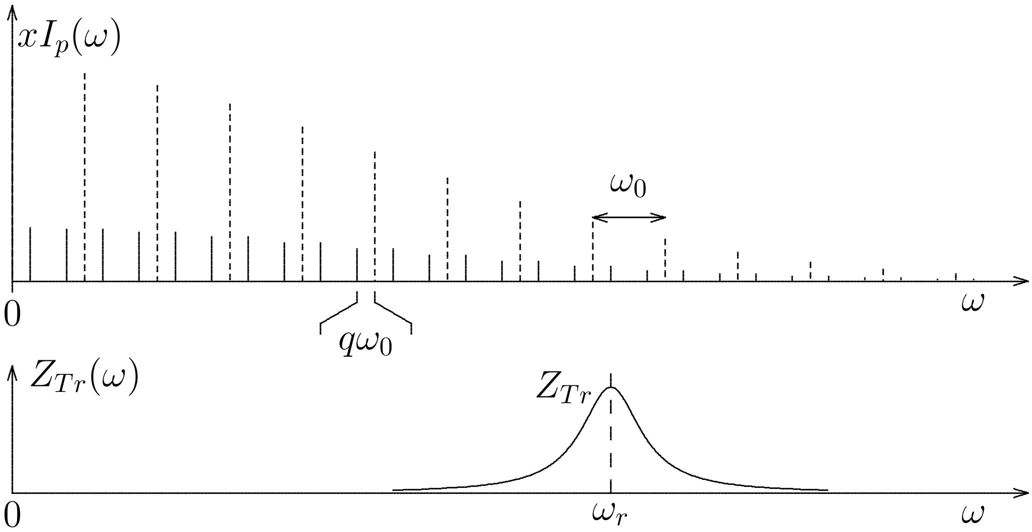
An inductive impedance $Z_{Ti} > 0$ is defocusing giving negative tune shift.

Instability due to the resistive impedance

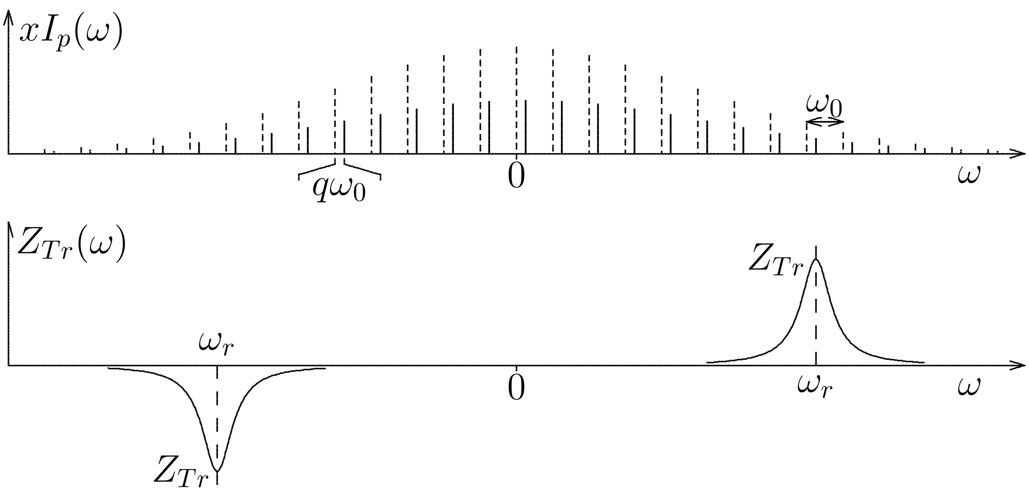
$$x = x_0 \mathrm{e}^{-at} \cos((Q_x\omega_0 + \Delta\omega_\beta)t + \phi) \;\; \text{if} \;\; a \ll Q_x\omega_0$$

$$a = \frac{1}{\tau} = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(I_{p+}^2 Z_{Tr}(\omega_{p+}) - I_{p-}^2 Z_{Tr}(\omega_{p-}) \right).$$

For a distributed impedance we replace local beta function by average $\beta_x \approx \langle \beta_x \rangle \approx R/Q$ with $R =$ average ring radius. Single strong impedances, RF-cavities, are best located at a small beta function.



$$a = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{p=-\infty}^{\infty} I_{p+}^2 Z_{Tr}(\omega_{p+}).$$

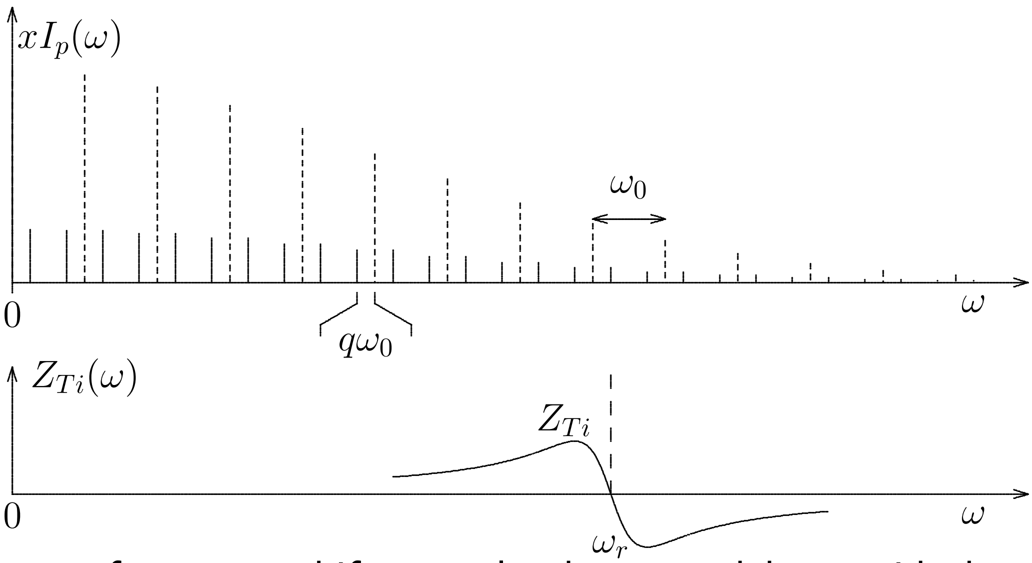


To drive this instability we need a narrow band impedance with a memory lasting at least for one turn.

Frequency shift due to the reactive impedance

$$x = x_0 \mathrm{e}^{-at} \cos((Q_x\omega_0 + \Delta\omega_\beta)t + \phi) \;\; \text{if} \;\; a \ll Q_x\omega_0$$

$$\begin{aligned}\Delta\omega_\beta = \omega_0\Delta Q_x &= -\frac{e\omega_0\beta_x}{4\pi m_0 c^2\gamma I_0}\sum_{\omega_\pm>0}\left(I_{p+}^2 Z_{Ti}(\omega_{p+}) + I_{p-}^2 Z_{Ti}(\omega_{p-})\right) \\ &= -\frac{e\omega_0\beta_x}{4\pi m_0 c^2\gamma I_0}\sum_{p=-\infty}^{\infty} I_{p+}^2 Z_{Ti}(\omega_{p+}).\end{aligned}$$

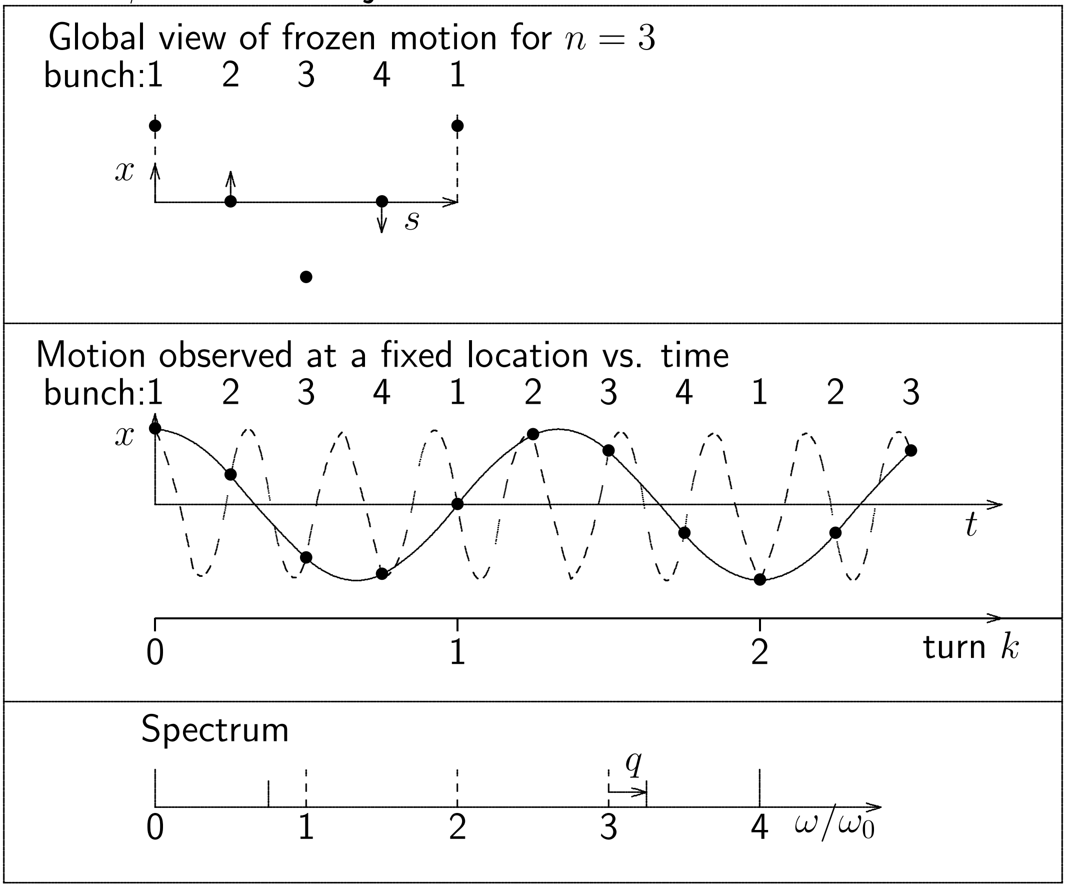


The betatron frequency shift can also be caused by a wide band impedance since there is no cancelation between the upper and lower side band. A measurement of this shift is often used to obtain a convolution between the impedance and the bunch spectrum. Doing this for different bunch lengths some information on the impedance itself can be extracted

This frequency shift acts only on the coherent (center of mass) motion of the bunch and has little influence on the incoherent motion of the individual particles and there frequencies. The reactive impedance can cause a separation between the coherent betatron frequency in the incoherent frequency distribution which can lead to a loss of Landau damping.

Transverse instability of many bunches

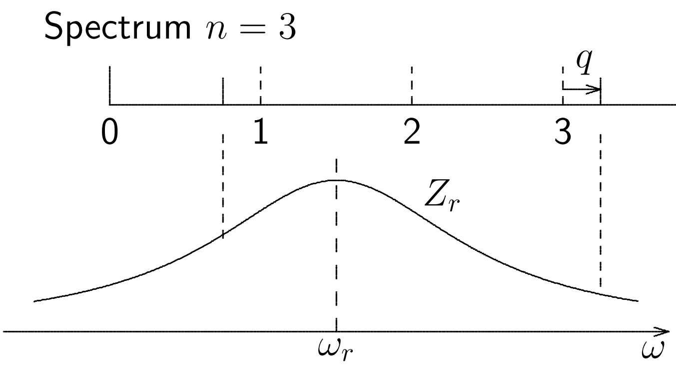
M bunches can oscillate in M different modes $n = M\Delta\phi/(2\pi)$ with the phase $\Delta\phi$ between adjacent bunches

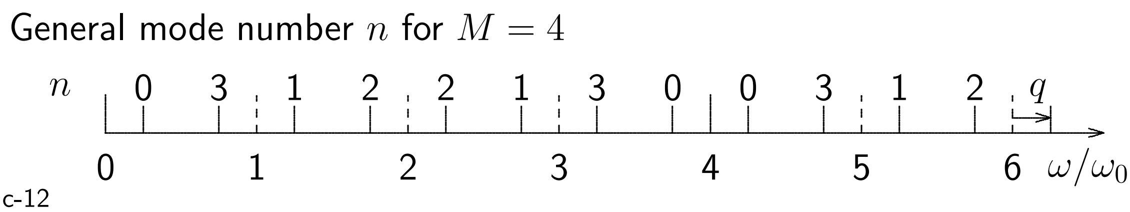


M bunches oscillating with mode n have frequencies and growth rate

$$\omega_{p\pm} = \omega_0 \left(pM \pm (n + q)\right)$$

$$a = \frac{1}{\tau} = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(I_{p+}^2 Z_{Tr}(\omega_{p+}) - I_{p-}^2 Z_{Tr}(\omega_{p-}) \right).$$



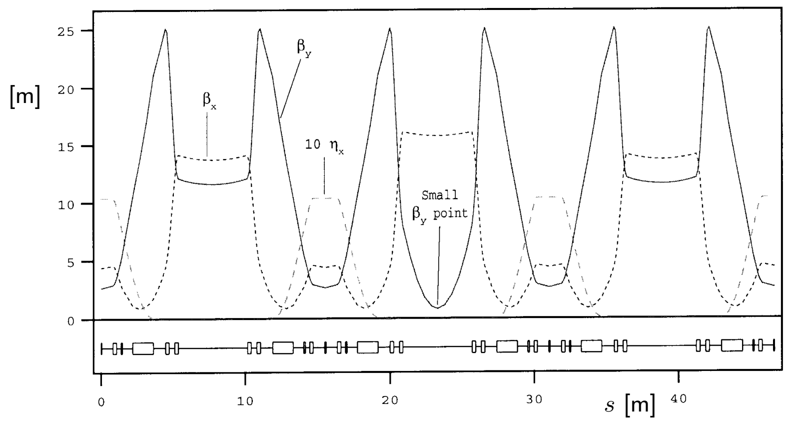


Dependance of the transverse instability on β_y at LNLS

The transverse instability growth rate is $\propto \beta_y$ since a deflection at low β gives lower oscillation energy than one at a large β

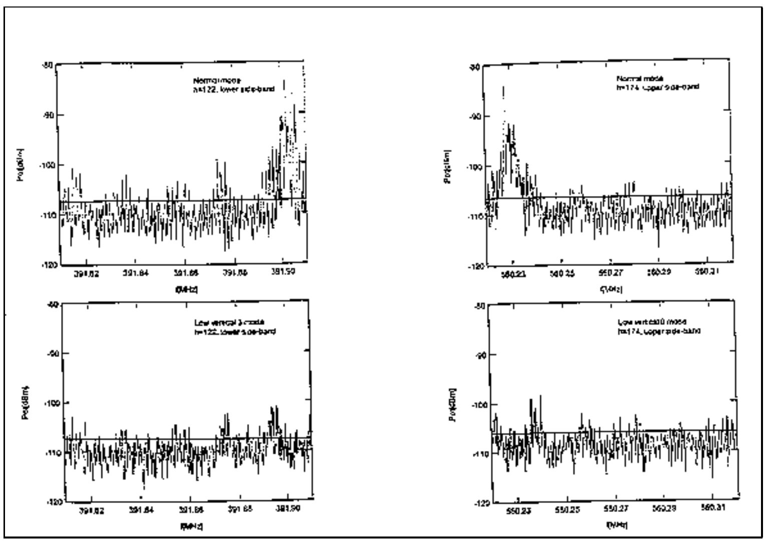
$$a = \frac{1}{\tau} = \frac{e\omega_0\beta_y}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(I_{p+}^2 Z_{Tr}(\omega_{p+}) - I_{p-}^2 Z_{Tr}(\omega_{p-}) \right).$$

This was measured at LNLS (Laboratòrio Nacional de Luz Sínchrotron, Brasil) where β_y can be reduced in the straight section of the RF-cavity. An exponential instability growth is only seen if a large intensity is injected or a feed-back system is used during accumulation and than turned off. For slow accumulation the instability saturates unstable betatron lines at a finite amplitude. Reducing β_y makes this line disappear indicating that the offending impedance is in this section, i.e. in the RF-cavity.



normal beta

low beta

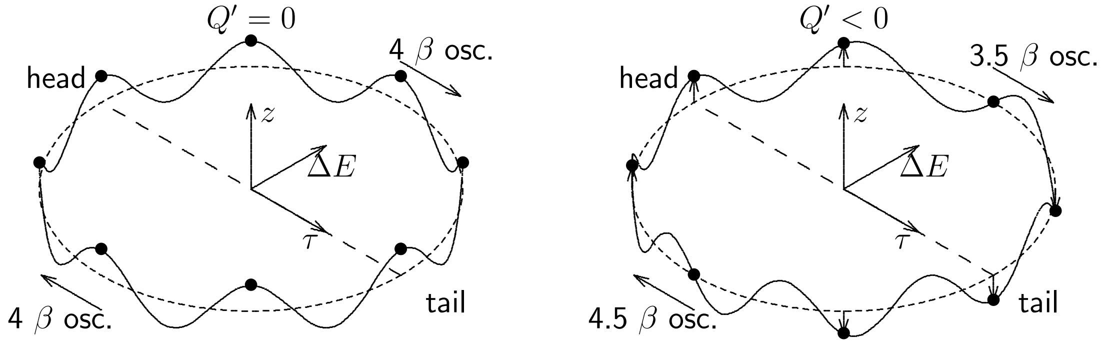


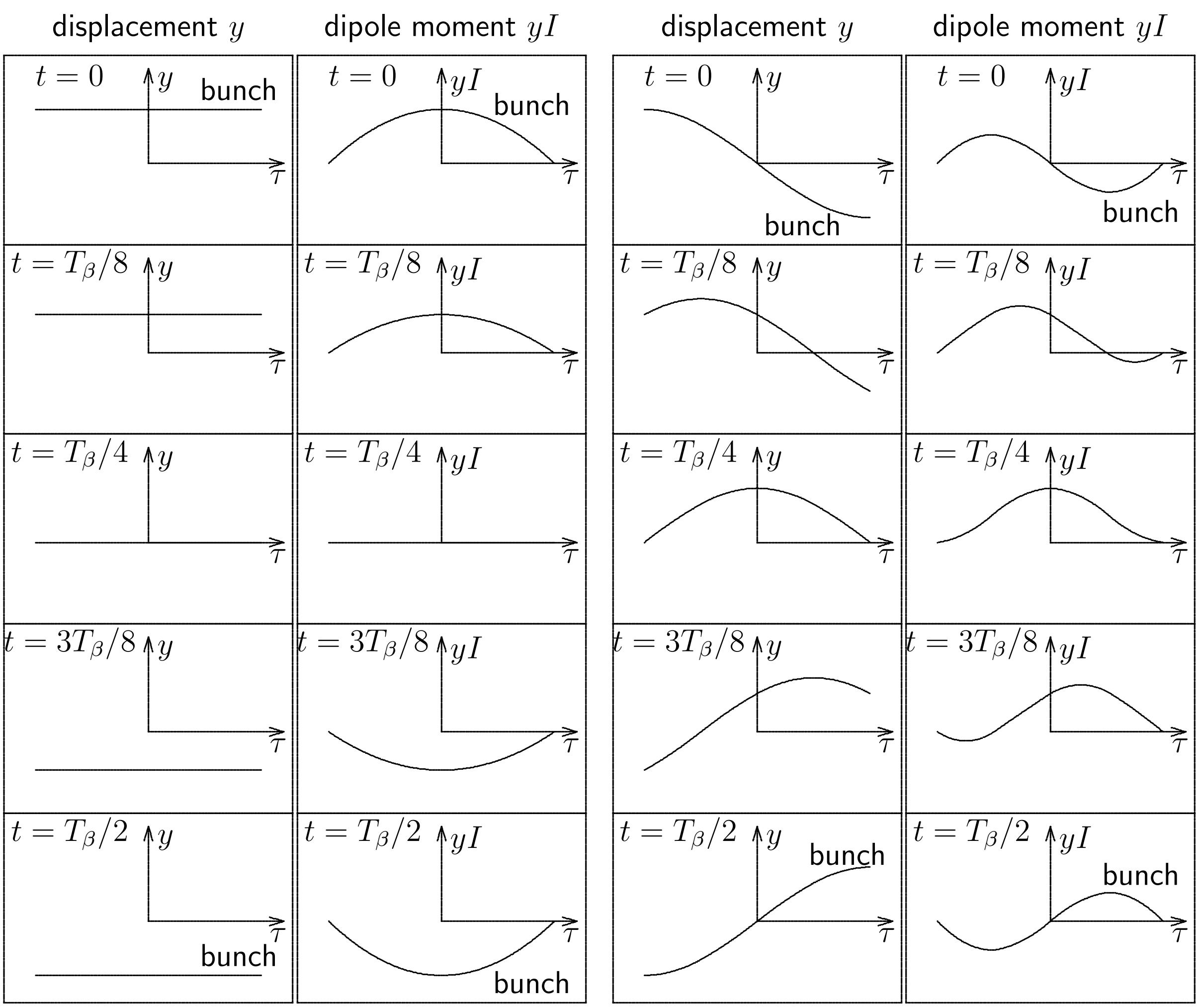
$n + q$ $\qquad$ $n - q$

HEAD-TAIL Instability

Head-tail mode oscillations

The synchrotron motion in energy and time deviation ΔE and τ affects transverse motion via chromaticity $Q' = dQ/(dp/p)$. For $\gamma > \gamma_T$ it has an excess energy moving from head to tail and an energy lack moving from tail to head. For $Q' > 0$, this gives a phase advance in the first and a phase lag in the second step and vice versa for $Q' < 0$ or $\gamma < \gamma_T$.

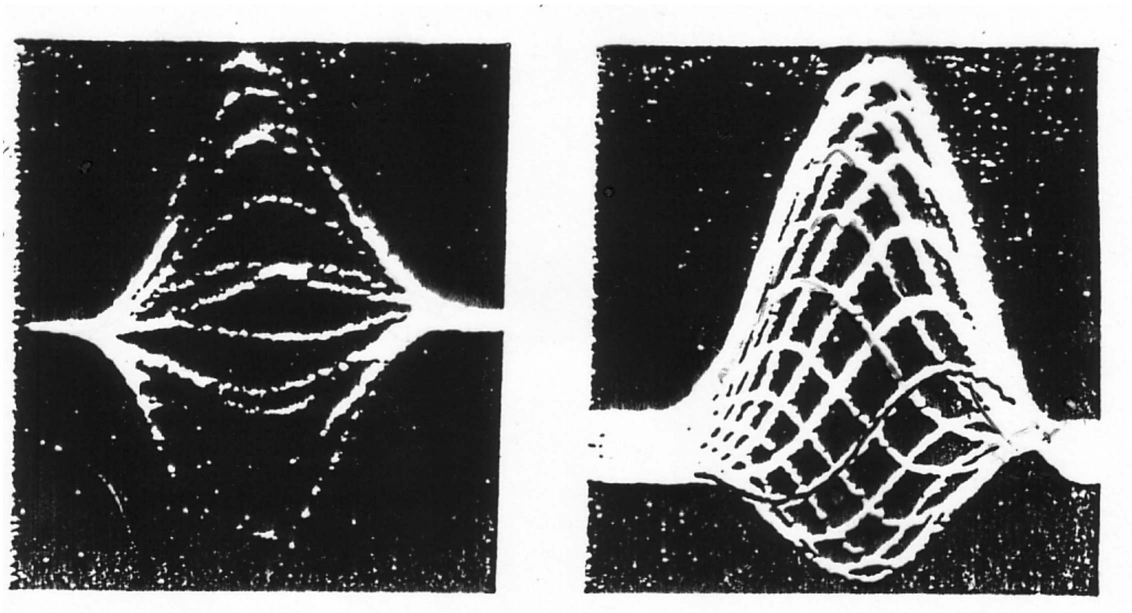




Betatron motion observed in steps of its period $T_\beta = T_0/q$

Observation of the head-tail mode in the CERN Booster

The head-tail mode oscillation of relatively long bunches can be observed directly with a fast position monitor. The figure shows such a measurement of the head-tail mode at vanishing and a finite chromaticity taken by J. Gareyte and F. Sacherer in the CERN Booster. I shows several traces each corresponding to a turn of the oscillating bunch passing through the transverse position monitor which gives a signal proportional the instantaneous dipole moment $x(t)I(t)$.



$Q' = 0$ $\qquad\qquad$ $Q' > 0$

Head-tail mode $m = 0$

Head-tail instability

A broad band impedance is excited by oscillating particles A at the bunch head which in turn excite particles B at the tail with a phase shifted by $\Delta\phi$ compared to the head. Half a synchrotron oscillation later particles B are at the head and while particles A are at the tail oscillating with phase $-\Delta\phi$ compared to B (assuming $Q' = 0$). The excitation by the head has the wrong phase to keep oscillation growing unless $Q' \neq 0$ producing a phase shift during a motion from head to tail or vice versa. The wake field excited by the head of the bunch will affect the tail later. The tail oscillates therefore with a phase lag compared to the tail. To keep the oscillation growing the head particle must undergo a relative phase delay while moving to the tail and the tail particle a relative phase advance moving to the head. We expect a possible instability if $Q' < 0$ for $\gamma > \gamma_T$ or if $Q' > 0$ for $\gamma < \gamma_T$.

The 'wiggle' of the head-tail motion is seen by a stationary observer (impedance) as an oscillation with the chromatic frequency ω_ξ which has to be considered in calculating the head-tail instability.

$$\Delta p/p = \Delta\hat{p}/p \sin(\omega_s t) \ \ , \ \ \tau = -\hat{\tau}\cos(\omega_s t) \quad \text{with} \quad \hat{\tau} = \frac{\omega_s}{\eta_c}\frac{\Delta\hat{p}}{p}$$

$\omega_s = Q_s\omega_0$ is the synchrotron frequency and $\eta_c = \alpha_c - 1/\gamma^2$ with $\alpha_c =$ momentum compaction. The relative betatron phase shift of a particle executing part of a synchrotron oscillation is

$$\begin{aligned}
\Delta\phi_\beta &= \omega_0 \int_{t_1}^{t_2} \Delta Q dt = \omega_0 Q' \frac{\Delta\hat{p}}{p}\int_{t_1}^{t_2} \sin(\omega_s t) dt \\
&= -\omega_0 Q' \frac{\Delta\hat{p}}{p}\left(\cos(\omega_s t_2) - \cos(\omega_s t_2)\right) = \frac{\omega_0 Q'}{\eta_c}(\tau_2 - \tau_1)
\end{aligned}$$

This gives for the chromatic frequency

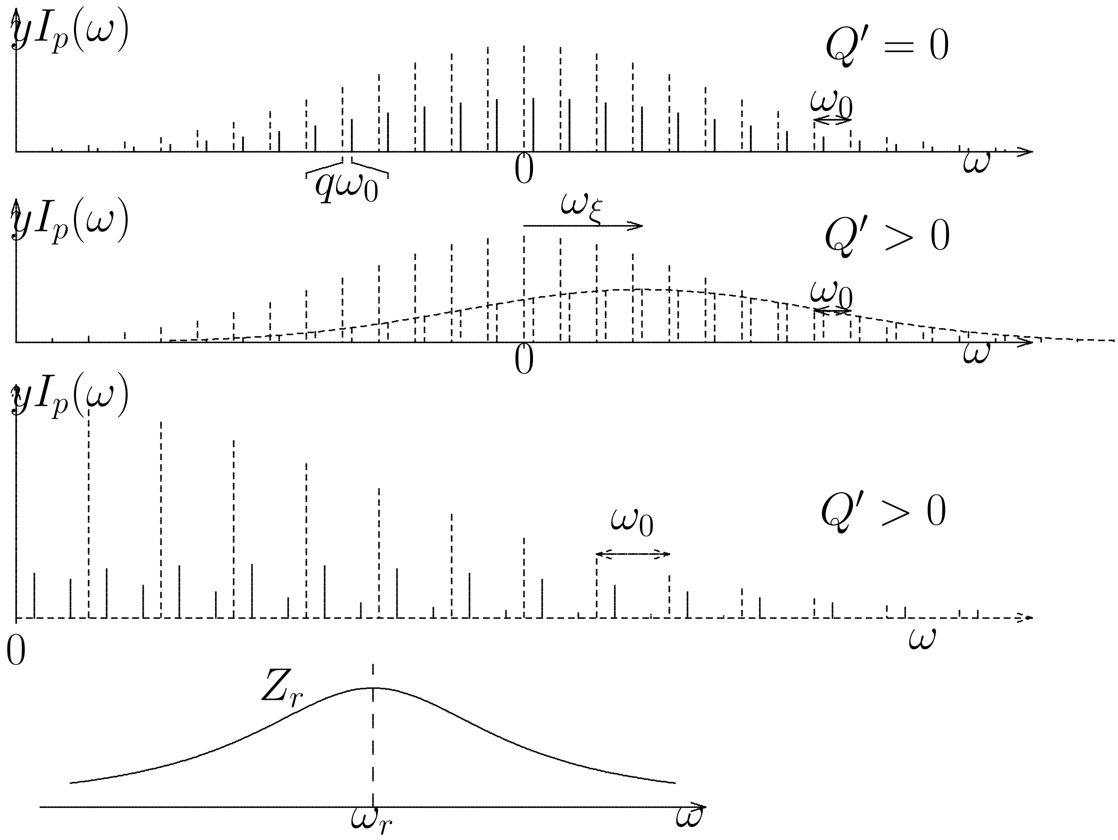
$$\omega_\xi = \frac{\Delta\phi_\beta}{\Delta\tau} = \frac{\omega_0 Q'}{\eta_c}.$$

The 'wiggle' of the head-tail mode shifts the envelope of the sidebands by the chromatic frequency $\omega_\xi = Q'\omega_0/\eta_c$ and we have current components

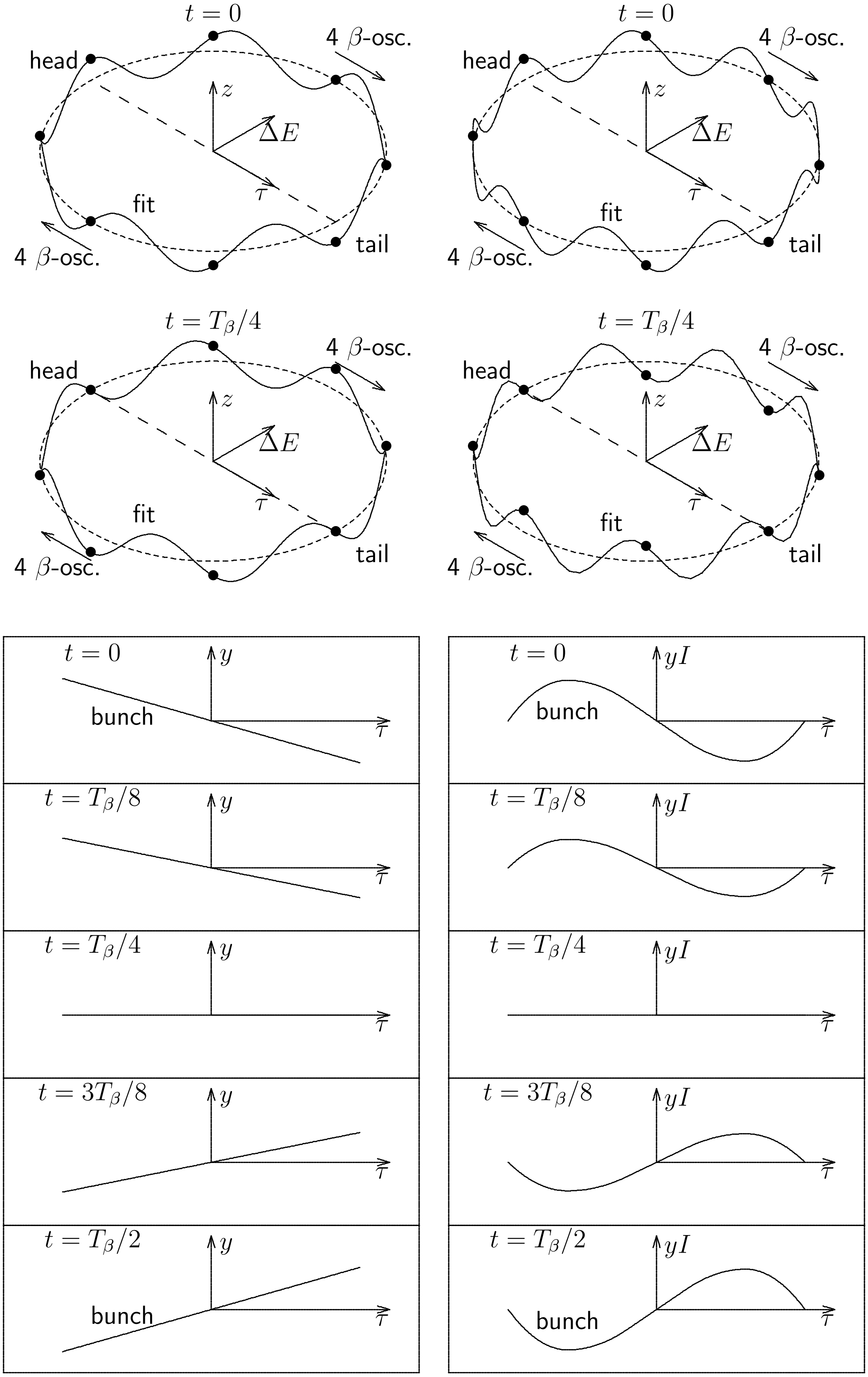
$$I_{p\xi\pm} = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(\omega_{p\pm} + \omega_\xi),\ \omega_{p\pm} = \omega_0\,(pM \pm (n+q))$$

which can be very different adjacent sidebands. Even a broad band impedance can lead to an instability with growth (or damping) rate

$$a = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0}\sum_{\omega>0}\left(I^2_{p\xi+}Z_{Tr}(\omega_{p+}) - I^2_{p\xi-}Z_{Tr}(\omega_{p-})\right).$$



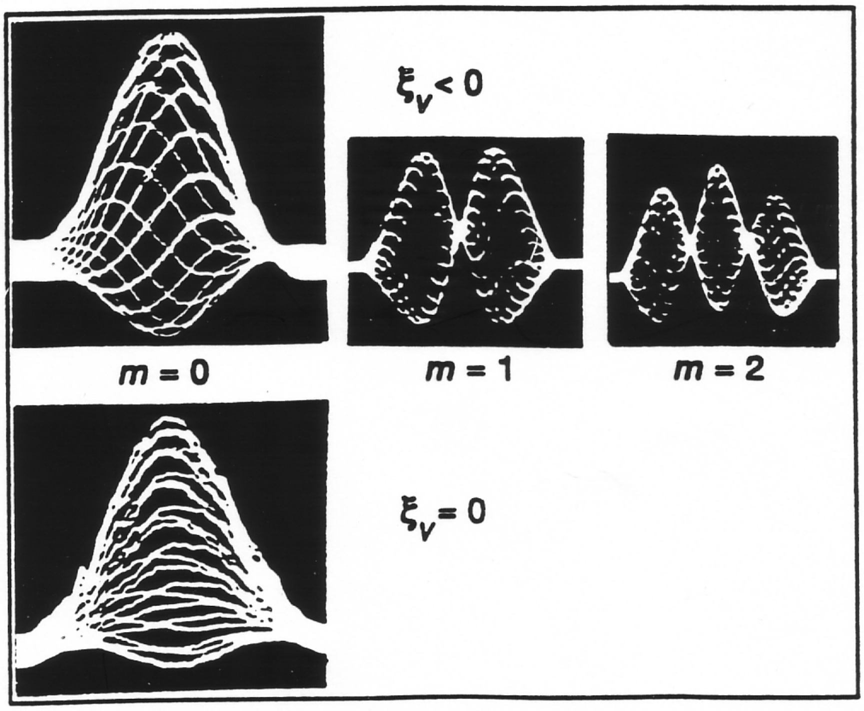
Higher head-tail mode; m=1, Q'=0



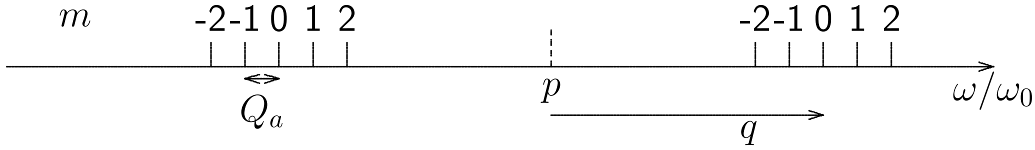
Head-tail mode $m = \pm 1$ seen in steps of $T_\beta = T_0/q$, it has node in center and comes in pair of frequencies $\omega_\beta = \omega_0(p \pm q \pm mQ_s)$.

Observation of higher head-tail modes

The head-tail mode oscillation of relatively long bunches can be observed directly with a fast position monitor. The figure shows measurements at the CERN Booster ring by J. Gareyte and F. Sacherer at chromaticities $\xi = Q'/Q$. Each trace corresponds to the bunch passage through the transverse position monitor giving a signal proportional the instantaneous dipole moment $x(t)I(t)$. This captures the oscillation at different phases. The higher modes show $m > 0$ show a number m of nodes.



Higher head-tail modes $m = 0,\ 1,\ $ and 2



Detailed spectrum of higher head-tail modes $\omega = \omega_0(p \pm q \pm Q_s)$