Electron Dynamics
with radiation

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Radiation effects in electron storage rings

Average radiated power restored by RF

$U_0 \cong 10^{-3}$ of E_0

- **Electron loses energy each turn**
- **RF cavities provide voltage to accelerate electrons back to the nominal energy**

$V_{RF} > U_0$

Radiation damping

- **Average rate of energy loss produces DAMPING of electron oscillations in all three degrees of freedom (if properly arranged!)**

Quantum fluctuations

- **Statistical fluctuations in energy loss (from quantised emission of radiation) produce RANDOM EXCITATION of these oscillations**

Equilibrium distributions

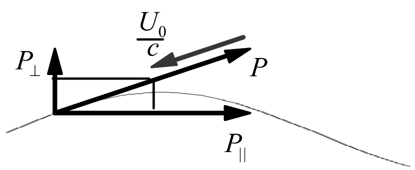
- **The balance between the damping and the excitation of the electron oscillations determines the equilibrium distribution of particles in the beam**

Average energy loss per turn

- Every turn electron radiates small amount of energy

$$E_1 = E_0 - U_0 = E_0\left(1 - \frac{U_0}{E_0}\right)$$

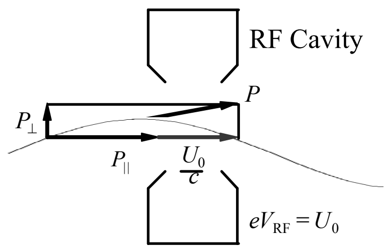
- Since the radiation is emitted along the tangent to the trajectory, only the amplitude of the momentum changes



$$P_1 = P_0 - \frac{U_0}{c} = P_0\left(1 - \frac{U_0}{E_0}\right)$$

Energy gain in the RF cavities

- Only the longitudinal component of the momentum is increased in the RF cavity



- The transverse momentum, or the amplitude of the betatron oscillation remains small

Energy of betatron oscillation

- Transverse momentum corresponds to the energy of the betatron oscillation

$$E_\beta \propto A^2$$

$$A_1^2 = A_0^2\left(1 - \frac{U_0}{E_0}\right) \quad \text{or} \quad A_1 \cong A_0\left(1 - \frac{U_0}{2E_0}\right)$$

- The relative change in the betatron oscillation amplitude that occurs in one turn (time T_0)

$$\frac{\Delta A}{A} = -\frac{U_0}{2E}$$

Exponential damping

- But this is just the exponential decay law!

$$\frac{\Delta A}{A} = -\frac{U_0}{2E}$$

- The amplitudes are exponentially **damped**

$$A = A_0 \cdot e^{-t/\tau}$$

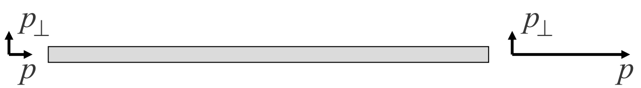
with the damping decrement

$$\frac{1}{\tau} = \frac{U_0}{2E\,T_0}$$

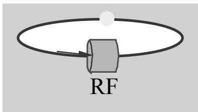
Adiabatic damping in linear accelerators

In a **linear accelerator**:

$$x' = \frac{p_\perp}{p} \quad \text{decreases} \quad \propto \frac{1}{E}$$

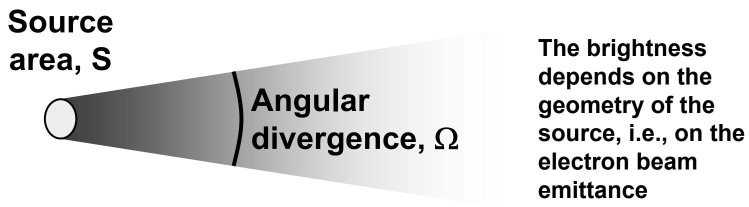


In a **storage ring** beam passes many times through same RF cavity



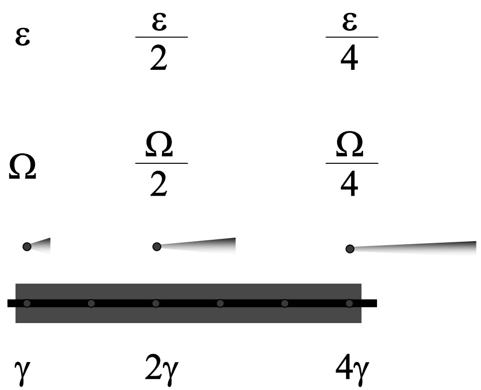
- **Clean loss of energy every turn (no change in x')**
- **Every turn is re-accelerated by RF (x' is reduced)**
- **Particle energy on average remains constant**

The electron beam "emittance":



Emittance = S x Ω

Emittance damping in linacs:



$\varepsilon \propto \frac{1}{\gamma}$

or

$\gamma\varepsilon = \text{const.}$

Damping time

- the time it would take particle to lose all of its energy

$$\tau_\varepsilon = \frac{E\,T_0}{U_0}$$

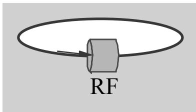
- or in terms of radiated power

$$\tau_\varepsilon = \frac{E\,T_0}{U_0} = \frac{E}{P_\gamma}$$

remember that $P_\gamma \propto E^4$

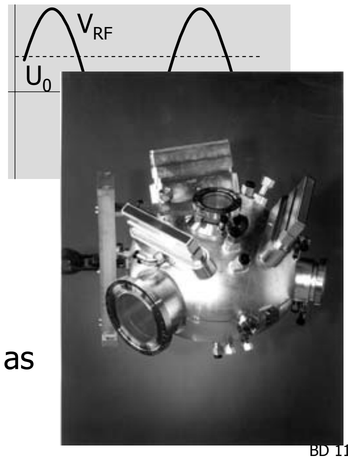
$$\tau_\varepsilon \propto \frac{1}{E^3}$$

Longitudinal motion: compensating radiation loss U_0



- RF cavity provides accelerating field with frequency
  - h – harmonic number

$$f_{RF} = h \cdot f_0$$

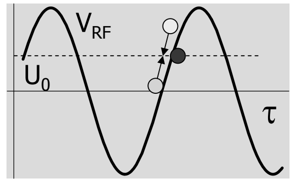


- The energy gain:

$$U_{RF} = eV_{RF}(\tau)$$

- Synchronous particle:
  - has design energy
  - gains from the RF on the average as much as it loses per turn U_0

Longitudinal motion: phase stability



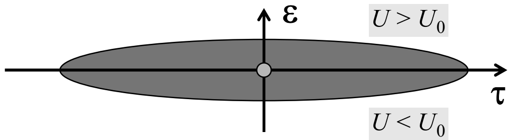
- Particle ahead of synchronous one
  - gets too much energy from the RF
  - goes on a longer orbit (not enough B)
    >> takes longer to go around
  - comes back to the RF cavity closer to synchronous part.
- Particle behind the synchronous one
  - gets too little energy from the RF
  - goes on a shorter orbit (too much B)
  - catches-up with the synchronous particle

Longitudinal motion: damping of synchrotron oscillations

$P_\gamma \propto \boldsymbol{E}^2 B^2$

During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces



- when the particle is in the lower half-plane, it loses less energy per turn, but receives U_0 on the average, so its energy deviation gradually reduces

The synchrotron motion is damped

- the phase space trajectory is spiraling towards the origin

Radiation loss

$\mathbf{P_\gamma \propto E^2 B^2}$

Displaced off the design orbit particle sees fields that are different from design values

- betatron oscillations: zero on *average*
  - linear term in B^2 - *averages to zero*
  - quadratic term - *small*
- energy deviation
  - different energy: $P_\gamma \propto E^2$
  - different magnetic field
    particle moves on a different orbit, defined by the *off-energy* or *dispersion* function D_x

$\Rightarrow$ both contribute to linear term in $P_\gamma(\varepsilon)$

Radiation loss

$\mathbf{P_\gamma \propto E^2 B^2}$

To first order in ε

$$\mathbf{U_{rad} = U_0 + U' \cdot \varepsilon}$$

electron energy changes slowly, at any instant it is moving on an orbit defined by $\mathbf{D_x}$

$$\mathbf{U' \equiv \left. \frac{dU_{rad}}{dE} \right|_{E_0}}$$

after some algebra one can write

$$U' = \frac{U_0}{E_0}\left(2 + \mathcal{D}\right)$$

$$\mathcal{D} \neq 0 \quad \textit{only when} \quad \frac{k}{\rho} \neq 0$$

Energy balance

Energy gain from the RF system: $U_{RF} = eV_{RF}(\tau) = U_0 + e\dot{V}_{RF} \cdot \tau$

- synchronous particle ($\tau = 0$) will get exactly the energy loss per turn
- we consider only linear oscillations $\dot{V}_{RF} = \left. \frac{dV_{RF}}{d\tau} \right|_{\tau=0}$
- Each turn electron gets energy from RF and loses energy to radiation within one revolution time T_0

$$\Delta\varepsilon = \left(U_0 + e\dot{V}_{RF} \cdot \tau\right) - \left(U_0 + U' \cdot \varepsilon\right) \qquad \frac{d\varepsilon}{dt} = \frac{1}{T_0}\left(e\dot{V}_{RF} \cdot \tau - U' \cdot \varepsilon\right)$$

- An electron with an energy deviation will arrive after one turn at a different time with respect to the synchronous particle

$$\frac{d\tau}{dt} = -\alpha \frac{\varepsilon}{E_0}$$

Synchrotron oscillations: damped harmonic oscillator

Combining the two equations $$\frac{d^2\varepsilon}{dt^2} + 2\alpha_\varepsilon \frac{d\varepsilon}{dt} + \Omega^2 \varepsilon = 0$$

- where the oscillation frequency $\Omega^2 \equiv \frac{\alpha e \dot{V}_{RF}}{T_0 E_0}$
- the damping is slow: $\alpha_\varepsilon \equiv \frac{U'}{2T_0}$ typically $\alpha_\varepsilon << \Omega$
- the solution is then:

$$\varepsilon(t) = \hat{\varepsilon}_0 e^{-\alpha_\varepsilon t} \cos(\Omega t + \theta_\varepsilon)$$

- similarly, we can get for the time delay:

$$\tau(t) = \hat{\tau}_0 e^{-\alpha_\varepsilon t} \cos(\Omega t + \theta_\tau)$$

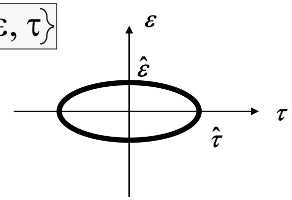
Synchrotron (time - energy) oscillations

The ratio of amplitudes at any instant $$\hat{\tau} = \frac{\alpha}{\Omega E_0}\hat{\varepsilon}$$

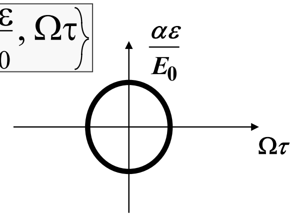
Oscillations are 90 degrees out of phase $$\theta_\varepsilon = \theta_\tau + \frac{\pi}{2}$$

The motion can be viewed in the phase space of conjugate variables

$\{\varepsilon, \tau\}$



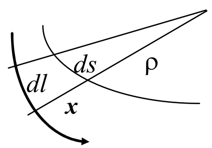
$\left\{\frac{\alpha\varepsilon}{E_0}, \Omega\tau\right\}$



Orbit Length

Length element depends on x

$$dl = \left(1 + \frac{x}{\rho}\right) ds$$



Horizontal displacement has two parts:

$$x = x_\beta + x_\varepsilon$$

- To first order x_β does not change L
- x_ε – has the same sign around the ring

Length of the off-energy orbit $$L_\varepsilon = \oint dl = \oint \left(1 + \frac{x_\varepsilon}{\rho}\right) ds = L_0 + \Delta L$$

$$\Delta L = \delta \cdot \oint \frac{D(s)}{\rho(s)} ds \quad \text{where} \quad \delta = \frac{\Delta p}{p} = \frac{\Delta E}{E}$$

$$\frac{\Delta L}{L} = \alpha \cdot \delta$$

Momentum compaction factor $\alpha \equiv \frac{1}{L}\oint \frac{D(s)}{\rho(s)} ds$

Like the tunes Q_x, Q_y - α depends on the whole optics

- A quick estimate for separated function guide field:

$$\alpha = \frac{1}{L_0 \rho_0} \int_{mag} D(s) ds = \frac{1}{L_0 \rho_0} \langle D \rangle \cdot L_{mag}$$

$\rho = \rho_0$ in dipoles
$\rho = \infty$ elsewhere

- But $L_{mag} = 2\pi\rho_0$ $\qquad \alpha = \frac{\langle D \rangle}{R}$
- Since dispersion is approximately

$$D \approx \frac{R}{Q^2} \Rightarrow \alpha \approx \frac{1}{Q^2} \text{ typically } < 1\%$$

and the orbit change for ~ 1% energy deviation

$$\frac{\Delta L}{L} = \frac{1}{Q^2} \cdot \delta \approx 10^{-4}$$

Something funny happens on the way around the ring...

Revolution time changes with energy $T_0 = \frac{L_0}{c\beta}$

$$\frac{\Delta T}{T} = \frac{\Delta L}{L} - \frac{\Delta\beta}{\beta}$$

- Particle goes faster (not much!) $\frac{d\beta}{\beta} = \frac{1}{\gamma^2} \cdot \frac{dp}{p}$ (relativity)
- while the orbit length increases (more!) $\frac{\Delta L}{L} = \alpha \cdot \frac{dp}{p}$
- The "slip factor" $\eta \cong \alpha$ since $\alpha >> \frac{1}{\gamma^2}$

$$\frac{\Delta T}{T} = \left(\alpha - \frac{1}{\gamma^2}\right) \cdot \frac{dp}{p} = \eta \cdot \frac{dp}{p}$$

- Ring is above "transition energy" $\alpha \equiv \frac{1}{\gamma_{tr}^2}$

isochronous ring: $\eta = 0 \text{ or } \gamma = \gamma_{tr}$

Not only accelerators work above transition

Dante, Paradiso

Robinson theorem
Damping partition numbers

- Transverse betatron oscillations are damped with $\frac{1}{\tau_x} = \frac{1}{\tau_z} = \frac{U_0}{2ET_0}$
- Synchrotron oscillations are damped twice as fast $\frac{1}{\tau_\varepsilon} = \frac{U_0}{ET_0}$

- The total amount of damping (Robinson theorem) depends only on energy and loss per turn

$$\frac{1}{\tau_x} + \frac{1}{\tau_y} + \frac{1}{\tau_\varepsilon} = \frac{2U_0}{ET_0} = \frac{U_0}{2ET_0}(J_x + J_y + J_\varepsilon)$$

the sum of the partition numbers $J_x + J_z + J_\varepsilon = 4$

Quantum nature of synchrotron radiation

Damping only

- If damping was the whole story, the beam emittance (size) would shrink to microscopic dimensions!*
- Lots of problems! (e.g. coherent radiation)

Quantum fluctuations

- Because the radiation is emitted in quanta, radiation itself takes care of the problem!
- It is sufficient to use quasi-classical picture:
  - » *Emission time is very short*
  - » *Emission times are statistically independent (each emission - only a small change in electron energy)*

Purely stochastic (Poisson) process

Quantum nature of synchrotron radiation

Damping only

- If damping was the whole story, the beam emittance (size) would shrink to microscopic dimensions!*
- Lots of problems! (e.g. coherent radiation)

* How small? On the order of electron wavelength

$$E = \gamma mc^2 = h\nu = \frac{hc}{\lambda_e} \quad \Rightarrow \quad \lambda_e = \frac{1}{\gamma}\frac{h}{mc} = \frac{\lambda_C}{\gamma}$$

$\lambda_C = 2.4\cdot 10^{-12} m$ – Compton wavelength

Diffraction limited electron emittance

$$\varepsilon \ge \frac{\lambda_C}{4\pi\gamma}\left(\times N^{1/3} - \text{fermions}\right)$$

Visible quantum effects

I have always been somewhat amazed that a purely quantum effect can have gross macroscopic effects in large machines;

and, even more,

that Planck's constant has just the right magnitude needed to make practical the construction of large electron storage rings.

A significantly larger or smaller value of $\hbar$

would have posed serious -- perhaps insurmountable -- problems for the realization of large rings.

Mathew Sands

Quantum excitation of energy oscillations

Photons are emitted with typical energy $u_{ph} \approx \hbar\omega_{typ} = \hbar c\,\frac{\gamma^3}{\rho}$

at the rate (photons/second) $\mathcal{N} = \frac{P_\gamma}{u_{ph}}$

Fluctuations in this rate excite oscillations

During a small interval Δt electron emits photons $N = \mathcal{N}\cdot\Delta t$

losing energy of $N\cdot u_{ph}$

Actually, because of fluctuations, the number is $N \pm \sqrt{N}$

resulting in **spread in energy loss** $\pm\sqrt{N}\cdot u_{ph}$

For large time intervals RF compensates the energy loss, providing damping towards the design energy E_0

Steady state: typical deviations from E_0
$\approx$ typical fluctuations in energy during a damping time τ_ε

Equilibrium energy spread: rough estimate

We then expect the rms energy spread to be $\sigma_\varepsilon \approx \sqrt{N\cdot\tau_\varepsilon}\cdot u_{ph}$

and since $\tau_\varepsilon \approx \frac{E_0}{P_\gamma}$ and $P_\gamma = N\cdot u_{ph}$

$\sigma_\varepsilon \approx \sqrt{E_0\cdot u_{ph}}$ geometric mean of the electron and photon energies!

Relative energy spread can be written then as:

$$\frac{\sigma_\varepsilon}{E_0} \approx \gamma\sqrt{\frac{\lambda\!\!\!^{-}_e}{\rho}} \qquad \lambda\!\!\!^{-}_e = \frac{\hbar}{m_e c} \approx 4\cdot 10^{-13} m$$

it is roughly constant for all rings

$$\frac{\sigma_\varepsilon}{E_0} \sim const \sim 10^{-3}$$

- typically $E \propto \rho^2$

Equilibrium energy spread

More detailed calculations give

- for the case of an 'isomagnetic' lattice $\rho(s) = \begin{cases} \rho_0 & \text{in dipoles} \\ \infty & \text{elsewhere} \end{cases}$

$$\left(\frac{\sigma_\varepsilon}{E}\right)^2 = \frac{C_q E^2}{J_\varepsilon \rho_0}$$

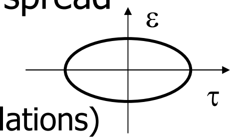
with $C_q = \frac{55}{32\sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \left[\frac{\text{m}}{\text{GeV}^2}\right]$

It is difficult to obtain energy spread < 0.1%

- limit on undulator brightness!

Equilibrium bunch length

Bunch length is related to the energy spread



- Energy deviation and time of arrival (or position along the bunch) are conjugate variables (synchrotron oscillations)

$$\hat{\tau} = \frac{\alpha}{\Omega_s}\left(\frac{\hat{\varepsilon}}{E}\right)$$

$$\sigma_\tau = \frac{\alpha}{\Omega_s}\left(\frac{\sigma_\varepsilon}{E}\right)$$

- recall that $\Omega_s \propto \sqrt{V_{RF}}$

Two ways to obtain short bunches:

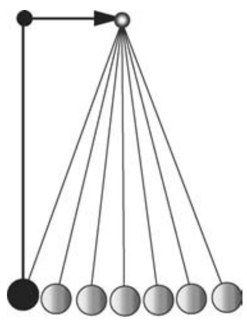
- RF voltage (power!) $\sigma_\tau \propto 1/\sqrt{V_{RF}}$
- Momentum compaction factor in the limit of $\alpha = 0$ isochronous ring: particle position along the bunch is frozen $\sigma_\tau \propto \alpha$

Horizontal oscillations: equilibrium

After an electron emits a photon

- its energy decreases: $E = E_0 - u_{ph}$ $\qquad E = E_0\left(1 - \frac{u_{ph}}{E_0}\right) = E_0(1+\delta)$
- Neither its position nor angle change after emission
- its reference orbit has smaller radius (Dispersion)

$$x_{ref} = D \cdot \delta$$



It will start a betatron oscillation around this new reference orbit

$$x_\beta = D \cdot \delta$$

Horizontal oscillations excitation

Emission of photons is a random process

- Again we have random walk, now in **x.** How far particle will wander away is limited by the radiation damping
- The balance is achieved on the time scale of the damping time $\tau_x = 2\,\tau_\varepsilon$

$$\sigma_{x\beta} \approx \sqrt{\mathcal{N} \cdot \tau_x} \cdot D \cdot \delta = \sqrt{2} \cdot D \cdot \frac{\sigma_\varepsilon}{E}$$

- In smooth approximation for D

or, typically 10^{-3} of R,
reduced further by Q^2 focusing!
In large rings $Q^2 \sim R$, so D ~ 1m

$$\sigma_{x\beta} \approx \frac{\sqrt{2}R}{Q^2} \cdot \frac{\sigma_E}{E}$$

Typical horizontal beam size ~ 1 mm

Quantum effect visible to the naked eye!

Vertical size - determined by coupling

Equilibrium horizontal emittance

Detailed calculations for isomagnetic lattice

$$\varepsilon_{x0} \equiv \frac{\sigma_{x\beta}^2}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{\langle \mathcal{H} \rangle_{mag}}{\rho}$$

where

$$\mathcal{H} = \gamma D^2 + 2\alpha D D' + \beta D'^2 = \frac{1}{\beta}\left[D^2 + (\beta D' + \alpha D)^2\right]$$

and $\langle \mathcal{H} \rangle_{mag}$ is average value in the bending magnets

$$\mathcal{H} \sim \frac{D^2}{\beta} \sim \frac{R}{Q^3}$$

For simple lattices (smooth approximation)

$$\varepsilon_{x0} \approx \frac{C_q E^2}{J_x} \cdot \frac{R}{\rho} \cdot \frac{1}{Q^3}$$

Beam emittance

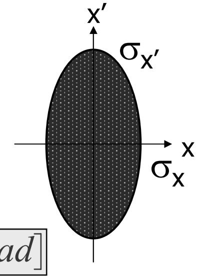
$$\text{Area} = \pi \cdot \varepsilon$$

Betatron oscillations

- Particles in the beam execute betatron oscillations with different amplitudes.

Transverse beam distribution

- Gaussian (electrons)
- "Typical" particle: 1 - σ ellipse (in a place where $\alpha = \beta' = 0$)



Units of ε $[m \cdot rad]$

$$\text{Emittance} \equiv \frac{\sigma_x^2}{\beta}$$

$$\sigma_x = \sqrt{\varepsilon\,\beta}$$
$$\sigma_{x'} = \sqrt{\varepsilon / \beta}$$

$$\varepsilon = \sigma_x \cdot \sigma_{x'}$$

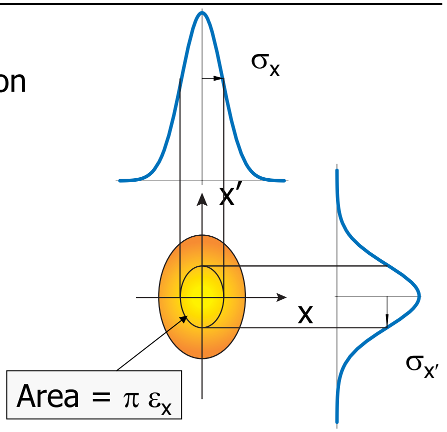
$$\beta = \frac{\sigma_x}{\sigma_{x'}}$$

2-D Gaussian distribution

Electron rings emittance definition

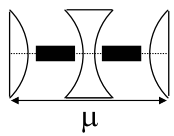
- 1 - σ ellipse

$$n(x)dx = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2} dx$$

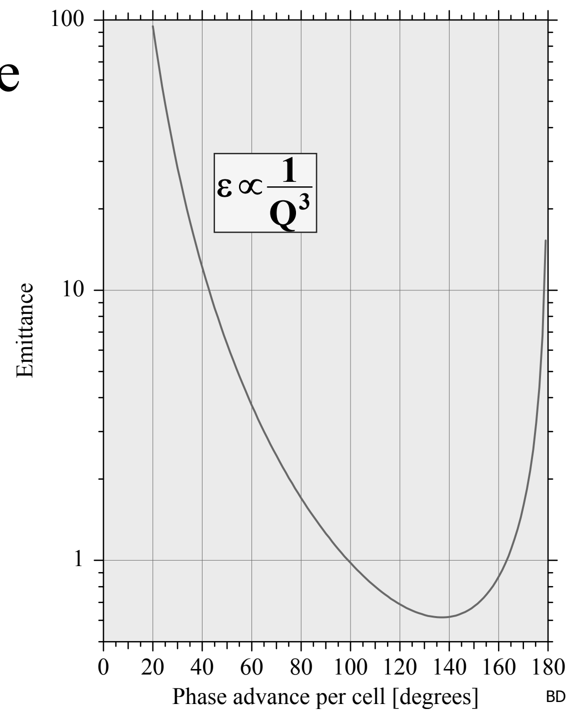


- Probability to be inside 1-σ ellipse $P_1 = 1 - e^{-1/2} = 0.39$
- Probability to be inside n-σ ellipse $P_n = 1 - e^{-n^2/2}$

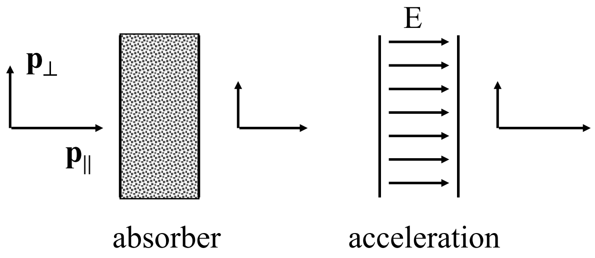
FODO Lattice emittance



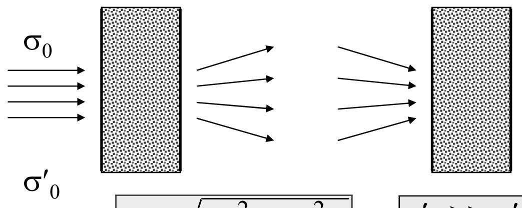
$$\varepsilon \propto \frac{\mathbf{E}^2}{\mathbf{J_x}} \theta^3 \mathrm{F}_{\mathrm{FODO}}(\mu)$$



Ionization cooling



similar to radiation damping, but there is multiple scattering in the absorber that blows up the emittance



$$\sigma' = \sqrt{\sigma'^2_0 + \sigma'^2_{MS}}$$

$$\sigma'_0 >> \sigma'_{MS}$$

to minimize the blow up due to multiple scattering in the absorber we can **focus** the beam

Summary of radiation integrals

Momentum compaction factor

$$\alpha = \frac{I_1}{2\pi R}$$

Energy loss per turn

$$U_0 = \frac{1}{2\pi} C_\gamma E^4 \cdot I_2$$

$$C_\gamma = \frac{4\pi}{3} \frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5} \left[\frac{\text{m}}{\text{GeV}^3}\right]$$

$$I_1 = \oint \frac{D}{\rho}\, ds$$
$$I_2 = \oint \frac{ds}{\rho^2}$$
$$I_3 = \oint \frac{ds}{|\rho^3|}$$
$$I_4 = \oint \frac{D}{\rho}\left(2k + \frac{1}{\rho^2}\right) ds$$
$$I_5 = \oint \frac{\mathcal{H}}{|\rho^3|}\, ds$$

Summary of radiation integrals (2)

Damping parameter $\mathcal{D} = \frac{I_4}{I_2}$

Damping times, partition numbers

$$J_\varepsilon = 2 + \mathcal{D}, \quad J_x = 1 - \mathcal{D}, \quad J_y = 1$$

$$\tau_i = \frac{\tau_0}{J_i} \qquad \tau_0 = \frac{2ET_0}{U_0}$$

Equilibrium energy spread

$$\left(\frac{\sigma_\varepsilon}{E}\right)^2 = \frac{C_q E^2}{J_\varepsilon} \cdot \frac{I_3}{I_2}$$

Equilibrium emittance

$$\varepsilon_{x0} = \frac{\sigma^2_{x\beta}}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{I_5}{I_2}$$

$$I_1 = \oint \frac{D}{\rho}\, ds$$
$$I_2 = \oint \frac{ds}{\rho^2}$$
$$I_3 = \oint \frac{ds}{|\rho^3|}$$
$$I_4 = \oint \frac{D}{\rho}\left(2k + \frac{1}{\rho^2}\right) ds$$
$$I_5 = \oint \frac{\mathcal{H}}{|\rho^3|}\, ds$$

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \left[\frac{\text{m}}{\text{GeV}^2}\right]$$

$$\mathcal{H} = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

Smooth approximation

Betatron oscillation approximated by harmonic oscillation

$$x(s) = a\sqrt{\beta(s)}\cos\left[\varphi(s) - \varphi_0\right]$$
$$\varphi(s) = \int_0^s \frac{d\dot{s}}{\beta(\dot{s})}$$

$$x \approx a\sqrt{\beta_n}\cos\left(\frac{s}{\beta_n} - \varphi_0\right) \quad \Leftarrow \quad x'' + k_{eff} \cdot x = 0, \quad k_{eff} = \frac{1}{\beta_n^2}$$

$$\beta(s) = \beta_n = \text{const}$$

- Phase advance around the ring $\quad 2\pi Q = \oint \frac{ds}{\beta_n} = \frac{1}{\beta_n} \cdot 2\pi R \quad \Rightarrow \quad \beta_n = \frac{R}{Q}$
- Dispersion obeys the equation $\quad D'' + k_{eff} D = \frac{1}{R} \quad \Rightarrow \quad D_n = \frac{\beta_n^2}{R} = \frac{R}{Q^2}$
- Momentum compaction factor α $\quad \alpha = \frac{\langle D \rangle}{R} = \frac{\beta_n^2}{R^2} \quad \Rightarrow \quad \alpha \approx \frac{1}{Q_x^2}$