Computational Methods in Accelerator Physics

An introduction (1)

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(http://cern.ch/Werner.Herr/lectures/baden_comp1.pdf)

Where are computational methods needed ?

- Studies of Beam Dynamics
  - Design and simulation of an accelerator
  - Control and operation

- Design of accelerator equipment
  - Magnets, RF cavities ...
  - Vacuum components, cryogenics

Steps of accelerator design

- Definition of basic parameters according to requirements
- Design of machine layout and optics
- Analysis of its (local and global) properties, evaluation of performance
- Stability of beams:
  - → Single particle stability
  - → Multi particle behaviour
- Process may be iterative

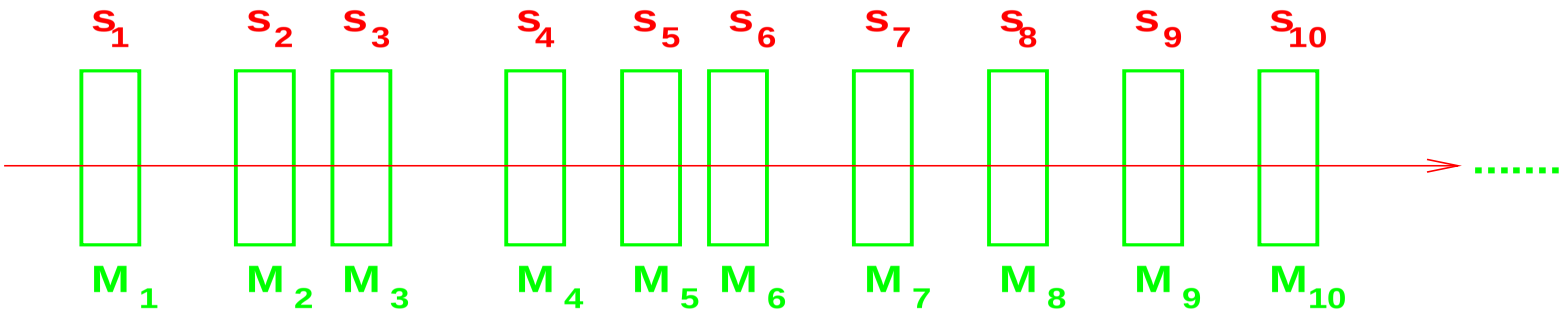
Accelerator design with an optics program

- It needs: Description of machine in standard format
- It does: Optics calculations
  - → Linear and non-linear optics computations
  - → Linear corrections
  - → Non-linear and chromatic corrections
- Parameter matching

How does an accelerator look like to a computer ?

- Basically as it looks like to a particle:

→ A (finite) sequence of machine elements $\mathcal{M}$ at longitudinal positions $s_1, s_2, s_3, \ldots$:



- All elements together make the ring or a beam line

How does an element look like to a computer ?

- Each element $\mathcal{M}$ acts on the beam in a deterministic way

- In general: $\vec{z_2} \neq \vec{z_1}$

What is $\mathcal{M}$? It can represent:

- A single machine element
  - magnet: dipole, quadrupole
  - vacuum chamber
  - collimators
  - RF cavity
  - monitors
  - drift
  - .....

It can also represent:

- **Several machine elements combined, (i.e. part of a ring made of n elements: $\mathcal{M}_{part}$)**

$$\mathcal{M}_{part} = \mathcal{M}_1 \circ \mathcal{M}_2 \circ ..... \circ \mathcal{M}_n$$

- **The whole machine (e.g. complete turn in a ring with N elements, $\mathcal{M}_{ring}$)**

$$\mathcal{M}_{ring} = \mathcal{M}_1 \circ \mathcal{M}_2 \circ ..... \circ \mathcal{M}_N$$

- **Many turns in a ring (M $= {\mathcal{M}_{ring}}^n$)**

How is an element described to a computer ?

- Let $\vec{z_1}, \vec{z_2}$ describe a quantity (coordinates, beam sizes ...) before and after the element
- Take an machine element (e.g. magnet) and build a mathematical object $\mathcal{M}$ for this quantity
  - → $\mathcal{M}$ describes the element in terms of this quantity
  - → In general: $\vec{z_2} = \mathcal{M} \circ \vec{z_1}$
  - → $\mathcal{M}$ is a so-called MAP
- All MAPS connect the pieces together to make a ring (or beam line)

MAPS transform coordinates through an element

- **4 coordinates needed for 2 dimensions (off momentum effects ignored)**
- **Coordinate vector:** $\vec{z} = (x, \quad x' = \frac{\delta x}{\delta s}, \quad y, \quad y' = \frac{\delta y}{\delta s})$
- $\mathcal{M}$ **transforms the coordinates** $\vec{z_1}(s_1)$ **at position** s_1 **to new coordinates** $\vec{z_2}(s_2)$ **at position** s_2**:**

$$\vec{z_2}(s_2) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \mathcal{M} \circ \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} = \mathcal{M} \circ \vec{z_1}(s_1)$$

... or OPTICAL functions

6 optical functions needed for 2 dimensions:

$$\vec{\nu_2}(s_2) = \begin{pmatrix} \beta_x \\ \alpha_x \\ \gamma_x \\ \beta_y \\ \alpha_y \\ \gamma_y \end{pmatrix}_{s_2} = \mathcal{M} \circ \begin{pmatrix} \beta_x \\ \alpha_x \\ \gamma_x \\ \beta_y \\ \alpha_y \\ \gamma_y \end{pmatrix}_{s_1} = \mathcal{M} \circ \vec{\nu_1}(s_1)$$

How does $\mathcal{M}$ look like ?

The map $\mathcal{M}$ describes local properties of a machine element and can be:

- **A simple linear matrix or transformation**
- **High order integration algorithm**
- **Derived from local Hamiltonian**
- **A computer program, subroutine etc.**
- **Any "description" to go from $\vec{z_1}$ to $\vec{z_2}$**
- **Can in principle be VERY abstract !**

Simple examples (linear, one dimensional)
(Matrix formulation for linear* elements)

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

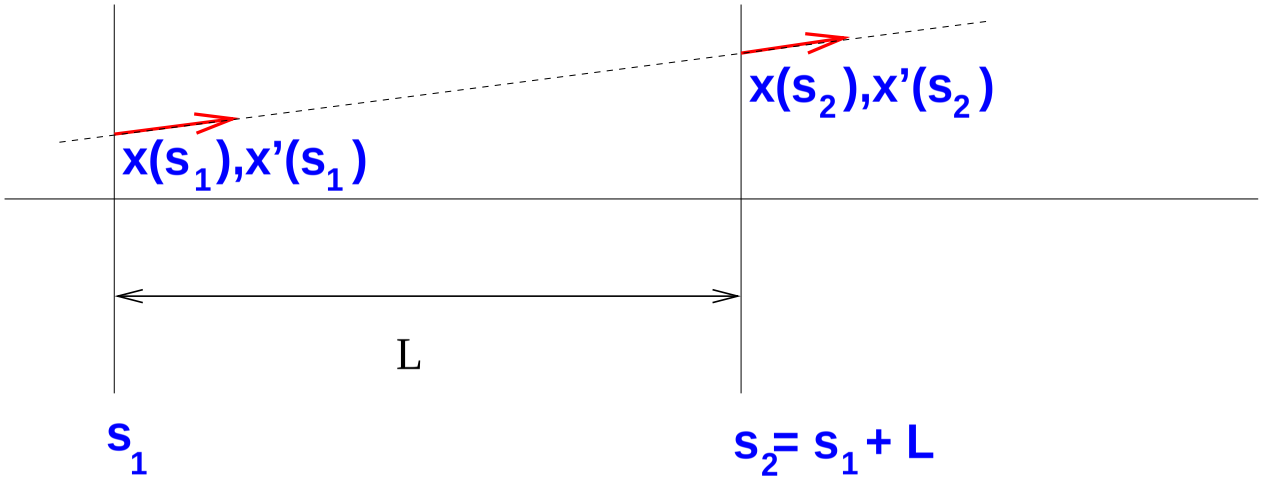
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22}+m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \circ \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_1}$$

→ **The maps become so-called transport matrices**

* **Its effect depends on x or x' only or is constant**

Transformation of coordinates (one dimension)

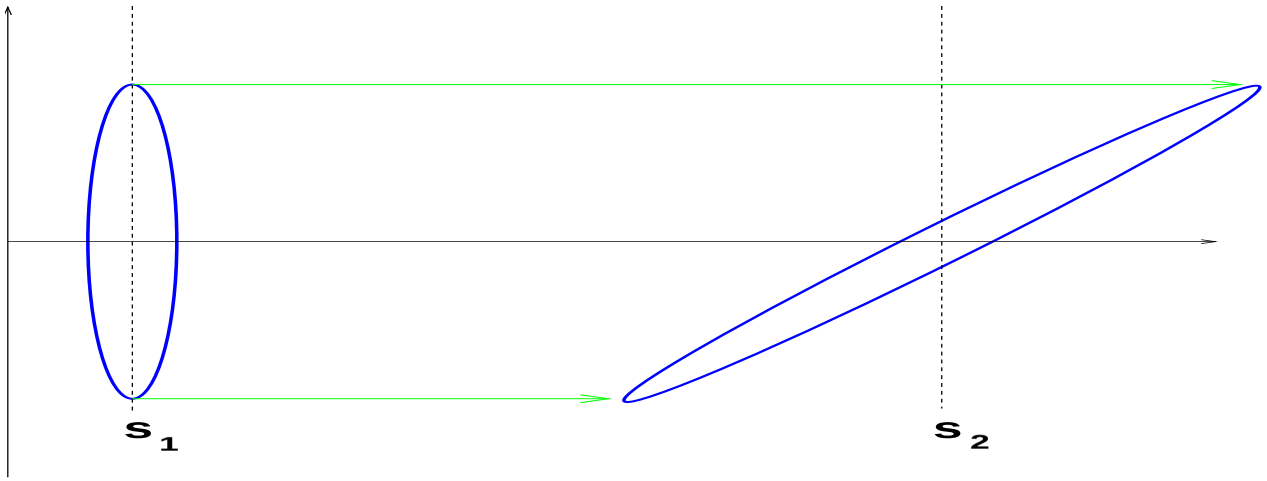
Drift space of length $L = s_2 - s_1$:



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} + \begin{pmatrix} x' \cdot L \\ 0 \end{pmatrix}_{s_1} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

Transformation of beam ellipse

Drift space of length $L = s_2 - s_1$:



$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_2} = \begin{pmatrix} 1 & -2L & L^2 \\ 0 & 1 & -L \\ 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_1} = \begin{pmatrix} \beta_0 - 2L\alpha_0 + L^2\gamma_0 \\ \alpha_0 - L\gamma_0 \\ \gamma_0 \end{pmatrix}_{s_2}$$

Simple examples (one dimensional)

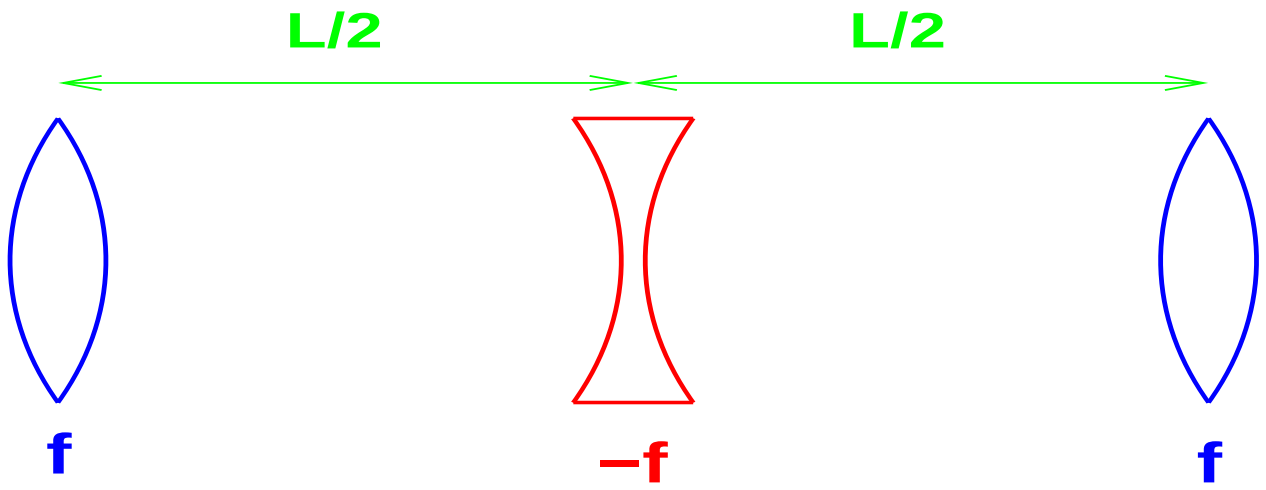
Focusing quadrupole of length L and strength K:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} \cos(L \cdot K) & \frac{1}{K} \cdot \sin(L \cdot K) \\ K \cdot \sin(L \cdot K) & \cos(L \cdot K) \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

Quadrupole with short length L (i.e.: $1 \gg L^2 \cdot K^2$)

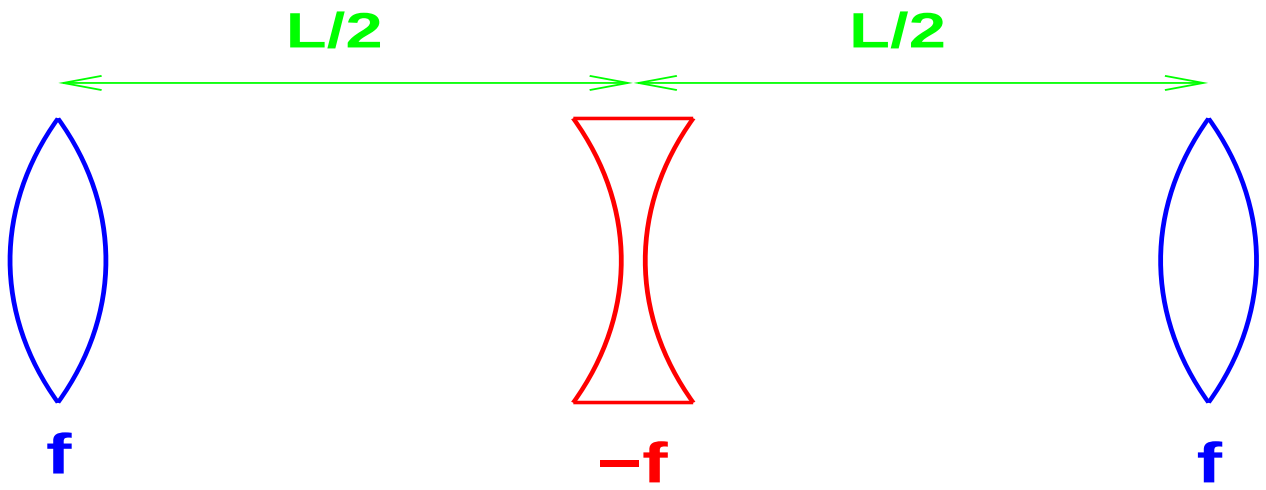
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} 1 & 0 \\ K^2 \cdot L(= -\frac{1}{f}) & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

Composition of elements (FODO cell)



$$\mathcal{M}_{cell} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}$$

Composition of elements (FODO cell)



$$\mathcal{M}_{cell} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$$

Computer program for optics calculation

- It reads the sequence of elements of a machine (their order, their positions ..)
- It reads properties of the elements, i.e. type (dipole, quadrupole, drift ...)
- It reads strength of the elements
- It sets up the maps (matrices)

→ An example is Methodical Accelerator Design (from CERN) but there are many more !

Simplest machine description (MAD format)

```
// description of elements and their strengths
// dipoles and quadrupoles only ...
   mb: dipole,     l=6.0, angle=0.03570;
   qf: quadrupole, l=3.0, k1= 0.013426;
   qd: quadrupole, l=3.0, k1=-0.013426;

// centre position of elements in the ring
   start:    at=0;
   qf.1: qf, at=1.5000e+00;
   mb: mb,   at=9.0000e+00;
   mb: mb,   at=1.9000e+01;
   qd.1: qd, at=2.6500e+01;
   mb: mb,   at=3.4000e+01;
   mb: mb,   at=4.4000e+01;
   qf.2: qf, at=5.1500e+01;
             ...
   end:      at=2.2000e+03;
```

→ **Construct a MAP for each element from this data**

The next step

Set up a sequence of MAPS:

$\mathcal{M}_1$[**QUADRUPOLE, L=3.0, STR=K1, START=0.0**]
$\mathcal{M}_2$[**DRIFT, L=3.0, START=3.0**]
$\mathcal{M}_3$[**DIPOLE, L=6.0, STR=ANGLE, START=6.0**]
$\mathcal{M}_4$[**DRIFT, L=4.0, START=12.0**]
$\mathcal{M}_5$[**DIPOLE, L=6.0, STR=ANGLE, START=16.0**]
$\mathcal{M}_6$[**DRIFT, L=3.0, START=22.0**]
$\mathcal{M}_7$[**QUADRUPOLE, L=3.0, STR=-K1, START=25.0**]
$\mathcal{M}_8$[**DRIFT, L=3.0, START=28.0**]
$\mathcal{M}_9$[**DIPOLE, L=6.0, STR=ANGLE, START=31.0**]
$\mathcal{M}_{10}$[**DRIFT, L=4.0, START=37.0**]
$\mathcal{M}_{11}$[**DIPOLE, L=6.0, STR=ANGLE, START=41.0**]
.....

Putting the "pieces" together

Starting from a position s_0 and applying all maps (for N elements) in sequence around a ring with circumference C to get the One-Turn-Map (OTM) for the position s_0 (for one dimension only):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 \,+\, C} = \mathcal{M}_1 \circ \mathcal{M}_2 \circ \ldots \circ \mathcal{M}_N \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\Longrightarrow \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 \,+\, C} = \mathcal{M}_{ring}(s_0) \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

Putting the "pieces" together

If all maps are matrices (i.e. only linear elements)

- **Always true in the first step !**
- **The One-Turn-Map is a MATRIX:**

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 \,+\, C} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

- **After all multiplications we get the One-Turn-Map which depends on the starting point s_0.**

The analysis of the results

What can we do with the One-Turn-Matrix ?

We can express the One-Turn-Matrix $\mathcal{M}_{ring}(s_0)$ in terms of Courant-Snyder parameters:

We know that $\mathcal{M}_{ring}(s_0)$ for one dimension must be:

$$\mathcal{M}_{ring}(s_0) \equiv \begin{pmatrix} \cos\mu + \alpha(s_0)\sin\mu & \beta(s_0)\sin\mu \\ -\gamma(s_0)\sin\mu & \cos\mu - \alpha(s_0)\sin\mu \end{pmatrix}$$

and we also know that (for a ring):

$$\alpha(s_0+C) \equiv \alpha(s_0), \quad \beta(s_0+C) \equiv \beta(s_0), \quad \gamma(s_0+C) \equiv \gamma(s_0)$$

The first optics calculations

Comparison of:

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \mathcal{M}_1 \circ \mathcal{M}_2 \circ \ldots \circ \mathcal{M}_N$$

and :

$$\mathcal{M}_{ring}(s_0) = \begin{pmatrix} \cos\mu + \alpha(s_0)\sin\mu & \beta(s_0)\sin\mu \\ -\gamma(s_0)\sin\mu & \cos\mu - \alpha(s_0)\sin\mu \end{pmatrix}$$

gives optical functions at position s_0:

- $\beta(s_0)$, $\alpha(s_0)$, $\gamma(s_0)$ **(depend on position s_0)**
- μ **is independent of s_0:** **(2πQ)**

We have now:

- Values for β_x, β_y, α_x, ... etc. at the position s_0
- Tunes for both planes

The next step:

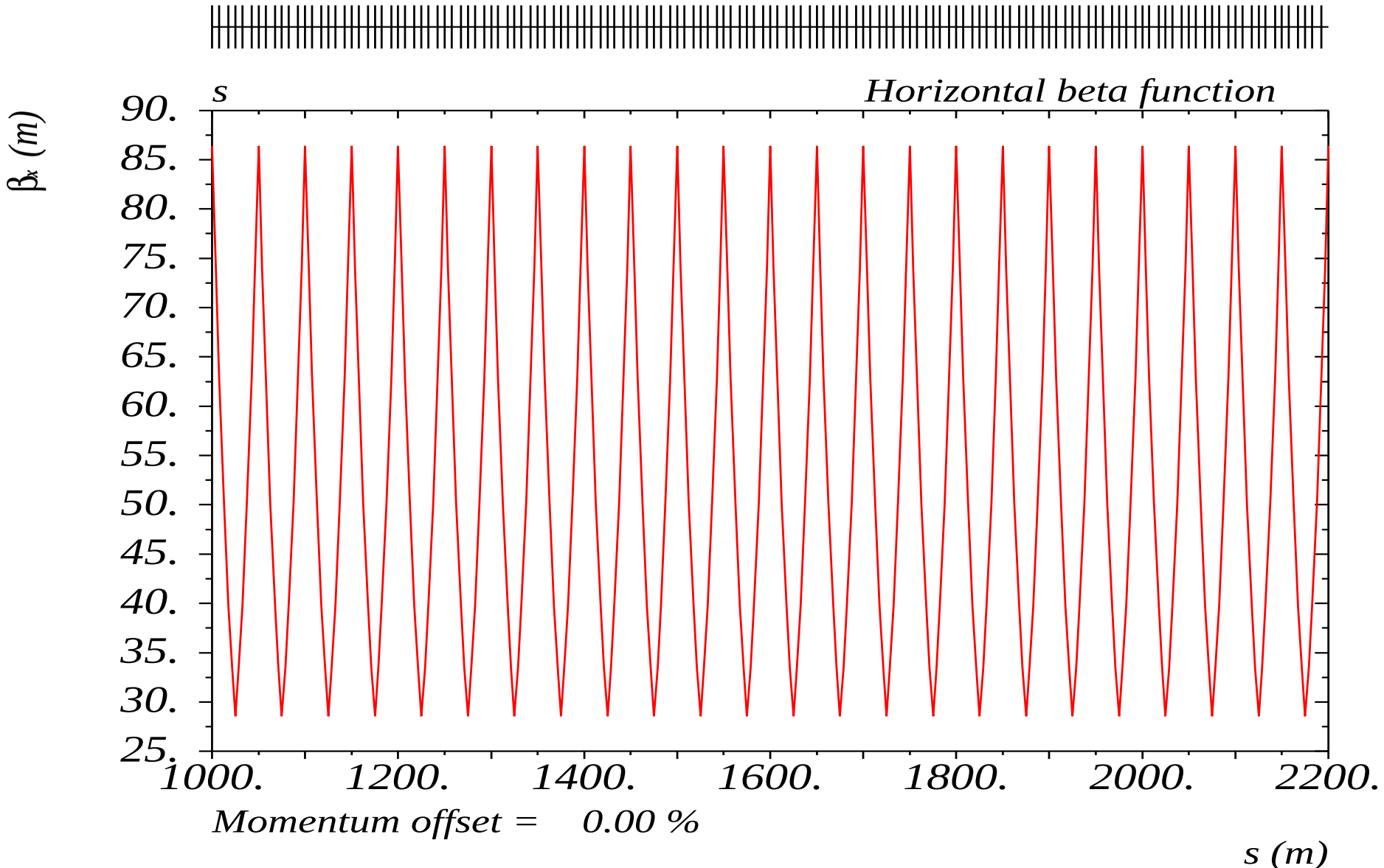
- Starting from initial optical (Twiss) functions at s_0, transforming β_x, β_y, α_x, ... through the lattice gives functions at all positions s.
- Question: what are the β-functions etc. of a linear accelerator or a beam line ???

Computation of optical functions around the ring

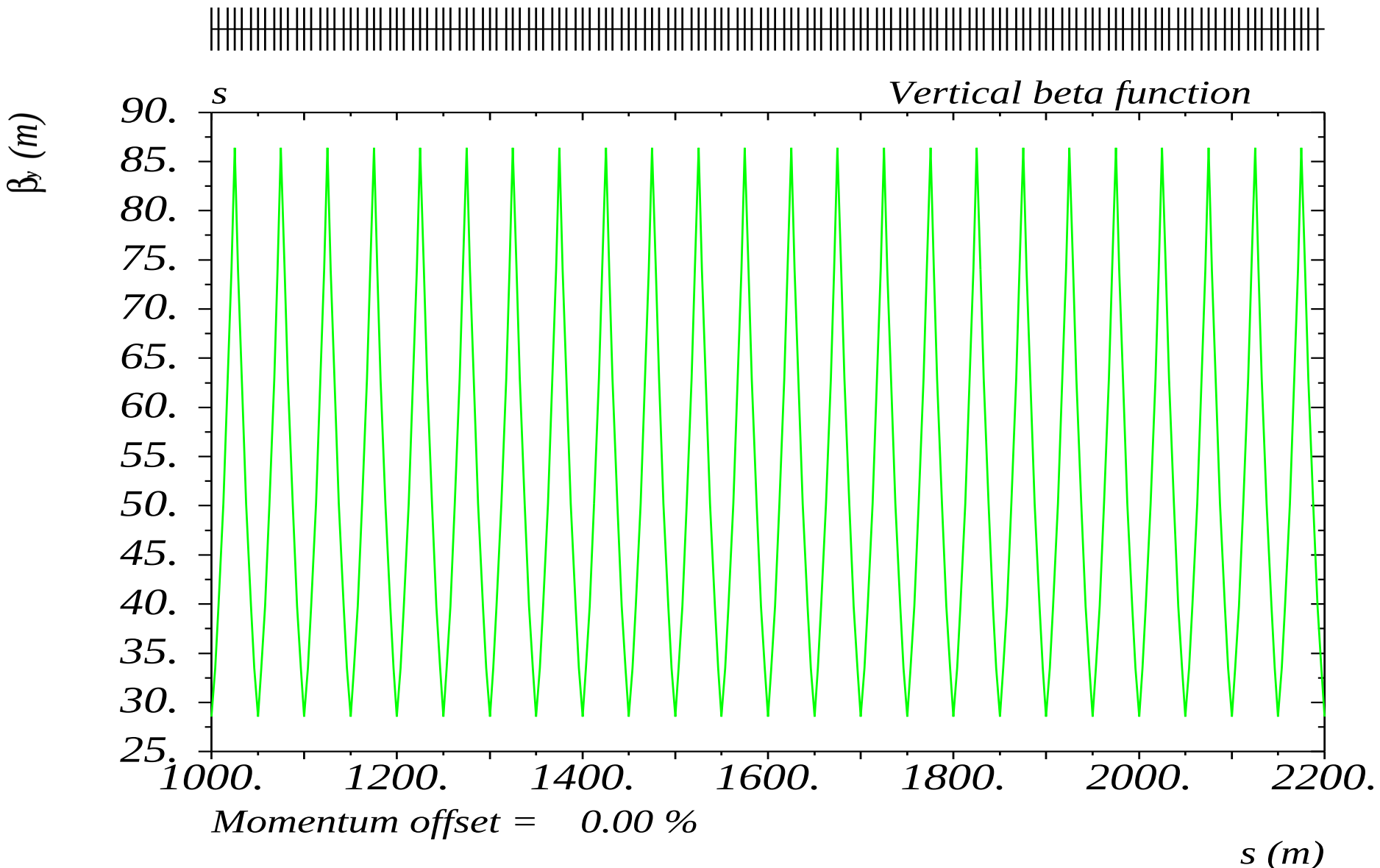
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22}+m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \circ \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_0}$$

Successive application of matrices give Twiss functions at each element around the ring and at each position s ➡

Optical functions (horizontal β):



Optical functions (vertical β):



Why do I spend so much time on $\mathcal{M}_{ring}$?

- Because it is very useful:
  - Courant-Snyder ansatz (formalism) assumes motion is linearly stable, periodic, confined, and has a closed orbit.
  - The OTM $\mathcal{M}_{ring}$ contains all information about global behaviour in the ring, i.e. stability, tune, β, closed orbit etc.
- No need for assumptions

Why do I spend so much time on $\mathcal{M}_{ring}$?

- $\mathcal{M}_{ring}$ or $\mathcal{M}_{part}$ allow the analysis of imperfections (and their correction !)
- "Straightforward" to formally extend it to complicated (e.g. non-linear) problems - additional tools and concepts needed (invariants, fixpoints, normal forms etc.)

(Interlude I: Fixed Points)

- **Certain points in phase space $\vec{z_1}$ repeat itself after n completed turns**

$$\mathcal{M}_{ring}^{n} \circ \vec{z_1} = \vec{z_2} \equiv \vec{z_1}$$

- **Defines a Fixed Point of order n**
- **Fixed Point of order 1 is the closed orbit**
- **Stability requires existence of such a fixed point**
- **Closed orbit is found (or not !) in optics programs by searching for the first order fixed point**

Extension to two dimensions

- **Can be written as separate equations, or:**
- **Extend vectors for coordinates or optical parameters**
- **Extend transfer maps/matrices**

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1}$$

Extension to two dimensions

- **The horizontal and vertical motion can be coupled:**

→ **Additional elements in matrix**

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1}$$

Off momentum effects

- **Introduces longitudinal motion and off momentum trajectories**

Strength k of element modified by non-zero $\frac{\Delta p}{p}$:

$$k \Longrightarrow k/(1 + \frac{\Delta p}{p})$$

- **Closed orbit and tune are usually different for non-zero $\frac{\Delta p}{p}$**

→ **Dispersion**

→ **Chromatic effects**

Going to three dimensions

- **Formally extended by adding two new variables:**
  - $\Delta s = c\Delta t$**: longitudinal displacement with respect to reference particle**
  - $\frac{\Delta p}{p}$**: relative momentum difference with respect to reference particle**

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ c\Delta t \\ \frac{\Delta p}{p} \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{32} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \\ m_{42} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\ m_{32} & m_{32} & m_{33} & m_{34} & m_{55} & m_{56} \\ m_{62} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ c\Delta t \\ \frac{\Delta p}{p} \end{pmatrix}_{s_1}$$

(Interlude II: Symplecticity)

- **Not all possible maps are allowed !**
- **Requires for a matrix** $\mathcal{M} \rightarrow \mathcal{M}^t \cdot S \cdot \mathcal{M} = \mathcal{S}$

  with:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

- **It basically means:** $\mathcal{M}$ **is area preserving and**

$$\lim_{n \to \infty} \mathcal{M}^n = \text{finite}$$

Introducing non-linear elements

Effect of a (short) quadrupole depends linearly on amplitude (re-written from the matrix form):

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} + \begin{pmatrix} 0 \\ k_1 \cdot x_{s_1} \\ 0 \\ k_1 \cdot y_{s_1} \end{pmatrix}$$

→ $\vec{z}(s_2) = \mathbf{M} \cdot \vec{z}(s_1)$

→ **M is a matrix**

Non-linear elements (e.g. sextupole)

Effect of a (thin) sextupole with strength k_2 is:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} + \begin{pmatrix} 0 \\ k_2 \cdot (x_{s_1} \cdot y_{s_1}) \\ 0 \\ \frac{1}{2} k_2 \cdot (x_{s_1}^2 - y_{s_1}^2) \end{pmatrix}$$

→ $\vec{z}(s_2) = \mathcal{M} \circ \vec{z}(s_1)$

→ $\mathcal{M}$ **is not a matrix, i.e. cannot be expressed by matrix multiplication**

Non-linear elements

Cannot be written in linear matrix form !

We need something like:

$$
\begin{aligned}
x \;=\; & R_{11}\cdot x \;+ R_{12}\cdot x' + \\
& + T_{111}\cdot x^2 + T_{112}\cdot xx' + T_{122}\cdot x'^2 + \\
& + T_{113}\cdot xy + T_{114}\cdot xy' + ... \\
& + \; ...
\end{aligned}
$$

and the equivalent for all other variables ...

Higher order MAPS:

Definition of a second order map $\mathcal{A}_2$ (Taylor expansion):

Let's call it : $\mathcal{A}_2 = [R, T]$, defined as (for : j = 1...4) :

$$z_j(s_2) = \sum_{k=1}^{4} R_{jk} z_k(s_1) + \sum_{k=1}^{4}\sum_{l=1}^{4} T_{jkl} z_k(s_1) z_l(s_1)$$

Higher orders can be defined as needed ...

$$\mathcal{A}_3 = [R, T, U] \quad \Longrightarrow \quad + \sum_{k=1}^{4}\sum_{l=1}^{4}\sum_{m=1}^{4} U_{jklm} z_k(s_1) z_l(s_1) z_m(s_1)$$

Second order MAPS composition:

Assume now 2 maps of second order:

$\mathcal{A}_2 = [R^A, T^A]$ **and** $\mathcal{B}_2 = [R^B, T^B]$

the combined second order map

$\mathcal{C}_2 = \mathcal{A}_2 \circ \mathcal{B}_2$ **is** $\mathcal{C}_2 = [R^C, T^C]$ **with:**

$$R^C = R^A \cdot R^B$$

and (after truncation of higher order terms !!):

$$T^C_{ijk} = \sum_{l=1}^{4} R^B_{il} T^A_{ljk} + \sum_{l=1}^{4} \sum_{m=1}^{4} T^B_{ilm} R^A_{lj} R^A_{mk}$$

(Interlude II: Symplecticity)

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  with:

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$$\lim_{n \to \infty} \mathcal{M}^n = \text{finite}$$

Symplecticity for higher order MAPS

- **Truncated Taylor expansions are not matrices !!**
- **It is the associated Jacobian matrix $\mathcal{J}$ which must fulfil the symplecticity condition:**

$$\mathcal{J}_{ik} = \frac{\delta z_2^i}{\delta z_1^k}$$

$\mathcal{J}$ **must fulfil:** $\mathcal{J}^t \cdot S \cdot \mathcal{J} = \mathcal{S}$

- **In general:** $\mathcal{J}_{ik} \neq$ **const** → **for truncated Taylor map can be difficult to fulfil for all** z

(Interlude III: Other higher order MAPS)

There are other types of higher order maps:

- Lie transformations (always symplectic for any order, ideal for tracking, easy to analyse)
- Symplectic integration algorithms
- whatever ...

→ Intermediate level school ... !

Matching optical functions

- Modify machine optics to get desired properties around the machine or in specific places
- For example you may want special conditions
  - for equipment: RF, collimators, diagnostics
  - for experiments: in colliding beam machines
- Algorithms to adjust parameters and layout

→ This process is called MATCHING !

→ Available in most optics programs (for lines and circular machines)

Matching optical functions

In general: $\vec{\nu}(\beta_x(s), \alpha_x(s)...) = f(\mathcal{M}_1, \mathcal{M}_2, ...\mathcal{M}_n)$
optical functions depend on all maps, i.e. strengths and layout.

- **Matching can change all parameters (strengths, positions ...)**
- **Additional constraints may be:**
  - **Tunes, chromaticities**
  - **Hardware parameters**
  - $\hat{\beta}_x$**, etc.**

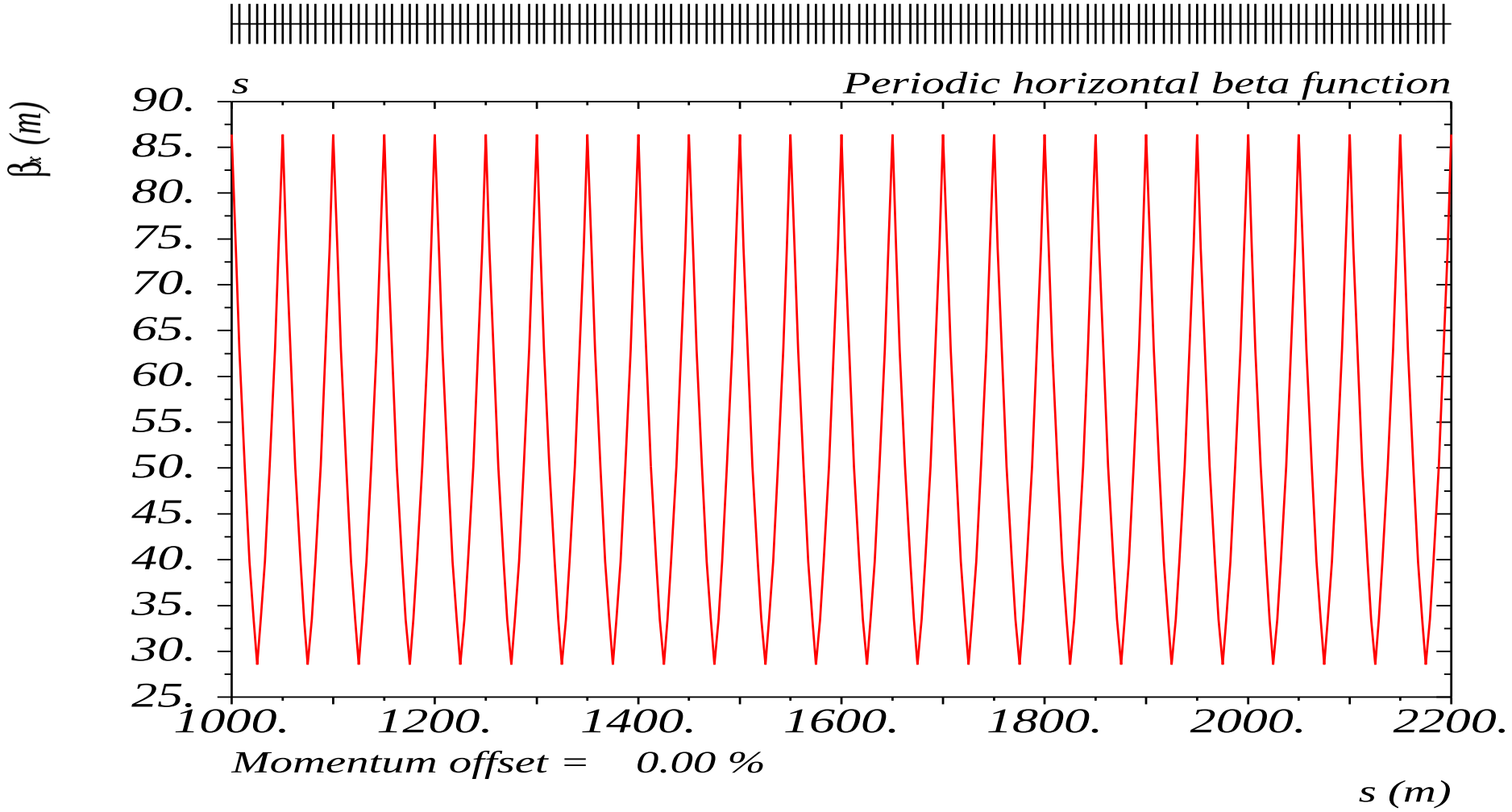
Example: matching of tunes Q (MAD language)

```
match;
   vary,name=kqf, step=0.00001;
   vary,name=kqd, step=0.00001;
   global,QX=7.420;
   global,QY=7.380;
endmatch;
```

- The strengths of main quadrupoles are varied
- Computes the new strengths

Local adjustment of optical functions

- **Small β_x needed for an experiment**



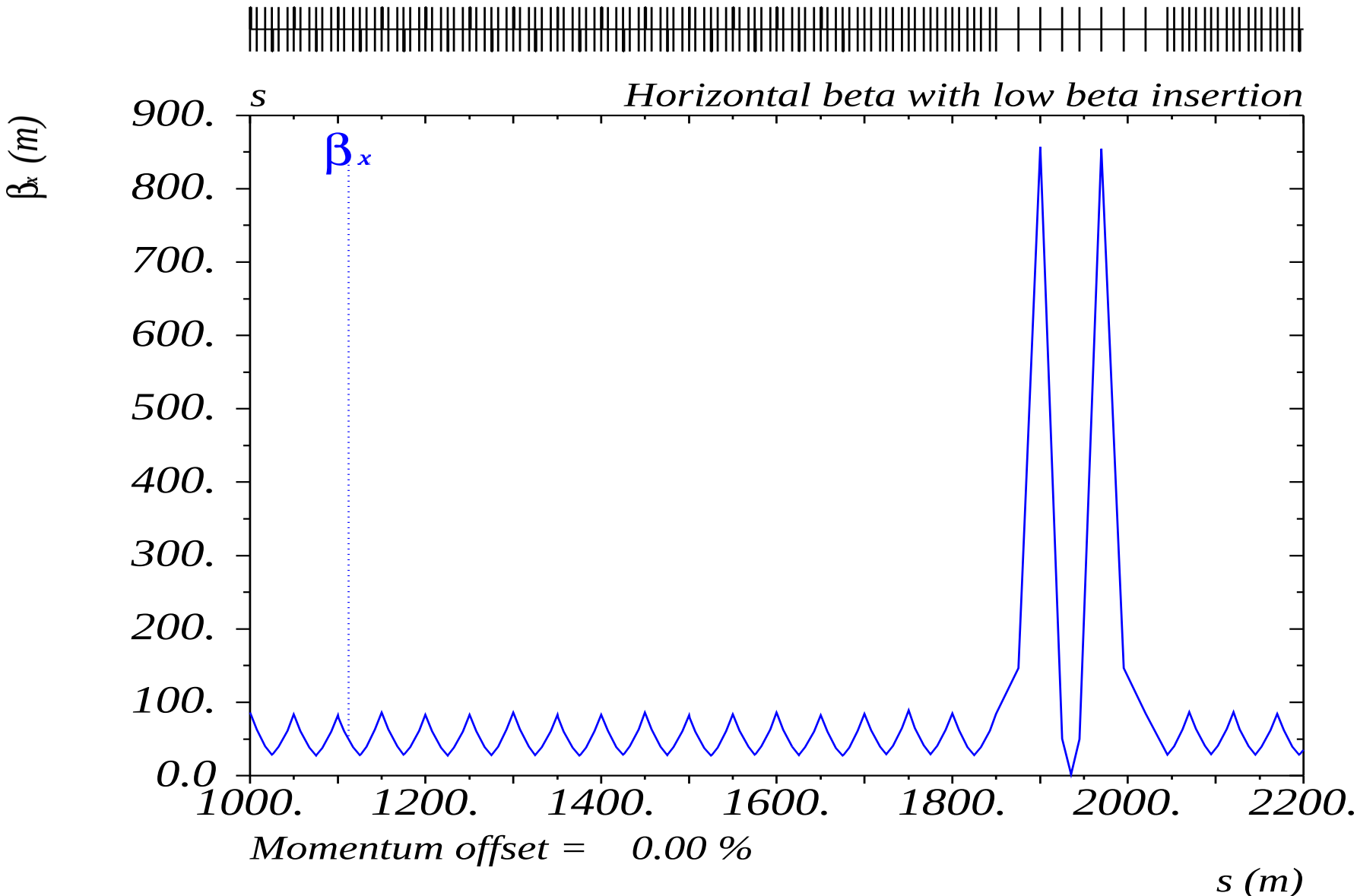
Example: matching of β-functions (MAD language - simplified !)

- **Allow a few quadrupoles to be changed**

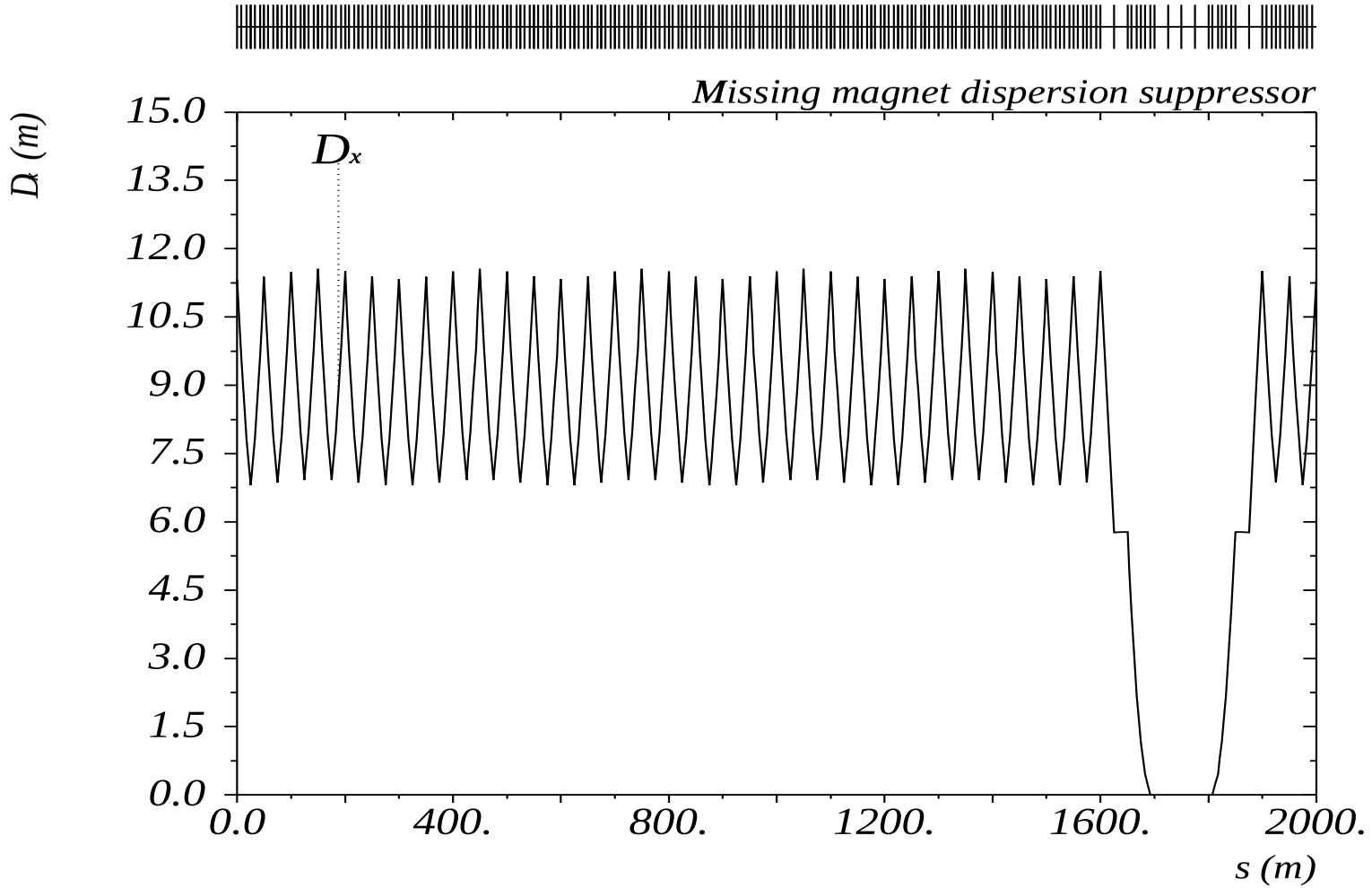
```
match;
   vary,name=kq1.l, step=0.00001;
   vary,name=kq2.l, step=0.00001;
   vary,name=kq3.l, step=0.00001;
   vary,name=kq4.l, step=0.00001;
   constraint,place=IP,betx= 2.0, alfx=0.0;
endmatch;
```

- **The strengths of a small number of additional quadrupoles are varied**

Optical functions (reduced β-function)



Optical functions (reduced dispersion)



General purpose optics programs

- Always allow to:
  - Compute optical parameters (Twiss functions)
  - Match the required properties
- But also:
  - Simulate machine imperfections
  - Correct imperfections

→ Example again Methodical Accelerator Design

(Interlude IV: Course on optics design)

- Intermediate level CAS 2003 offered a course on optics design
  - Purpose was to develop a realistic accelerator optics
  - Included correction elements, optical matching, dispersion suppressors ...
  - MAD was used for practical implementation
- The course is available on CD-ROM (on request) or from the web

Simulation of an accelerator

- Purpose is to imitate the world of a particle or a beam in an accelerator
- Use local properties of the machine element to describe its interaction with a particle
- The grand scheme is to derive global behaviour

Evaluation by simulation (1)

- Single particle behaviour
- Usually concerns long term behaviour such as beam loss
- The effect of the accelerator components on a single particle
  - → Non-linear elements
  - → Machine imperfections
  - → External distortions

Evaluation by simulation (2)

- Multi particle behaviour
- Usually concerns collective behaviour: coherent motion, emittance increase, damping etc.
  - → The effect of the accelerator components on an ensemble of particles (e.g. impedances)
  - → Interactions of particles between each other. These are usually dictated by the properties of the accelerator (e.g. space charge, beam-beam effects ...)

Single particle tracking

- The motion of a test particle through the elements of a machine is simulated for a defined number of turns → "tracking"
- Use appropriate coordinates, start with $\vec{z_0}$.
- In each element (or part of the machine), the coordinates are transformed by $\vec{z_{n+1}} = \mathcal{M} \circ \vec{z_n}$
- $\mathcal{M}$ must be symplectic
- We must distinguish thick and thin elements

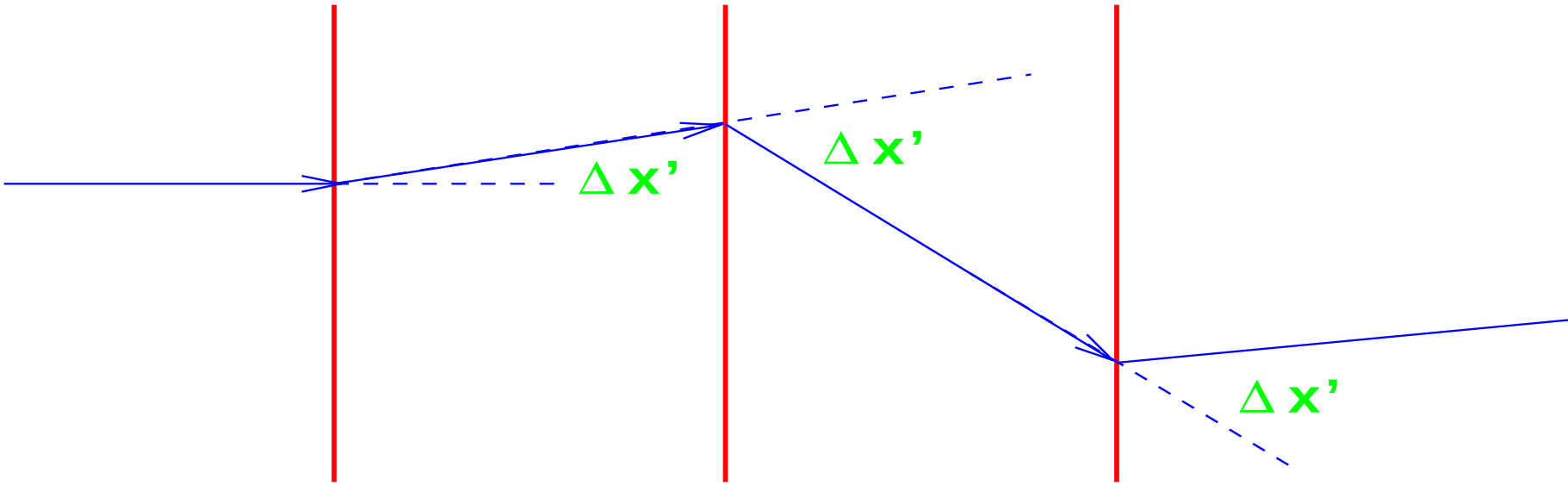
Tracking through thin elements

- **Thin "magnet": let go length to zero, but keep field integral finite (constant)**
- **No change of amplitudes x and y**

→ $x \rightarrow x$ **and** $y \rightarrow y$

- **The momenta x' and y' receive an amplitude dependent deflection (kick)**

→ $x' \rightarrow x' + \Delta x'$ **and** $y' \rightarrow y' + \Delta y'$



- **No change of amplitudes x and y**
- **The momenta x' and y' receive an amplitude dependent deflection (kick)**

→ $x' \rightarrow x' + \Delta x'$ **and** $y' \rightarrow y' + \Delta y'$

Tracking through thin elements

- So-called kick-codes (thin lens tracking)
- Always symplectic ! (homework)
- Usually rather fast on computers
- → A "thick" element can be sliced into several thin elements

Analysis tools

- Fourier analysis, diffusion coefficients, chaos detection ...
- Phase space plots (simple example):
  - Start with a "particle" with initial coordinates x and x' at a position s_0
  - Pass through the One-Turn-Map (for position s_0 !)
  - Plot the new x and x' coordinates at position s_0 after every turn

A simple example

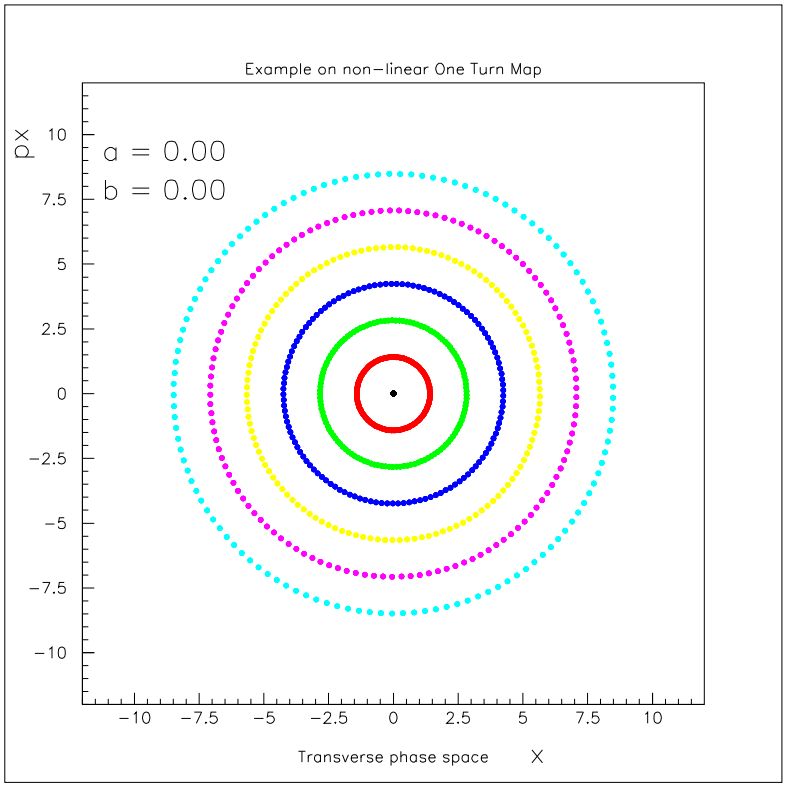
Linear transformation plus a constant deflection (i.e. orbit kick from displaced quadrupole)

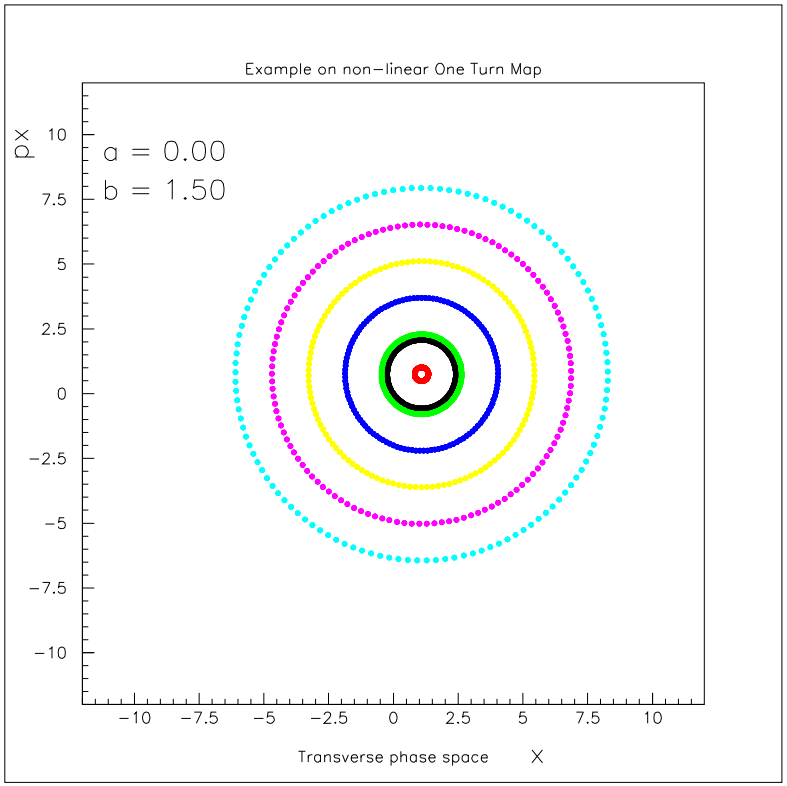
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{n+1} = \begin{pmatrix} \cos(\mu) & \sin(\mu) \\ -\sin(\mu) & \cos(\mu) \end{pmatrix}_{s_0} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{n} + \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$\mu = 2\pi \mathrm{Q}_x = 2\pi \cdot 0.19$,
constant b is a free parameter

➞ **Find the fixed point(s) (closed orbit)**

A simple example ..





- **Start at different amplitudes and ”observe” x and x’ at position s_0**

A (still) simple example

Linear transformation plus a quadratic non-linearity (e.g. (thin) sextupole) plus a constant deflection

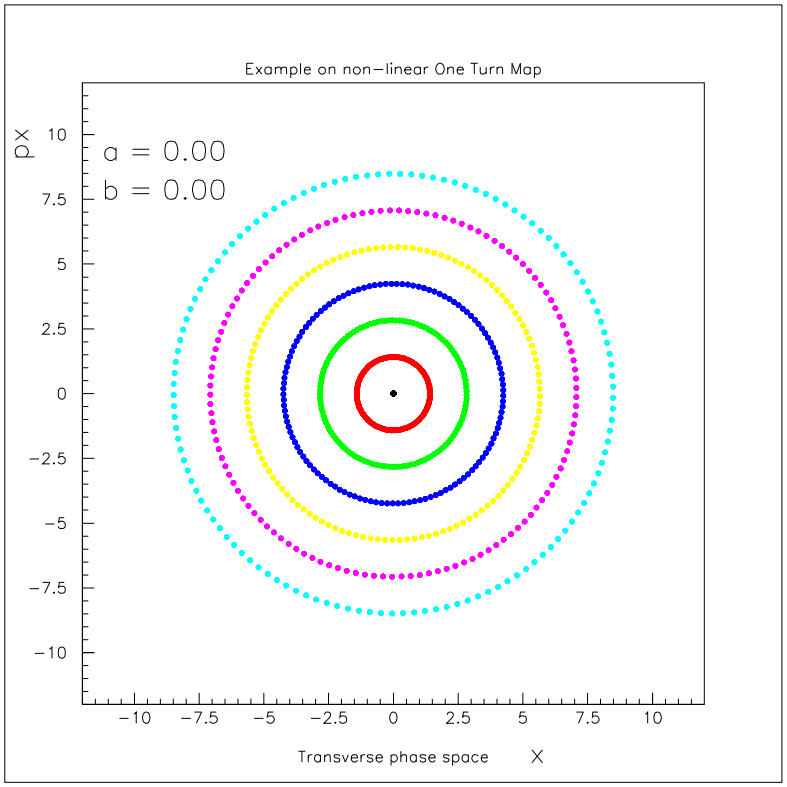
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{n+1} = \begin{pmatrix} \cos(\mu) & \sin(\mu) \\ -\sin(\mu) & \cos(\mu) \end{pmatrix}_{s_0} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{n} + \begin{pmatrix} 0 \\ ax^2 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix}$$

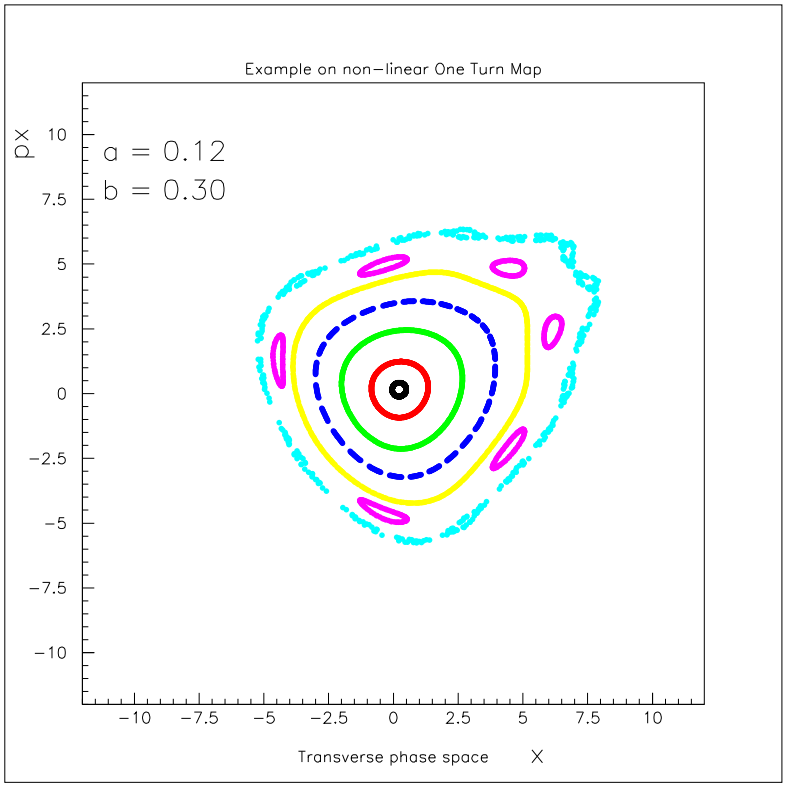
$\mu = 2\pi \mathrm{Q}_x = 2\pi \cdot 0.19$,

constants a and b are free parameters

➞ **Find the fixed points (how many do you see ?)**

A (still) simple example ..





- **Motion at different amplitudes distorted**
- **Stability region reduced by non-linear effect**

Tracking through thick elements

- Real magnets are NOT thin !
- Consequence: they are NOT linear elements (also not dipoles, quadrupoles ..)
- Especially important for "small" machines (e.g. large deflections, fringe fields etc.)
- Constructing a MAP for a thick element is more involved

MAPS for thick elements

- Non-linear, i.e. not matrices (even for quadrupoles)
- Need integration of equation of motion
- Special techniques have been developped, some keywords:
  - Lie transforms
  - Symplectic integration
  - Differential Algebra ...
- In the end: we get again a One-Turn-Map

Single particle dynamics in a nutshell

or: things are simpler than you thought

- Try to compute a (one turn) MAP
- It contains everything
- Its analysis will tell you what you need to know
- It doesn't matter how you got it !
- Use the Lorentz force (or equivalent) for the construction of maps (no need to "apply" accelerator physics concepts !)

You can derive:

- Tunes (we already did)
- Betatron functions (we already did)
- Stability borders, dynamic aperture
- Detuning with amplitudes
- Fixpoints, closed orbit, resonance strength
- ... and much more !

Try to construct a global, time dependent Hamiltonian to get all that !

You do not get:

- Rules of thumb for simple calculations
- "Handy" formulae
- Scaling laws for estimates (indirectly)

Complications: light particles

- Light particles (e^-, e^+ etc.) emit synchrotron radiation and motion is damped
  - → Stochastic component
  - → No symplecticity, no invariants (but equilibrium parameters, e.g. emittance)
- Synchrotron motion must be simulated
- Computation of damping properties

Simulation of multi particle effects

- Often requires the simulation of a beam: simulate many (10^4 or more) particles simultaneously and study their behaviour:
  - Beam shape (density distribution)
  - Centre of mass motion of all particles
- Fields generated by the beam need to be computed
- Advanced algorithms for field calculation have been developped and are used.

Complications: many particles

- Particles have different amplitudes
- Particles have different tunes
- Particles have different momenta !
- Definition of emittance becomes more complicated

Strategy for multi-particle simulations (1)

- **Generate and simulate many particles** $(10^4 - 10^5)$ **simultaneously**
- **Every particle interacts with the machine elements individually**
- **The whole ensemble interacts with the machine elements**
- **Every particle interacts with other particles !**
- → **Feed back into motion of individual particles**

Strategy for multi-particle simulations (2)

- **All particles must be treated in parallel**
- → **For realistic LHC:** 10^7 **to** 10^8 **particles to simulate**
- → **Already for storage requires** $\approx$ **10 Gb memory**
- → **Parallel processing needed for reasonable computing time**
- **Often requires (intelligent) simplifications**

Simulation of interactions with environment

This means: interaction of the individual particles and the whole beam with:

- **Machine elements (e.g. magnets, RF, ...)**
- **Wake fields**
- **Impedances**
- **Electron cloud**
- **Intercepting elements (e.g. collimators, ...)**

→ **Strategies have changed with fast computers ...**

Simulation of interactions with itself

Particles inside a beam interact with other particles from the same beam:

- Space charge effects
- Intra-beam scattering
- Multi-bunch effects

Simulation of interactions with other beams

So-called beam-beam effects

- Other beam acts like a (very) non-linear lens
- Incoherent beam-beam effects (on each individual particle)
- Coherent beam-beam effects (on ensemble of particles)
- Requires the knowledge of fields generated by the other beam

Matrix formulation (linear models)

- **One-Turn-Maps can be written for two bunches or two beams (e.g. 1 and 2)**

→ **Additional elements couple two beams**

$$\begin{pmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}_{n+1} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}_{n}$$

→ **Allows computation of eigenmodes, eigenfrequencies etc.**

Field computation

- **Some simulations require the computation of fields (or forces) produced by a beam from Poisson equation (here 2-dimensional):**

$$\Delta V = -4\pi \cdot \rho(x, y)$$

- → **The density of the beam is $\rho(x, y)$**
- → **For simple distributions (Gaussian, uniform ...) can be solved analytically**
- → **In general (i.e. in the interesting cases !) it is done numerically**

Some basic methods

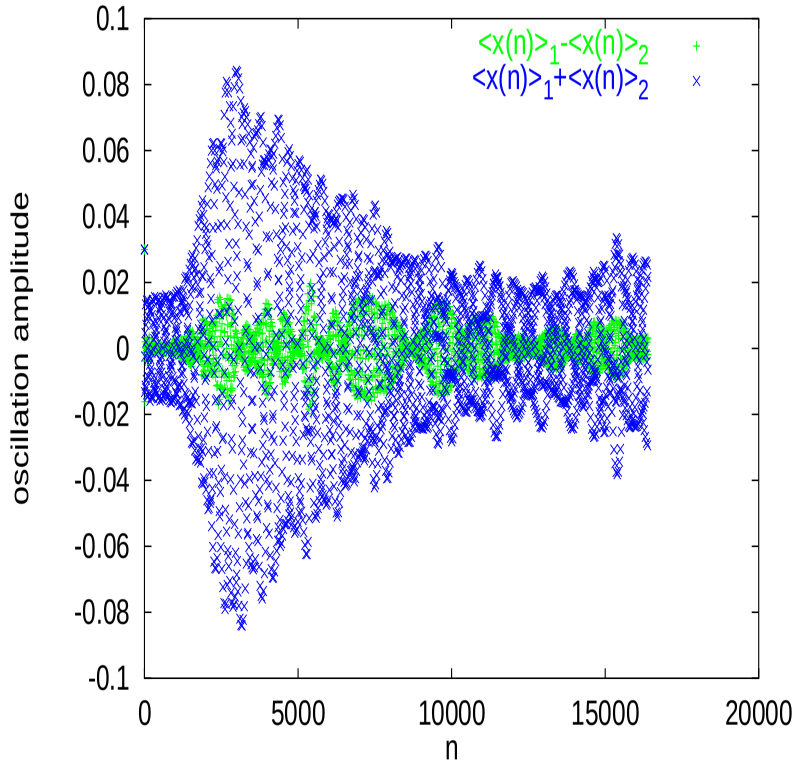
Particle - particle methods: compute field between each particle pair and add up (unpractical for large number of particles, sometimes used in celestial mechanics)

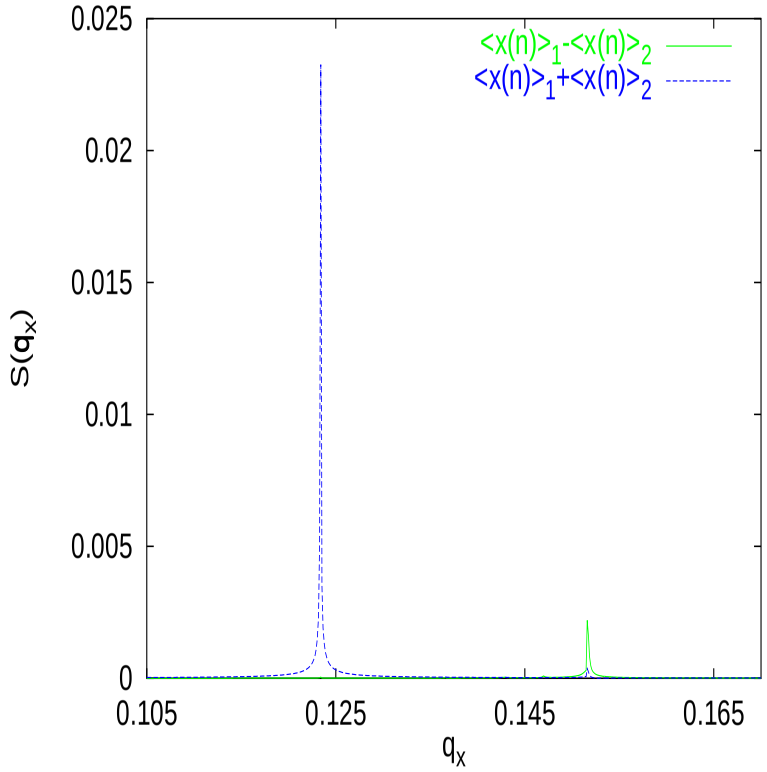
Particle - mesh methods: distribute particles on a mesh (grid) and solve the Poisson equation for discrete points

Multipole methods: develope potentials/fields as multipole expansion

→ Choice depends on application and parameters

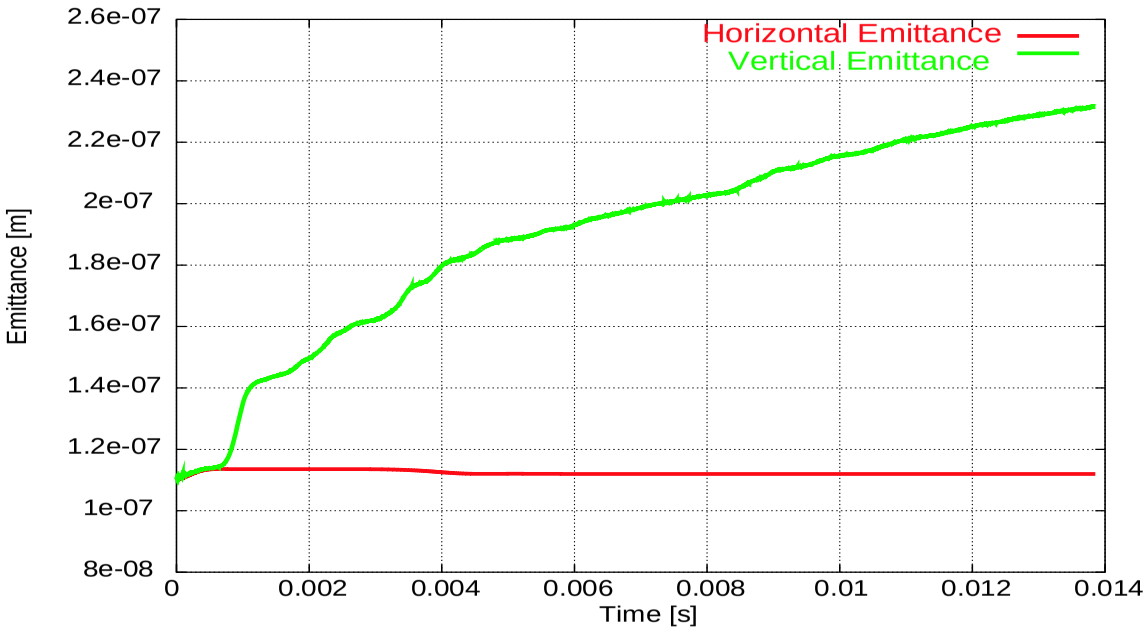
Example: Centre of mass motion as function of tim





→ From beam-beam simulations: Bunch oscillations and frequency spectrum

Example: Beam size as function of time

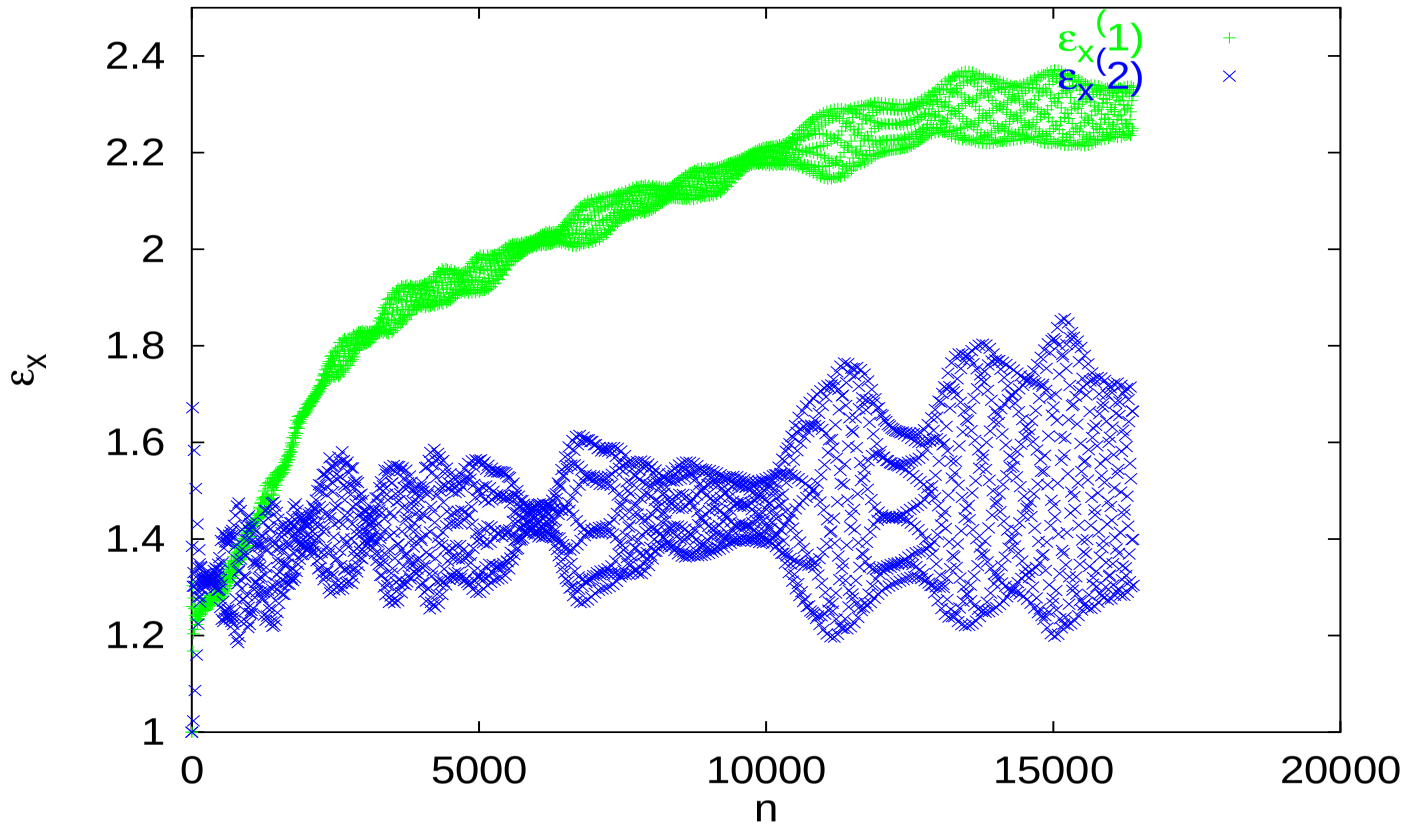


➡ **From electron-cloud simulation (courtesy: E. Benedetto)**

Alternatives

- Sometimes multi-particle simulations are too time consuming
- Numerical solution of the Vlasov-equation
- → E.g. finite difference methods
- → Intermediate level school

Example: Beam size as function of time



→ **Beam-beam simulation: numerical solution of Vlasov equation gives evolution of beam sizes**

(Personal) Comments on simulations:

- Here I gave only a selective overview of what can be done
- Techniques and tools in dedicated schools and courses (some in Intermediate CAS Course)
- What can be done has changed a lot in the last decade
- It is easy to write a program !
- Analysis and interpretation is usually the difficult part

Control and operation

- Basic aim: optimize performance

As operator or accelerator physicist:

- Provide and improve model of machine
- Measure and interprete beam parameters
- Correct and control beam parameters
- Conduct machine experiments

Control and operation of an accelerator

Basic problem: measure and control beam parameters

- **Control (orbit, chromaticities ..) depend on machine model which may be incomplete**
- **Feedback from measurements improves the model and simulations**
- **Should use the same strategies and methods as during design (Remember: matching !)**

→ **May be an iterative process**

Control and operation of an accelerator

- Very similar to simulation or design of a machine except:
  - → Interface to hardware and control (e.g. power converters)
  - → Beam instrumentation !
  - → Communication (networks, etc.)
  - → Issues such as: timing, alarms, interlocks,
- Treated in dedicated workshops and schools