Foundations of Analytical and Numerical Field Computation

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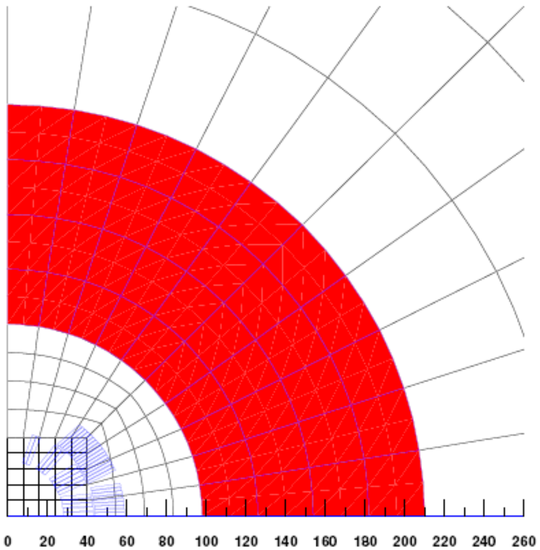
Reduced Vector Potential Formulation

$$\mathbf{A} = \mathbf{A}_\mathrm{s} + \mathbf{A}_\mathrm{r} \qquad \mathbf{B} = \mu_0 \mathbf{H}_\mathrm{s} + \operatorname{curl} \mathbf{A}_\mathrm{r}$$

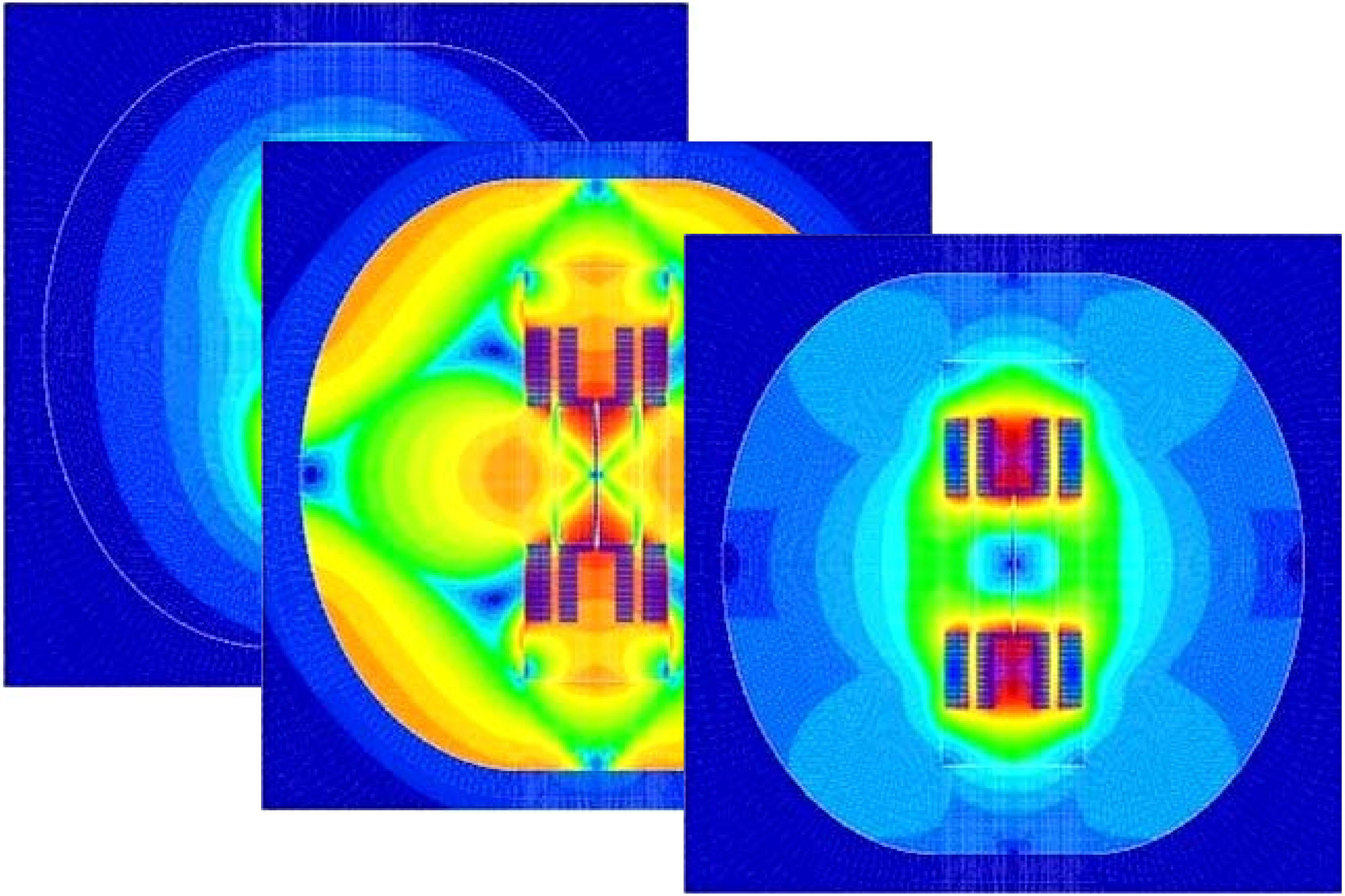
$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} (\mathbf{A}_\mathrm{r} + \mathbf{A}_\mathrm{s}) - \operatorname{grad} \frac{1}{\mu} \operatorname{div} (\mathbf{A}_\mathrm{r} + \mathbf{A}_\mathrm{s}) = \mathbf{J}$$



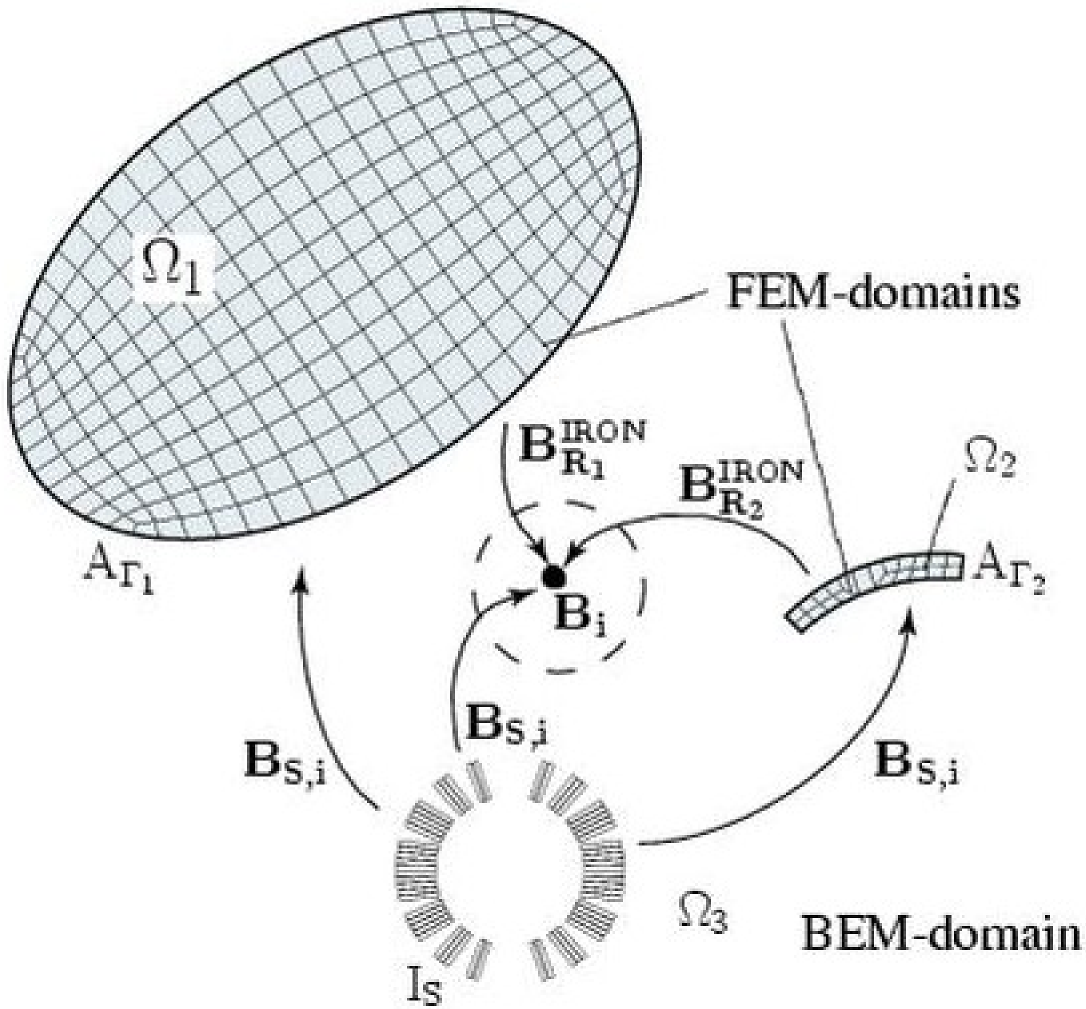
$$\begin{aligned} \operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A}_\mathrm{r} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A}_\mathrm{r} &= \mathbf{J} - \operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A}_\mathrm{s} \\ &= \operatorname{curl} \mathbf{H}_\mathrm{s} - \operatorname{curl} \frac{\mu_0}{\mu} \mathbf{H}_\mathrm{s} \\ &= \operatorname{curl} \left(\mathbf{H}_\mathrm{s} - \frac{\mu_0}{\mu} \mathbf{H}_\mathrm{s} \right) \end{aligned}$$

Advantages: No meshing of the coil, no cancellation errors, distinction between source field and iron magnetization

Source, Reduced, Total Field



BEM-FEM Coupling (Elementary Model Problem)



The FEM Part (Vector Laplace Equation)

$$\boxed{-\frac{1}{\mu_0}\nabla^2\mathbf{A} = \mathbf{J} + \operatorname{curl}\mathbf{M}} \qquad \text{in } \Omega_i,$$

$$\mathbf{A}\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_H,$$

$$\frac{1}{\mu_0}\operatorname{div}\mathbf{A} = 0 \qquad \text{on } \Gamma_B,$$

$$\mathbf{n}\times(\mathbf{A}\times\mathbf{n}) = \mathbf{0} \qquad \text{on } \Gamma_B,$$

$$\frac{1}{\mu}(\operatorname{curl}\mathbf{A})\times\mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma_H,$$

$$\left[\frac{1}{\mu_0}\operatorname{div}\mathbf{A}_\mathrm{a}\right]_\mathrm{ai} = 0 \qquad \text{on } \Gamma_\mathrm{ai},$$

$$\boxed{\frac{1}{\mu_0}(\operatorname{curl}\mathbf{A}_\mathrm{i} - \mu_0\mathbf{M})\times\mathbf{n}_\mathrm{i} + \frac{1}{\mu_0}(\operatorname{curl}\mathbf{A}_\mathrm{a})\times\mathbf{n}_\mathrm{a} = \mathbf{0}} \qquad \text{on } \Gamma_\mathrm{ai},$$

$$[\mathbf{A}]_\mathrm{ai} = \mathbf{0} \qquad \text{on } \Gamma_\mathrm{ai}.$$

$$\frac{1}{\mu_0}\int_{\Omega_\mathrm{i}} \mathrm{grad}\,(\mathbf{A}\cdot\mathbf{e}_a)\cdot\,\mathrm{grad}\,w_a\,\mathrm{d}\Omega_\mathrm{i} - \boxed{\frac{1}{\mu_0}\oint_{\Gamma_\mathrm{ai}}\left(\frac{\partial\mathbf{A}}{\partial n_\mathrm{i}} - (\mu_0\mathbf{M}\times\mathbf{n}_\mathrm{i})\right)\cdot\mathbf{w}_a\,\mathrm{d}\Gamma_\mathrm{ai}} = \int_{\Omega_\mathrm{i}}\mathbf{M}\cdot\,\mathrm{curl}\,\mathbf{w}_a\,\mathrm{d}\Omega_\mathrm{i}$$

$$\mathbf{Q}_{\Gamma_\mathrm{ai}} := -\frac{\partial\mathbf{A}^\mathrm{BEM}_{\Gamma_\mathrm{ai}}}{\partial n_\mathrm{a}}$$

$$\frac{\partial\mathbf{A}^\mathrm{FEM}_\mathrm{i}}{\partial n_\mathrm{i}} - (\mu_0\mathbf{M}\times\mathbf{n}_\mathrm{i}) + \frac{\partial\mathbf{A}^\mathrm{BEM}_\mathrm{a}}{\partial n_\mathrm{a}} = \mathbf{0} \quad \text{on } \Gamma_\mathrm{ai}$$

$$\frac{1}{\mu_0}\left(\mathrm{curl}\,\mathbf{A}^\mathrm{FEM}_\mathrm{i} - \mu_0\mathbf{M}\right)\times\mathbf{n}_\mathrm{i} + \frac{1}{\mu_0}\left(\mathrm{curl}\,\mathbf{A}^\mathrm{BEM}_\mathrm{a}\right)\times\mathbf{n}_\mathrm{a} = \mathbf{0} \quad \text{on } \Gamma_\mathrm{ai}$$

$$\frac{1}{\mu_0}\int_{\Omega_\mathrm{i}} \mathrm{grad}\,(\mathbf{A}\cdot\mathbf{e}_a)\cdot\,\mathrm{grad}\,w_a\,\mathrm{d}\Omega_\mathrm{i} - \frac{1}{\mu_0}\oint_{\Gamma_\mathrm{ai}}\mathbf{Q}_{\Gamma_\mathrm{ai}}\cdot\mathbf{w}_a\,\mathrm{d}\Gamma_\mathrm{ai} = \int_{\Omega_\mathrm{i}}\mathbf{M}\cdot\,\mathrm{curl}\,\mathbf{w}_a\,\mathrm{d}\Omega_\mathrm{i}$$

$$[K]\{A_x\} - [T]\{Q_x\} = \{F_x(\mathbf{M})\}$$

Accelerator Technology Department

Vector Laplace

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}\,, \qquad \text{in } \Omega_\mathrm{a}$$

Weighted Residual

$$\int_{\Omega_\mathrm{a}} \nabla^2 A w \, \mathrm{d}\Omega_\mathrm{a} = -\int_{\Omega_\mathrm{a}} \mu_0 J w \, \mathrm{d}\Omega_\mathrm{a}$$

From Green's second theorem:

$$\int_{\Omega_\mathrm{a}} A \nabla^2 w \mathrm{d}\Omega_\mathrm{a} = -\int_{\Omega_\mathrm{a}} \mu_0 J w \, \mathrm{d}\Omega_\mathrm{a} + \int_{\Gamma_\mathrm{ai}} A \frac{\partial w}{\partial n_a} \mathrm{d}\Gamma_\mathrm{ai} - \int_{\Gamma_\mathrm{ai}} \frac{\partial A}{\partial n_a} w \mathrm{d}\Gamma_\mathrm{ai}$$

$$u^*(\mathbf{r}, \mathbf{r}') := w = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

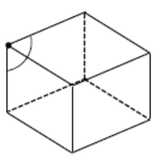
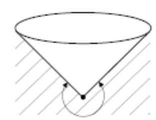
$$\nabla^2 w = -\delta(|\mathbf{r} - \mathbf{r}'|)$$

$$q^*(\mathbf{r}, \mathbf{r}') := \frac{\partial u^*}{\partial n_a} = \frac{\partial w}{\partial n_a} = -\frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{n}_a}{4\pi |\mathbf{r} - \mathbf{r}'|^3}$$

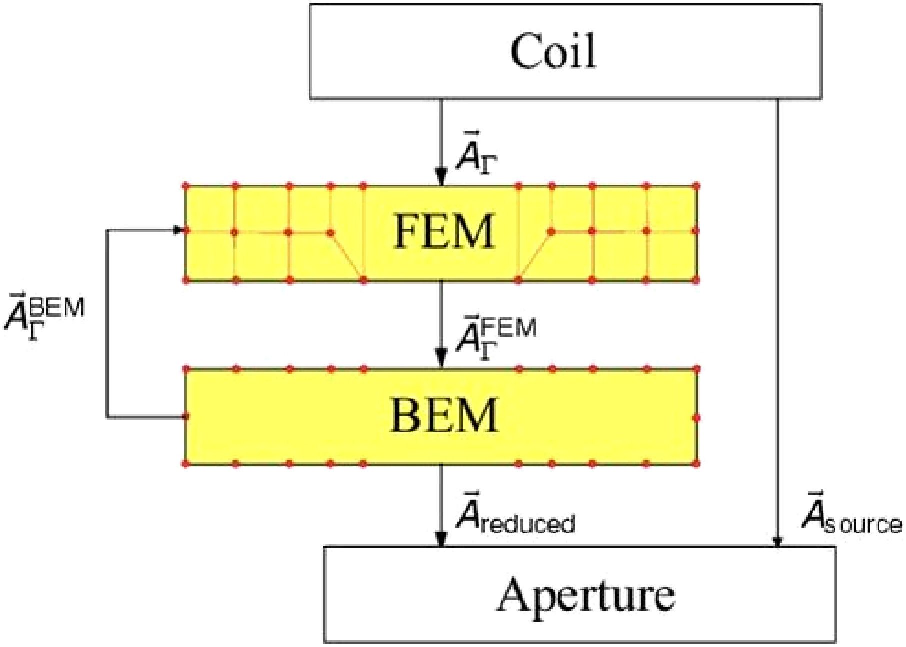
$$\int_\Omega A(\mathbf{r}) \nabla^2 w \mathrm{d}\Omega = \int_\Omega A(\mathbf{r}) \delta(|\mathbf{r} - \mathbf{r}'|) \mathrm{d}\Omega = A(\mathbf{r}')$$

BEM Part

$$\frac{\Theta}{4\pi}A(\mathbf{r}) + \int\limits_{\Gamma_{\mathrm{ai}}} Q_{\Gamma_{\mathrm{ai}}}\, u^*(\mathbf{r},\mathbf{r}')\ \mathrm{d}\Gamma_{\mathrm{ai}} + \int\limits_{\Gamma_{\mathrm{ai}}} A_{\Gamma_{\mathrm{ai}}}\, q^*(\mathbf{r},\mathbf{r}')\ \mathrm{d}\Gamma_{\mathrm{ai}} = \int\limits_{\Omega_{\mathrm{a}}} \mu_0 J u^*(\mathbf{r},\mathbf{r}')\,\mathrm{d}\Omega_{\mathrm{a}}.$$

Ω_{a}	 90°corner	 90° cone inner	 half space	 90°cone outer
Θ	$\frac{1}{2}\,\pi$	$(2-\sqrt{2})\pi$	2π	$(2+\sqrt{2})\,\pi$
$\frac{\Theta}{4\pi}$	$\frac{1}{8}$	$\frac{2-\sqrt{2}}{4}$	$\frac{1}{2}$	$\frac{2+\sqrt{2}}{4}$

BEM-FEM Coupling



BEM

$$[G]\{Q_z\} + [H]\{A_z\} = \{A_{z,\mathrm{s}}\}$$

FEM

$$[K]\{A_z\} - [T]\{Q_z\} = \{F_z(\mathbf{M})\}$$

$$\{Q\} = -[G]^{-1}[H]\{A\} + [G]^{-1}\{A\}$$

$$\boxed{\left([K] + [T][G]^{-1}[H]\right)\{A\} = \{F(\mathbf{M})\} + [T][G]^{-1}\{A_\mathrm{s}\}}$$

$$[K]\{A\} = \{F(A_s, M)\}$$

yields an iteration scheme

$$\{A_{k+1}\} = [K]^{-1}\{F(A_s, M_k)\}$$

Subtracting $\{A_k\}$ from both sides yields:

$$\{\Delta A_k\} := \{A_{k+1}\} - \{A_k\} = [K]^{-1}\{R_k\}$$

where the residual is defined by

$$\{R_k\} = \{F(A_s, M_k)\} - [K]\{A_k\}$$

Introduce a *relaxation parameter* ω.

$$\{A_{k+1}\} = \{A_k\} + \omega\{\Delta A_k\}$$

with $\omega_0 = 1$ and

$$\omega_k = \frac{\omega_{k-1}}{1 - \frac{\{\Delta A_k\}\cdot\{\Delta A_{k-1}\}}{\|\{\Delta A_{k-1}\}\|^2}}$$

1. Set iteration index $k = 0$, and initialize vector potentials $\{A_0\} = \{0\}$.
2. For $k = 0, 1, 2, ...,$ unit convergence Do:
3. Compute the force vector $\{F(A_s, M_k)\}$ and the residual $\{R_k\}$.
4. If $\frac{\|\{R_k\}\|}{\|\{F(A_{sk}, M_k)\}\|} < \varepsilon$ Goto 8.
5. Calculate the step sizes $\{\Delta A_k\} = [\mathrm{K}]^{-1}\{R_k\}$.
6. Choose the relaxation parameter.
7. $\{A_{k+1}\} = \{A_k\} + \omega\{\Delta A_k\}$
8. End Do.

Advantages

- If direct solvers are used, the stiffness matrix needs to be inverted only once.
- The method is globally convergent
- No derivative of the M(B) curve is required.

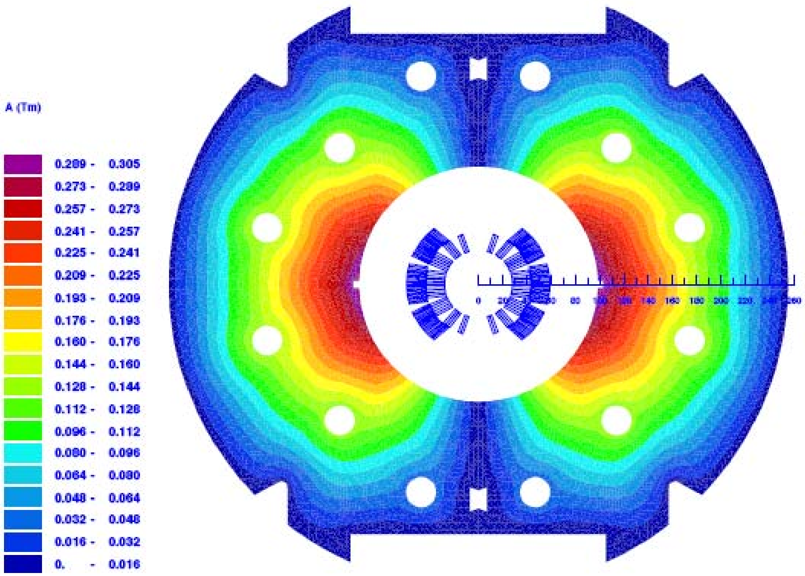
and disadvantages

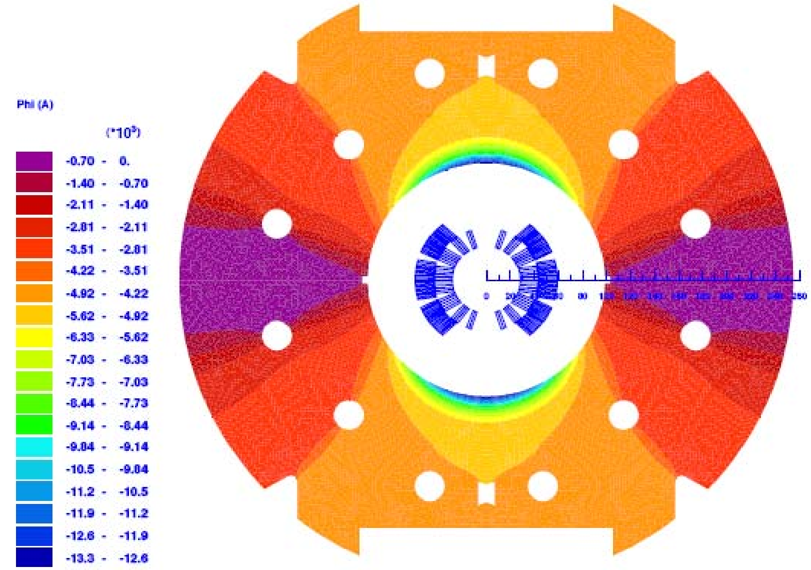
- The number of iteration steps is high.
- Even in the case of linear media the M(B)-iteration is necessary.

```
               GMRES STEP 23, RESIDUAL= 0.1014E-05 ( -59.52 DB)
               GMRES STEP 24, RESIDUAL= 0.2402E-06 ( -65.78 DB)
INFO 240:      GMRES STEP 24, RESIDUAL= 0.2402E-06 ( -65.78 DB).
               >>>>>>>>>>>> NEWTON STEP 4, RESIDUAL= 0.3917E-03 ( -38.38 DB)
COMPUTING THE GLOBAL MATRIX...
               GMRES STEP 25, RESIDUAL= 0.9083E+00 (   0.00 DB)
               GMRES STEP 26, RESIDUAL= 0.1915E-04 ( -46.76 DB)
               GMRES STEP 27, RESIDUAL= 0.6527E-05 ( -51.44 DB)
               GMRES STEP 28, RESIDUAL= 0.2510E-05 ( -55.59 DB)
               GMRES STEP 29, RESIDUAL= 0.4689E-06 ( -62.87 DB)
INFO 240:      GMRES STEP 29, RESIDUAL= 0.4689E-06 ( -62.87 DB).
               >>>>>>>>>>>> NEWTON STEP 5, RESIDUAL= 0.1299E-03 ( -43.22 DB)
COMPUTING THE GLOBAL MATRIX...
               GMRES STEP 30, RESIDUAL= 0.9083E+00 (   0.00 DB)
               GMRES STEP 31, RESIDUAL= 0.3846E-05 ( -53.73 DB)
               GMRES STEP 32, RESIDUAL= 0.1670E-05 ( -57.36 DB)
               GMRES STEP 33, RESIDUAL= 0.7543E-06 ( -60.81 DB)
INFO 240:      GMRES STEP 33, RESIDUAL= 0.7543E-06 ( -60.81 DB).
               >>>>>>>>>>>> NEWTON STEP 6, RESIDUAL= 0.3413E-04 ( -49.05 DB)
COMPUTING THE GLOBAL MATRIX...
               GMRES STEP 34, RESIDUAL= 0.9083E+00 (   0.00 DB)
               GMRES STEP 35, RESIDUAL= 0.2110E-05 ( -56.34 DB)
               GMRES STEP 36, RESIDUAL= 0.1095E-05 ( -59.19 DB)
               GMRES STEP 37, RESIDUAL= 0.3836E-06 ( -63.74 DB)
INFO 240:      GMRES STEP 37, RESIDUAL= 0.3836E-06 ( -63.74 DB).
               >>>>>>>>>>>> NEWTON STEP 7, RESIDUAL= 0.7897E-05 ( -55.43 DB)
INFO 240:      >>>>>>>>>>>> NEWTON STEP 7, RESIDUAL= 0.7897E-05 ( -55.43 DB).
INFO 296:      SOLVER INFORMATION
               NUMBER OF TIME STEPS              = 1
               NUMBER OF NON-LINEAR STEPS        = 7
                   AVERAGE PER TIME STEP         = 7.0
               NUMBER OF LINEAR STEPS            = 37
```

Always check convergence of your computation

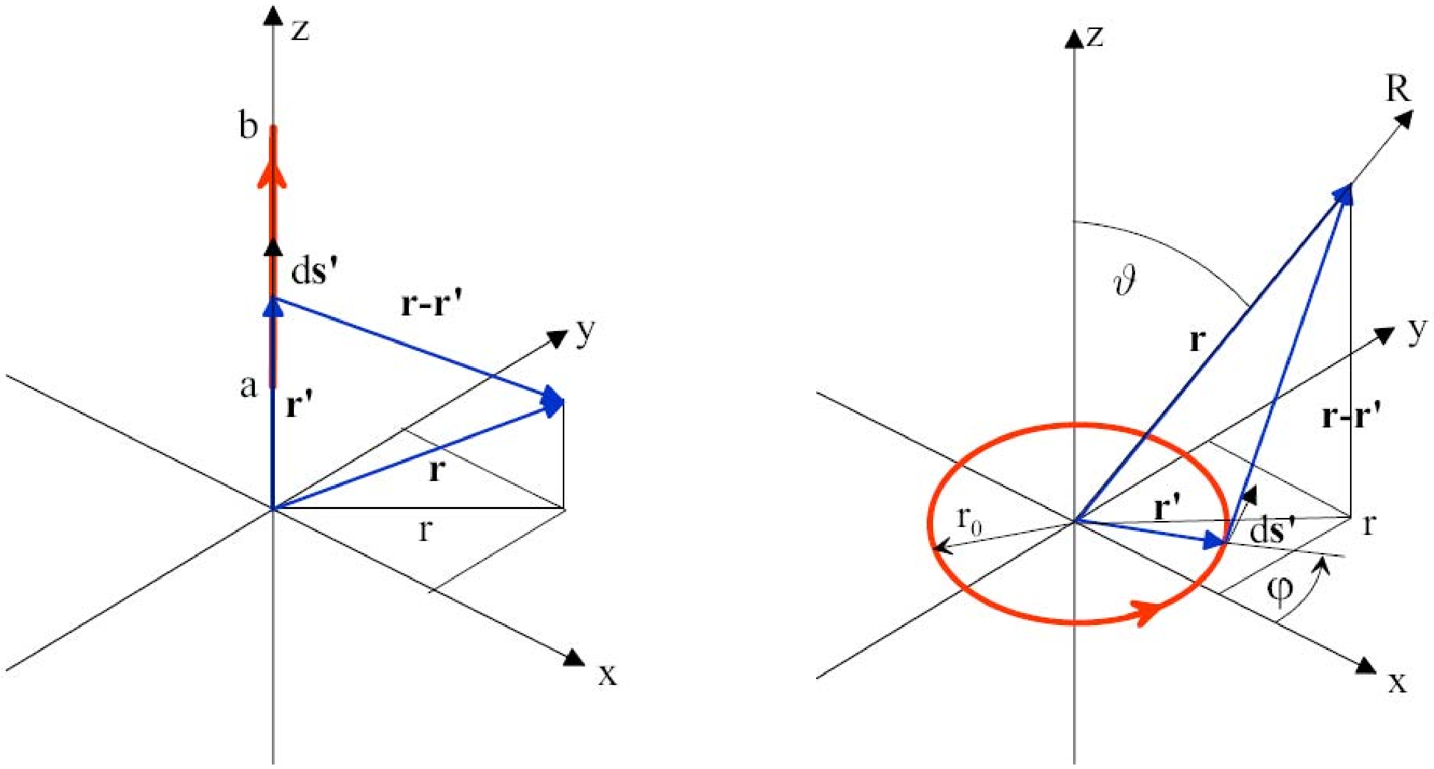
Vector Potential and Total Scalar Potential





Number of finite elements	60	178	449	787	2799	6233
Total scalar potential	65.8	72.1	13.0	5.0	3.8	15.7
Vector potential	-40.5	-27.4	-7.4	-4.8	-3.8	25.0

Pre-processing (Current sources)



Biot Savart's Law for Line Currents

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \qquad \nabla^2\mathbf{A} = -\mu_0\mathbf{J}$$

$$A_{x,y,z}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{J_{x,y,z}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'\,.$$

$$\begin{aligned}
\mathbf{B}(\mathbf{r}) &= \operatorname{curl}\mathbf{A}(\mathbf{r}) \\
&= \frac{\mu_0}{4\pi}\int_V \operatorname{curl}_{\mathbf{r}}\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_V \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{curl}_{\mathbf{r}}\mathbf{J}(\mathbf{r}') - \mathbf{J}(\mathbf{r}')\times\operatorname{grad}_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right)\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\mathrm{d}V'\,.
\end{aligned}$$

$$\begin{aligned}
\operatorname{div}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V \operatorname{div}\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_V \left(\mathbf{J}(\mathbf{r}')\cdot\operatorname{grad}_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) + \frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{div}_{\mathbf{r}}\mathbf{J}(\mathbf{r}')\right)\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\cdot\operatorname{grad}_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' = -\frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\cdot\operatorname{grad}_{\mathbf{r}'}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi}\int_V \left(\operatorname{div}_{\mathbf{r}'}\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{div}_{\mathbf{r}'}\mathbf{J}(\mathbf{r}')\right)\mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi}\int_V \operatorname{div}_{\mathbf{r}'}\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' = -\frac{\mu_0}{4\pi}\int_{\partial V}\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\cdot\mathrm{d}\mathbf{a}' = 0\,. \qquad (13.2
\end{aligned}$$

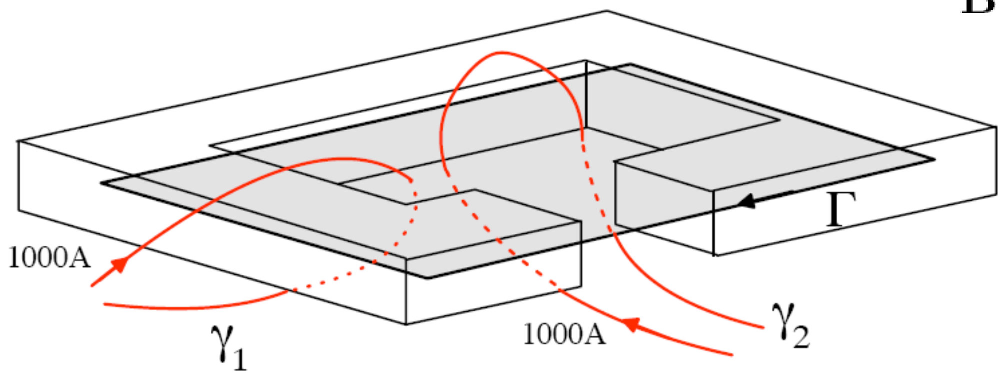
Current loops have always be closed, must not leave the problem domain

Biot-Savart's Law and the Linking Number

$$\mathbf{J}(\mathbf{r}')\mathrm{d}V' = \mathbf{J}(\mathbf{r}')\mathrm{d}a'\mathrm{d}s' = I\mathrm{d}s'$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint_\gamma \frac{\mathrm{d}\mathbf{s}'}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint_\gamma \frac{\mathrm{d}\mathbf{s}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$



$$\int_{\partial a} \mathbf{H}\cdot\mathrm{d}\mathbf{s} = \int_a \mathbf{J}\cdot\mathrm{d}\mathbf{a} = N\,I \text{ where } \Gamma := \partial a$$

$$N\,I = I\sum_{k=1}^{K} \mathrm{link}(\Gamma,\gamma_k) = I\sum_{k=1}^{K}\mathrm{int}(a,\gamma_k)$$

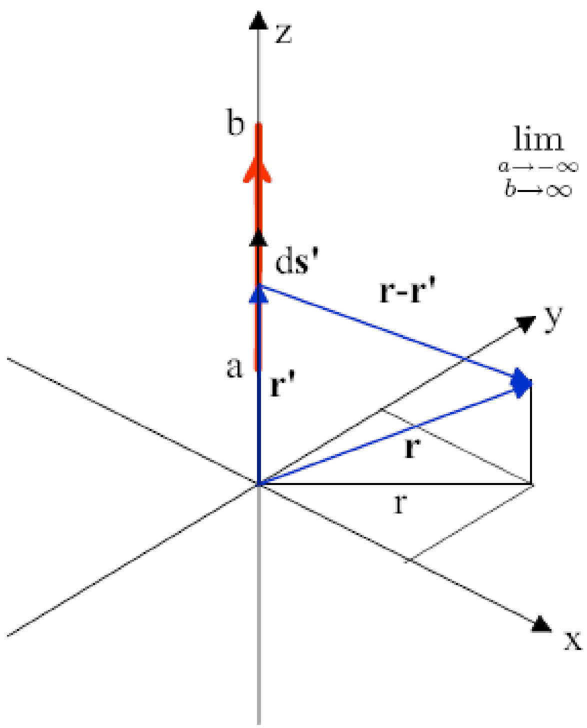
$$\mathrm{link}(\Gamma,\gamma_k) = \frac{1}{4\pi}\oint_\Gamma\oint_{\gamma_k}\frac{\mathrm{d}\mathbf{s}'\times(\mathbf{r}-\mathbf{r}')\cdot\mathrm{d}\mathbf{s}}{|\mathbf{r}-\mathbf{r}'|^3}$$

Vector Potential of a Line Current

$$A_z = \frac{\mu_0 I}{4\pi}\int_a^b \frac{\mathrm{d}z'}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 I}{4\pi}\int_a^b \frac{\mathrm{d}z'}{\sqrt{x^2+y^2+(z-z')^2}}$$

$$= \frac{-\mu_0 I}{4\pi}\left[\ln\left((z-z')+\sqrt{x^2+y^2+(z-z')^2}\right)\right]_a^b$$

$$= \frac{\mu_0 I}{4\pi}\ln\frac{z-a+\sqrt{x^2+y^2+(z-a)^2}}{z-b+\sqrt{x^2+y^2+(z-b)^2}}\,.$$



$$\lim_{\substack{a\to-\infty\\ b\to\infty}} \ln\frac{z-a+\sqrt{x^2+y^2+(z-a)^2}}{z-b+\sqrt{x^2+y^2+(z-b)^2}} = \lim_{\substack{a\to-\infty\\ b\to\infty}} \ln\frac{-a+|a|\sqrt{1+\frac{x^2+y^2}{a^2}}}{-b+|b|\sqrt{1+\frac{x^2+y^2}{b^2}}}$$

$$= \lim_{\substack{a\to-\infty\\ b\to\infty}} \ln\frac{-a-a(1+\frac{x^2+y^2}{2a^2}+\cdots)}{-b+b(1+\frac{x^2+y^2}{2b^2}+\cdots)}$$

$$= \lim_{\substack{a\to-\infty\\ b\to\infty}} \ln\frac{-2a}{-b+b+\frac{x^2+y^2}{2b}}$$

$$= \lim_{\substack{a\to-\infty\\ b\to\infty}} \ln\frac{-4ab}{x^2+y^2}\,,$$

Caution: Infinitely long line currents have infinite energy

Vector Potential of two Line Currents

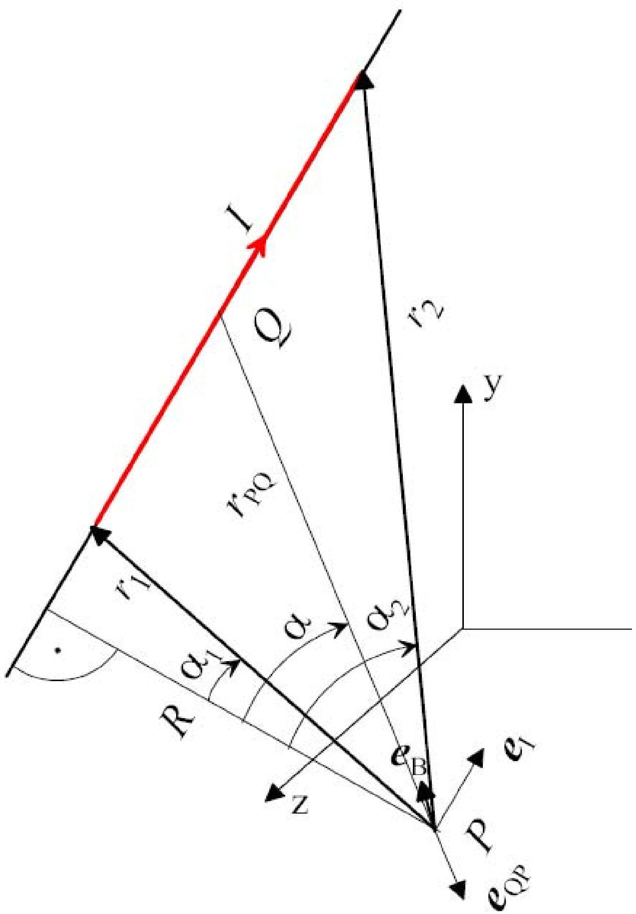
$$A_z = \lim_{\substack{a \to -\infty \\ b \to \infty}} \frac{\mu_0 I}{4\pi} \ln \frac{-4ab}{x^2 + y^2}$$

$$= \lim_{\substack{a \to -\infty \\ b \to \infty}} \frac{\mu_0 I}{4\pi} \ln \left(\frac{-4ab}{x_0^2 + y_0^2} \right) - \frac{\mu_0 I}{4\pi} \ln \left(\frac{x^2 + y^2}{x_0^2 + y_0^2} \right)$$

$$\boxed{\mathbf{A} = -\frac{\mu_0 I}{4\pi} \ln\left(\frac{x^2 + y^2}{x_0^2 + y_0^2}\right) \mathbf{e}_z = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{r}{R_{\mathrm{ref}}}\right) \mathbf{e}_z}$$

Problem solved, but reference radius has physical significance: Return path for sum-currents

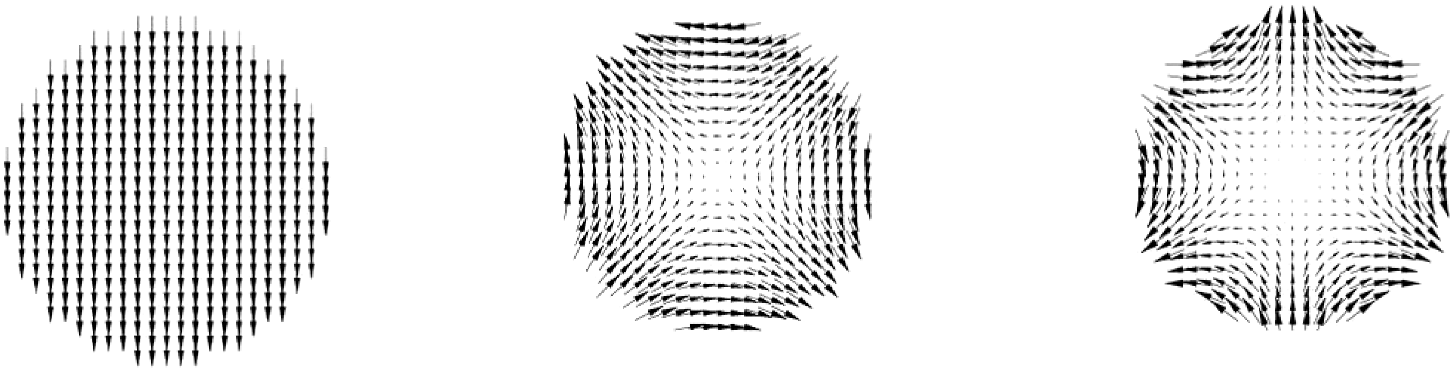
$$r_{\mathrm{PQ}} = \frac{R}{\cos\alpha}, \; \mathrm{d}s = \frac{R}{\cos^2\alpha}\mathrm{d}\alpha, \; \mathbf{e}_{\mathrm{I}} \times \mathbf{e}_{\mathrm{QP}} = \cos\alpha\,\mathbf{e}_{\mathrm{B}}$$



$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I}{4\pi R}\left[\sin\alpha_2 - \sin\alpha_1\right]\mathbf{e}_{\mathrm{B}} \\
&= \frac{\mu_0 I}{4\pi}\,\frac{\cos\alpha_2 + \cos\alpha_1}{R}\,\frac{\sin\alpha_2 - \sin\alpha_1}{\cos\alpha_2 + \cos\alpha_1}\,\mathbf{e}_{\mathrm{B}} \\
&= \frac{\mu_0 I}{4\pi}\left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|}\right)\frac{\sin(\alpha_2 - \alpha_1)}{1 + \cos(\alpha_2 - \alpha_1)}\,\mathbf{e}_{\mathrm{B}} \\
&= \frac{\mu_0 I}{4\pi}\left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|}\right)\frac{\sin(\alpha_2 - \alpha_1)}{1 + \frac{\mathbf{r}_1\cdot\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}}\,\frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|\sin(\alpha_2 - \alpha_1)} \\
&= \frac{\mu_0 I}{4\pi}\,\frac{|\mathbf{r}_1| + |\mathbf{r}_2|}{|\mathbf{r}_1|\,|\mathbf{r}_2| + \mathbf{r}_1\cdot\mathbf{r}_2}\,\frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}\,.
\end{aligned}$$

Easy to program equation. Still remember: Closed loops

Harmonic Analysis



$$\varphi_k = \frac{2\pi k}{N}, \qquad k = 0, 1, 2, ..., N-1$$

$$A_n(r_0) \approx \frac{2}{N} \sum_{k=0}^{N-1} B_r(r_0, \varphi_k) \cos n\varphi_k$$

$$B_n(r_0) \approx \frac{2}{N} \sum_{k=0}^{N-1} B_r(r_0, \varphi_k) \sin n\varphi_k .$$

Discrete Fourier Transform

$$B_r(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi)$$

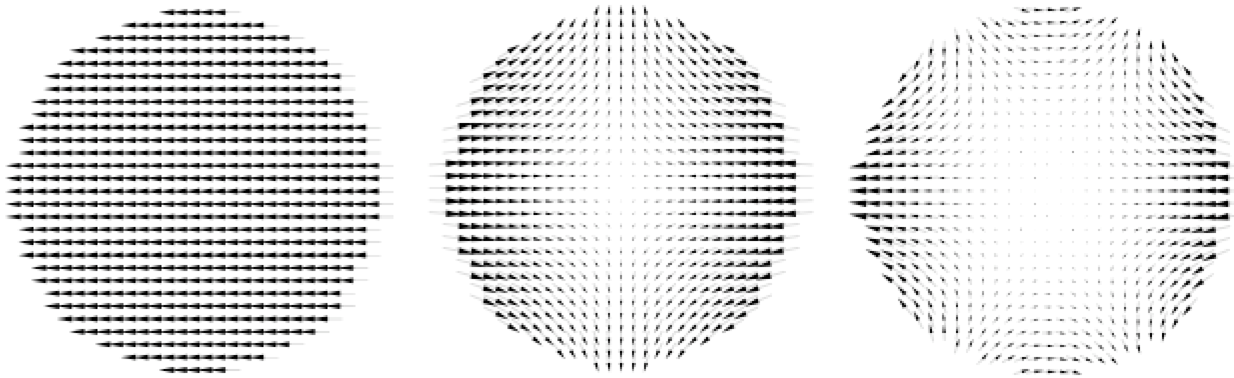
Don't bother with the Fast Fourier Transform, but consider to use the vector-potential directly

$$A_z(r_0, \varphi) = \sum_{n=1}^{\infty} (\mathcal{F}_n(r_0) \cos n\varphi + \mathcal{E}_n(r_0) \sin n\varphi)$$

$$B_n(r_0) = \frac{-n\,\mathcal{F}_n}{r_0}, \qquad A_n(r_0) = \frac{n\,\mathcal{E}_n}{r_0}$$

And please: Never use holomorphic continuation, it's just to inaccurate

The Case of Solenoids



$$B_R(R_0,\vartheta) = \sum_{n=1}^{\infty} A_n P_n(\cos\vartheta)$$

$$A_n(R_0) = \frac{2n+1}{2}\int_0^{\pi} B_R(R_0,\vartheta) P_n(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta$$

$$P_n(x) = \sum_{m=0}^{M}(-1)^m \frac{(2n-2m)!}{2^n r!(n-m)!(n-2r)!} x^{n-2m}$$

To derive this result would take 5 pages. So take it from the "horse's mouth" (and don't do Fourier series analysis).

Separation in Cartesian Coordinates

$$X_p(x) = \mathcal{A}_p \cosh px + \mathcal{B}_p \sinh px$$
$$Y_p(y) = \mathcal{C}_p \cos py + \mathcal{D}_p \sin py\,,$$

$$\phi_{\mathrm{m}}(x,y) = \sum_{p=0}^{\infty} X_p(x) Y_p(y)$$

$$B_y(-x,y) = B_y(x,y) \quad \rightarrow \quad X_p(x) = X_p(-x)$$

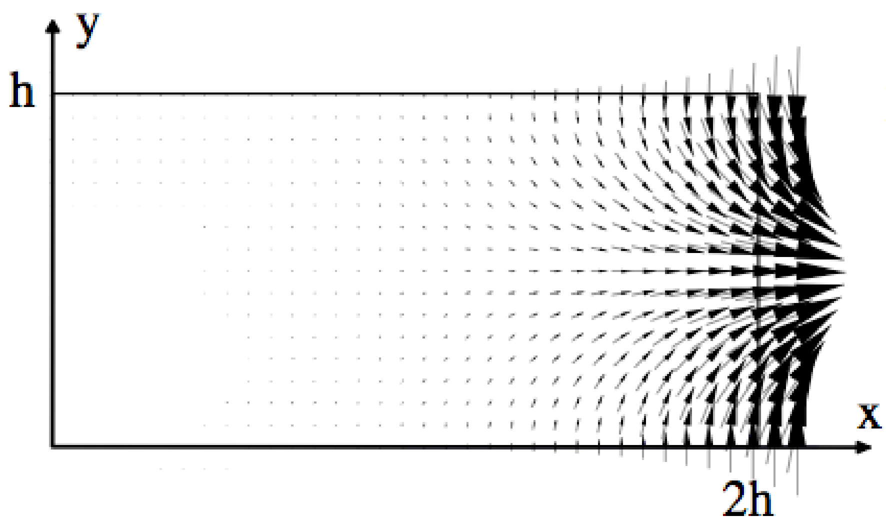
$$B_y(x,-y) = B_y(x,y) \quad \rightarrow \quad \left.\frac{\mathrm{d}Y_p(y)}{\mathrm{d}y}\right|_{y=-y_0} = \left.\frac{\mathrm{d}Y_p(y)}{\mathrm{d}y}\right|_{y=y_0}$$

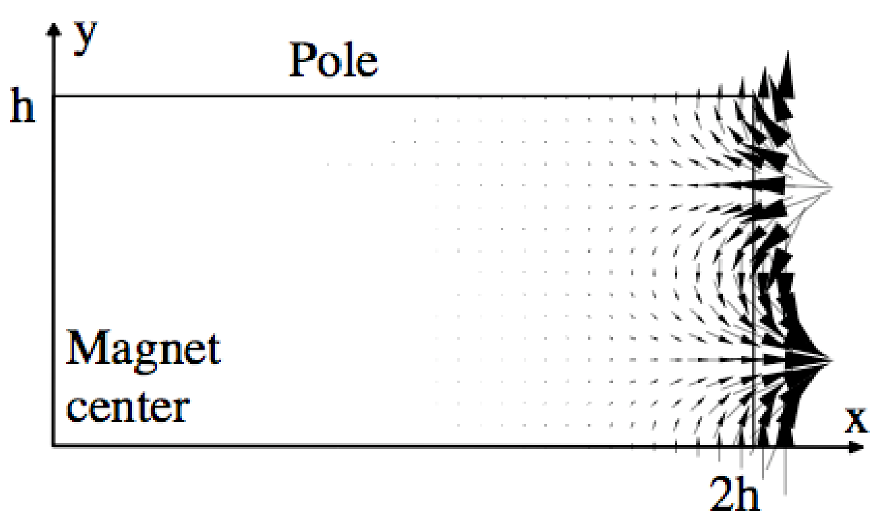
$$B_x = \sum_{p=1}^{\infty} \mathcal{U}_p \sinh px \, \sin py \qquad B_x(x,\pm h) = 0, \qquad p = \frac{n\pi}{h}$$

$$B_x(x,y) = \sum_{n=1}^{\infty} \mathcal{U}_n \sinh\left(\frac{n\pi}{h}x\right) \sin\left(\frac{n\pi}{h}y\right),$$

$$B_y(x,y) = U_0 + \sum_{n=1}^{\infty} \mathcal{U}_n \cosh\left(\frac{n\pi}{h}x\right) \cos\left(\frac{n\pi}{h}y\right)$$

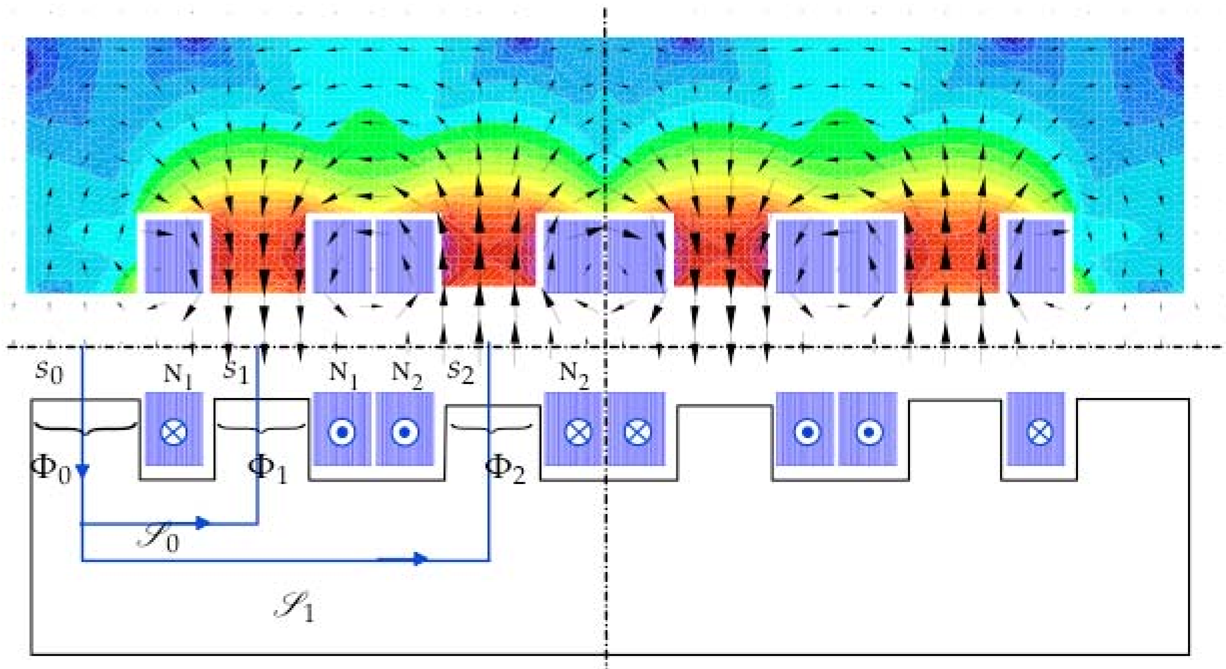
Higher order Harmonics in Cartesian Coordinates





Apparent Inductance

$$\Phi_i = \sum_{j=1}^{n} L_{ij} I_j.$$



$$U = L^{\mathrm{d}} \frac{\mathrm{d}I}{\mathrm{d}t}$$

$$U(t) = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}(L\,I)}{\mathrm{d}t} = L\frac{\mathrm{d}I}{\mathrm{d}t} + I\frac{\mathrm{d}L}{\mathrm{d}t}$$

$$\mathrm{d}L = \frac{\partial L}{\partial I}\mathrm{d}I + \frac{\partial L}{\partial t}\mathrm{d}t$$

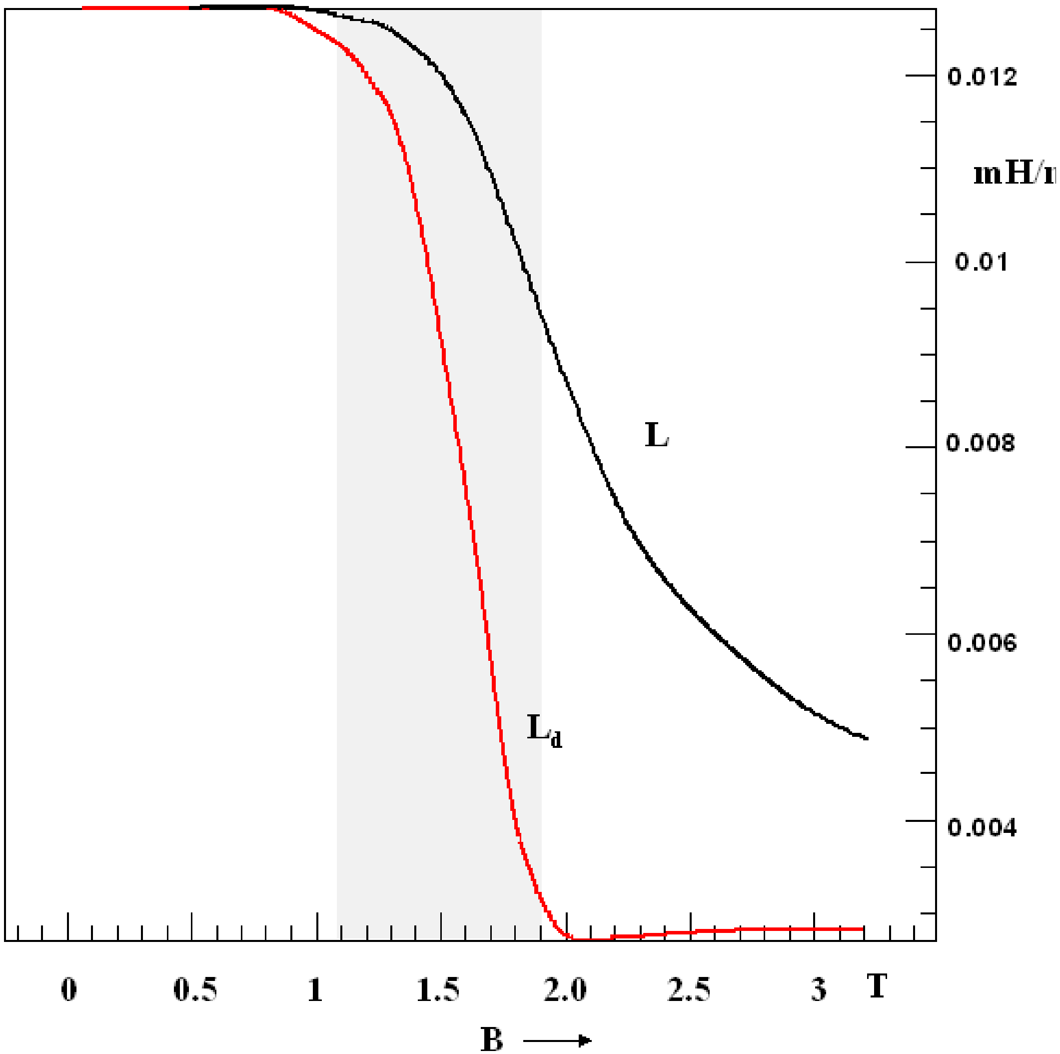
$$U(t) = \left(\frac{\partial L}{\partial I}I + L\right)\frac{\mathrm{d}I}{\mathrm{d}t} + I\frac{\partial L}{\partial t}$$

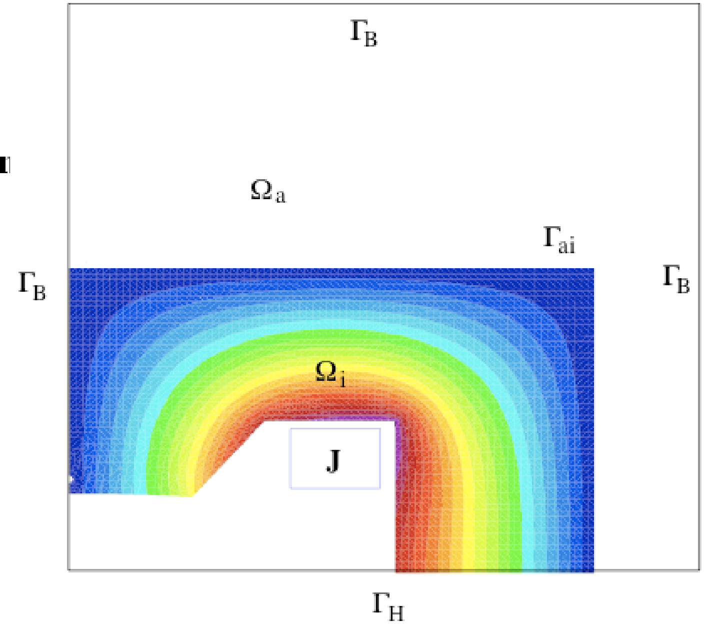
$$L^{\mathrm{d}} = L + I\frac{\partial L}{\partial I} = \frac{\mathrm{d}\Phi}{\mathrm{d}I}$$

$$W = \frac{1}{2}L^{\mathrm{W}}I^2$$

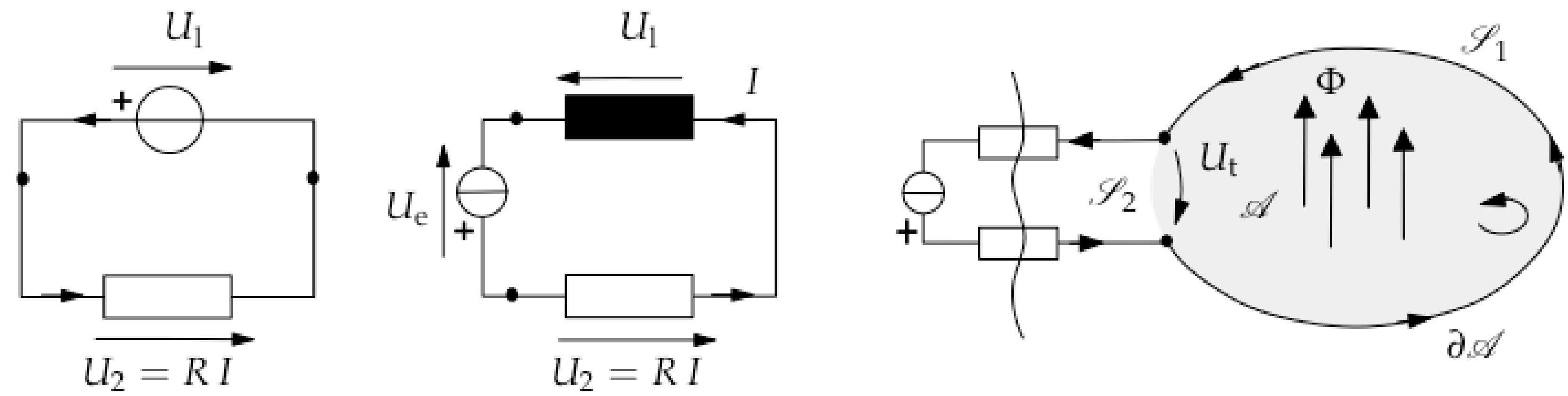
$$W = \int_0^t U\,I\,\mathrm{d}\tau = \int_0^t \frac{\mathrm{d}\Phi(I(\tau))}{\mathrm{d}\tau}I(\tau)\,\mathrm{d}\tau = \int_0^t \frac{\mathrm{d}\Phi(I(\tau))}{\mathrm{d}I}\frac{\mathrm{d}I(\tau)}{\mathrm{d}\tau}I(\tau)\,\mathrm{d}\tau$$
$$= \int_0^{I(t)} \frac{\mathrm{d}\Phi}{\mathrm{d}I}I\,\mathrm{d}I = \int_0^{I(t)} L^{\mathrm{d}}\,I\,\mathrm{d}I\,,$$

$$L^{\mathrm{W}} = \frac{2}{I(t)^2}\int_0^{I(t)} L^{\mathrm{d}}\,I\,\mathrm{d}I = \int_0^1 2\lambda L^{\mathrm{d}}(\lambda I)\,\mathrm{d}\lambda$$


0.012
mH/m
0.01
0.008
0.006
0.004
L
L_d
0
0.5
1
1.5
2.0
2.5
3
T
B


Γ_B
Ω_a
Γ_{ai}
Γ_B
Γ_B
Ω_i
J
Γ_H

Sign conventions



$$U(\partial\mathscr{A}) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathscr{A})$$

Faradays law

Generator case

$$U_1 = U_2 = U_t = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathscr{A})$$

Load case

$$U_1 = U_e - U_2 = -U_t = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathscr{A})$$

Maxwell stresses

$$F_{\mathrm{m}} = \int_{\mathscr{V}} \frac{1}{\mu_0} (\operatorname{curl} \mathbf{B}) \times \mathbf{B} \, \mathrm{d}V$$

$$(\operatorname{curl} \mathbf{B}) \times \mathbf{B} = \operatorname{div} \mathbf{S}_{\mathrm{m}} - \mathbf{B} \operatorname{div} \mathbf{B}$$

$$\mathbf{S}_{\mathrm{m}} := (\sigma_{ij}) = \begin{pmatrix} B_x^2 - \frac{1}{2}|\mathbf{B}|^2 & B_x B_y & B_x B_z \\ B_y B_x & B_y^2 - \frac{1}{2}|\mathbf{B}|^2 & B_y B_z \\ B_z B_x & B_z B_y & B_z^2 - \frac{1}{2}|\mathbf{B}|^2 \end{pmatrix}$$

$$\mathbf{F}_{\mathrm{m}} = \int_{\mathscr{V}} \frac{1}{\mu_0} \operatorname{div} \mathbf{S}_{\mathrm{m}} \, \mathrm{d}V = \int_{\partial \mathscr{V}} \frac{1}{\mu_0} \mathbf{S}_{\mathrm{m}} \cdot \mathbf{n} \, \mathrm{d}a$$

$$\mathbf{S}_{\mathrm{m}} \cdot \mathbf{n} = (\mathbf{B} \cdot \mathbf{n})\mathbf{B} - \frac{1}{2}|\mathbf{B}|^2 \mathbf{n}$$

$$\mathbf{F}_{\mathrm{m}} = \int_{\partial \mathscr{V}} \left(\frac{1}{\mu_0} (\mathbf{B} \cdot \mathbf{n})\mathbf{B} - \frac{1}{2\mu_0} |\mathbf{B}|^2 \mathbf{n} \right) \mathrm{d}a$$

$$X \subseteq \mathbb{R}^n \qquad (x_1, x_2, ..., x_n)^T \in X$$

$$\min\{f(\mathbf{x})\} \qquad f : X \rightarrow \mathbb{R}$$

Subject to

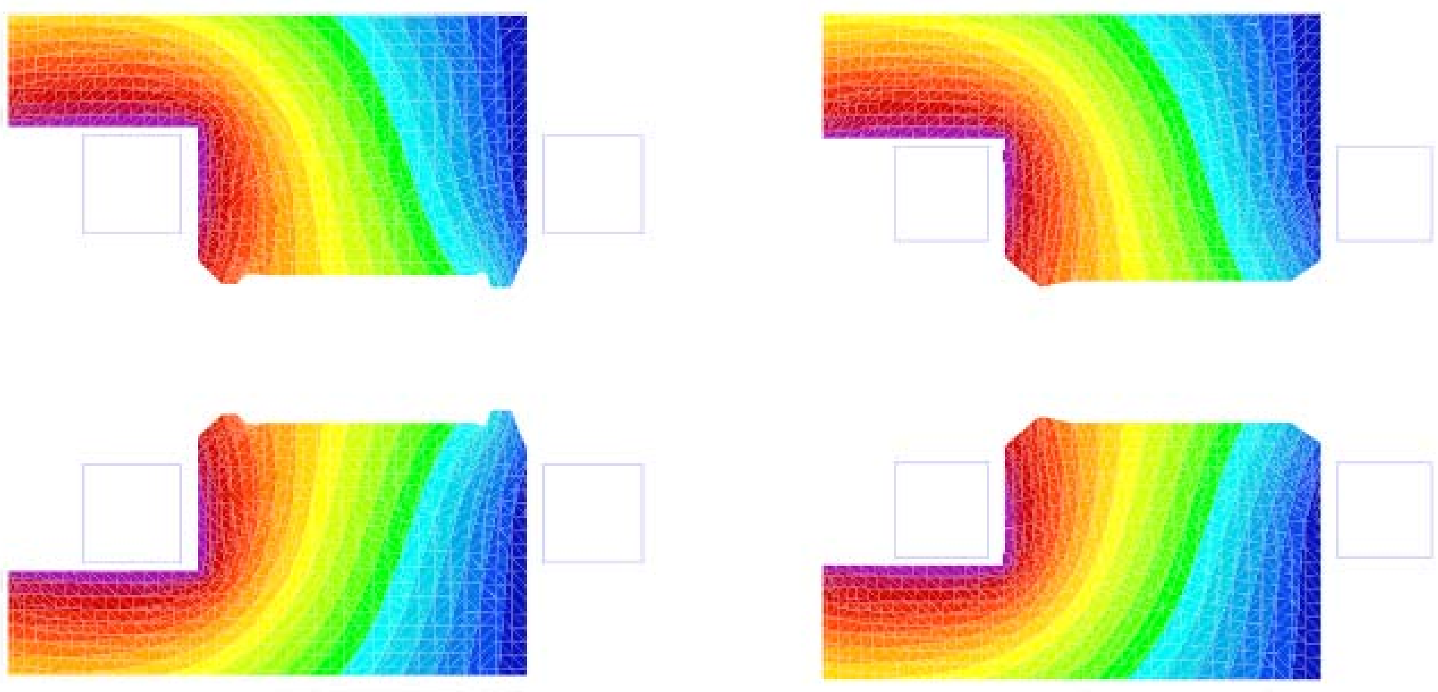
$$g_i(\mathbf{x}) \leq 0, \qquad i = 1, 2, ..., m,$$

$$h_j(\mathbf{x}) = 0, \qquad j = 1, 2, ..., p,$$

$$x_{l,\text{lower}} < x_l < x_{l,\text{upper}}, \qquad l = 1, 2, ..., n$$

$$M = \{\mathbf{x} \in X | g_i(\mathbf{x}) \leq 0;\ h_j(\mathbf{x}) = 0;\ x_{l,\text{lower}} < x_l < x_{l,\text{upper}}; \\ \forall\, i = 1, .., m;\ j = 1, .., p;\ l = 1, ..n\}.$$

$$\min\{f(\mathbf{x}) | \mathbf{x} \in M\}$$



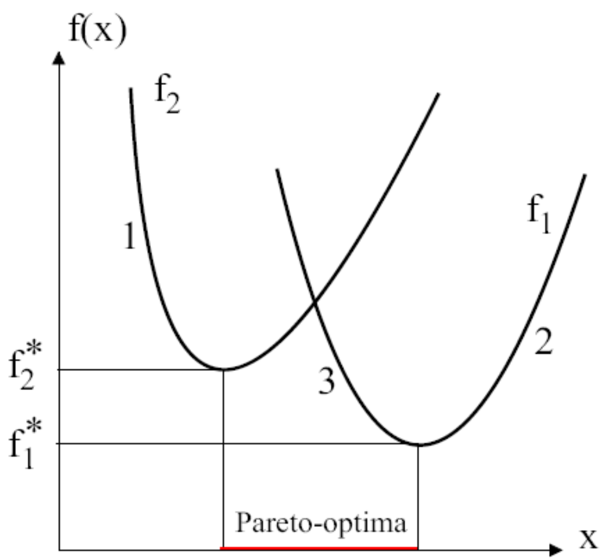
Pareto Optimality

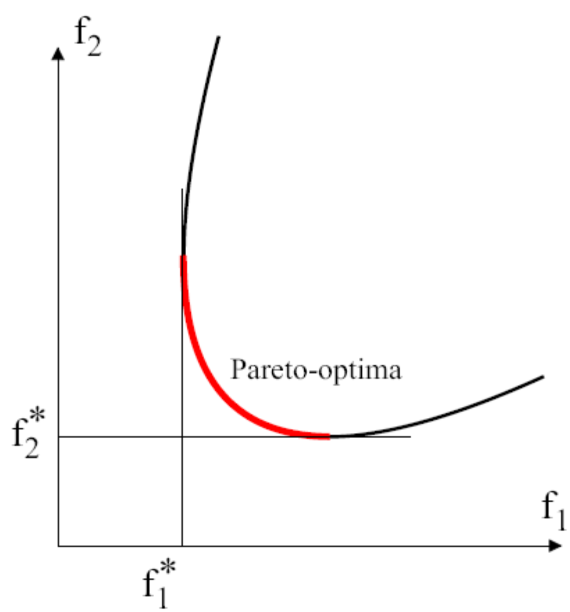
$$\mathrm{MIN}\,\{\mathbf{f}(\mathbf{x})\} = \mathrm{MIN}\,\{f_1(\mathbf{x}), f_2(\mathbf{x}), .., f_K(\mathbf{x})\,\}$$

A Pareto optimal solution X* is given if there exists no solution with

$$f_k(\mathbf{x}) \leq f_k(\mathbf{x}^*) \qquad \forall\, k \in [1, K],$$

$$f_k(\mathbf{x}) < f_k(\mathbf{x}^*) \qquad \text{for at least one } k \in [1, K]$$





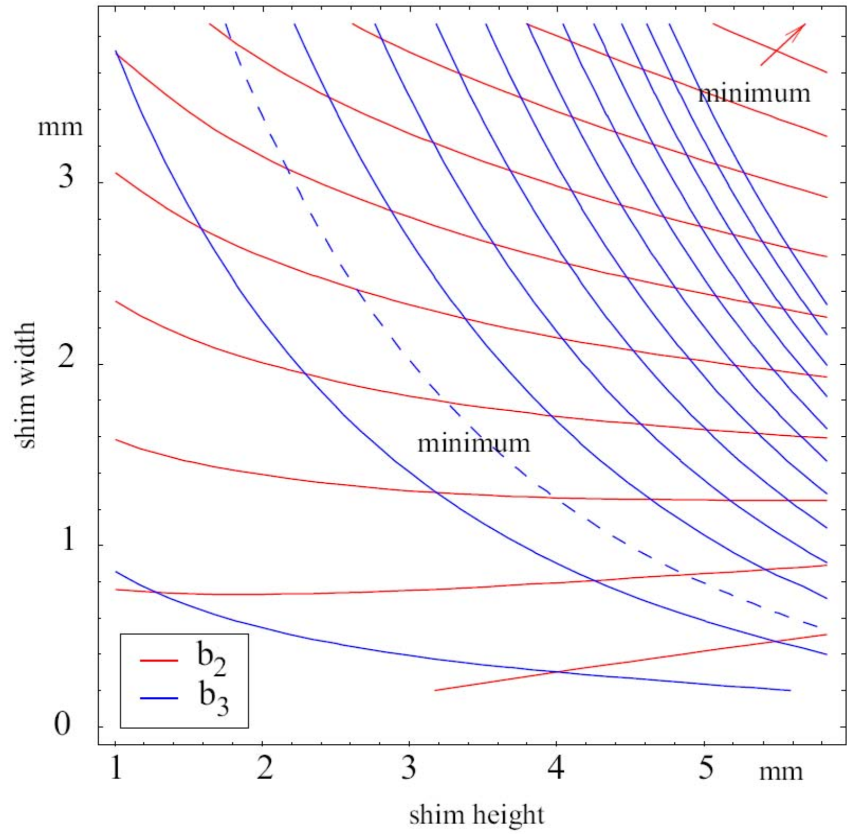
Real World Optimization Problems

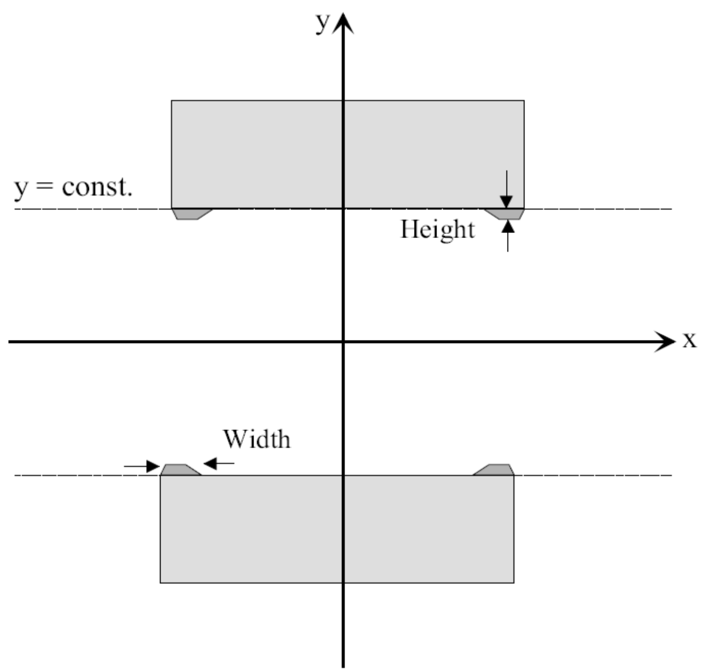
- There are only Pareto-optimal solutions
  - Decision making
  - Treatment of nonlinear constraints
  - Optimization algorithms

- The objective conflict is the characteristic of real world optimization problems

- Fuzzy objectives in the concept phase

Objective Conflict

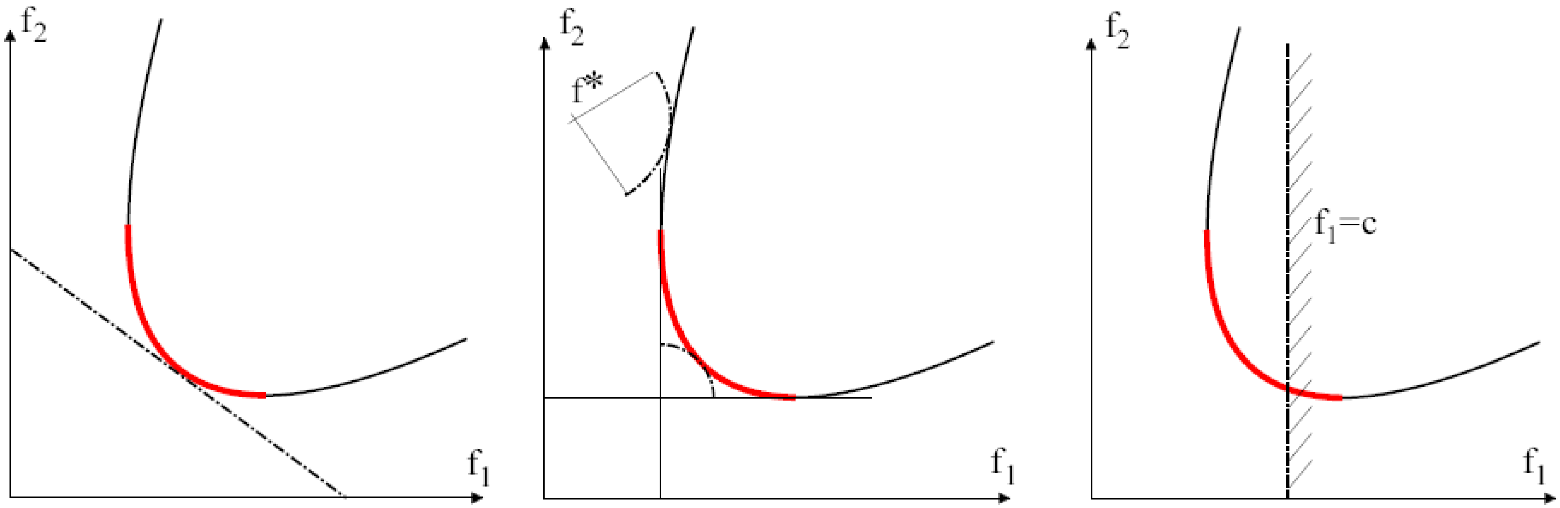




$$\min\left\{u(\mathbf{f}(\mathbf{x})) := \sum_{k=1}^{K} t_k\, f_k(\mathbf{x}) \;|\, \mathbf{x} \in M\right\}$$

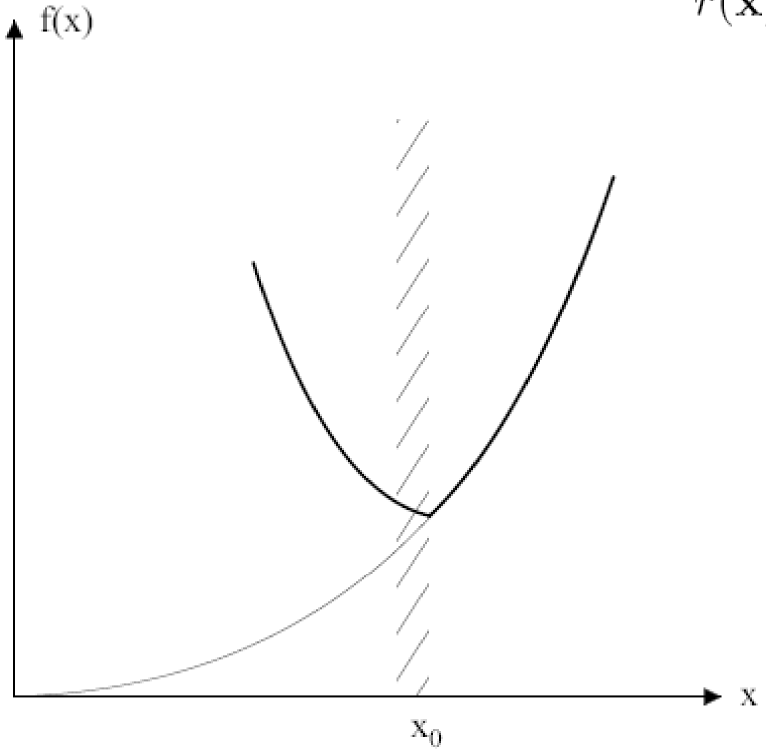
$$\min\left\{\|\,\mathbf{z}(\mathbf{x})\,\|_2^2 := \sum_{k=1}^{K} (t_k(f_k^*(\mathbf{x}) - f_k(\mathbf{x})))^2 \;|\, \mathbf{x} \in M\right\}$$

$$\min\{f_i(\mathbf{x})\} \quad \text{s.t.} \quad f_k(\mathbf{x}) - r_k \le 0$$

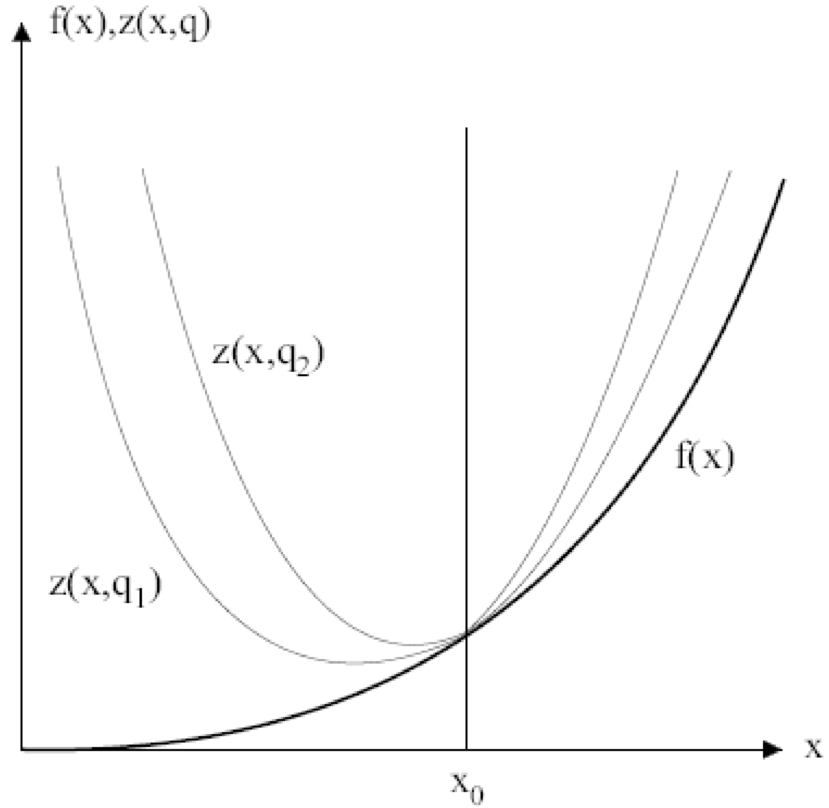


$$p(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{no bound violated} \\ f(\mathbf{x}^*) + r(\mathbf{x}) & \text{bound violated} \end{cases}$$

$$r(\mathbf{x}) = \sum_l r_l \begin{cases} (x_l - x_{l,\text{upper}})^2 & \text{if } x_l > x_{l,\text{upper}} \\ (x_{l,\text{lower}} - x_l)^2 & \text{if } x_l < x_{l,\text{lower}} \\ 0 & \text{otherwise} \end{cases}$$



Penalty Transformation



$$z(\mathbf{x},\mathbf{p},\mathbf{q}) = f_i(\mathbf{x}) + \sum_{k=1}^{m+K-1} p_k \cdot \max{}^2\{(0.\,,\ g_k(\mathbf{x}) - d_k)\} + \sum_{j=1}^{p} q_j (h_j(\mathbf{x}) - c_j)^2$$

Optimization Algorithms

Search methods		
Direct search	Gauss-Seidel	
EXTREM	Jacob	1982
Rosenbrock	Rosenbrock	1960
Powell	Powell	1965
Flexible Polyhedron search	Nelder-Mead	1964
Hooke-Jeeves	Hooke-Jeeves	1962
Gradient methods		
Steepest descend	Cauchy	1847
Newton's method	Newton	1700
Levenberg-Marquard	Levenberg Marquard	1963
Conjugate gradient (CG)	Fletcher-Reeves	1964
Quasi-Newton	Davidon-Fletcher-Powell	1959
Stochastic and neural computing		
Evolutionary	Rechenberg	1964
Genetic algorithms	Fogel-Holland	1987
Neural computing (ANN)	Aarts-Korst	1989

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