Foundations of Analytical and Numerical Field Computation

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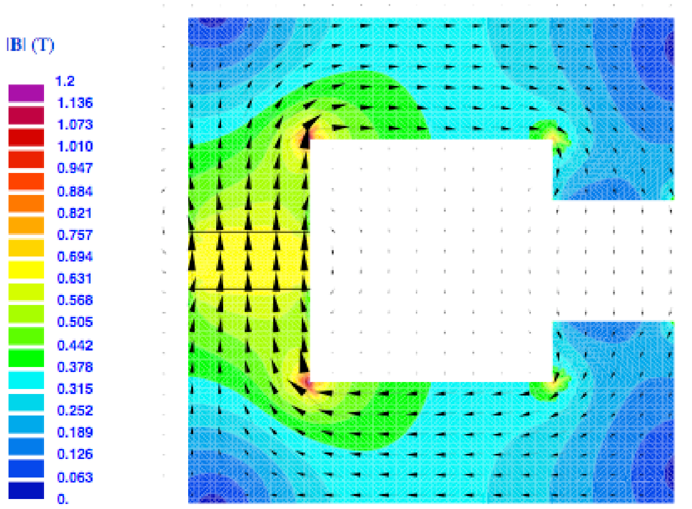
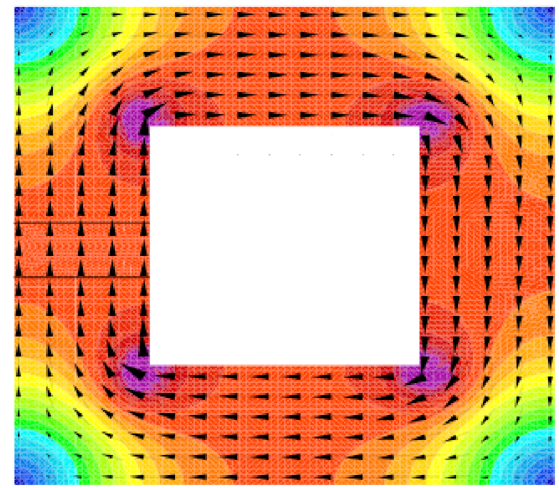
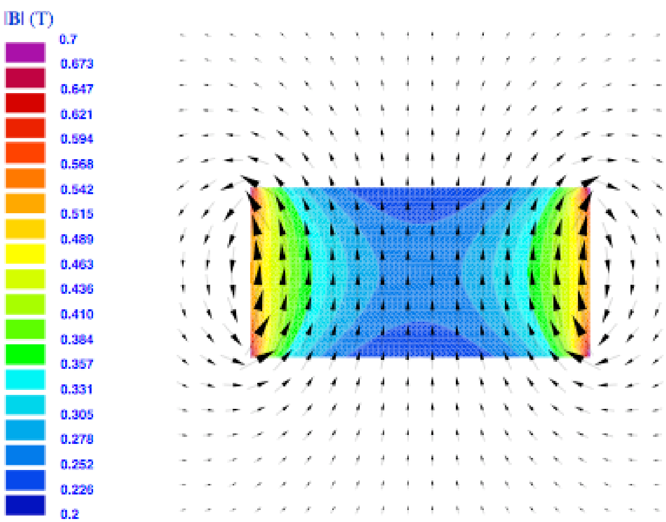
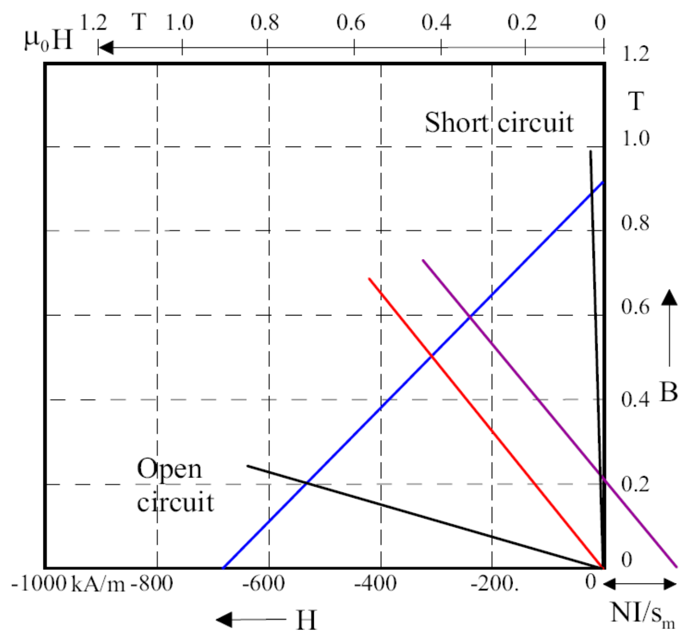
Permanent Magnet Circuits

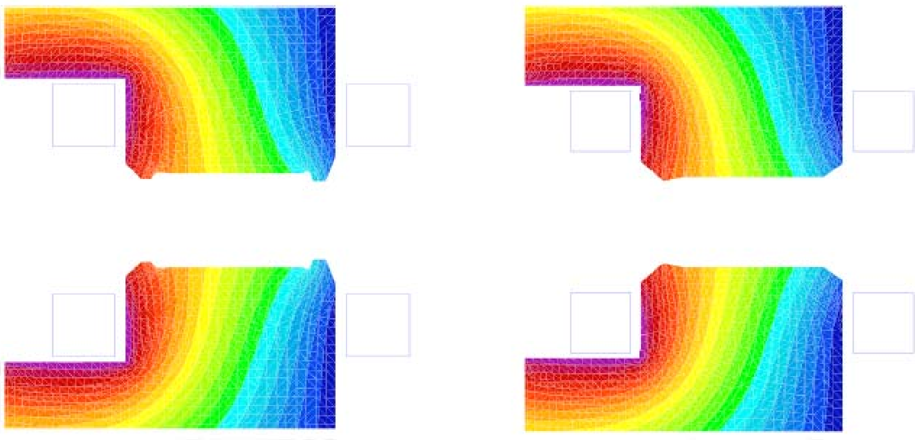




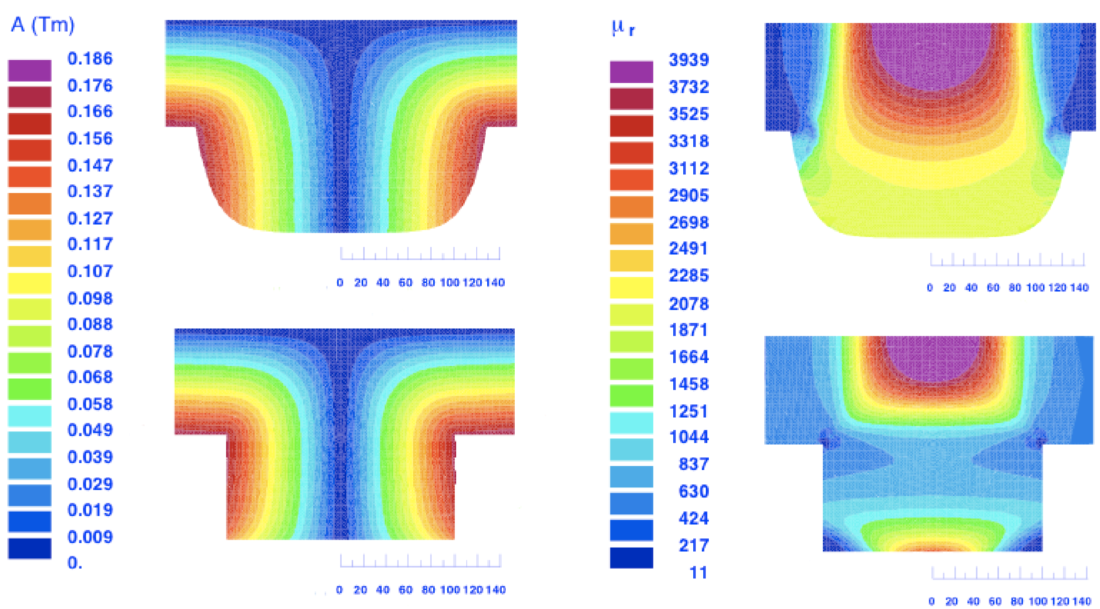




$$s = \frac{B_\mathrm{m}}{H_\mathrm{m}} = \mu_0 P = -\mu_0 \frac{s_\mathrm{m}}{s_0} \frac{a_0}{a_\mathrm{m}}$$



Pole shimming



Rogowski profiles

Different Incarnations of Maxwell's Equations

$$\int_{\partial a} \vec{H}\cdot \mathrm{d}\vec{s} = \int_a \vec{J}\cdot \mathrm{d}\vec{a} + \frac{\mathrm{d}}{\mathrm{d}t}\int_a \vec{D}\cdot \mathrm{d}\vec{a}$$
$$\int_{\partial a} \vec{E}\cdot \mathrm{d}\vec{s} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_a \vec{B}\cdot \mathrm{d}\vec{a}$$
$$\int_{\partial V} \vec{B}\cdot \mathrm{d}\vec{a} = 0$$
$$\int_{\partial V} \vec{D}\cdot \mathrm{d}\vec{a} = \int_V \rho \,\mathrm{d}V$$

$$\begin{aligned} \mathrm{d}H &= J + \partial_t D \\ \mathrm{d}E &= -\partial_t B \\ \mathrm{d}B &= 0 \\ \mathrm{d}D &= \rho \end{aligned}$$

$$V_{\mathrm{m}}(\partial a) = I(a) + \frac{\mathrm{d}}{\mathrm{d}t}\Psi(a)$$
$$U(\partial a) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(a)$$
$$\Phi(\partial V) = 0$$
$$\Psi(\partial V) = Q(V)$$

Integral form

Global form

$$\mathrm{curl}\,\vec{H} = \vec{J} + \partial_t \vec{D}$$
$$\mathrm{curl}\,\vec{E} = -\partial_t \vec{B}$$
$$\mathrm{div}\,\vec{B} = 0$$
$$\mathrm{div}\,\vec{D} = \rho$$

Local form

$$a = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}$$
$$b = \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}$$
$$c = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}$$

$$p + \frac{\partial f}{\partial t} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}$$
$$q + \frac{\partial g}{\partial t} = \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}$$
$$r + \frac{\partial h}{\partial t} = \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}$$

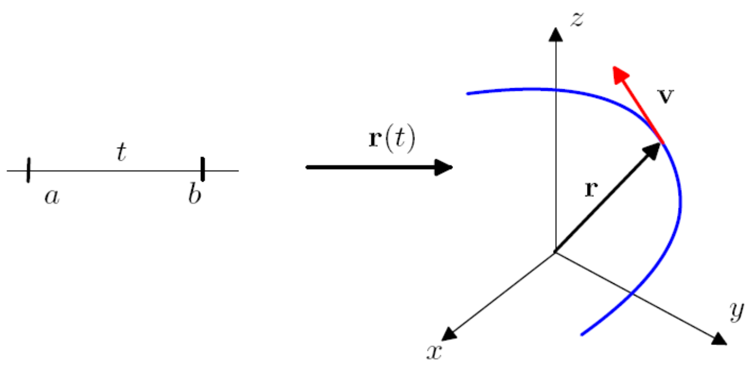
$$P = -\frac{\partial F}{\partial t} - \frac{\partial \varphi}{\partial x}$$
$$Q = -\frac{\partial G}{\partial t} - \frac{\partial \varphi}{\partial y}$$
$$R = -\frac{\partial H}{\partial t} - \frac{\partial \varphi}{\partial z}$$

$$\rho = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\begin{aligned} \mathfrak{B} &= V.\nabla\mathfrak{U} \\ \mathfrak{E} &= V.\dot{\rho}\mathfrak{B} - \dot{\mathfrak{U}} - \nabla\psi \\ \mathfrak{C} &= C\mathfrak{E} + \dot{\mathfrak{D}} \\ \mathfrak{B} &= \mathfrak{H} + 4\pi\mathfrak{J} \\ 4\pi\mathfrak{C} &= V.\nabla\mathfrak{H} \\ \mathfrak{D} &= \frac{1}{4\pi}\varkappa\mathfrak{E} \end{aligned}$$

Directional Derivative

Space curve with $\mathbf{r}(t) = (x(t), y(t), z(t))$ parametrized such that $\mathbf{r}(0) = P$.



1-smooth scalar field $\phi : E_3 \to R : \mathbf{r} \mapsto \phi(\mathbf{r})$ expressed as $\phi(x, y, z)$, then $\phi(\mathbf{r}(t))$ at parameter (time) t.

$$\partial_{\mathbf{v}}\phi = \frac{\partial \phi}{\partial v} = \frac{\mathrm{d}}{\mathrm{d}t}[\phi(\mathbf{r} + t\mathbf{v})]_{t=0} = \lim_{t \to 0} \frac{\phi(\mathbf{r} + t\mathbf{v}) - \phi(\mathbf{r})}{t}$$

$$\partial_{\mathbf{v}}\phi = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\mathbf{r}(t)) = \frac{\partial \phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{grad}\,\phi \cdot \mathbf{v}$$

$$\mathrm{grad}\phi = \frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

The Differential Operators

$$\nabla = \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y + \frac{\partial}{\partial z}\mathbf{e}_z$$

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial y} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla \phi = \mathsf{grad}\,\phi = \frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

$$\nabla \cdot \mathbf{a} = \mathsf{div}\,\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

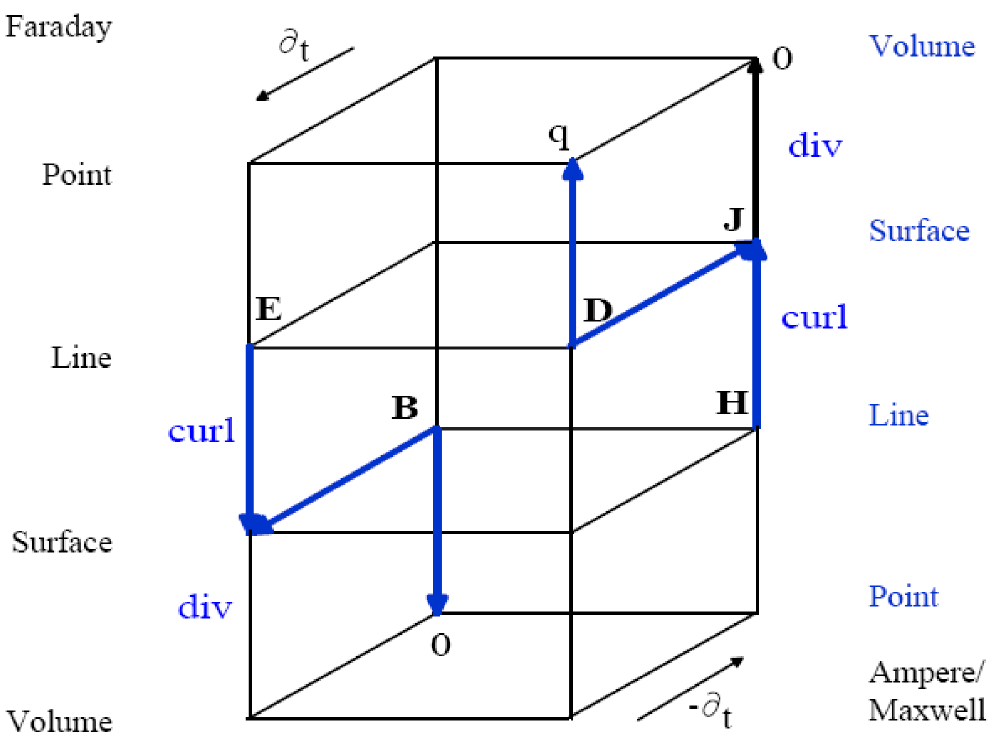
$$\nabla \times \mathbf{a} = \mathsf{curl}\,\mathbf{a} = (\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z})\mathbf{e}_x + (\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x})\mathbf{e}_y + (\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y})\mathbf{e}_z$$

$$\nabla^2 \mathbf{A} = (\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2})\mathbf{e}_x + (\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2})\mathbf{e}_y + (\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2})\mathbf{e}_z = (\nabla^2 A_x)\mathbf{e}_x + (\nabla^2 A_y)\mathbf{e}_y + (\nabla^2 A_z)\mathbf{e}_z$$

Conclusion: This is horrible, so let's try the geometrical approach

Maxwell's House

$$\mathbf{v} \cdot \operatorname{grad} \phi = \lim_{s \to 0} \frac{\phi(P_2) - \phi(P_1)}{s}$$



$$\operatorname{div} \mathbf{g} = \lim_{V \to 0} \frac{\int_{\partial V} \mathbf{g} \cdot \mathbf{da}}{V}$$

$$\mathbf{n} \cdot \operatorname{curl} \mathbf{g} = \lim_{a \to 0} \frac{\int_{\partial a} \mathbf{g} \cdot \mathrm{ds}}{a}$$

Maxwell's Equations in Differential Form

$$\int_a \operatorname{curl}\vec{g} \cdot d\vec{a} = \int_{\partial a} \vec{g} \cdot d\vec{s}$$

$$\int_V \operatorname{div} \vec{g}\, dV = \int_{\partial V} \vec{g} \cdot d\vec{a}$$

$$\int_{\partial a} \vec{H} \cdot d\vec{s} = \int_a \vec{J} \cdot d\vec{a} + \frac{d}{dt} \int_a \vec{D} \cdot d\vec{a}$$

$$\int_{\partial a} \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int_a \vec{B} \cdot d\vec{a}$$

$$\int_{\partial V} \vec{B} \cdot d\vec{a} = 0$$

$$\int_{\partial V} \vec{D} \cdot d\vec{a} = \int_V \rho\, dV$$

$$\int_a \operatorname{curl}\vec{H} \cdot d\vec{a} = \int_a (\vec{J} + \frac{\partial}{\partial t}\vec{D}) \cdot d\vec{a}$$

$$\int_a \operatorname{curl}\vec{E} \cdot d\vec{a} = -\int_a \frac{\partial}{\partial t}\vec{B} \cdot d\vec{a}$$

$$\int_V \operatorname{div} \vec{B}\, dV = 0$$

$$\int_V \operatorname{div} \vec{D}\, dV = \int_V \rho\, dV$$

$$\operatorname{curl}\vec{H} = \vec{J} + \partial_t \vec{D}$$

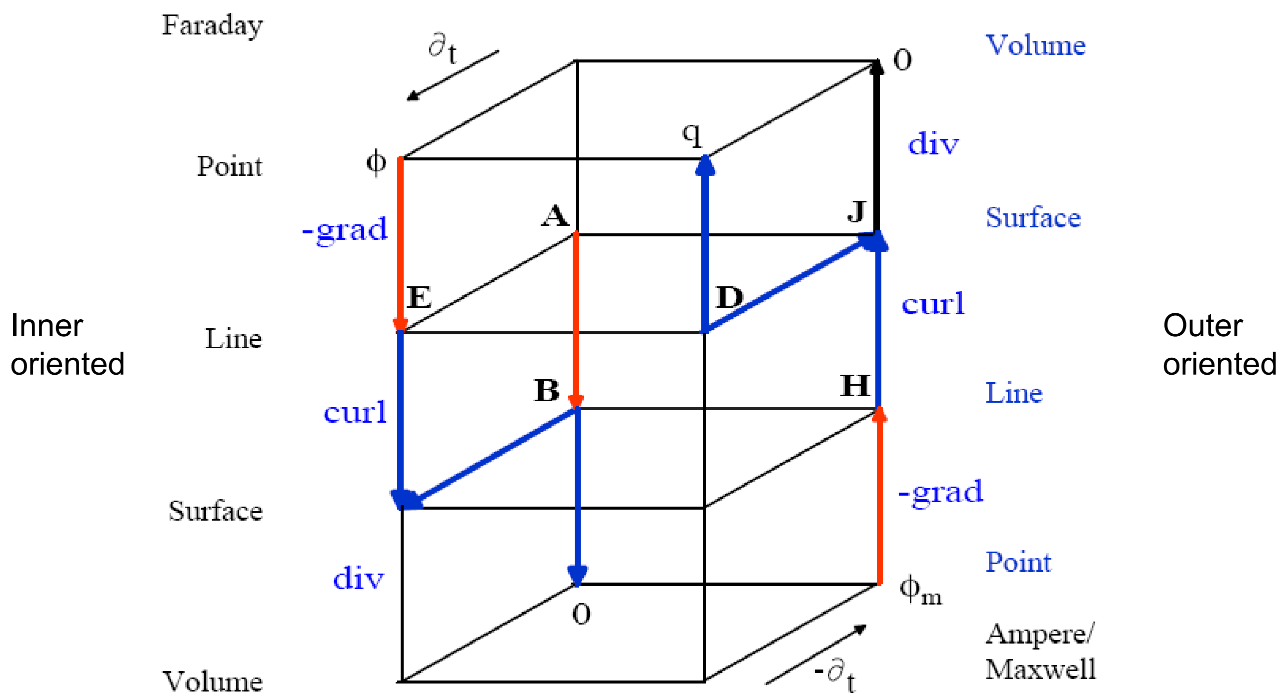
$$\operatorname{curl}\vec{E} = -\partial_t \vec{B}$$

$$\operatorname{div} \vec{B} = 0$$

$$\operatorname{div} \vec{D} = \rho$$

Maxwell's House

$$\int_V \mathrm{div}\ \mathrm{curl}\, \mathbf{g} dV = \int_{\partial V} \mathrm{curl}\, \mathbf{g} \cdot \mathbf{da} = \int_{\partial(\partial V)} \mathbf{g} \cdot \mathbf{ds} = 0$$



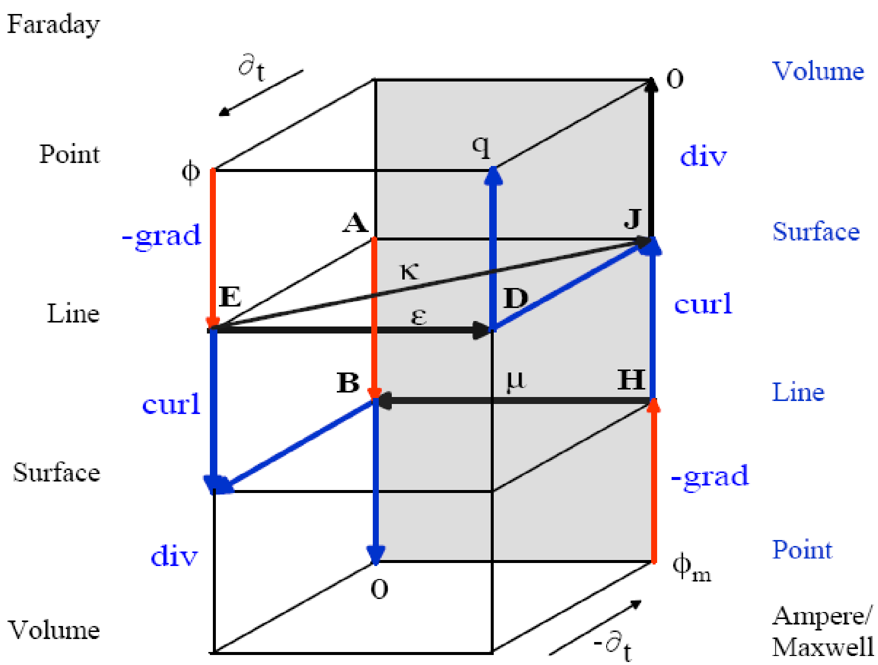
$$\int_a \mathrm{curl}\ \mathrm{grad}\, \phi \cdot \mathbf{da} = \int_{\partial a} \mathrm{grad}\, \phi \cdot \mathbf{ds} = \phi|_{\partial(\partial a)} = 0$$

Would be even more symmetric with magnetic monopoles

$$\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A} = \mathbf{J} \qquad \frac{1}{\mu}\,\mathrm{curl}\;\mathrm{curl}\,\mathbf{A} = 0 \qquad \nabla^2\mathbf{A} - \;\mathrm{grad}\;\mathrm{div}\,\mathbf{A} = 0 \qquad \nabla^2 A_z = 0$$

Constant permeablity and no sources

Only for Cartesian components



$$\mathrm{div}\,\mu\,\mathrm{grad}\,\phi_\mathrm{m} = 0 \qquad \mu_0\,\mathrm{div}\;\mathrm{grad}\,\phi_\mathrm{m} = 0 \qquad \nabla^2\phi_\mathrm{m} = 0$$

No sources

$$\nabla^2 A_z = -\mu_0 J_z \quad \text{in } \Omega_\mathrm{a}$$

$$\nabla^2 A_z = 0 \quad \text{in } \Omega_\mathrm{a}, \mathbf{J} = 0$$

$$r^2 \frac{\partial^2 A_z}{\partial r^2} + r \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0 \quad \text{in } \Omega_\mathrm{a}, \mathbf{J} = 0$$

$$A_z = R(r)\phi(\varphi)$$

$$\frac{\partial A_z}{\partial r} = \frac{\partial R(r)}{\partial r}\phi(\varphi),$$

$$\frac{\partial^2 A_z}{\partial r^2} = \frac{\partial^2 R(r)}{\partial r^2}\phi(\varphi),$$

$$\frac{\partial^2 A_z}{\partial \varphi^2} = \frac{\partial^2 \phi(\varphi)}{\partial \varphi^2}R(r).$$

$$\underbrace{\frac{1}{R(r)}\left(r^2 \frac{\partial^2 R(r)}{\partial r^2} + r \frac{\partial R(r)}{\partial r}\right)}_{n^2} = \underbrace{-\frac{1}{\phi(\varphi)}\frac{\partial^2 \phi(\varphi)}{\partial \varphi^2}}_{n^2}$$

$$r^2 \frac{\mathrm{d}^2 R(r)}{\mathrm{d}r^2} + r \frac{\mathrm{d}R(r)}{\mathrm{d}r} - n^2 R(r) = 0$$

$$\frac{\mathrm{d}^2 \phi(\varphi)}{\mathrm{d}\varphi^2} + n^2 \phi(\varphi) = 0$$

How do you solve differential equations: Look them up in a book

$$R(r) = \mathcal{E}\, r^n + \mathcal{F}\, r^{-n},$$
$$\phi(\varphi) = \mathcal{G}\, \sin n\varphi + \mathcal{H}\, \cos n\varphi.$$

$$A_z(r,\varphi) = \sum_{n=1}^{\infty} (\mathcal{E}_n r^n + \mathcal{F}_n r^{-n})(\mathcal{G}_n \sin n\varphi + \mathcal{H}_n \cos n\varphi)$$
$$= \sum_{n=1}^{\infty} r^n (\mathcal{C}_n \sin n\varphi - \mathcal{D}_n \cos n\varphi)$$

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{C}_n \cos n\varphi + \mathcal{D}_n \sin n\varphi),$$
$$B_\varphi(r,\varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} n r^{n-1} (\mathcal{C}_n \sin n\varphi - \mathcal{D}_n \cos n\varphi).$$

What have we won? If we know the field at a reference radius, we know it everywhere inside

Multipoles and Scaling Laws

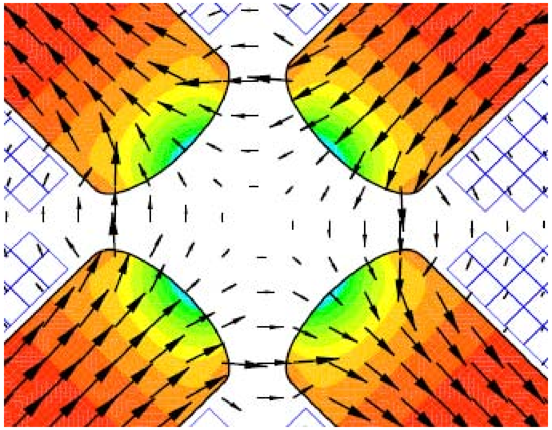
$$A_n = n r_0^{n-1} \mathcal{C}_n \qquad \text{and} \qquad B_n = n r_0^{n-1} \mathcal{D}_n$$

$$\boxed{\begin{aligned} B_r(r_0,\varphi) &= \sum_{n=1}^{\infty} (B_n \sin n\varphi + A_n \cos n\varphi) = B_N \sum_{n=1}^{\infty} (b_n \sin n\varphi + a_n \cos n\varphi) \\ B_\varphi(r_0,\varphi) &= \sum_{n=1}^{\infty} (B_n \cos n\varphi - A_n \sin n\varphi) = B_N \sum_{n=1}^{\infty} (b_n \cos n\varphi - a_n \sin n\varphi) \end{aligned}}$$

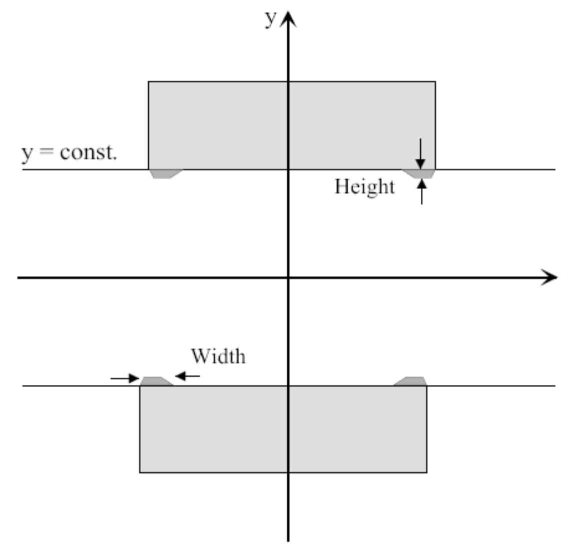
$$A_n(r_1) = (\frac{r_1}{r_0})^{n-1} A_n(r_0), \qquad B_n(r_1) = (\frac{r_1}{r_0})^{n-1} B_n(r_0).$$

$$b_n(r_1) = \frac{B_n(r_1)}{B_N(r_1)} = \frac{(\frac{r_1}{r_0})^{n-1} B_n(r_0)}{(\frac{r_1}{r_0})^{N-1} B_N(r_0)} = (\frac{r_1}{r_0})^{n-N} b_n(r_0),$$

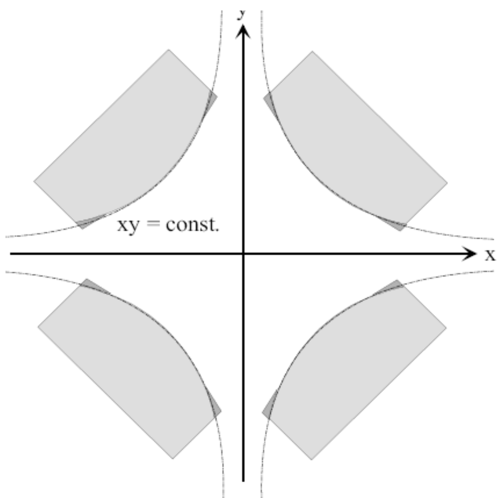
Ideal Pole Shape of Conventional Magnets

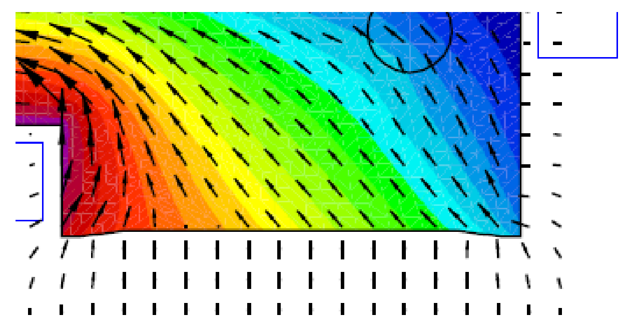


$$\phi_{\mathrm{m}} = \mathcal{C}_1 x + \mathcal{D}_1 y$$



$$\phi_{\mathrm{m}} = \mathcal{C}_2(x^2 - y^2) + 2\mathcal{D}_2 xy$$



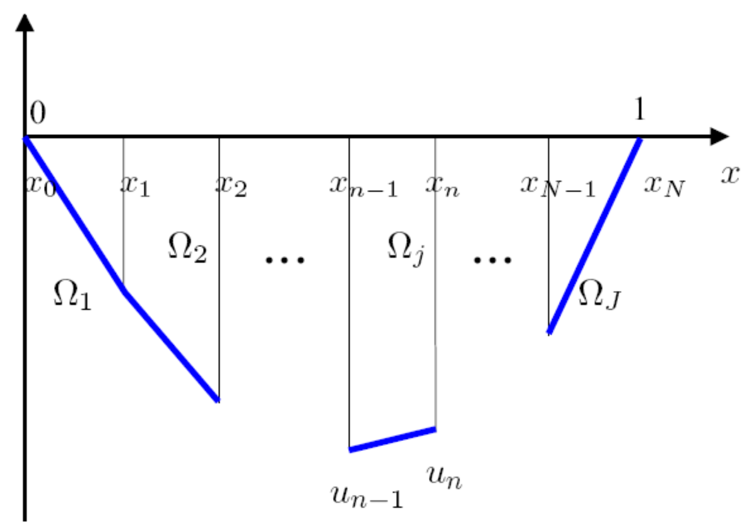


- Principles of numerical field computation
  - Formulation of the Problem
  - Weighted residual
  - Weak form
  - Discretization
  - Numerical example
- Total vector potential formulation
  - Weak form in 3-D
- Element shape functions
  - Global shape functions
  - Barycentric coordinates
- Mesh generation

Accelerator Technology Department

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = f(x), \qquad x \in \Omega$$

$$u(x)|_{x=0} = \overline{u}_0 \qquad u(x)|_{x=1} = \overline{u}_1 \quad \text{or} \quad \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=1} = \overline{q}_1$$



$$u(x) = \frac{C}{2}\left(x^2 - x\right)$$

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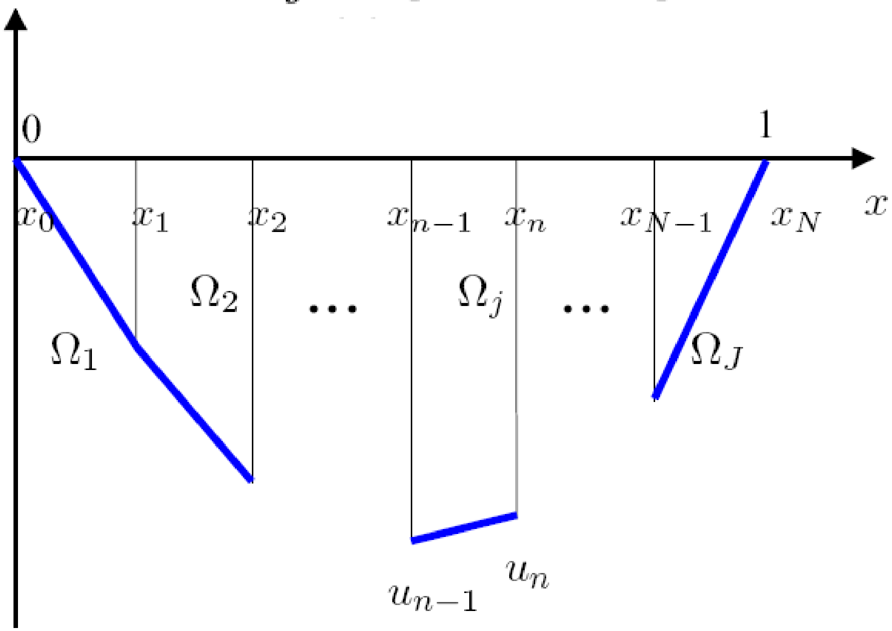
$$\Omega_j = [x_{n-1}, x_n]$$



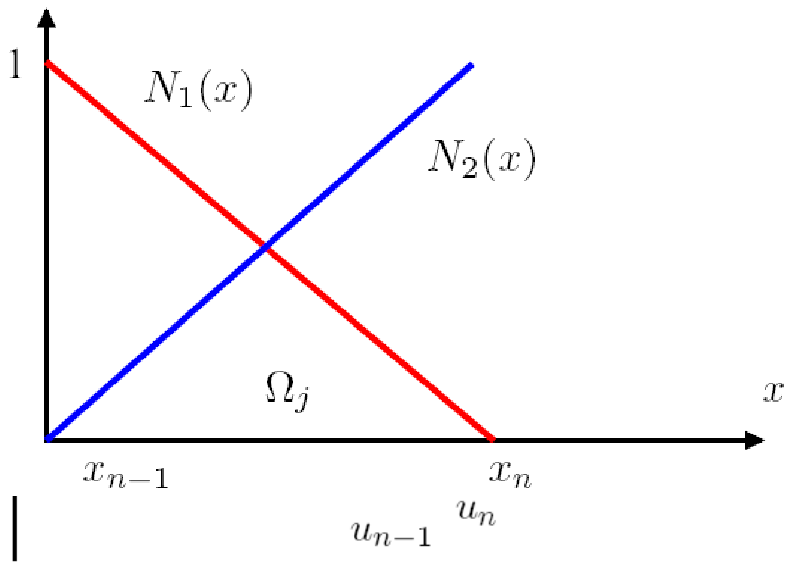
$$\Omega = \bigcup_{j=1}^{J} \Omega_j$$

$$u_j(x) = \alpha_{j1} + \alpha_{j2} x \qquad x \in \Omega_j$$

$$u_{n-1} = \alpha_{j1} + \alpha_{j2} x_{n-1}$$

$$u_n = \alpha_{j1} + \alpha_{j2} x_n$$

Shape Functions



$$\alpha_{j1} = \frac{\begin{vmatrix} u_{n-1} & x_{n-1} \\ u_n & x_n \end{vmatrix}}{\begin{vmatrix} 1 & x_{n-1} \\ 1 & x_n \end{vmatrix}}$$

Cramer's rule

$$\alpha_{j1} = \frac{x_n u_{n-1} - x_{n-1} u_n}{x_n - x_{n-1}}$$

$$\alpha_{j2} = \frac{u_n - u_{n-1}}{x_n - x_{n-1}}$$

$$u_j(x) = \alpha_{j1} + \alpha_{j2} x = \frac{x_n - x}{x_n - x_{n-1}} u_{n-1} + \frac{-x_{n-1} + x}{x_n - x_{n-1}} u_n$$

$$\boxed{N_{j1}(x) = \frac{x_n - x}{x_n - x_{n-1}} \qquad N_{j2}(x) = \frac{-x_{n-1} + x}{x_n - x_{n-1}}}$$

What have we won? We can express the field in the element as a function of the node potentials using known polynomials in the spatial coordinates

Accelerator Technology Department

The Weighted Residual

$$R(x) := \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - f(x)$$

$$\int_\Omega w(x) R(x)\,\mathrm{d}\Omega = \int_\Omega w(x) \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2}\,\mathrm{d}\Omega - \int_\Omega w(x) f(x)\,\mathrm{d}\Omega = 0$$

$$\int_a^b \phi\psi'\,\mathrm{d}x = [\phi\psi]_a^b - \int_a^b \phi'\psi\,\mathrm{d}x \qquad w(x) = \phi \qquad \frac{\mathrm{d}u(x)}{\mathrm{d}x} = \psi$$

$$-\int_\Omega \frac{\mathrm{d}w(x)}{\mathrm{d}x}\frac{\mathrm{d}u(x)}{\mathrm{d}x}\,\mathrm{d}\Omega + \left[w(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right]_0^1 - \int_\Omega w(x) f(x)\,\mathrm{d}\Omega = 0$$

What have we won? Removal of the second derivative, a way to incorporate Neumann boundary conditions

Galerkin's Method

$$\int_\Omega \frac{\mathrm{d}w(x)}{\mathrm{d}x}\frac{\mathrm{d}u(x)}{\mathrm{d}x}\,\mathrm{d}\Omega = -\int_\Omega w(x)f(x)\,\mathrm{d}\Omega$$

$$\int_{\Omega_j} \frac{\mathrm{d}w_l(x)}{\mathrm{d}x}\sum_{k=1,2}\frac{\mathrm{d}N_{jk}(x)}{\mathrm{d}x}u^{(k)}\,\mathrm{d}\Omega_j = -\int_{\Omega_j} w_l(x)f(x)\,\mathrm{d}\Omega_j\,, \qquad l=1,2.$$

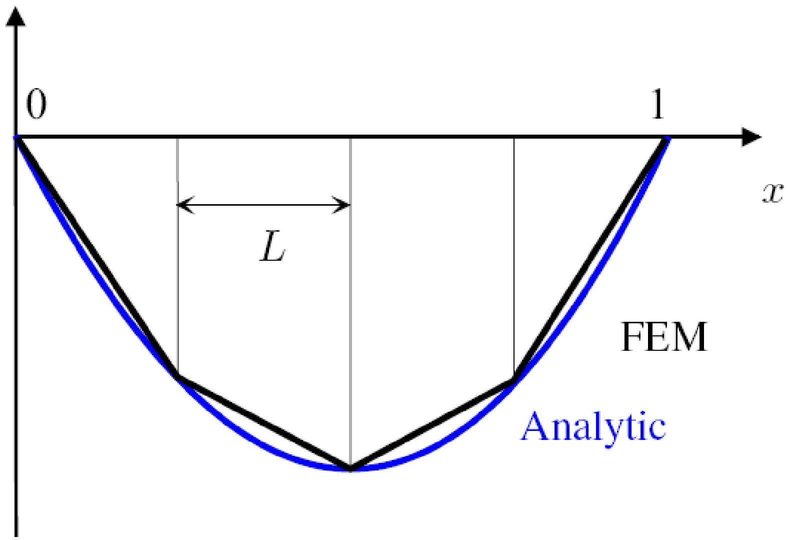
$$\int_{\Omega_j} \frac{\mathrm{d}N_{jl}(x)}{\mathrm{d}x}\sum_{k=1,2}\frac{\mathrm{d}N_{jk}(x)}{\mathrm{d}x}u^{(k)}\,\mathrm{d}\Omega_j = -\int_{\Omega_j} N_{jk}(x)f(x)\,\mathrm{d}\Omega_j\,, \qquad l=1,2$$

$$\int_{x_{n-1}}^{x_n}\left(\frac{\mathrm{d}N_{j1}}{\mathrm{d}x}\frac{\mathrm{d}N_{j1}}{\mathrm{d}x}u_{n-1} + \frac{\mathrm{d}N_{j1}}{\mathrm{d}x}\frac{\mathrm{d}N_{j2}}{\mathrm{d}x}u_n\right)\mathrm{d}x = -\int_{x_{n-1}}^{x_n} N_{j1}f(x)\,\mathrm{d}x$$
$$\int_{x_{n-1}}^{x_n}\left(\frac{\mathrm{d}N_{j2}}{\mathrm{d}x}\frac{\mathrm{d}N_{j1}}{\mathrm{d}x}u_{n-1} + \frac{\mathrm{d}N_{j2}}{\mathrm{d}x}\frac{\mathrm{d}N_{j2}}{\mathrm{d}x}u_n\right)\mathrm{d}x = -\int_{x_{n-1}}^{x_n} N_{j2}f(x)\,\mathrm{d}x$$

$$\boxed{[k_j]\{u_j\} = \{f_j\}}$$

Linear equation system for the node potentials

4 finite elements $\Omega_j, j = 1, ..., 4$ of equidistant length L

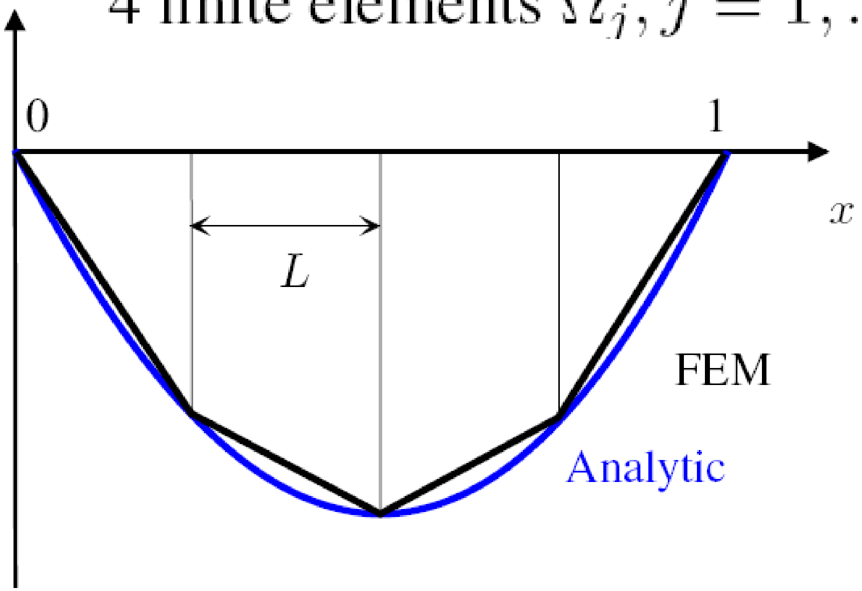


$$[k_j] = \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{dN_{j1}}{dx}\frac{dN_{j1}}{dx} & \frac{dN_{j1}}{dx}\frac{dN_{j2}}{dx} \\ \frac{dN_{j2}}{dx}\frac{dN_{j1}}{dx} & \frac{dN_{j2}}{dx}\frac{dN_{j2}}{dx} \end{pmatrix} dx = \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{1}{(x_n - x_{n-1})^2} & \frac{-1}{(x_n - x_{n-1})^2} \\ \frac{-1}{(x_n - x_{n-1})^2} & \frac{1}{(x_n - x_{n-1})^2} \end{pmatrix} dx = \begin{pmatrix} \frac{1}{L} & \frac{-1}{L} \\ \frac{-1}{L} & \frac{1}{L} \end{pmatrix}$$

$$\{f_j\} = -\int_{x_{n-1}}^{x_n} \begin{pmatrix} N_{j1} \\ N_{j2} \end{pmatrix} C dx = -C \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{x_n - x}{x_n - x_{n-1}} \\ \frac{-x_{n-1} + x}{x_n - x_{n-1}} \end{pmatrix} dx$$

$$= -\frac{C}{2L} \begin{pmatrix} 2x_n x - x^2 \\ -2x_{n-1}x + x^2 \end{pmatrix} \Bigg|_{x_{n-1}}^{x_n} = -\frac{C}{2L} \begin{pmatrix} (x_n - x_{n-1})^2 \\ (x_{n-1} - x_n)^2 \end{pmatrix} = -\begin{pmatrix} 0.5\,CL \\ 0.5\,CL \end{pmatrix}$$

$$4 \text{ finite elements } \Omega_j, j = 1, \ldots, 4 \text{ of equidistant length } L$$



$$\begin{pmatrix} \frac{1}{L} & \frac{-1}{L} & 0 & 0 & 0 \\ \frac{1}{L} & \frac{2}{L} & \frac{-1}{L} & 0 & 0 \\ 0 & \frac{-1}{L} & \frac{2}{L} & \frac{-1}{L} & 0 \\ 0 & 0 & \frac{-1}{L} & \frac{2}{L} & \frac{-1}{L} \\ 0 & 0 & 0 & \frac{-1}{L} & \frac{1}{L} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = - \begin{pmatrix} 0.5\,CL \\ CL \\ CL \\ CL \\ 0.5\,CL \end{pmatrix}$$

Essential boundary conditions (Dirichlet)

$$\begin{pmatrix} \frac{2}{L} & \frac{-1}{L} & 0 \\ \frac{-1}{L} & \frac{2}{L} & \frac{-1}{L} \\ 0 & \frac{-1}{L} & \frac{2}{L} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} CL \\ CL \\ CL \end{pmatrix}$$

$$\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} \frac{3L}{4} & \frac{L}{2} & \frac{L}{4} \\ \frac{L}{2} & L & \frac{L}{2} \\ \frac{L}{4} & \frac{L}{2} & \frac{2L}{4} \end{pmatrix} \begin{pmatrix} CL \\ CL \\ CL \end{pmatrix} = \begin{pmatrix} -0.375 \\ -0.5 \\ -0.375 \end{pmatrix}$$

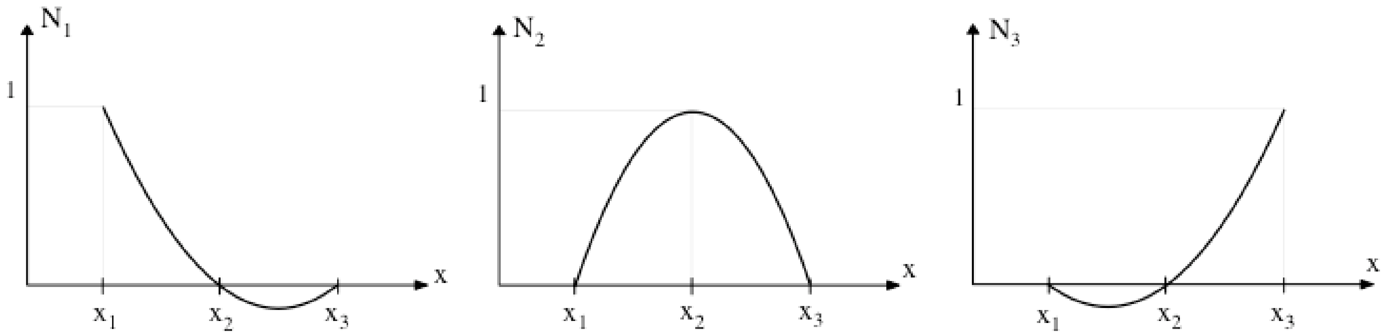
Higher order elements

$$u^{(1)} = \alpha_{j1} + \alpha_{j2}x_1 + \alpha_{j3}x_1^2$$
$$u^{(2)} = \alpha_{j1} + \alpha_{j2}x_2 + \alpha_{j3}x_2^2$$
$$u^{(3)} = \alpha_{j1} + \alpha_{j2}x_3 + \alpha_{j3}x_3^2$$

$$u_j(x) = \sum_{k=1}^{3} N_{jk}(x)u^{(k)}$$

$$N_{j1}(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}, \qquad N_{j2}(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$N_{j3}(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}.$$



Two Quadratic Elements

$$[k_j] = \int_{x_1}^{x_3} \begin{pmatrix} \frac{\mathrm{d}N_{j1}}{\mathrm{d}x}\frac{\mathrm{d}N_{j1}}{\mathrm{d}x} & \frac{\mathrm{d}N_{j1}}{\mathrm{d}x}\frac{\mathrm{d}N_{j2}}{\mathrm{d}x} & \frac{\mathrm{d}N_{j1}}{\mathrm{d}x}\frac{\mathrm{d}N_{j3}}{\mathrm{d}x} \\ \frac{\mathrm{d}N_{j2}}{\mathrm{d}x}\frac{\mathrm{d}N_{j1}}{\mathrm{d}x} & \frac{\mathrm{d}N_{j2}}{\mathrm{d}x}\frac{\mathrm{d}N_{j2}}{\mathrm{d}x} & \frac{\mathrm{d}N_{j2}}{\mathrm{d}x}\frac{\mathrm{d}N_{j3}}{\mathrm{d}x} \\ \frac{\mathrm{d}N_{j3}}{\mathrm{d}x}\frac{\mathrm{d}N_{j1}}{\mathrm{d}x} & \frac{\mathrm{d}N_{j3}}{\mathrm{d}x}\frac{\mathrm{d}N_{j2}}{\mathrm{d}x} & \frac{\mathrm{d}N_{j3}}{\mathrm{d}x}\frac{\mathrm{d}N_{j3}}{\mathrm{d}x} \end{pmatrix} \mathrm{d}x$$

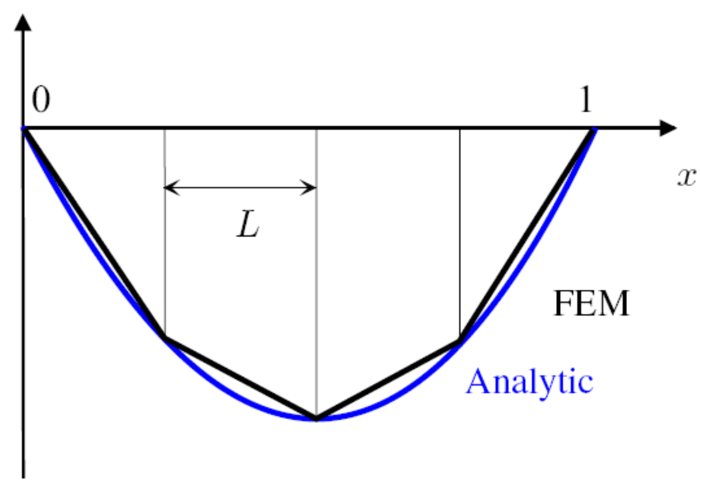
$$[k_j] = \begin{pmatrix} \frac{7}{6l} & \frac{-8}{6l} & \frac{1}{6l} \\ \frac{-8}{6l} & \frac{16}{6l} & \frac{-8}{6l} \\ \frac{1}{6l} & \frac{-8}{6l} & \frac{7}{6l} \end{pmatrix}$$

$$\{f_j\} = -\int_{x_1}^{x_3} \begin{pmatrix} N_{j1} \\ N_{j2} \\ N_{j3} \end{pmatrix} f(x)\,\mathrm{d}x$$

$$\{f_j\} = -\frac{1}{3}c \begin{pmatrix} l \\ 4l \\ l \end{pmatrix}$$

$$\begin{pmatrix} \frac{2}{l} & \frac{-1}{l} & 0 \\ \frac{-1}{l} & \frac{2}{l} & \frac{-1}{l} \\ 0 & \frac{-1}{l} & \frac{2}{l} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} cl \\ cl \\ cl \end{pmatrix}$$

$$\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} \frac{3l}{4} & \frac{l}{2} & \frac{l}{4} \\ \frac{l}{2} & l & \frac{l}{2} \\ \frac{l}{4} & \frac{l}{2} & \frac{3l}{4} \end{pmatrix} \begin{pmatrix} cl \\ cl \\ cl \end{pmatrix} = \begin{pmatrix} -0.375 \\ -0.5 \\ -0.375 \end{pmatrix}$$



Curl-Curl Equation

$$\mathbf{B} = \mathrm{curl}\,\mathbf{A} \qquad \mathrm{in}\,\Omega$$

$$\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A} = \mathbf{J} \qquad \mathrm{in}\,\Omega$$

$$\mathbf{H}_{\mathrm{t}} = \mathbf{0} \quad \rightarrow \quad \frac{1}{\mu}(\mathrm{curl}\,\mathbf{A}) \times \mathbf{n} \;=\; \mathbf{0} \;\; \mathrm{on}\;\Gamma_H$$

$$B_{\mathrm{n}} = 0 \quad \rightarrow \quad \mathbf{B}\cdot\mathbf{n} = \mathrm{curl}\,\mathbf{A}\cdot\mathbf{n} \;=\; 0 \;\; \mathrm{on}\;\Gamma_B$$

$$\left[\frac{1}{\mu}\,(\mathrm{curl}\,\mathbf{A})\times\mathbf{n}\right]_{\mathrm{ai}} \;=\; \mathbf{0} \;\; \mathrm{on}\;\Gamma_{\mathrm{ai}}$$

$$[\mathbf{A}]_{\mathrm{ai}} \;=\; \mathbf{0} \;\; \mathrm{on}\;\Gamma_{\mathrm{ai}}$$

Problem in 3-D: Gauging

$$\mathbf{A} \rightarrow \mathbf{A}' : \mathbf{A}' = \mathbf{A} + \mathrm{grad}\,\psi$$

$$\mathrm{div}\,\mathbf{A}' = q$$

$$q = \mathrm{div}\,\mathbf{A} + \nabla^2\psi$$

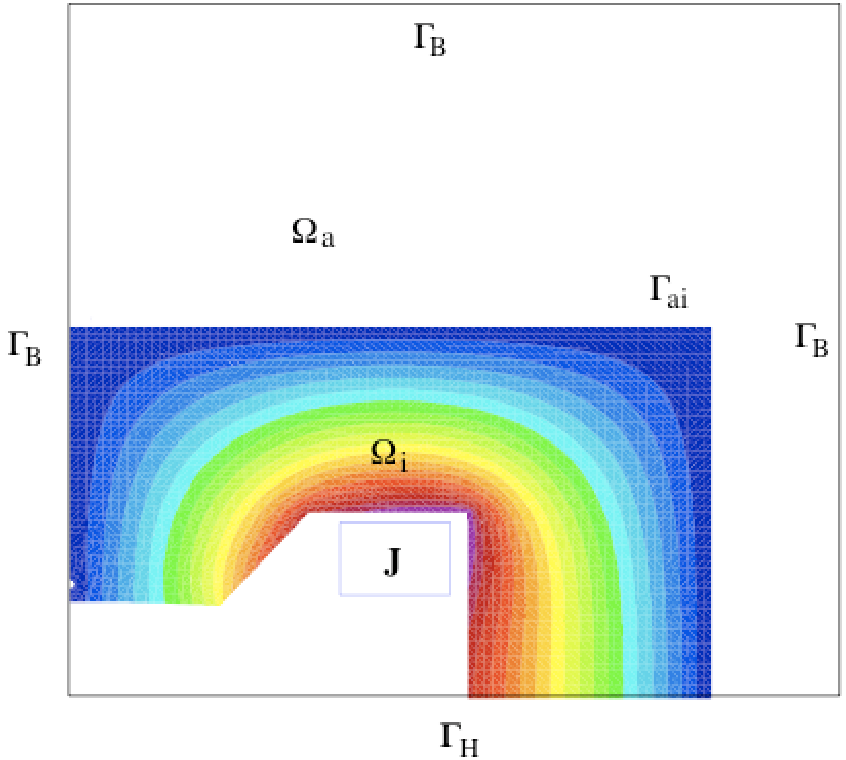
$$\frac{1}{\mu}\mathrm{div}\,\mathbf{A} = 0 \;\; \mathrm{in}\;\Omega$$

$$\mathbf{A}\cdot\mathbf{n} = 0 \;\; \mathrm{on}\;\Gamma_H$$

$$\boxed{\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A} - \;\mathrm{grad}\,\frac{1}{\mu}\,\mathrm{div}\;\mathbf{A} = \mathbf{J} \;\; \mathrm{in}\,\Omega}$$

Weak Form in the FEM Problem

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} = \mathbf{J} \quad \text{in}\, \Omega$$



$$\begin{aligned}
\mathbf{A} \cdot \mathbf{n} &= 0 && \text{on } \Gamma_H, \\
\frac{1}{\mu} \operatorname{div} \mathbf{A} &= 0 && \text{on } \Gamma_B, \\
\mathbf{n} \times (\mathbf{A} \times \mathbf{n}) &= \mathbf{0} && \text{on } \Gamma_B, \\
\mathbf{n} \times \left(\frac{1}{\mu} (\operatorname{curl} \mathbf{A}) \times \mathbf{n} \right) &= \mathbf{0} && \text{on } \Gamma_H, \\
\left[\frac{1}{\mu} \operatorname{div} \mathbf{A} \right]_{\mathrm{ai}} &= 0 && \text{on } \Gamma_{\mathrm{ai}}, \\
\left[\frac{1}{\mu} (\operatorname{curl} \mathbf{A}) \times \mathbf{n} \right]_{\mathrm{ai}} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{ai}}, \\
[\mathbf{A}]_{\mathrm{ai}} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{ai}}.
\end{aligned}$$

Weak Form in the FEM Problem

$$\operatorname{curl}\frac{1}{\mu}\operatorname{curl}\mathbf{A}-\operatorname{grad}\frac{1}{\mu}\operatorname{div}\mathbf{A}=\mathbf{J}\quad \text{in}\,\Omega$$

$$\operatorname{curl}\frac{1}{\mu}\operatorname{curl}\mathbf{A}-\operatorname{grad}\frac{1}{\mu}\operatorname{div}\mathbf{A}-\mathbf{J}=\mathbf{R}$$

$$\int_\Omega \mathbf{w}_a\cdot\left(\operatorname{curl}\frac{1}{\mu}\operatorname{curl}\mathbf{A}-\operatorname{grad}\frac{1}{\mu}\operatorname{div}\mathbf{A}\right)\mathrm{d}\Omega=\int_\Omega \mathbf{w}_a\cdot\mathbf{J}\,\mathrm{d}\Omega\,,\qquad a=1,2,3.$$

$$\int_\Omega\frac{1}{\mu}\operatorname{curl}\mathbf{A}\cdot\operatorname{curl}\mathbf{w}_a\mathrm{d}\Omega-\int_{\Gamma_\mathrm{H}}\frac{1}{\mu}(\operatorname{curl}\mathbf{A}\times\mathbf{n})\cdot\mathbf{w}_a\mathrm{d}\Gamma_\mathrm{H}+\int_\Omega\frac{1}{\mu}\operatorname{div}\mathbf{A}\operatorname{div}\mathbf{w}_a\mathrm{d}\Omega-$$
$$\int_{\Gamma_\mathrm{B}}\frac{1}{\mu}\operatorname{div}\mathbf{A}(\mathbf{n}\cdot\mathbf{w}_a)\mathrm{d}\Gamma_\mathrm{B}-\int_{\Gamma_\mathrm{ai}}\left(\frac{1}{\mu}\operatorname{div}\mathbf{A}_\mathrm{i}(\mathbf{n}_\mathrm{i}\cdot\mathbf{w}_a)+\frac{1}{\mu_0}\operatorname{div}\mathbf{A}_\mathrm{a}(\mathbf{n}_\mathrm{a}\cdot\mathbf{w}_a)\right)\mathrm{d}\Gamma_\mathrm{ai}-$$
$$\int_{\Gamma_\mathrm{ai}}\left(\frac{1}{\mu}(\operatorname{curl}\mathbf{A}_\mathrm{i}\times\mathbf{n}_\mathrm{i})+\frac{1}{\mu_0}(\operatorname{curl}\mathbf{A}_\mathrm{a}\times\mathbf{n}_\mathrm{a})\right)\cdot\mathbf{w}_a\mathrm{d}\Gamma_\mathrm{ai}=\int_\Omega\mathbf{w}_a\cdot\mathbf{J}\,\mathrm{d}\Omega,$$

$$\boxed{\int_\Omega\frac{1}{\mu}\operatorname{curl}\mathbf{w}_a\cdot\operatorname{curl}\mathbf{A}\,\mathrm{d}\Omega+\int_\Omega\frac{1}{\mu}\operatorname{div}\mathbf{w}_a\operatorname{div}\mathbf{A}\,\mathrm{d}\Omega=\int_\Omega\mathbf{w}_a\cdot\mathbf{J}\,\mathrm{d}\Omega}$$

Conclusion: 3-D is more complicated than addition just one dimension in space; it’s a different mathematics, and thus often a separate software package

Weak Form in the FEM Problem

$$\boxed{\int_{\Omega} \frac{1}{\mu} \operatorname{curl} \mathbf{w}_a \cdot \operatorname{curl} \mathbf{A} \, \mathrm{d}\Omega + \int_{\Omega} \frac{1}{\mu} \operatorname{div} \mathbf{w}_a \operatorname{div} \mathbf{A} \, \mathrm{d}\Omega = \int_{\Omega} \mathbf{w}_a \cdot \mathbf{J} \, \mathrm{d}\Omega}$$

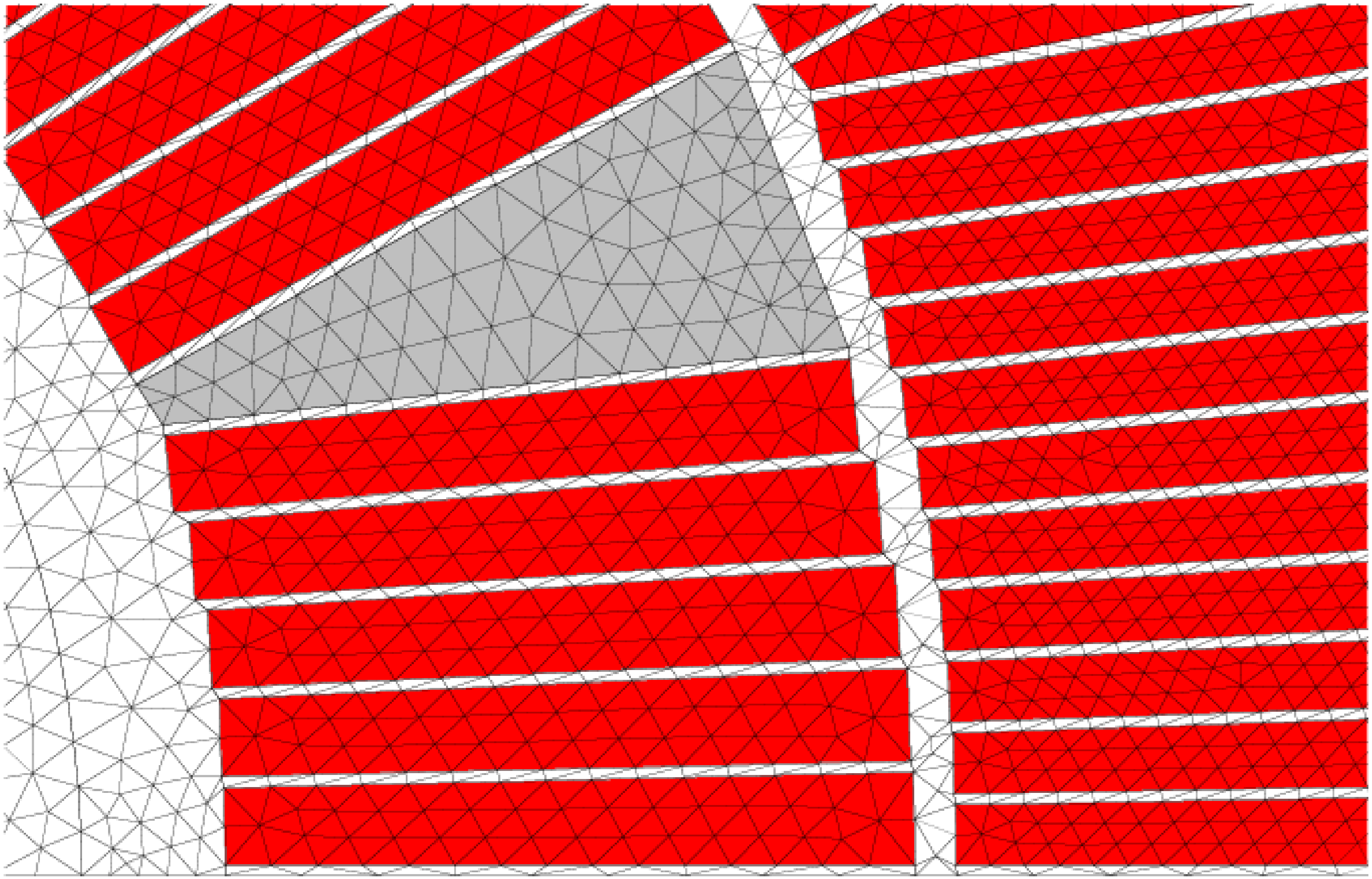
$$\mathbf{A}_j(\mathbf{r}) = \sum_{k=1}^{K} N_k(\mathbf{r}) \mathbf{A}^{(k)} \qquad \mathbf{r} \in \Omega_j$$

$$\mathbf{A}^{(k)} := \begin{pmatrix} A_x^{(k)} \\ A_y^{(k)} \\ A_z^{(k)} \end{pmatrix}$$

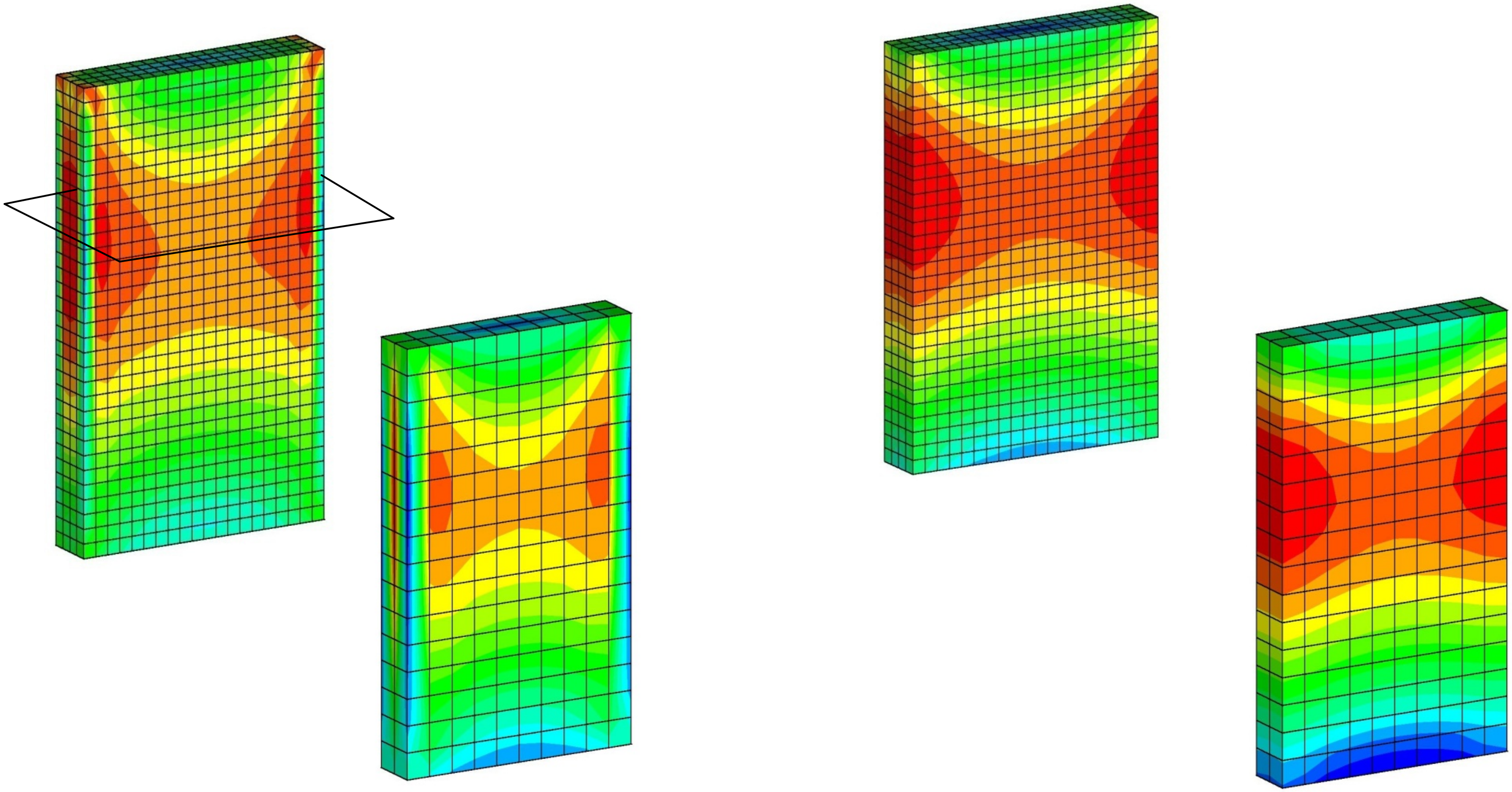
$$\mathbf{N}_{l1} := \begin{pmatrix} N_l \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{N}_{l2} := \begin{pmatrix} 0 \\ N_l \\ 0 \end{pmatrix}, \qquad \mathbf{N}_{l3} := \begin{pmatrix} 0 \\ 0 \\ N_l \end{pmatrix}$$

$$\boxed{\int_{\Omega_j} \frac{1}{\mu} \operatorname{curl} \mathbf{N}_{la} \cdot \operatorname{curl} \left(\sum_{k=1}^{K} N_k \mathbf{A}^{(k)} \right) \mathrm{d}\Omega + \int_{\Omega_j} \frac{1}{\mu} \operatorname{div} \mathbf{N}_{la} \operatorname{div} \left(\sum_{k=1}^{K} N_k \mathbf{A}^{(k)} \right) \mathrm{d}\Omega = \int_{\Omega_j} \mathbf{N}_{la} \cdot \mathbf{J} \, \mathrm{d}\Omega}$$

Meshing the Coil



Nodal versus Edge-Elements



Notice: Finer discretization does not help! Use edge-elements, or a different formulation (scalar potential, whenever possible. Remember: This problem does not exist in 2-D

Total Scalar Potential / Reduced Scalar Potential

$$\operatorname{curl}\mathbf{H} = 0$$

$$\mathbf{H} = -\operatorname{grad}\phi_\mathrm{m} \quad \text{in}\,\Omega,\ \mathbf{J} = 0$$

$$\operatorname{div}(\mu\operatorname{grad}\phi_\mathrm{m}) = 0 \quad \text{in}\,\Omega,\ \mathbf{J} = 0$$

$$\begin{aligned} \mu_0 \operatorname{div}\operatorname{grad}\phi_\mathrm{m} &= 0 \quad \text{in}\,\Omega_\mathrm{a},\ \mathbf{J} = 0 \\ \nabla^2\phi_\mathrm{m} &= 0 \quad \text{in}\,\Omega_\mathrm{a},\ \mathbf{J} = 0 \end{aligned}$$

$$\begin{aligned} \mathbf{H}_\mathrm{t} = \mathbf{0} \quad \rightarrow \quad \mathbf{n}\times(\operatorname{grad}\phi_\mathrm{m}\times\mathbf{n}) &= \mathbf{0} \quad \text{on}\,\Gamma_H \\ B_\mathrm{n} = 0 \quad \rightarrow \quad \mu\mathbf{n}\cdot\operatorname{grad}\phi_\mathrm{m} &= 0 \quad \text{on}\,\Gamma_B \end{aligned}$$

$$\begin{aligned} [\mu\mathbf{n}\cdot\operatorname{grad}\phi_\mathrm{m}]_\mathrm{ai} &= 0 \quad \text{on}\,\Gamma_\mathrm{ai} \\ [\operatorname{grad}\phi_\mathrm{m}\times\mathbf{n}]_\mathrm{ai} &= \mathbf{0} \quad \text{on}\,\Gamma_\mathrm{ai} \end{aligned}$$

$$\mathbf{H} = \mathbf{H}_\mathrm{s} + \mathbf{H}_\mathrm{m}$$

$$\mathbf{H} = -\operatorname{grad}\phi_\mathrm{m}^\mathrm{red} + \mathbf{H}_\mathrm{s} \qquad \text{in}\,\Omega$$

$$\begin{aligned} \operatorname{div}\mathbf{B} &= 0 \\ \operatorname{div}\mu(-\operatorname{grad}\phi_\mathrm{m}^\mathrm{red} + \mathbf{H}_\mathrm{s}) &= 0 \\ \operatorname{div}(\mu\operatorname{grad}\phi_\mathrm{m}^\mathrm{red}) &= \operatorname{div}(\mu\mathbf{H}_\mathrm{s}) \quad \text{in}\,\Omega \end{aligned}$$

$$\mathbf{H}_\mathrm{t} = \mathbf{0} \quad \rightarrow \quad \operatorname{grad}\phi_\mathrm{m}^\mathrm{red}\times\mathbf{n} = \mathbf{H}_\mathrm{s}\times\mathbf{n} \quad \text{on}\,\Gamma_H$$

$$B_\mathrm{n} = 0 \quad \rightarrow \quad \mu\mathbf{n}\cdot\operatorname{grad}\phi_\mathrm{m}^\mathrm{red} = \mu\mathbf{H}_\mathrm{s}\cdot\mathbf{n} \quad \text{on}\,\Gamma_B$$

$$\begin{aligned} \left[-\mu\mathbf{n}\cdot\operatorname{grad}\phi_\mathrm{m}^\mathrm{red} + \mu\mathbf{H}_\mathrm{s}\cdot\mathbf{n}\right]_\mathrm{ai} &= 0 \quad \text{on}\,\Gamma_\mathrm{ai} \\ \left[-\operatorname{grad}\phi_\mathrm{m}^\mathrm{red}\times\mathbf{n} + \mathbf{H}_\mathrm{s}\times\mathbf{n}\right]_\mathrm{ai} &= \mathbf{0} \quad \text{on}\,\Gamma_\mathrm{ai} \end{aligned}$$

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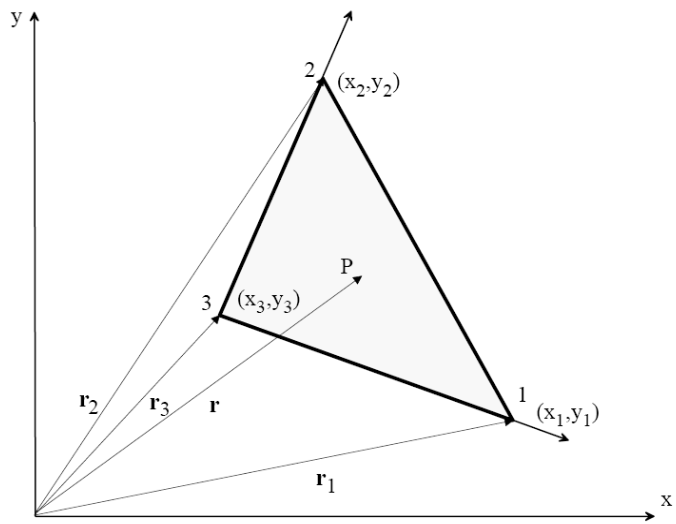
Shape Functions

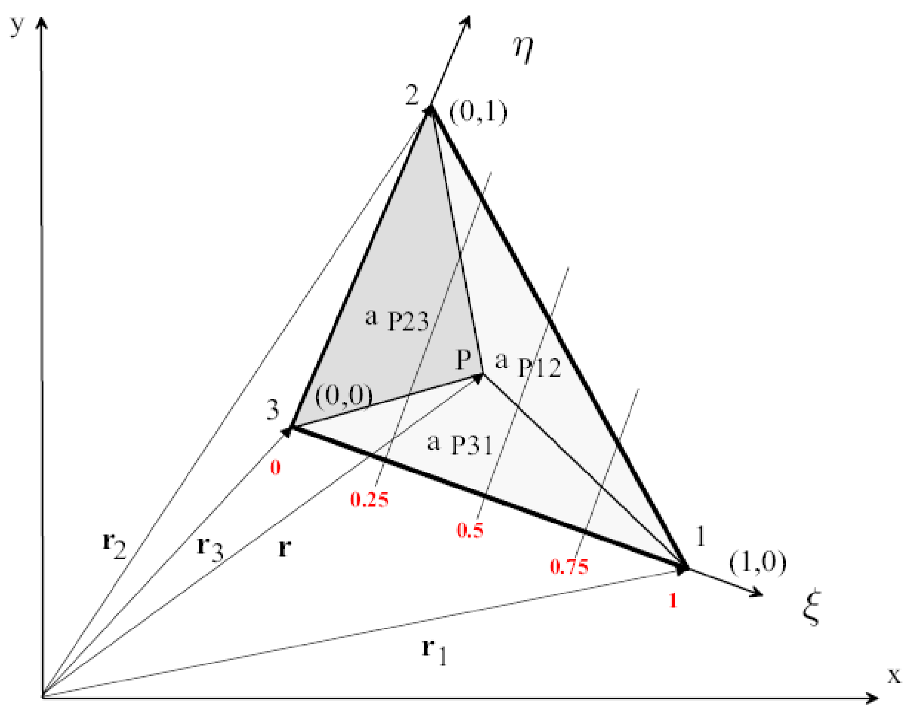
$$A_j(\mathbf{x}) = \alpha_1 + \alpha_2 x + \alpha_3 y, \qquad \mathbf{x} \in \Omega_j \qquad A_j(\mathbf{x}) = A_{z_j}(x, y)$$



$$A^{(1)} = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$
$$A^{(2)} = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$
$$A^{(3)} = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$\begin{pmatrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\{\alpha\} = [C]^{-1}\{A\} \quad \longleftarrow \quad \{A\} = [C]\{\alpha\}$$



Then for each point $P \in R^2$ there is one and only one set $\{\lambda_1, \lambda_2, \lambda_3\}$ of real numbers for which

$$\mathbf{r} = \lambda_1 \mathbf{r}_1 + \lambda_2 \mathbf{r}_2 + \lambda_3 \mathbf{r}_3$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

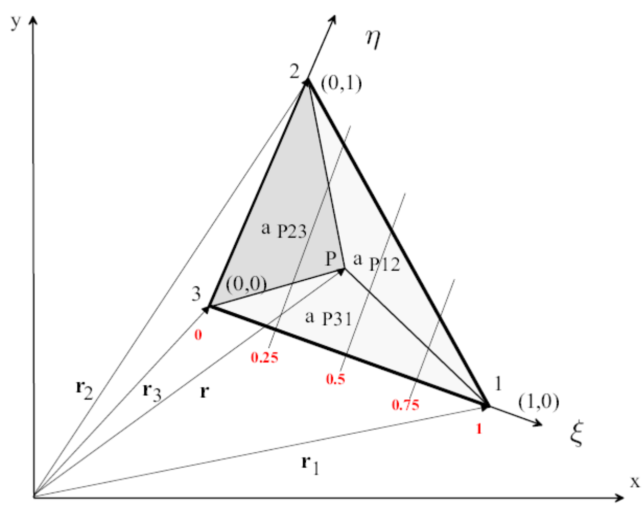
$$\begin{aligned} x &= \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \,, \\ y &= \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 \,, \\ 1 &= \lambda_1 + \lambda_2 + \lambda_3 \end{aligned}$$

$$\lambda_1 = \frac{a_{P23}}{a_{123}} \qquad \lambda_2 = \frac{a_{P31}}{a_{123}} \qquad \lambda_3 = \frac{a_{P12}}{a_{123}}$$

$$\begin{aligned} x &= x_3 + (x_1 - x_3)\xi + (x_2 - x_3)\eta \,, \\ y &= y_3 + (y_1 - y_3)\xi + (y_2 - y_3)\eta \end{aligned}$$

$$\lambda_1 = \xi, \qquad \lambda_2 = \eta, \qquad \lambda_3 = 1 - \xi - \eta$$

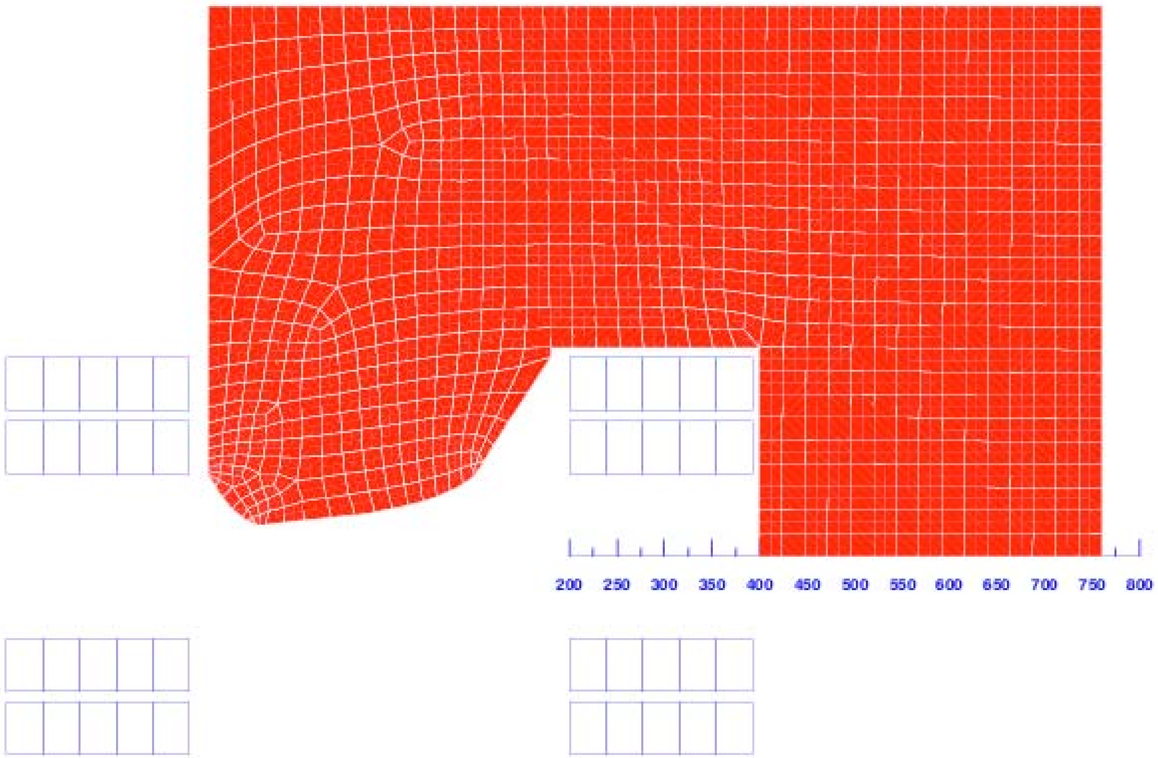
$$\xi = \frac{2a_{P23}}{2a_{123}} \qquad \eta = \frac{2a_{P31}}{2a_{123}}$$



$$\xi = \frac{x_2y_3 - x_3y_2 + (y_2 - y_3)x + (x_3 - x_2)y}{x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1} \,,$$

$$\eta = \frac{x_3y_1 - x_1y_3 + (y_3 - y_1)x + (x_1 - x_3)y}{x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1}$$

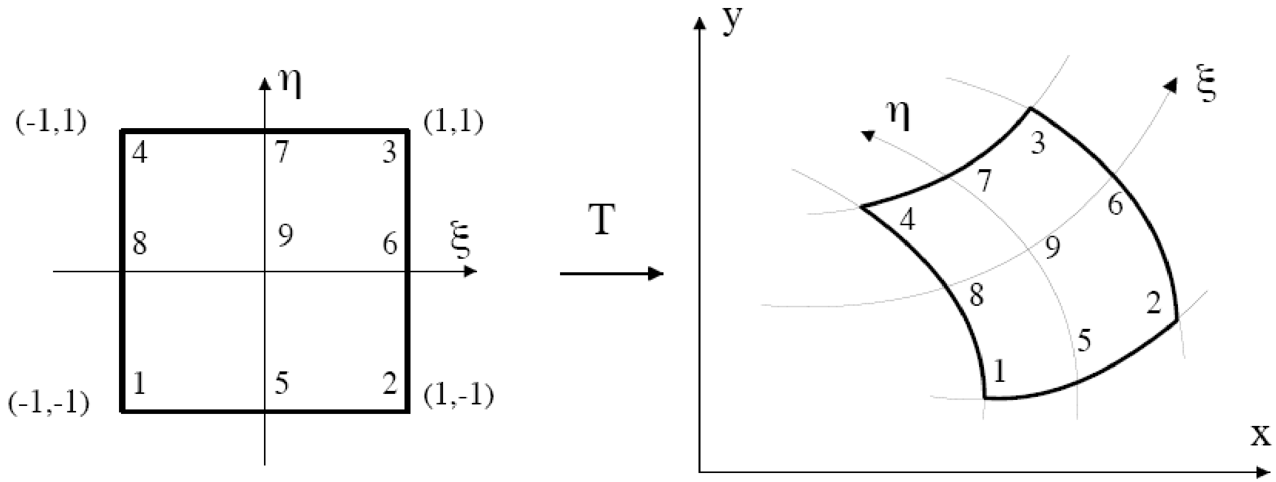
Higher Order Elements



Higher accuracy of the field solution, but also better modeling of the iron contour

Mapped Elements

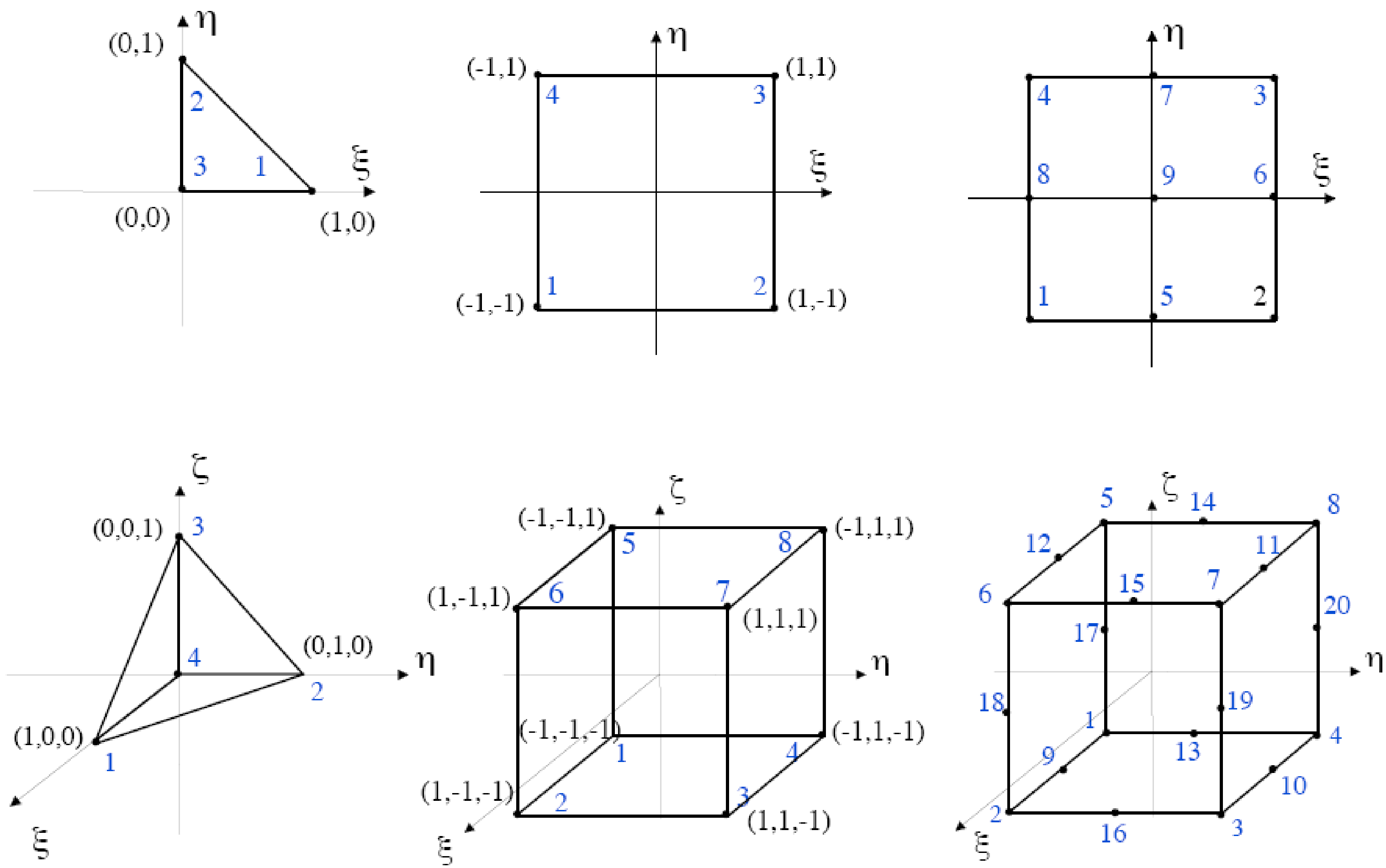
$$x = x(\xi, \eta, \zeta), \qquad y = y(\xi, \eta, \zeta), \qquad z = z(\xi, \eta, \zeta)$$



$$A_j(\boldsymbol{\xi}) = \sum_{k=1}^{K} N_k(\boldsymbol{\xi}) A^{(k)} \qquad x_j(\boldsymbol{\xi}) = \sum_{k=1}^{K} N_k(\boldsymbol{\xi}) x^{(k)} \qquad y_j(\boldsymbol{\xi}) = \sum_{k=1}^{K} N_k(\boldsymbol{\xi}) y^{(k)}$$

Use of the same shape functions for the transformation of the elements

Mapped Elements



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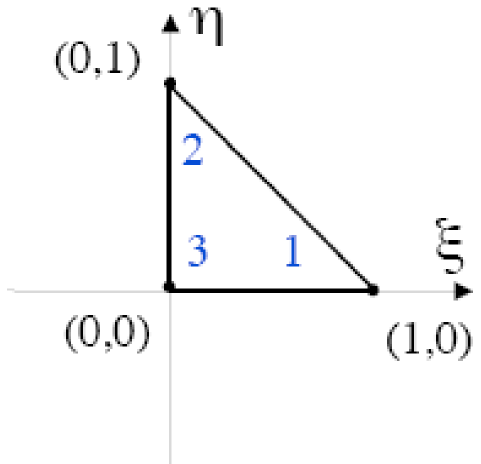
Transformation of Differential Operators

$$\frac{\partial N_k}{\partial x} = \frac{\partial N_k}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_k}{\partial \eta}\frac{\partial \eta}{\partial x}$$



Complicated **Easy**

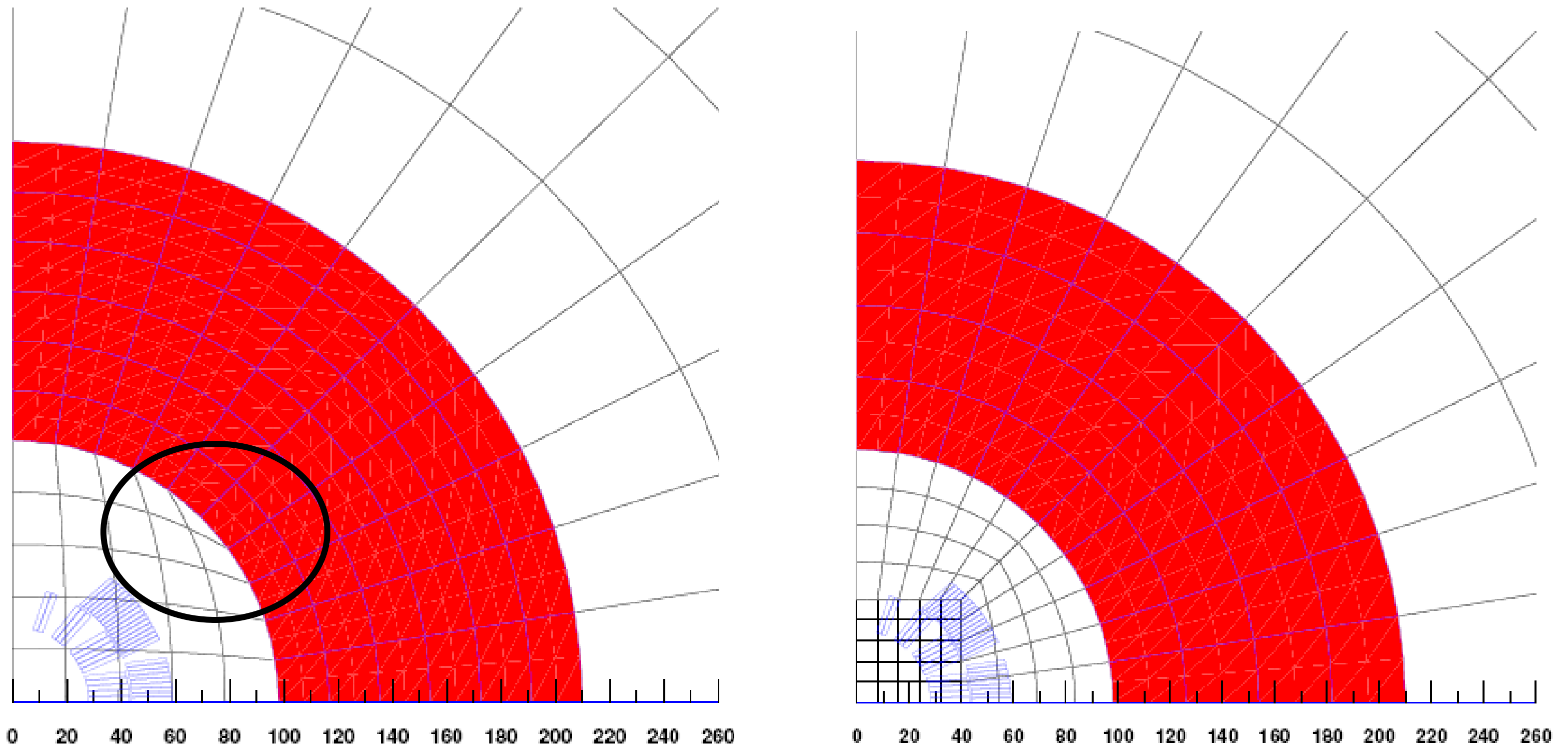
$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} N_k = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} N_k = [J]_{T^{-1}} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} N_k$$

$$[J]_{T^{-1}} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}^{-1} = [J]_T^{-1}$$

$$[J]_T = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{K} \frac{\partial N_k}{\partial \xi} x^{(k)} & \sum_{k=1}^{K} \frac{\partial N_k}{\partial \xi} y^{(k)} \\ \sum_{k=1}^{K} \frac{\partial N_k}{\partial \eta} x^{(k)} & \sum_{k=1}^{K} \frac{\partial N_k}{\partial \eta} y^{(k)} \end{pmatrix} = \begin{pmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \cdots & \frac{\partial N_K}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \cdots & \frac{\partial N_K}{\partial \eta} \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_k & y_k \end{pmatrix}$$

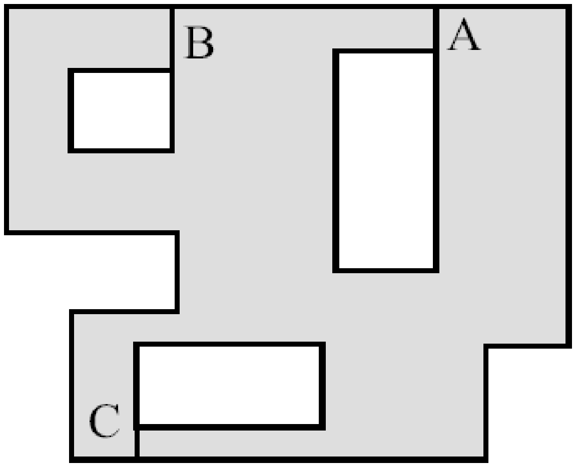
But how about the Jacobian being singular?

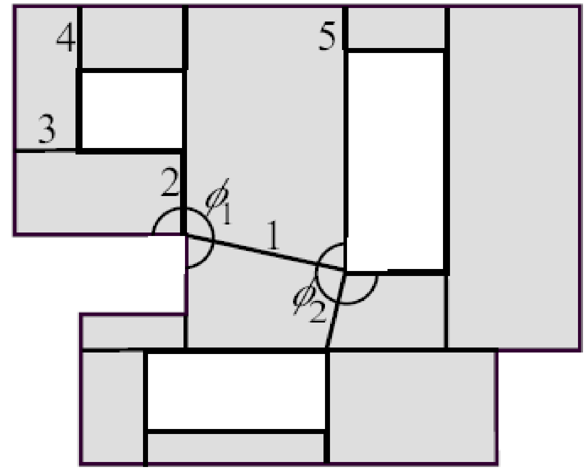
Collinear Sides yield Singular Jacobi Matrices

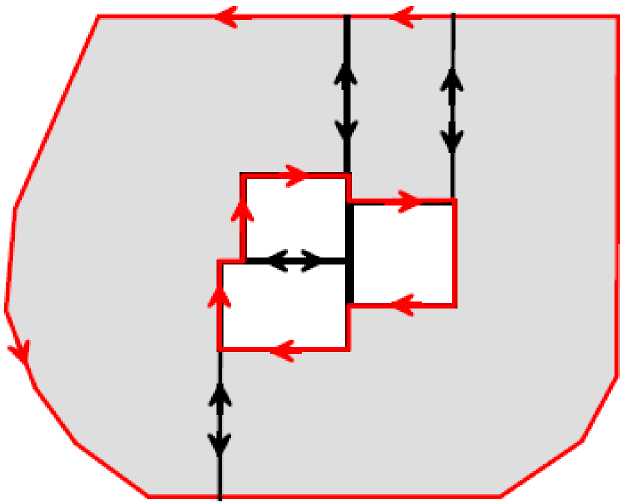


Note: Bad meshing is not a trivial offence

Topology Decomposition



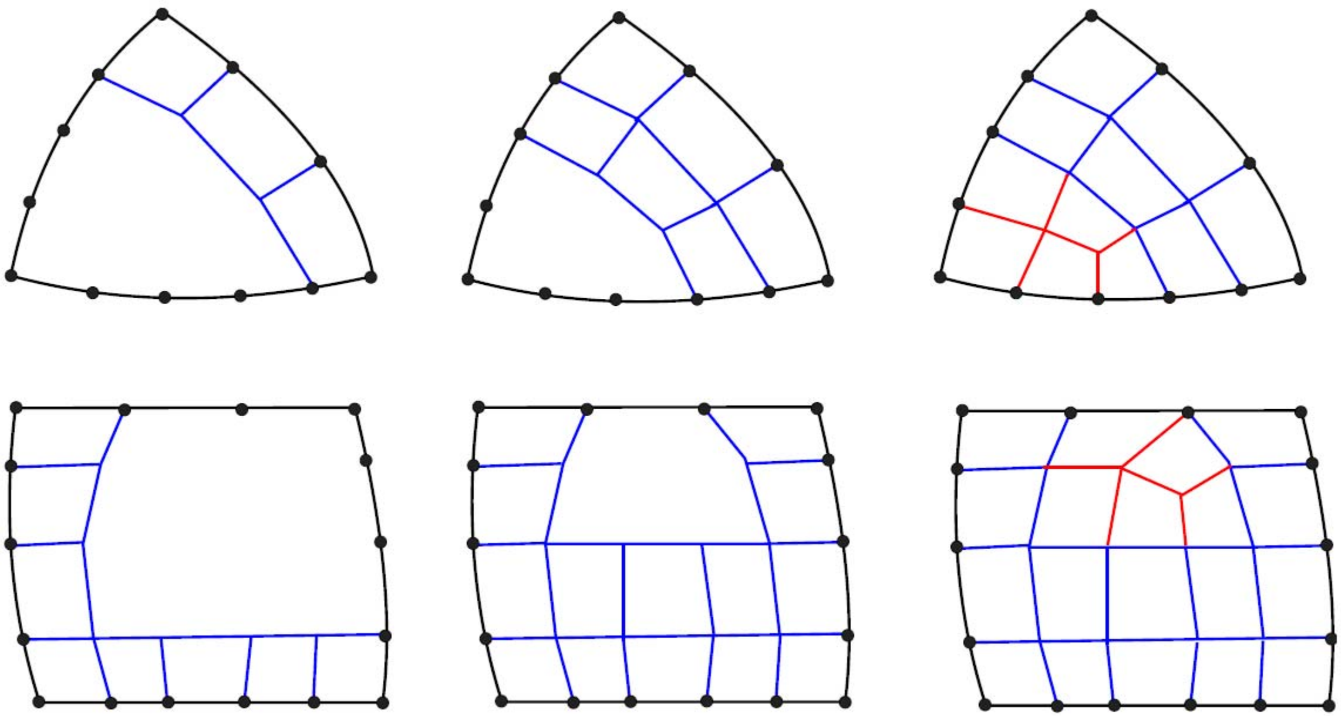




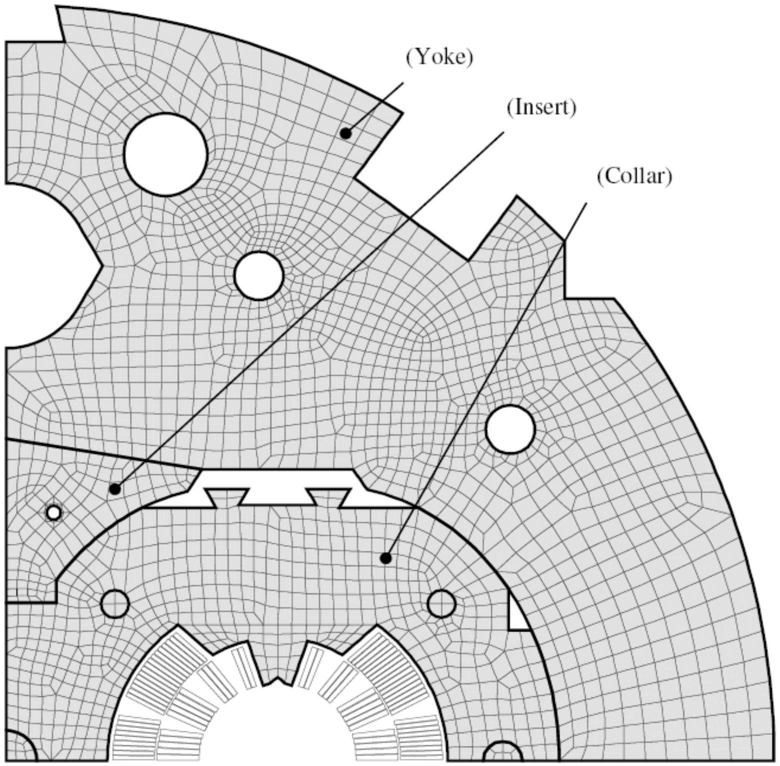
$$\Gamma^* = \min f(\Gamma) = \min \left\{ \lambda_1 \frac{L^2}{\min\{a_1, a_2\}} + \lambda_2 \frac{L}{\min\{C_1, C_2\}} + \lambda_3 (4\pi - \phi_1 - \phi_2) \right\}$$

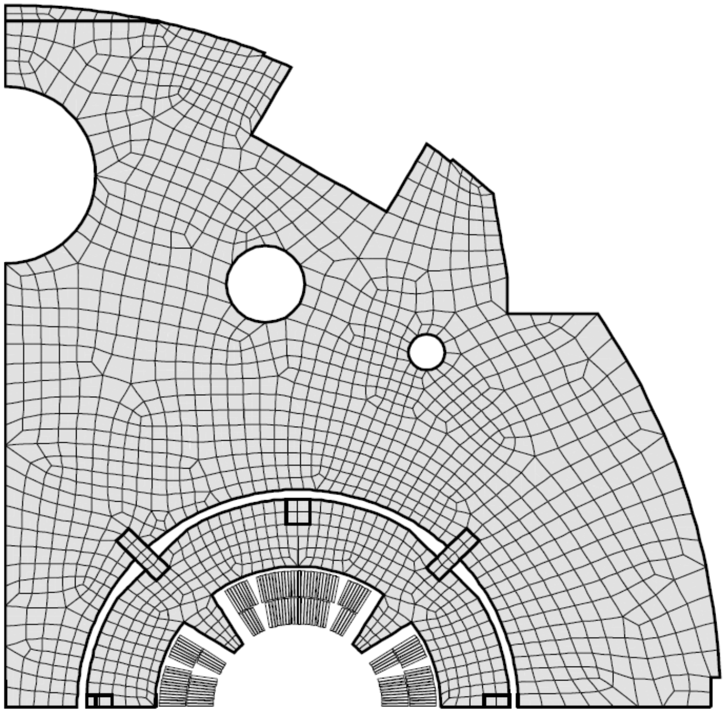
Paving and Mesh Closing in Simple Domains

- ➔ The number of nodes is less than 6
- ➔ The domian does not contain “bottlenecks” , i.e., C^2/a approaches 4π
- ➔ The biggest inner angle is less then π
- ➔ For triangles: a+b < c

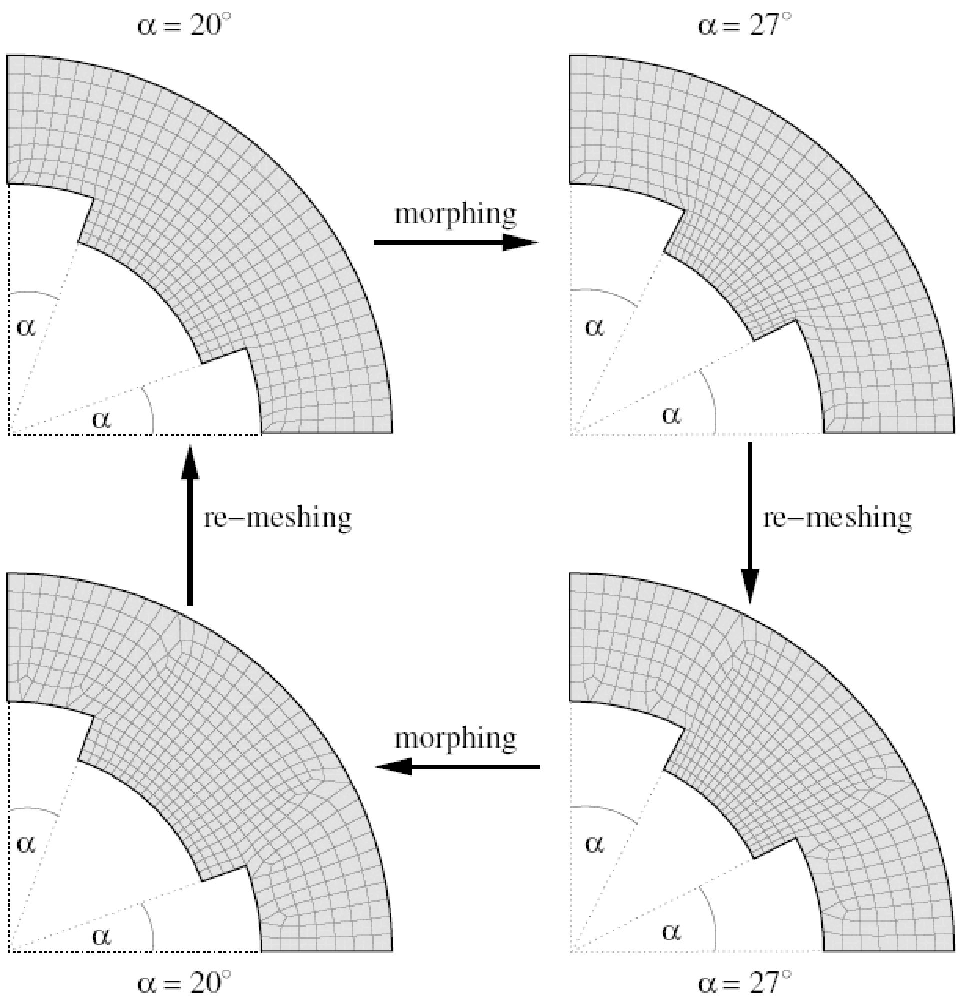


Examples for FEM Meshes

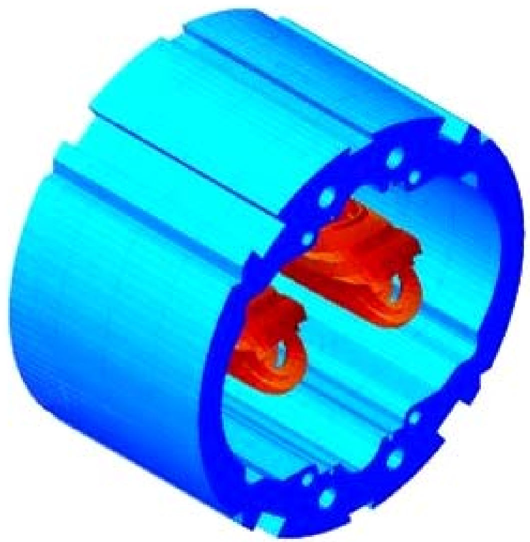


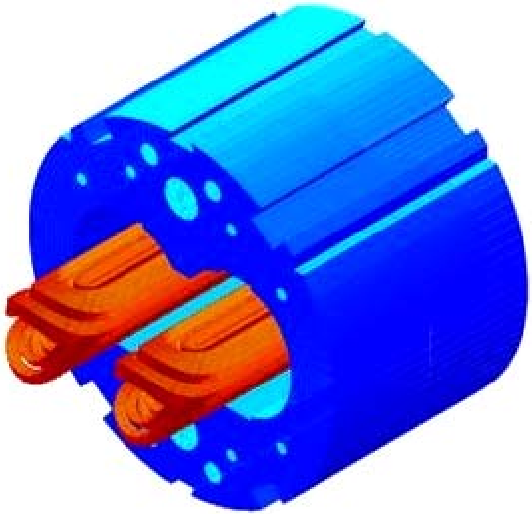


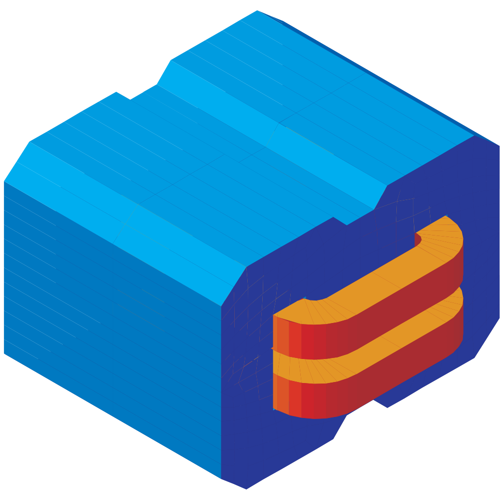
Point Based Morphing

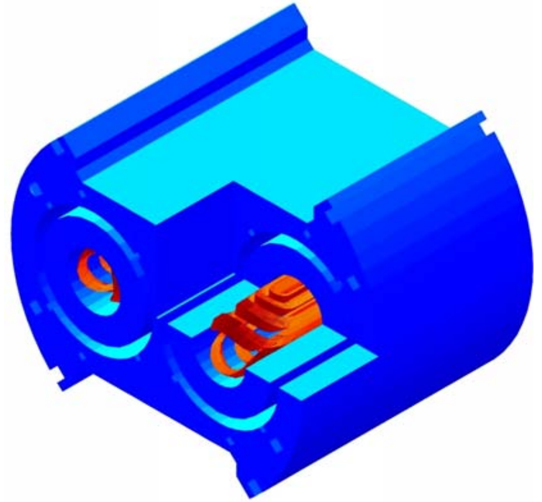


Always use morphing (if available) for sensitivity analysis





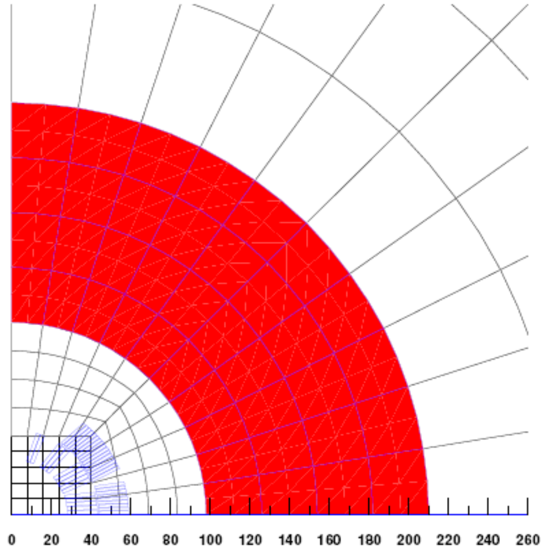




Reduced Vector Potential Formulation

$$\mathbf{A} = \mathbf{A}_\mathrm{s} + \mathbf{A}_\mathrm{r} \qquad \mathbf{B} = \mu_0 \mathbf{H}_\mathrm{s} + \operatorname{curl} \mathbf{A}_\mathrm{r}$$

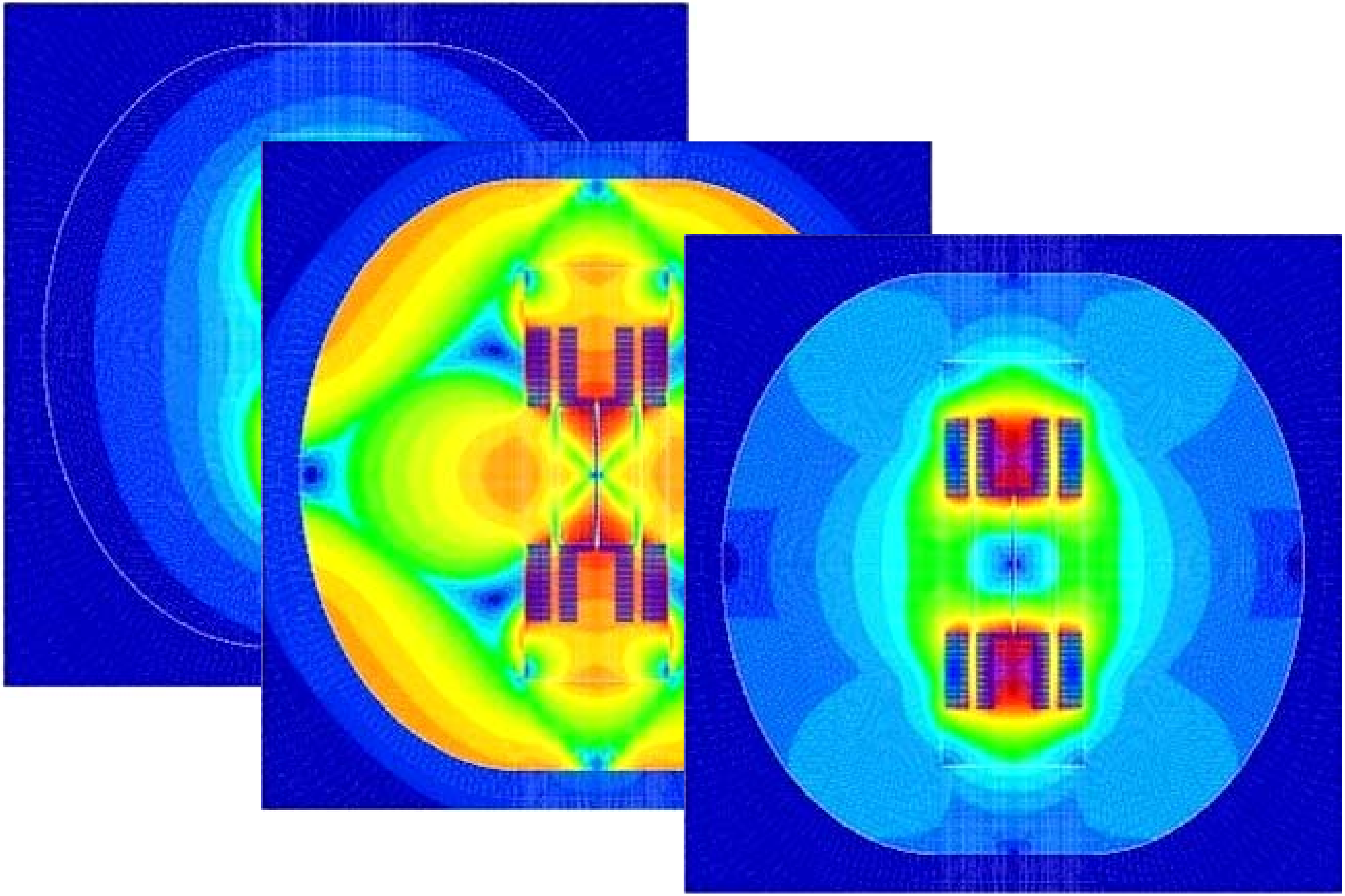
$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} (\mathbf{A}_\mathrm{r} + \mathbf{A}_\mathrm{s}) - \operatorname{grad} \frac{1}{\mu} \operatorname{div} (\mathbf{A}_\mathrm{r} + \mathbf{A}_\mathrm{s}) = \mathbf{J}$$



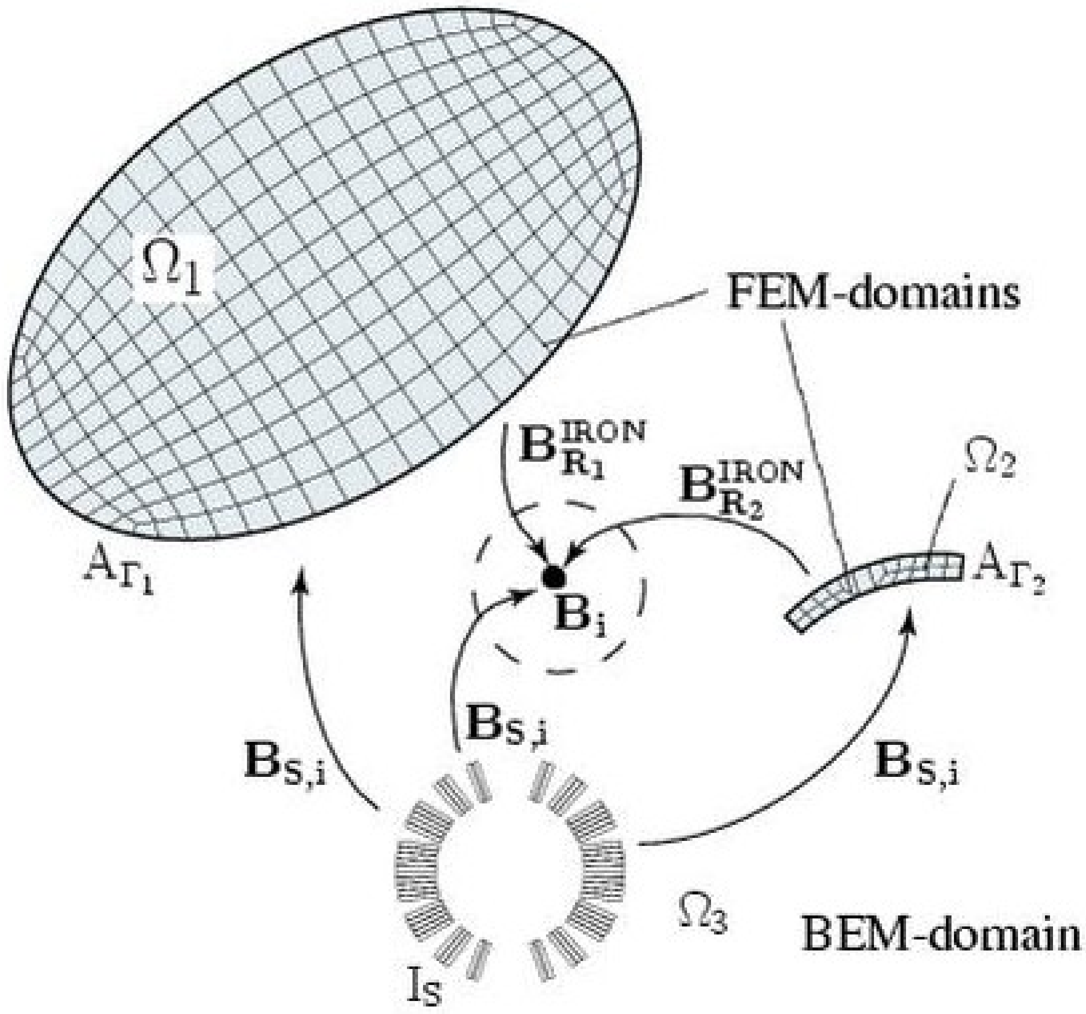
$$\begin{aligned} \operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A}_\mathrm{r} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A}_\mathrm{r} &= \mathbf{J} - \operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A}_\mathrm{s} \\ &= \operatorname{curl} \mathbf{H}_\mathrm{s} - \operatorname{curl} \frac{\mu_0}{\mu} \mathbf{H}_\mathrm{s} \\ &= \operatorname{curl} \left(\mathbf{H}_\mathrm{s} - \frac{\mu_0}{\mu} \mathbf{H}_\mathrm{s} \right) \end{aligned}$$

Advantages: No meshing of the coil, no cancellation errors, distinction between source field and iron magnetization

Source, Reduced, Total Field



BEM-FEM Coupling (Elementary Model Problem)



The FEM Part (Vector Laplace Equation)

$$\boxed{-\frac{1}{\mu_0}\nabla^2 \mathbf{A} = \mathbf{J} + \operatorname{curl}\mathbf{M}} \qquad \text{in } \Omega_i,$$

$$\mathbf{A}\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_H,$$

$$\frac{1}{\mu_0}\operatorname{div}\mathbf{A} = 0 \qquad \text{on } \Gamma_B,$$

$$\mathbf{n}\times(\mathbf{A}\times\mathbf{n}) = \mathbf{0} \qquad \text{on } \Gamma_B,$$

$$\frac{1}{\mu}(\operatorname{curl}\mathbf{A})\times\mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma_H,$$

$$\left[\frac{1}{\mu_0}\operatorname{div}\mathbf{A}_{\mathrm{a}}\right]_{\mathrm{ai}} = 0 \qquad \text{on } \Gamma_{\mathrm{ai}},$$

$$\boxed{\frac{1}{\mu_0}(\operatorname{curl}\mathbf{A}_{\mathrm{i}} - \mu_0\mathbf{M})\times\mathbf{n}_{\mathrm{i}} + \frac{1}{\mu_0}(\operatorname{curl}\mathbf{A}_{\mathrm{a}})\times\mathbf{n}_{\mathrm{a}} = \mathbf{0}} \qquad \text{on } \Gamma_{\mathrm{ai}},$$

$$[\mathbf{A}]_{\mathrm{ai}} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{ai}}.$$

FEM Part

$$\frac{1}{\mu_0}\int_{\Omega_\mathrm{i}} \mathrm{grad}\,(\mathbf{A}\cdot\mathbf{e}_a)\cdot\mathrm{grad}\,w_a\,\mathrm{d}\Omega_\mathrm{i} - \boxed{\frac{1}{\mu_0}\oint_{\Gamma_\mathrm{ai}}\left(\frac{\partial\mathbf{A}}{\partial n_\mathrm{i}} - (\mu_0\mathbf{M}\times\mathbf{n}_\mathrm{i})\right)\cdot\mathbf{w}_a\,\mathrm{d}\Gamma_\mathrm{ai}} = \int_{\Omega_\mathrm{i}} \mathbf{M}\cdot\mathrm{curl}\,\mathbf{w}_a\,\mathrm{d}\Omega_\mathrm{i}$$

$$\mathbf{Q}_{\Gamma_\mathrm{ai}} := -\frac{\partial\mathbf{A}^\mathrm{BEM}_{\Gamma_\mathrm{ai}}}{\partial n_\mathrm{a}} \qquad \frac{\partial\mathbf{A}_\mathrm{i}^\mathrm{FEM}}{\partial n_\mathrm{i}} - (\mu_0\mathbf{M}\times\mathbf{n}_\mathrm{i}) + \frac{\partial\mathbf{A}_\mathrm{a}^\mathrm{BEM}}{\partial n_\mathrm{a}} = \mathbf{0} \quad \text{on } \Gamma_\mathrm{ai}$$

$$\frac{1}{\mu_0}\left(\mathrm{curl}\,\mathbf{A}_\mathrm{i}^\mathrm{FEM} - \mu_0\mathbf{M}\right)\times\mathbf{n}_\mathrm{i} + \frac{1}{\mu_0}\left(\mathrm{curl}\,\mathbf{A}_\mathrm{a}^\mathrm{BEM}\right)\times\mathbf{n}_\mathrm{a} = \mathbf{0} \quad \text{on } \Gamma_\mathrm{ai}$$

$$\frac{1}{\mu_0}\int_{\Omega_\mathrm{i}} \mathrm{grad}\,(\mathbf{A}\cdot\mathbf{e}_a)\cdot\mathrm{grad}\,w_a\,\mathrm{d}\Omega_\mathrm{i} - \frac{1}{\mu_0}\oint_{\Gamma_\mathrm{ai}} \mathbf{Q}_{\Gamma_\mathrm{ai}}\cdot\mathbf{w}_a\,\mathrm{d}\Gamma_\mathrm{ai} = \int_{\Omega_\mathrm{i}} \mathbf{M}\cdot\mathrm{curl}\,\mathbf{w}_a\,\mathrm{d}\Omega_\mathrm{i}$$

$$[K]\{A_x\} - [T]\{Q_x\} = \{F_x(\mathbf{M})\}$$

Vector Laplace

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}\,, \qquad \text{in}\ \Omega_\mathrm{a}$$

Weighted Residual

$$\int_{\Omega_\mathrm{a}} \nabla^2 A w \,\mathrm{d}\Omega_\mathrm{a} = -\int_{\Omega_\mathrm{a}} \mu_0 J w \,\mathrm{d}\Omega_\mathrm{a}$$

From Green's second theorem:

$$\int_{\Omega_\mathrm{a}} A \nabla^2 w \mathrm{d}\Omega_\mathrm{a} = -\int_{\Omega_\mathrm{a}} \mu_0 J w \,\mathrm{d}\Omega_\mathrm{a} + \int_{\Gamma_\mathrm{ai}} A \frac{\partial w}{\partial n_a} \mathrm{d}\Gamma_\mathrm{ai} - \int_{\Gamma_\mathrm{ai}} \frac{\partial A}{\partial n_a} w \mathrm{d}\Gamma_\mathrm{ai}$$

$$u^*(\mathbf{r}, \mathbf{r}') := w = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

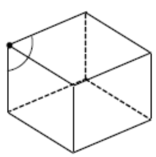
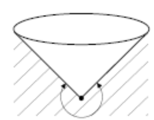
$$\nabla^2 w = -\delta(|\mathbf{r} - \mathbf{r}'|)$$

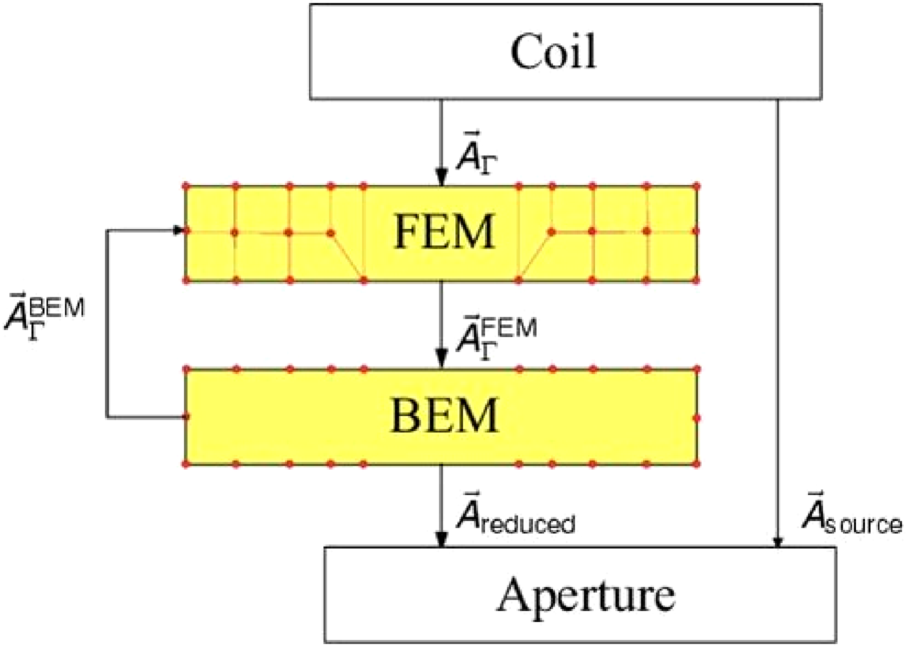
$$q^*(\mathbf{r}, \mathbf{r}') := \frac{\partial u^*}{\partial n_a} = \frac{\partial w}{\partial n_a} = -\frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{n}_a}{4\pi |\mathbf{r} - \mathbf{r}'|^3}$$

$$\int_{\Omega} A(\mathbf{r}) \nabla^2 w \mathrm{d}\Omega = \int_{\Omega} A(\mathbf{r}) \delta(|\mathbf{r} - \mathbf{r}'|) \mathrm{d}\Omega = A(\mathbf{r}')$$

BEM Part

$$\boxed{\frac{\Theta}{4\pi}A(\mathbf{r}) + \int_{\Gamma_{\mathrm{ai}}} Q_{\Gamma_{\mathrm{ai}}}\, u^*(\mathbf{r},\mathbf{r}')\;\mathrm{d}\Gamma_{\mathrm{ai}} + \int_{\Gamma_{\mathrm{ai}}} A_{\Gamma_{\mathrm{ai}}}\, q^*(\mathbf{r},\mathbf{r}')\;\mathrm{d}\Gamma_{\mathrm{ai}} = \int_{\Omega_{\mathrm{a}}} \mu_0 J u^*(\mathbf{r},\mathbf{r}')\,\mathrm{d}\Omega_{\mathrm{a}}.}$$

Ω_{a}	 90°corner	 90° cone inner	 half space	 90°cone outer
Θ	$\frac{1}{2}\,\pi$	$(2-\sqrt{2})\pi$	2π	$(2+\sqrt{2})\,\pi$
$\frac{\Theta}{4\pi}$	$\frac{1}{8}$	$\frac{2-\sqrt{2}}{4}$	$\frac{1}{2}$	$\frac{2+\sqrt{2}}{4}$



BEM

$$[G]\{Q_z\} + [H]\{A_z\} = \{A_{z,\mathrm{s}}\}$$

FEM

$$[K]\{A_z\} - [T]\{Q_z\} = \{F_z(\mathbf{M})\}$$

$$\{Q\} = -[G]^{-1}[H]\{A\} + [G]^{-1}\{A\}$$

$$\boxed{\left([K] + [T][G]^{-1}[H]\right)\{A\} = \{F(\mathbf{M})\} + [T][G]^{-1}\{A_\mathrm{s}\}}$$

$$[K]\{A\} = \{F(A_s, M)\}$$

yields an iteration scheme

$$\{A_{k+1}\} = [K]^{-1}\{F(A_s, M_k)\}$$

Subtracting $\{A_k\}$ from both sides yields:

$$\{\Delta A_k\} := \{A_{k+1}\} - \{A_k\} = [K]^{-1}\{R_k\}$$

where the residual is defined by

$$\{R_k\} = \{F(A_s, M_k)\} - [K]\{A_k\}$$

Introduce a *relaxation parameter* ω.

$$\{A_{k+1}\} = \{A_k\} + \omega\{\Delta A_k\}$$

with $\omega_0 = 1$ and

$$\omega_k = \frac{\omega_{k-1}}{1 - \frac{\{\Delta A_k\}\cdot\{\Delta A_{k-1}\}}{\|\{\Delta A_{k-1}\}\|^2}}$$

1. Set iteration index $k = 0$, and initialize vector potentials $\{A_0\} = \{0\}$.
2. For $k = 0, 1, 2, \ldots$, unit convergence Do:
3. Compute the force vector $\{F(A_s, M_k)\}$ and the residual $\{R_k\}$.
4. If $\frac{\|\{R_k\}\|}{\|\{F(A_{sk}, M_k)\}\|} < \varepsilon$ Goto 8.
5. Calculate the step sizes $\{\Delta A_k\} = [\mathrm{K}]^{-1}\{R_k\}$.
6. Choose the relaxation parameter.
7. $\{A_{k+1}\} = \{A_k\} + \omega\{\Delta A_k\}$
8. End Do.

Advantages

- If direct solvers are used, the stiffness matrix needs to be inverted only once.
- The method is globally convergent
- No derivative of the M(B) curve is required.

and disadvantages

- The number of iteration steps is high.
- Even in the case of linear media the M(B)-iteration is necessary.

```
          GMRES STEP 23, RESIDUAL= 0.1014E-05 ( -59.52 DB)
          GMRES STEP 24, RESIDUAL= 0.2402E-06 ( -65.78 DB)
INFO 240: GMRES STEP 24, RESIDUAL= 0.2402E-06 ( -65.78 DB).
          >>>>>>>>>>>> NEWTON STEP 4, RESIDUAL= 0.3917E-03 ( -38.38 DB)
COMPUTING THE GLOBAL MATRIX...
          GMRES STEP 25, RESIDUAL= 0.9083E+00 (   0.00 DB)
          GMRES STEP 26, RESIDUAL= 0.1915E-04 ( -46.76 DB)
          GMRES STEP 27, RESIDUAL= 0.6527E-05 ( -51.44 DB)
          GMRES STEP 28, RESIDUAL= 0.2510E-05 ( -55.59 DB)
          GMRES STEP 29, RESIDUAL= 0.4689E-06 ( -62.87 DB)
INFO 240: GMRES STEP 29, RESIDUAL= 0.4689E-06 ( -62.87 DB).
          >>>>>>>>>>>> NEWTON STEP 5, RESIDUAL= 0.1299E-03 ( -43.22 DB)
COMPUTING THE GLOBAL MATRIX...
          GMRES STEP 30, RESIDUAL= 0.9083E+00 (   0.00 DB)
          GMRES STEP 31, RESIDUAL= 0.3846E-05 ( -53.73 DB)
          GMRES STEP 32, RESIDUAL= 0.1670E-05 ( -57.36 DB)
          GMRES STEP 33, RESIDUAL= 0.7543E-06 ( -60.81 DB)
INFO 240: GMRES STEP 33, RESIDUAL= 0.7543E-06 ( -60.81 DB).
          >>>>>>>>>>>> NEWTON STEP 6, RESIDUAL= 0.3413E-04 ( -49.05 DB)
COMPUTING THE GLOBAL MATRIX...
          GMRES STEP 34, RESIDUAL= 0.9083E+00 (   0.00 DB)
          GMRES STEP 35, RESIDUAL= 0.2110E-05 ( -56.34 DB)
          GMRES STEP 36, RESIDUAL= 0.1095E-05 ( -59.19 DB)
          GMRES STEP 37, RESIDUAL= 0.3836E-06 ( -63.74 DB)
INFO 240: GMRES STEP 37, RESIDUAL= 0.3836E-06 ( -63.74 DB).
          >>>>>>>>>>>> NEWTON STEP 7, RESIDUAL= 0.7897E-05 ( -55.43 DB)
INFO 240: >>>>>>>>>>>> NEWTON STEP 7, RESIDUAL= 0.7897E-05 ( -55.43 DB).
INFO 296: SOLVER INFORMATION
          NUMBER OF TIME STEPS              = 1
          NUMBER OF NON-LINEAR STEPS        = 7
              AVERAGE PER TIME STEP         = 7.0
          NUMBER OF LINEAR STEPS            = 37
```

Always check convergence of your computation