Short Introduction to

(Classical) Electromagnetic Theory

(.. and applications to accelerators)

(http://cern.ch/Werner.Herr/CAS2018_Archamps/em1.pdf)

READING MATERIAL

- **J.D. Jackson,** *Classical Electrodynamics* **(Wiley, 1998 ..)**
- **R.P. Feynman,** *Feynman lectures on Physics*, **Vol2.**
- **J. Slater, N. Frank,** *Electromagnetism*, **(McGraw-Hill, 1947, and Dover Books, 1970)**
- **A. Wolski,** *Theory of electromagnetic fields*, **Proc. CAS: "RF for accelerators", CERN-2011-007.**

Variables, units and (CERN) conventions

Maxwell's equations relate Electric and Magnetic fields from charge and current distributions (SI units).

$\vec{E}$	=	electric field [V/m]
$\vec{H}$	=	magnetic field [A/m]
$\vec{D}$	=	electric displacement [C/m^2]
$\vec{B}$	=	magnetic field [T]
q	=	electric charge [C]
ρ	=	electric charge density [C/m^3]
$\vec{j}$	=	current density [A/m^2]
μ_0	=	permeability of vacuum, $4\,\pi \cdot 10^{-7}$ [H/m or N/A^2]
ϵ_0	=	permittivity of vacuum, $8.854 \cdot 10^{-12}$ [F/m]
c	=	speed of light, $2.99792458 \cdot 10^{8}$ [m/s]

It is all about interactions ....

Electromagnetic fields reveal themselves only through their interaction with particles

Conveniently described by abstract models, i.e. vectors and potentials:
(Classical) fields are models of "reality"

Used here: fields in vacuum:

Electric phenomena: $\vec{D}$

Magnetic phenomena: $\vec{B}$

Electric and Magnetic potentials: ϕ and $\vec{A}$

These are well described by Maxwell's equations

Note: theory of electromagnetism and electrodynamics is not complete without Special Relativity ...

Objectives:

- **Establish (non-relativistic) Maxwell's equations**
  Using a physical and relaxed/informal approach
- **Use them in these lectures to get relevant physics**
  (wave guides, cavities, magnet and cable design, energy ramping ...)

  Note: a simple list of formulae won't do (but there are quite a number of slides as a reference) !

Look at some mathematics first

- **Extension of previous lecture (R.S.), more related to this lecture**
- **Required to formulate Maxwell's equations but no need to understand details (just follow me step by step)**

Initially look abstract, but provide a much better intuitive picture

Recap: Vector Products (sometimes cross product)

Define a vector product for (usual) vectors like: $\vec{a} \times \vec{b}$,

$$\vec{a} \;=\; (x_a, y_a, z_a) \qquad\qquad \vec{b} \;=\; (x_b, y_b, z_b)$$

$$\begin{aligned} \vec{a} \times \vec{b} \;&=\; (x_a, y_a, z_a) \;\times\; (x_b, y_b, z_b) \\ &= (\underbrace{y_a \cdot z_b - z_a \cdot y_b}_{x_{ab}}, \quad \underbrace{z_a \cdot x_b - x_a \cdot z_b}_{y_{ab}}, \quad \underbrace{x_a \cdot y_b - y_a \cdot x_b}_{z_{ab}}) \end{aligned}$$

This product of two vectors is a "vector", not a number

Example:

$$(-2, 2, 1) \;\times\; (2, 4, 3) \;=\; (2, 8, -12)$$

Need also Scalar Products (essential for Relativity)

Define a scalar product for (usual) vectors like: $\vec{a} \cdot \vec{b}$,

$$\vec{a} \;=\; (x_a, y_a, z_a) \qquad\qquad \vec{b} \;=\; (x_b, y_b, z_b)$$

Multiplication element by (corresponding) element:

$$\vec{a} \cdot \vec{b} \;=\; (x_a, y_a, z_a) \;\cdot\; (x_b, y_b, z_b) = (\underbrace{x_a \cdot x_b} \;+\; \underbrace{y_a \cdot y_b} \;+\; \underbrace{z_a \cdot z_b})$$

This product is a "<u>scalar</u>" (single value), not a vector.

Example:

$$(-2, 2, 1) \;\cdot\; (2, 4, 3) \;=\; -2 \cdot 2 \;+\; 2 \cdot 4 \;+\; 1 \cdot 3 \;=\; 7$$

Differentiation with vectors - several options:

Input or output can be a vector

Input or output can be a scalar (e.g. a function like: ϕ(x, y, z))

One defines a special vector ∇

called the "gradient":

$$\nabla \stackrel{def}{=} \left(\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}\right)$$

Can be used like a vector (e.g. in vector and scalar products), for example:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial z}\right)$$

Specific example for the last operation on a <u>scalar function</u> $\phi(x, y, z)$:

$$\nabla\phi \quad = \quad (\frac{\partial}{\partial x}, \; \frac{\partial}{\partial y}, \; \frac{\partial}{\partial z} \;) \; \phi \quad = \quad (\frac{\partial\phi}{\partial x}, \; \frac{\partial\phi}{\partial y}, \; \frac{\partial\phi}{\partial z}) \quad = \; (G_x, G_y, G_z)$$

and we get a <u>vector</u> $\vec{G}$. It is a "slope" in the 3 directions.

Example: $\phi(x, y, z) \; = \; 0.1x^2 \; - 0.2xy \; + z^2$

$$\nabla \; \phi(x, y, z) \quad = \quad \begin{pmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial z} \end{pmatrix} \quad = \quad \begin{pmatrix} 0.2x - 0.2y \\ -0.2x \\ 2z \end{pmatrix}$$

$\vec{\nabla}$ is very versatile !

$\vec{\nabla}$ can be used like a "normal" vector in all products and can act on a scalar function ϕ (e.g. Potential) as well as on a vector $\vec{F}$ (e.g. Force)

but the results are very different:

$\nabla \cdot \vec{F}$ is a scalar (e.g. "density" of a source, see later)

$\nabla \cdot \phi$ is a vector (e.g. electric field $\vec{E}$, force)

$\nabla \times \vec{F}$ is a pseudo-vector (e.g. magnetic induction $\vec{B}$)

$\nabla \cdot \nabla$ is called Δ (another important one !)

also contraptions like: $\nabla \times (\nabla \times \vec{F}), \quad \nabla \cdot (\nabla \times \vec{F}), \quad \nabla \times (\vec{F}_1 \times \vec{F}_2)$ ..

If bored, prove: $\nabla \times (\nabla \times \vec{F}) \;=\; \nabla \cdot (\nabla \cdot \vec{F}) \;-\; \Delta F$

Two operations with ∇ have special names:

DIVERGENCE (scalar product of ∇ with a vector):

$$\mathrm{div}(\vec{F}) \stackrel{def}{=} \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Physical significance: "amount of density", (see later)

CURL (vector product of ∇ with a vector):

$$\mathrm{curl}(\vec{F}) \stackrel{def}{=} \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Physical significance: "amount of circulation", (see later)

Example: Coulomb field of a point charge Q

A charge Q generates a field $\vec{E}(r)$ according to :

$$\vec{E}(r) \quad = \quad \frac{Q}{4\pi\epsilon_0} \, \frac{\vec{r}}{r^3}$$

depends only on r (rather : $\frac{1}{r^2}$)

<u>all</u> field lines pointing <u>away</u> from charge (source) : $\vec{r}$

expect that divergence div $\vec{E}$ has something to tell ➡

We can do the (non-trivial*)) computation of the divergence:

$$\text{div } \vec{E} = \nabla \vec{E} = \frac{dE_x}{dx} + \frac{dE_y}{dy} + \frac{dE_z}{dz} = \frac{\rho}{\epsilon_0}$$

(negative charges) $\nabla \cdot \vec{E} < 0$

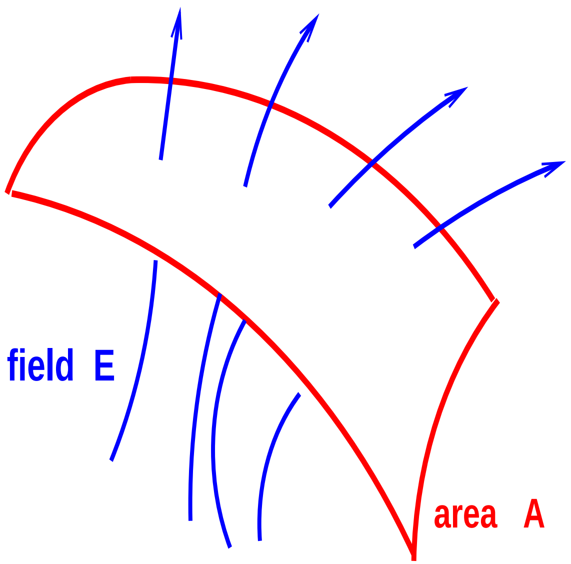
(positive charges) $\nabla \cdot \vec{E} > 0$

Divergence related to <u>charge density ρ</u> generating the field $\vec{E}$

Charge density ρ is <u>charge per volume</u>: $\rho = \frac{Q}{V} \implies \iiint \rho \, dV = Q$

*) **see backup slides if interested**

How to quantify electric (or magnetic) fields ?



Count field vectors (somehow)

passing through an area

Counting is Integrating

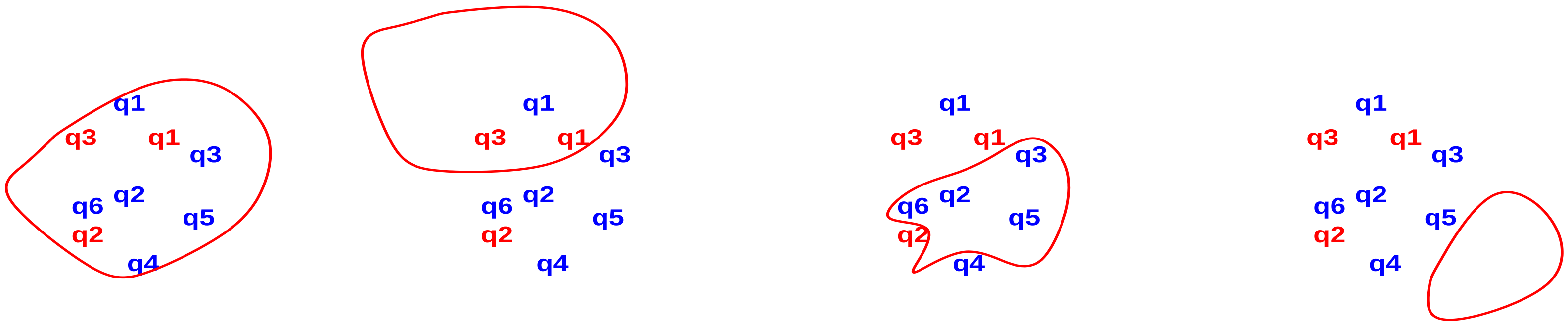
$$\Omega = \iint \vec{E} \cdot d\vec{A}$$

Ω is the flux through the area A

- **Larger field - more flux**
- **Larger area - more flux**

What if the area is closed, i.e. a surface ?

Integrating fields (2D) - add field lines through the boundary:

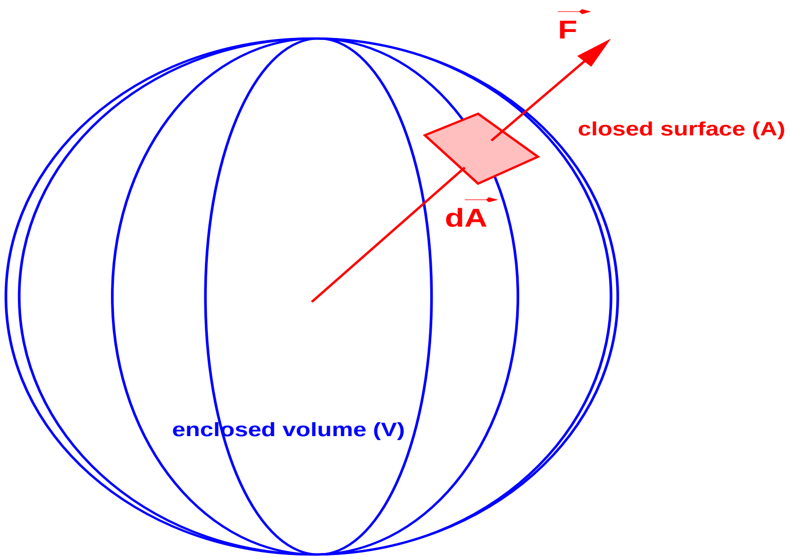


$\int_{A_1} \vec{E}\, d\vec{A} = +3q$ $\int_{A_2} \vec{E}\, d\vec{A} = -1q$ $\int_{A_3} \vec{E}\, d\vec{A} = +4q$ $\int_{A_4} \vec{E}\, d\vec{A} = 0$

Any closed surface around charges "counts" the charges enclosed (independent of shape !!!) !

- **If positive: total net charge enclosed positive**
- **If negative: total net charge enclosed negative**
- **If zero: no charges enclosed**

If the shape does not matter - make it a sphere (because it is easy to compute):



$$\Omega = \iint_A \vec{E} \cdot d\vec{A}$$

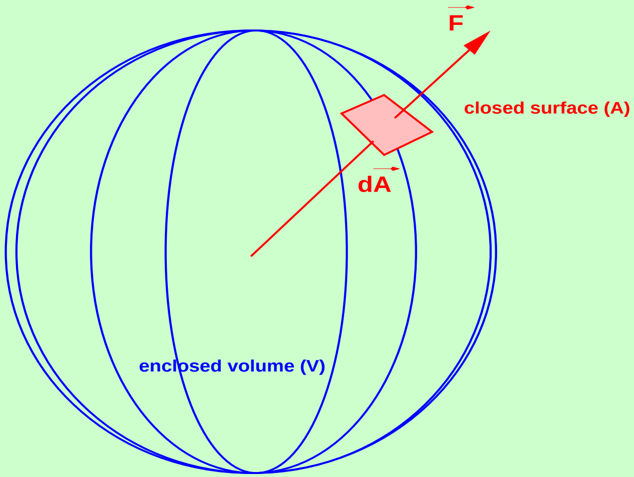
Count how many go in ϕ_{in} and how many go out ϕ_{out}

→ **Difference is the flux through the sphere**

Measures somehow what is <u>diverging</u> from the inside ...

Sounds like we should make some use of div ??

Used in the following: Gauss' theorem to evaluate flux integral:



$$\iint_A \vec{E} \cdot d\vec{A} = \iiint_V \nabla \cdot \vec{E} \cdot dV \quad \text{or}$$

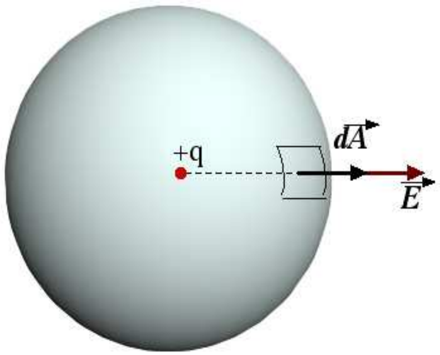
$$\iint_A \vec{E} \cdot d\vec{A} = \iiint_V \text{div } \vec{E} \cdot dV$$

Integral through closed surface (flux) is integral of divergence in the enclosed volume

Surface integral related to the divergence from the enclosed volume

Sum of all sources inside the volume gives the flux out of this region (remember the pictures a few slides ago)

More formal: Maxwell's first equation using Gauss's



charge density ρ to charge Q : $\int_V \frac{\rho}{\epsilon_0} \cdot dV = \frac{Q}{\epsilon_0}$

$$\int_A \vec{E} \cdot d\vec{A} = \int_V \nabla \cdot \vec{E} \cdot dV = \frac{Q}{\epsilon_0} = \left[\int_V \frac{\rho}{\epsilon_0} \cdot dV \right]$$

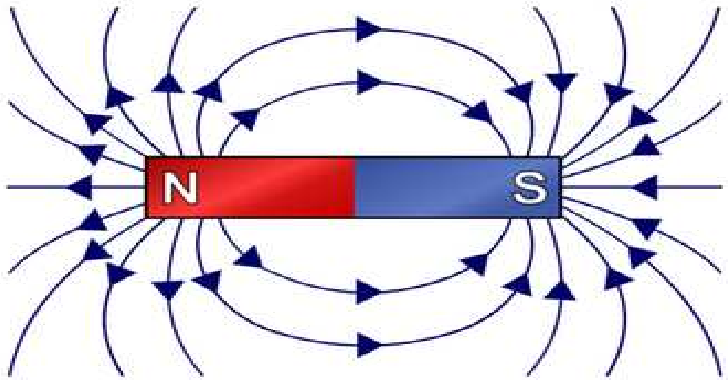
Written with <u>charge density</u> ρ we get Maxwell's <u>first</u> equation:

$$\text{div}\vec{E} = \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

The higher the charge density:

- **The larger the divergence of the field**
- **The more comes out/diverges**

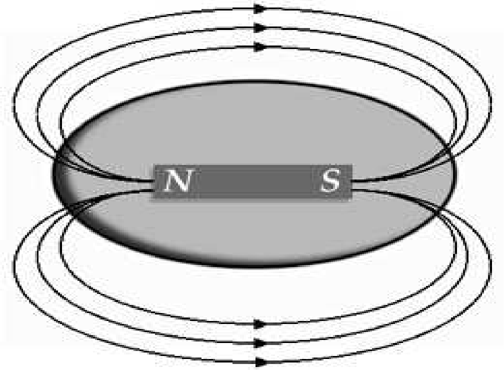
What about magnetic fields ? ...



- Field lines of $\vec{B}$ are always closed
- They have a direction (by definition): magnetic field lines from North to South
- $\mathbf{Q}_{fcb}$: which is the direction of the earth magnetic field lines ?

What about divergence of magnetic fields ?

Enclose it again in a surface:



$$\int\int_A \vec{B} \; \mathrm{d}\vec{A} = \int\int\int_V \nabla\vec{B} \; \mathrm{d}V = 0$$

Volume (thus dV) is <u>never</u> $= 0$

➡ $\nabla\vec{B} = \text{div } \vec{B} = 0$

What goes into the closed surface also goes out

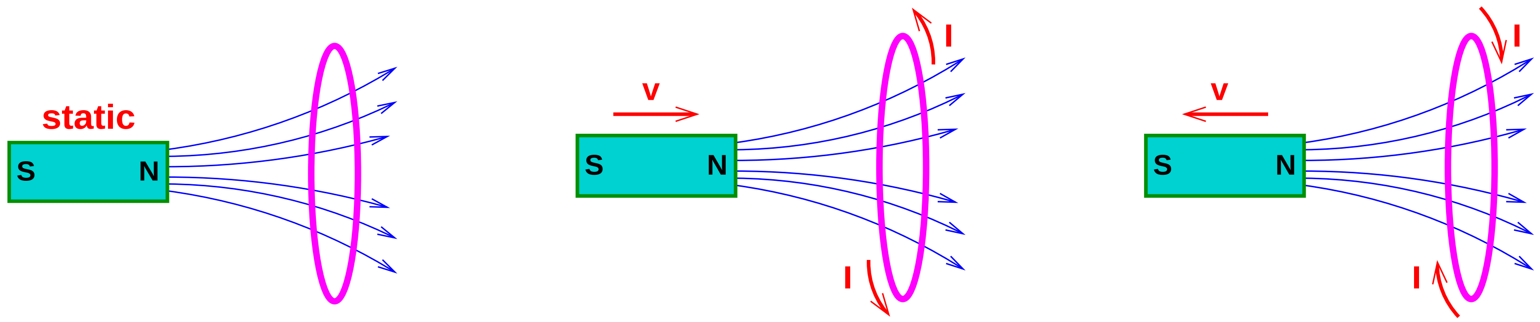
➡ **<u>Maxwell's second equation:</u>** $\nabla\vec{B} = \text{div } \vec{B} = 0$

➡ **Physical significance: (probably) no Magnetic Monopoles**

Enter Faraday

Again look at the flux through an area (enclosed by a coil)

static flux : $\Omega = \int_A \vec{B} \cdot \mathrm{d}\vec{A}$ changing flux : $\frac{\partial \Omega}{\partial t} = \int_A \frac{\partial (\vec{B})}{\partial t} \cdot \mathrm{d}\vec{A}$



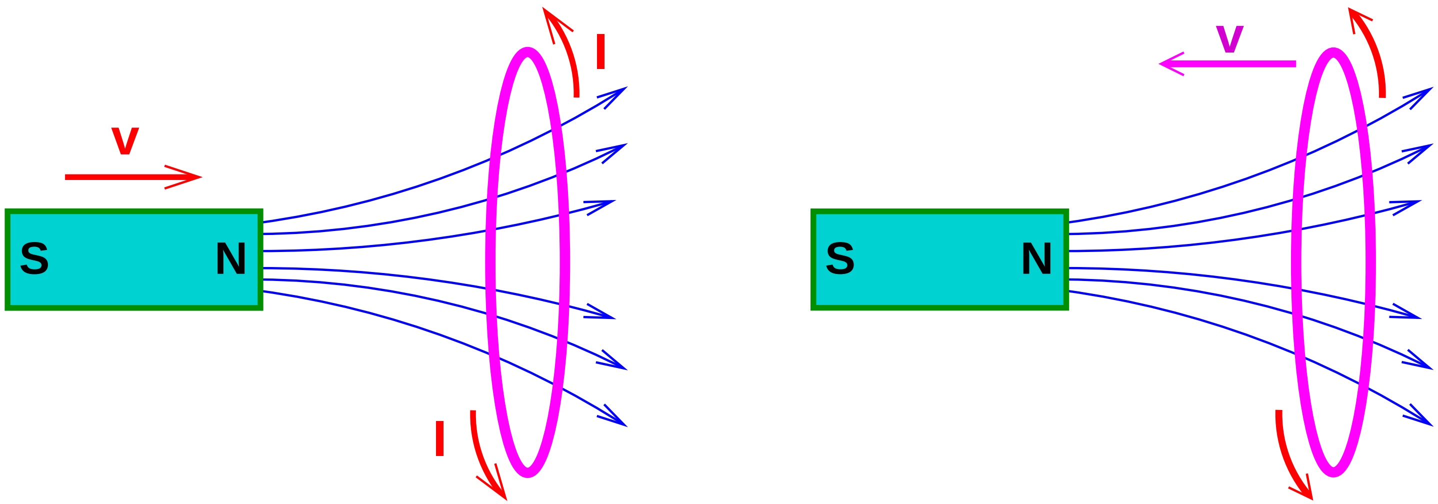
Moving the magnet changes the flux (density or number of lines) through the area $\Longrightarrow$

Induces a <u>circulating</u> (curling) electric field $\vec{E}$ in the coil which "pushes" charges around the coil $\Longrightarrow$

Current I in the coil (observe its direction ..)

Experimental evidence:

It does not matter whether the magnet or the coil is moved:



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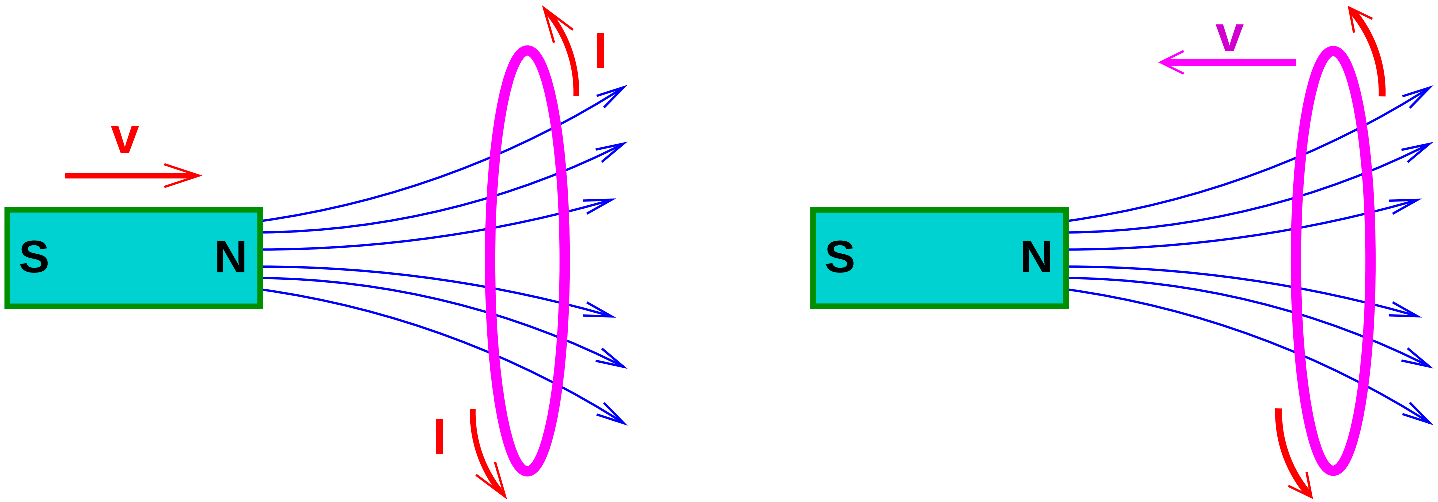


If you think it is obvious - not for everybody :

This was the reason for Einstein to develope special relativity !!!

A changing flux Ω through an area A produces circular electric field $\vec{E}$, "pushing" charges $\Longrightarrow$ a current I

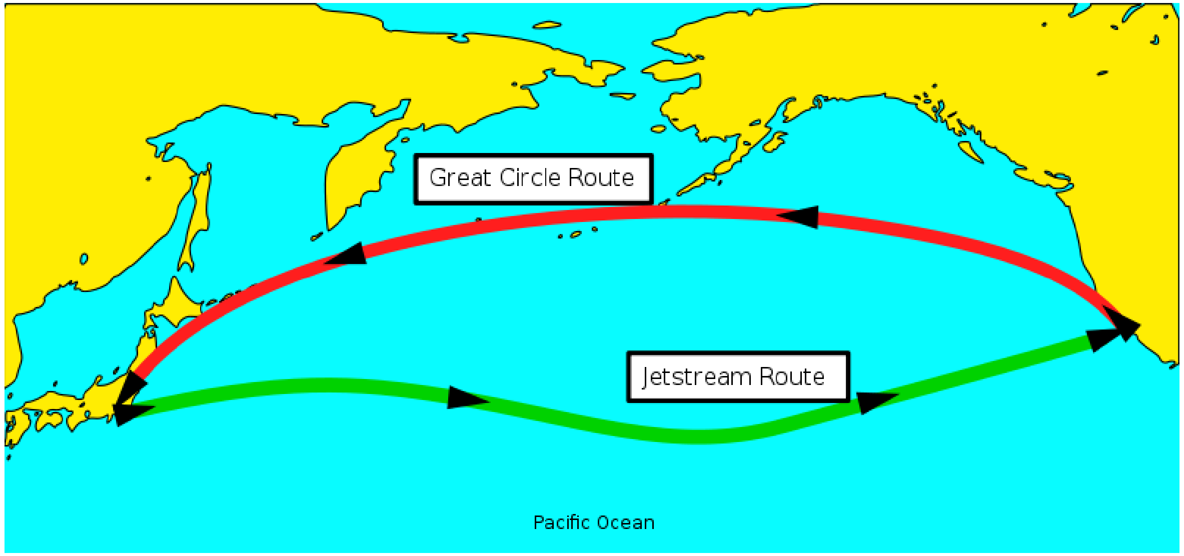
$$-\frac{\partial\Omega}{\partial t} = \frac{\partial}{\partial t}\overbrace{\int_A \vec{B}\cdot d\vec{A}}^{\text{flux }\Omega} = \underbrace{\oint_C \vec{E}\cdot d\vec{r}}_{\text{pushed charges}}$$

Flux can be changed by:

- **Change of magnetic field $\vec{B}$ with time t (e.g. transformers)**
- **Change of area A with time t (e.g. dynamos)**

How to count "pushed charges" $\left[\int_C \vec{E}\cdot d\vec{r} \quad \text{is a } \underline{\text{line integral}} \right]$

Everyday example ..

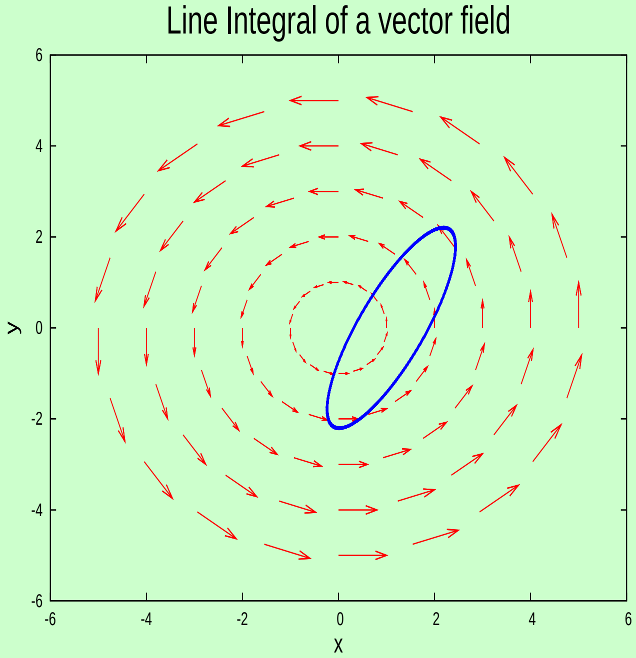


Line integrals: sum up "pushes" along the two Lines/Routes

Optimize: e.g. fuel consumption, time of flight

$$\text{Note}: \quad \int_C \vec{F} \cdot d\vec{r} \ \text{ can be written as } \ \iint_A \nabla \times \vec{F} \cdot d\vec{A} \quad \text{or} \quad \iint_A \text{curl } \vec{F} \cdot d\vec{A}$$

Used in the following: Stoke's theorem



$$\oint_C \vec{F} \cdot d\vec{r} = \iint_A \nabla \times \vec{F} \cdot d\vec{A} \quad \text{or}$$

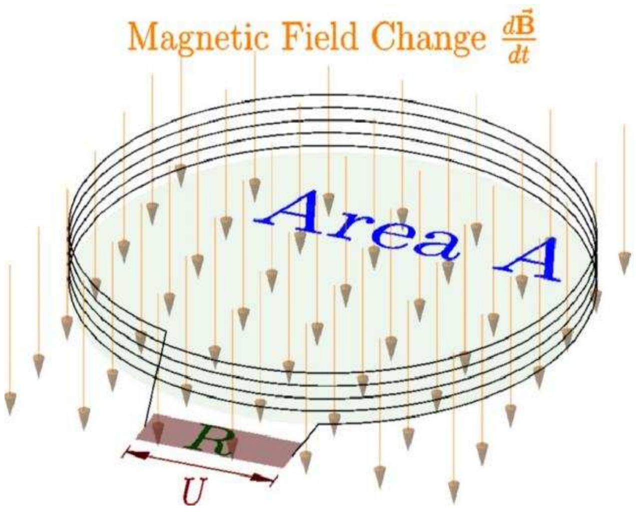
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_A \text{curl } \vec{F} \cdot d\vec{A}$$

$$\text{obviously : div } \vec{F} = 0$$

Summing up all vectors inside the area: net effect is the sum along the closed curve

➡ **measures something that is "curling" inside and how strongly**

One case use this theorem for a coil enclosing a closed area



$$\int_A -\frac{\partial \vec{B}}{\partial t} d\vec{A} = \underbrace{\int_A \nabla \times \vec{E}\, d\vec{A} = \oint_C \vec{E} \cdot d\vec{r}}_{\text{Stoke's formula}}$$

$$\underbrace{\int_A -\frac{\partial \vec{B}}{\partial t} d\vec{A} = \int_A \nabla \times \vec{E}\, d\vec{A}}_{\text{same Integration}}$$

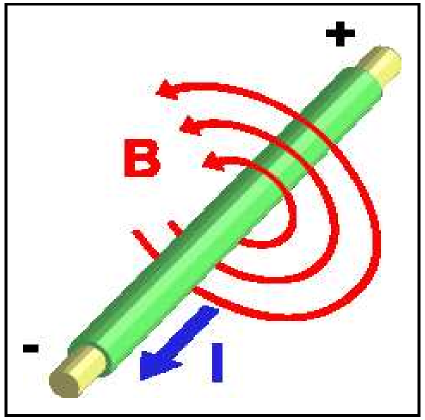
Re-written: changing magnetic field through an area induces curling electric field around the area (Faraday)

Maxwell' 3rd equation

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \text{curl } \vec{E}$$

Next: Maxwell's fourth equation (part 1) ...

From Ampere's law, for example current density $\vec{j}$:



$$\int_A \nabla \times \vec{B} \;\; d\vec{A} \;=\; \oint_C \vec{B} \cdot d\vec{r} \;=\; \int_A \mu_0 \vec{j} \;\; d\vec{A}$$

$\vec{j}$: "amount" of charges through area $\vec{A}$

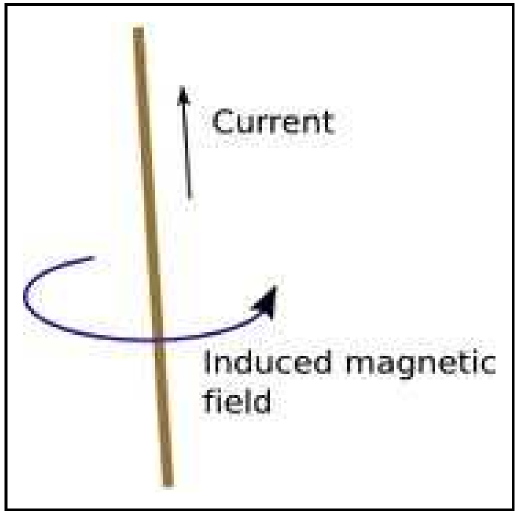
$$\int_A \mu_0 \vec{j} \;\; d\vec{A} \;=\; \mu_0 \; I \qquad \text{(total current)}$$

Static electric current induces circular magnetic field (magnets !)

Using the same argument as before:

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{j}}$$

For a static electric current I in a single wire we get Biot-Savart law (using the area of a circle $A = r^2 \cdot \pi$, we can easily do the integral):



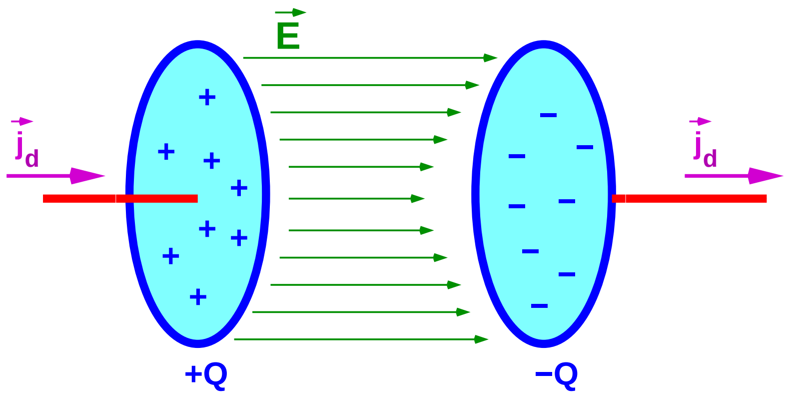
$$\vec{B} = \frac{\mu_0}{4\pi} \oint \vec{j} \cdot \frac{\vec{r} \cdot d\vec{r}}{r^3}$$

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{\vec{j}}{r}$$

Application: magnetic field calculations in wires

Part 2: Maxwell's fourth equation

Charging capacitor: Current enters left plate - leaves from right plate, builds up an electric field between plates ➡ produces a "current" <u>during</u> the charging process



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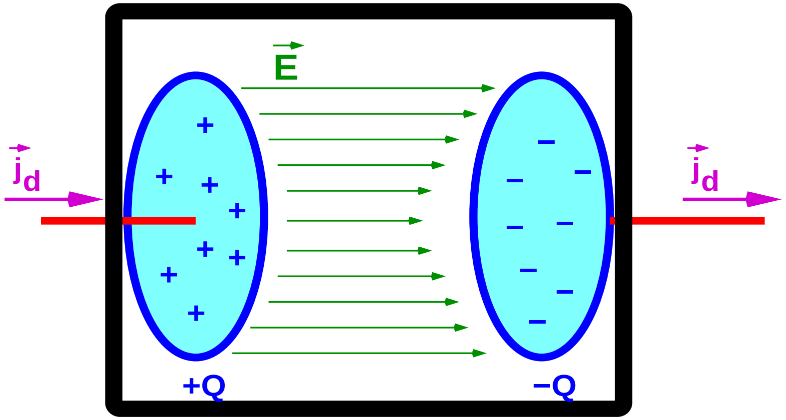


Displacement Current :

$$\vec{j}_d = \frac{d\vec{E}}{dt}$$

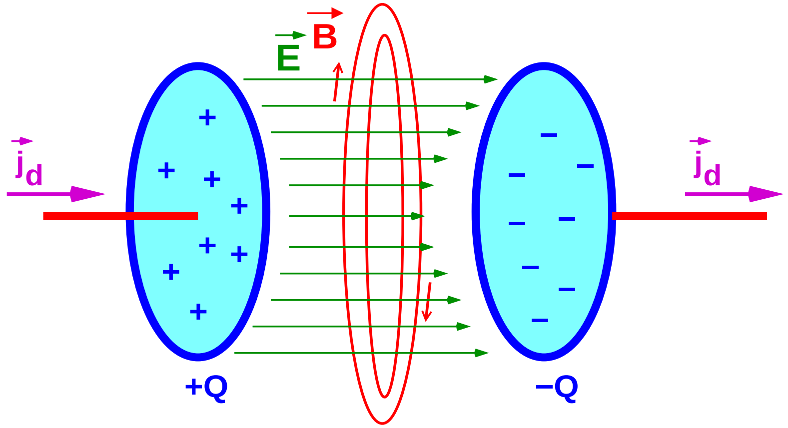
This is <u>not</u> a current from charges moving through a wire

This is a "current" from <u>time varying electric fields</u>

Once charged: fields are constant, (displacement) "current" stops

Cannot distinguish the origin of a current - apply Ampere's law to j_d

- Displacement current j_d produces magnetic field, just like "real currents" do ...



- Time varying electric field induces circular magnetic field (using the current density $\vec{j}_d$)

$$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{d\vec{E}}{dt}$$

Magnetic fields $\vec{B}$ can be generated by two different currents:

$$\nabla \times \vec{B} = \mu_0 \vec{j} \qquad \textbf{(electrical current)}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \qquad \textbf{(changing electric field)}$$

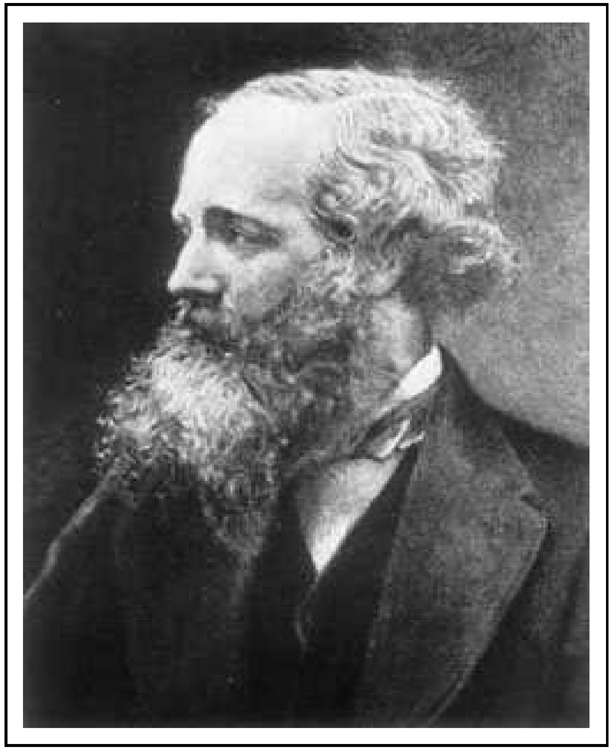
or putting them together to get Maxwell's fourth equation:

$$\nabla \times \vec{B} = \mu_0(\vec{j} + \vec{j}_d) = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

or in integral form:

$$\int_A \nabla \times \vec{B} \cdot d\vec{A} = \int_A \left(\mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

SUMMARY: MAXWELL'S EQUATIONS



$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

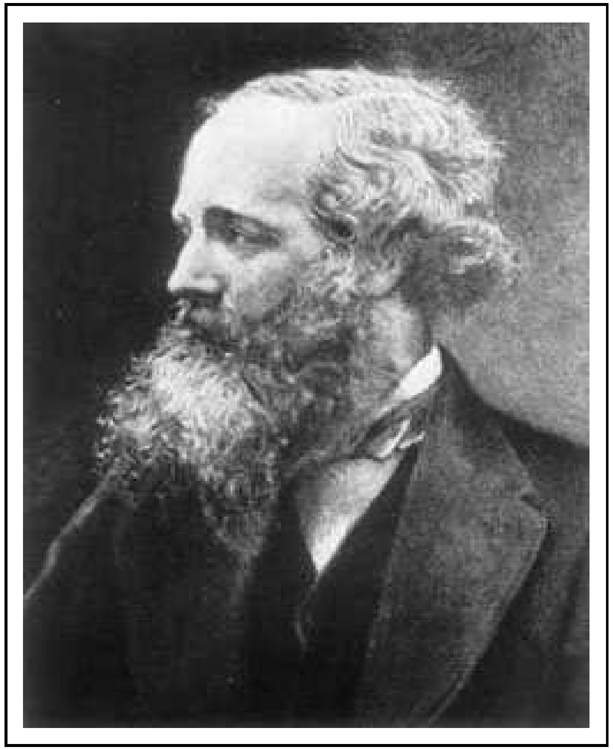
$$\int_A \vec{B} \cdot d\vec{A} = 0$$

$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt}\right) \cdot d\vec{A}$$

$$\oint_C \vec{B} \cdot d\vec{r} = \int_A \left(\mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}\right) \cdot d\vec{A}$$

Written in Integral form

SUMMARY: MAXWELL'S EQUATIONS



$$\nabla \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} \; + \; \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

Written in Differential form (my preference)

V.G.F.A.Q:

Why :

$$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\text{curl } \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\text{div } \vec{B} = 0$$

$$\text{curl } \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

Why Not :

$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt} \right) \cdot d\vec{A}$$

$$\int_A \vec{B} \cdot d\vec{A} = 0$$

$$\oint_C \vec{B} \cdot d\vec{r} = \int_A \left(\mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} \right) \cdot d\vec{A}$$

$$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$$

something $(\vec{E})$ spreading out

$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

???

$$\text{curl } \vec{E} = -\frac{d\vec{B}}{dt}$$

something $(\vec{E})$ circulating

$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt}\right) \cdot d\vec{A}$$

???

Maxwell's Equations - compact

1. **Electric fields $\vec{E}$ are generated by charges and proportional to total charge**

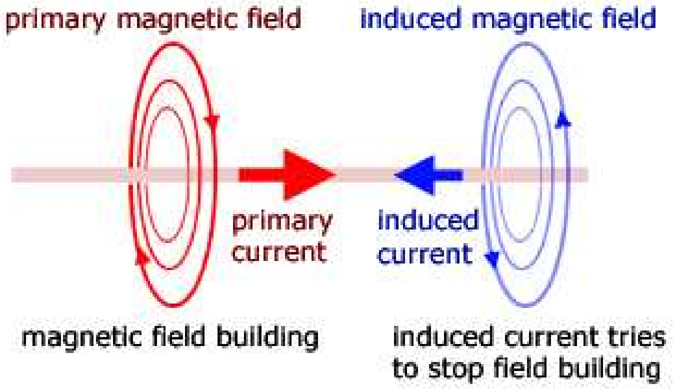
2. **Magnetic monopoles do (probably) <u>not</u> exist**

3. **Changing magnetic <u>flux</u> generates circular electric fields/currents**

4.1 **Changing electric <u>flux</u> generates circular magnetic fields**

4.2 **Static electric current generates circular magnetic fields**

Changing fields: Powering and self-induction



- Primary magnetic flux $\vec{B}$ changes with changing current

➡ Induces an electric field, resulting in a current and induced magnetic field $\vec{B}_i$

➡ Induced current will oppose a change of the primary current

➡ If we want to change a current to ramp a magnet ...

Have to overcome this counteraction, applying a sufficient Voltage: if pushed, push harder

Ramp rate determines required Voltage:

$$U = -L\frac{\partial I}{\partial t}$$

Inductance L in Henry (H)

Example:

- **Required ramp rate: 10 A/s**
- **With $L = 15.1\ H$ per powering sector**

➡ **Required Voltage is $\approx$ 150 V**

Lorentz force on charged particles

Note:

Lorentz force is an <u>ad hoc addition</u> to Maxwell equations !

Can not be derived/understood without Relativity (but then it comes out easily !)

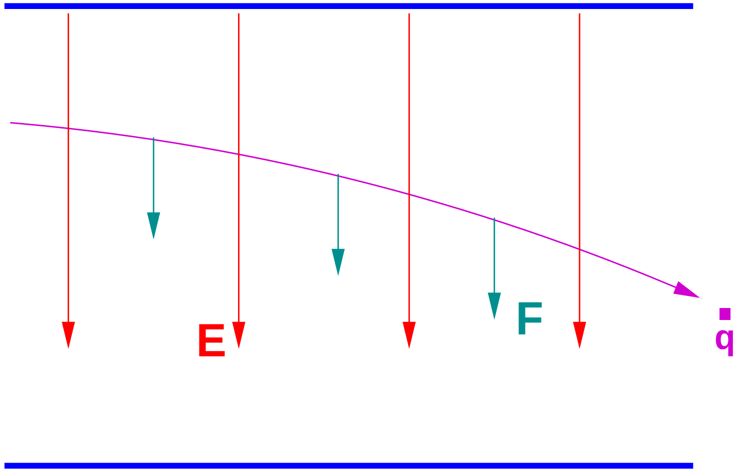
Moving ($\vec{v}$) charged (q) particles in electric ($\vec{E}$) and magnetic ($\vec{B}$) fields experience the Lorentz force $\vec{f}$:

$$\vec{f} \;=\; q \cdot (\vec{E} + \vec{v} \times \vec{B})$$

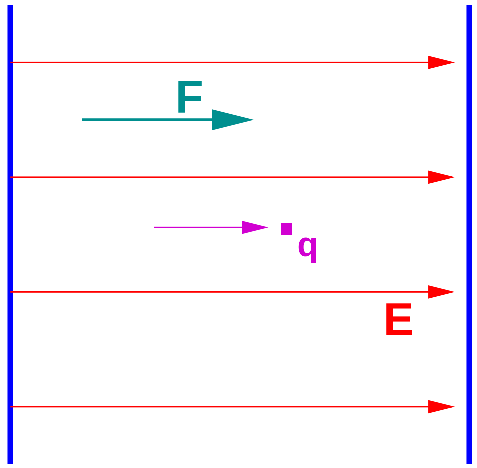
for the equation of motion we get (using Newton's law);

$$\frac{d}{dt}(m\vec{v}) \;=\; \vec{f} \;=\; q \cdot (\vec{E} + \vec{v} \times \vec{B})$$

Motion in electric fields



$\vec{v} \perp \vec{E}$



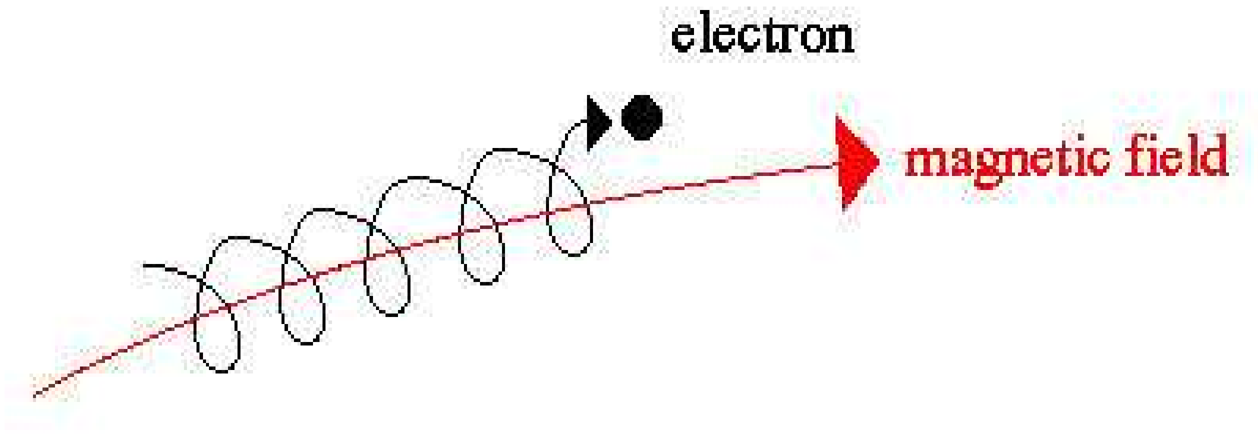
$\vec{v} \parallel \vec{E}$

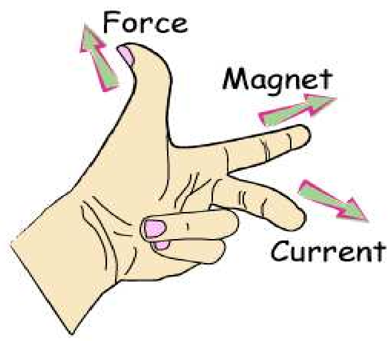
Assume no magnetic field:

$$\frac{d}{dt}(m\vec{v}) \quad = \quad \vec{f} \quad = \quad q \cdot \vec{E}$$

Force always in direction of field $\vec{E}$, also for particles at rest.

Motion in magnetic fields





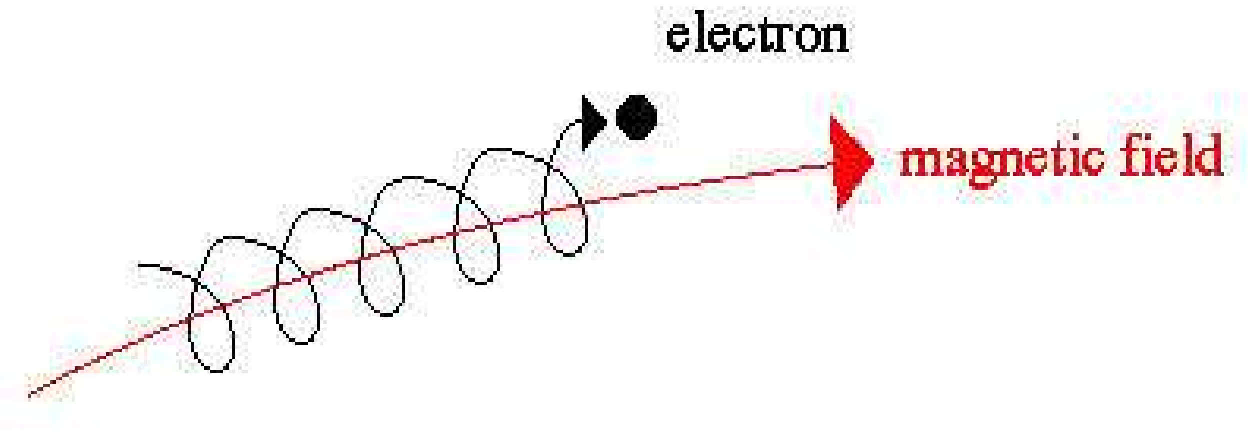
Without electric field : $$\frac{d}{dt}(m\vec{v}) = \vec{f} = q \cdot \vec{v} \times \vec{B}$$

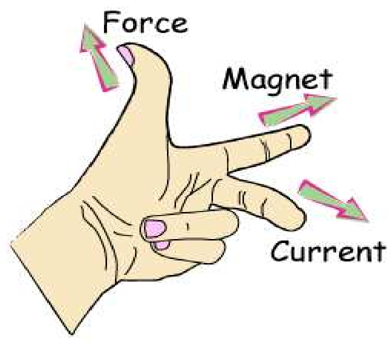
Force is perpendicular to both, $\vec{v}$ and $\vec{B}$

<u>No</u> force on particles at rest - do we understand that ?

Or is it just a fabricated story to get the right answer ?

Motion in magnetic fields





Without electric field : $$\frac{d}{dt}(m\vec{v}) \quad = \quad \vec{f} \quad = \quad q \cdot \vec{v} \times \vec{B}$$

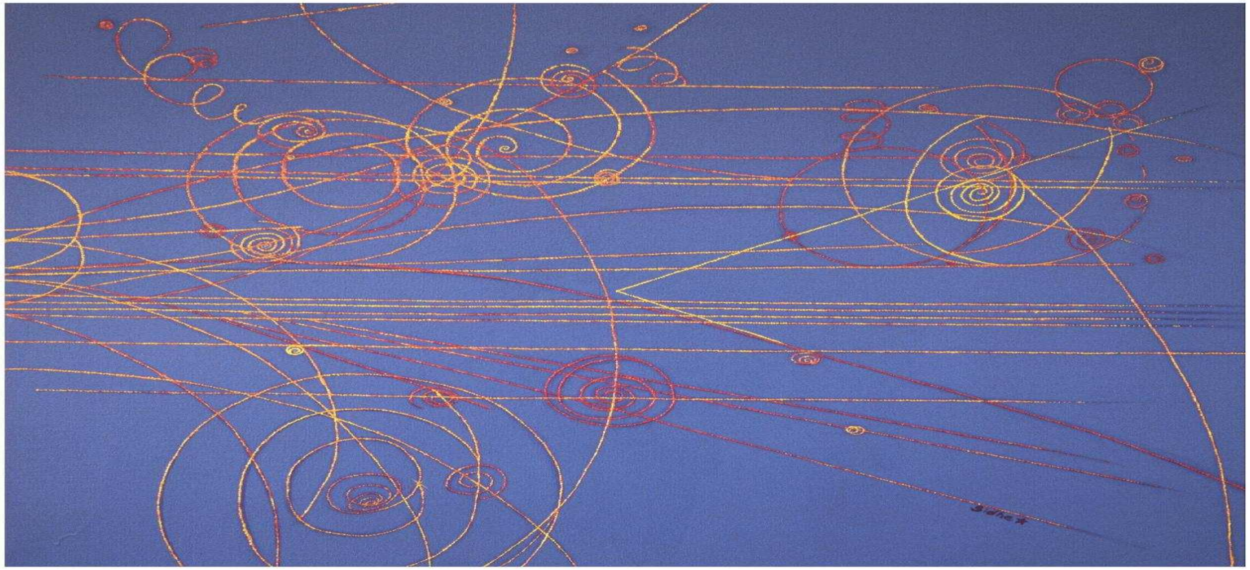
Force is perpendicular to both, $\vec{v}$ and $\vec{B}$

No force on particles at rest - do we understand that ?

Or is it just a fabricated story to get the right answer ?

Yes, but see next lecture ...

Particle motion in magnetic fields - made visible



- Magnetic field perpendicular to motion
- Bending radius depends on momentum
- Bending radius depends on charge

➢ Direction of the magnetic field $\vec{B}$???

Practical units:

$$B\ [T] \cdot \rho\ [m] \quad = \quad \frac{p\ [eV/c]}{c\ [m/s]}$$

Example LHC:

B = 8.33 T, p = 7^{12} eV/c ➡ ρ **= 2804 m**

More - bending angle α of a dipole magnet of length L:

$$\alpha \ = \ \frac{B\ [T] \cdot L\ [m] \cdot 0.3}{p\ [GeV/c]}$$

Example LHC:

B = 8.33 T, p = 7000 GeV/c, L = 14.3 m ➡ α **= 5.11 mrad**

En passant: Energy in electric and magnetic fields

Energy density in electric field:

$$U_E \quad = \quad \frac{\text{energy}}{\text{volume}} \quad = \quad \frac{1}{2}\epsilon_0 E^2$$

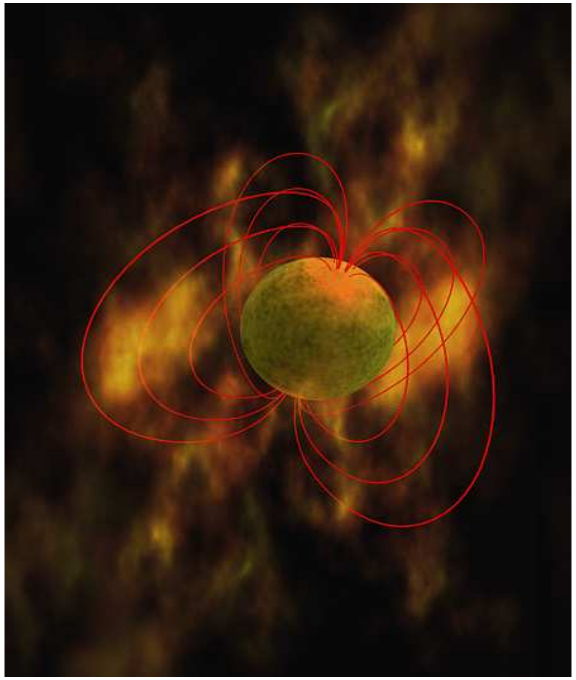
Energy density in magnetic field:

$$U_B \quad = \quad \frac{\text{energy}}{\text{volume}} \quad = \quad \frac{1}{2}\frac{B^2}{\mu_0}$$

Everyday example: B **=** **5·10**$^{-5}$ **T (= 0.5 Gauss)**

➡ $U_B \approx$ **1 mJ/m**^{3}

… and some really strong magnetic fields



Example : CXOUJ164710.2 – 45516

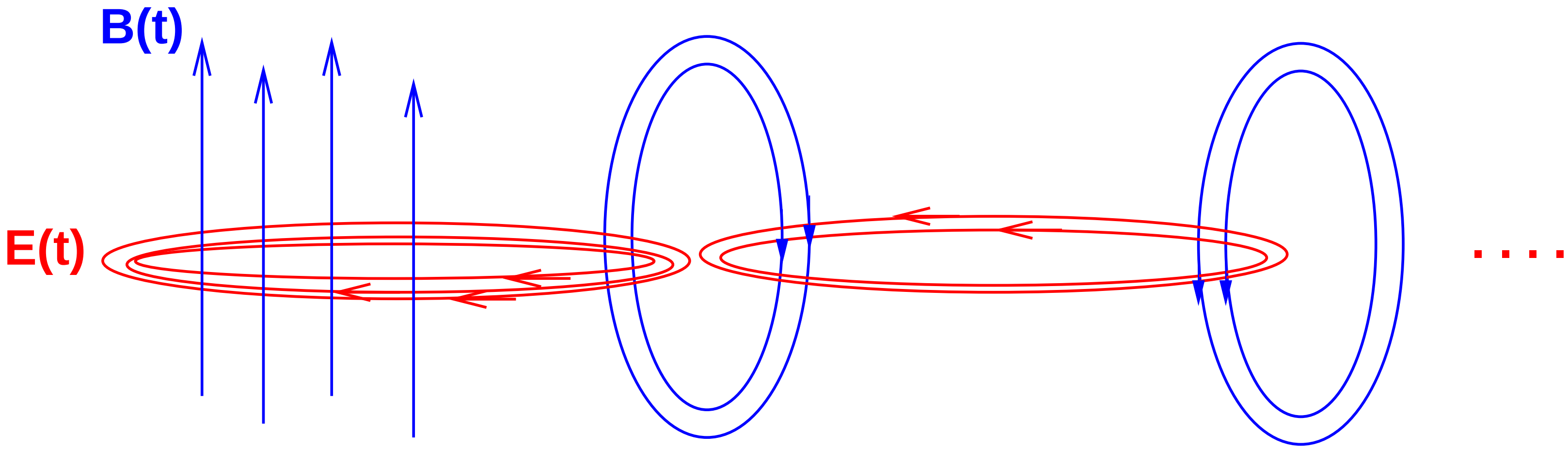
Diameter : 10 – 20 km

Field : $\approx 10^{12}$ Tesla

As accelerator : $\approx 10^{12}$ TeV

Very fast time varying electromagnetic fields - γ-ray bursts up to 10^{40} W

Time Varying Fields - (Maxwell 1864)



Time varying magnetic fields produce circular electric fields

Time varying electric fields produce circular magnetic fields

- **Can produce self-sustaining, propagating fields (i.e. waves)**
- **Rather useful picture (but without "Relativity": BIG problems)**

In vacuum: only fields, no charges ($\rho = 0$), no current ($j = 0$) ...

From $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ **educated guess and juggling with ∇:**

$$\begin{aligned}
\Longrightarrow \quad \nabla \times (\nabla \times \vec{E}) &= -\nabla \times (\frac{\partial \vec{B}}{\partial t}) \\
\Longrightarrow \quad -(\nabla^2 \vec{E}) &= -\frac{\partial}{\partial t}(\nabla \times \vec{B}) \\
\Longrightarrow \quad -(\nabla^2 \vec{E}) &= -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}
\end{aligned}$$

$$\nabla^2 \vec{E} \;=\; \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{E}}{\partial t^2} \;=\; \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

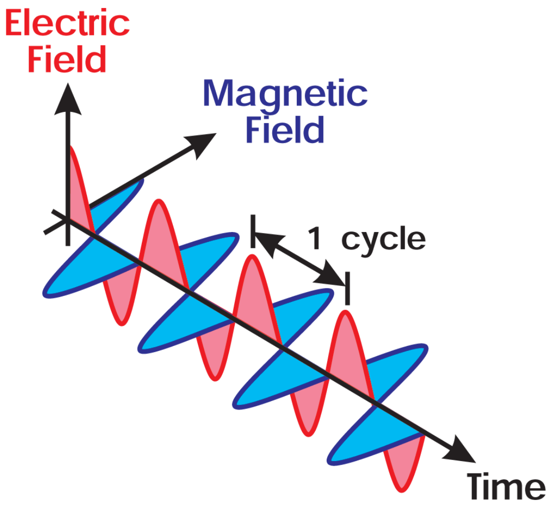
(same equation for $\vec{B}$)

➡ **Equation for a wave with velocity:** $c \;=\; \frac{1}{\sqrt{\mu_0 \cdot \epsilon_0}}$

Challenge: try to derive the wave equation from the Integral Form

Electromagnetic waves

$$\vec{E} = \vec{E_0} e^{i(\omega t - \vec{k}\cdot\vec{x})}$$
$$\vec{B} = \vec{B_0} e^{i(\omega t - \vec{k}\cdot\vec{x})}$$



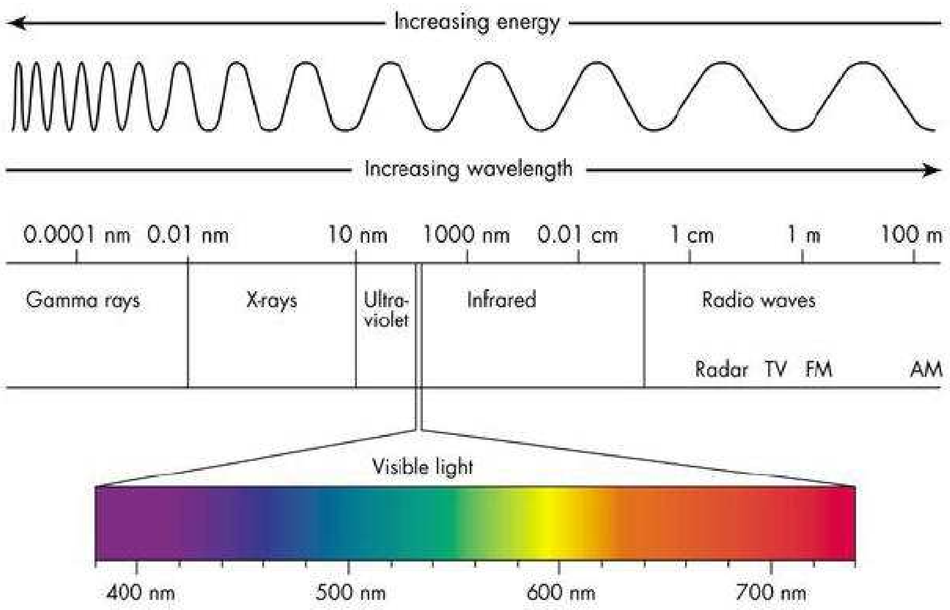
Important quantities :

$$|\vec{k}| = \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad (\textbf{propagation vector})$$
$$\lambda = (\textbf{wave length, 1 cycle})$$
$$\omega = (\textbf{frequency} \cdot 2\pi)$$

Magnetic and electric fields are transverse to direction of propagation:
$\vec{E} \perp \vec{B} \perp \vec{k}$

Short wave length $\rightarrow$ high frequency $\rightarrow$ high energy

Spectrum of Electromagnetic waves



Example: yellow light → $\approx 5 \cdot 10^{14}$ **Hz (i.e.** $\approx$ **2 eV !)**

LEP (SR) → $\leq 2 \cdot 10^{20}$ **Hz (i.e.** $\approx$ **0.8 MeV !)**

gamma rays → $\leq 3 \cdot 10^{21}$ **Hz (i.e.** $\leq$ **12 MeV !)**

(For estimates using temperature: 3 K $\approx$ **0.00025 eV)**

Waves in material → **Index of refraction: n**

Speed of electromagnetic waves in vacuum: $c = \frac{1}{\sqrt{\mu_0 \cdot \epsilon_0}}$

$$\text{n} = \frac{\text{Speed of light in vacuum}}{\text{Speed of light in material}}$$

For water n ≈ 1.33

Depends on wavelength

n $\approx 1.32 - 1.39$

Waves impacting material

Need to look at the behaviour of electromagnetic fields at boundaries between different materials

Important for highly conductive materials in accelerators, e.g.:

- **RF systems**
- **Wave guides**
- **Impedance calculations**

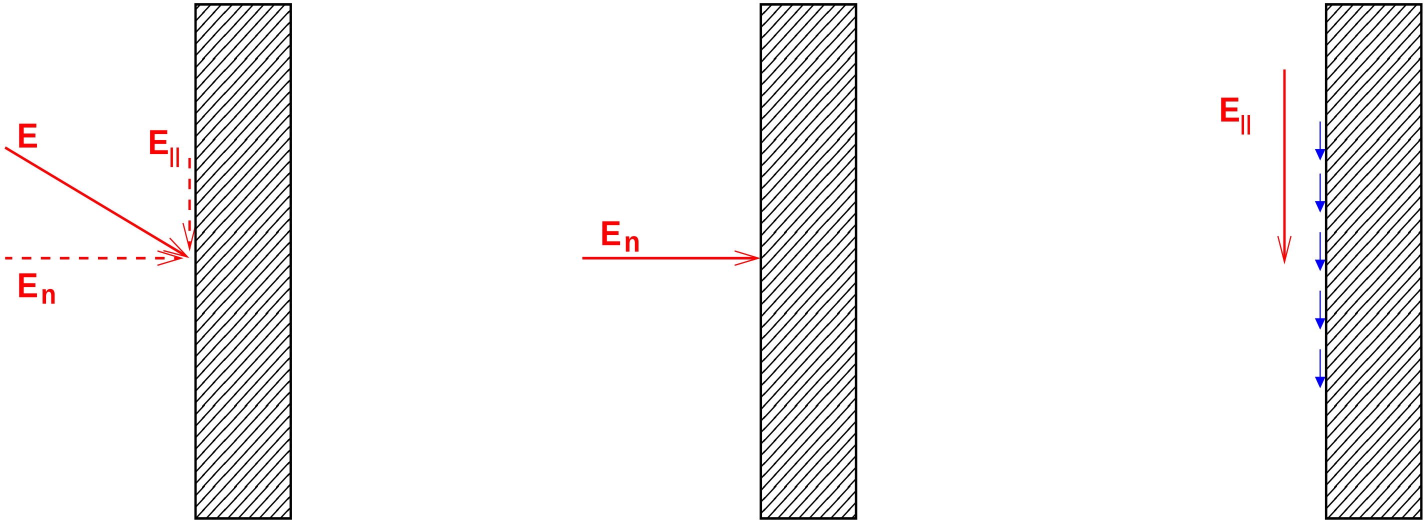
Can be derived immediately from Maxwell's equations

(using all $\mathrm{div}\vec{E},\ \mathrm{div}\vec{B},\ \mathrm{curl}\vec{E},\ \mathrm{curl}\vec{B}$**)**

Here only the results !

Boundary conditions: air/vacuum and conductor

A simple case ($\vec{E}$-fields on a conducting surface):



Field parallel to surface $E_{\|}$ cannot exist (it would move charges and we get a surface current): $E_{\|} \ = \ 0$

- **Only a field normal (orthogonal) to surface E_n is possible**

Extreme case: ideal conductor

For an ideal conductor (i.e. no resistance) we must have:

$$\vec{E}_{\|} \;=\; 0, \qquad \vec{B}_n \;=\; 0$$

otherwise the surface current becomes infinite

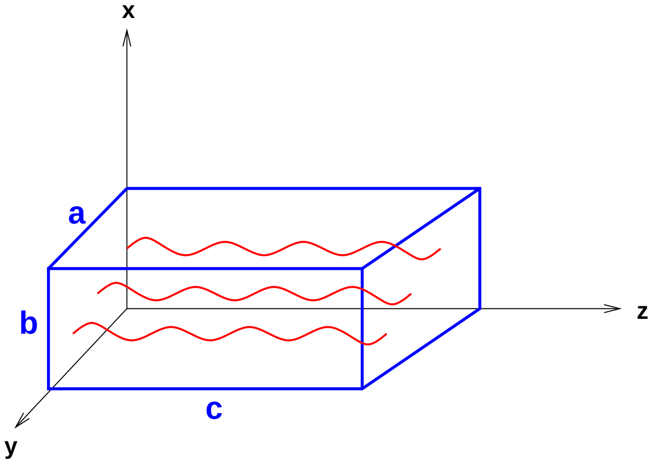
This implies:

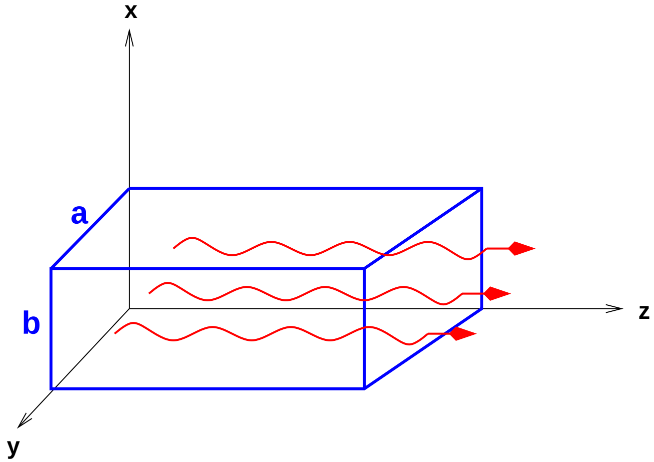
- **All energy of an electromagnetic wave is reflected from the surface of an ideal conductor.**
- **Fields at any point in the ideal conductor are zero.**
- **Only some fieldpatterns are allowed in waveguides and RF cavities**

A very nice lecture in R.P.Feynman, Vol. II

Examples: cavities and wave guides

Rectangular, conducting cavities and wave guides (schematic) with dimensions $a \times b \times c$ and $a \times b$:





- **RF cavity, fields can persist and be stored (reflection !)**
- **Plane waves can propagate along wave guides, here in z-direction**

(here just the basics, many details in "RF Systems" by Frank Tecker)

Fields in RF cavities - as reference

Assume a rectangular RF cavity ($a,\ b,\ c$), ideal conductor.

Without derivations, the components of the fields are:

$$E_x = E_{x0} \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$

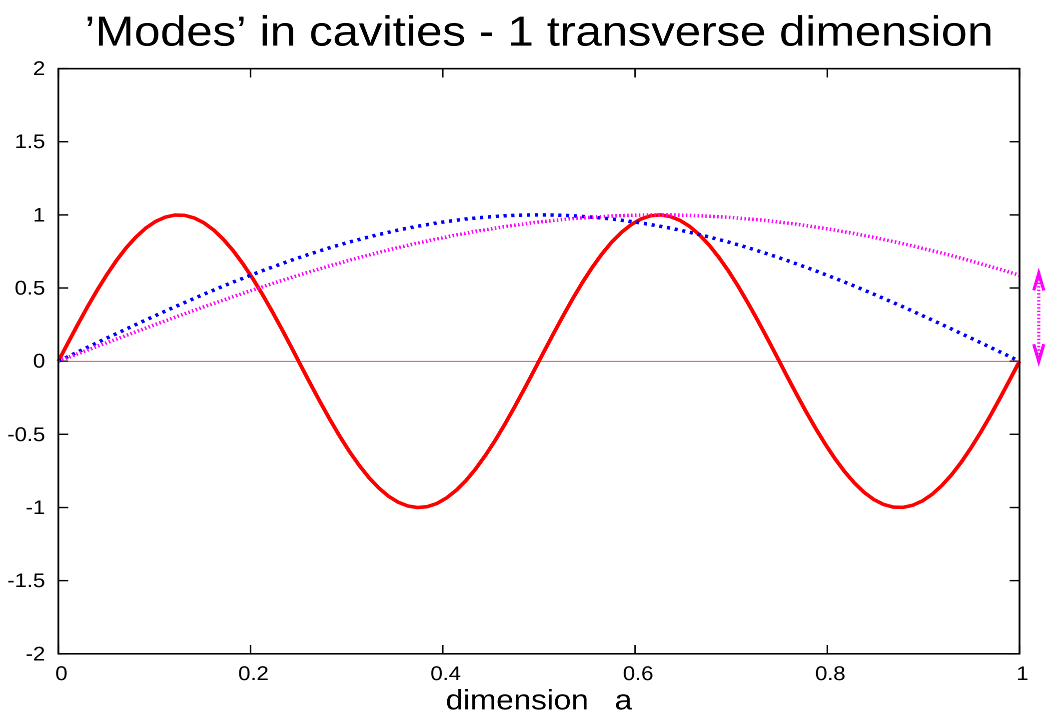
$$E_y = E_{y0} \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$

$$E_z = E_{z0} \cdot \sin(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$

$$B_x = \frac{i}{\omega}(E_{y0}k_z - E_{z0}k_y) \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$

$$B_y = \frac{i}{\omega}(E_{z0}k_x - E_{x0}k_z) \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$

$$B_z = \frac{i}{\omega}(E_{x0}k_y - E_{y0}k_x) \cdot \cos(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$



No electric field at boundaries, wave must have "nodes" = zero fields at the boundaries

Only modes which 'fit' into the cavity are allowed

In the example: $\frac{\lambda}{2} = \frac{a}{4}$, $\frac{\lambda}{2} = \frac{a}{1}$, $\frac{\lambda}{2} = \frac{a}{0.8}$

(then either "sin" or "cos" is 0)

Consequences for RF cavities

Field must be zero at conductor boundary, only possible if:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

and for k_x, k_y, k_z **we can write, (then they all fit):**

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \frac{m_z \pi}{c},$$

The integer numbers m_x, m_y, m_z **are called mode numbers, important for design of cavity !**

$\rightarrow$ **half wave length** $\lambda/2$ **must always fit exactly the size of the cavity.**

(For cylindrical cavities: use cylindrical coordinates)

Similar considerations lead to (propagating) solutions in (rectangular) wave guides:

$$E_x = E_{x0} \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$E_y = E_{y0} \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$E_z = i \cdot E_{z0} \cdot \sin(k_x x) \cdot \sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_x = \frac{1}{\omega}(E_{y0} k_z - E_{z0} k_y) \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_y = \frac{1}{\omega}(E_{z0} k_x - E_{x0} k_z) \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_z = \frac{1}{i \cdot \omega}(E_{x0} k_y - E_{y0} k_x) \cdot \cos(k_x x) \cdot \cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

This part is new: $e^{i(k_z z)}$ $\Longrightarrow$ **something moving in z direction**

In z direction: No Boundary - No Boundary Condition ...

Consequences for wave guides

Similar considerations as for cavities, no field at boundary.
We must satisfy again the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

This leads to modes like (no boundaries in direction of propagation z):

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b},$$

The numbers m_x, m_y are called mode numbers for planar waves in wave guides !

Re-writing the condition as:

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2 \quad \rightarrow \quad k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

Propagation without losses requires k_z to be real, i.e.:

$$\frac{\omega^2}{c^2} > k_x^2 + k_y^2 = (\frac{m_x \pi}{a})^2 + (\frac{m_y \pi}{b})^2$$

which defines a cut-off frequency ω_c. For lowest order mode:

$$\omega_c = \frac{\pi \cdot c}{a}$$

- **Above cut-off frequency: propagation without loss**
- **At cut-off frequency: standing wave**
- **Below cut-off frequency: attenuated wave (it does not "fit in").**

There is a very easy way to show that very high frequencies easily propagate !!!

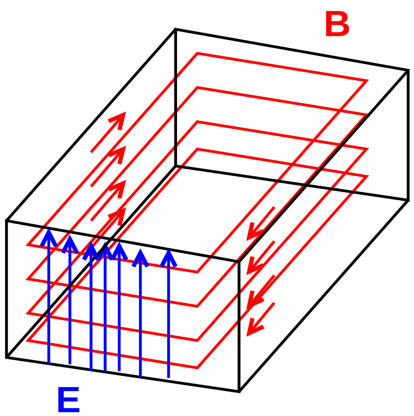
Classification of modes:

Transverse electric modes (TE): $E_z = 0 \quad H_z \neq 0$

Transverse magnetic modes (TM): $E_z \neq 0 \quad H_z = 0$

Transverse electric-magnetic modes (TEM): $E_z = 0 \quad H_z = 0$

(Not all of them can be used for acceleration ... !)



Note (here a TE mode) :

Electric field lines end at boundaries

Magnetic field lines appear as "loops"

Other case: finite conductivity

Starting from Maxwell equation:

$$\nabla \times \vec{B} = \mu \vec{j} \; + \; \mu\epsilon \frac{d\vec{E}}{dt} \; = \; \underbrace{\overbrace{\sigma \cdot \vec{E}}^{\vec{j}}}_{Ohm's\ law} + \mu\epsilon \frac{d\vec{E}}{dt}$$

Wave equations:

$$\vec{E} \; = \; \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \qquad \vec{B} \; = \; \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

We want to know k with this new contribution:

$$k^2 \; = \; \frac{\omega^2}{c^2} \; - \; \underbrace{i\omega\sigma\mu}_{new}$$

Consequence → Skin Depth

Electromagnetic waves can now penetrate into the conductor !

For a good conductor $\sigma \gg \omega\epsilon$:

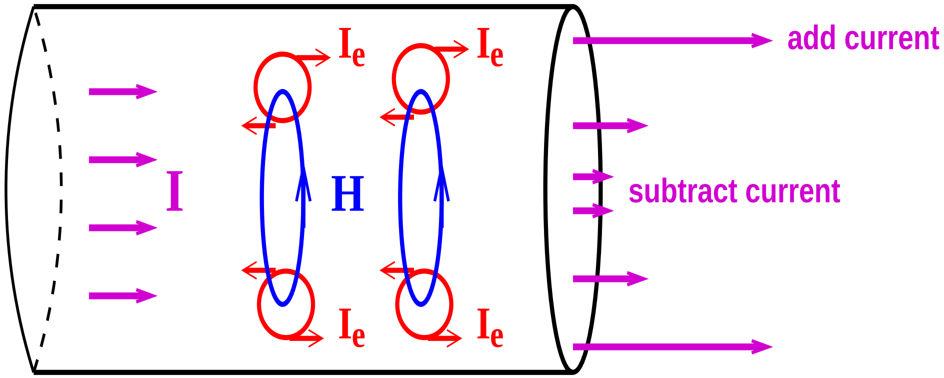
$$k^2 \approx -i\omega\mu\sigma \quad \rightarrow \quad k \approx \sqrt{\frac{\omega\mu\sigma}{2}}(1+i) = \frac{1}{\delta}(1+i)$$

$$\boxed{\delta = \sqrt{\frac{2}{\omega\mu\sigma}}}$$

is the <u>Skin Depth</u>

- **High frequency currents "avoid" penetrating into a conductor, flow near the surface**
- **Penetration depth small for large conductivity**

"Explanation" - inside a conductor (very schematic)



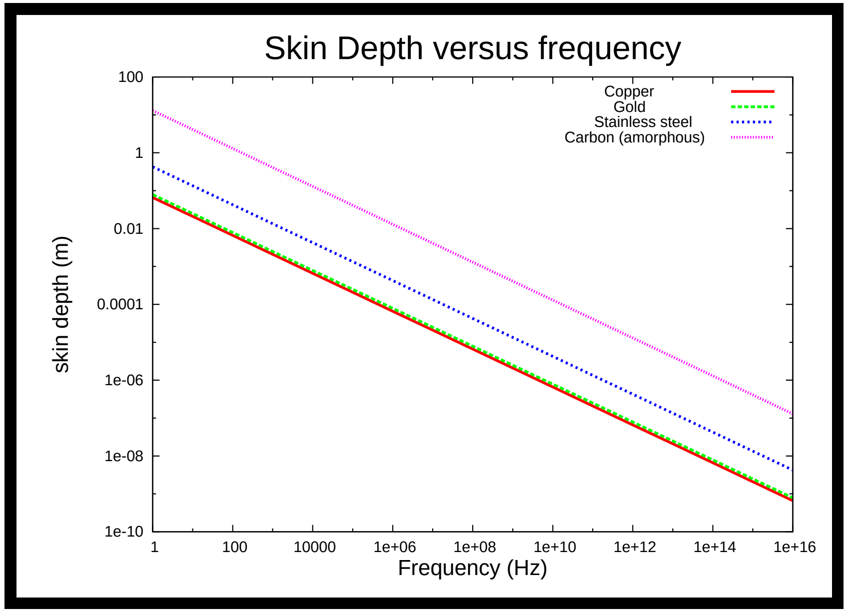
Eddy currents I_E from changing $\vec{H}$-field: $\nabla \times \vec{E} \;=\; \mu_0 \frac{d\vec{H}}{dt}$

Cancel current flow in the centre of the conductor $\quad \mathbf{I} - \mathbf{I_e}$

Enforce current flow near the "skin" (surface) $\quad \mathbf{I} + \mathbf{I_e}$

Q: Why are high frequency cables thin ??

Attenuated waves - penetration depth



Waves incident on conducting material are attenuated

Basically by the Skin depth :

(attenuation to 1/e)

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Wave form:

$$e^{i(kz-\omega t)} = e^{i((1+i)z/\delta-\omega t)} = e^{\frac{-z}{\delta}} \cdot e^{i(\frac{z}{\delta}-\omega t)}$$

Values of δ can have a very large range ..

- **Skin depth Copper ($\sigma \approx 6 \cdot 10^7$ S/m):**

  2.45 GHz: $\delta \approx$ 1.5 μm, 50 Hz: $\delta \approx$ 10 mm

  (there is an easy way to waste your money ...)

- **Penetration depth Glass (strong variation, σ typically $6 \cdot 10^{-13}$ S/m):**

  2.45 GHz: $\delta >$ km

- **Penetration depth pig (strong variation, σ typically $3 \cdot 10^{-2}$ S/m):**

  2.45 GHz: $\delta \approx$ 6 cm

- **Penetration depth Seawater ($\sigma \approx 4$ S/m):**

  76 Hz: $\delta \approx$ 25 - 30 m

Done list:

1. **Review of basics and write down Maxwell's equations**
2. **Add Lorentz force and motion of particles in EM fields**
3. **Electromagnetic waves in vacuum**
4. **Electromagnetic waves in conducting media**
   - **Waves in RF cavities**
   - **Waves in wave guides**
   - **Important concepts: mode numbers, cut-off frequency, skin depth**

But still a few (important) problems to sort out →

- BACKUP SLIDES -

For a point charge Q with the field : $\vec{E}(x,y,z) \;=\; \vec{E}(r) \;=\; \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$

one can write all the derivaties (used for DIV and CURL):

$$\frac{\partial E_x}{\partial x} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3x^2}{R^5}\right) \qquad \frac{\partial E_x}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \qquad \frac{\partial E_x}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5}$$

$$\frac{\partial E_y}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \qquad \frac{\partial E_y}{\partial y} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3y^2}{R^5}\right) \qquad \frac{\partial E_y}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5}$$

$$\frac{\partial E_z}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5} \qquad \frac{\partial E_z}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5} \qquad \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3z^2}{R^5}\right)$$

(it does not get any worse than this horror ..)

$$\frac{\partial E_x}{\partial x} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3x^2}{R^5}\right) \qquad \frac{\partial E_x}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \qquad \frac{\partial E_x}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5}$$

$$\frac{\partial E_y}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \qquad \frac{\partial E_y}{\partial y} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3y^2}{R^5}\right) \qquad \frac{\partial E_y}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5}$$

$$\frac{\partial E_z}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5} \qquad \frac{\partial E_z}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5} \qquad \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3z^2}{R^5}\right)$$

$$\text{div } \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\epsilon_0}\left(\frac{3}{R^3} - \frac{3}{R^5}(x^2 + y^2 + z^2)\right)$$

$$\frac{\partial E_x}{\partial x} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3x^2}{R^5}\right) \qquad \frac{\partial E_x}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \qquad \frac{\partial E_x}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5}$$

$$\frac{\partial E_y}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \qquad \frac{\partial E_y}{\partial y} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3y^2}{R^5}\right) \qquad \frac{\partial E_y}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5}$$

$$\frac{\partial E_z}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5} \qquad \frac{\partial E_z}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5} \qquad \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3z^2}{R^5}\right)$$

$$\text{curl } \vec{E} \quad = \quad \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial z}\right) \quad = \quad (0, \quad 0, \quad 0)$$

(there is nothing circulating)

Interlude and Warning !!

Maxwell's equation can be written in other forms.
Often used: cgs (Gaussian) units instead of SI units, example:

Starting from (SI):

$$\nabla \cdot \vec{E} \ = \ \frac{\rho}{\epsilon_0}$$

we would use:

$$\vec{E}_{cgs} \ = \ \frac{1}{c} \ \cdot \vec{E}_{SI} \qquad \text{and} \qquad \epsilon_0 \ = \ \frac{1}{4\pi \cdot c}$$

and arrive at (cgs):

$$\nabla \cdot \vec{E} \ = \ 4\pi \cdot \rho$$

Beware: there are more different units giving: $\nabla \cdot \vec{E} \ = \ \rho$

Electromagnetic fields in material

In vacuum:

$$\vec{D} = \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_0 \cdot \vec{H}$$

In a material:

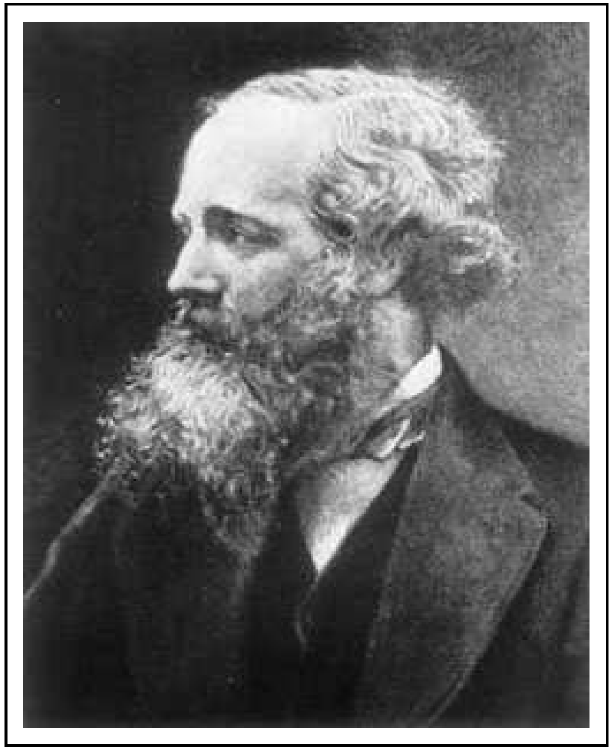
$$\vec{D} = \epsilon_r \cdot \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_r \cdot \mu_0 \cdot \vec{H}$$

ϵ_r **is relative permittivity** $\approx \quad [1 - 10^5]$

μ_r **is relative permeability** $\approx \quad [0(!) - 10^6]$

Origin: polarization and Magnetization

Once more: Maxwell's Equations



$$
\begin{aligned}
\nabla \vec{D} &= \rho \\
\nabla \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{d\vec{B}}{dt} \\
\nabla \times \vec{H} &= \vec{j} + \frac{d\vec{D}}{dt}
\end{aligned}
$$

Re-factored in terms of the free current density $\vec{j}$ and free charge density ρ ($\mu_0 = 1, \epsilon_0 = 1$):

Some popular confusion ..

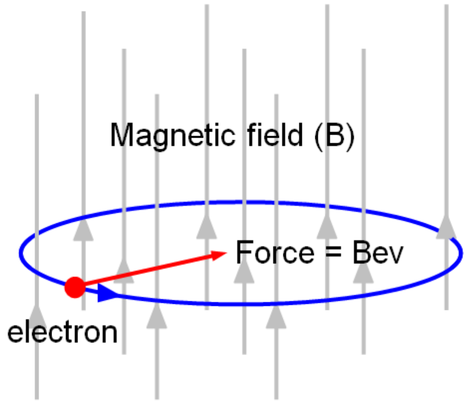
V.F.A.Q: why this strange mixture of $\vec{E}, \vec{D}, \vec{B}, \vec{H}$??

Materials respond to an applied electric E field and an applied magnetic B field by producing their own internal charge and current distributions, contributing to E and B. Therefore H and D fields are used to re-factor Maxwell's equations in terms of the free current density $\vec{j}$ and free charge density ρ:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$
$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$\vec{M}$ **and** $\vec{P}$ **are** $Magnetization$ **and** $Polarisation$ **in material**

Is that the full truth ?



If we have a circular E-field along the circle of radius R ?

➡ **should get acceleration !**

Remember Maxwell's third equation:

$$\oint_C \vec{E} \cdot d\vec{r} \quad = \quad -\frac{d}{dt} \int_A \vec{B} \cdot d\vec{A}$$

$$\Rightarrow 2\pi R E_\theta \quad = \quad -\frac{d\phi}{dt}$$

Motion in magnetic fields

- This is the principle of a Betatron
  - Time varying magnetic field creates circular electric field !
  - Time varying magnetic field deflects the charge !

For a constant radius we need:

$$-\frac{m \cdot v^2}{R} = e \cdot v \cdot B \quad \Rightarrow \quad B = -\frac{p}{e \cdot R}$$

$$\frac{\partial}{\partial t} B(r,t) = -\frac{1}{e \cdot R}\frac{dp}{dt}$$

$$\Rightarrow \quad B(r,t) = \frac{1}{2}\frac{1}{\pi R^2}\int\int B dS$$

B-field on orbit must be half the average over the circle ➡ Betatron condition

Other case: finite conductivity

Assume conductor with finite conductivity ($\sigma_c = \rho_c^{-1}$) , waves will penetrate into surface. Order of the skin depth is:

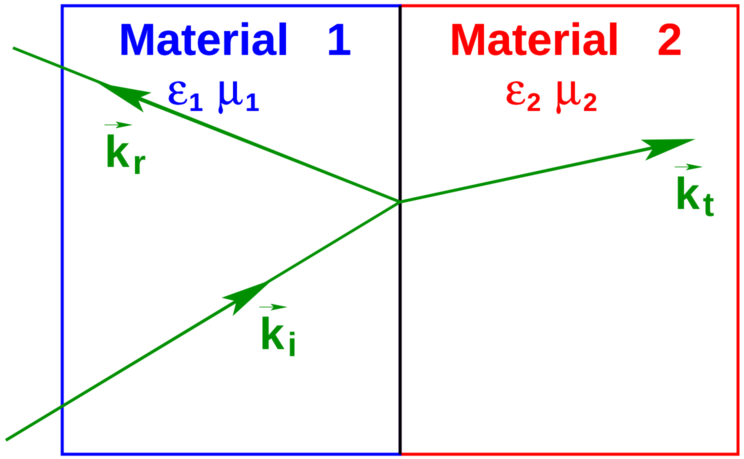
$$\delta_s = \sqrt{\frac{2\rho_c}{\mu\omega}}$$

i.e. depend on resistivity, permeability and frequency of the waves (ω).
We can get the surface impedance as:

$$Z = \sqrt{\frac{\mu}{\epsilon}} = \frac{\mu\omega}{k}$$

the latter follows from our definition of k and speed of light.
Since the wave vector k is complex, the impedance is also complex. We get a phase shift between electric and magnetic field.

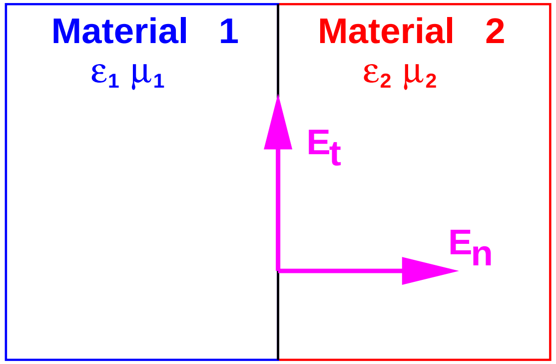
Boundary conditions for fields

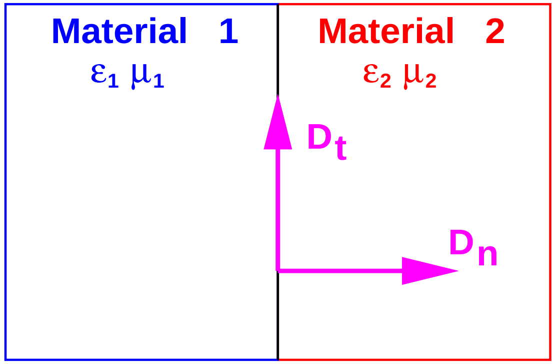


What happens when an incident wave $(\vec{K}_i)$ encounters a boundary between two different media ?

- **Part of the wave will be reflected $(\vec{K}_r)$, part is transmitted $(\vec{K}_t)$**
- **What happens to the electric and magnetic fields ?**

Boundary conditions for fields

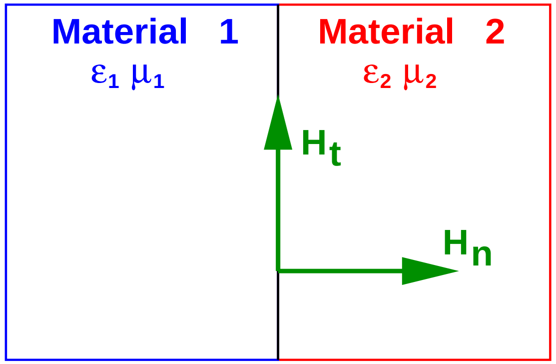


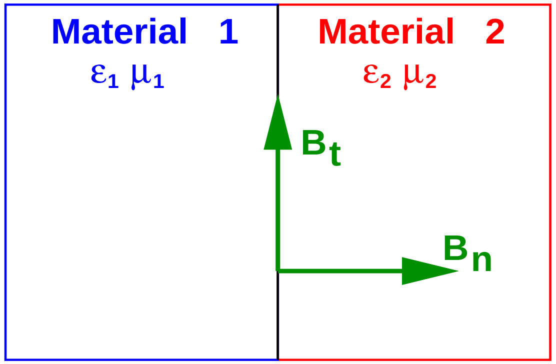


Assuming no surface charges:

- **tangential $\vec{E}$-field constant across boundary** $(E_{1t} = E_{2t})$
- **normal $\vec{D}$-field constant across boundary** $(D_{1n} = D_{2n})$

Boundary conditions for fields





Assuming no surface currents:

- **tangential $\vec{H}$-field constant across boundary** $(H_{1t} = H_{2t})$
- **normal $\vec{B}$-field constant across boundary** $(B_{1n} = B_{2n})$