Kinematics of Particle Beams
An introduction to Special Relativity

Contents:

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Course-Title: Kinematics of Particle Beams - Relativity

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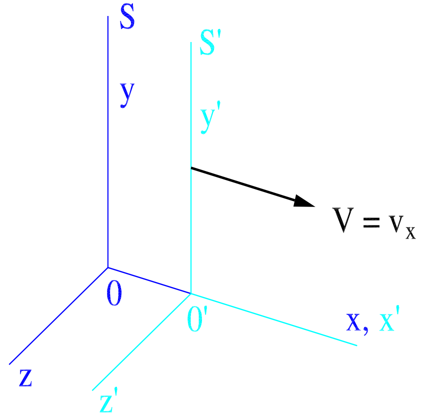
Preamble

- We will use orthogonal frames (3 axes at $90^{\mathbf{0}}$) where the cartesian (or rectangular) coordinates of a point in space are specified.

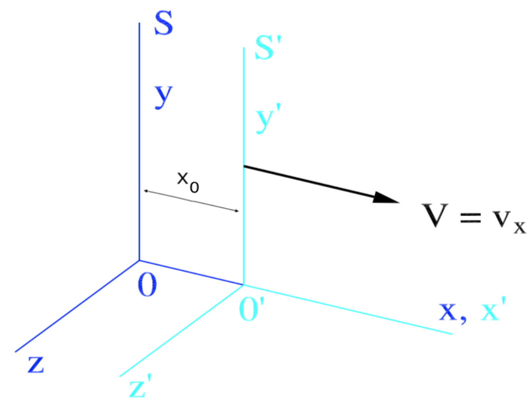
- Any frame may be made to coincide with any other by translations and rotations.
  - For this reason when considering frames attached to moving observers we will just consider translational motion along one common axis.
    This simplifies the mathematics.



WHY THE "THEORY OF RELATIVITY"?

Quantitive description of physical events needs a *frame of reference*, where the coordinates of the observed object are specified. Euclidean geometry specifies how coordinates of points in different frames are related. For instance, if S' is *translated* by x_0 wrt S along the common $\hat{x}$-axis it is

$$x' = x - x_0 \qquad y' = y \qquad z' = z \tag{1}$$

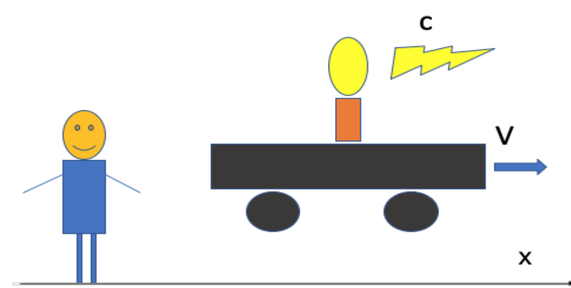


Suppose S' is *moving* wrt S along the $\hat{x}$-axis with velocity $\vec{V}$. Thus $x_0 = Vt$ (assuming O and O' coincide at $t=0$) and making the first and second derivates wrt time

$$\dot{x}' = \dot{x} - V \qquad \dot{y}' = \dot{y} \qquad \dot{z}' = \dot{z} \tag{2}$$

$$\ddot{x}' = \ddot{x} \qquad \ddot{y}' = \ddot{y} \qquad \ddot{z}' = \ddot{z} \tag{3}$$

Eqs. 1, 2 and 3 are the Galilean transformations for coordinates, velocity and acceleration. We implicitly assumed that $t' = t$ and that the lengths were invariant in the two frames.



- Eq.2 means that velocities add.
- Eq.3 says that the acceleration is invariant.

The basis of the classical mechanics are the three laws[a] of dynamics.

- The first dynamics law is the principle of inertia (Galileo) which states

  "A free body remains in a state of rest or of uniform motion"

  - A reference frame where the principle of inertia holds good is said inertial.
  - Because of Galilean transformations, any frame in uniform motion wrt an inertial one is inertial too.

- The second law (Newton) states

  "In an inertial frame it holds good $\vec{F} = m\vec{a}$ "

  The variation of velocity with time (acceleration), $\vec{a}$, is proportional to the applied force, $\vec{F}$, through a constant, m ("inertial mass").

  - Implicitly it is assumed that m is a *characteristic* of the body which doesn't depend upon its motion.

- The third Newton law states

  " Whenever two bodies interact they apply equal and opposite forces to each other."

Third law combined with the second one gives the momentum conservation law[b] for a closed system.

[a] Physics laws are not mathematical axioms but statements based on reproducible observations.

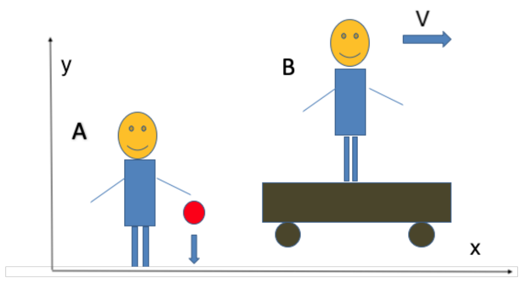
[b] Momentum: $\vec{p} \equiv m\vec{v}$

The three laws of dynamics hold good in inertial frames. As those frames are all equivalent it is reasonable to assume that mechanics laws are the same for inertial observers.

This is expressed by the principle of relativity:

"Mechanics laws have the same form for all inertial observers".

Suppose that Alex (A) is studying the motion of a ball let to fall under the earth gravitational force. A measures that the object is subject to a constant acceleration of $a \approx 9.8$ ms^{-2}. By using different balls he finds that the acceleration is always the same, g. A concludes that there must be a force acting on the balls which is directed towards the center of the earth and has magnitude mg.



Assuming that Galilean transformations hold good, observer Beth (B) on a train moving with uniform velocity $\vec{V}=\hat{x}V$ wrt A will describe the ball motion as

$$\ddot{x}' = \ddot{x} = 0 \qquad \ddot{y}' = \ddot{y}$$

and as the mass, m is a constant, will agree with A on magnitude and direction of the force.

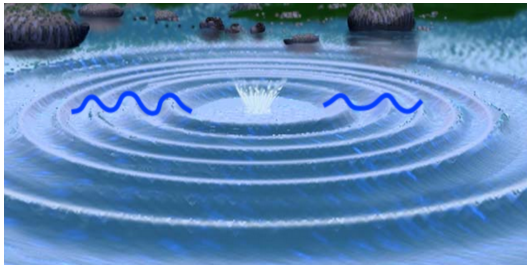
Galilean transformations satisfy the the principle of relativity!

Is EM invariant under Galilean transformations?

In the second half of the XIX century Maxwell had summarized the whole EM phenomena into 4 differential equations containing the constant $c = \sqrt{\epsilon_0 \mu_0}$. From these equations far from sources one finds the wave equation for fields and potentials.

Simplest case: $$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi = 0$$

The constant c is the velocity of propagation of the wave and is numerically equal to the speed of light in vacuum. Because of the addition of the velocities it is weird that it is a constant, unless we assume it is the velocity wrt a propagation medium and that the equation is written in a frame connected with that medium.



This medium would permeate the whole space and has to be extremely rarefied to be undetectable directly. The supporting medium was named "*ether*".

Under Galilean transformation the wave equation becomes

$$\left[\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{V^2}{c^2}\frac{\partial^2}{\partial x'^2} - 2\frac{V}{c^2}\frac{\partial^2}{\partial x'\partial t'}\right]\Phi = 0$$

Indeed the equation is <u>not</u> invariant.

- EM laws are written in a frame connected to the ether.
- According to Galilean transformations, the speed of light for an observer moving wrt ether wouldn't be c.

Hypotheses

1. The relativity principle holds good only for the mechanics; for the EM exists a preferred frame of reference where the speed of the light is c (the reference system where the ether is at rest).
   - A bunch of experiments (starting with the famous Michelson-Morley experiment in 1887) aiming to prove the existence of the ether failed. Their results suggested instead that the speed of the light was a constant non dependent upon the status of motion of source or observer.
2. The at the time relatively young EM laws are wrong.
   - Attempts at modifying the EM in such a way that it would be invariant under Galilean transformations led to predictions of new phenomena which couldn't be proven by experiments.
3. EM laws are correct, but the Galilean transformations (and mechanics laws) must be modified.

Relativity of time

In the “Annus Mirabilis” 1905 Einstein published 4 fundamental papers. The third of them contained the idea of relativity of time and the basis of the theory of special relativity.

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3. *Zur Elektrodynamik bewegter Körper;*
von A. Einstein.

Bern, Juni 1905.

(Eingegangen 30. Juni 1905.)

At that time the “mainstream” idea was that the relativity principle did not apply to EM laws. However the negative results for instance of the first Michelson-Morley experiments were already known.

Einstein starts, on the basis of the experimental evidence, by giving up the existence of ether and introducing instead a “Principle of Relativity” based on two postulates

1) Physics laws are the same in all inertial reference systems, there is no preferred reference frame.

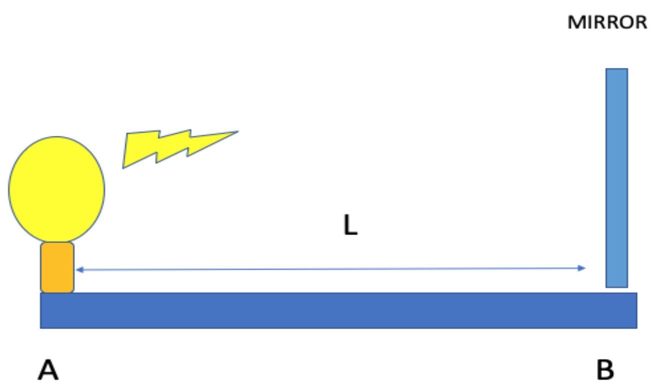
2) The speed of the light in the empty space has the same finite value c in all inertial reference frames.

The paper goes on demonstrating that it is not possible to synchronize clocks attached to frames in relative motion.

To find out whether two clocks at rest in different locations of an inertial frame $\boldsymbol{S}$ run synchronously we proceed as follows. The observer $\boldsymbol{A}$ has a clock and sends a light ray at time $\boldsymbol{t_A}$ to observer $\boldsymbol{B}$ which receives it at $\boldsymbol{t_B}$.

A mirror reflects the light back to $\boldsymbol{A}$ which receive it at $\boldsymbol{t'_A}$.

Because of the second postulate, the clocks are synchronized if



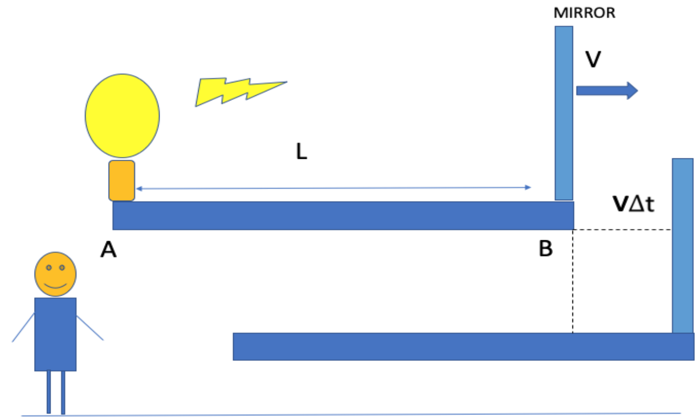
$$\boldsymbol{t_B - t_A = t'_A - t_B}$$

If the clocks are identical they stay synchronized.

All clocks attached to $\boldsymbol{S}$ can be synchronized in this way.

Any other inertial observer $\boldsymbol{S'}$ can synchronize its clocks by the same procedure. How the time measured by observers in relative motion are related?

Let's look for instance to the synchronizing operation for two clocks attached to the ends, A and B, of a rod moving along the x-axis as seen by a stationary observer.
While the light moves to B, B moves further and once reflected back to A, A moves toward the light.



Therefore for the resting observer the time needed to reach B is obtained by setting

$$c(t_B - t_A) = L + V(t_B - t_A) \quad \rightarrow \quad t_B - t_A = L/(c - V)$$

while the time needed to reach A is obtained from

$$c(t'_A - t_B) = L - V(t'_A - t_B) \quad \rightarrow \quad t'_A - t_B = L/(c + V)$$

$$\Delta t_{B\rightarrow A} - \Delta t_{A\rightarrow B} = \frac{2VL}{c^2[1 - (V/c)^2]} \neq 0$$ consequence of c being finite!

If the clocks in the moving frame would be synchronous with the stationary ones they wouldn't be synchronous in their own frame. The "stationary" frame would dictate the timing. However stationarity is relative, the inertial frames are all equivalent: if there exist no privileged frame, we must abandon the idea of universal time.

Lorentz transformations “abridged”

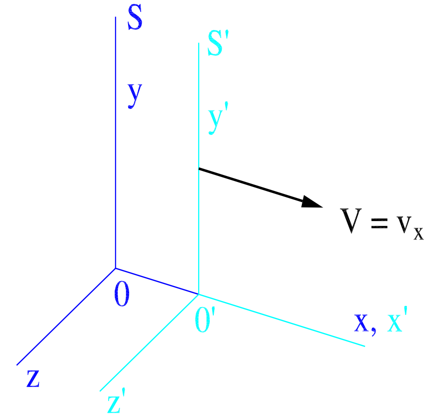
By assuming the speed of light constant in all reference frames, the Galilean transformations, implying the velocity addition rule, must be modified. The new transformations must reduce to the Galilean ones when the relative motion is slow ($V \ll c$). According to the first Einstein postulate, the empty space is **isotropic** (all direction are equivalent) and **homogeneous** (all points are equivalent); it would make no sense to postulate that the laws are invariant in a space which is not homogeneous and isotrope. Homogeneity implies **linearity**:

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ y' &= a_{21}x + a_{22}y + a_{23}z + a_{24}t \\ z' &= a_{31}x + a_{32}y + a_{33}z + a_{34}t \\ t' &= a_{41}x + a_{42}y + a_{43}z + a_{44}t \end{aligned}$$

16 unknowns!

For the case we are considering of motion along the common x-axis the coordinates y and z do not play a role and therefore it is reasonable to write [a]

$$
\begin{aligned}
x' &= a_{11}x + a_{14}t \\
y' &= y \\
z' &= z \\
t' &= a_{41}x + a_{44}t
\end{aligned}
$$



The origin of the S' frame is described in S by $x_0 = Vt$ and by definition it is x'_0=0 at any time. Therefore

$$0 = x'_0 = a_{11}x_0 + a_{14}t = a_{11}Vt + a_{14}t$$

that is $a_{14}/a_{11} = -V$ and

$$x' = a_{11}(x + a_{14}t/a_{11}) = a_{11}(x - Vt)$$

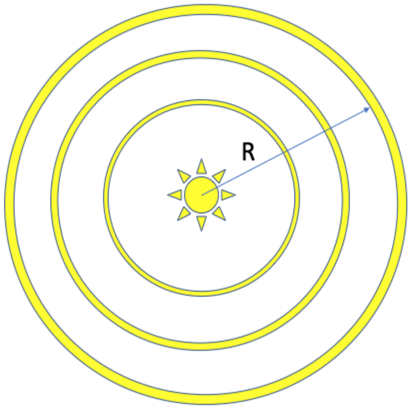
We are left with a_{11}, a_{41} and a_{44}.

[a] see script for complete derivation.

For finding the remaining 3 coefficients we resort to the fact that, according to the two postulates, the speed of light is the same in S and S' and that the wave equation is invariant in form.

Suppose an EM spherical wave leaves the origin of the frame S at t=0.

The propagation is described in S by the equation of a sphere which radius increases with time as



$$R^2 = x^2 + y^2 + z^2 = c^2t^2 \tag{4}$$

In S' the wave propagates with the same speed c and therefore

$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

which expressing the primed coordinates in terms of the un-primed ones becomes

$$a_{11}^2x^2 + a_{11}^2V^2t^2 - 2a_{11}xVt + y^2 + z^2 = c^2a_{41}^2x^2 + c^2a_{44}^2t^2 + 2a_{41}a_{44}xt$$

$$(a_{11}^2 - c^2a_{41}^2)x^2 + y^2 + z^2 - 2(Va_{11}^2 + c^2a_{41}a_{44})xt = (c^2a_{44}^2 - V^2a_{11}^2)t^2 \tag{5}$$

Comparing Eqs. 4 and 5 we get a system of 3 equations in the 3 unknown a_{11}, a_{41} and a_{44}.

Solution:

$$a_{11} = a_{44} = \gamma$$

$$a_{41} = -\gamma\beta/c$$

with

$$\beta \equiv V/c \quad \text{and} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

The coordinate transformation for a translational motion along $\hat{x}$ with (constant) velocity $\vec{V} = \hat{x}V$ are (Lorentz transformations)

$$t' = \frac{t - xV/c^2}{\sqrt{1 - V^2/c^2}} \equiv \gamma(t - Vx/c^2)$$

$$x' = \frac{x - Vt}{\sqrt{1 - V^2/c^2}} \equiv \gamma(x - Vt)$$

$$y' = y \qquad z' = z$$

The inverse transformations are obtained replacing V with $-V$.

The transformation can be written also in matrix form

$$\begin{pmatrix} \boldsymbol{ct'} \\ \boldsymbol{x'} \\ \boldsymbol{y'} \\ \boldsymbol{z'} \end{pmatrix} = \gamma \begin{pmatrix} \mathbf{1} & -\boldsymbol{\beta} & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{\beta} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \boldsymbol{ct} \\ \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} \equiv \boldsymbol{\mathcal{L}} \begin{pmatrix} \boldsymbol{ct} \\ \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix}$$

Some comments

$$t' = \frac{t - xV/c^2}{\sqrt{1 - V^2/c^2}} \equiv \gamma(t - Vx/c^2)$$

$$x' = \frac{x - Vt}{\sqrt{1 - V^2/c^2}} \equiv \gamma(x - Vt)$$

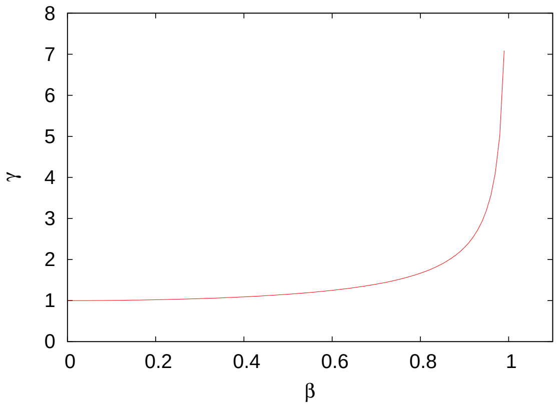
$$y' = y \qquad z' = z$$

- $V \ll c$ the Lorentz transformations reduce to the Galilean ones. Good!

- For $V > c$ the transformations are meaningless because the argument of the square root, $1 - V^2/c^2$, becomes negative!
  Therefore

$$0 \leq \beta \leq 1$$

$$1 \leq \gamma \leq +\infty$$



- The existence of a signal with $V > c$ would yield to a violation of the causality principle, as we will see.

Time is one of the 4 coordinates describing an event and as the space coordinates is subject to a (Lorentz) transformation between moving frames.

For space coordinates it is always possible if for instance $x_2 > x_1$ to find a new coordinates frame such that $x_2' < x_1'$.

Is it possible to find a Lorentz transformation which inverts the temporal order of events?

Assume an event happening at the time t_1 at the location x_1 in S and a second event happens at t_2 in x_2 with $t_2 > t_1$. Is it possible to find a Lorentz transformation such that $t_2' < t_1'$? In S' it is

$$ct_1' = \gamma(ct_1 - \beta x_1) \quad \text{and} \quad ct_2' = \gamma(ct_2 - \beta x_2)$$

$$c(t_2' - t_1') = \gamma[c(t_2 - t_1) - \beta(x_2 - x_1)]$$

Therefore $t_2' < t_1'$ if $\beta(x_2 - x_1) > c(t_2 - t_1)$, that is if $V(x_2 - x_1)/(t_2 - t_1) > c^2$. This may be possible depending on the values of $x_2 - x_1$ and $t_2 - t_1$. However if the first event in S *drives* the second one, x_2 and t_2 are not arbitrary.

If w is the speed of the signal which triggers the second event from the first one it is

$$x_2 - x_1 = w(t_2 - t_1)$$

$w \leq c$ and $V < c$

$$c(t_2' - t_1') = \gamma[c(t_2 - t_1) - \beta w(t_2 - t_1)] = \gamma c(t_2 - t_1)\left(1 - \frac{Vw}{c^2}\right) > 0$$

Causality is not violated.

Some consequences of Lorentz transformations: time dilation and relativity of simultaneity

Suppose a clock at rest in S measuring a time interval $t_2 - t_1$ between two events happening at the same location, $x_1 = x_2$. The time interval in the moving frame S' is measured by two different clocks because according to Lorentz transformations, the events happen in S' in different locations. The time difference in S' is

proper time (time measured by the same clock)

$$t'_2 - t'_1 = \gamma(t_2 - t_1) \geq t_2 - t_1$$

time dilation

The proper time interval is always the shortest.

Events which in S are simultaneous ($t_1 = t_2$), but happen in different places ($x_1 \neq x_2$), will be no more simultaneous in the moving frame S'

$$c(t'_2 - t'_1) = \gamma\beta(x_1 - x_2) \neq 0$$

Some consequences of Lorentz transformations: length contraction

Consider a rod of length L' along the x-axis and at rest in the moving frame S'.
The length in S is determined by the position of the rod ends at the same time $(t_1{=}t_2)$ and therefore

$$\underbrace{L'}_{\text{length at rest}} = x_2' - x_1' = \gamma(x_2 - x_1) = \gamma L \quad \rightarrow \quad L = L'/\gamma \qquad \text{length contraction}$$

However the length of a rod aligned with one of the two axis perpendicular to the direction of motion is invariant. Angle are in general not invariant.

The transformation for the components of the velocity, $\vec{v}$, are obtained from the coordinate transformations

$$v'_x \equiv \frac{dx'}{dt'} = \frac{dx - V\,dt}{dt - V\,dx/c^2} = \frac{v_x - V}{1 - v_x\beta/c}$$
$$v'_y \equiv \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - V\,dx/c^2)} = \frac{v_y}{\gamma(1 - v_x\beta/c)} \tag{6}$$
$$v'_z \equiv \frac{dz'}{dt'} = \frac{dz}{\gamma(dt - V\,dx/c^2)} = \frac{v_z}{\gamma(1 - v_x\beta/c)}$$

with $\beta \equiv V/c$, $v_x \equiv dx/dt$, $v_y \equiv dy/dt$ and $v_z \equiv dz/dt$.
Remember β $(=V/c)$ refers to the motion of the frame.

- As time is not invariant, despite the lengths perpendicular to the motion direction being unchanged, the time needed to cover them is different.
  - Unlike classical kinematics, the velocity components perpendicular to the motion, unless vanishing, are affected by the motion of the frame.
- For $v_x{=}c$ and $v_y{=}v_z{=}0$ it is

$$v'_x = \frac{c - V}{1 - V/c} = c\,\frac{c - V}{c - V} = c \quad \text{and} \quad v'_y = v'_z = 0$$

  For $v_y{=}c$ and $v_x{=}v_z{=}0$ it is $v'_x{=}-V$, $v'_y{=}c/\gamma$, $v'_z{=}0$ and

$$v'^2_x + v'^2_y = V^2 + c^2(1 - (V/c)^2) = c^2$$

In a similar way as for the velocity, it is possible to find the transformations for the acceleration $\vec{a}$

$$a'_x = \frac{a_x}{\gamma^3(1 - v_x\beta/c)^3}$$

$$a'_y = \frac{a_y}{\gamma^2(1 - v_x\beta/c)^2} + \frac{a_x v_y \beta/c}{\gamma^2(1 - v_x\beta/c)^3}$$

$$a'_z = \frac{a_z}{\gamma^2(1 - v_x\beta/c)^2} + \frac{a_x v_z \beta/c}{\gamma^2(1 - v_x\beta/c)^3}$$

- Acceleration in general is not invariant under Lorentz transformations.

A historical curiosity

The relativistic transformations were named by Poincaré after the dutch physicist Hendrik Lorentz who introduced them, before Einstein paper.

Lorentz had discovered that those transformations leave Maxwell equations invariant.

He had also introduced the notion of "local time" and of "contraction of bodies" for explaining the negative results of the Michelson-Morley experiment because he was convinced, as many other leading scientists, of the validity of the ether theory.

It seems that Einstein was not aware of Lorentz work... Anyway Einstein gave to the transformations a deep physical meaning making them extendable also to mechanics and causing a revolution of classical dynamics.

Experimental evidence of relativistic kinematics

Light aberration

Light aberration is the apparent motion of a light source due to the movement of the observer. It was first discovered in astronomy.
Source emitting photons at an angle θ wrt to the x-axis in the S frame where $v_y = c\sin\theta$ and $v_x = c\cos\theta$.
In S' it is $v'_y = c'\sin\theta'$ and $v'_x = c'\cos\theta'$.
Using Galilean transformations for the velocity components

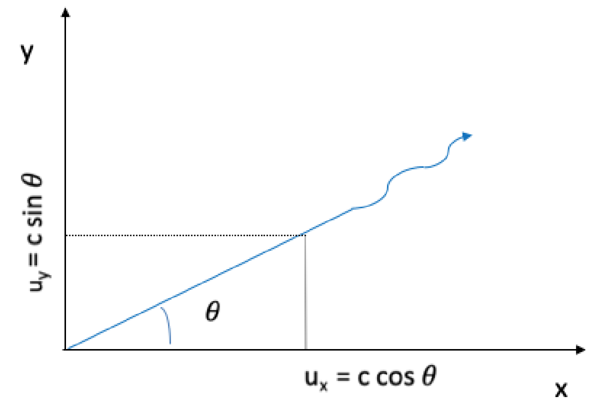
$$v'_y = v_y \text{ and } v'_x = v_x - V$$

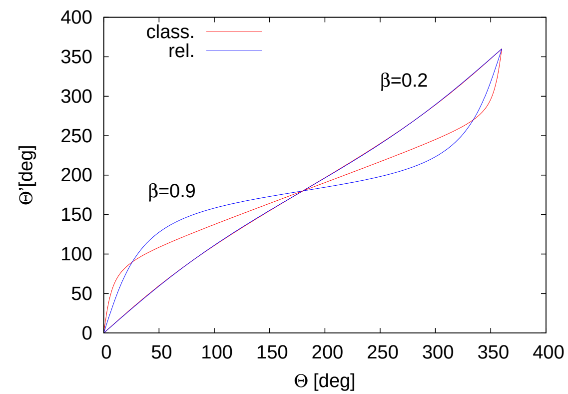
$$\tan\theta' = v'_y/v'_x = v_y/(v_x - V)$$

$$\tan\theta' = \frac{\sin\theta}{\cos\theta - \beta}$$

Using instead Lorentz transformations

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$



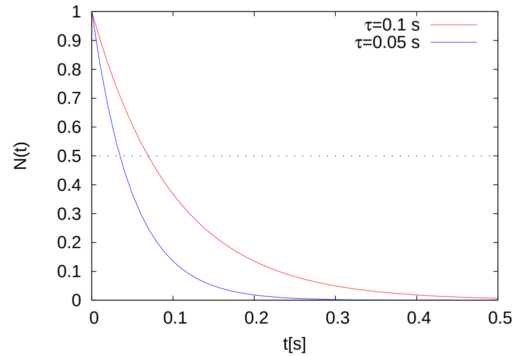


- High energy experiments involving emission of photons confirm the relativistic expression.

Lifetime of unstable particles

Beside e^-, p and n, in nature there are particles which are produced by scattering process and unlike e^+, $\bar{p}$ and $\bar{n}$, are "short-living". Their number decays in time as

$$N(t) = N_0 e^{-t/\tau}$$



The lifetime of charged pions at rest is τ_0=26×10^{-9} s. Time needed for the pions at rest to decay by half

$$N(t) = N_0 e^{-t/\tau} = \frac{N_0}{2} \quad \rightarrow \quad t = 18\ \mathrm{ns}$$

They are produced by bombarding a proper target by high energy protons and leave the target with $v \approx$2.97×10^8 m/s that is β=0.99 and $\gamma \approx$ 7. It is observed that they are reduced to the half after 37 m from the target. If their lifetime would be as at rest they should become the half already after about 5 m.

The experimental observation is explained if the pion lifetime in the laboratory frame is

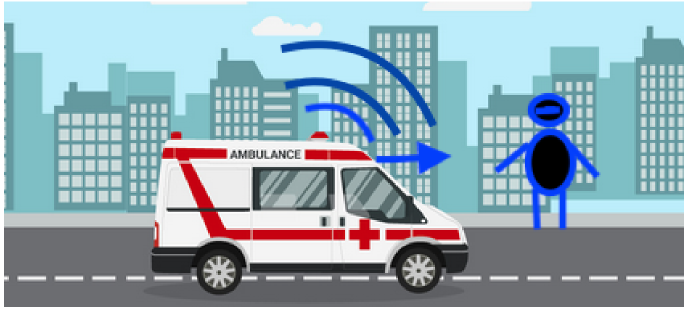
$$\tau = \gamma \tau_0$$

as predicted by time dilation.

Time dilation may allow us realizing future colliders smashing unstable particles like muons!

Doppler shift of light

Doppler effect exists also classically: we experience it when we hear the siren of a police car or an ambulance. The frequency perceived by an observer at rest is higher when the car is approaching because the number of the acoustic wave knots per unit time is larger, while the frequency decreases when the source is moving away. Classically there is no "transverse" (wave propagation direction perpendicular to motion) Doppler effect: in the moment the car is at the minimum distance it is $\boldsymbol{\Delta}\omega$=0.



Relativistically for a light wave the situation source ($\boldsymbol{S}$) or receiver ($\boldsymbol{R}$) in motion are identical. When the angle, $\boldsymbol{\theta}$, between wave propagation direction and motion is 0 the angular frequency is

$$\omega = \omega_0 \sqrt{\frac{1-\beta}{1+\beta}} \quad \text{with } \beta < 0 \text{ for } \boldsymbol{R} \text{ and } \boldsymbol{S} \text{ approaching, } \beta > 0 \text{ when they move away}$$

In addition because of the time dilation there is also a transverse Doppler effect.

For $\boldsymbol{\theta}$=90$^{\mathbf{0}}$ in the source frame

$$\boldsymbol{\Delta}\omega = \omega_0(\gamma - 1)$$

This was predicted by Einstein who suggested an experiment using hydrogen ions for measuring it. The experiment realized for the first time by Ives and Stilwell in 1938 proved the correctness of Einstein prediction.

Relativistic Dynamics

Assuming $\vec{F}$ invariant and m constant, Newton law, $\vec{F} = m\vec{a}$, is not invariant under Lorentz transformations because we have seen that $\vec{a}$ is not invariant.

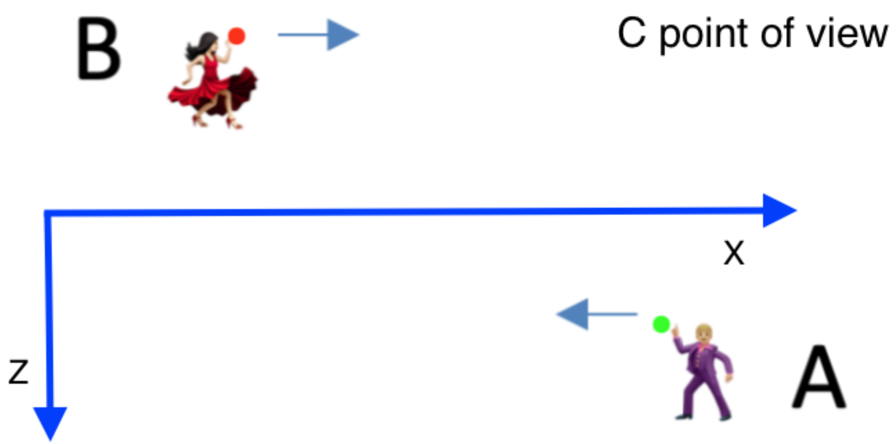
In addition mass can't be a constant because by applying a constant force to an object its speed would increase indefinitely becoming larger than c.

- Classical mechanics must be modified to achieve invariance under Lorentz transformations.
- The new expressions must reduce to the classical ones for $v/c \ll 1$.

In the 1905 paper, Einstein used the Lorentz force and the electro-magnetic field transformations to achieve the generalization of the definition of momentum and energy.

In 1909 two MIT professors of chemistry, Lewis and Tolman, suggested a different more straightforward approach involving purely mechanical arguments.

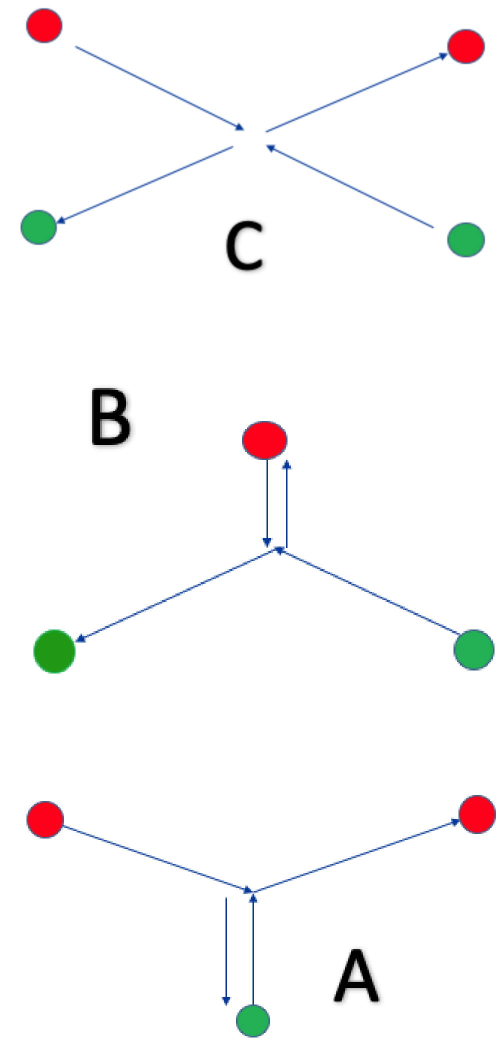
Let's assume there are two observers, Alex and Betty, moving towards each other with the same speed as seen by a third observer, Charlie.
Betty sits in S and Alex in S'.
Alex and Betty have identical elastic balls.



Betty releases the red ball with v_x^B=0 and v_z^B=$u \neq 0$, while Alex releases the green ball with speed $v_x'^A$=0 and $v_z'^A$ numerically equal and opposite to the red ball velocity, that is

$$v_z'^A = -u$$

Green ball in S' Red ball in S



The experiment is set so up that the two balls collide and rebound.

Now let's consider Betty point of view. For Betty it is

$$\Delta p_x^B = 0 \quad \Delta p_z^B = 2m_B u$$

$$\Delta p_x^A = 0 \quad \Delta p_z^A = 2m_A v_z^A$$

We need here the inverse velocity transformation because we know the numerical value of the z direction component in the moving frame S'

$$v_z = \frac{v_z'}{\gamma(1 + v_x'\beta/c)}$$

In our case v_x'=0 and v_z'=$-u$ and therefore

$$v_z^A = v_z'^A/\gamma = -u/\gamma \qquad \text{with } \gamma = \frac{1}{\sqrt{1 - (v_x^A/c)^2}}$$

Momentum is conserved if

$$\Delta p_z^B = -\Delta p_z^A$$

that is

$$2m_B u = -2m_A v_z^A = 2m_A \frac{u}{\gamma} \quad \rightarrow \quad m_A = \gamma m_B$$

We may assume that u is small so that m_B is the mass at rest, m_0, and $m_A = m(v)$.
So we have found that

$$m(v) = \gamma m_0$$

We can keep the momentum definition from classical dynamics by giving up the invariance of mass.
Relativistically mass is not conserved.
A clear example is the annihilation of a e^+e^- pair into 2 photons.

Let's try modifying the classical Newton law

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

into

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m_0 \vec{v}) = m_0 \vec{v}\frac{d\gamma}{dt} + m_0\gamma\frac{d\vec{v}}{dt}$$

$\vec{F}$ and $\vec{a}$ are not parallel!

By scalar multiplication by $\vec{v}$ it is

$$\vec{F} \cdot \vec{v} = \vec{v} \cdot \frac{d\vec{p}}{dt}$$

work/unit time $= dE/dt$

$$\vec{v} \cdot \frac{d\vec{p}}{dt} = m_0 \gamma \vec{v} \cdot \frac{d\vec{v}}{dt} + m_0 \frac{v^2}{c^2} \gamma^3 v \frac{dv}{dt} = m_0 \gamma v (1 + \frac{v^2 \gamma^2}{c^2}) = m_0 \gamma^3 v \frac{dv}{dt}$$

that is

$$\frac{dE}{dt} = \vec{F} \cdot \vec{v} = m_0 \gamma^3 v \frac{dv}{dt}$$

It is easy to verify that this equation is satisfied by defining the energy as

$$E = mc^2 = \gamma m_0 c^2$$

For v=0 it is E_0=$m_0 c^2$ which is the energy at rest.

The (relativistic) kinetic energy is obtained by subtracting the rest energy from the total energy

$$T = mc^2 - m_0 c^2 = m_0 c^2 (\gamma - 1) \neq \frac{1}{2} \gamma m_0 v^2$$

which gives the classical kinetic energy $T \simeq m_0 v^2/2$ for $v \ll c$.

Measurement of T and $\vec{v}$ relationship

Experiments confirmed the validity of the relativistic relationship between T and $\vec{v}$.
Bertozzi experiment measured directly the velocity of e^- accelerated in a linear accelerator.

- e^- speed was measured through the time of flight.
- Kinetic energy computation relied on the knowledge of the accelerating field and on the measurement of the heat deposited at the aluminum target.

The results also show clearly the presence of a limit speed, c.

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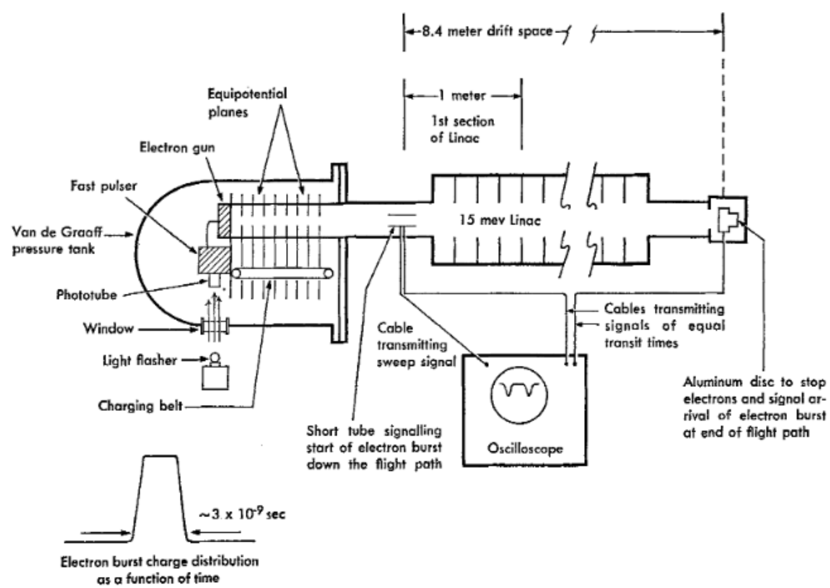
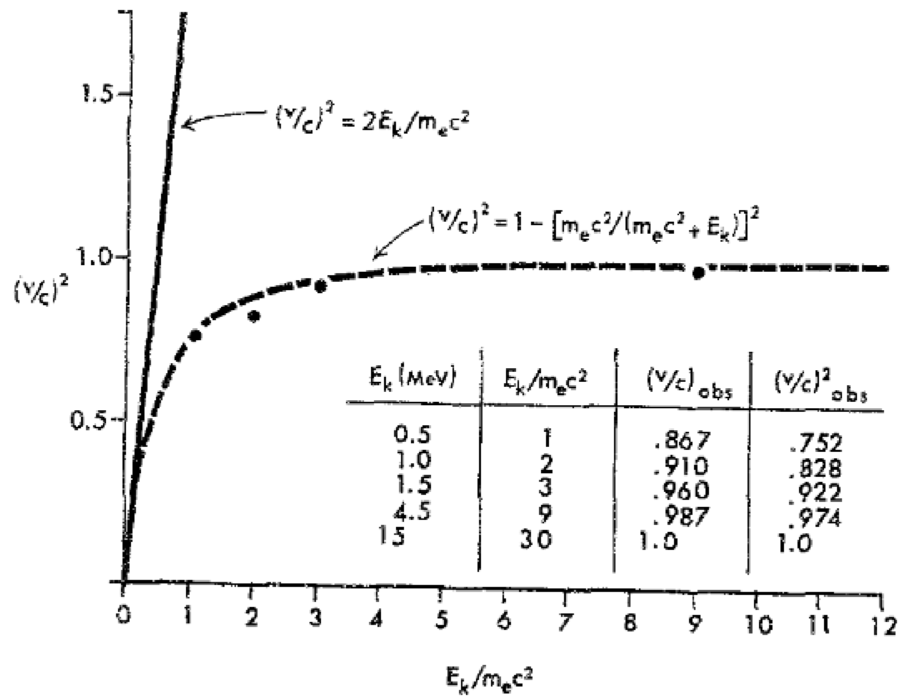


FIG. 1. Schematic diagram of the experiment set up for measuring the time of flight of the electron burst from the Van de Graaff.



E_k (MeV)	E_k/m_ec^2	$(v/c)_{obs}$	$(v/c)^2_{obs}$
0.5	1	.867	.752
1.0	2	.910	.828
1.5	3	.960	.922
4.5	9	.987	.974
15	30	1.0	1.0

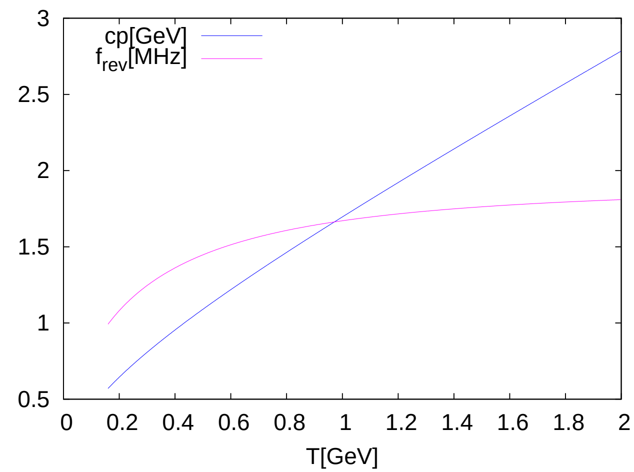
$m_e = e^-$ rest mass

Importance of relativity for accelerators

Example of CERN PS Booster.

Circumference: L=157 m.

Particles are injected from Linac4 with $\boldsymbol{T}$=160 MeV and accelerated to $\boldsymbol{T}$=2 GeV.



- The dipole field must be ramped up according to momentum for keeping the particles on the design orbit ($\boldsymbol{\rho = p/eB}$).
- $\boldsymbol{f_{rf} = h f_{rev}}$. For large γ it is $\boldsymbol{f_{rf} \approx h \frac{c}{L}(1 - \frac{1}{2\gamma^2})}$
  $\rightarrow$ almost constant at high energy as the speed approaches $\boldsymbol{c}$.
  - Particularly true for $\boldsymbol{e^{\pm}}$ which have 1836 larger γ for the same energy.

Relativity has basic relevance for accelerators!

Transformations of momentum, energy and force

The transformations for momentum and energy follow from the definition $\vec{p}=m\vec{v}$ and from the transformations of the velocity. The result is

$$p'_x = \gamma_V (p_x - E\, V/c^2) \qquad \text{with} \quad \gamma_V = \frac{1}{\sqrt{1 - V^2/c^2}}$$

$\gamma_v m_0 c^2$ frame speed

$$p'_y = p_y \qquad p'_z = p_z$$

$$E' = \frac{E - V\, p_x}{\sqrt{1 - V^2/c^2}}$$

The transformations have the same form as the coordinates transformations with

$$\vec{r} \to \vec{p} \qquad \text{and} \quad t \to E/c^2$$

- Vectors which 4 components transform according to Lorentz transformations are called *4-vectors*.
  - $(E/c, \vec{p})$ is therefore a 4-vector.

4-vectors

Classically time intervals and distances are invariant. This is not true in relativity. However the interval defined as

$$(ds)^2 \equiv [d(ct)]^2 - (dx)^2 - (dy)^2 - (dz)^2$$

is invariant under Lorentz transformations, ie it has the same value in any frame. It can be easily proven by using the Lorentz transformations.

In general for any 4-vector the quantities

$$A^\nu B_\nu \equiv A_0 B_0 - (A_x B_x + A_y B_y + A_z B_z)$$

and in particular

$$A^\nu A_\nu = A_0^2 - (A_x^2 + A_y^2 + A_z^2)$$

are invariant.

Let us consider a particle moving with velocity $\vec{v}(t)$, non necessarily uniform, in S. The time interval $d\tau$ evaluated in a inertial frame S' where the particle is instantaneously at rest is called proper time. It is related to the time measured in S by

$$d\tau = \sqrt{1 - v^2/c^2}dt \equiv \frac{dt}{\gamma}$$

and for a finite time interval

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1 - v^2/c^2}}$$

The proper time is *by definition* an invariant. This results also from the fact that $c^2 d\tau^2$ is the invariant ds^2 evaluated in the frame where the particle is instantaneously at rest (dx=dy=dz=0).

Owing to the fact that the proper time interval $d\tau = dt/\gamma$ is an invariant and that (cdt, dx, dy, dz) transforms obviously as (ct, x, y, z), the quantity defined as

$$\left(\frac{c\,dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = \left(\gamma\frac{c\,dt}{dt}, \gamma\frac{dx}{dt}, \gamma\frac{dy}{dt}, \gamma\frac{dz}{dt}\right) \equiv (\gamma c, \gamma\vec{v}) \tag{7}$$

transforms according to Lorentz transformations (*4-velocity*). Multiplying the 4-velocity by the rest mass we get

$$m_0(\gamma c, \gamma\vec{v}) = (E/c, \vec{p})$$

which is also a 4-vector (*energy-momentum* or *4-momentum vector*) and therefore transforms according to Lorentz transformation.

Relativistic energy and momentum are closely connected, they are components of the same 4-vector.

- The quantity $(E/c)^2 - (p_x^2 + p_y^2 + p_z^2)$ is invariant.
- If energy and momentum are conserved in one inertial frame of reference they are conserved in all inertial frames (a vector which components are all zero has zero components in any frame).
- If momentum in conserved in two inertial frames, energy too is conserved in both frames.

Newton and Minkowski force

We may write the relativistic Newton law $\vec{F}=d\vec{p}/dt$ in terms of 4-vectors. In the particle proper frame

$$\frac{dp^\nu}{d\tau} = f^\nu$$

with $(p^0, p^1, p^2, p^3) \equiv (E/c, p_x, p_y, p_z)$ and $(f^0, f^1, f^2, f^3) \equiv (f^0, f_x, f_y, f_z)$.
The l.h.s. is a 4-vector and therefore also $\vec{f}$, the Minkowski force, on the r.h.s. must be a 4-vector related to the Newton force $\vec{F}$.

The space part of the equation of motion is

$$\frac{d\vec{p}}{d\tau} = \vec{f} \qquad \rightarrow \qquad \gamma\frac{d\vec{p}}{dt} = \vec{f} \qquad \rightarrow \qquad \vec{f} = \gamma\vec{F}$$

The time part of the equation is

$$f^0 = \frac{dp^0}{d\tau} = \frac{1}{2p^0}\frac{d(p^0)^2}{d\tau} = \frac{1}{2p^0}\frac{d(E/c)^2}{d\tau}$$

The invariance of $(E/c)^2 - \vec{p}\cdot\vec{p} = (m_0 c)^2$ implies that

$$0 = \frac{d}{d\tau}\left[\left(\frac{E}{c}\right)^2 - \vec{p}\cdot\vec{p}\right] = \frac{d}{d\tau}\left(\frac{E}{c}\right)^2 - \frac{d}{d\tau}(\vec{p}\cdot\vec{p})$$

$$\rightarrow \quad \frac{d}{d\tau}\left(\frac{E}{c}\right)^2 = 2\vec{p}\cdot\frac{d\vec{p}}{d\tau}$$

which inserted in the equation for f_0 gives

$$f^0 = \frac{1}{2p^0}\frac{d(E/c)^2}{d\tau} = \frac{1}{2p^0}2\vec{p}\cdot\frac{d\vec{p}}{d\tau} = \frac{m_0\gamma\vec{v}}{E/c}\cdot(\gamma\vec{F}) = \gamma\vec{\beta}\cdot\vec{F}$$

The Minkowski force is therefore

$$(f^0, f^1, f^2, f^3) = (\gamma\vec{\beta}\cdot\vec{F}, \gamma\vec{F})$$

In absence of external forces ($\vec{F}$=0) the Minkowski force components are vanishing and momentum and energy are conserved.

Being a 4-vector, Minkowski force transforms following Lorentz transformations.
Attention must be paid to distinguish between the particle velocity, $\vec{v}$, in the S frame and the frames relative speed $\vec{V}$.
Using the general expression of Lorentz transformations (see lecture script) we have

$$f'^0 = \gamma_V \left(f^0 - \vec{\beta}_V \cdot \vec{f}\right)$$
$$\vec{f}\,' = \vec{f} + \frac{\gamma_V - 1}{\beta_V^2}(\vec{\beta}_V \cdot \vec{f})\vec{\beta}_V - \gamma_V f^0 \vec{\beta}_V$$

The Newton force transformation writes

$$\gamma\,'\vec{F}' = \gamma\vec{F} + \frac{\gamma_V - 1}{\beta_V^2}\left[\vec{\beta}_V \cdot (\gamma\vec{F})\right]\vec{\beta}_V - \gamma_V \vec{\beta}_V (\gamma\vec{\beta} \cdot \vec{F})$$

The inverse transformation is obtained by replacing $\vec{\beta}_V$ with $-\vec{\beta}_V$

For $V \ll c$ ($\beta_V \to 0$ and $\gamma_V \to 1$) it is $\vec{F}'$=$\vec{F}$ which is the classical result.

For the translational motion along $\hat{x}$ the transformations write

$$\boldsymbol{F'_x = F_x - \frac{v_y V}{c^2 - v_x V} F_y - \frac{v_z V}{c^2 - v_x V} F_z}$$

$$\boldsymbol{F'_{y,z} = \frac{\sqrt{1 - V^2/c^2}}{1 - v_x V/c^2} F_{y,z}}$$

If the force $\vec{\boldsymbol{F}}$ is acting on a particle which is instantaneously at rest in $\boldsymbol{S}$ ($\boldsymbol{v}$=0), the transformations simplify

$$\boldsymbol{F'_x = F_x \qquad F'_y = \frac{1}{\gamma} F_y \qquad F'_z = \frac{1}{\gamma} F_z}$$

Transformations of EM fields

The transformations are found by applying the force transformations to the force experienced by a charged particle moving with velocity $\vec{v}$ in an EM field

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \qquad \text{(Lorentz force)}$$

The corresponding Minkowski force is

$$f^{\nu} = \left(\gamma\vec{\beta}\cdot\vec{F}, \gamma\vec{F}\right) = q\left[\gamma\vec{\beta}\cdot\left(\vec{E} + \vec{v}\times\vec{B}\right), \gamma\left(\vec{E} + \vec{v}\times\vec{B}\right)\right]$$

with $\gamma = 1/\sqrt{1-(v/c)^2}$. This equation can be written in matrix form

$$\begin{pmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{pmatrix} = \frac{q}{c}\begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix}\begin{pmatrix} \gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$$

In the moving frame S' the Minkowski force will be expressed in the same form in terms of primed quantities.

Minkowski force and the 4-velocity $(\gamma c, \gamma\vec{v})$ are 4-vectors. Using the Lorentz transformation $\mathcal{L}$ from S to S' and $\mathcal{L}^{-1}$ from S' to S we get

$$\begin{pmatrix} f'^0 \\ f'^1 \\ f'^2 \\ f'^3 \end{pmatrix} = \mathcal{L} \begin{pmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{pmatrix} = \frac{q}{c}\mathcal{L} \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix} \mathcal{L}^{-1} \begin{pmatrix} \gamma' c \\ \gamma' v'_x \\ \gamma' v'_y \\ \gamma' v'_z \end{pmatrix}$$

Requiring that the Minkowski force in S' has the same form as in S, it must be

$$\begin{pmatrix} 0 & E'_x & E'_y & E'_z \\ E'_x & 0 & cB'_z & -cB'_y \\ E'_y & -cB'_z & 0 & cB'_x \\ E'_z & cB'_y & -cB'_x & 0 \end{pmatrix} = \mathcal{L} \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix} \mathcal{L}^{-1}$$

By carrying out the matrix multiplications and equating the same indices elements, the field components in S' are found:

$$
\begin{aligned}
E'_x &= E_x & B'_x &= B_x \\
E'_y &= \gamma_V (E_y - V B_z) & B'_y &= \gamma_V \left(B_y + \frac{V}{c^2} E_z\right) \\
E'_z &= \gamma_V (E_z + V B_y) & B'_z &= \gamma_V \left(B_z - \frac{V}{c^2} E_y\right)
\end{aligned}
$$

Denoting by "parallel" and "normal" the fields components wrt to direction of motion the field transformations can be written in the general form

$$
\begin{aligned}
E'_{\|} &= E_{\|} & B'_{\|} &= B_{\|} \\
E'_{\perp} &= \gamma(\vec{E} + \vec{V} \times \vec{B})_{\perp} & B'_{\perp} &= \gamma(\vec{B} - \vec{V} \times \vec{E}/c^2)_{\perp}
\end{aligned}
$$

For the inverse transformations $\vec{V}$ mus be replaced by $-\vec{V}$.

Transformation of Source Distributions

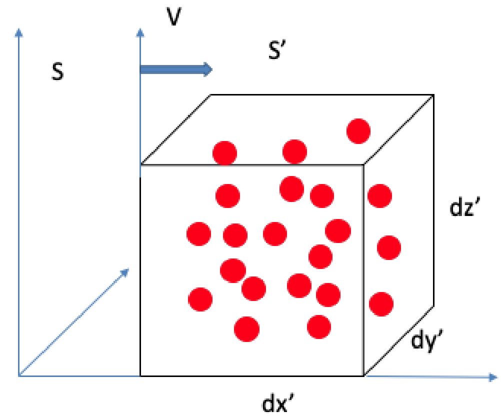
Let us consider a distribution of charges at rest in S'. The charge density is given by

$$\rho'(x', y', z', t') = \frac{qN}{dx' dy' dz'}$$

In S, moving with velocity $-V$ wrt S', the volume element is

$$dx\,dy\,dz = \frac{dx'}{\gamma} dy' dz'$$

length contraction



Charge density in S

$$\rho = \frac{qN}{dx\,dy\,dz} = \gamma \frac{qN}{dx'\,dy'\,dz'} = \gamma\rho'$$

As the charge distribution moves in S with velocity $+\hat{x}V$, in S there is a current with density

$$j_x = \rho V = \gamma\rho' V \qquad \text{(in general: } \vec{j} = \rho\vec{V} = \gamma\rho'\vec{V}\text{)}$$

Multiplying the 4-velocity by the charge density at rest ρ_0 we get the 4-vector

$$\rho_0(\gamma c, \gamma\vec{V}) = (\rho c, \rho\vec{V}) = (\rho c, \vec{j})$$

(charge-current 4-vector).

Invariance of Maxwell Equations

Knowing how fields and sources transform one can prove that Maxwell equations are invariant under Lorentz transformation. This was demonstrated by Lorentz before Einstein formulated the special relativity theory.

We want to show that if the Maxwell equations hold good in S, they hold with the same form also in S'.

For example let'us prove that

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \Rightarrow \qquad \nabla' \cdot \vec{E}' = \frac{\rho'}{\epsilon_0}$$

The partial derivatives in S' and in S are related by the cyclic rule

$$\frac{\partial}{\partial ct'} = \frac{\partial ct}{\partial ct'}\frac{\partial}{\partial ct} + \frac{\partial x}{\partial ct'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial ct'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial ct'}\frac{\partial}{\partial z} = \gamma\left(\frac{\partial}{\partial ct} + \beta\frac{\partial}{\partial x}\right)$$

$$\frac{\partial}{\partial x'} = \frac{\partial ct}{\partial x'}\frac{\partial}{\partial ct} + \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x'}\frac{\partial}{\partial z} = \gamma\left(\beta\frac{\partial}{\partial ct} + \frac{\partial}{\partial x}\right)$$

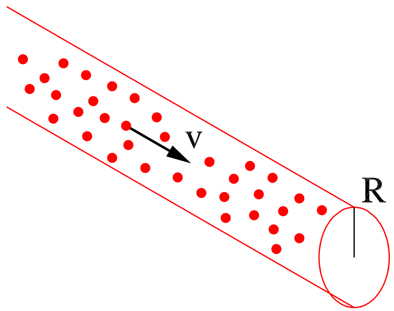
$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \qquad\qquad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$$

By using these expressions, the field transformations and the fact that Maxwell equation hold good in $\boldsymbol{S}$, we find

$$
\begin{aligned}
\nabla' \cdot \vec{E}' &= \frac{\partial E'_x}{\partial x'} + \frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'} \\
&= \gamma \frac{\partial E'_x}{\partial x} + \frac{\partial E'_y}{\partial y} + \frac{\partial E'_z}{\partial z} + \gamma\beta \frac{\partial E'_x}{\partial ct} \\
&= \gamma \frac{\partial E_x}{\partial x} + \gamma \frac{\partial E_y}{\partial y} + \gamma \frac{\partial E_z}{\partial z} - \gamma V \frac{\partial B_z}{\partial y} + \gamma V \frac{\partial B_y}{\partial z} + \gamma\beta \frac{\partial E_x}{\partial ct} \\
&= \gamma \nabla \cdot \vec{E} - \gamma V \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \gamma\beta \frac{\partial E_x}{\partial ct} \\
&= \gamma \frac{\rho}{\epsilon_0} - \gamma V \left(\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right)_x \\
&= \gamma \frac{\rho}{\epsilon_0} - \gamma V \frac{j_x}{\epsilon_0 c^2} \\
&= \frac{\gamma}{\epsilon_0 c} \left(\rho c - \beta j_x \right) \\
&= \frac{\rho'}{\epsilon_0}
\end{aligned}
$$

Forces between moving charges

Let us consider an uniform cylindrical beam of radius R of equally charged particles moving with velocity $\vec{v}=\hat{x}v$ in S.
In the reference frame S' where the particles are at rest there is no magnetic field and the electric force acting on each charge is purely radial and repulsive.



Inside the beam ($r' \leq R$) it is

$$F'_{r'} = qE'_{r'} = \frac{1}{2\pi\epsilon_0 R^2} q^2 \lambda' r' \qquad \lambda' = \text{ line density in } S'$$

By using the Newton force transformation

$$F_x = F'_x \quad F_y = F'_y/\gamma \qquad F_z = F'_z/\gamma$$

we get

$$F_{\|} = 0 \qquad F_r = \frac{1}{\gamma} F'_{r'} = \frac{1}{2\pi\epsilon_0 R^2} q^2 \lambda r \frac{1}{\gamma^2} \qquad \lambda = \gamma\lambda' \quad \text{(length contraction!)}$$

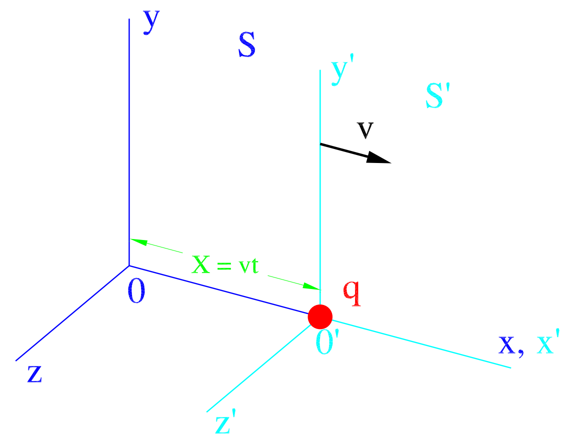
In the reference frame S the force is still radial and repulsive, but it is reduced by a factor $1/\gamma^2$.

Beam in accelerators may be approximated by a uniform cylindrical charge distribution, the repulsive force between the equally charged particles of the beam becomes smaller at high energy.

The field of a moving charge

The EM fields generated by a charge at rest in the origin of the S' frame is

$$\vec{E}' = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}'}{r'^3}$$
$$\vec{B}' = 0$$



The electric field components in S where the particle is uniformly moving are (inverse field transformations with $\vec{B}'$=0)

$$E_x = E'_x = \frac{q}{4\pi\epsilon_0}\frac{x'}{r'^3} \qquad E_y = \gamma E'_y = \gamma\frac{q}{4\pi\epsilon_0}\frac{y'}{r'^3} \qquad E_z = \gamma E'_z = \gamma\frac{q}{4\pi\epsilon_0}\frac{z'}{r'^3}$$

Primed particle coordinates in terms of the unprimed coordinates in S

$$x' = \gamma(x - vt) \qquad y' = y \qquad z' = z$$

Electric field components in S in terms of S coordinates are

$$E_x = \frac{q}{4\pi\epsilon_0} \frac{\gamma(x - vt)}{[\gamma^2(x - vt)^2 + y^2 + z^2]^{3/2}}$$
$$E_y = \frac{q}{4\pi\epsilon_0} \frac{\gamma y}{[\gamma^2(x - vt)^2 + y^2 + z^2]^{3/2}}$$
$$E_z = \frac{q}{4\pi\epsilon_0} \frac{\gamma z}{[\gamma^2(x - vt)^2 + y^2 + z^2]^{3/2}}$$

In S the particle is moving, in S there is also a magnetic field.
Using the magnetic field transformations

$$0 = \vec{B}' = \vec{B}_{||} + \gamma_V \left(\vec{B}_{\perp} - \frac{\vec{V}}{c^2} \times \vec{E}\right)$$

$\perp$ to motion

which means

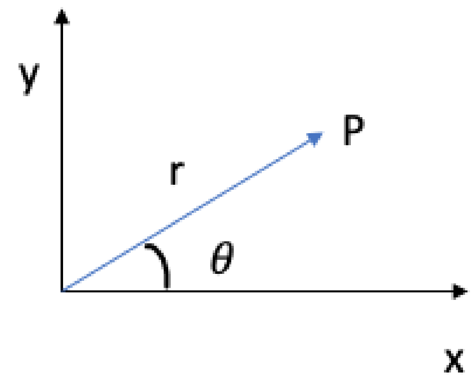
$$\vec{B}_{||} = 0$$
$$\vec{B}_{\perp} = \frac{1}{c^2} \vec{v} \times \vec{E}$$

We may evaluate the electric field at the time $t = 0$ [a]

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\gamma \vec{r}}{[\gamma^2 x^2 + y^2 + z^2]^{3/2}}$$

Denoting with θ the angle between the $\hat{x}$-axis and $\vec{r}$ and using the relationship

$$\gamma^2 x^2 + y^2 + z^2 = \gamma^2 r^2 (1 - \beta^2 \sin^2 \theta)$$



we get

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2 \theta)^{3/2}} \frac{\hat{r}}{r^2}$$

- The electric field is still radial and follows the $1/r^2$ law, but has no more a spherical symmetry.
- The magnetic field is perpendicular to the plane defined by $\vec{r}$ and $\vec{v}$.

In accelerators, particles are often "ultra-relativistic" that is their speed in the laboratory frame is almost c. For $\beta \to 1$ it is $\vec{E} \to 0$, unless θ=90^0 or 270^0 where the field is enhanced by a factor γ

$$\frac{1 - \beta^2}{(1 - \beta^2)^{3/2}} = \gamma$$

[a] At a different time $\bar{t}$ the fields take at (x, y, z) the same values as at $(x - v\bar{t}, y, z)$ for t=0.

The CM Energy

The center of momentum for an isolated ensemble of particles is defined as the inertial frame where it holds

$$\sum_i \vec{p}_i = \sum_i \frac{m_{0,i}\vec{v}_i}{\sqrt{1 - V^2/c^2}} = 0$$

frame velocity

We have seen that $(E/c)^2 - |\vec{p}|^2 = m_0^2 c^2$.
For the total energy and momentum of the ensemble

$$E = \sum_i E_i \quad \text{and} \quad \vec{P} = \sum_i \vec{p}_i$$

total energy

total momentum

the invariant in the CM frame is just the total energy in the CM

energy in CM

$$\left(\sum_i E_i/c\right)^2 - \sum_i \vec{p}_i \cdot \sum_i \vec{p}_i = \left(\sum_i E_i'/c\right)^2$$

Let us consider two simple cases:

a) two ultra-relativistic particles colliding "head-on";
b) one ultra-relativistic particle hitting a particle at rest.

For the system of two particles it is

$$\frac{(E_1' + E_2')^2}{c^2} = \frac{(E_1 + E_2)^2}{c^2} - (\vec{p}_1 + \vec{p}_2) \cdot (\vec{p}_1 + \vec{p}_2)$$
$$= \frac{(E_1 + E_2)^2}{c^2} - p_1^2 - p_2^2 - 2\vec{p}_1 \cdot \vec{p}_2$$

Moreover for ultra-relativistic particles it is

$$p = mv \simeq mc = \frac{E}{c}$$

a) $\vec{p}_1/p_1 = -\vec{p}_2/p_2$

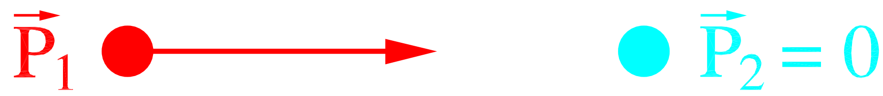
$$\frac{(E_1' + E_2')^2}{c^2} = \frac{E_1^2}{c^2} + \frac{E_2^2}{c^2} + 2\frac{E_1 E_2}{c^2} - \frac{E_1^2}{c^2} - \frac{E_2^2}{c^2} + 2\frac{E_1 E_2}{c^2} = 4\frac{E_1 E_2}{c^2}$$

and thus

$$E_1' + E_2' = 2\sqrt{E_1 E_2}$$

LHC (p/p): E_1=E_2=6.5 TeV $\rightarrow$ energy in the center of mass $E_1' + E_2'$=2×6.5=13 TeV.
HERA ($p/e^{\pm}$): E_1=920 GeV and E_2=27.5 GeV $\rightarrow$ $E_1' + E_2'$=318 GeV.

b) $\vec{p}_2 = 0$ and $E_2 = m_{0,2}c^2$



$$\frac{(E'_1 + E'_2)^2}{c^2} = \frac{(E_1 + E_2)^2}{c^2} - p_1^2 - p_2^2 - 2\vec{p}_1 \cdot \vec{p}_2$$

$$\frac{(E'_1 + E'_2)^2}{c^2} = \frac{E_1^2}{c^2} + \frac{E_2^2}{c^2} + 2\frac{E_1 E_2}{c^2} - \frac{E_1^2}{c^2} = \frac{E_2^2}{c^2} + 2\frac{E_1 E_2}{c^2}$$

and therefore

$$(E'_1 + E'_2) = \sqrt{E_2(E_2 + 2E_1)} = \sqrt{E_2(m_{0,2}c^2 + 2E_1)} \simeq \sqrt{2E_1 E_2}$$

For example, with $E_2 = 0.938$ GeV (proton rest mass) to get in the CM an energy of 318 GeV must be E_1 =54 TeV.

From this example we see the advantage of collider experiments wrt. fixed target ones, beam intensity permitting...

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