Beam Dynamics in Linacs II

CERN Accelerator School
High Power Hadron Machines
Bilbao
26th May 2011

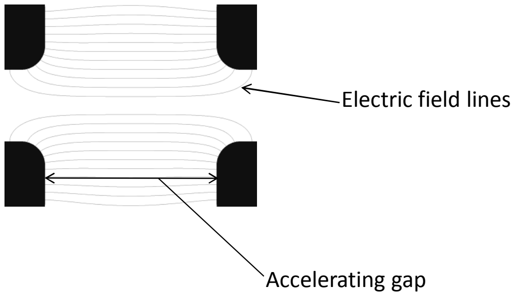
Alan Letchford
ISIS, Rutherford Appleton Laboratory

Lecture II: Linac Longitudinal Dynamics

- Acceleration by an RF field
- Phase stability
- Acceleration by a series of gaps
- Equations of motion
- Separatrix
- Synchrotron oscillation
- Phase Damping
- Thin Gap Transformation

Acceleration by an RF Field

A proton linac typically consists of a series of RF cavities with gaps designed to efficiently accelerate the particles.



For the longitudinal dynamics only the on-axis component of the field is considered.

Energy gain in an RF gap

For a generic RF gap of frequency ω and length L with an axial electric field

$$E_z(r, z, t)$$

the field experienced by an on axis particle is given by

$$E_z(r=0, z, t) = E(0, z)\cos[\omega t(z) + \phi]$$

where $$t(z) = \int_0^z \frac{dz}{v(z)}$$

Taking the origin to be the centre of the gap when t = 0 and RF phase = ϕ then the energy gain is

$$\Delta W = q\int_{-L/2}^{L/2} E(0, z)\cos[\omega t(z) + \phi]dz$$

Average Field

Using a trigonometric identity the energy gain can be written

$$\Delta W = q\int_{-L/2}^{L/2} E(0,z)[\cos\omega t\cos\phi - \sin\omega t\sin\phi]dz$$

Which is expressed in the conventional form as

$$\Delta W = qE_0TL\cos\phi$$

where $$E_0 = \frac{1}{L}\int_{-L/2}^{L/2} E(0,z)dz$$

is the average on-axis electric field and

$$T = \frac{\int_{-L/2}^{L/2} E(0,z)\cos\omega t(z)dz}{\int_{-L/2}^{L/2} E(0,z)dz} - \tan\phi\frac{\int_{-L/2}^{L/2} E(0,z)\sin\omega t(z)dz}{\int_{-L/2}^{L/2} E(0,z)dz}$$

is the Transit Time Factor.

Transit Time Factor

If $E(0,z)$ is symmetric about z=0 then

$$\int_{-L/2}^{L/2} E(0,z)\sin\omega t(z)dz = 0$$

and $$T = \frac{\int_{-L/2}^{L/2} E(0,z)\cos\omega t(z)dz}{\int_{-L/2}^{L/2} E(0,z)dz}$$

Further, if the change in particle velocity across the gap is small

$$\omega t \approx \frac{\omega z}{\beta c} = \frac{2\pi z}{\beta\lambda}$$

giving $$T \approx \frac{\int_{-L/2}^{L/2} E(0,z)\cos(2\pi z/\beta\lambda)dz}{\int_{-L/2}^{L/2} E(0,z)dz}$$

Synchronous Phase

$$\Delta W = qE_0TL\cos\phi$$

The value of ϕ at which the cavity is designed to operate is called the synchronous phase. A particle arriving at the cavity with the synchronous energy and synchronous phase will also arrive at all subsequent cavities at the synchronous energies and phases. Acceleration only occurs when $\cos(\phi_s)$ is positive:

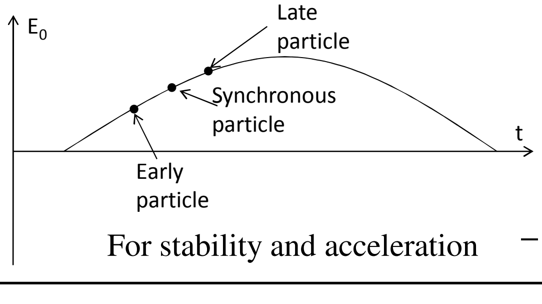
$$-\frac{\pi}{2} < \phi_s < \frac{\pi}{2}$$

Phase Stability

Phase stability occurs when the accelerating field is rising in time

$$-\pi < \phi_s < 0$$

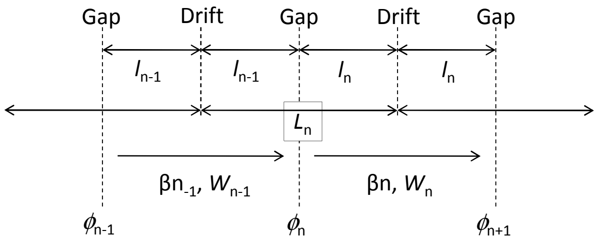
A particle that arrives earlier than the synchronous phase receive less acceleration than the synchronous particle. A later particle receives more acceleration. The effect is longitudinal focusing which drives the particles towards the synchronous phase.



For stability and acceleration $-\frac{\pi}{2} < \phi_s < 0$

Acceleration by a series of gaps

Longitudinal motion is studied by treating the linac as a series of thin gaps separated by field free drifts



$$\phi_n = \phi_{n-1} + \frac{2\omega l_{n-1}}{\beta_{n-1} c} + \begin{cases} 0 \text{ for } 0 \text{ mode} \\ \pi \text{ for } \pi \text{ mode} \end{cases}$$

$$l_{n-1} = \frac{N\beta_{s,n-1}\lambda}{2} \text{ where } \begin{cases} N = 1 \text{ for } 0 \text{ mode} \\ \mathrm{N} = \frac{1}{2} \text{ for } \pi \text{ mode} \end{cases}$$

Relative Particle Phase

If the synchronous velocity is given by $\beta_{s,n}$ then

$$L_n = \frac{N}{2}\left(\beta_{s,n-1} + \beta_{s,n}\right)\lambda$$

and $$\Delta\left(\phi - \phi_s\right)_n = 2\pi N \beta_{s,n-1}\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}}\right)$$

If $$\beta - \beta_s = \frac{W - W_s}{mc^2 \beta_s \gamma_s^3} << 1$$

then $$\frac{1}{\beta} - \frac{1}{\beta_s} \approx -\frac{\beta - \beta_s}{\beta_s^2}$$

and $$\Delta\left(\phi - \phi_s\right)_n = -2\pi N \frac{W_{n-1} - W_{s,n-1}}{mc^2 \beta_{s,n-1}^2 \gamma_{s,n-1}^3}$$

Relative Particle Energy

The difference in the particle energy is simply the difference in the effective voltage

$$\Delta(W - W_s)_n = qE_0 T L_n (\cos\phi_n - \cos\phi_{s,n})$$

leading to a pair of coupled difference equations in relative energy and phase.

$$\Delta(\phi - \phi_s)_n = -2\pi N \frac{W_{n-1} - W_{s,n-1}}{mc^2 \beta_{s,n-1}^2 \gamma_{s,n-1}^3}$$

Equations of Motion

To study the particle motion analytically the difference equations can be converted to differential equations noting that

$$s = nN\beta_s \lambda$$

and letting $\Delta(\phi - \phi_s) = \dfrac{d(\phi - \phi_s)}{dn}$

and $\Delta(W - W_s) = \dfrac{d(W - W_s)}{dn}$

giving $\dfrac{d(\phi - \phi_s)}{ds} = -2\pi \dfrac{W - W_s}{mc^2 \beta_s^3 \gamma_s^3 \lambda}$

$$\frac{d(W - W_s)}{ds} = qE_0 T (\cos\phi - \cos\phi_s)$$

Hamiltonian

Combining the coupled equations gives a second order differential equation

$$\frac{d^2(\phi-\phi_s)}{ds^2} = -\frac{2\pi q E_0 T}{mc^2 \beta_2^3 \gamma_s^3 \lambda}(\cos\phi - \cos\phi_s)$$

Which leads to the Hamiltonian for the longitudinal motion

$$\frac{Aw^2}{2} + B(\sin\phi - \phi\cos\phi_s) = H_\phi$$

$$A = \frac{2\pi}{\beta_s^3 \gamma_s^3 \lambda} \quad B = \frac{qE_0 T}{mc^2} \quad w = \frac{W - W_s}{mc^2}$$

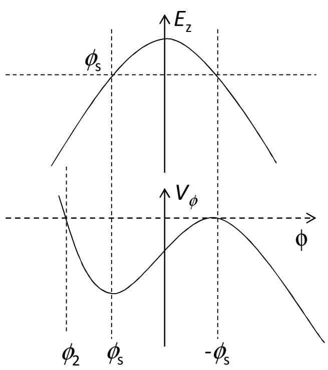
kinetic energy + potential energy = constant

Potential Well

The potential energy term

$$V_\phi = B(\sin\phi - \phi\cos\phi_s)$$

indicates a potential well for $-\pi < \phi_s < 0$



Separatrix

The potential well defines the region of stable phase motion which covers

$$\phi_2 < \phi < -\phi_s$$

At the upper limit

$$\frac{d\phi}{ds} = -Aw = 0$$

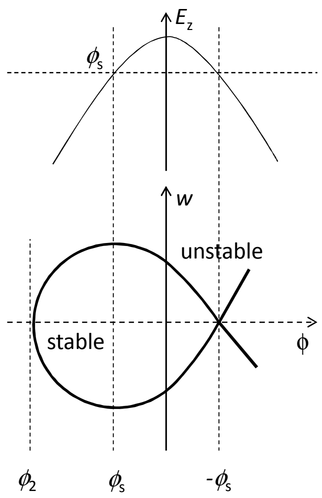
which defines the constant as

$$H_\phi = B\left(\sin(-\phi_s) - (-\phi_s \cos\phi_s)\right)$$

leading to the equation for the separatrix

$$\frac{Aw^2}{2} + B\left(\sin\phi - \phi\cos\phi_s\right) = B\left(\sin\phi_s - \phi_s\cos\phi_s\right)$$

Separatrix



The separatrix separates longitudinal phase space into stable and unstable regions.

The boundary can be plotted if A, B & ϕ_s are known.

The lower limit is given by

$$\sin\phi_2 - \phi_2\cos\phi_s = \phi_s\cos\phi_s - \sin\phi_s$$

Phase Width

The total phase width of the separatrix is

$$\psi = |\phi_s| + |\phi_2| = -\phi_s - \phi_2$$

Leading to

$$\tan\phi_s = \frac{\sin\psi - \psi}{1 - \cos\psi}$$

Energy Width

The maximum energy extent of the separatrix occurs at $\phi = \phi_s$.

Solving the separatrix equation gives

$$w_{\max} = \frac{(W - W_s)_{\max}}{mc^2} = \sqrt{\frac{2qE_0T\beta_s^3\gamma_s^3\lambda}{\pi mc^2}(\phi_s \cos\phi_s - \sin\phi_s)}$$

Small Amplitude Oscillations

For a phase difference which is small compared to the synchronous phase, trigonometric approximations allow the equation of phase motion to be written

$$\frac{d^2\phi}{ds^2} + k_{l0}^2\left[(\phi-\phi_s) - \frac{(\phi-\phi_s)^2}{2\tan(-\phi_s)}\right] = 0$$

Where $$k_{l0}^2 = \left(\frac{2\pi}{\beta_s \lambda_{l0}}\right)^2 = \frac{2\pi q E_0 T \sin(-\phi_s)}{mc^2 \beta_s^3 \gamma_s^3 \lambda}$$

is the square of the longitudinal wave number.

The quadratic term reduces the focusing at large excursions.

Ellipse Representation

Similar approximations on the condition that

$$|\phi-\phi_s| << 1$$

lead to

$$Aw^2 + B\sin(-\phi_s)(\phi-\phi_s)^2 = 2(H_\phi + \phi_s \cos\phi_s - \sin\phi_s)$$

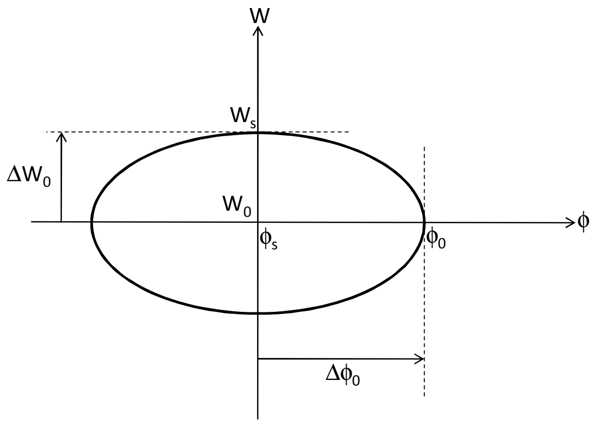
For $\phi_s < 0$ this is the equation of an upright ellipse with centre at w=0 and $\phi = \phi_s$.

If $$\phi_0 = \phi\big|_{w=0} \qquad \Delta\phi_0 = \phi_0 - \phi_s$$

then $$\frac{w^2}{w_0^2} + \frac{(\phi-\phi_s)^2}{\Delta\phi_0^2} = 1$$

where $$w_0 = w\big|_{\phi=\phi_s} = \sqrt{\frac{qE_0 T \beta_s^3 \gamma_s^3 \lambda \Delta\phi_0^2 \sin(-\phi_s)}{2\pi mc^2}}$$

Ellipse Representation



Phase Damping

If the rate of acceleration is small Liouville's Theorem applies and the area of the ellipse in phase space is an adiabatic invariant.

$$\text{area} = \pi\Delta\phi_0\Delta W_0 = \pi\Delta\phi_0^2\sqrt{\frac{qE_0Tmc^2\beta_s^3\gamma_s^3\lambda\sin(-\phi_2)}{2\pi}}$$

If the accelerating field and synchronous phase are fixed

$$\Delta\phi_0 = \frac{\text{constant}}{(\beta_s\gamma_s)^{3/4}} \qquad \Delta W_0 = \text{constant}\times(\beta_s\gamma_s)^{3/4}$$

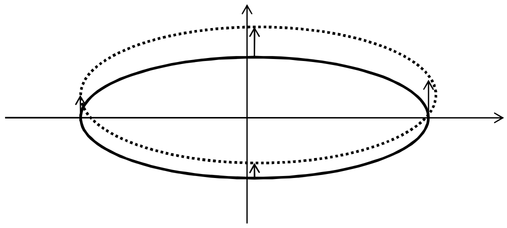
During acceleration the amplitude of the phase oscillations decrease while the amplitude of energy oscillations increases.

Thin Gap Transformation

Treating the accelerating cavity as a thin gap the energy and phase transform as

$$W_f = W_i + qE_0TL\cos\phi_i$$
$$\phi_f = \phi_i$$

where *i* and *f* refer to the coordinates before and after the transformation.
It can be shown that this doesn't satisfy Liouville's Theorem as the phase space area increases.



Liouvillian Thin Gap

By introducing a phase jump into the transformation the thin gap comes closer to satisfying Liouville.

$$W_f = W_i + qE_0TL\cos\phi_i$$
$$\phi_f = \phi_i + \frac{qE_0L}{mc^2\beta^2\gamma^3}kT'\sin\phi_i$$
$$T' = \frac{dT}{dk} = \frac{\int_{-L/2}^{L/2} zE(z)\sin(kz)dz}{\int_{-L/2}^{L/2} E(z)dz}$$
$$k = \frac{2\pi}{\beta\lambda}$$