



THE CERN ACCELERATOR SCHOOL

Maxwell's Equations for Magnets

Part I: "Ideal" Multipole Fields

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Maxwell's Equations for Magnets

In these lectures, we shall discuss solutions to Maxwell's equations for magnetostatic fields:

1. in two dimensions (multipole fields);
2. in three dimensions (fringe fields, insertion devices...)

In the first lecture, we will see how to construct multipole fields in two dimensions, using electric currents and magnetic materials, considering idealised situations.

In the second lecture, we will consider three dimensional fields, and some of the effects of non-ideal geometries and materials.

Unfortunately, we have no time to discuss pulsed magnets, septa...

Maxwell's Equations for Magnets

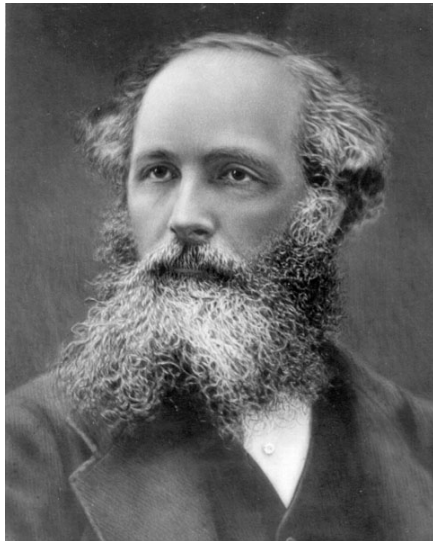
I shall assume some familiarity with the following topics:

- vector calculus in Cartesian and polar coordinate systems;
- Stokes' and Gauss' theorems;
- Maxwell's equations and their physical significance;
- types of magnets commonly used in accelerators.

The fundamental physics and mathematics is presented in many textbooks. I shall (try) to follow the notation used in:

A. Chao and M. Tigner, "Handbook of Accelerator Physics and Engineering," World Scientific (1999).

Maxwell's equations



James Clerk Maxwell
1831 – 1879

Maxwell's equations are:

$$\begin{aligned}\nabla \cdot \vec{D} &= \rho & \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} & \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t}\end{aligned}$$

where $\vec{D} = \epsilon \vec{E}$ is the electric displacement; $\vec{B} = \mu \vec{H}$ is the magnetic flux density; ρ is the electric charge density; and \vec{J} is the current density.

Maxwell's equations

In these lectures, I shall consider only magnetostatic fields.

Maxwell's equations for the magnetic field become:

$$\nabla \cdot \vec{B} = 0, \quad (1)$$

$$\nabla \times \vec{H} = \vec{J}. \quad (2)$$

In this first lecture, we shall show that multipole fields provide solutions to these equations in two dimensions, i.e. where the fields and currents are independent of one coordinate (z). We shall also deduce the current distributions and material property and geometries that can generate fields with specified multipole components.

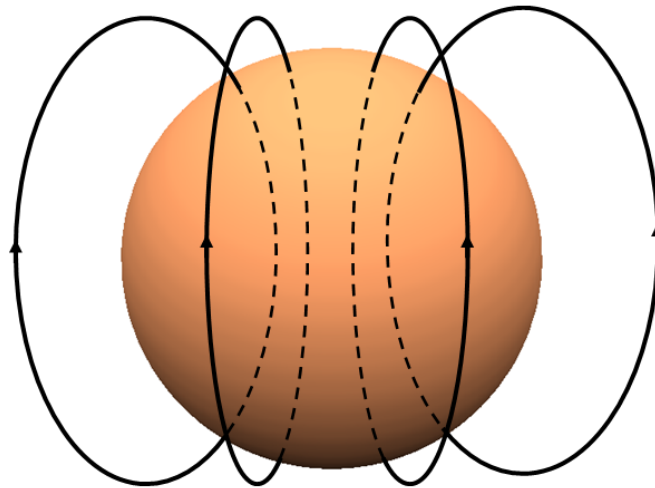
Physical interpretation of $\nabla \cdot \vec{B} = 0$

Gauss' theorem tells us that for *any* smooth vector field \vec{B} :

$$\int_V \nabla \cdot \vec{B} dV = \oint_S \vec{B} \cdot d\vec{S}, \quad (3)$$

where the closed surface S bounds the region V .

Applied to Maxwell's equation $\nabla \cdot \vec{B} = 0$, Gauss' theorem tells us that the total flux entering a bounded region equals the total flux leaving the same region.



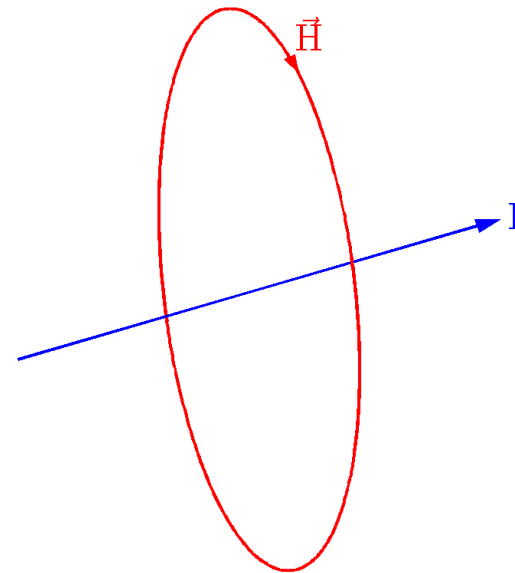
Stokes' theorem tells us that for *any* smooth vector field \vec{H} :

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint_C \vec{H} \cdot d\vec{\ell}, \quad (4)$$

where the closed loop C bounds the surface S .

Applied to Maxwell's equation $\nabla \times \vec{H} = \vec{J}$, Stokes' theorem tells us that the magnetic field \vec{H} integrated around a closed loop equals the total current passing through that loop:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \vec{J} \cdot d\vec{S} = I. \quad (5)$$



Maxwell's equations are *linear*:

$$\nabla \cdot (\vec{B}_1 + \vec{B}_2) = \nabla \cdot \vec{B}_1 + \nabla \cdot \vec{B}_2, \quad (6)$$

and:

$$\nabla \times (\vec{H}_1 + \vec{H}_2) = \nabla \times \vec{H}_1 + \nabla \times \vec{H}_2. \quad (7)$$

This means that if two fields \vec{B}_1 and \vec{B}_2 satisfy Maxwell's equations, so does their sum $\vec{B}_1 + \vec{B}_2$.

As a result, we can apply the *principle of superposition* to construct complicated magnetic fields just by adding together a set of simpler fields.

Let us first consider fields that satisfy Maxwell's equations in free space, e.g. the interior of an accelerator vacuum chamber. Here, we have $\vec{J} = 0$, and $\vec{B} = \mu_0 \vec{H}$; hence, Maxwell's equations (1) and (2) become:

$$\nabla \cdot \vec{B} = 0, \quad \text{and} \quad \nabla \times \vec{B} = 0. \quad (8)$$

Consider the field given by $B_z = \text{constant}$, and:

$$B_y + iB_x = C_n (x + iy)^{n-1}, \quad (9)$$

where n is a positive integer, and C_n is a complex number.

Note that the field components B_x , B_y and B_z are all *real*; we are only using complex numbers for convenience.

Now consider the differential operator:

$$\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}. \quad (10)$$

Applying this operator to the left hand side of (9) gives:

$$\begin{aligned} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (B_y + iB_x) &= \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) + i \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \right), \\ &= \left[\nabla \times \vec{B} \right]_z + i \nabla \cdot \vec{B}. \end{aligned} \quad (11)$$

In the final step, we have used the fact that B_z is constant.

Also using this fact, and the fact that B_x and B_y are independent of z , we see that the x and y components of $\nabla \times \vec{B}$ vanish.

Applying the operator (10) to the right hand side of (9) gives:

$$\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (x + iy)^{n-1} = (n-1)(x + iy)^{n-2} + i^2(n-1)(x + iy)^{n-2} = 0. \quad (12)$$

Multipole fields

Hence, applying the operator (10) to both sides of equation (9), we find that:

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = 0. \quad (13)$$

Therefore, the field (9) satisfies Maxwell's equations for a magnetostatic system in free space.

Of course, this analysis simply tells us that the field (9):

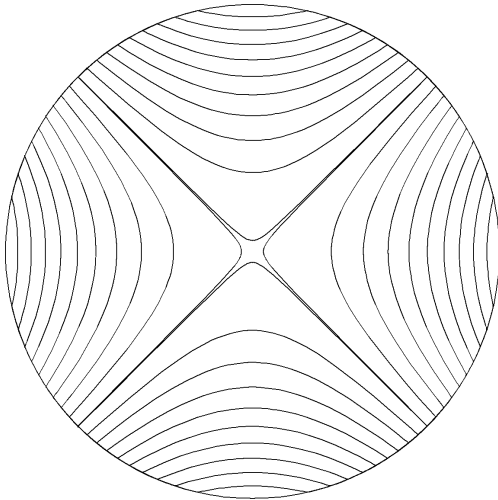
$$B_y + iB_x = C_n (x + iy)^{n-1}$$

is a possible solution to Maxwell's equations in the situation we have described: it does not tell us how to generate such a field.

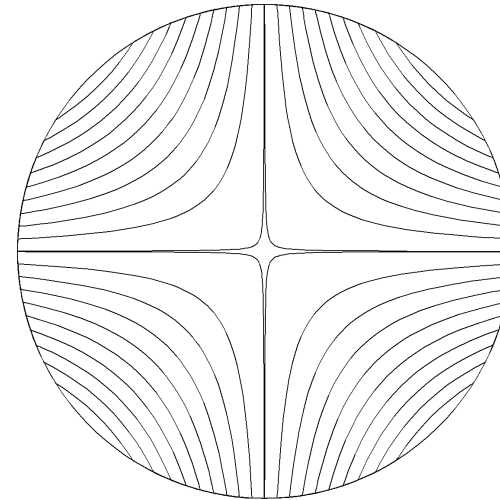
Fields given by (9) are called *multipole fields*. Note that, since Maxwell's equations are linear, we can superpose any number of multipole fields, and obtain a valid solution to Maxwell's equations.

Multipole fields

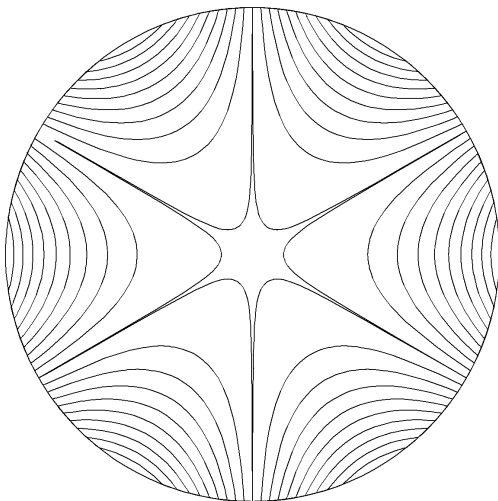
$C_2 = \text{real}$, normal quadrupole



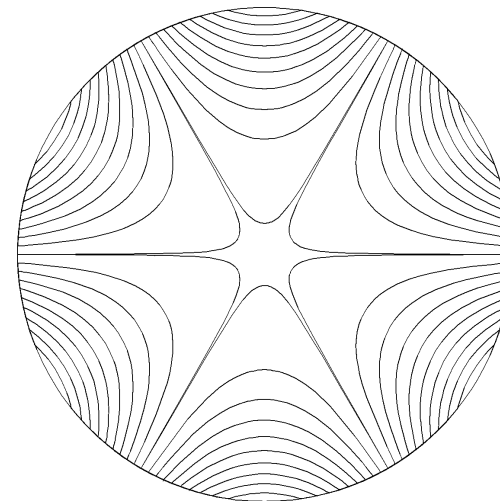
$C_2 = \text{imaginary}$, skew quadrupole



$C_3 = \text{real}$, normal sextupole



$C_3 = \text{imaginary}$, skew sextupole



For $C_n = 0$ for all n , we have:

$$B_x = B_y = 0, \quad B_z = \text{constant.} \quad (14)$$

This is a solenoid field, and is not generally regarded as a multipole field.

In the conventional notation (see Chao and Tigner), we rewrite the field (9) as:

$$B_y + iB_x = B_{\text{ref}} \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{R_{\text{ref}}} \right)^{n-1}. \quad (15)$$

The b_n are the “normal multipole coefficients”, and the a_n are the “skew multipole coefficients”. B_{ref} and R_{ref} are a reference field strength and a reference radius, whose values may be chosen arbitrarily; however their values will affect the values of the multipole coefficients for a given field.

Multipole fields

The interpretation of the multipole coefficients is probably best understood by considering the field behaviour in the plane $y = 0$:

$$B_y = B_{\text{ref}} \sum_{n=1}^{\infty} b_n \left(\frac{x}{R_{\text{ref}}} \right)^{n-1}, \quad \text{and} \quad B_x = B_{\text{ref}} \sum_{n=1}^{\infty} a_n \left(\frac{x}{R_{\text{ref}}} \right)^{n-1}. \quad (16)$$

A single multipole component with $n = 1$ is a dipole field: $B_y = b_1 B_{\text{ref}}$ is constant, and $B_x = a_1 B_{\text{ref}}$ is also constant.

A single multipole component with $n = 2$ is a quadrupole field:

$$B_y = b_2 B_{\text{ref}} \frac{x}{R_{\text{ref}}}, \quad \text{and} \quad B_x = a_2 B_{\text{ref}} \frac{x}{R_{\text{ref}}}. \quad (17)$$

Both B_y and B_x vary linearly with x .

For $n = 3$ (sextupole), the field components vary as x^2 , etc.

To see how to generate a multipole field, we start with the magnetic field around a thin wire carrying a current I_0 .

Generally, the magnetic field in the presence of a current density \vec{J} is given by Maxwell's equation (2):

$$\nabla \times \vec{H} = \vec{J}.$$

Consider a thin straight wire of infinite length, oriented along the z axis. Let us integrate Maxwell's equation (2) over a circular disc of radius r centered on the wire, and normal to the wire:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \int_S \vec{J} \cdot d\vec{S} = I_0, \quad (18)$$

where we have used the fact that the integral of the current density over the cross section of the wire equals the total current flowing in the wire.

Generating multipole fields from a current distribution

Now we apply Stokes' theorem, which tells us that for *any* smooth vector field F :

$$\int_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{\ell}, \quad (19)$$

where C is the closed curve bounding the surface S .

Applied to equation (18), Stokes' theorem gives us:

$$\oint_C \vec{H} \cdot d\vec{\ell} = I_0, \quad (20)$$

By symmetry, the magnetic field must be the same magnitude at equal distances from the wire. We also know, from Gauss' theorem applied to $\nabla \cdot \vec{B} = 0$, that there can be no radial component to the magnetic field.

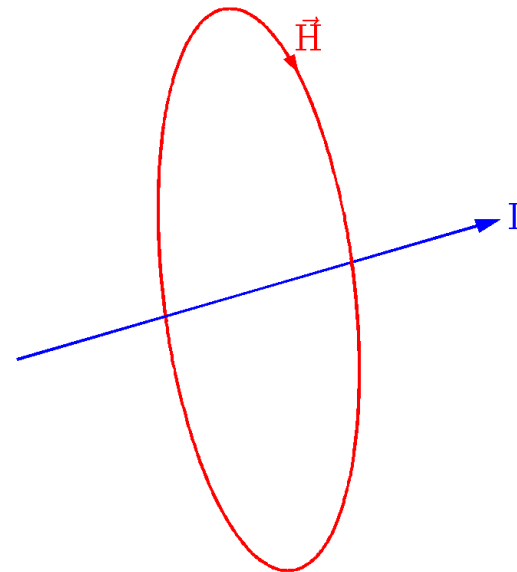
Generating multipole fields from a current distribution

Hence, the magnetic field at any point is tangential to a circle centered on the wire and passing through that point. We also find, by performing the integral in (20), that the magnitude of the magnetic field at distance r from the wire is given by:

$$\vec{H} = \frac{I_0}{2\pi r}. \quad (21)$$

If there are no magnetic materials present, $\mu = \mu_0$, so:

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I_0}{2\pi r}. \quad (22)$$



Generating multipole fields from a current distribution

Now, let us work out the field at a point $\vec{r} = (x, y, z)$ from a current parallel to the z axis, but displaced from it. The line of current is defined by $x = x_0, y = y_0$.

The magnitude of the field is given, from (22) by:

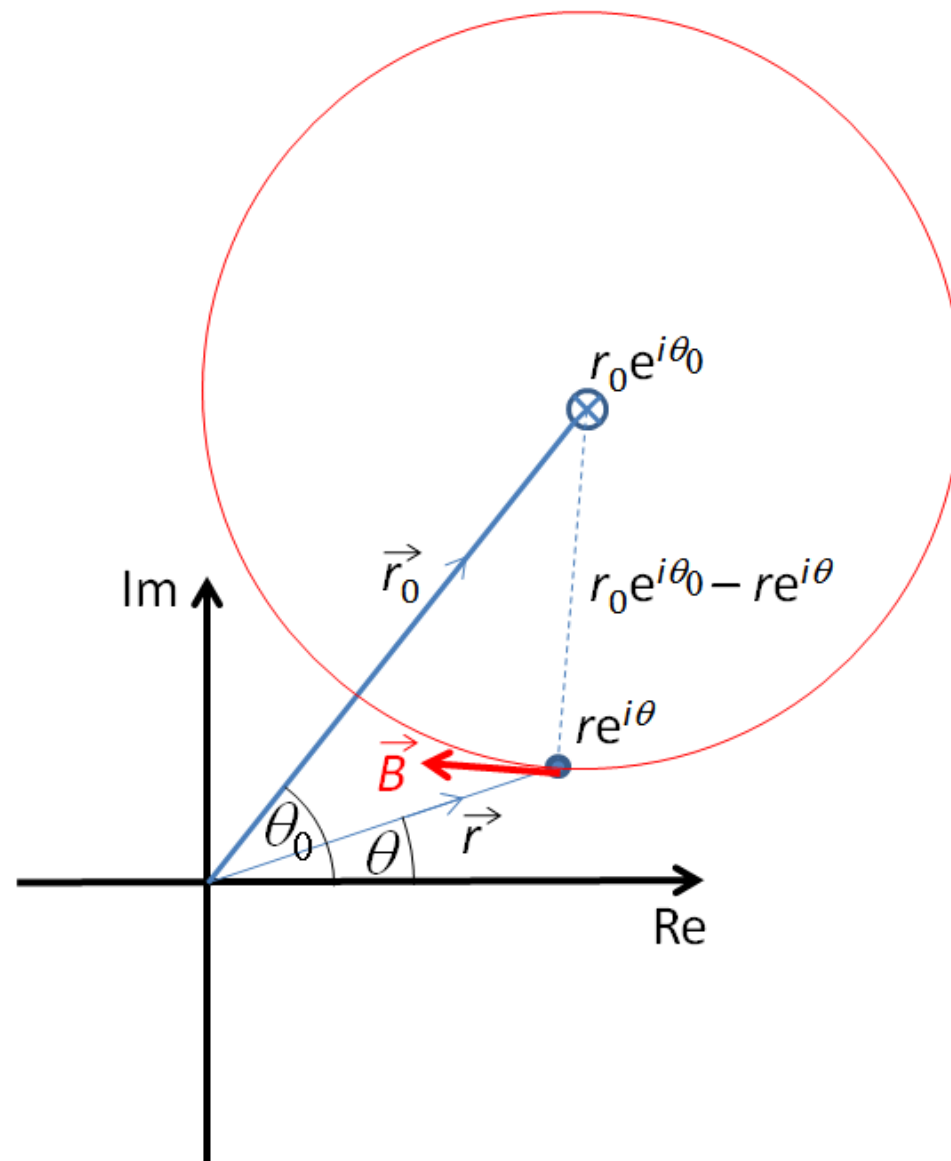
$$B = \frac{\mu_0 I_0}{2\pi |\vec{r} - \vec{r}_0|}, \quad (23)$$

where the vector \vec{r}_0 has components $\vec{r}_0 = (x_0, y_0, z)$.

Since the field at \vec{r} is perpendicular to $\vec{r} - \vec{r}_0$, the field vector is given by:

$$\vec{B} = \frac{\mu_0 I_0}{2\pi} \frac{(y - y_0, -x + x_0, 0)}{|\vec{r} - \vec{r}_0|^2}. \quad (24)$$

Generating multipole fields from a current distribution



Generating multipole fields from a current distribution

It is convenient to express the field (24) in complex notation. Writing:

$$x + iy = re^{i\theta}, \quad \text{and} \quad x_0 + iy_0 = r_0e^{i\theta_0}, \quad (25)$$

we find that:

$$B_y + iB_x = \frac{\mu_0 I_0}{2\pi} \frac{(r_0 e^{-i\theta_0} - r e^{-i\theta})}{|r_0 e^{i\theta_0} - r e^{i\theta}|^2}. \quad (26)$$

Using the fact that for any complex number ζ , we have $|\zeta|^2 = \zeta\zeta^*$:

$$\begin{aligned} B_y + iB_x &= \frac{\mu_0 I_0}{2\pi} \frac{1}{r_0 e^{i\theta_0} - r e^{i\theta}} \\ &= \frac{\mu_0 I_0}{2\pi r_0} \frac{e^{-i\theta_0}}{\left(1 - \frac{r}{r_0} e^{i(\theta - \theta_0)}\right)}. \end{aligned} \quad (27)$$

Generating multipole fields from a current distribution

Using the Taylor series expansion:

$$(1 - \zeta)^{-1} = \sum_{n=0}^{\infty} \zeta^n, \quad (28)$$

(valid for $|\zeta| < 1$) we can express the magnetic field (27) as:

$$B_y + iB_x = \frac{\mu_0 I_0}{2\pi r_0} e^{-i\theta_0} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} e^{i(n-1)(\theta-\theta_0)}, \quad (29)$$

which is valid for $r < r_0$.

Generating multipole fields from a current distribution

The advantage of writing the field in the form (29) is that by comparing with equation (15) we immediately see that the field is a sum over an infinite number of multipoles, with coefficients given by:

$$\frac{B_{\text{ref}}}{R_{\text{ref}}^{n-1}} (b_n + ia_n) = \frac{\mu_0 I_0}{2\pi r_0} \frac{e^{-in\theta_0}}{r_0^{n-1}}. \quad (30)$$

If we choose:

$$B_{\text{ref}} = \frac{\mu_0 I_0}{2\pi r_0}, \quad \text{and} \quad R_{\text{ref}} = r_0, \quad (31)$$

we see that:

$$b_n + ia_n = e^{-in\theta_0}. \quad (32)$$

Generating multipole fields from a current distribution

Now, let us consider the total field generated by a set of wires distributed around a cylinder of radius r_0 , such that the current flowing in a region at angle θ_0 and subtending angle $d\theta_0$ at the origin is:

$$I_0 = I_m \cos m(\theta_0 - \theta_m) d\theta_0, \quad (33)$$

where m is an integer.

The total field is found by integrating over all θ_0 . From (29):

$$\begin{aligned} B_y + iB_x &= \frac{\mu_0 I_m}{2\pi r_0} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} e^{i(n-1)\theta} \int_0^{2\pi} e^{-in\theta_0} \cos m(\theta_0 - \theta_m) d\theta_0 \\ &= \frac{\mu_0 I_m}{2\pi r_0} \left(\frac{r}{r_0}\right)^{m-1} e^{i(m-1)\theta} \pi e^{-im\theta_m}. \end{aligned} \quad (34)$$

We see that the cosine current distribution (33) generates a pure $2m$ -pole field within the cylinder on which the current flows.

Choosing the reference field and radius (31) as we did above:

$$B_{\text{ref}} = \frac{\mu_0 I_m}{2\pi r_0}, \quad \text{and} \quad R_{\text{ref}} = r_0,$$

we find that the multipole coefficients for the field generated by the cosine current distribution (33) are:

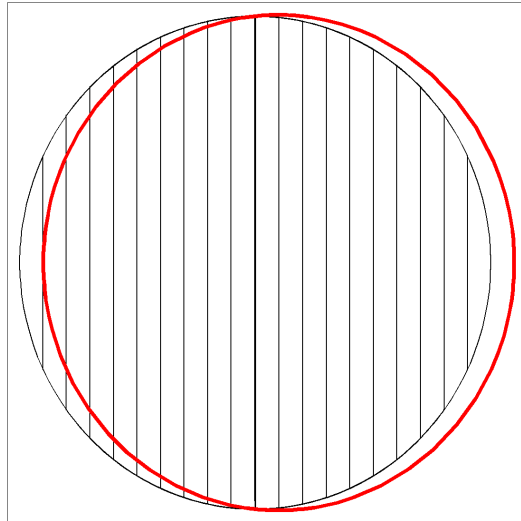
$$b_m + ia_m = \pi e^{-im\theta_m}. \quad (35)$$

For $\theta_m = 0$ or $\theta_m = \pi$, we have a normal $2m$ -pole field.

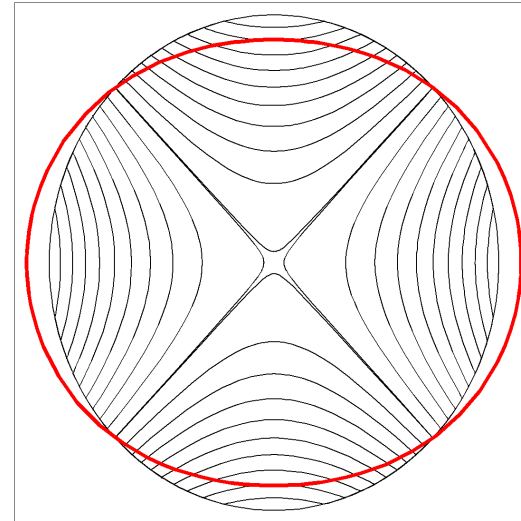
For $\theta_m = \pm\pi/2$, we have a skew $2m$ -pole field.

Generating multipole fields from a current distribution

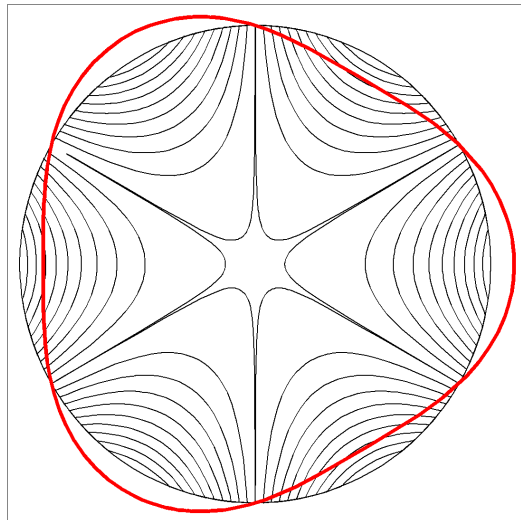
Dipole



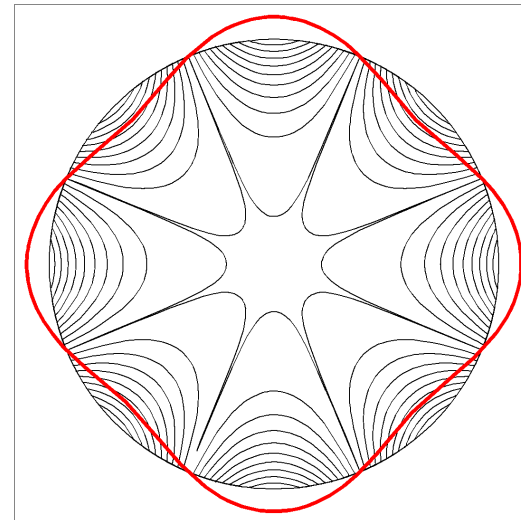
Quadrupole



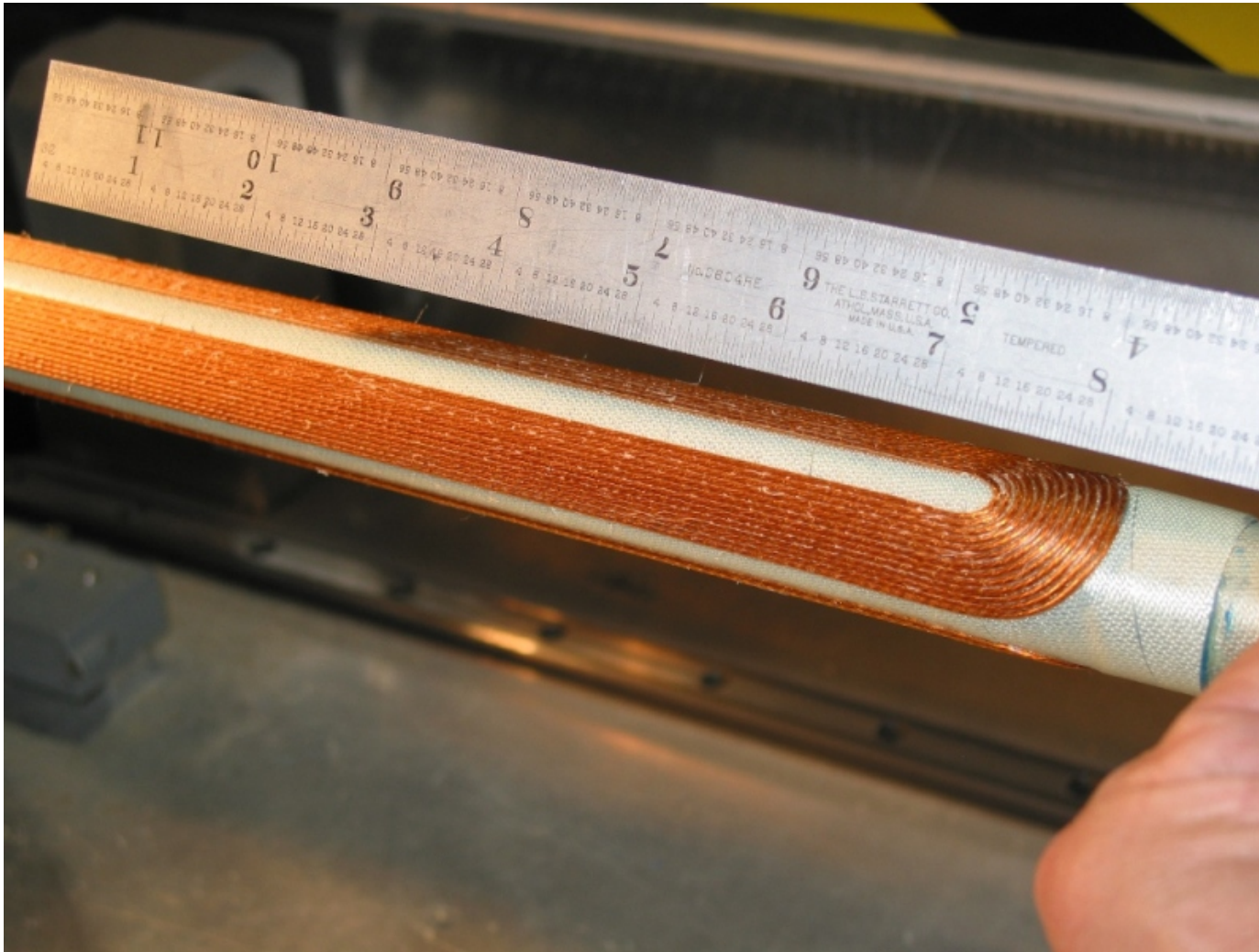
Sextupole



Octupole



A superconducting quadrupole for a (linear) collider final focus



Second layer of a six-layer superconducting quadrupole developed by Brookhaven National Laboratory for a linear collider. The design goal is a gradient of 140 T/m.

Generating multipole fields in an iron-cored magnet

To generate magnetic fields of the strengths often required in accelerators using only a current distribution, the size of the current needs to be large. Usually, this means using superconductors to carry the current.

Magnetic fields of reasonable strength can also be generated using resistive conductors to drive magnetic flux in high-permeability materials.

We shall finish this lecture with a discussion of the required geometry for an iron-cored magnet to generate a pure $2m$ -pole field, and the relationship between current and field strength. To keep things simple, we assume that the magnet is infinitely long in the z direction, and that the core has infinite permeability.

Generating multipole fields in an iron-cored magnet

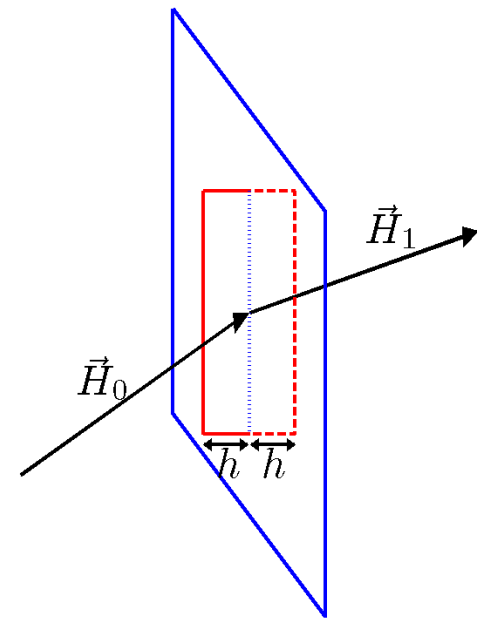
First of all, we note that the magnetic flux lines in free space must meet a material with infinite permeability normal to the surface. This we shall now show.

Consider a thin rectangular loop spanning the surface of the material. If we integrate Maxwell's equation:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (36)$$

across the surface bounded by the loop, and apply Stokes' theorem, we obtain:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint_C \vec{H} \cdot d\vec{\ell} = \int_S \vec{J} \cdot d\vec{S} + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}. \quad (37)$$



Generating multipole fields in an iron-cored magnet

Now, if we take the limit where the width of the loop tends to zero, then assuming there is no surface current, and that the time derivative of the electric displacement remain finite, we obtain:

$$H_{0t} - H_{1t} = 0, \quad (38)$$

where H_{0t} is the tangential component of the magnetic field just outside the boundary to the material, and H_{1t} is the tangential component of the magnetic field just inside the boundary.

We see that the tangential component of the magnetic field H is continuous across the boundary. Writing $B = \mu H$:

$$\frac{B_{0t}}{\mu_0} = \frac{B_{1t}}{\mu_1}. \quad (39)$$

For a material with infinite permeability, assuming that the magnetic field B remains finite within the material, we see that:

$$B_{0t} = 0. \quad (40)$$

Thus, the tangential component of the field at the surface of the material vanishes; in other words, the magnetic field at the surface must be normal to the surface.

If we can shape a material (with infinite permeability) such that its surface is everywhere normal to a given $2m$ -pole field, then the only field that can exist around the material will be the $2m$ -pole field.

Generating multipole fields in an iron-cored magnet

To derive an explicit expression for the shape of the magnetic material in a pure $2m$ -pole magnetic field, it is helpful to introduce the magnetic scalar potential, Φ . This is defined so that:

$$\vec{B} = -\nabla\Phi. \quad (41)$$

For static fields in free space, Maxwell's equation:

$$\nabla \times \vec{B} = 0 \quad (42)$$

is satisfied for *any* scalar field Φ ; and the other Maxwell equation:

$$\nabla \cdot \vec{B} = 0 \quad (43)$$

gives, in terms of the potential, Laplace's equation:

$$\nabla^2\Phi = 0. \quad (44)$$

Since the vector $\nabla\Phi$ is always normal to a surface of constant Φ , the surface of the magnetic material of infinite permeability is always a surface of constant magnetic scalar potential.

To find the geometry for the magnetic material in a pure $2m$ -pole field, we simply have to determine the appropriate magnetic scalar potential Φ , and then the equation

$$\Phi = \text{constant}$$

determines the geometry.

Let us hazard a guess at the potential:

$$\Phi = -|C_m| \frac{r^m}{m} \sin(m\theta - \varphi_m). \quad (45)$$

Taking the gradient in cylindrical polar coordinates:

$$\begin{aligned} -\nabla\Phi &= \hat{r} \frac{\partial\Phi}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial\Phi}{\partial\theta} \\ &= \hat{r} |C_m| r^{m-1} \sin(m\theta - \varphi_m) - \hat{\theta} |C_m| r^{m-1} \cos(m\theta - \varphi_m). \end{aligned} \quad (46)$$

Using:

$$\hat{r} = \hat{x} \cos\theta + \hat{y} \sin\theta, \quad \text{and} \quad \hat{\theta} = -\hat{x} \sin\theta + \hat{y} \cos\theta, \quad (47)$$

we find:

$$-\nabla\Phi = \hat{x} |C_m| r^{m-1} \sin[(m-1)\theta - \varphi_m] + \hat{y} |C_m| r^{m-1} \cos[(m-1)\theta - \varphi_m]. \quad (48)$$

Generating multipole fields in an iron-cored magnet

For a pure $2m$ -pole field:

$$B_y + iB_x = C_m r^{m-1} e^{i(m-1)\theta}, \quad (49)$$

so:

$$B_x = |C_m| r^{m-1} \sin[(m-1)\theta - \varphi_m], \quad (50)$$

$$B_y = |C_m| r^{m-1} \cos[(m-1)\theta - \varphi_m]. \quad (51)$$

Comparing with equation (48), we conclude that the scalar potential (45):

$$\Phi = -|C_m| \frac{r^m}{m} \sin(m\theta - \varphi_m) \quad (52)$$

generates the pure $2m$ -pole field:

$$B_y + iB_x = -\nabla\Phi = C_m (x + iy)^{m-1}. \quad (53)$$

Since the surface of the magnetic material must be surface of constant potential (assuming infinite permeability), we see that the surface of the material in a pure $2m$ -pole field must be given by:

$$r^m \sin(m\theta - \varphi_m) = \text{constant}, \quad (54)$$

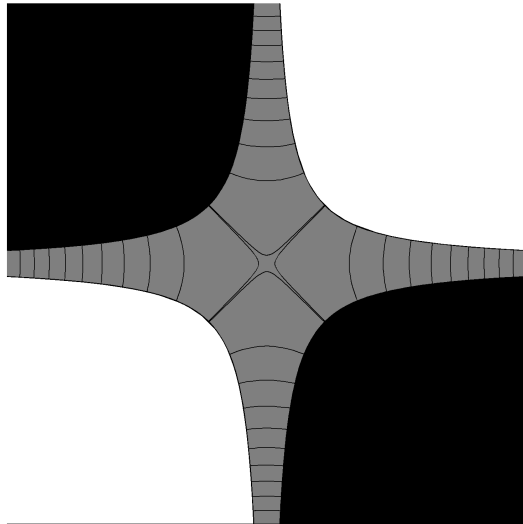
or:

$$r = \sqrt[m]{\frac{\text{constant}}{\sin(m\theta - \varphi_m)}}. \quad (55)$$

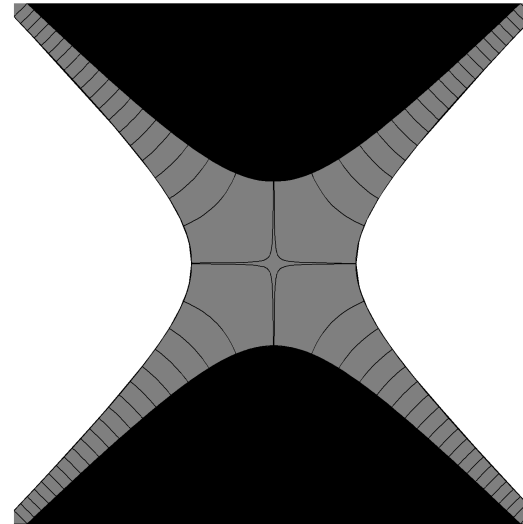
φ_m is the phase angle of C_m^* . If $\varphi_m = 0$, then C_m is real, and we generate a normal $2m$ -pole field. If $\varphi_m = \pi/2$, then C_m is imaginary, and we generate a skew $2m$ -pole field.

Multipole fields in an iron-cored magnet

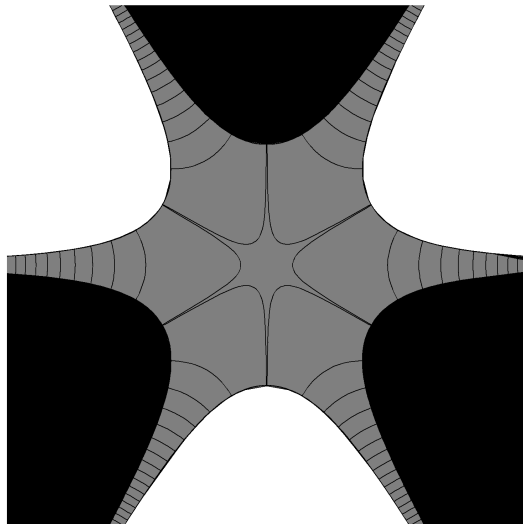
Normal Quadrupole



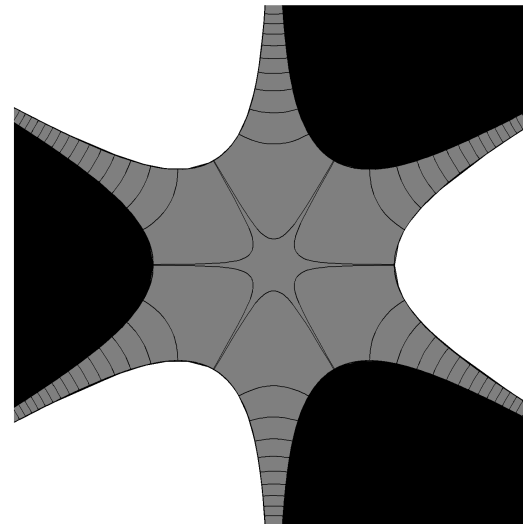
Skew Quadrupole



Normal Sextupole



Skew Sextupole



Generating multipole fields in an iron-cored magnet

Our final task is to calculate the field strength in an iron-cored magnet for a given number of ampere-turns around each pole. To do this, we can consider just a normal $2m$ -pole, since skew $2m$ -poles are simply rotations of normal $2m$ -poles.

We assume that the magnetic field is generated by wires carrying currents between the poles, with the wires parallel to the z axis, and positioned a large distance from the axis. Since the distance from the centre of the magnet to the currents is large, we can neglect the field arising “directly” from the current, and consider only the field arising from magnetisation of the iron.

Furthermore, we maintain symmetry by placing equal currents between each pair of poles, alternating in direction from one set of wires to the next.

Generating multipole fields in an iron-cored magnet

We again use Maxwell's equation:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \quad (56)$$

Now we integrate across a surface in the x - y plane, bounded by the contour C defined by the lines:

$$\theta = 0, \quad \text{and} \quad \theta = \frac{\pi}{2m}, \quad (57)$$

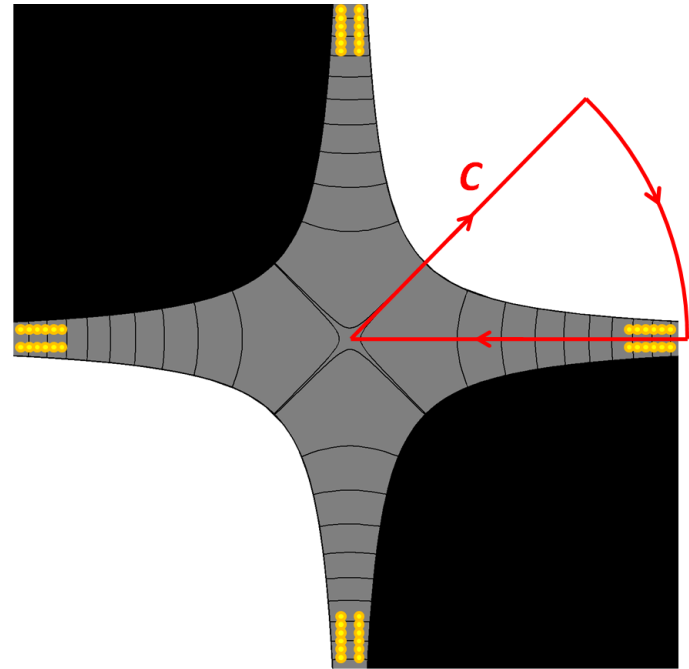
and closed at $r \rightarrow \infty$.

Again applying Stokes' theorem, we obtain:

$$\oint_C \vec{H} \cdot d\vec{\ell} = NI, \quad (58)$$

where there are $2N$ wires carrying current I between each pair of poles.

Note that conventionally, the current is supplied by a coil of N turns around each pole; thus the total number of wires between each adjacent pair of poles is $2N$.



Generating multipole fields in an iron-cored magnet

We can break the path integral into two segments: C_0 in vacuum with permeability μ_0 , and C_1 inside the magnetic material with permeability μ :

$$\int_{C_0} \frac{\vec{B}}{\mu_0} \cdot d\vec{\ell} + \int_{C_1} \frac{\vec{B}}{\mu} \cdot d\vec{\ell} = NI. \quad (59)$$

In the limit $\mu \rightarrow \infty$, the segment of the integral inside the magnetic material vanishes. Furthermore along the segment $\theta = 0$, the field is perpendicular to the path, so makes no contribution to the path integral.

We are left with:

$$\int_0^{r_0} B_r dr = \mu_0 NI, \quad (60)$$

where r_0 is the radius of the largest circle that can be inscribed within the pole tips of the magnet.

The radial field component along $\theta = \pi/2m$ is given by:

$$B_r = B_{\text{ref}} b_m \left(\frac{r}{R_{\text{ref}}} \right)^{m-1}. \quad (61)$$

Let us choose $R_{\text{ref}} = r_0$, and $B_{\text{ref}} = B_r(r_0, \pi/2m) = B_0$. Then, $b_m = 1$:

$$B_r = B_0 \left(\frac{r}{r_0} \right)^{m-1}, \quad (62)$$

and we obtain:

$$\int_0^{r_0} B_r dr = \frac{B_0 r_0}{m} = \mu_0 N I. \quad (63)$$

Therefore, the field is given by:

$$B_y + iB_x = \frac{m\mu_0 NI}{r_0} \left(\frac{x + iy}{r_0} \right)^{m-1}. \quad (64)$$

The multipole gradient is given by:

$$\frac{\partial^{m-1} B_y}{\partial x^{m-1}} = \frac{m! \mu_0 NI}{r_0^m}. \quad (65)$$

For example, for a quadrupole magnet ($m = 2$), the gradient is given by:

$$\frac{\partial B_y}{\partial x} = \frac{2\mu_0 NI}{r_0^2}. \quad (66)$$

We have shown that:

- multipole fields satisfy Maxwell's equations in free space;
- a pure $2m$ -pole field can be generated by a $\cos(m\theta)$ current distribution on the surface of a cylinder;
- a pure $2m$ -pole field can be generated by an iron-cored magnet, whose pole tips follow surfaces of constant magnetic scalar potential.

Of course, the expressions we have derived here are only exactly correct with ideal (and rather impractical) conditions on the geometry and material properties.

In the next lecture, we shall consider three dimensional fields, and the effects of imperfections in the magnet construction.