Regulation Theory

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Introduction

Power converters:

Convert electric energy from one form to another that is optimally suited for user loads

⇨ Regulation is an important part of the design and construction of any power converter

Course objective:

- ✓ Recall common continuous-time control techniques
- ✓ Present digital control: Associated tools and more specifically the Z-transform, concept of discrete-time model, main methods to synthesize digital controllers, choice of the sampling frequency
- ✓ This course does not aim to describe control theory in a systematic and exhaustive way

  It will be limited to the control of single-input single-output linear time-invariant systems

  Non-linear control theory will not be presented here

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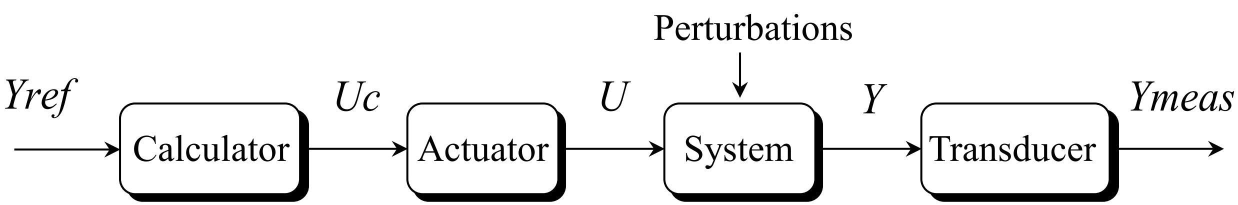
Basic recalls

- **Control theory**

  - Interdisciplinary branch of engineering and mathematics that deals with the behavior of dynamical systems with inputs

  - Usual objective: Provide the input(s) to a system to obtain the desired effect on its output(s)



If the system is a single-input single-output linear and time-invariant system, then its input and output are related by a differential equation with constant coefficients:

$$a_n \cdot \frac{d^n}{dt^n} y + \ldots + a_0 \cdot y = b_0 \cdot u + \ldots + b_m \cdot \frac{d^m}{dt^m} u \qquad m \leq n \qquad \text{n = system order}$$

= Time domain representation

Basic recalls

Taking the Laplace transform of both sides and assuming zero initial conditions:

$$a_n \cdot s^n Y + \ldots + a_0 \cdot Y = b_0 \cdot U + \ldots + b_m \cdot s^m U$$

where s is the Laplace operator

=> $$\frac{Y(s)}{U(s)} = \frac{\sum_{i=0}^{m} b_i \cdot s^i}{\sum_{j=0}^{n} a_i \cdot s^j} = \frac{Num(s)}{Den(s)} = H(s)$$

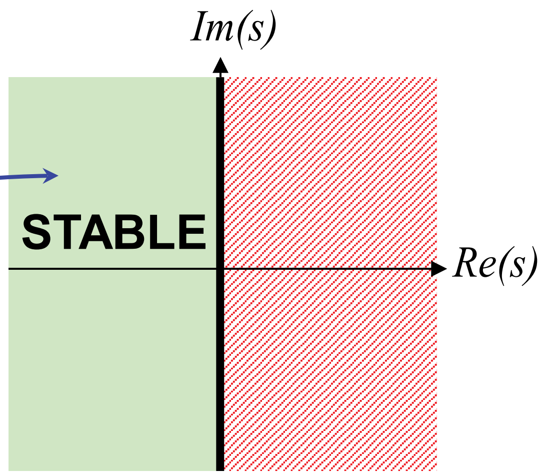
H(s) = system transfer function

= Frequency domain representation

NB :

- poles = roots of the polynomial $Den(s)$
  zeros = roots of the polynomial $Num(s)$

- Stability condition:
  determined by the location of the poles of H(s)



- if H(s) behaves like K/s^{α} for $s \to 0$: α = system class
  K = H(0) = static gain

- System with delay:
  u(t) => u(t – t0) Reflects the fact that the input will act with a delay of t0
  H(s) => H(s).$e^{-s.t0}$ = Laplace time shifting property

Basic recalls

- **Why use feedback control ?**

What if we design a controller that equals the plant inverse:

$C = H^{-1}$ (Feedforward correction)

Then in theory (assuming a unit gain for the actuator TF): Y = Yref

Impossible in practice because of:

- Uncertainty or variation of the model parameters
- Random disturbances

⇨ A feedback correction is needed

Advantages:

- Guaranteed performance even with model uncertainties
- Reduced sensitivity to parameter variations
- Disturbance rejection
- Stabilization of unstable open-loop systems

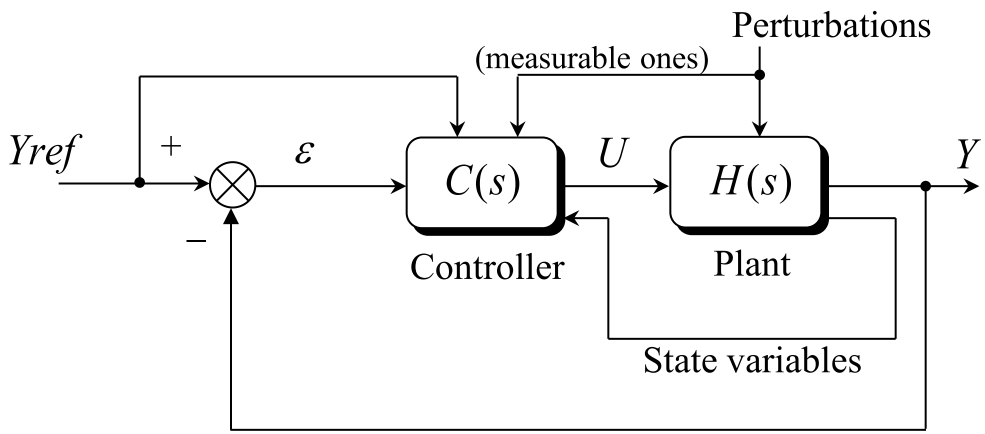
Drawback (contrary to feedforward correction):

Reaction after the error has arisen → Slowness

Basic recalls

- **Typical control structure**

= Combination of feedback and feedforward control



Closed control loop performance criteria:

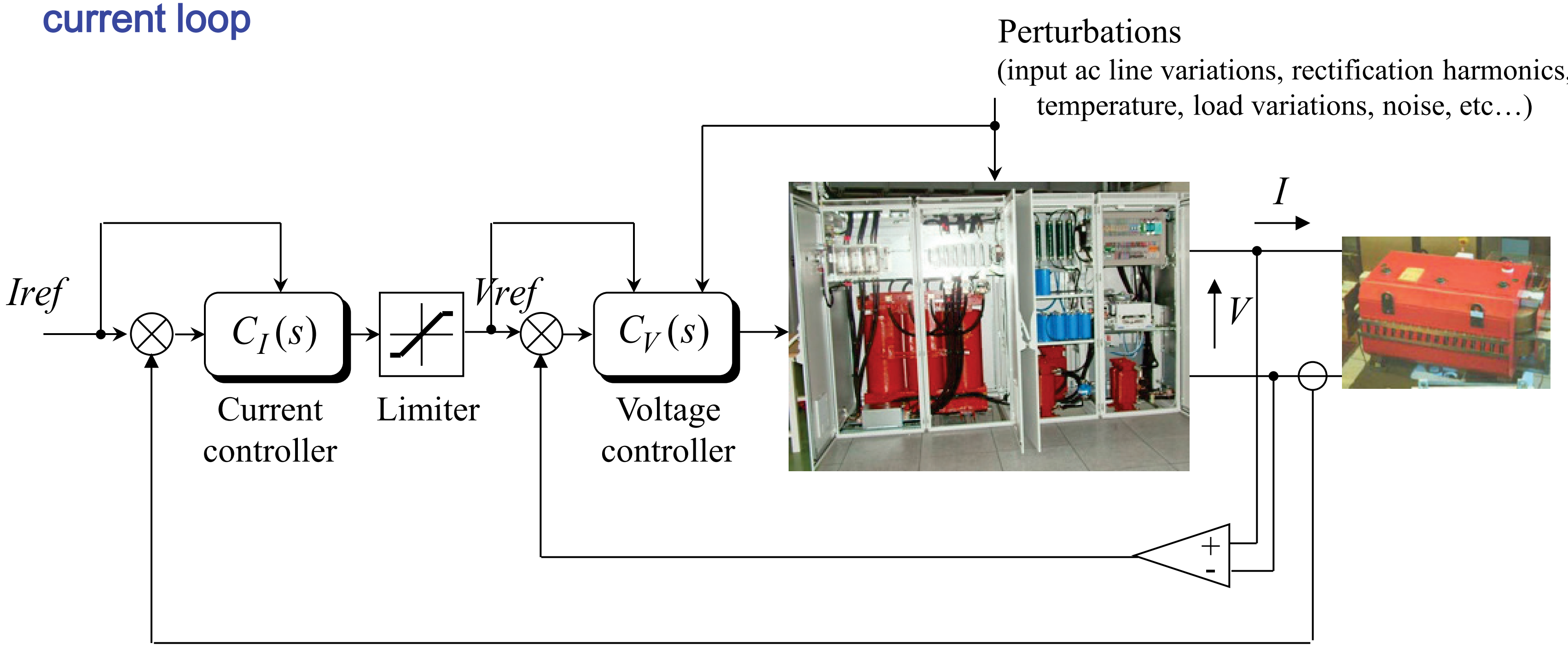
- Static & dynamic behavior
- Stability & robustness

⇨ Translate into constraints on the frequency response of the compensated system in open-loop (= Controller + Plant)

Basic recalls

- **Typical control structure for current-regulated power supplies**

= Cascade structure: Nested connection of a fast inner voltage loop and a slower outer current loop

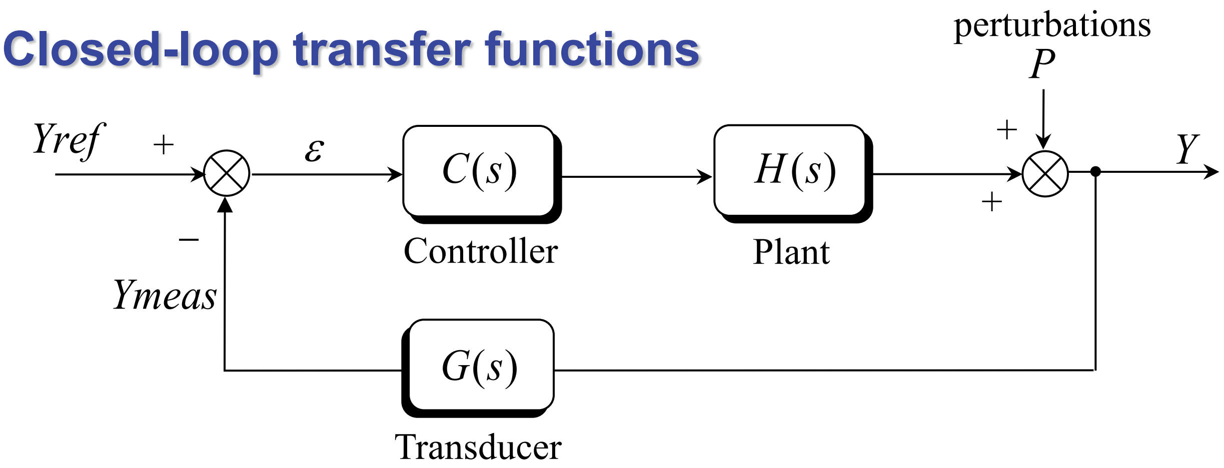


- **Fast inner voltage loop:** Acts as an active filter to reject the output voltage ripple and the output voltage fluctuations due to float of input mains + simplifies the design of the current loop

- **Outer current loop:** Ensures the overall stability of the power supply + provides an inherent over-voltage limitation

Analysis of closed-loop systems

- **Open-loop / Closed-loop transfer functions**



Open-loop: $OL(s) = \dfrac{Ymeas(s)}{\varepsilon(s)} = C(s) \cdot H(s) \cdot G(s)$

Open-loop bandwidth: Interval of pulsatances for which $\left|C(jw) \cdot H(jw) \cdot G(jw)\right| > 1$

Closed-loop: $CL(s) = \dfrac{Y(s)}{Yref(s)} = \dfrac{C(s) \cdot H(s)}{1 + C(s) \cdot H(s) \cdot G(s)}$

Error vs. input & perturbation: $\varepsilon(s) = \varepsilon_{Yref}(s) + \varepsilon_P(s)$

$$\frac{\varepsilon_{Yref}(s)}{Yref(s)} = \frac{1}{1 + C(s) \cdot H(s) \cdot G(s)}$$

$$\frac{\varepsilon_P(s)}{P(s)} = -\frac{G(s)}{1 + C(s) \cdot H(s) \cdot G(s)}$$

NB: Defining transfer functions in Matlab → Function ‘tf’

Analysis of closed-loop systems

- **Precision of closed-loop systems**
  - Static error

    - Precision versus the input

Final value theorem: $\lim\limits_{t\to\infty} \varepsilon_{Yref}(t) = \lim\limits_{s\to 0} s \cdot \varepsilon_{Yref}(s) = \lim\limits_{s\to 0} \frac{s}{1+C(s)\cdot H(s)\cdot G(s)} \cdot Yref(s)$

=> To achieve zero steady-state error, we require

- at least 1 integrator (pole @ s =0) in the open-loop TF for a step input ($Yref(s) = K/s$)
- at least 2 integrators in the open-loop TF for a ramp input ($Yref(s) = K/s^2$)
- …
- Sinusoidal input: $K \cdot \sin(w_0 \cdot t)$

At steady state, ε_{Yref} is a harmonic signal which module $|\varepsilon_{Yref}|$ is such that

$$\frac{|\varepsilon_{Yref}|}{K} = \left|\frac{1}{1+C(s)\cdot H(s)\cdot G(s)}\right|_{s=jw_0}$$

=> if w_0 is inside the OL bandwidth: $\frac{|\varepsilon_{Yref}|}{K} \approx \left|\frac{1}{C(s)\cdot H(s)\cdot G(s)}\right|_{s=jw_0}$

Error amplitude inversely proportional to OL gain @ w_0

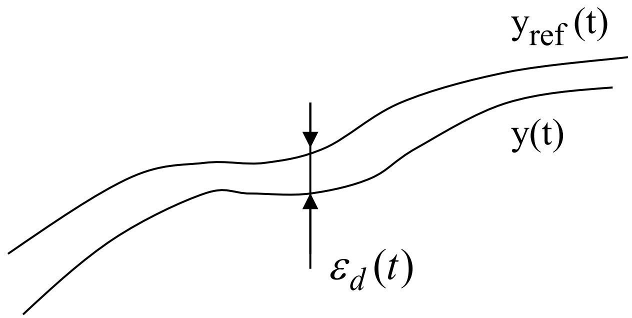
- Perturbation rejection

$$\lim_{t\to\infty} \varepsilon_P(t) = \lim_{s\to 0} s \cdot \varepsilon_P(s) = \lim_{s\to 0} \frac{-s\cdot G(s)}{1+C(s)\cdot H(s)\cdot G(s)} \cdot P(s)$$

To reject disturbances of class N → at least N integrators in $C(s)\cdot H(s)$

Analysis of closed-loop systems

- **Precision of closed-loop systems**
  - Dynamic error



Pb: Limit ε_d to $\varepsilon_{d\max}$

Assumption: Velocity v and acceleration γ of the input are limited
The input signal is then defined by the following constraints: $v < v_{\max}$ $\gamma < \gamma_{\max}$

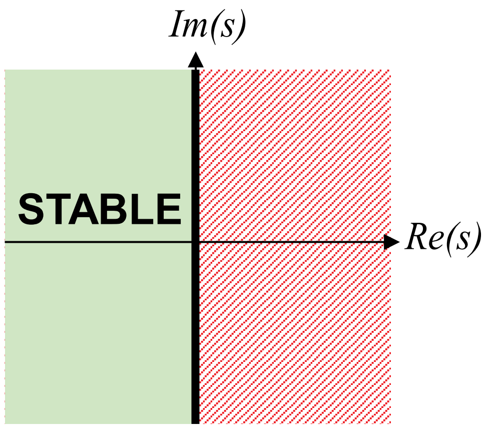
It can be demonstrated that

$$\varepsilon_d < \varepsilon_{d_{\max}} \quad \Rightarrow \quad \left|OL(s)\right|_{s=j\frac{\gamma_{\max}}{v_{\max}}} > \frac{v_{\max}{}^2}{\gamma_{\max} \cdot \varepsilon_{d_{\max}}}$$

Analysis of closed-loop systems

- **Stability & robustness of closed-loop systems**

$$CL(s) = \frac{Y(s)}{Yref(s)} = \frac{C(s)\cdot H(s)}{1 + C(s)\cdot H(s)\cdot G(s)}$$



s-plane stable pole location

Closed-loop stability ⇔

Real part of the closed-loop TF poles (= Roots of the characteristic equation $1 + C(s)\cdot H(s)\cdot G(s) = 0$) < 0

Phase Margin Φ_M : $\boxed{\Phi_M = 180° + \arg\left[OL(j\cdot w_{cr})\right]}$

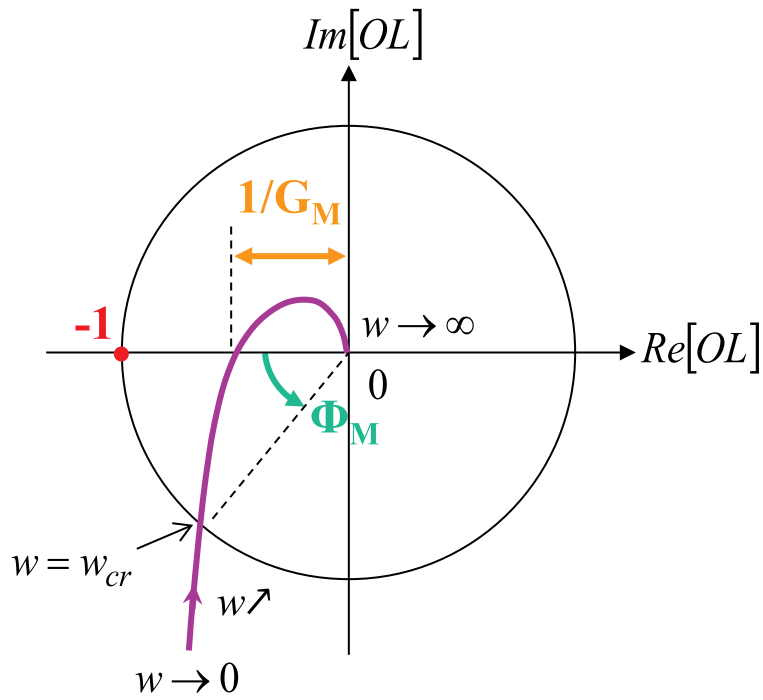
where w_{cr} is such that $\left|OL(j\cdot w_{cr})\right| = 1$

Gain margin G_M : $\boxed{G_M = \frac{1}{\left|OL(j\cdot w_{\pi})\right|}}$

where w_{π} is such that

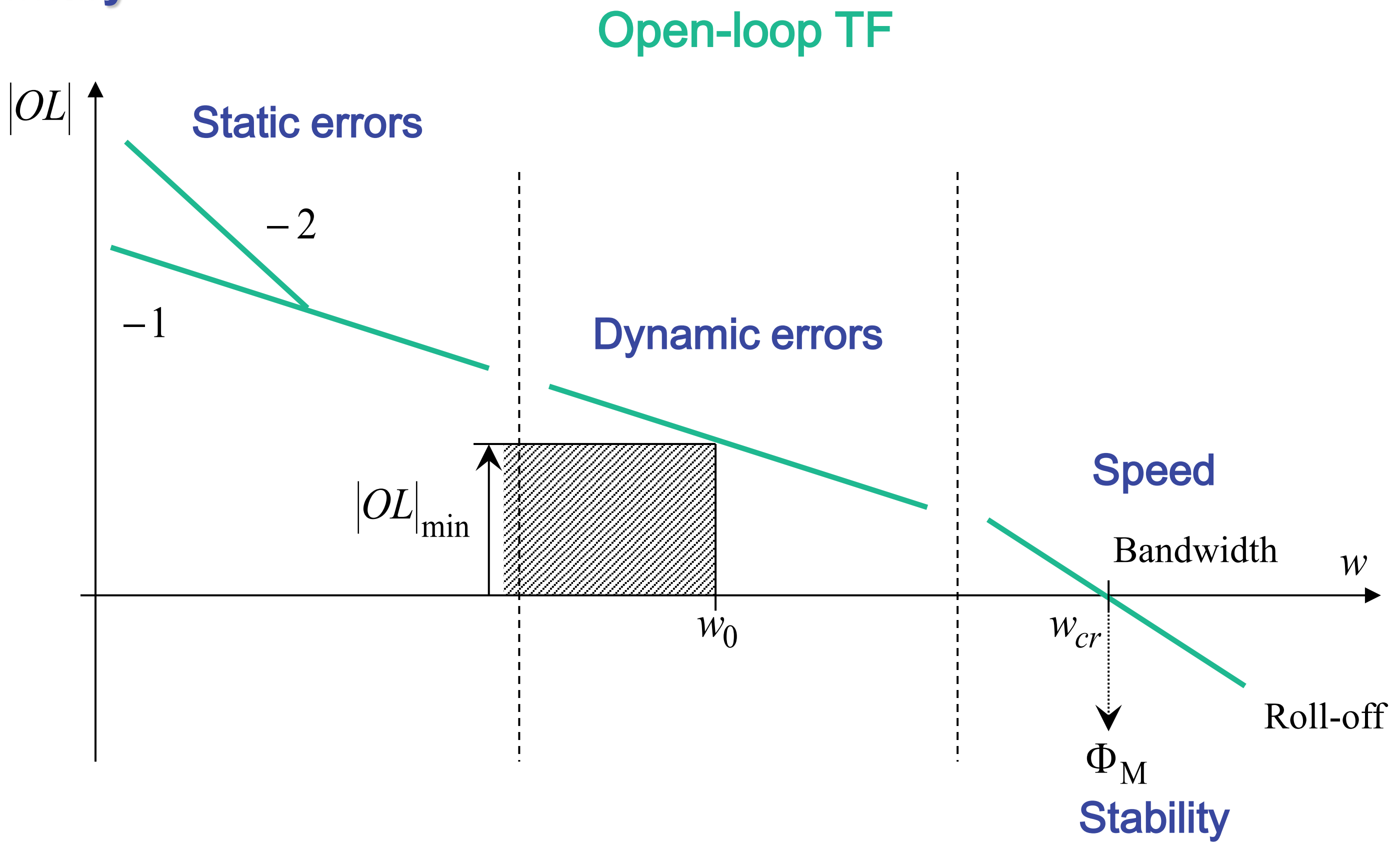
$$\arg\left[OL(j\cdot w_{\pi})\right] = -180°$$

Typically: $30° < \Phi_M < 60°$, $G_M > 6\text{dB}$



Analysis of closed-loop systems

- **Summary**

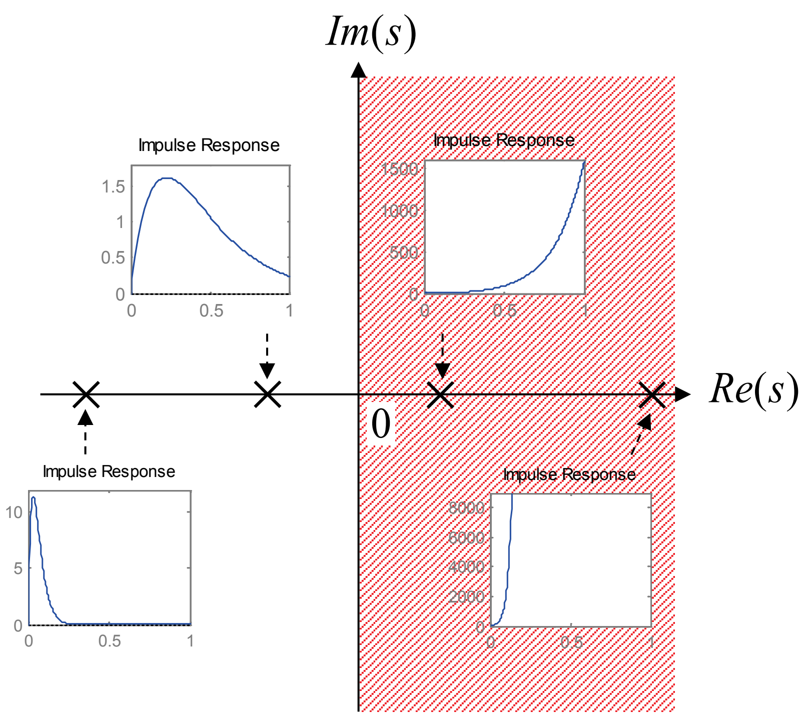


Contradiction between precision and stability ⇨ **Tradeoff**

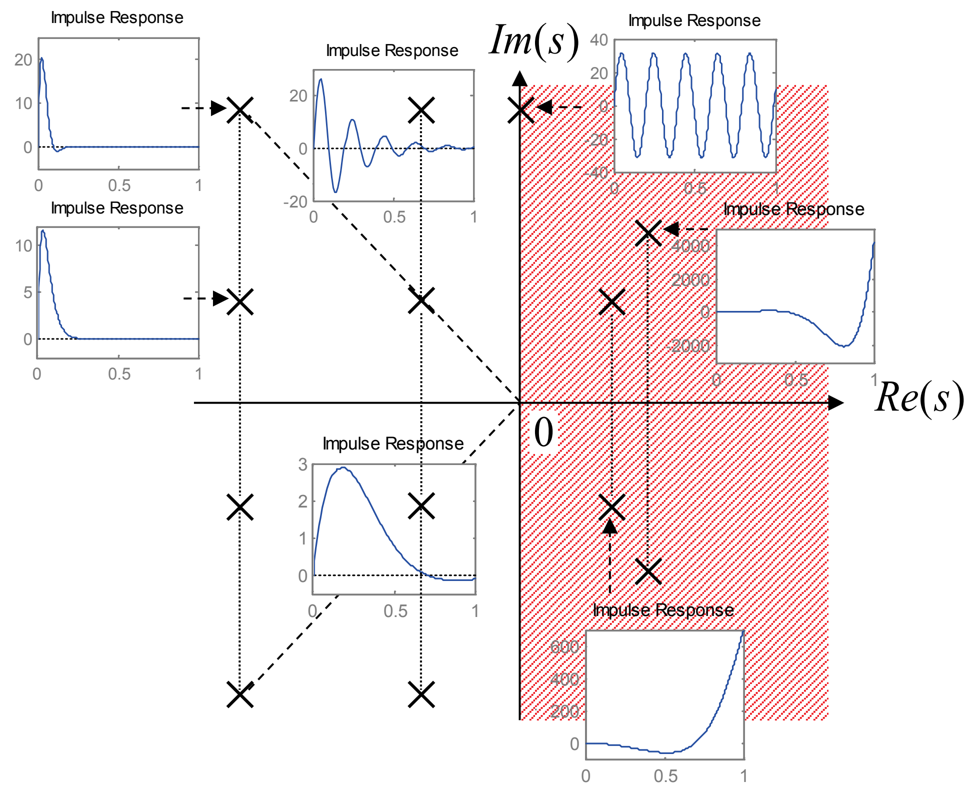
Analysis of closed-loop systems

- **Influence of the poles on the transient behavior**

Contribution of real poles



Contribution of complex poles



NB: Poles farther to the left → Faster transient regime

=> The poles closest to the imaginary axis are the ones that tend to dominate the response since their contribution takes a longer time to die out: Called dominant poles if the ratio of their real part to the one of any other poles < typically 1/5

Enables to simplify the TF by keeping the dominant pole(s) (and the static gain unchanged)

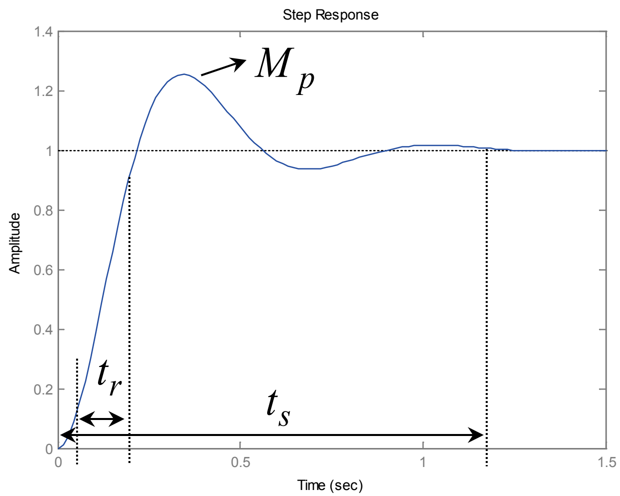
Analysis of closed-loop systems

- **Particular case: 2nd order systems with complex conjugate poles**

A common strategy for controller design consists to derive its parameters from a pole placement such that the closed-loop behaves like a 1rst order or a 2nd order system

2nd order Σ: The design specifications imply constraints on the dominant poles $p_1, \overline{p_1}$ => on the cut-off frequency w_n and the damping ratio ζ of the TF CL_{des}

$$CL_{des}(s) = \frac{w_n^{\ 2}}{s^2 + 2\cdot\zeta\cdot w_n\cdot s + w_n^{\ 2}}$$



Step input response

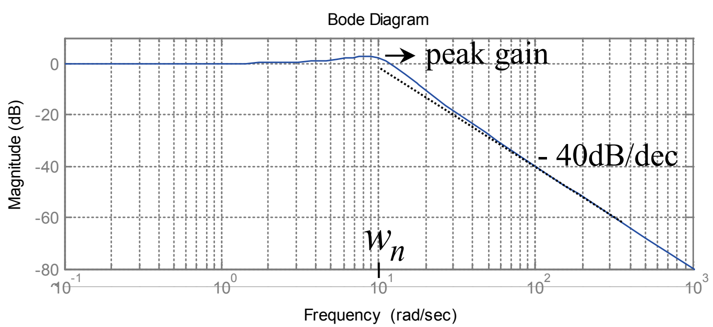
Rise time (10% → 90%): $t_r \approx (2.6\cdot\zeta^2 - 0.45\cdot\zeta + 1.2)/w_n$

Peak overshoot: $M_p = e^{-\pi\cdot\zeta/\sqrt{1-\zeta^2}}$

Settling time (to 1%): $t_s \approx 4.6/\zeta\cdot w_n$

$$\boxed{t_r,\ M_p,\ t_s \ \Rightarrow\ \zeta,\ w_n}$$

ζ	M_p in %	Φ_M in degrees
0,1	73	11
0,3	37	33
0,5	16	52
0,7	4,6	65
0,9	0,15	73



NB: For 2nd order systems, a good phase margin guarantees a good gain margin

Analysis of closed-loop systems

- **Particular case: 2nd order systems**

Influence of a zero

$$CL(s) = K \cdot \frac{s+z_0}{(s-p_1)\cdot(s-\overline{p_1})}$$

where $z_0 \in \mathbb{R}\_^*$ and $K = p_1 \cdot \overline{p_1}/z_0$ (→ unit static gain)

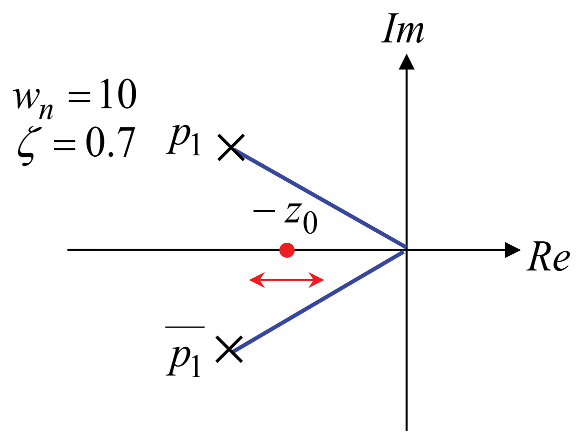
The unit step response of the above TF can be written as:

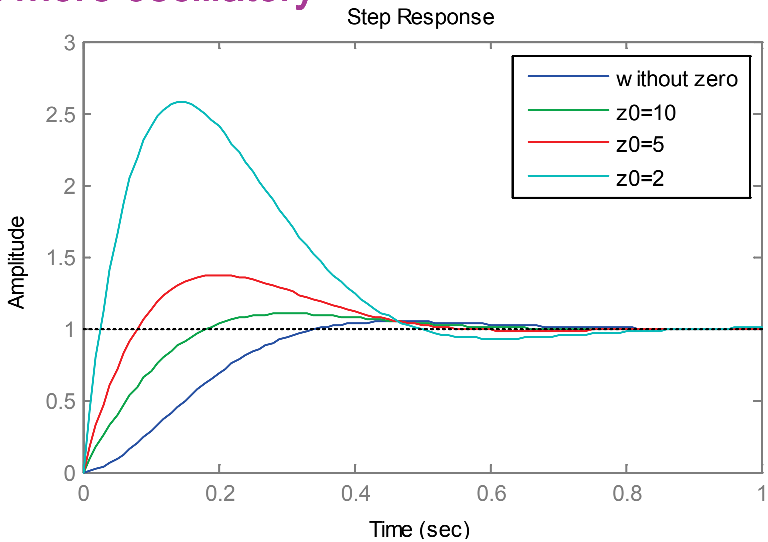
$$Y(s) = \frac{K}{s} \cdot \frac{s+z_0}{(s-p_1)\cdot(s-\overline{p_1})} = \frac{K \cdot z_0}{s\cdot(s-p_1)\cdot(s-\overline{p_1})} + \frac{K \cdot s}{s\cdot(s-p_1)\cdot(s-\overline{p_1})}$$

=> $$y(t) = y_{2^{nd}\,order}(t) + \frac{1}{z_0} \cdot \frac{d}{dt} y_{2^{nd}\,order}(t)$$

The additional zero makes the system faster and more oscillatory

= more prominent effect as z_0 decreases





Continuous-time controller synthesis

Controller design process:

1. Specification of the desired closed control loop performance => Tradeoff (cf. above)
   = Linked to plant dynamics & power availability of the actuator
   during transients (Prevent actuator saturation ⇨ Loss of controllability)
2. Choice of the controller type and its design method
3. Modelling of the plant to be controlled => Transfer function, state-space equations

To get the plant dynamic model:

1. Use physic laws to derive the differential equations used to represent it mathematically
   Power converters:
   - Construct equivalent averaged circuit model
   - Determine large-signal averaged model
   - Perturb and linearize about quiescent operating point to obtain small-signal averaged model
   - Simplify the transfer function (by keeping the dominant poles)

   If hysteretic control is to be used rather than PWM: Model this non-linear element using the 1rst order harmonic approximation method → Complex equivalent gain

2. Other method: Given generic model structure, estimate parameters from experimental data (= plant model identification)

Continuous-time controller synthesis

Proportional-integral-derivative (PID) controller

- By far the most widely used control algorithm
- Involves only 3 separate constant parameters to tune the control loop
  → Simple and intuitive (many controllers do not even use derivative action = PI)
  → Well-suited for systems exhibiting dominant 1rst or 2nd order behavior, for which the desired performance of the CL compared to the OL response of the Σ is not too demanding
  → For systems with higher order dominant dynamics, or systems including high delay or several oscillation modes, the PID is no longer adequate and a more complex regulator (with more parameters) has to be used

PID algorithm:

$$u(t) = K_p \cdot \varepsilon(t) + K_i \int_0^t \varepsilon(t) + K_d \cdot \frac{d}{dt}\varepsilon(t)$$

Controller TF standard form:

$$\boxed{C_{PID}(s) = K_p + \frac{K_i}{s} + \frac{K_d \cdot s}{1 + \frac{K_d}{N \cdot K_p} \cdot s}}$$

$$10 \le N_{typ.} \le 20$$

NB: Pure derivative amplifies noise => LP filter

Effects of PID tuning on closed control loop					
Param. change	**Rise time**	**Overshoot**	**Settling time**	**Steady-state error**	**System stability**
Increase *Kp*	Decrease	Increase	Small change	Decrease	Degrade
Increase *Ki*	Decrease	Increase	Increase	Eliminate	Degrade
Increase *Kd*	Small change	Decrease	Decrease	No effect	Improve if *Kd* small

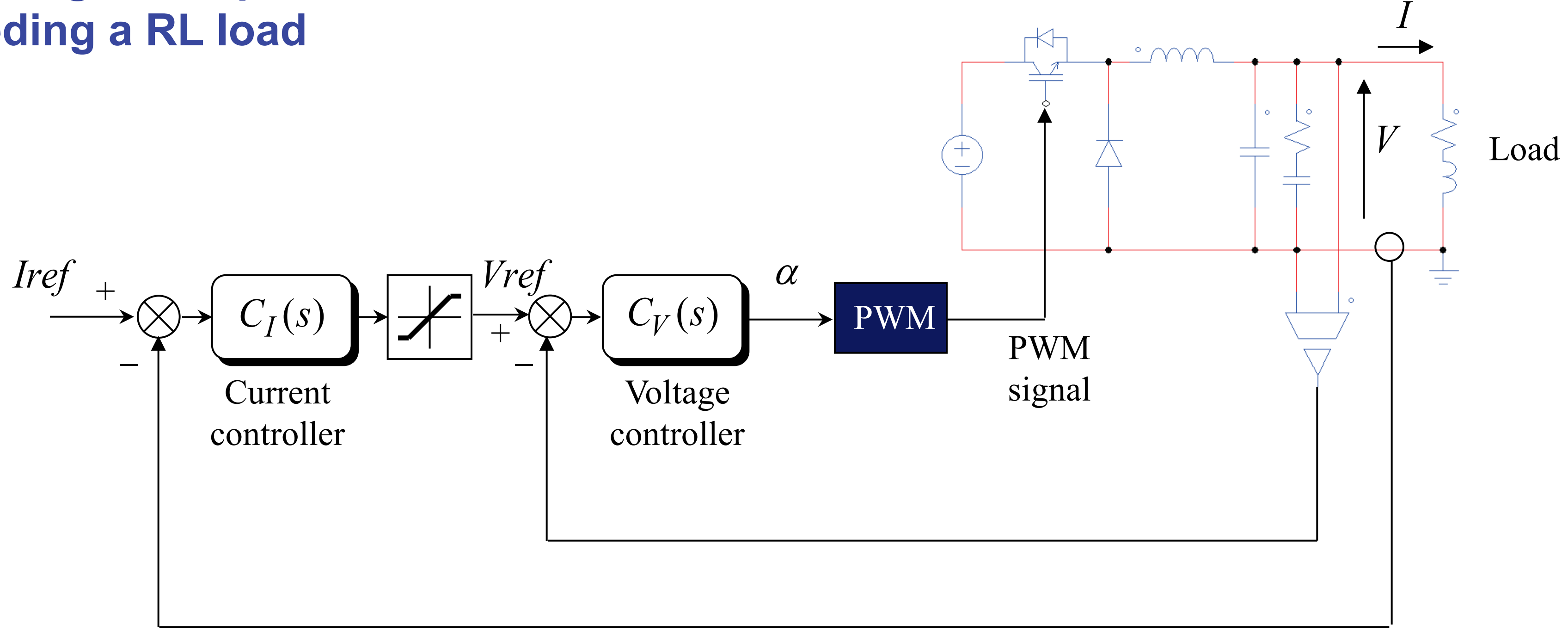
Continuous-time controller synthesis

PID synthesis methods

A number of alternative approaches for PID tuning are available:

- Heuristic PID tuning procedures: Ziegler-Nichols, Cohen-Coon, …
- Graphical methods: Loop shaping, root locus, …
- Pole placement
- Minimization of integral type criterion
- …

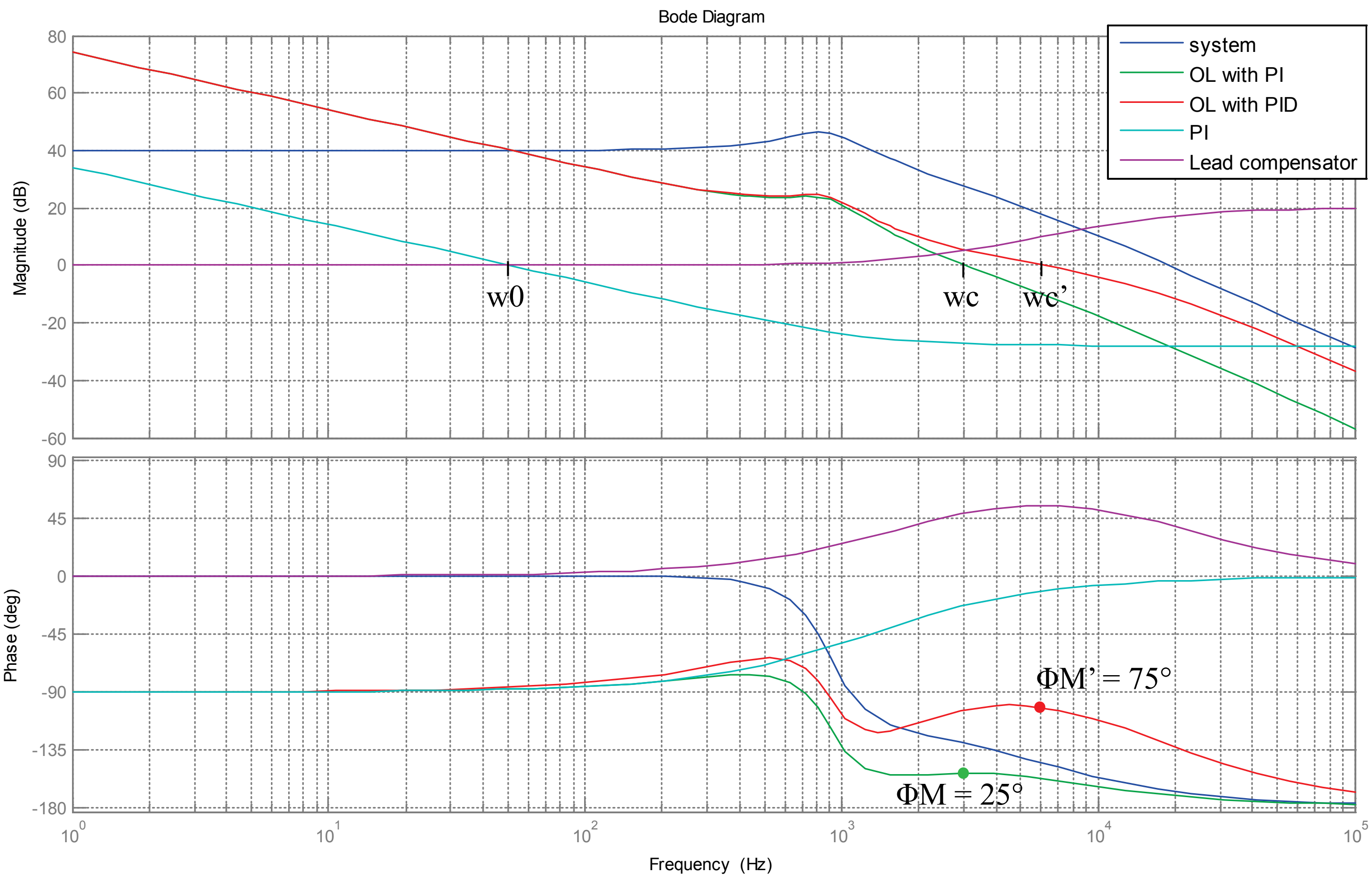
Putting it into practice: Control of a Buck converter feeding a RL load



Continuous-time controller synthesis

Design of the voltage loop controller using the loop shaping method

Loop shaping is one of the primary methodologies used for designing classical controllers such as PIDs => The controller structure and gains are selected such that the magnitude of the OL frequency response has particular characteristics - or a particular shape



Assume the following specs:

1/ Zero static error

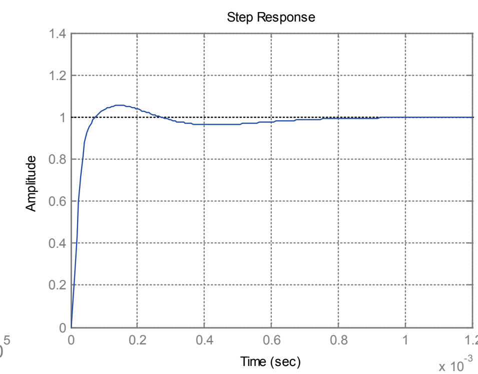
2/ Dynamic precision: |OL| > 40dB @ w0

3/ Bandwidth: {0, wc}

4/ $\Phi M \geq 50°$

A PI controller is first tried for Cv(s) but it leads to insufficient phase margin

→ A lead compensator is then added to correct ΦM



Continuous-time controller synthesis

Design of the current loop controller using pole placement

Open current loop transfer function:

$$OL_I(s) = C_I(s) \cdot CL_V(s) \cdot \frac{b_0}{1 + a_1 \cdot s}$$ where $b_0 = 1/R_{load}$ $a_1 = L_{load}/R_{load}$

With a PI controller for Ci(s) and after Σ simplification by keeping the dominant pole:

$$OL_I(s) \approx k_p \cdot \left(1 + \frac{k_i}{s}\right) \cdot \frac{b_0}{1 + a_1 \cdot s}$$

Setting $k_I = 1/a_1$ (pole cancellation) the closed loop can be written as

$$CL_I(s) = \frac{1}{1 + (a_1/k_p \cdot b_0) \cdot s}$$

Then choosing $k_p = a_1 \cdot w_n / b_0$ with $w_n = 2.2/t_r$, the CL behaves like a 1st order system with a rise time equal to t_r

This pole cancelation method requires a good knowledge of the process. If a_1 is likely to vary ($L_{load} = f(I)$) and especially when the pole $-1/a_1$ is close to the origin, the following pole placement gives better results (Cf. next slide): $k_p = (a_1 \cdot 2 \cdot \xi \cdot w_n - 1)/b_0$ $k_i = a_1 \cdot w_n{}^2 / (k_p \cdot b_0)$

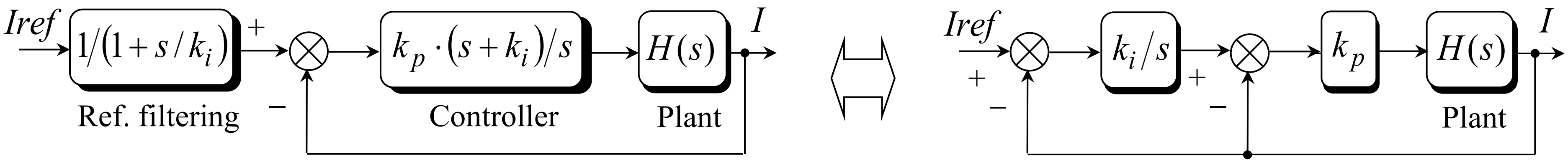
The CL then behaves like a 2nd order system which characteristic eq. = $s^2 / w_n{}^2 + (2 \cdot \xi / w_n) \cdot s + 1$

Pb: This controller setting gives rise to a zero $-k_i$ in the CL TF which may affect the transient response => Solution to cancel this zero = filter the reference

Continuous-time controller synthesis

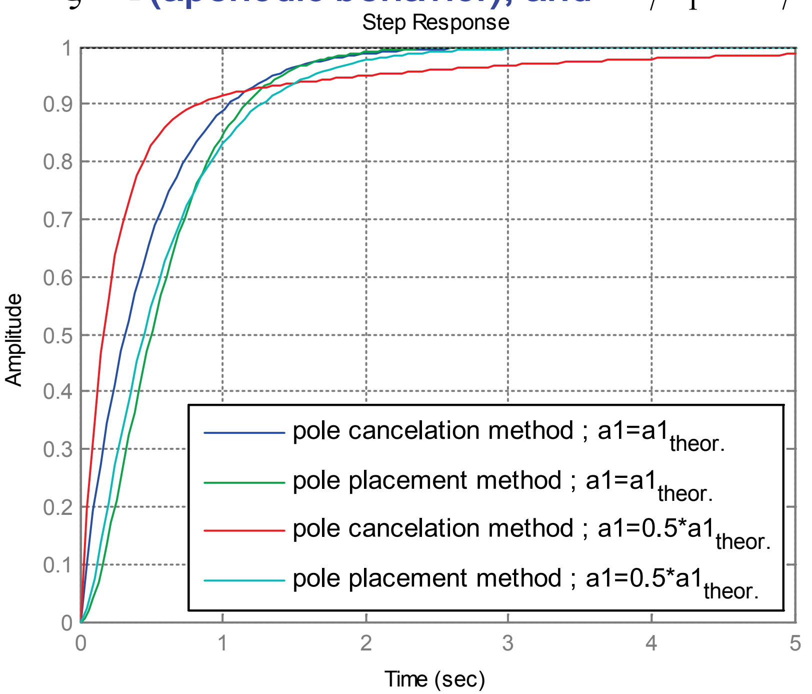
Design of the current loop controller using pole placement

Solution to cancel the zero of the CL TF:



IP controller

Results after zero cancellation, for $t_r=1$, $\xi=1$ (aperiodic behavior), and $-1/a_1=-1/2$



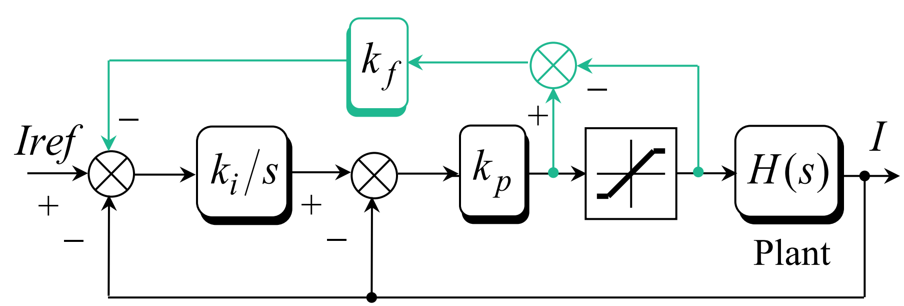
Integral windup issue:

When the output of the current or voltage controller reaches its saturation value, the integral part of the controller gets 'overcharged'

At the end of the saturated mode of operation, a negative error is needed to remove the accumulated positive error , which may give large transients

=> Anti-windup mechanism

Example:



Digital Control

Benefits and consequences of using digital control over analog

- Development of digital technology over the past 2 decades
  => Improvement in performance, cost and usability
- Increasing demands for higher performance
  and monitoring capabilities

=> Growing use of digital control in power converters

Benefits of using digital control:

- Performance enhancement, as digital control allows more complex regulation schemes (Ex: Non linear, predictive, adaptive control strategies, …)
- Improved flexibility
- Better reliability and reproducibility (no ageing effects, thermal drifts, …)
- Better noise immunity
- Provides system monitoring and archiving capability
- More compact and lightweight
- Implementation of human-machine interface & external communication requires some kind of embedded processor
- …

One major issue: Time delays introduced to do computations of control algorithm in the processor

Other drawbacks = Aliasing, quantization errors, limit cycling, software bugs, …

NB: Interesting alternative = Mix of analog and digital

Introducing the Z-transform

Z transform = Major mathematical tool for analysis in such topics as digital control and digital signal processing

- Reminder: The Laplace transform X(s) of a continuous-time causal signal x(t) is given by

$$X(s) = \int_{0-}^{+\infty} x(t) \cdot e^{-st} \cdot dt \qquad s = \sigma + j \cdot w$$ such that X(s) converges

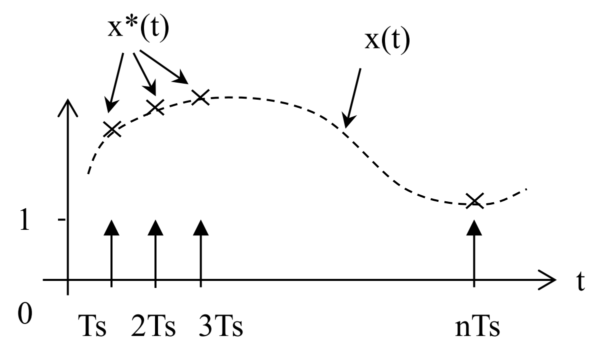
- Case of discrete-time causal signals:

$$x^*(t) = \sum_{k=0}^{+\infty} x(k) \cdot \delta(t - k \cdot Ts) \qquad x(k) = x(t)\big|_{t=k \cdot Ts}$$

{x(k)}, $k \in \mathbb{Z}$: Sequence of sampled values (= 0 $\forall$ k<0)

Ts: Sampling period (assumed constant)

δ: Dirac delta function



=> The Laplace transform X*(s) of a discrete-time signal x*(t) is given by

$$X^*(s) = \sum_{k=0}^{+\infty} x(k) \cdot e^{-s \cdot k \cdot Ts} \qquad (1)$$

Not a polynomial form…

Introducing the Z-transform

With the change of variable $z = e^{s \cdot Ts}$ in eq. (1), we derive the following expression
= definition of the Z-transform:

$$\boxed{X(z) = \sum_{k=0}^{+\infty} x(k) \cdot z^{-k}}$$

$\forall$ z $\in \mathbb{C}$ for which X(z) converges

=> Takes the form of a polynomial of the complex variable z

The Z-transform is the discrete-time counter-part of the Laplace transform
⇒ Essential tool for the analysis and design of discrete-time systems

Interpretation of the variable z^{-1}

From Laplace time shifting property, we know that $e^{-s \cdot Ts}$ is time delay by Ts second
Therefore $z^{-1} = e^{-s \cdot Ts}$ corresponds to unit sample period delay

Introducing the Z-transform

- **Properties of Z-transforms**

  - Linearity $$Z[\lambda \cdot x(k) + \mu \cdot y(k)] = \lambda \cdot X(z) + \mu \cdot Y(z)$$

  - Shifting property $$Z[x(k-n)] = z^{-n} \cdot X(z)$$

  - Convolution $$Z[x(k) * y(k)] = Z\left[\sum_{n=-\infty}^{n=+\infty} x(n) \cdot y(k-n)\right] = X(z) \cdot Y(z)$$

  - Multiply by k property $$Z[k \cdot x(k)] = -z \cdot \frac{d}{dz}(X(z))$$

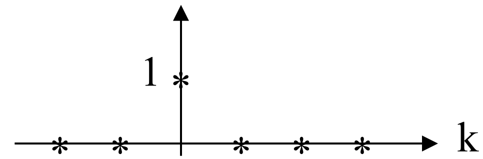
  - Final value $$\lim_{k \to \infty} x(k) = \lim_{z \to 1} (z-1) \cdot X(z)$$

Introducing the Z-transform

- **Examples of Z-transforms**

- Discrete impulse

$$x(k)=\delta(k)=\begin{cases}1 & k=0\\ 0 & k\neq 0\end{cases}$$

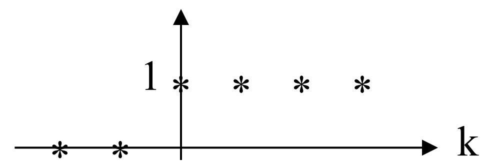


$$X(z)=\sum_{k=0}^{\infty}x(k)\cdot z^{-k}=x(0)+x(1)\cdot z^{-1}+x(2)\cdot z^{-2}+\ldots=x(0)$$

$$\boxed{\Rightarrow X(z)=1}$$

- Discrete step

$$x(k)=\begin{cases}1 & k\geq 0\\ 0 & k<0\end{cases}$$

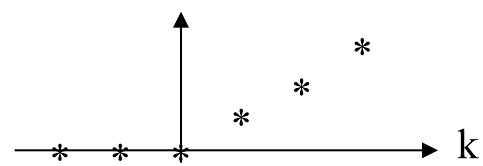


$$X(z)=x(0)+x(1)\cdot z^{-1}+x(2)\cdot z^{-2}+\ldots=1+z^{-1}+z^{-2}+\ldots$$

$$\boxed{\Rightarrow X(z)=\frac{1}{1-z^{-1}}\qquad |z|>1}$$

- Discrete ramp

$$x(k)=\begin{cases}k & k\geq 0\\ 0 & k<0\end{cases}$$



$$X(z)=-z\cdot\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right)\qquad \boxed{\Rightarrow X(z)=\frac{z^{-1}}{\left(1-z^{-1}\right)^2}\qquad |z|>1}$$

Introducing the Z-transform

- **Z-transform Table**

Time function	Laplace Transform	Discrete Time function	Z transform
$\delta(t)$	1	$\delta(nT)$	1
$u(t)$	$\frac{1}{s}$	$u(nT)$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	nT	$\frac{zT}{(z-1)^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$	$\frac{(nT)^2}{2}$	$\frac{z(z+1)T^2}{2(z-1)^3}$
e^{-at}	$\frac{1}{s+a}$	e^{-anT}	$\frac{z}{z-e^{-aT}}$
$t\,e^{-at}$	$\frac{1}{(s+a)^2}$	$nT\,e^{-anT}$	$\frac{zT\,e^{-aT}}{(z-e^{-aT})^2}$
$a^{t/T}$	$\frac{1}{s-(1/T)\ln(a)}$	a^n	$\frac{z}{z-a} \quad (a>0)$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega nT)$	$\frac{z\sin(\omega T)}{z^2-2z\cos(\omega T)+1}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\cos(\omega nT)$	$\frac{z^2-z\cos(\omega T)}{z^2-2z\cos(\omega T)+1}$

Relation between Laplace and Z-transforms

X(s) → X(z) ?

- **Case of signals having only simple poles**

$$X(s) = \sum_{i=1}^{N} \frac{A_i}{s - s_i} \qquad \Rightarrow x(t) = \sum_{i=1}^{N} A_i \cdot e^{s_i \cdot t} \quad t \geq 0$$

By sampling x(t), we obtain the following discrete sequence

$$x(k) = \sum_{i=1}^{N} A_i \cdot e^{s_i \cdot k \cdot Ts} \quad k \geq 0$$

From line 5 of the Z-transform table: $X(z) = \sum_{i=1}^{N} \frac{A_i \cdot z}{z - e^{si \cdot Ts}}$

$$\boxed{\Rightarrow X(s) = \sum_{i=1}^{N} \frac{A_i}{s - s_i} \xrightarrow{Z} X(z) = \sum_{i=1}^{N} \frac{A_i}{1 - e^{si \cdot Ts} \cdot z^{-1}}} \quad (2)$$

=> A pole s_i in X(s) yields a pole $z_i = e^{s_i \cdot Ts}$ in X(z)

$$\boxed{s_i \xrightarrow{Z} z_i = e^{s_i \cdot Ts}}$$

Relation between Laplace and Z-transforms

- **General case**

$$\boxed{X(z) = \sum_{s_i = poles\ of\ X(s)} \text{Residues}\left\{ X(s) \cdot \frac{1}{1 - e^{s \cdot Ts} \cdot z^{-1}} \right\}_{s = s_i}} \qquad (3)$$

Calculation of the residue at the pole s_j of multiplicity m:

$$\text{Residue}\left\{ X(s) \cdot \frac{1}{1 - e^{s \cdot Ts} \cdot z^{-1}} \right\}_{s = s_j} = \frac{1}{(m-1)!} \cdot \lim_{s \to s_j} \frac{d^{m-1}}{ds^{m-1}} \left[(s - s_j)^m \cdot \mathrm{X(s)} \cdot \frac{1}{1 - e^{s \cdot Ts} \cdot z^{-1}} \right]$$

For a simple pole (m = 1):

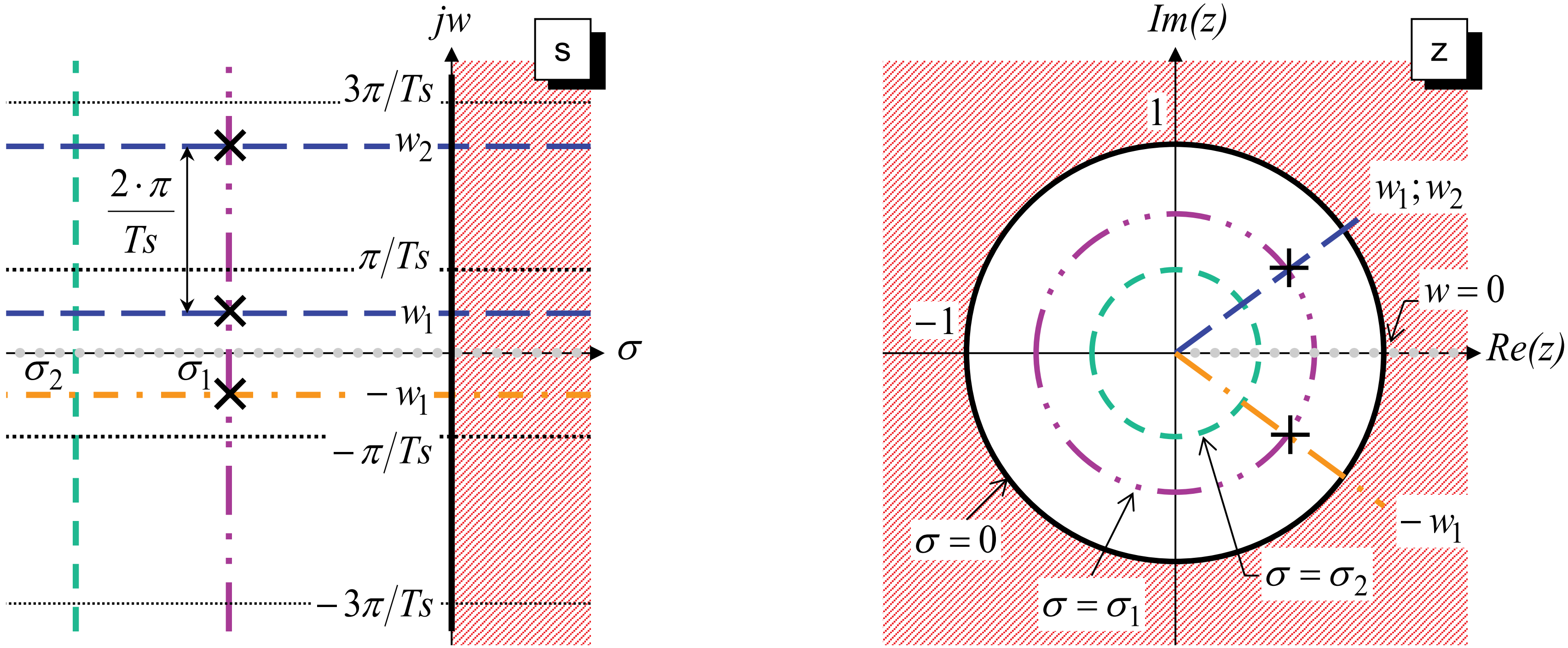
$$\text{Residue}\left\{ X(s) \cdot \frac{1}{1 - e^{s \cdot Ts} \cdot z^{-1}} \right\}_{s = s_j} = \lim_{s \to s_j} (s - s_j) \cdot \mathrm{X(s)} \cdot \frac{1}{1 - e^{s \cdot Ts} \cdot z^{-1}}$$

An example of calculation will be given further on in this document

Relation between Laplace and Z-transforms

- **Mapping from s-plane to z-plane**

Since $z_i = e^{s_i \cdot Ts} = e^{\sigma_i \cdot Ts} \cdot e^{j \cdot w_i \cdot Ts}$ we can map the s-plane to the z-plane as below:



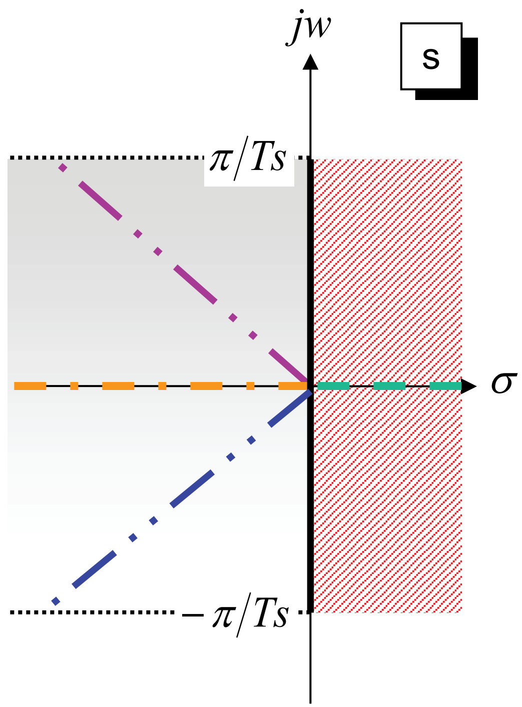
NB: 2 poles in the s-plane which imaginary part differ by $2\pi/Ts$ map to the same pole in the z-plane

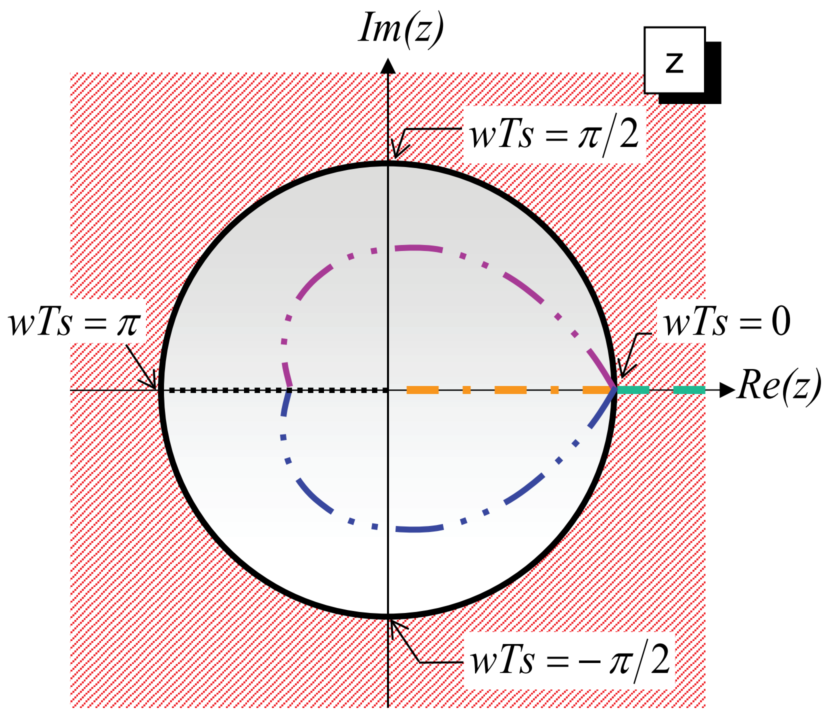
Bijective mapping between both planes => $Im(X(s)) \in [-\pi/Ts; +\pi/Ts]$

$$\Rightarrow \frac{\pi}{Ts} > \max_i \left\{ \left| Im(s_i) \right| \right\}$$

Relation between Laplace and Z-transforms

- **Mapping from s-plane to z-plane**



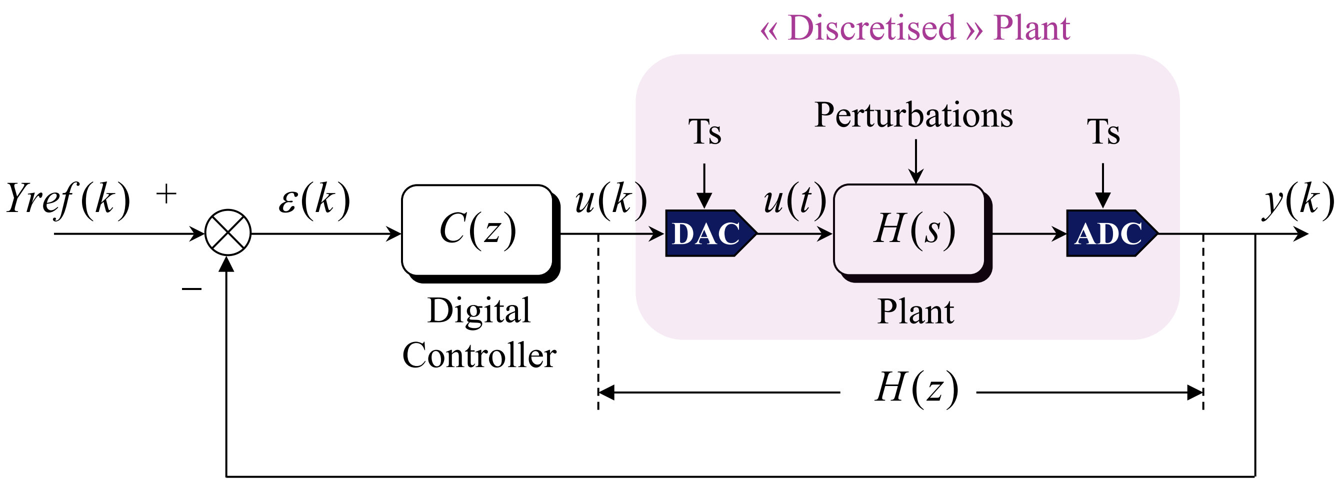


Legend:

———	$s = jw$	$\lvert z\rvert = 1$
– – –	$s = \sigma \geq 0$	$z = r, \quad r \geq 1$
– · – · –	$s = \sigma \leq 0$	$z = r, \quad 0 \leq r \leq 1$
– · · –	$\begin{cases} s = -\zeta w_n \pm j w_n \sqrt{1-\zeta^2} \\ \zeta = \text{Constant}, \quad w_n \text{ varies} \end{cases}$	Logarithmic spiral
············	$s = \sigma \pm j\pi/Ts$	$z = -r$

Digitally controlled continuous-time systems

- **Modelling of the plant to be controlled**



Model of the DAC = Zero-order hold (ZOH)

→ Converts u(k) to u(t) by holding each sample value for one sample interval

$$u(t) = u(k), \quad k \cdot Ts \le t \le (k+1) \cdot Ts$$

=> Delay introduced by the ZOH = Ts/2

The Laplace transform transfer function of the ZOH is

$$H_{ZOH}(s) = \frac{1 - e^{-s \cdot Ts}}{s}$$

$$\boxed{\Rightarrow H(z) = \left(1 - z^{-1}\right) \cdot Z\left[\frac{H(s)}{s}\right]}$$

If H(s) has poles $s = s_i$, then H(z) has poles $z = e^{s_i \cdot Ts}$. But the zeros are unrelated

Digitally controlled continuous-time systems

Calculation of $Z\left[\frac{H(s)}{s}\right]$:

- Partial fraction decomposition + use z-transform table

- If $\frac{H(s)}{s}$ has only simple poles, use Eq. (2):

$$\frac{H(s)}{s} = \frac{A_1}{s} + \sum_{i=2}^{N} \frac{A_i}{s - s_i} \xrightarrow{Z} X(z) = \frac{A_1}{1 - z^{-1}} + \sum_{i=2}^{N} \frac{A_i}{1 - e^{si \cdot Ts} \cdot z^{-1}}$$

- Use Eq. (3): $H(z) = \sum_{s_i = poles\ of\ H(s)/s} \text{Residues} \left\{ \frac{H(s)}{s} \cdot \frac{1}{1 - e^{s \cdot Ts} \cdot z^{-1}} \right\}_{s = s_i}$

- $H(z)$ → Ask Matlab: Function 'c2d'

Syntax

sysd = c2d(sys,Ts)

Description

sysd = c2d(sys,Ts) discretizes the continuous-time LTI model sys using zero-order hold on the inputs and a sample time of Ts seconds.

Digitally controlled continuous-time systems

Let $Y(z) = \mathrm{Z}[y(k)]$; $Yref(z) = \mathrm{Z}[Yref(k)]$; $E(z) = \mathrm{Z}[\varepsilon(k)]$; $U(z) = \mathrm{Z}[u(k)]$

- **Open-loop transfer function:** $\boxed{\dfrac{Y(z)}{E(z)} = C(z)\cdot H(z)}$

- **Closed-loop transfer function:** $\boxed{\dfrac{Y(z)}{Yref(z)} = \dfrac{C(z)\cdot H(z)}{1+C(z)\cdot H(z)}}$

- **Controller algorithm:**

Transfer function of the digital controller

$$C(z) = \frac{U(z)}{E(z)} = \frac{b_0 + b_1 \cdot z^{-1} + \ldots + b_p \cdot z^{-p}}{1 + a_1 \cdot z^{-1} + \ldots + a_n \cdot z^{-n}}$$

$$\Rightarrow \left(1 + a_1 \cdot z^{-1} + \ldots + a_n \cdot z^{-n}\right)\cdot U(z) = \left(b_0 + b_1 \cdot z^{-1} + \ldots + b_p \cdot z^{-p}\right)\cdot E(z)$$

Hence $\quad u(k) + a_1 \cdot u(k-1) + \ldots + a_n \cdot u(k-n) = b_0 \cdot \varepsilon(k) + b_1 \cdot \varepsilon(k-1) + \ldots + b_p \cdot \varepsilon(k-p)$

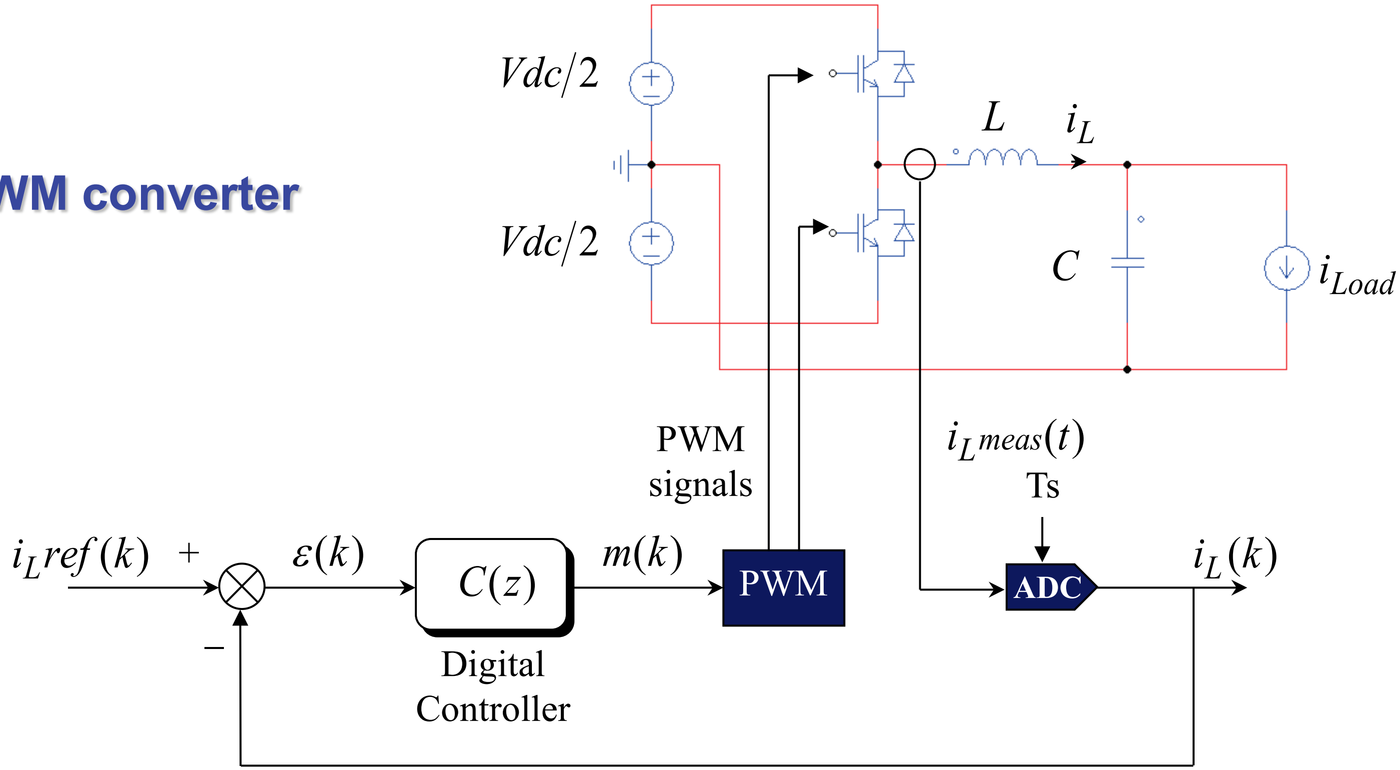
$\left(\mathrm{Z}[x(k-n)] = z^{-n} \cdot X(z)\right)$

$$\boxed{\Rightarrow u(k) = b_0 \cdot \varepsilon(k) + b_1 \cdot \varepsilon(k-1) + \ldots + b_p \cdot \varepsilon(k-p) - a_1 \cdot u(k-1) - \ldots - a_n \cdot u(k-n)}$$

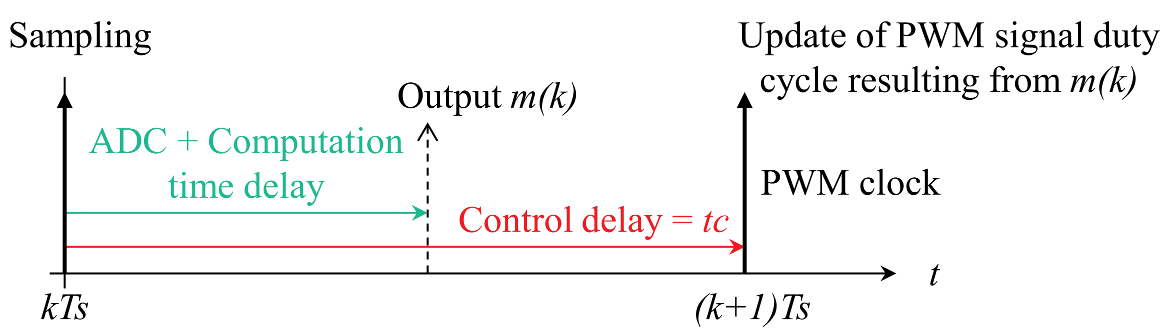
= Difference equation, where present output is dependent on present input and past inputs and outputs

Digitally controlled continuous-time systems

Example:
Half bridge PWM converter



Controller output =
modulation index (-1 < m < 1)

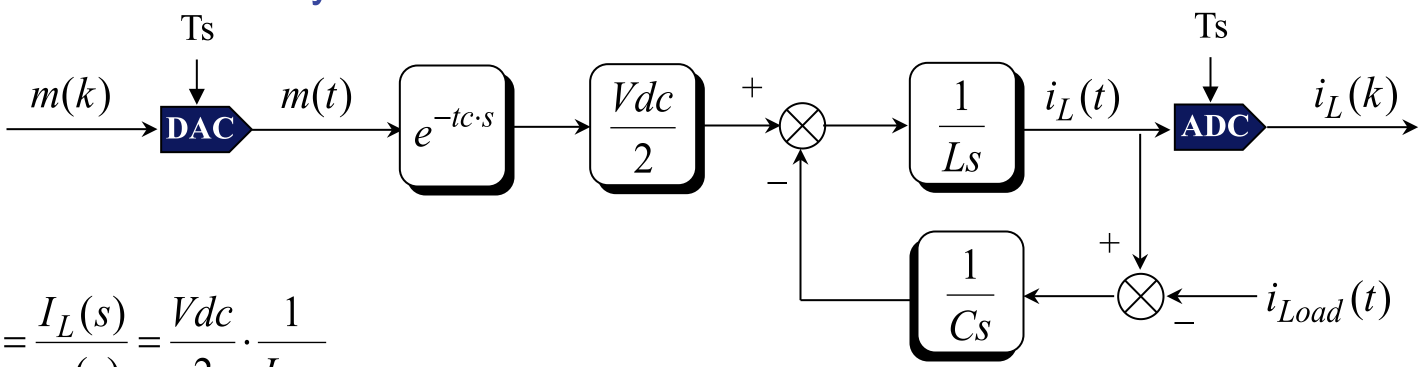


H(z) ?

Digitally controlled continuous-time systems

Modelling of the PWM converter

The PWM can be modelled by a DAC



$$H(s)=\frac{I_L(s)}{m(s)}=\frac{Vdc}{2}\cdot\frac{1}{L\cdot s}$$

$$\Rightarrow H(z)=\left(1-z^{-1}\right)\cdot \mathrm{Z}\left[\frac{H(s)}{s}\right]=\left(1-z^{-1}\right)\cdot\frac{Vdc}{2\cdot L}\cdot \mathrm{Z}\left[\frac{e^{-tc\cdot s}}{s^2}\right]=z^{-n}\cdot\left(1-z^{-1}\right)\cdot\frac{Vdc}{2\cdot L}\cdot \mathrm{Z}\left[\frac{e^{\theta\cdot s}}{s^2}\right]$$

with $tc=n\cdot Ts-\theta$, $0\leq\theta<Ts$

Using Eq. (3), with a double pole:

$$\mathrm{Z}\left[\frac{e^{\theta\cdot s}}{s^2}\right]=\frac{1}{(2-1)!}\cdot\lim_{s\to 0}\frac{d^{2-1}}{ds^{2-1}}\left[(s-0)^2\cdot\frac{e^{\theta\cdot s}}{s^2}\cdot\frac{1}{1-e^{s\cdot Ts}\cdot z^{-1}}\right]$$

$$=\lim_{s\to 0}\frac{d}{ds}\left[e^{\theta\cdot s}\cdot\frac{1}{1-e^{s\cdot Ts}\cdot z^{-1}}\right]=\frac{\theta\cdot\left(1-z^{-1}\right)+Ts\cdot z^{-1}}{\left(1-z^{-1}\right)^2}$$

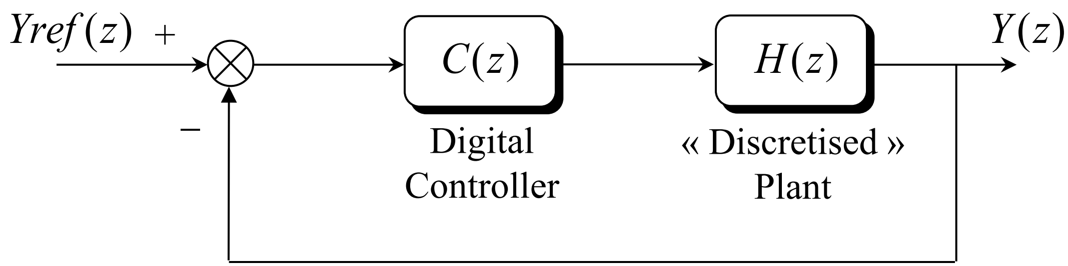
In this example $n=1$ and $\theta=0$

$$\boxed{\Rightarrow H(z)=\frac{Vdc\cdot Ts}{2\cdot L}\cdot\frac{z^{-2}}{1-z^{-1}}}$$

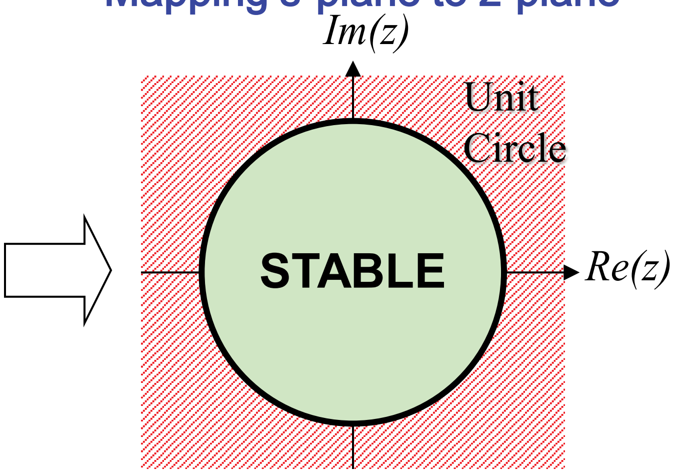
Analysis of digital closed-loops

System behavior & stability
= determined by the roots of the closed-loop TF polynomials

- **Stability of digital closed-loops**



Mapping s-plane to z-plane



z-plane stable pole location

Closed-loop transfer function:

$$CL(z) = \frac{Y(z)}{Yref(z)} = \frac{C(z)\cdot H(z)}{1+C(z)\cdot H(z)}$$

Closed-loop stability ⇔

Modulus of the closed-loop TF poles (= Roots of the characteristic equation $1+C(z)\cdot H(z)=0$) < 1

Analysis of digital closed-loops

- **Robustness**

Open-loop transfer function: $OL(z) = C(z) \cdot H(z)$

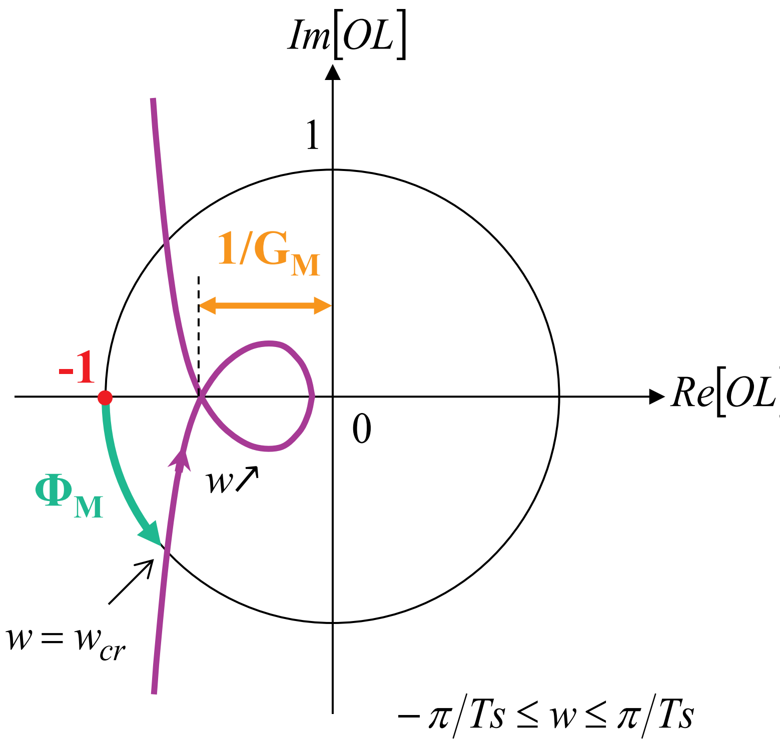
Phase margin Φ_M : $\boxed{\Phi_M = 180° + \arg\left[OL(e^{j \cdot w_{cr} \cdot Ts})\right]}$

where w_{cr} is such that $\left|OL(e^{j \cdot w_{cr} \cdot Ts})\right| = 1$

Gain margin G_M : $\boxed{G_M = \dfrac{1}{\left|OL(e^{j \cdot w_\pi \cdot Ts})\right|}}$

where w_π is such that

$$\arg\left[OL(e^{j \cdot w_\pi \cdot Ts})\right] = -180°$$



NB: Matlab plots & margins → Functions 'nyquist' , 'bode' , 'margin'

Analysis of digital closed-loops

- **Influence of the poles on the transient behavior**

Illustration with the step response analysis of a system CL(z) having only simple poles:

$$y(k) = CL(1) + \underbrace{\sum_{i=1}^{n} c_i \cdot z_i^{\,k}}_{\substack{real \\ poles}} + \underbrace{\sum_{j=1}^{m} \left|c_j\right| \cdot \left|z_j\right|^k \cdot \cos\left(k \cdot \theta_j + \varphi_j\right)}_{\substack{complex \\ conjugate \\ poles}}$$

- Steady-state:

$$\lim_{k \to \infty} y(k) = CL(1)$$

- Contribution of real poles z_i

Sum of exponential terms
$$\begin{cases} \xrightarrow[k\to\infty]{} 0 & if \ \left|z_i\right| < 1 \\ \xrightarrow[k\to\infty]{} \pm\infty & if \ \left|z_i\right| > 1 \end{cases}$$

- Contribution of complex conjugate poles z_j

Oscillating regime
$$\begin{cases} \xrightarrow[k\to\infty]{} 0 & if \ \left|z_j\right| < 1 \\ \xrightarrow[k\to\infty]{} \pm\infty & if \ \left|z_j\right| > 1 \end{cases}$$

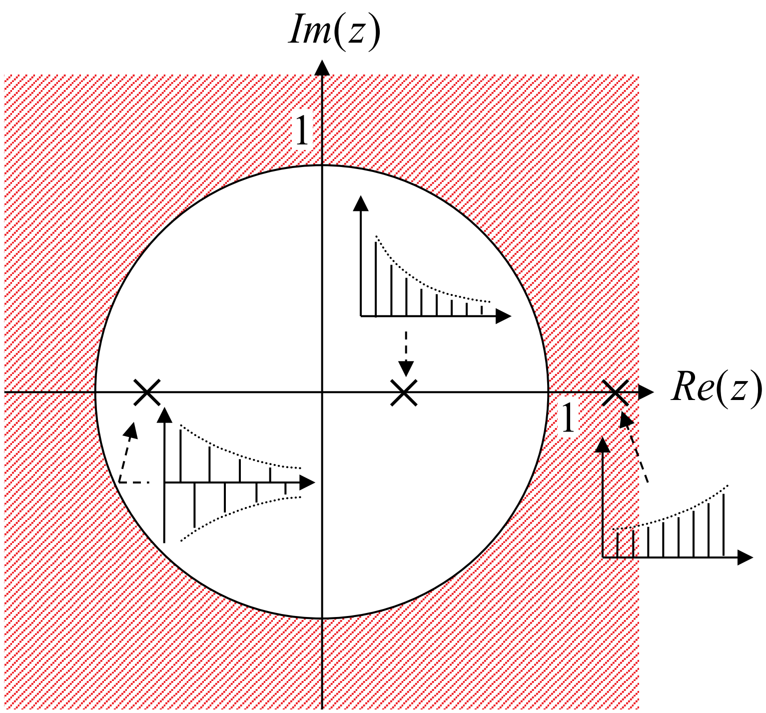
=> Damped oscillations (if $\left|z_j\right| < 1$)

=> Undamped oscillations (if $\left|z_j\right| > 1$)

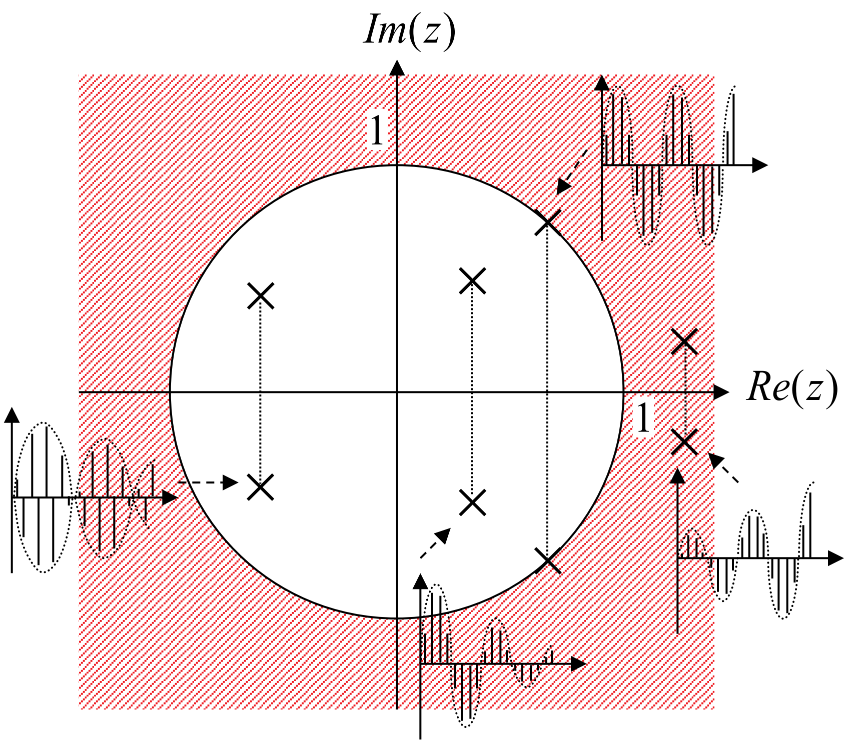
Analysis of digital closed-loops

- Influence of the poles on the transient behavior

Contribution of real poles



Contribution of complex poles



NB: Poles closer to origin → Faster transient regime

Analysis of digital closed-loops

- **Particular case: 2nd order systems**

As seen previously, a common controller design method consists to derive the controller parameters from a pole placement such that the dominant closed-loop dynamics is of second order

$$Den_{CL_{des}}(s) = \underbrace{\left(s^2 + 2\cdot\zeta\cdot w_n\cdot s + w_n{}^2\right)}_{\text{Dominant poles}}\cdot \underbrace{P_{aux}(s)}_{\text{Additional poles, faster}}$$

Reminder: Continuous-time theory

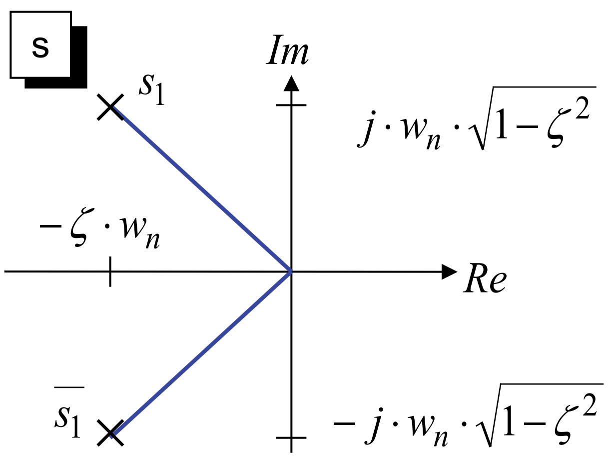
The design specifications imply constraints on the cut-off frequency w_n and the damping ratio ζ $\Rightarrow$

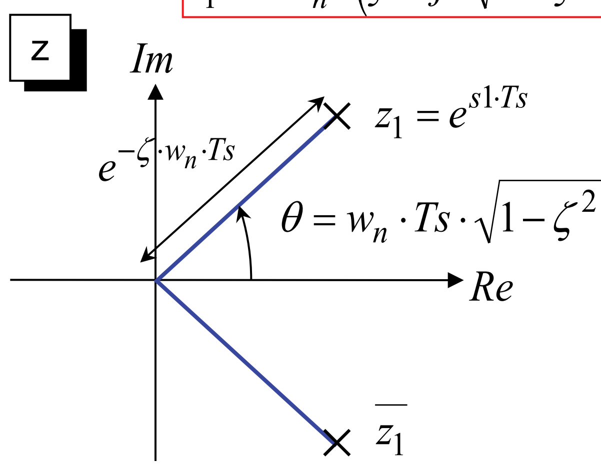
- Rise time (10% → 90%): $t_r \approx (2.6\cdot\zeta^2 - 0.45\cdot\zeta + 1.2)/w_n$
- Peak overshoot: $M_p = e^{-\pi\cdot\zeta/\sqrt{1-\zeta^2}}$
- Settling time (to 1%): $t_s \approx 4.6/\zeta\cdot w_n$

Discrete closed-loops:

Pole mapping from s-plane to z-plane:

$$\boxed{s_1 = -w_n\cdot\left(\zeta - j\cdot\sqrt{1-\zeta^2}\right) \quad\Rightarrow\quad z_1 = e^{-w_n\cdot Ts\cdot\left(\zeta - j\cdot\sqrt{1-\zeta^2}\right)}}$$





$$\boxed{t_r\ ,\ M_p \Rightarrow \zeta, w_n \Rightarrow z_1, \bar{z}_1}$$

$$\boxed{\begin{array}{l} t_s = 4.6/\zeta\cdot w_n \\ \Rightarrow\ |z_1| < e^{-4.6\cdot Ts/t_s} \approx 0.01^{Ts/t_s} \end{array}}$$

=> condition for Ts

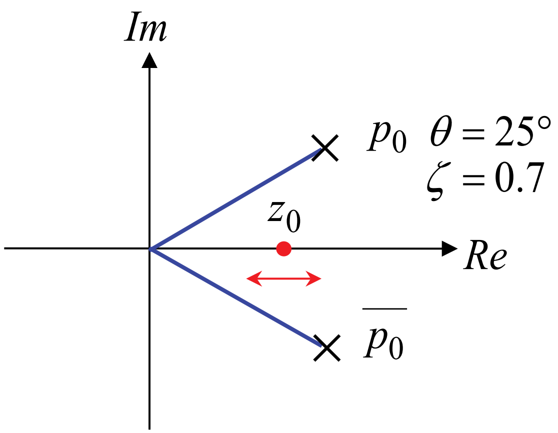
Analysis of digital closed-loops

- **Particular case: 2nd order systems**

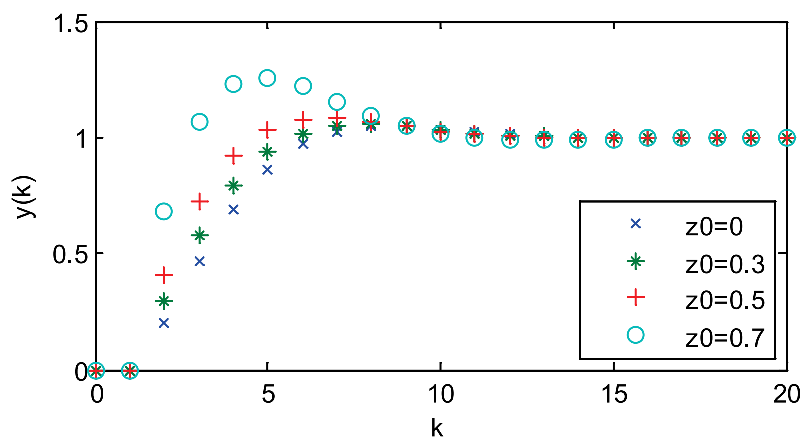
Influence of a zero

$$CL(z) = K \cdot \frac{z - z_0}{(z - p_0)\cdot(z - \overline{p_0})}$$

where $z_0 \in \mathbb{R}$ and $K = (1 - p_0)\cdot(1 - \overline{p_0})/(1 - z_0)$
($\rightarrow$ unit static gain)



Discrete step response

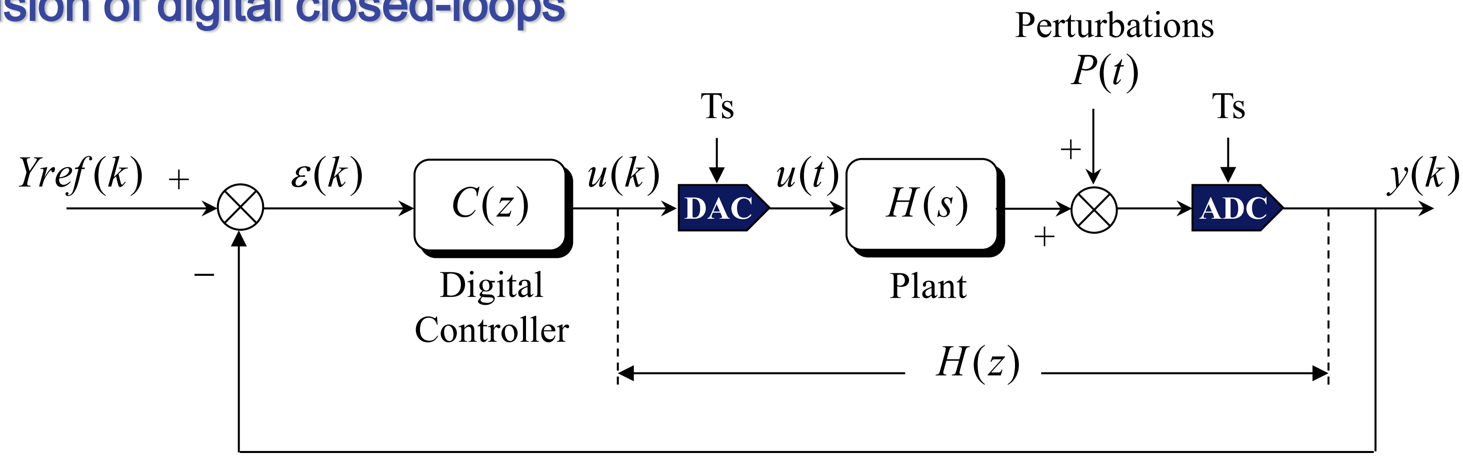


- Increasing overshoot when the zero is moving towards +1 $\rightarrow$ Take care…

- The reference tracking performance can be improved by designing appropriate zeros in the closed-loop transfer function

Analysis of digital closed-loops

- **Precision of digital closed-loops**



Same conclusions as for continuous-time closed-loops

- Precision versus the input

To achieve zero steady-state error, we require

→ at least 1 integrator (pole @ z =1) in the open-loop TF $C(z) \cdot H(z)$ for a step input

→ at least 2 integrators in the open-loop TF for a ramp input

→ …

- Perturbation rejection

To reject disturbances of class N → at least N integrators in $C(z) \cdot H(z)$

Discrete-time controller synthesis

2 main ways to synthesize discrete-time controllers:

1. Emulation design

Continuous-time design → Controller discretization → Digital implementation

2. Direct discrete-time design

Plant discretization → Controller design → Digital implementation

Discrete-time controller synthesis

- **Emulation design**

1rst step: Continuous-time controller design. At this stage the sampling is ignored (But the impact on the phase margin of the control delay & anti-aliasing filter should preferably be taken into account → preserve stability margin)

2nd step: Discretization of the continuous-time controller (Followed by simulations to check performance)

≠ **Methods:**

- Approximate s, i.e. $C(s) \rightarrow C(z)$
- Pole-zero matching

3rd step: Derivation of the controller algorithm (difference equation)

Approximation methods:

- **Euler** $\boxed{s \rightarrow \frac{1}{Ts} \cdot \left(1 - z^{-1}\right)}$

- **Tustin's or bilinear approximation** $\boxed{s \rightarrow \frac{2}{Ts} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}}$

Example: Discretization of a PI controller using Tustin's approximation

$$C(s) = Kp \cdot \left(1 + \frac{1}{Ti \cdot s}\right) \quad \Rightarrow \quad C(z) = C(s)\Big|_{s = \frac{2}{Ts} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}} = Kp \cdot \left(\frac{1 + Ts/(2 \cdot Ti) + \left(-1 + Ts/(2 \cdot Ti)\right) \cdot z^{-1}}{1 - z^{-1}}\right)$$

Matlab: sysd = c2d(sys,Ts,'tustin')

Discrete-time controller synthesis

- **Comparison between Euler and Tustin's approx.**

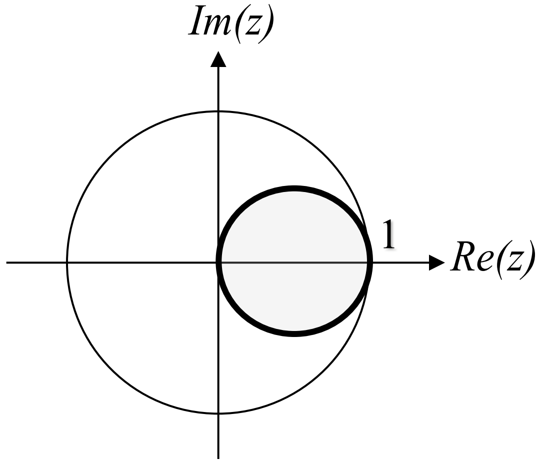
- Stability

= preserved

Half-plane

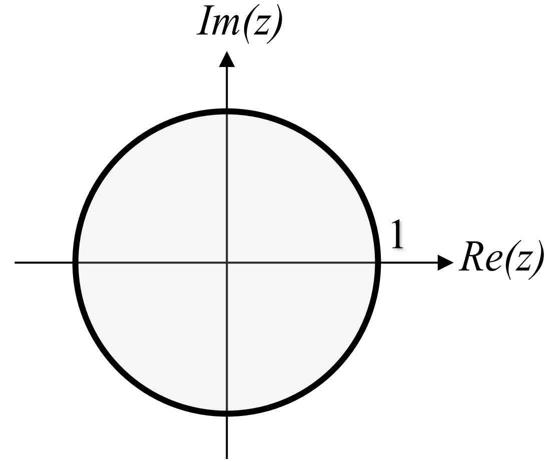
$Re[s] < 0$

Euler



An unstable continuous-time system can be mapped to a stable discrete system

Tustin



Perfect correspondence

- Mapping of the poles

Euler: $s - s_0 \rightarrow \left(z \cdot \left(1 - s_0 \cdot Ts\right) - 1\right) / Ts \cdot z$ **=>** $z_0 = 1/1 - s_0 \cdot Ts = 1 + \left(s_0 \cdot Ts\right) + \left(s_0 \cdot Ts\right)^2 + \left(s_0 \cdot Ts\right)^3 + \dots$

Tustin: $s - s_0 \rightarrow \left(z \cdot \left(2 - s_0 \cdot Ts\right) - \left(2 + s_0 \cdot Ts\right)\right) / Ts \cdot (z+1)$

=> $z_0 = \left(1 + s_0 \cdot Ts/2\right) / \left(1 - s_0 \cdot Ts/2\right) = 1 + \left(s_0 \cdot Ts\right) + \frac{1}{2} \cdot \left(s_0 \cdot Ts\right)^2 + \frac{1}{4} \cdot \left(s_0 \cdot Ts\right)^3 + \dots$

To be compared to $z_0 = e^{s_0 \cdot Ts} = 1 + \left(s_0 \cdot Ts\right) + \frac{1}{2} \cdot \left(s_0 \cdot Ts\right)^2 + \frac{1}{6} \cdot \left(s_0 \cdot Ts\right)^3 + \dots$ **≠ poles**

Discrete-time controller synthesis

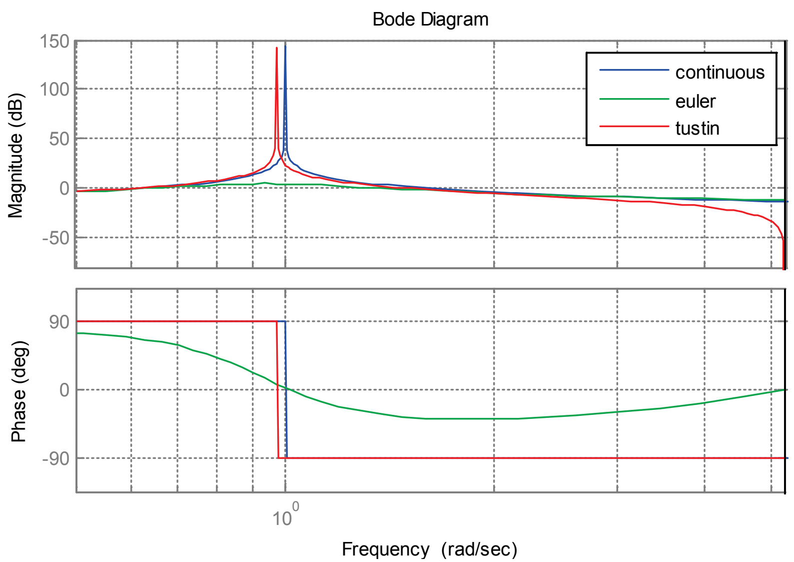
- **Comparison between Euler and Tustin's approx.**

  - Pole and zero locations not preserved → Frequency response is changed

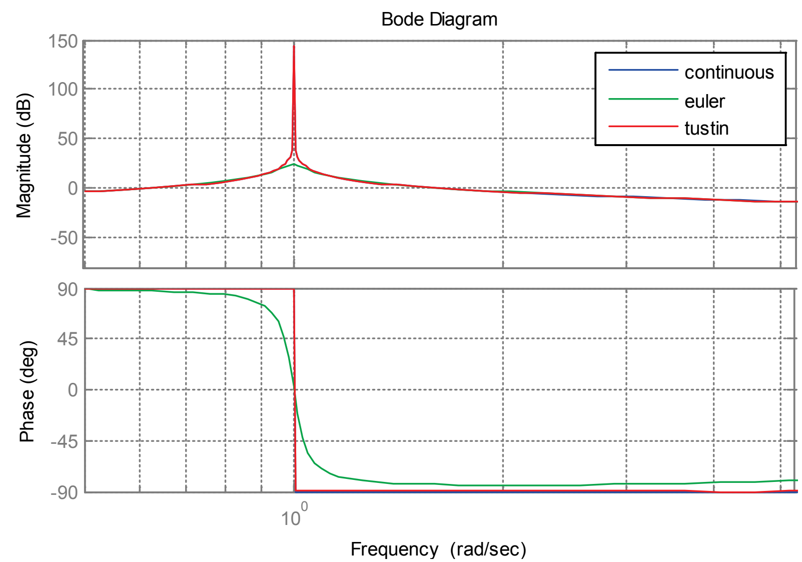
  - Increasing the sampling frequency → Smaller approximation errors

Example 1: $C(s) = \dfrac{s}{1+s^2}$ TF with resonance @ w_0 = 1rad/s

Ts = 0.6



Ts = 0.06

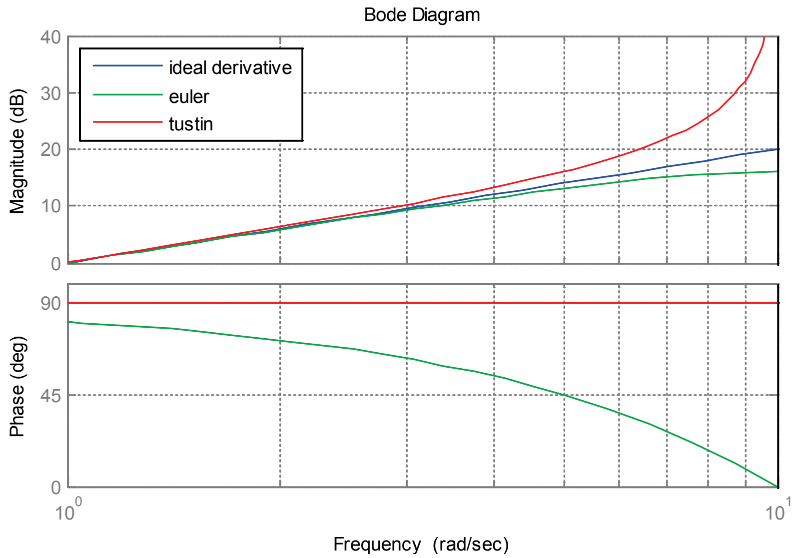


=> Better result with Tustin

Discrete-time controller synthesis

- **Comparison between Euler and Tustin's approx.**

Example 2: Ideal derivative



Euler: Filtering effect @ high frequencies

Tustin:
Magnitude → ∞ when w → π/Ts
Noise amplification @ high frequencies

=> Euler more appropriate for discretization of high-pass filters

=> Tustin more appropriate for discretization of low-pass filters

- **Other discretization method = Matched transform**

Matlab:
sysd = c2d(sys,Ts,'matched')

$$C(s) = K \cdot \frac{\prod_j (s - r_j)}{\prod_j (s - \sigma_j)} \xrightarrow{Z} C(z) = Kc \cdot \frac{\prod_j (z - e^{r_j \cdot Ts})}{\prod_j (z - e^{\sigma_j \cdot Ts})}$$

Kc is set so to obtain the same static gain

No frequency distortion => Well-adapted for the discretization of transfer functions including resonances (ex: notch filter, …)

Discrete-time controller synthesis

- **Direct discrete-time design**

- A system controlled using an emulation controller always suffer performance degradation compared with its continuous-time counter-part

- To reduce the degree of degradation, very fast sampling can be needed, as {ADC – Digital controller – DAC} should behave the same as the analogue controller (generally PID type = very simple control algorithm)

⇒ Bad use of the potentialities of the digital controller

In this case, direct discrete-time design offers an alternative solution, since in this design the sampling is considered from the beginning of the design process

1st step: Discretization of the continuous-time plant

2rd step: Choice of controller type and synthesis methodology

3rd step: Derivation of the controller algorithm (difference equation)

Discrete-time controller synthesis

- **Choice of the sampling period**

- Ts too small => Fast and expensive control hardware

=> Numerical issues: Recall the relation between poles in s-domain and z-domain: $z_i = e^{s_i \cdot Ts}$ => For $Ts \to 0$ we have $z_i \to 1 \quad \forall s_i$

→ Makes trouble when working in finite precision

=> Systems with control delays that are not multiples of the sampling period: Plant discretization may bring about unstable zeros

→ Limitation on the possible method to compute the regulator

=> If the sampling frequency of the outer current loop is small enough in comparison with the inner voltage loop bandwidth: No need to include the digital model of the voltage source → Lower order controller, less complexity

- Ts too large => Loss of information, aliasing (violation of the sampling theorem)

=> Regulation may not react enough readily to disturbances affecting the Σ

=> Plant discretization may give birth to poles having negative real part: Not desirable as the step response caused by such poles cycles back and forth between positive and negative deviations from the steady-state value

To prevent this: $\mathrm{Re}[z_i] > 0$ => $1/Ts > 2/\pi \cdot |\mathrm{Im}[s_i]|$

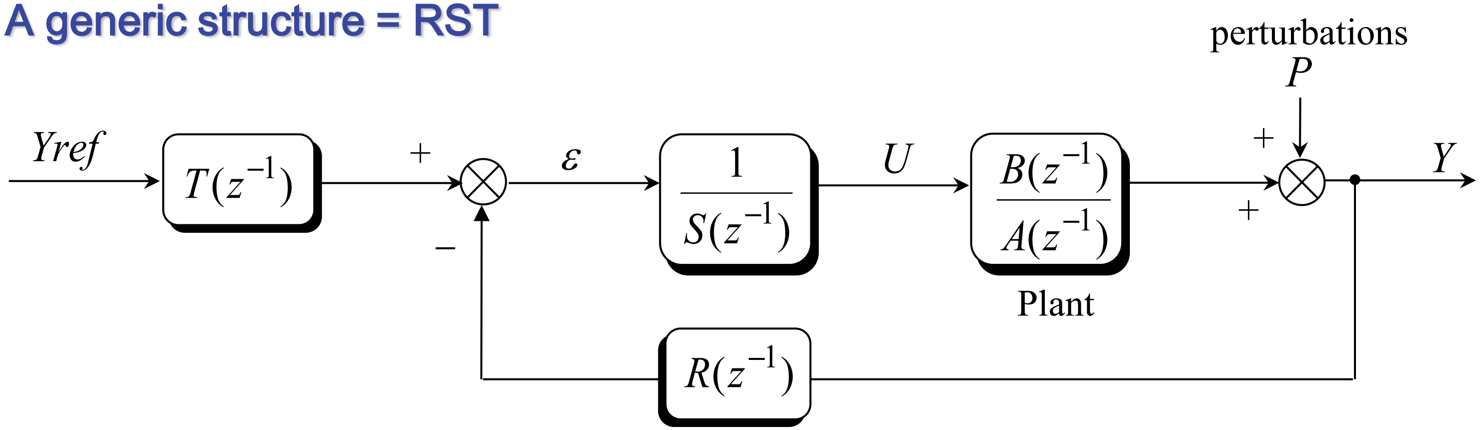
<u>Rule of thumb:</u>

Choice of Ts based on the closed-loop bandwidth $F^{CL}{}_B$

$$\boxed{\frac{1}{Ts} = (6 \;\; to \;\; 25) \cdot F^{CL}{}_B}$$

Discrete-time controller synthesis

- **RST controller structure**
  - Digital control → Enables implementation of new controller structures
  - A generic structure = RST

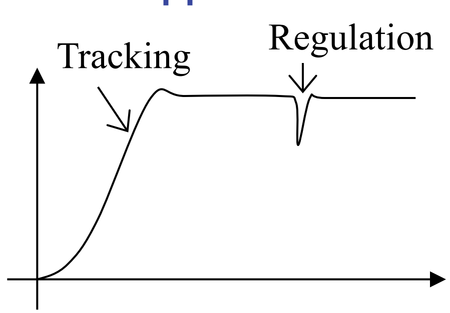


R, S and T are 3 polynomials to be determined (usually by pole placement)

- The control signal is calculated as $$U(z) = \frac{T(z^{-1})}{S(z^{-1})} \cdot Yref(z) - \frac{R(z^{-1})}{S(z^{-1})} \cdot Y(z)$$

=> Combination of Feedforward and Feedback that can be tuned separately

- General approach with RST: Decouple the regulation pb from the **tracking pb**



Polynomials R and S

Polynomial T

Discrete-time controller synthesis

- **RST controller structure**
  - Closed-loop system equation:

$$Y = \frac{B \cdot T}{A \cdot S + B \cdot R} \cdot Yref + \frac{A \cdot S}{A \cdot S + B \cdot R} \cdot P$$

  - Synthesis of the RST control law using pole-zero placement:

1st step: Choose arbitrarily a desired closed-loop transfer function or model

$$CL_{des}(z^{-1}) = B_m(z^{-1}) / A_m(z^{-1})$$

2nd step: Cancel poles and zeros of the plant TF (stable ones)

$$A(z^{-1}) = A^-(z^{-1}) \cdot A^+(z^{-1}) \quad \Rightarrow \quad R(z^{-1}) = A^+(z^{-1}) \cdot R_1(z^{-1})$$

$$B(z^{-1}) = B^-(z^{-1}) \cdot B^+(z^{-1}) \quad \Rightarrow \quad S(z^{-1}) = B^+(z^{-1}) \cdot S_1(z^{-1})$$

$A^-(z^{-1})$: *"non-compensable poles"* = unstable & poorly damped poles + poles with negative real part

$B^-(z^{-1})$: *"non-compensable zeros"* = unstable & poorly damped zeros + plant pure delay z^{-d} (d ≥ 1) + zeros having negative real part

Cannot be a factor of $A \cdot S + B \cdot R$ (=> CL unstable), thus: $B_m = B^- \cdot B_{m_1}$

3rd step: Perturbation rejection in steady-state

The open-loop TF must contain the classes of perturbation => Introduce an appropriate nb of integral terms in the controller by means of the polynomial S

$$S_1(z^{-1}) = (1 - z^{-1})^{n-l} \cdot S_2(z^{-1})$$

n : perturbation class $\quad$ l : plant class

Discrete-time controller synthesis

- **RST controller structure**

  - Synthesis of the RST control law (cont'd)

    4th step: Compute R and S = Solve the following Diophantine equation such that the poles of the CL regulated system are in the required position

    $$\boxed{A^-(z^{-1})\cdot(1-z^{-1})^{n-l}\cdot S_2(z^{-1})+B^-(z^{-1})\cdot R_1(z^{-1})=A_m(z^{-1})}$$

    5th step: Compute T

    $$\boxed{T(z^{-1})=A^+(z^{-1})\cdot B_{m_1}(z^{-1})}$$

    NB: To ensure unity gain to the CL TF, we must have $T(1)=R(1)$

    Particular case: System with stable zeros (apart from the pure delay z^{-1} systematically present = consequence of the ZOH) => $B^-=z^{-1}$

    Then the tracking TF is $\dfrac{Y}{Yref}=\dfrac{B\cdot T}{A\cdot S+B\cdot R}=\dfrac{z^{-1}\cdot B^+\cdot A^+\cdot B_{m1}}{A^+\cdot A^-\cdot B^+\cdot S_1+z^{-1}\cdot B^+\cdot A^+\cdot R_1}=\dfrac{z^{-1}\cdot B_{m_1}}{A_m}=\dfrac{B_m}{A_m}$

    Choosing $B_{m_1}(z^{-1})=A_m(z^{-1})$, thus $T(z^{-1})=A^+(z^{-1})\cdot A_m(z^{-1})$

    => $\dfrac{Y}{Yref}=z^{-1}$

    = Dead-beat control

    NB: Requires an accurate modelling of the system → Identification may be necessary. Moreover, if the Σ parameters are likely to vary (ex: magnet saturation → $L_{load}=f(I)$), adaptive control may be required

Discrete-time controller synthesis

- **RST controller structure**

  - Synthesis of the RST control law (cont'd)

Choice of A_m

Degree(A_m) ≤ degree of the first member of the Diophantine eq.

If CL desired behavior = 2nd order Σ : $A_m(z^{-1}) = (1 - z_1 \cdot z^{-1}) \cdot (1 - \bar{z}_1 \cdot z^{-1}) \cdot P_{aux}(z^{-1})$

where
$z_1, \bar{z}_1 = e^{-\zeta \cdot w_n \cdot Ts} \cdot e^{\pm j \cdot w_n \cdot Ts \cdot \sqrt{1-\zeta^2}}$ = dominant poles

$P_{aux}(z^{-1})$ contains the {degree(A_m) – 2} remaining poles z_i , chosen for example such that $|z_i| \leq 0.1$ (=> fast transient response as compared to the one due to $z_1, \bar{z}_1$)

  - PID = Special case of RST controller

$$Num_{PID}(z^{-1}) = R(z^{-1}) = r_0 + r_1 \cdot z^{-1} + r_2 \cdot z^{-2} \qquad T(z^{-1}) = R(z^{-1})$$
$$Den_{PID}(z^{-1}) = S(z^{-1}) = (1 - z^{-1}) \cdot (1 + s_1 \cdot z^{-1})$$

  - IP controller

$$Num_{PI}(z^{-1}) = R(z^{-1}) = r_0 + r_1 \cdot z^{-1} \qquad T(z^{-1}) = R(1) = r_0 + r_1$$
$$Den_{PI}(z^{-1}) = S(z^{-1}) = 1 - z^{-1}$$

Polynomial T = simple gain chosen to ensure unity gain to the closed-loop TF

Discrete-time controller synthesis

• RST controller structure

Example: Design a RST for the current loop of the previously presented Buck converter

System model:

$$\frac{B(z^{-1})}{A(z^{-1})} = CL_V(z^{-1}) \cdot \frac{z^{-n}}{R_{load}} \cdot \frac{(e^{-a\cdot\theta} - e^{-a\cdot Ts}) \cdot z^{-1} + 1 - e^{-a\cdot\theta}}{1 - z^{-1} \cdot e^{-a\cdot Ts}}$$

$CL_V(z^{-1})$ = Voltage closed-loop TF

$a = R_{load} / L_{load}$

$tc = n \cdot Ts - \theta$ = Control delay

Assume 1/Ts << voltage CL bandwidth, Ts << 1/a, tc << Ts

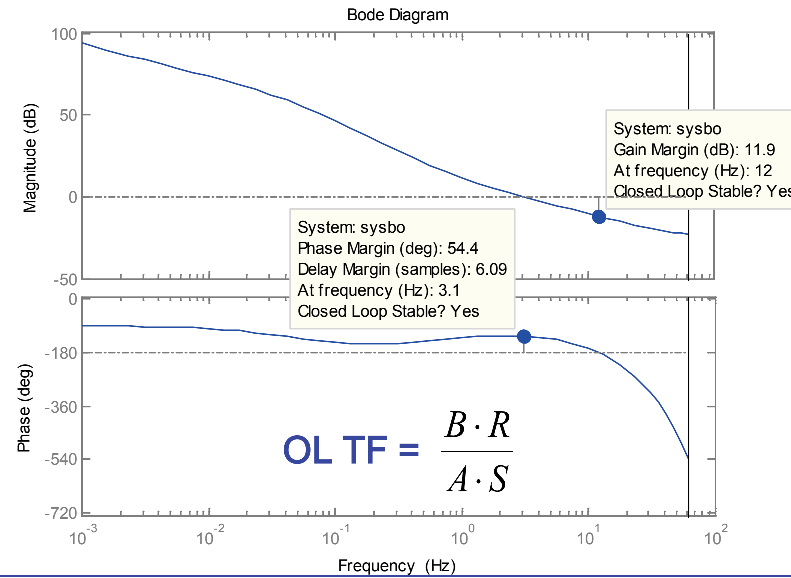
=> $$\frac{B(z^{-1})}{A(z^{-1})} \approx \frac{b_0 \cdot z^{-1}}{1 - z^{-1}} \qquad b_0 = \frac{Ts}{L_{load}}$$

Analog signals often sampled @ a rate > 1/Ts: If cut-off freq. of anti-aliasing filter >> $F^{CL}{}_B$, no need to take it into account in the model

Diophantine equation:

$$(1 - z^{-1})^2 + b_0 \cdot z^{-1} \cdot (r_0 + r_1 \cdot z^{-1}) = (1 - z_1 \cdot z^{-1}) \cdot (1 - \bar{z}_1 \cdot z^{-1})$$

=> $$r_0 = \frac{2 - (z_1 + \bar{z}_1)}{b_0} \qquad r_1 = \frac{z_1 \cdot \bar{z}_1 - 1}{b_0}$$



OL TF = $\frac{B \cdot R}{A \cdot S}$

Choice for T:

- If no tracking requirement: $T(z^{-1}) = r_0 + r_1$

=> No undesirable zero in the CL TF = no overshoot

- Fast tracking required:

$$T(z^{-1}) = (1 - z_1 \cdot z^{-1}) \cdot (1 - \bar{z}_1 \cdot z^{-1}) / b_0$$

Questions

A lot more… But time is over

Thank you for your attention

Questions?