Introduction to Transverse Beam Optics

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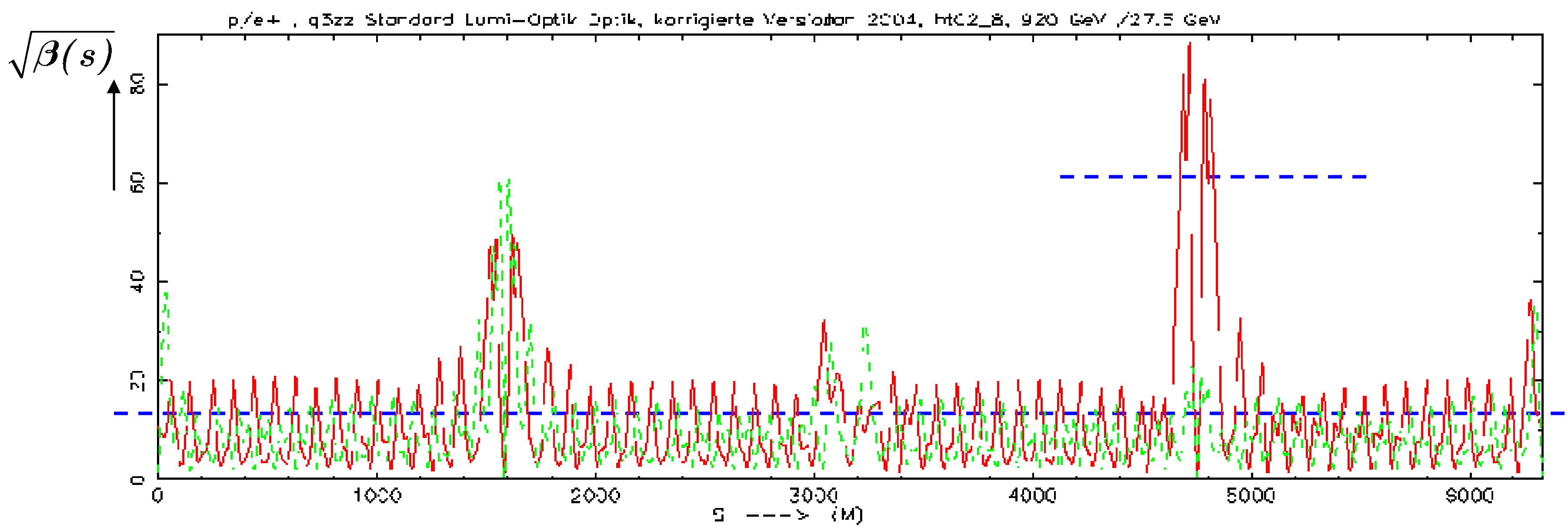
Part III: Errors in Field and Gradient



Optics error caused by a detuned quadrupole lens

I.) Dipole Errors: Closed Orbit Distortions

consider field error of a dipole: δB
located at s=0

→ *kick on the particle*

$$\Delta x' = \frac{e\,\delta B}{p} \cdot \Delta s$$

$$\Delta x' = \frac{1}{\rho} \cdot \Delta s$$

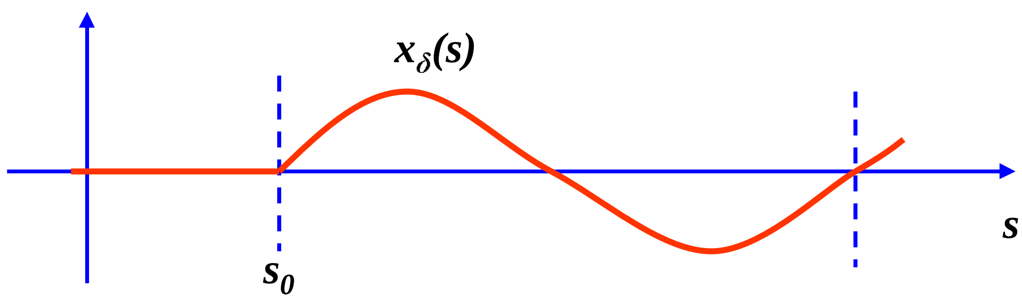
driving term to the equation of motion: $\Delta x'' = \Delta x' \,/\, \Delta s$

$$x'' = K(s) \cdot x + \frac{1}{\rho}$$

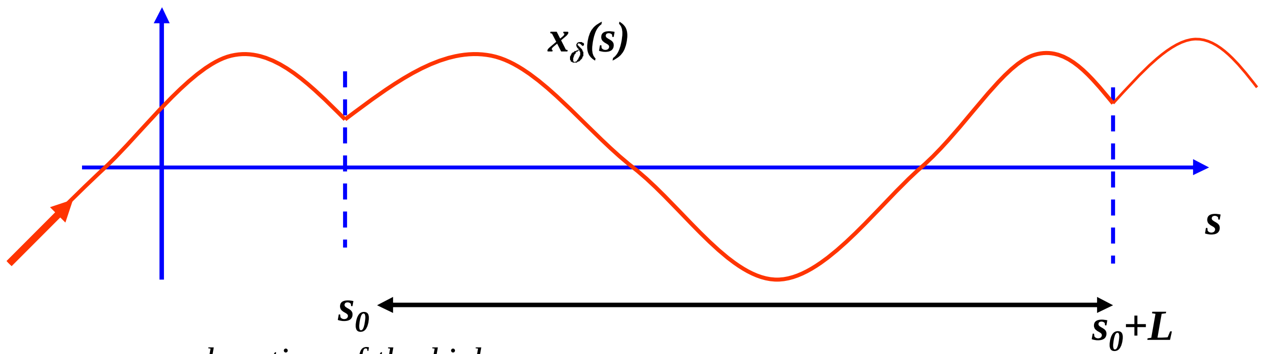
** general solution: solution of the homogeneous equation* → *β-tron oscillation*
& special solution of the inhomogeneous equation

** small displacements, small orbit kicks* → *linear approximation still valid*

$$x(s) = x_d + x_\beta$$



Problem: * *closed orbit = trajectory that closes itself after 1 turn*
(... the only closed trajectory)
* *the picture above is nonsense*



distorted orbit: $x(s) = a\sqrt{\beta(s)}\cos(\psi(s) - \vartheta)$ $a,\ \vartheta =$ const.

require: $x(s + L) = x(s)$ (i)

$$x'(s + L) + \frac{\Delta s}{\rho} = x'(s) \qquad (ii)$$

Gretchen Frage: *1.) rigorous treatment → lengthy, boaring, nasty*
(... Goethe) *2.) not so rigorous treatment → nice, easy to understand*

make your choice .....

$$x(s) = a\sqrt{\beta(s)}\cos(\psi(s) - \vartheta)$$

$$x'(s) = -\frac{a}{\sqrt{\beta(s)}}\sin(\psi(s) - \vartheta) + \frac{\beta'}{2\sqrt{\beta}}a\cos(\psi(s) - \vartheta)$$

condition (i): $x(s + L) = x(s)$

$$a\sqrt{\beta(s + L)}\cos(\psi(s) + 2\pi Q - \vartheta) = x(s) = a\sqrt{\beta(s)}\cos(\psi(s) - \vartheta)$$

deliberately: location of the distortion $s_0 = 0$, $\varphi(0) = 0$

$$\cos(2\pi Q - \vartheta) = \cos(-\vartheta) \qquad \rightarrow \vartheta = \pi Q$$

condition (ii): $x'(s + L) + \frac{\Delta s}{\rho} = x'(s)$

$$-\frac{a}{\sqrt{\beta(s_0 + L)}}\sin(\psi(s_0 + L) - \vartheta) + \frac{\beta'(s_0 + L)}{2\sqrt{\beta(s_0 + L)}}a\cos(\psi(s_0 + L) - \vartheta) + \frac{\Delta s}{\rho} =$$

$$= -\frac{a}{\sqrt{\beta(s_0)}}\sin(\psi(s_0) - \vartheta) + \frac{\beta'(s_0)}{2\sqrt{\beta(s_0)}}a\cos(\psi(s_0) - \vartheta)$$

using $\beta(s+L) = \beta(s)$ and $\varphi(s+L) = \varphi(s) + 2\pi Q$

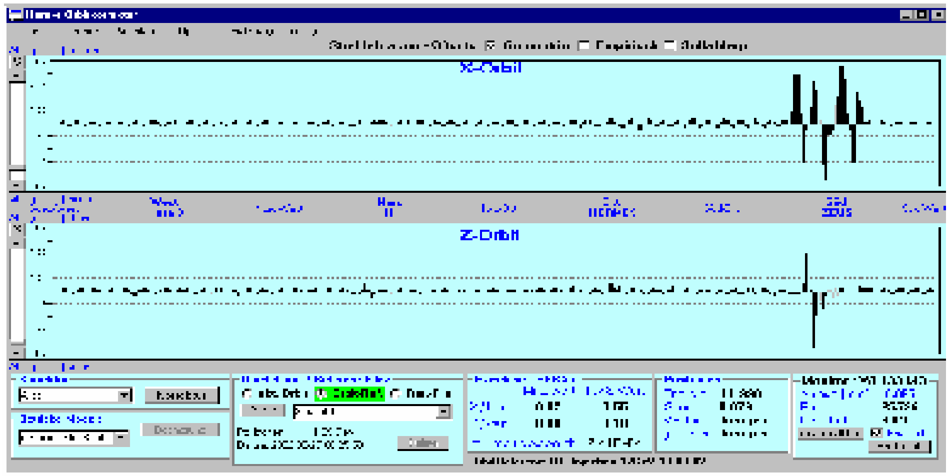
$$-\frac{a}{\sqrt{\beta(s_0)}}\,sin(\pi Q) + \frac{\beta'(s_0)}{2\sqrt{\beta(s_0)}}\,a\,cos(\pi Q) + \frac{\Delta s}{\rho} =$$

$$= -\frac{a}{\sqrt{\beta(s_0)}}\,sin(-\pi Q) + \frac{\beta'(s_0)}{2\sqrt{\beta(s_0)}}\,a\,cos(\pi Q)$$

→ ***amplitude factor a of the distorted orbit:*** $a = \dfrac{\Delta s / \rho \cdot \sqrt{\beta_0}}{2\,sin(\pi Q)}$

$$x(s) = \frac{\Delta s / \rho \cdot \sqrt{\beta_0}}{2\,sin(\pi Q)} \cdot \sqrt{\beta(s)}\,cos(\psi(s) - \pi Q)$$

Example: orbit distortion, deliberately applied in a certain section of a storage ring. (using 3 coils that form a closed bump).



general error distribution:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} \oint \frac{1}{\rho(\tilde{s})} \sqrt{\beta(\tilde{s})} \cos(|\varphi(\tilde{s}) - \varphi(s)| - \pi Q) d\tilde{s}$$

! orbit distortion is proportional to $\sqrt{\beta}$ ***at the place of the error***

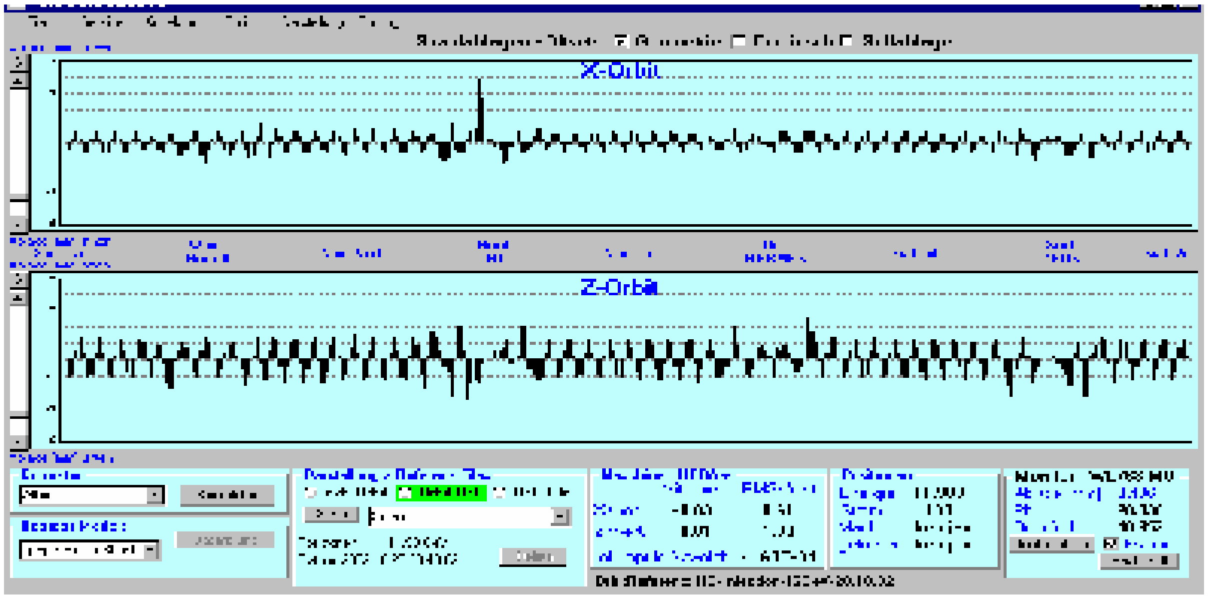
!! and at the place of observation

!!! distortion travels around the machine with the tune frequency φ(s)

!!!! attention: denominator can become zero

Example: orbit distortion, applied for the whole storage ring using 1 correction coil

... number of oscillations = tune

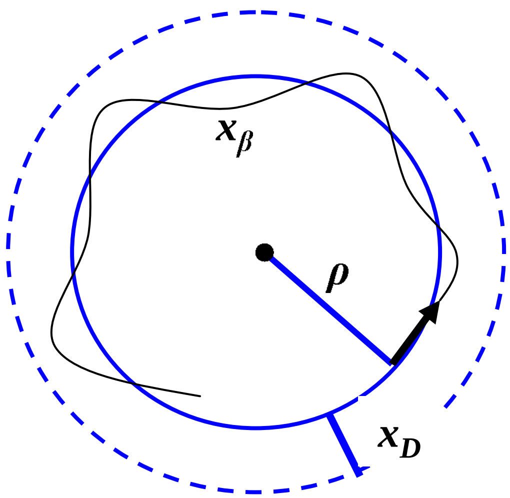


II.) Periodic Dispersion

closed orbit distortion → field error acts as driving term to the equation of motion:

$$\Delta x' = \frac{e\,\delta B}{p}\cdot\Delta s = \frac{1}{\rho}\cdot\Delta s$$

$$x'' = K(s)\cdot x + \frac{1}{\rho}$$



particle with momentum error → Δp/p acts as driving term to the equation of motion:

$$x'' + K(s)x = \frac{1}{\rho}\frac{\Delta p}{p}$$

remember the dispersion function D(s):

$$x_D(s) = D(s)\frac{\Delta p}{p}$$

Example: Assume weak focusing machine:

closed orbit given by D(s) and Δp/p

solution ... in linear approximation:

$$x(s) = x_D(s) + x_\beta(s)$$

where $x_D(s)$ describes the new closed orbit for $\Delta p/p \neq 0$:

differential equation for D(s) ... as usual:

$$D''(s) + K(s)D(s) = \frac{1}{\rho(s)}$$

... but now it has to be a periodic function:

$$D(s + L_0) = D(s)$$

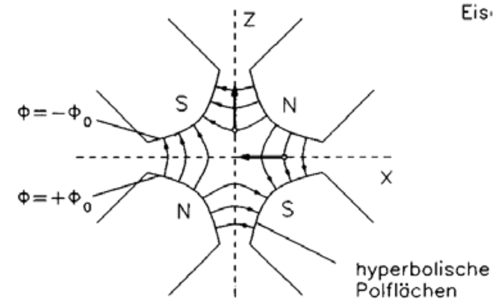
$$D'(s + L_0) = D'(s)$$

going through exactly the same calculation as in the case of the distorted closed orbit we get

$$D(s) \equiv \eta(s) = \frac{\sqrt{\beta(s)}}{2 \, sin \, \pi Q} \int_{s_0}^{s_0 + L_0} \frac{1}{\rho(\tilde{s})} \sqrt{\beta(\tilde{s})} \, cos(\left|\psi(s) - \psi(\tilde{s})\right| - \pi Q) d\tilde{s}$$

III.) Quadrupole Errors: Alignment

$$B_z = -g \cdot x$$

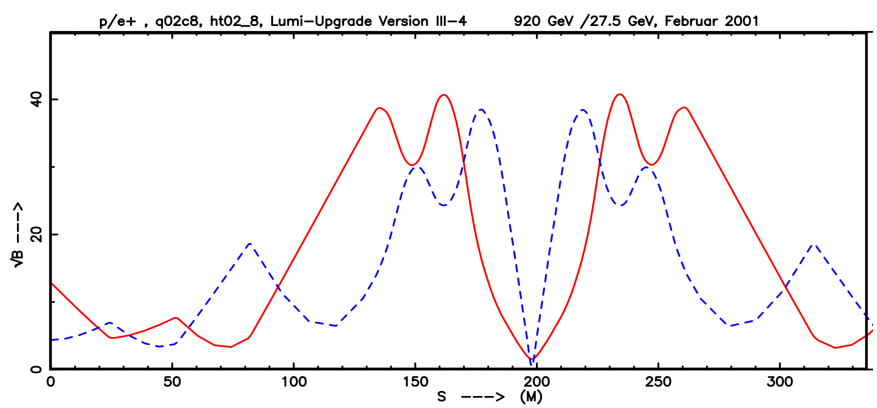


quadrupole lenses have a linear increasing magnetic field

offset in magnet alignment: $$\Delta B = g \cdot \Delta x$$

→ *leads to a kick angle* $$\Delta x' = l \cdot \frac{1}{\rho} = l \frac{B}{p/e} = l \frac{g \cdot \Delta x}{p/e}$$

$$\Delta x' = l \cdot k \cdot \Delta x$$



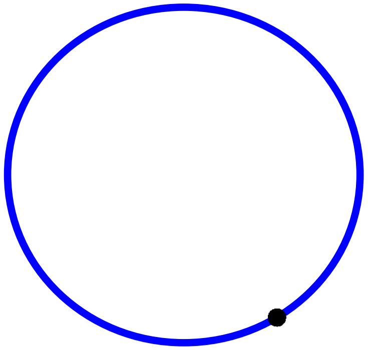
again: closed orbit distortion

$$x(s) = \frac{\sqrt{\beta(s)}}{2 \sin \pi Q} \Delta x' \sqrt{\beta(s_0)} \cos\left[\left| \psi(s) - \psi(s_0) \right| - \pi Q \right]$$

IV.) Quadrupole Errors: Gradient

matrix for 1 complete revolution

$$M = \begin{pmatrix} \cos\mu_0 + \alpha_0 \sin\mu_0 & \beta_0 \sin\mu_0 \\ -\gamma_0 \sin\mu_0 & \cos\mu_0 - \alpha_0 \sin\mu_0 \end{pmatrix}$$



location s_0 of the quadrupole error

remember: $\quad trace(M) = 2\cos\mu_0$

assume: small gradient error at position s_0

$$M_{error} = \begin{pmatrix} 1 & 0 \\ \Delta k ds & 1 \end{pmatrix}$$

$$\tilde{M} = \begin{pmatrix} 1 & 0 \\ \Delta k ds & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\mu_0 + \alpha_0 \sin\mu_0 & \beta_0 \sin\mu_0 \\ -\gamma_0 \sin\mu_0 & \cos\mu_0 - \alpha_0 \sin\mu_0 \end{pmatrix}$$

$$\tilde{M} = \begin{pmatrix} \cos\mu_0 + \alpha_0 \sin\mu_0 & \beta_0 \sin\mu_0 \\ \Delta k\, ds(\cos\mu_0 + \alpha_0 \sin\mu_0) - \gamma_0 \sin\mu_0 & \Delta k\, ds\beta_0 \sin\mu_0 + \cos\mu_0 - \alpha_0 \sin\mu_0 \end{pmatrix}$$

tune of the distorted optic:

$$trace(\tilde{M}) = 2\cos\tilde{\mu} = 2\cos\mu_0 + \Delta k ds \beta_0 \sin\mu_0$$

defining a tune shift $\mu = \mu_0 + \Delta\mu$ *and writing* $\mu_0 = 2\pi Q_0$

$$2\cos(2\pi Q_0 + dQ) = 2\cos 2\pi Q_0 + \Delta k ds \beta_0 \sin 2\pi Q_0$$

$$\cos 2\pi Q_0 \cdot \underbrace{\cos 2\pi dQ}_{\approx 1} - \sin 2\pi Q_0 \cdot \underbrace{\sin 2\pi dQ}_{\approx 2\pi dQ} = \cos 2\pi Q_0 + \frac{\Delta k ds \beta_0 \sin 2\pi Q_0}{2}$$

for a small error Δk we expect a small tune shift dQ

$$dQ = \frac{\Delta k ds \beta_0}{4\pi}$$

integrating over the length of the quadrupol error:

$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k \beta(s)}{4\pi} ds \approx \frac{\Delta k l_{quad} \bar{\beta}}{4\pi}$$

! *the tune shift is proportional to the β-function at the quadrupole*

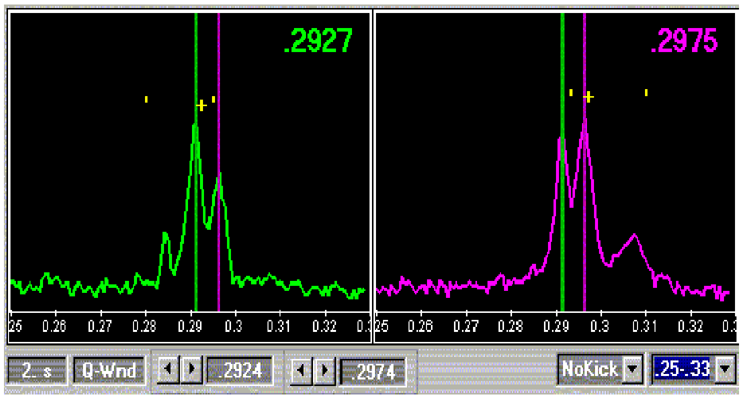
!! *field quality, power supply tolerances etc are much tighter at places where β is large*

!!! *mini beta quads: β ≈ 1900*

arc quads: β ≈ 80

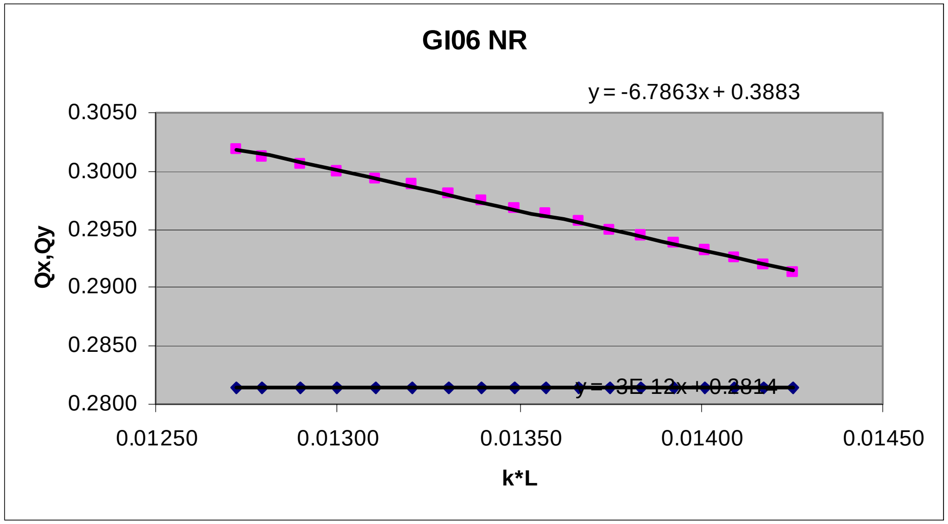
!!!! *β is a measure for the sensitivity of the beam*

Example: measurement of β in a storage ring:



tune spectrum ...

*

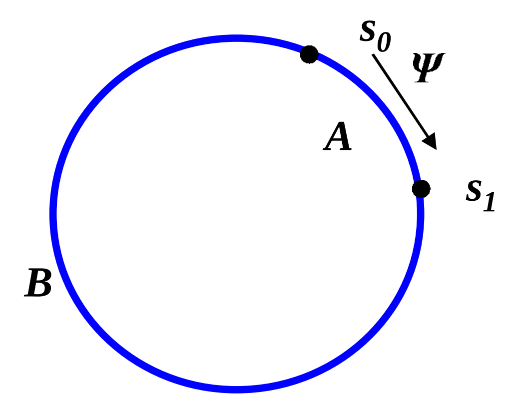


tune shift as a function of a gadient change

V.) Quadrupole Errors: Beta Function

$$M = \begin{pmatrix} \cos\mu_0 + \alpha_0 \sin\mu_0 & \beta_0 \sin\mu_0 \\ -\gamma_0 \sin\mu_0 & \cos\mu_0 - \alpha_0 \sin\mu_0 \end{pmatrix}$$

assume: distortion at s_1
observation point: s_0



matrix of unperturbed optics
... β is obtained via m_{12}

$$m_{12} = \beta_0 \sin 2\pi Q$$

introduce matrix with error:

$$\tilde{M}(s_0) = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} \\ \tilde{m}_{21} & \tilde{m}_{22} \end{pmatrix} = B \cdot \begin{pmatrix} 1 & 0 \\ -\Delta k\, ds & 1 \end{pmatrix} \cdot A$$

we expect a tune shift and an error in β:

$$\tilde{m}_{12} = (\beta_0 + d\beta) \sin 2\pi(Q + dQ) \qquad (i)$$

from the matrix multiplication we get the element m12 as a function of the error

$$\tilde{M}(s_0) = \begin{pmatrix} \sim & b_{11}a_{12} + b_{12}(-\Delta k\, ds\, a_{12} + a_{22}) \\ \sim & \sim \end{pmatrix}$$

$$\tilde{m}_{12} = \underbrace{b_{11}a_{12} + b_{12}a_{22}} - b_{12}a_{12}\Delta k\, ds$$

m_{12}, the element of the unperturbed transformation

$$\tilde{m}_{12} = \beta_0 \sin 2\pi Q - b_{12}a_{12}\Delta k\, ds \qquad (ii)$$

equalise (i) and (ii)

$$\beta_0 \sin 2\pi Q - b_{12}a_{12}\Delta k\, ds = (\beta_0 + d\beta) \sin 2\pi(Q + dQ)$$

$$= (\beta_0 + d\beta) \sin 2\pi Q \cdot \underbrace{\cos 2\pi\, dQ} + \cos 2\pi Q \cdot \underbrace{\sin 2\pi\, dQ}$$

... as we consider a small error → a small tune shift dQ .... ≈ 1 $\approx 2\pi dQ$

$$= (\beta_0 + d\beta) \sin 2\pi Q + \cos 2\pi Q \cdot 2\pi\, dQ$$

$$\beta_0 \sin 2\pi Q - b_{12}a_{12}\Delta k\, ds = \beta_0 \sin 2\pi Q + \beta_0 2\pi dQ \cos 2\pi Q +$$

$$+ d\beta \sin 2\pi Q + \underbrace{d\beta 2\pi dQ \cos 2\pi Q}_{\approx 0}$$

$$-b_{12}a_{12}\Delta k\, ds = \beta_0 2\pi dQ \cdot \cos 2\pi Q + d\beta \sin 2\pi Q$$

the tune shift dQ is related to the quadrupole error by

$$dQ = \frac{\Delta k \beta(s_1)\, ds}{4\pi}$$

$$-b_{12}a_{12}\Delta k\, ds = \frac{\beta_0 \Delta k \beta_1 ds}{2} \cdot \cos 2\pi Q + d\beta \sin 2\pi Q$$

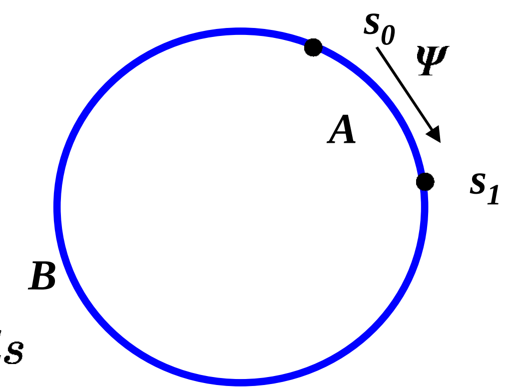
$$d\beta = \frac{-a_{12}b_{12}\Delta k\, ds - \frac{1}{2}\beta_0 \Delta k \beta_1\, ds \cos 2\pi Q}{\sin 2\pi Q}$$

$$d\beta = \frac{-1}{2 \sin 2\pi Q}\left\{2a_{12}b_{12} + \beta_0\beta_1 \cos 2\pi Q\right\}\Delta k\, ds$$

matrix elements a_{12}, b_{12}

$$a_{12} = \sqrt{\beta_0 \beta_1}\, \sin\psi$$

$$b_{12} = \sqrt{\beta_1 \beta_0}\, \sin(2\pi Q - \psi)$$



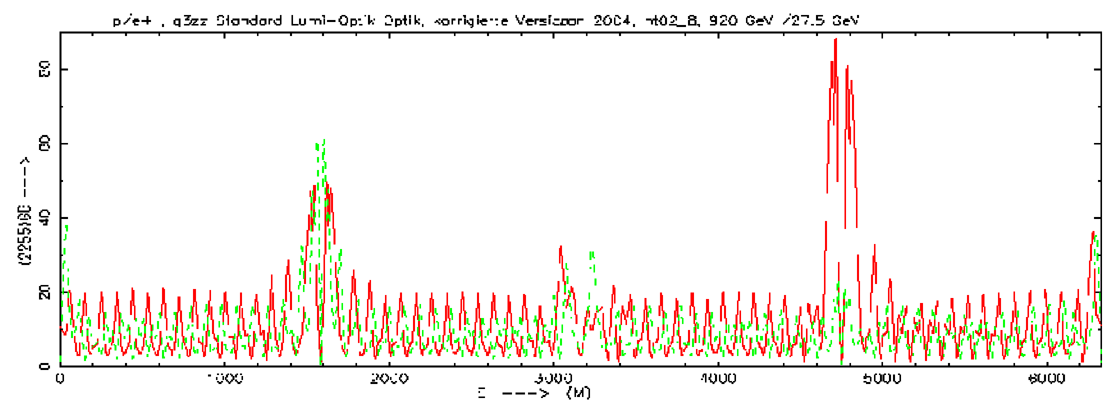
$$d\beta = \frac{-\beta_0 \beta_1}{2 \sin 2\pi Q} \underbrace{\left\{2 \sin\psi \cdot \sin(2\pi Q - \psi) + \cos 2\pi Q\right\}}_{= \cos(2\psi - 2\pi Q)} \Delta k\, ds$$

$$\Delta\beta_0 = \frac{-\beta_0}{2 \sin 2\pi Q} \int_{s1}^{s1+l} \beta(s) \Delta k(s) \cos\left\{2\left|\psi(s) - \psi_0\right| - 2\pi Q\right\} ds$$

**** the error depends on β at the location of the perturbation***

… and on β at the location of the observer

**** the error travels araound the machine at twice the tune !***



VI.) Resonances

Remember:

orbit distortion due to dipole field errors

$$x(s) = \frac{\sqrt{\beta(s)}}{2\,\sin \pi Q}\,\Delta x' \sqrt{\beta(s_0)}\, \cos\left[\left|\psi(s) - \psi(s_0)\right| - \pi Q\right]$$

optics perturbation due to quadrupole gradient errors

$$\Delta\beta_0 = \frac{-\beta_0}{2\,\sin 2\pi Q} \int_{s1}^{s1+l} \beta(s)\Delta k(s)\cos\left\{2\left|\psi(s) - \psi_0\right| - 2\pi Q\right\} ds$$

Tune may not be an integer, or half an integer or ...

including higher multipole terms ...

general condition for the working point:

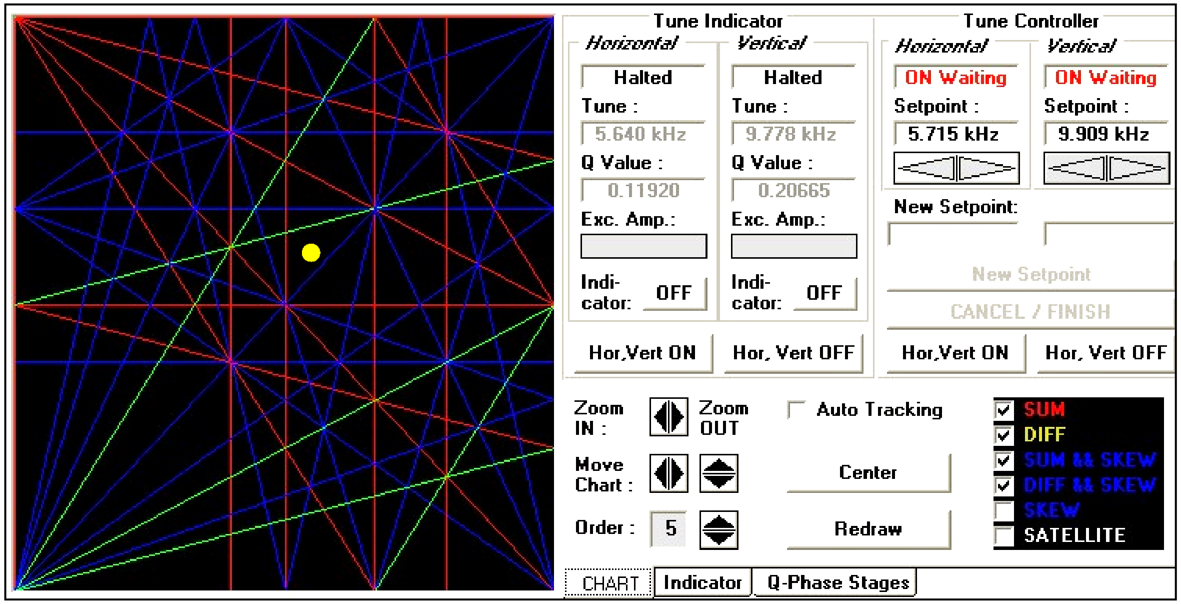
$$mQ_x + nQ_z \neq l$$

VI.) Resonances

HERA working diagram

including resonance lines up to 5th order

Qz=32.5

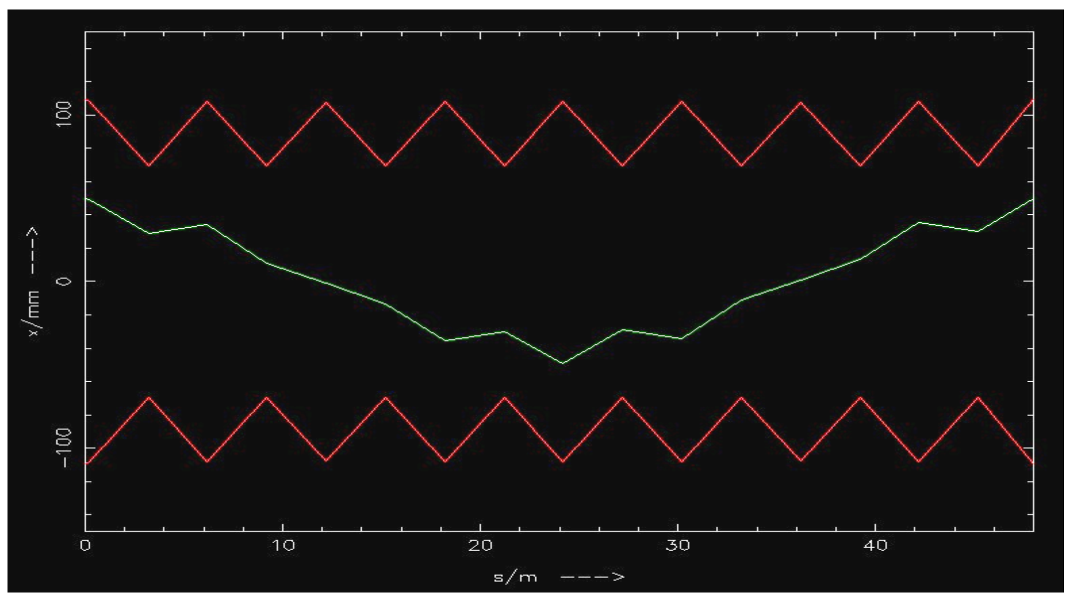


Qz=32.0

Qx = 31.0 31.5

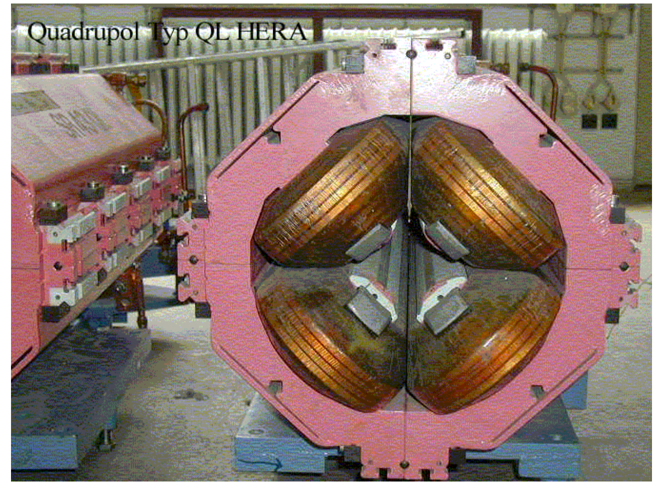
$Q_x = 31.292$
$Q_z = 32.297$

Example: qualitatively speaking ...



quantitatively: → Oliver Bruening

VII.) Chromaticity:



villain ...: the quadrupole lens

$$k = \frac{eg}{p_0}$$

consider a small momentum error: $p = p_0 + \Delta p$

$$k = -\frac{eg}{p_0 + \Delta p} \approx -\frac{e}{p_0}\left(1 - \frac{\Delta p}{p_0}\right)g = k_0 - \Delta k$$

we get a focusing error: $\Delta k = \frac{\Delta p}{p_0} k_0$

which leads to a tune change: $dQ = \frac{\Delta p}{p_0} \frac{1}{4\pi} \int k_0 \beta(s) ds$

integrating over all quadrupole lenses: $\Delta Q = \frac{\Delta p}{p_0} \frac{1}{4\pi} \oint k(s)\beta(s)\, ds$

$$\Delta Q = \frac{-1}{4\pi} \frac{\Delta p}{p_0} \oint k(s)\beta(s)\, ds$$

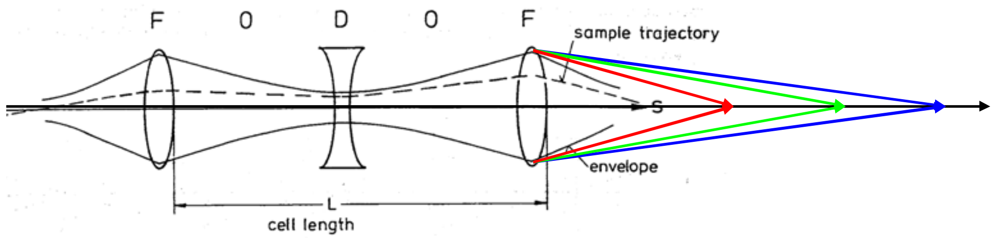


Figure 29: FODO cell

* *the tune change is highest for* ***strong quadrupoles***
* *„ „ „* ***at places where β is high***

Definition of Chromaticity:

$$\xi = \frac{\Delta Q}{\Delta p / p_0} = \frac{-1}{4\pi} \oint k(s)\beta(s)\, ds$$

ξ is a number that characterizes the chromatic focusing error of the quadrupole magnets

typical values: ξ ≈ -70 in large machines

$\Delta p/p \approx 10^{-3}$

$\Delta Q \approx 0.14$

Correction of ξ:

1.) sort the particles acording to their momentum

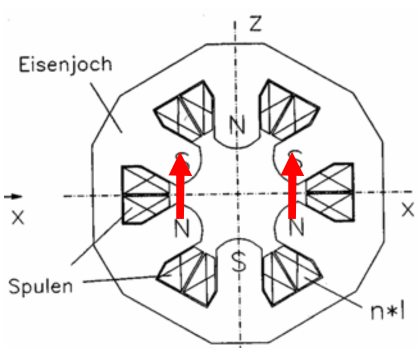
$$x_D(s) = D(s)\frac{\Delta p}{p}$$

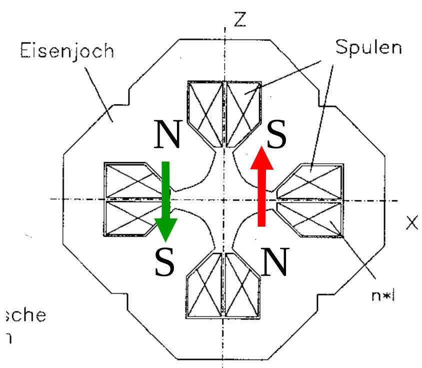
2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left.\begin{aligned} B_x &= \tilde{g}xz \\ B_z &= \frac{1}{2}\tilde{g}(x^2 - z^2) \end{aligned}\right\} \qquad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

linear rising „gradient“:

Sextupole Magnets:





normalised quadrupole strength:

$$k_{sext} = \frac{\tilde{g}x}{p/e} = m_{sext}.x$$

$$k_{sext} = m_{sext}.D\frac{\Delta p}{p}$$

corrected chromaticity:

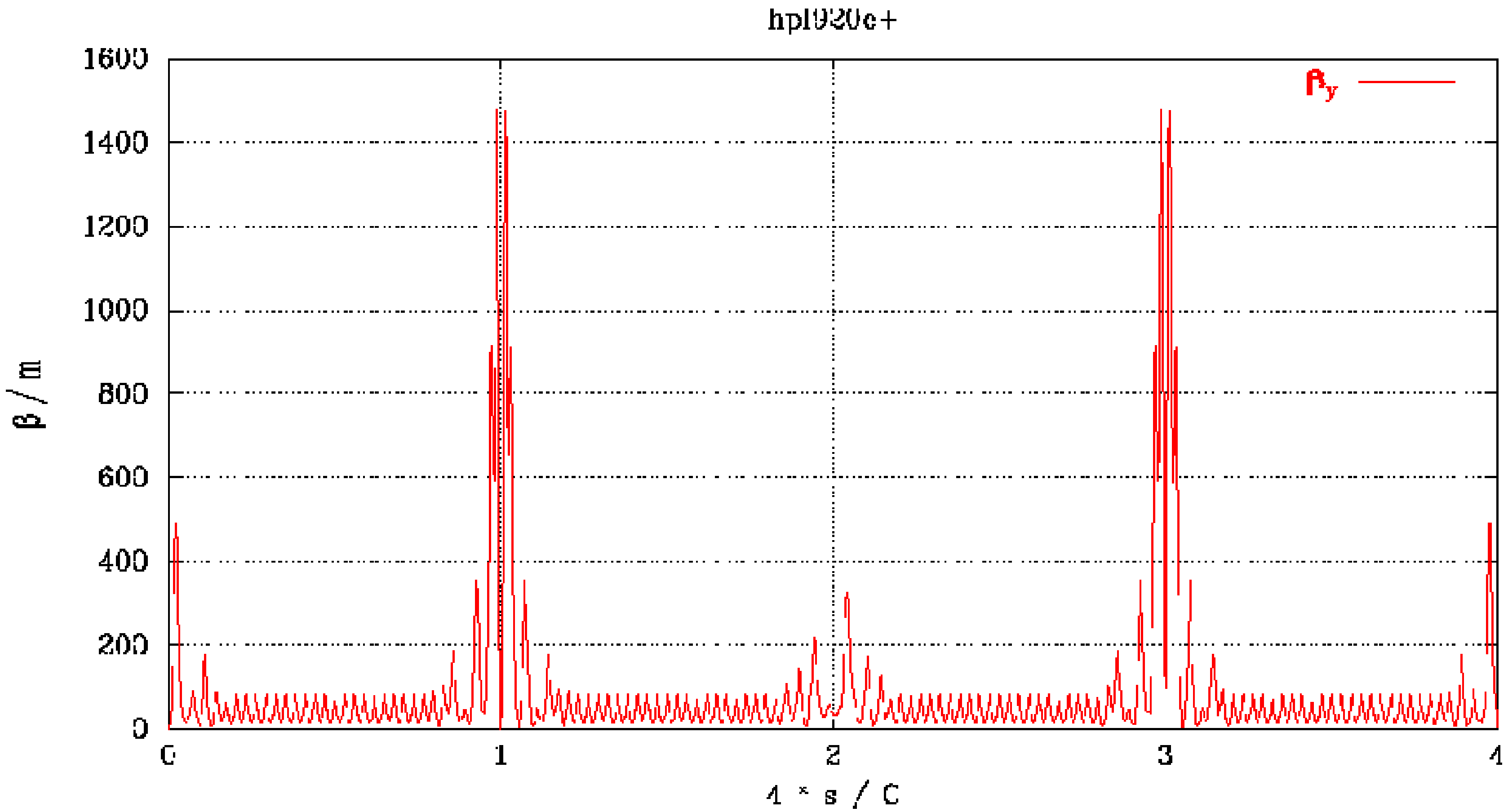
$$\xi = \frac{-1}{4\pi}\oint \{k(s) - mD(s)\}\beta(s)\,ds$$

Chromaticity

$$\xi = \frac{-1}{4\pi} \oint k(s)\beta(s)\,ds$$

question: main contribution to ξ in a lattice …

beam optics used for collision mode in a typical storage ring

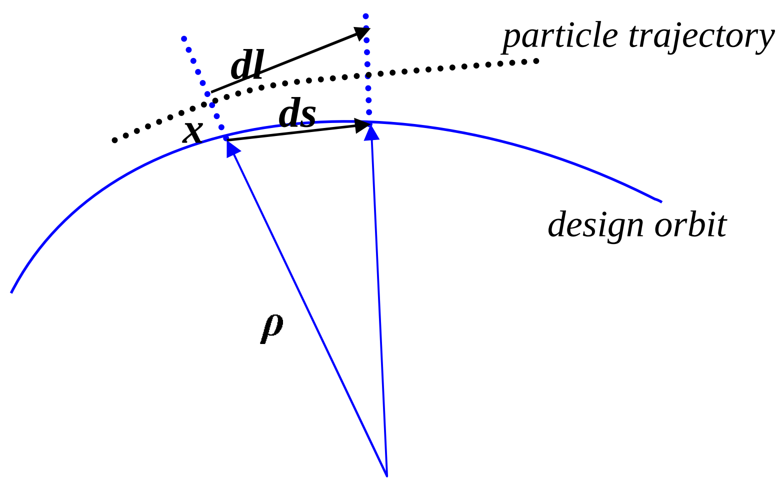


VIII.) Momentum Compaction Factor:

The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate.

$$x'' + K(s) * x = \frac{1}{\rho}\frac{\Delta p}{p} \qquad\qquad x(s) = x_\beta(s) + D(s)\frac{\Delta p}{p}$$

But it does much more:



particle with a displacement x to the design orbit
→ path length dl ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho} \qquad \rightarrow \qquad dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$

circumference of an off-energy closed orbit

$$l_\varepsilon = \oint dl = \oint \left(1 + \frac{x_\varepsilon}{\rho(s)}\right) ds$$

remember: $x_\varepsilon(s) = D(s)\frac{\Delta p}{p}$

$$\delta l_\varepsilon = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)} \right) ds$$

* The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.

Definition:

$$\frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p} \quad \rightarrow \quad \alpha_{cp} = \frac{1}{L} \oint \left(\frac{D(s)}{\rho(s)} \right) ds$$

For first estimates assume:

$$\frac{1}{\rho} = const \qquad l_{dipoles} \cdot \langle D \rangle_{dipole} = \int_{dipoles} D(s) ds$$

$$\alpha_{cp} = \frac{1}{L} l_{dipoles} \langle D \rangle \frac{1}{\rho} = \frac{1}{L} 2\pi\rho \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \alpha_{cp} \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Assume: $v \approx c$

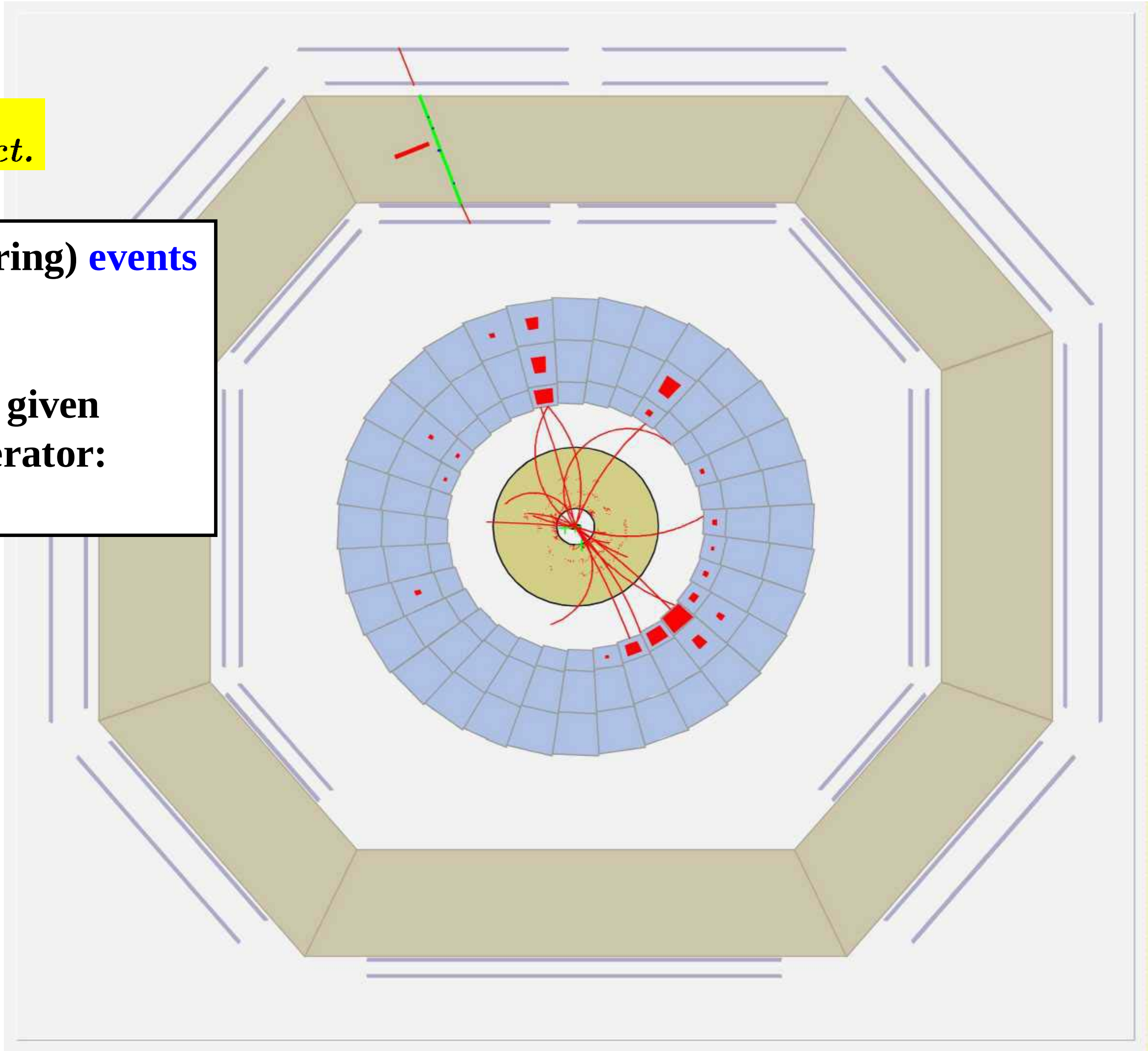
$$\rightarrow \frac{\delta T}{T} = \frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p}$$

α_{cp} combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

IX.) Luminosity

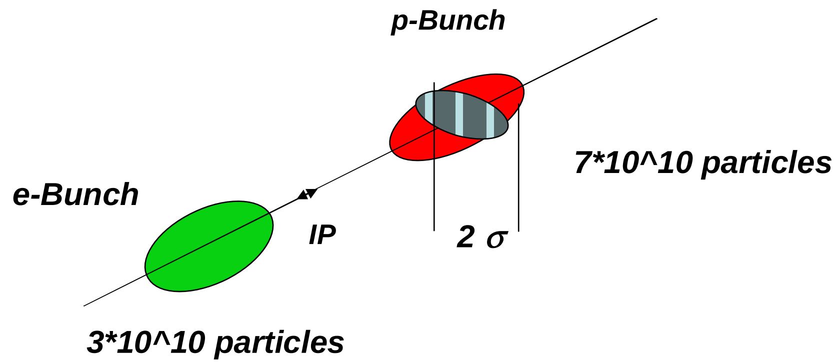
$$R = L * \sigma_{react.}$$

production rate of (scattering) events
is determined by the
cross section σ_{react}
and a parameter L that is given
by the design of the accelerator:
… the luminosity



ZEUS detector: inelastic scattering event of e+/p

Luminosity:



$$L = \frac{n_b * N_p * N_e * f_0}{2\pi * \sqrt{(\sigma_{x,p}^2 + \sigma_{x,e}^2)} * \sqrt{(\sigma_{y,p}^2 + \sigma_{y,e}^2)}}$$

comment: ... oh my goodnes... or in other words
... can we do a little bit easier ?

$$I = N * e * f_0 * n_b$$

$$\sigma_{x,p} = \sigma_{x,e}$$

$$\sigma_{y,p} = \sigma_{y,e}$$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_e I_p}{\sigma_x^* \sigma_y^*}$$

small β required at the collision point

... do you remember Liouville ?

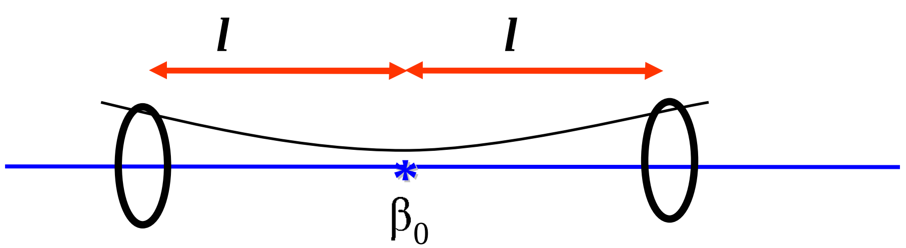
$$\beta(s) = \beta^* + \frac{s^2}{\beta^*}$$

Find the β at the center of the drift that leads to the lowest maximum β at the end:

$$\frac{d\hat{\beta}}{d\beta_0} = 1 - \frac{\ell^2}{\beta_0^2} = 0$$

$$\rightarrow \quad \beta_0 = \ell$$

$$\rightarrow \quad \hat{\beta} = 2\beta_0$$

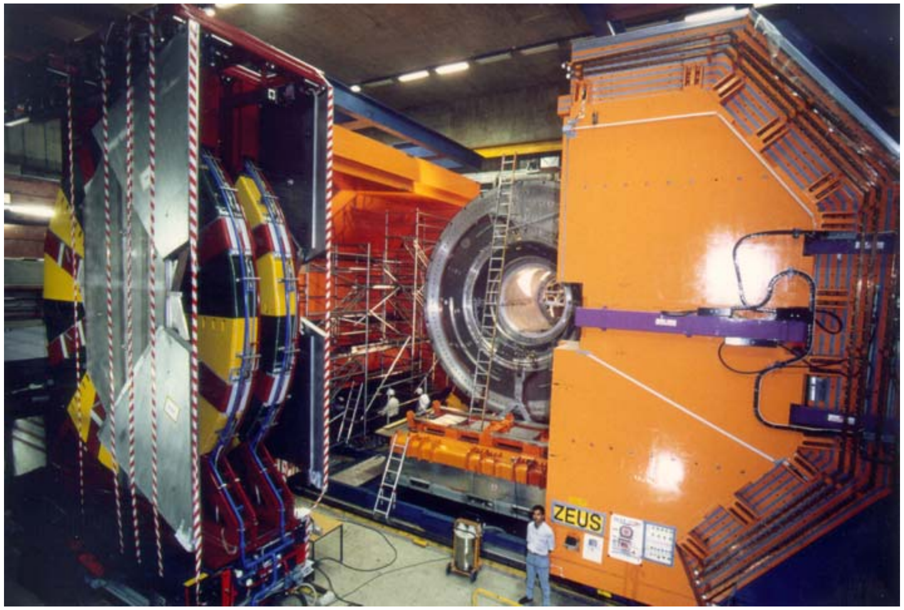


If we choose $\beta_0 = \ell$ we get the smallest β at the end of the drift and the maximum β is just twice the distance ℓ

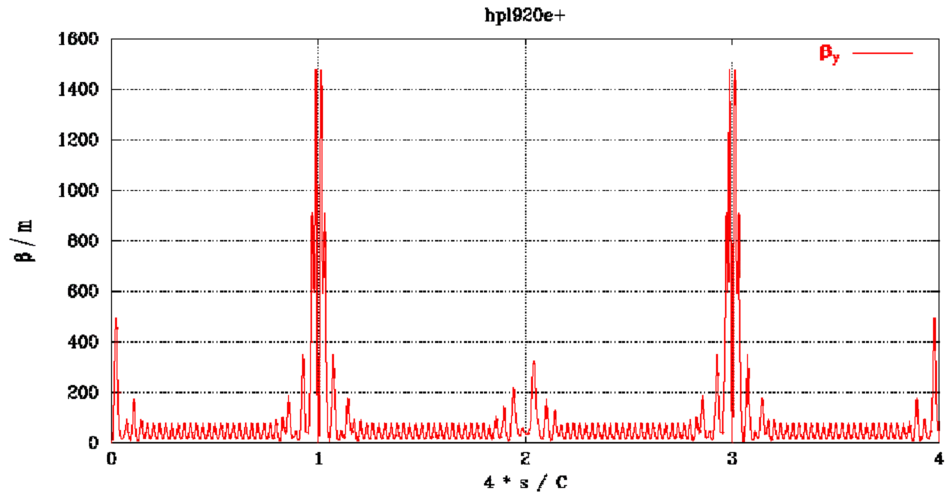
Example: HERA

βx = 2.45m, → ideal size of the detector: some "cm"

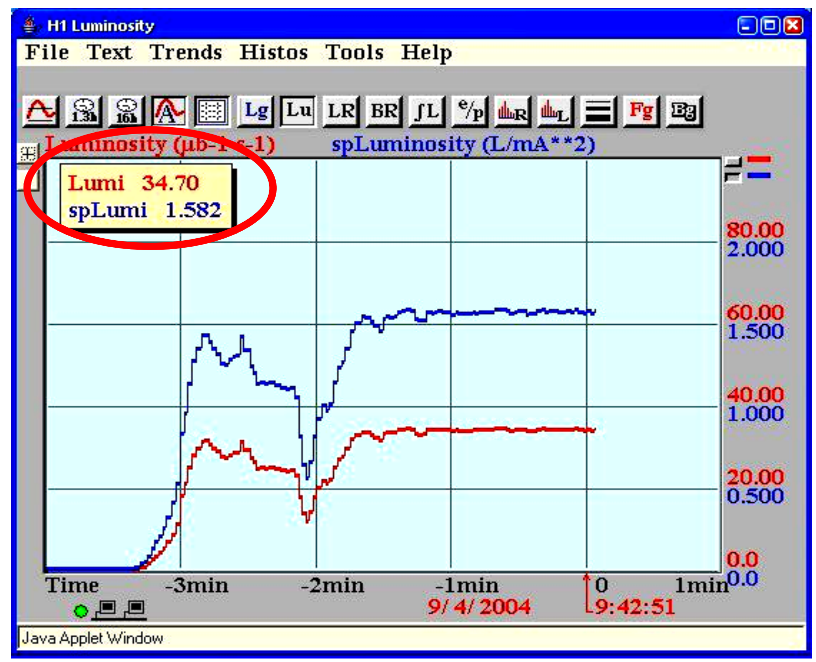
βy = 18 cm

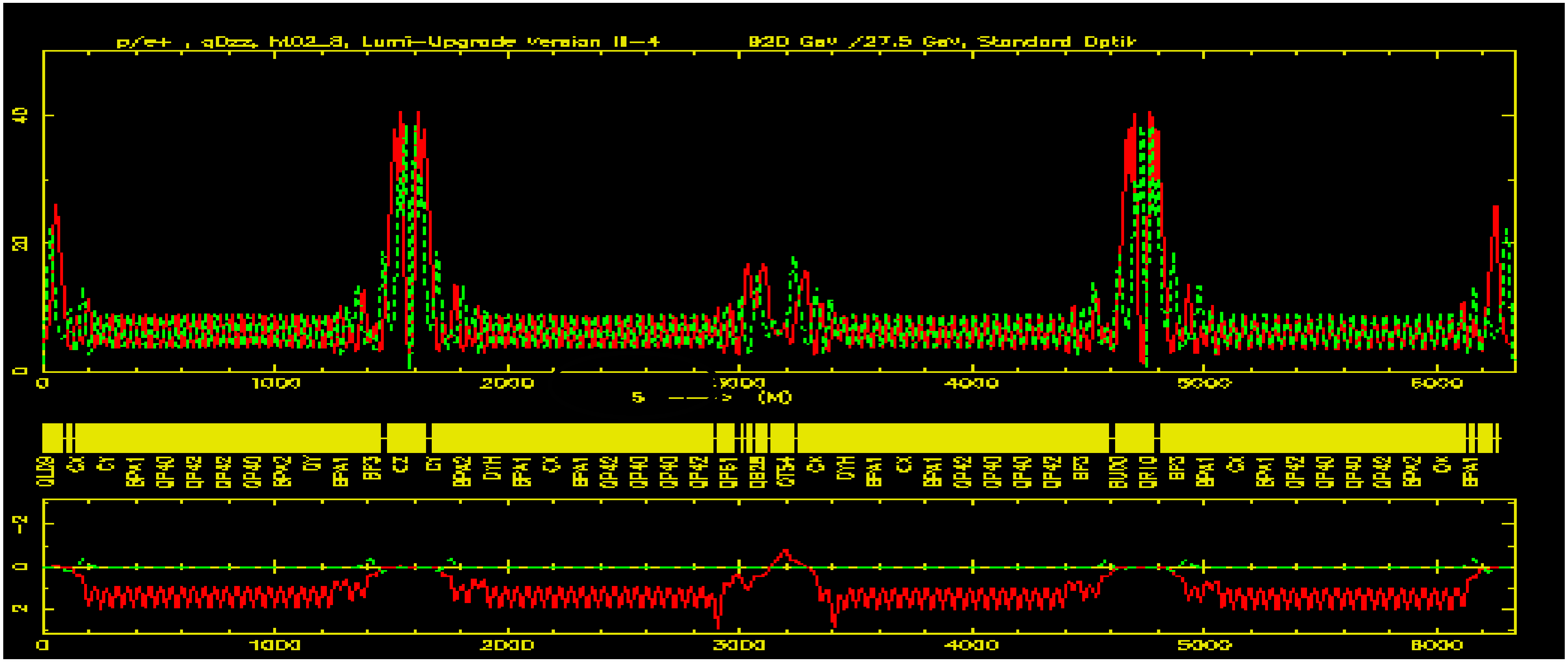


ZEUS detector at the HERA collider



value at IP	horizontal	vertical
β at IP	$\beta^*_x = 2.45m$	$\beta^*_z = 0.18m$
max β-function	$\hat{\beta}_x = 1700m$	$\hat{\beta}_z = 1500m$
emittance	$\varepsilon_x = 7 * 10^{-9} rad\ m$	$\varepsilon_z = \varepsilon_x$
beam size	$\sigma_x = 118\mu m$	$\sigma_z = 32\mu m$
beam currents	$I_e = 43mA$	$I_p = 84mA$
bunch rev. freq.	$f_0 = 47.3kHz$	$n_b = 180$
Luminosity	$L = 34.0 * 10^{30}\ {}^{1}/_{cm^2 s}$	





Lattice Design of a high energy storage ring:

Arc: ***regular (periodic) magnet structure:***

bending magnets → define the energy of the ring
main focusing & tune control, chromaticity correction,
multipoles for higher order corrections

Straight sections: ***drift spaces for injection, dispersion suppressors,***
low beta insertions, RF cavities, etc....
... and the high energy experiments if they cannot be avoided

IX.) Résumé:

Orbit distortion due to dipole error:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} \oint \frac{1}{\rho(\tilde{s})} \sqrt{\beta(\tilde{s})} \cos(\left|\varphi(\tilde{s}) - \varphi(s)\right| - \pi Q) d\tilde{s}$$

Tune shift due to quadrupole error:

$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k \beta(s)}{4\pi} ds \approx \frac{\Delta k l_{quad} \bar{\beta}}{4\pi}$$

Beta beat due to quadrupole error:

$$\Delta\beta_0 = \frac{-\beta_0}{2\sin 2\pi Q} \int_{s1}^{s1+l} \beta(s) \Delta k(s) \cos\left\{2\left|\psi(s) - \psi_0\right| - 2\pi Q\right\} ds$$

Natural chromaticity of a lattice:

$$\xi = \frac{-1}{4\pi} \oint k(s) \beta(s) \, ds$$

Momentum compaction factor:

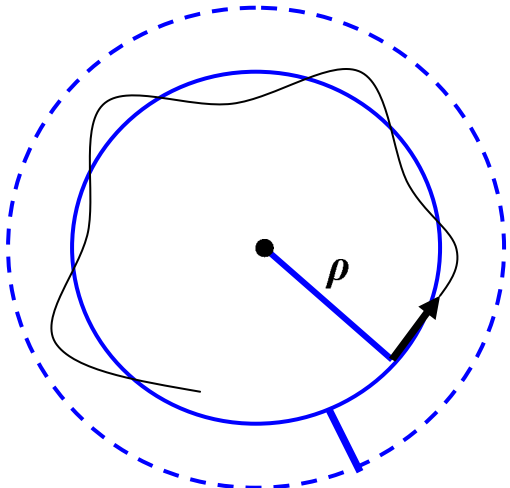
$$\alpha_{cp} = \frac{1}{L} \oint \left(\frac{D(s)}{\rho(s)} \right) ds \approx \frac{\langle D \rangle}{R}$$

APPENDIX:

periodic dispersion: ***closed orbit for a particle with $\Delta p/p \neq 0$***

particle with ideal energy: $$x'' + K(s)x = 0$$

→ betatron oscillations with respect to ideal closed orbit.



Assume weak focusing machine:
closed orbit given by D(s) and Δp/p

particle with momentum error:

$$x'' + K(s)x = \frac{1}{\rho}\frac{\Delta p}{p}$$

solution:

$$x(s) = x_D(s) + x_\beta(s)$$

where xD(s) describes the new closed orbit for $\Delta p/p \neq 0$:

$$x_D(s) = D(s)\frac{\Delta p}{p}$$

differential equation for D(s) ... as usual:

$$D'' + K(s)D = \frac{1}{\rho}$$

... but now it has to be periodic:

$$D(s + L_0) = D(s)$$

$$D'(s + L_0) = D'(s)$$

general solution of, starting from position $s_0 = 0$

$$D(s) = D_0 \cdot C(s) + D_0' \cdot S(s) + d(s)$$

$$d(s) = S(s) \cdot \int_0^s \frac{C(\tilde{s})}{\rho(\tilde{s})} d\tilde{s} - C(s) \cdot \int_0^s \frac{S(\tilde{s})}{\rho(\tilde{s})} d\tilde{s}$$

consider 1 turn from $s_0 \rightarrow s_0 + L_0 = s_1$

$$D_0 = D_0 C_1 + D_0{}' S_1 + d_1 \qquad (i)$$

$$D_0' = D_0 C_1' + D_0{}' S_1' + d_1' \qquad (ii)$$

boundary conditions for periodicity

solve (ii) for D'_0

$$D_0' = \frac{D_0 C_1' + d_1'}{1 - S_1'}$$

and put into (i) to get D_0

$$D_0 = D_0 C_1 + S_1 \frac{D_0 C_1' + d_1'}{1 - S_1} + d_1$$

solve for D_0

$$D_0 = \frac{S_1 d_1' + d_1(1 - S_1')}{(C_1 - 1)(S_1' - 1) - S_1 C_1'} = \frac{Nom}{Denom}$$

Denominator:

$$Denom = C_1 S_1' - C_1 - S_1' + 1 - S_1 C_1'$$

$$= 1 + \underbrace{(C_1 S_1' - S_1 C_1')}_{= \det M = 1} - \underbrace{(C_1 + S_1')}_{= trace\, M}$$

$$= 2 - 2\cos\mu = 4\sin^2\frac{\mu}{2}$$

remember the trigonometric gymnastics

$$\cos 2a = \cos^2\frac{a}{2} - \sin^2\frac{a}{2}$$

Nominator:

$$Nom = S_1 d_1' + d_1(1 - S_1')$$

where
$$d(s) = S(s)\cdot\int_0^s \frac{C(\tilde{s})}{\rho(\tilde{s})}d\tilde{s} - C(s)\cdot\int_0^s \frac{S(\tilde{s})}{\rho(\tilde{s})}d\tilde{s}$$

$$d'(s) = S'(s)\cdot\int_0^s \frac{C(\tilde{s})}{\rho(\tilde{s})}d\tilde{s} - C'(s)\cdot\int_0^s \frac{S(\tilde{s})}{\rho(\tilde{s})}d\tilde{s}$$

$$Nom = S_1 d_1' + d_1(1 - S_1')$$

$$= S_1 \left\{ S_1' \int_{s0}^{s1} \frac{C(\tilde{s})}{\rho(\tilde{s})} d\tilde{s} - C_1' \cdot \int_0^s \frac{S(\tilde{s})}{\rho(\tilde{s})} d\tilde{s} \right\} - (S_1' - 1) \left\{ S_1 \int_{s0}^{s1} \frac{C(\tilde{s})}{\rho(\tilde{s})} d\tilde{s} - C_1 \cdot \int_0^s \frac{S(\tilde{s})}{\rho(\tilde{s})} d\tilde{s} \right\}$$

$$= S_1 \int_{s0}^{s1} \frac{C(\tilde{s})}{\rho(\tilde{s})} d\tilde{s} - C_1 \cdot \int_0^s \frac{S(\tilde{s})}{\rho(\tilde{s})} d\tilde{s} + \underbrace{(C_1 S_1' - S_1 C_1')}_{= \det M = 1} \int_0^s \frac{S(\tilde{s})}{\rho(\tilde{s})} d\tilde{s}$$

$$= S_1 \int_{s0}^{s1} \frac{C(\tilde{s})}{\rho(\tilde{s})} d\tilde{s} + (1 - C_1) \cdot \int_0^s \frac{S(\tilde{s})}{\rho(\tilde{s})} d\tilde{s}$$

now, remember that the matrix elements C, S are related to the Twiss parameters by

$$C(s) = \sqrt{\frac{\beta(s)}{\beta_0}} \cos(\psi(s) - \psi_0) + \alpha_0 \sin(\psi(s) - \psi_0)$$

$$S(s) = \sqrt{\beta(s)\beta_0} \sin(\psi(s) - \psi_0)$$

and considering one turn $C_1 = \cos\mu + \alpha_0 \sin\mu$ $\quad S_1 = \beta_0 \sin\mu$

$$Nom = \beta_0 \sin\mu \int_{s0}^{s1} \frac{1}{\rho(\tilde{s})} \sqrt{\frac{\beta(\tilde{s})}{\beta_0}} (\cos\Delta\psi + \alpha_0 \sin\Delta\psi) d\tilde{s} +$$

$$(1 - \cos\mu - \alpha_0 \sin\mu) \int_{s0}^{s1} \frac{1}{\rho(\tilde{s})} \sqrt{\beta(\tilde{s})\beta_0} \sin\Delta\psi d\tilde{s}$$

$$Nom = 2\sqrt{\beta_0} \sin\frac{\mu}{2} \int_{s0}^{s1} \frac{1}{\rho(\tilde{s})} \sqrt{\beta(\tilde{s})} \cos(\psi(s) - \psi_0 - \frac{\mu}{2}) d\tilde{s}$$

in the end and after all ....:

$$D(s_0) = \frac{Nom}{Denom} = \frac{\sqrt{\beta(s_0)}}{2\sin\frac{\mu}{2}} \int_{s_0}^{s_0+L_0} \frac{1}{\rho(\tilde{s})} \sqrt{\beta(\tilde{s})} \cos(\psi(s) - \psi(s_0) - \frac{\mu}{2}) d\tilde{s}$$

or in general

$$D(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q} \int_{s_0}^{s_0+L_0} \frac{1}{\rho(\tilde{s})} \sqrt{\beta(\tilde{s})} \cos(|\psi(s) - \psi(\tilde{s})| - \pi Q) d\tilde{s}$$

Closed Orbit Distortion:

remember: particle with momentum error

$$x'' + K(s)x = \frac{1}{\rho}\frac{\Delta p}{p}$$

defining the function D(s) $x_D(s) = D(s)\frac{\Delta p}{p}$ *we get* $D'' + K(s)D = \frac{1}{\rho}$

assume: driving force is not Δp/p but a dipole field error:

$$\frac{1}{\rho} = \frac{e}{p_0}\Delta B$$

we can go through the same calculation – but for the periodic closed orbit $x_c(s)$ instead of D(s) and get:

$$x_c(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q}\int_{s_0}^{s_0+L_0}\frac{1}{\rho(\tilde{s})}\sqrt{\beta(\tilde{s})}\cos\left(\left|\psi(s)-\psi(\tilde{s})\right|-\pi Q\right)d\tilde{s}$$