A first taste of Non-Linear Beam Dynamics

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Disclaimer

- These lectures are largely based on the lectures of **A. Wolski** (University of Liverpool) from the CAS 2016 on “Introduction to Accelerator Physics” at Budapest, and on the lectures of **Y. Papaphilippou** on “A first taste of Non-Linear Beam Dynamics” from the CAS 2019 on “Introduction to Accelerator Physics” at Vysoké Tatry.

Purpose of the lecture

- Introducing aspects of non-linear dynamics
  - **Mathematical tools** for modelling nonlinear dynamics
    - Power series (Taylor) maps and symplectic maps
  - Effects of **nonlinear perturbations**
    - Resonances, tune shifts, dynamic aperture
  - **Analysis** methods
    - Normal forms, frequency map analysis

- Employ two types of accelerator systems for illustrating methods and tools
  - **Bunch compressor** (single-pass system)
  - **Storage ring** (multi-turn system)

Aim of the 1st Lecture

- Provide an introduction to some of the **key concepts** of nonlinear dynamics in particle accelerators
- Describe some of the **sources** of nonlinearities
- Outline some of the **tools** used for modelling
- Explain the significance and potential **impact** of nonlinear dynamics in some accelerator systems

From Linear to Non-linear

From Linear to Non-linear

- Particle motion through simple components such as drifts, dipoles and quadrupoles can be represented by **linear transfer maps**
- For example, in a drift space of length L, the **horizontal coordinate** and the (scaled) **horizontal momentum** from initial position 0 to a final position 1 are

$$
\begin{aligned}
x_1 &= x_0 + L p_{x0} \\
p_{x1} &= p_{x0}
\end{aligned}
$$

- Note that the horizontal momentum is

$$p_x = \frac{\gamma m v_x}{P_0} \approx \frac{dx}{ds}$$

where γ is the relativistic factor, m is the rest mass of the particle, v_x is the horizontal velocity, and P_0 is the reference momentum

From Linear to Non-linear

- **Linear transfer maps** can be written in terms of **matrices** and for example for a drift space of length L

$$\begin{pmatrix} x_1 \\ p_{x1} \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ p_{x0} \end{pmatrix}$$

- In general, a **linear transformation** can be written as

$$\vec{x}_1 = R\vec{x}_0 + \vec{A}$$

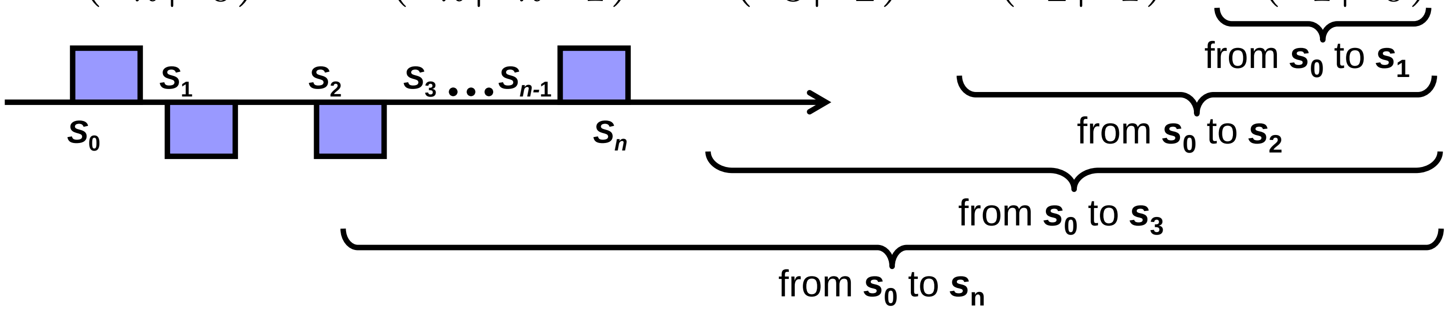
where the phase space vectors are $\vec{x} = (x, p_x)$

- The transfer matrix R and the vector $\vec{A}$ are constants, i.e. they do no depend on $\vec{x}_0$

From Linear to Non-linear

- The transfer matrix for a section of beamline can be found by **multiplying** the **transfer matrices** for the accelerator components within that section

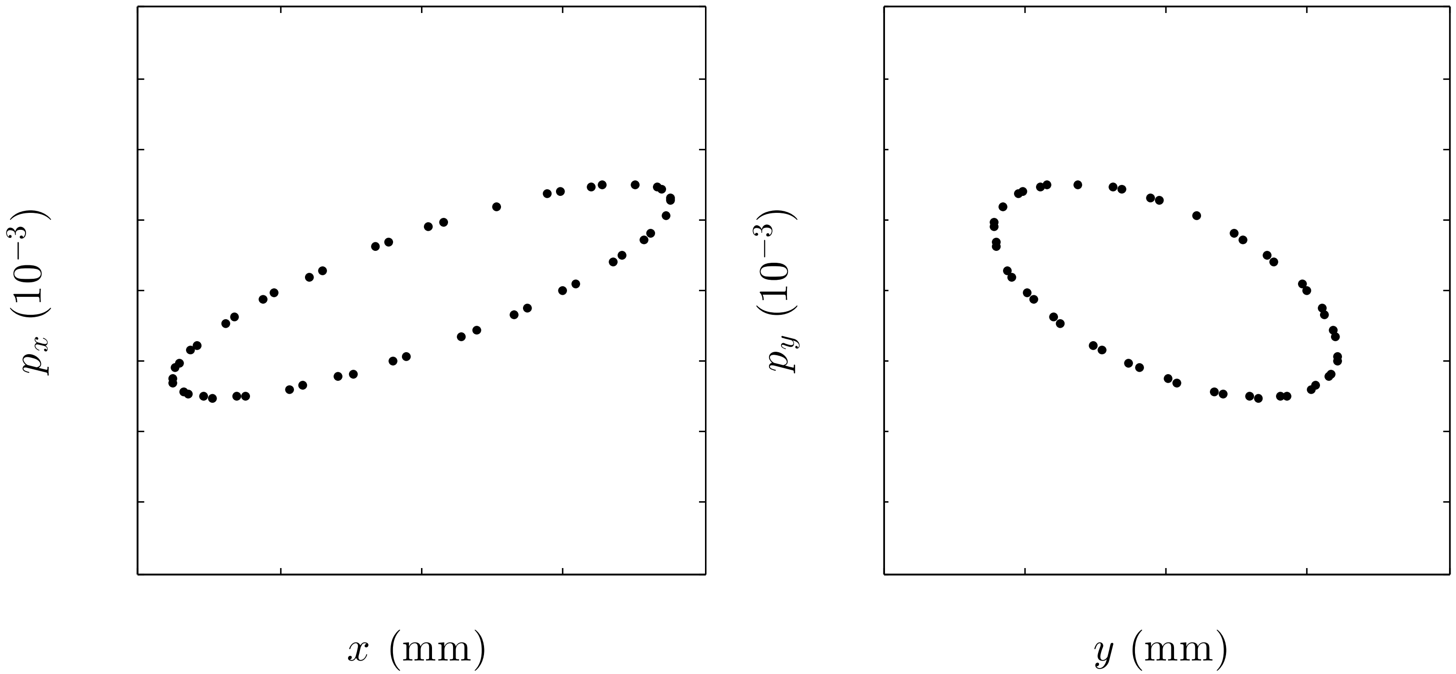
$$R(s_n|s_0) = R(s_n|s_{n-1})...R(s_3|s_2) \cdot R(s_2|s_1) \cdot R(s_1|s_0)$$



- For **periodic beamlines** (i.e. a beamline constructed from a repeated unit), the transfer matrix for a single period can be parameterised in terms of the **Courant–Snyder parameters** (α, β, γ) and the **phase advance** μ:

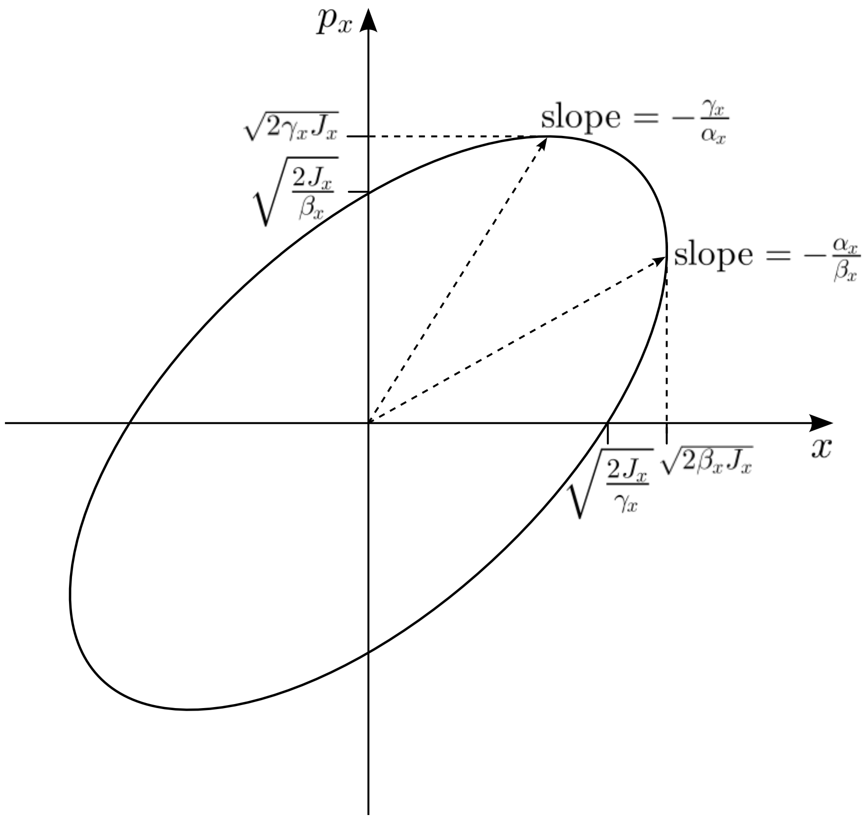
$$R = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- The characteristics of the particle motion can be represented by a **phase space portrait** showing the co-ordinates and momenta of a particle after an increasing number of passes through full periods of the beamline



From Linear to Non-linear

- If the transfer map for each period is linear, then the phase space portrait is an **ellipse** with area πJ_x
- The **action** J_x characterises the **amplitude** of the betatron oscillations
- The **shape** of the ellipse is described by the **Courant–Snyder parameters**
- The rate at which particles move around the ellipse (**phase advance** per period) is **independent** of the betatron **action**



From Linear to Non-linear

- Nonlinearities in particle dynamics can come from a number of different **sources**, e.g.
  - **Stray fields** in drift spaces
  - Higher-order **multipole components** in dipoles and quadrupoles
  - Higher-order **multipole magnets** (sextupoles, octupoles...) used to control various properties of the beam;
  - Effects of **fields** generated by a **bunch** of particles on individual particles within the same or another bunch (space-charge forces, beam-beam effects...)
- The effects of nonlinearities can be varied and quite dramatic
- It is paramount to have some understanding of nonlinear dynamics for **optimising** the **design** and **operation** of many accelerator systems

Non-linear transfer maps and effects of non-linearities

Nonlinear transfer map: sextupole

- As example, consider the **vertical field** component in a **sextupole** magnet:

$$\frac{B_y}{B\rho} = \frac{1}{2} k_2 x^2$$

with $B\rho$ the beam rigidity and k_2 the normalized sextupole gradient

- In the “thin lens” approximation, the **deflection** of a particle passing through the sextupole of length L is

$$\Delta p_x = -\int \frac{B_y}{B\rho} ds = -\frac{1}{2} k_2 L x^2$$

- The (thin lens) **transfer map** for the sextupole is

$$\begin{aligned} x_1 &= x_0 \, , \\ p_{x1} &= p_{x0} - \frac{1}{2} k_2 L x^2 \end{aligned}$$

Power series representation

- A **nonlinear transfer map** can be represented as a **power series**

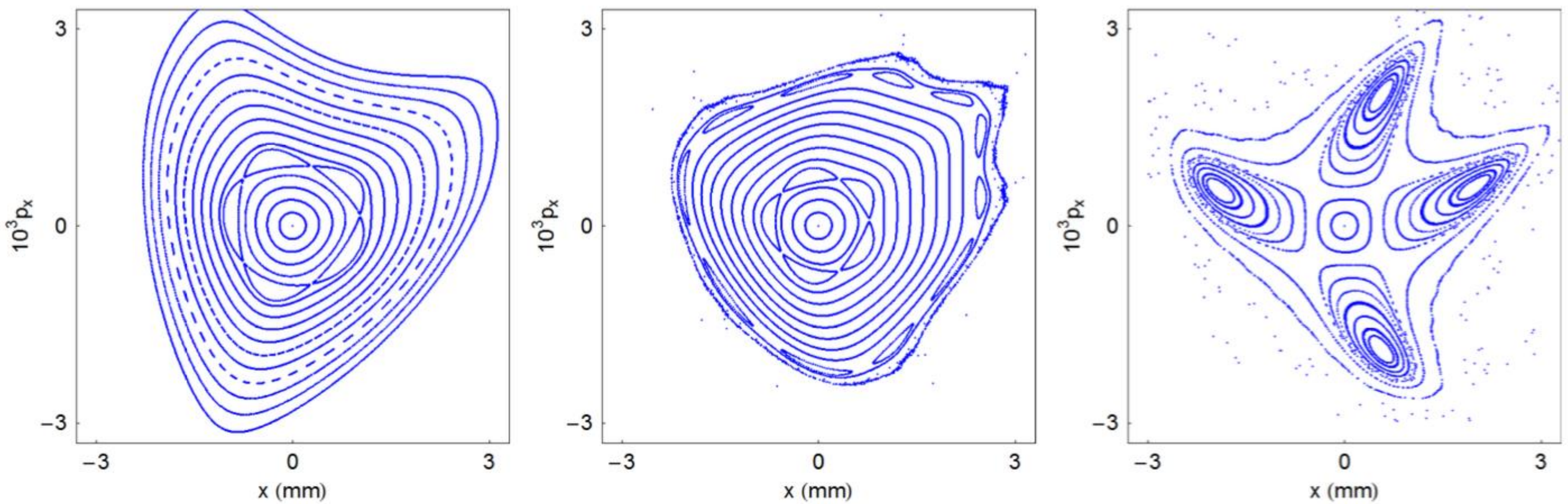
$$x_1 = A_1 + R_{11}x_0 + R_{12}p_{x0} + T_{111}x_0^2 + T_{112}x_0p_{x0} + T_{122}p_{x0}^2 + \ldots$$

$$p_{x1} = A_2 + R_{21}x_0 + R_{22}p_{x0} + T_{211}x_0^2 + T_{212}x_0p_{x0} + T_{222}p_{x0}^2 + \ldots$$

- The **coefficients** R_{ij} are components of the **transfer matrix** R
- The coefficients of the **higher-order** (nonlinear) terms are conventionally represented by T_{ijk} (2nd order), U_{ijkl} (3rd order) and so on…
- The values of the **indices** correspond to **components** of the phase space vector, thus:

index value	1	2	3	4	5	6
component	x	p_x	y	p_y	z	δ

Effects of nonlinearities

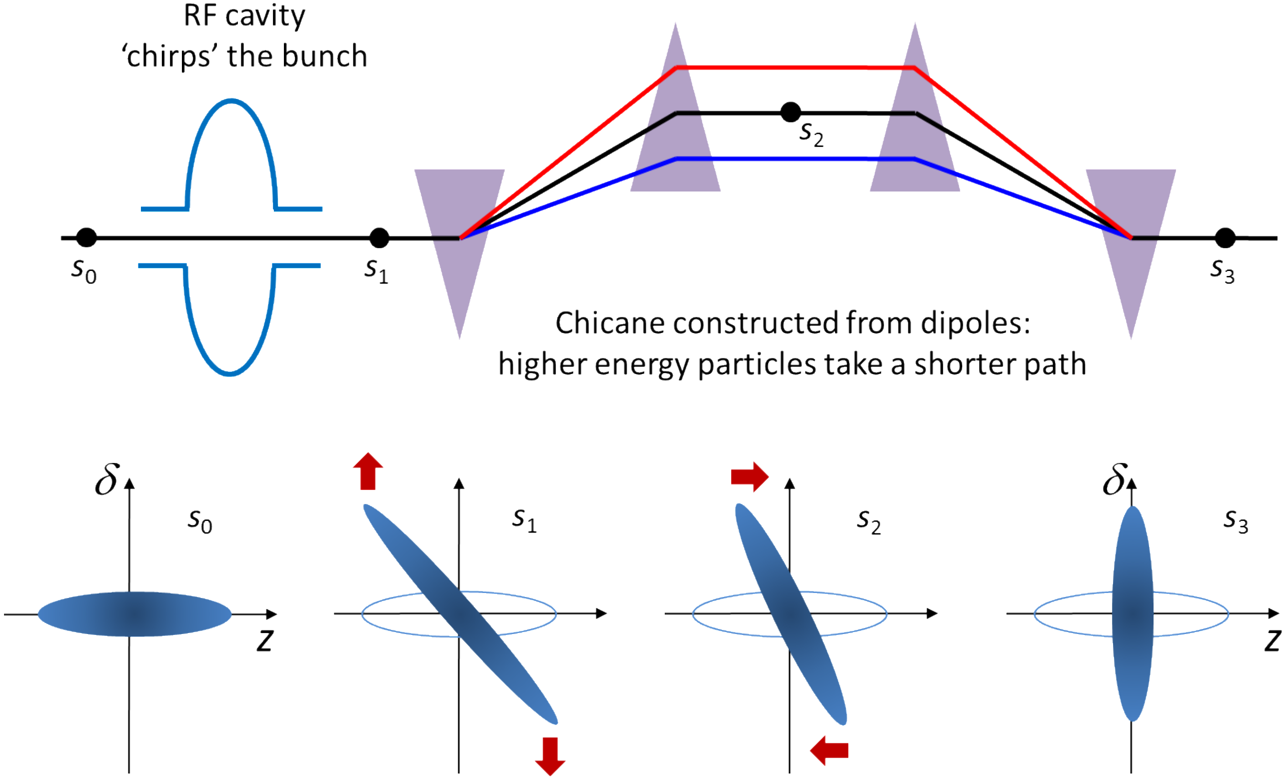


- Nonlinearities in a periodic beamline can have a number of **effects** (more during the 2nd lecture):
  - The shape of the phase space **ellipse** becomes **distorted**
  - Features such as **"islands"** (closed loops around points away from the origin) appear in phase space portrait
  - The **phase advance per period** can depend on the betatron amplitude, i.e. **depends** on the **action** J_x
  - The **motion** can be **stable** for **small amplitude**, but unstable (**chaotic**) at **large amplitude**

Nonlinear effects in a bunch compressor

Bunch compressors

- A **bunch compressor** reduces the **length** of a bunch, by performing a **rotation** in longitudinal phase space
- Bunch compressors are used, for example, in **free electron lasers** to increase the peak current



Distribution of particles 'rotates' in longitudinal phase space (area is conserved).

Bunch compressors

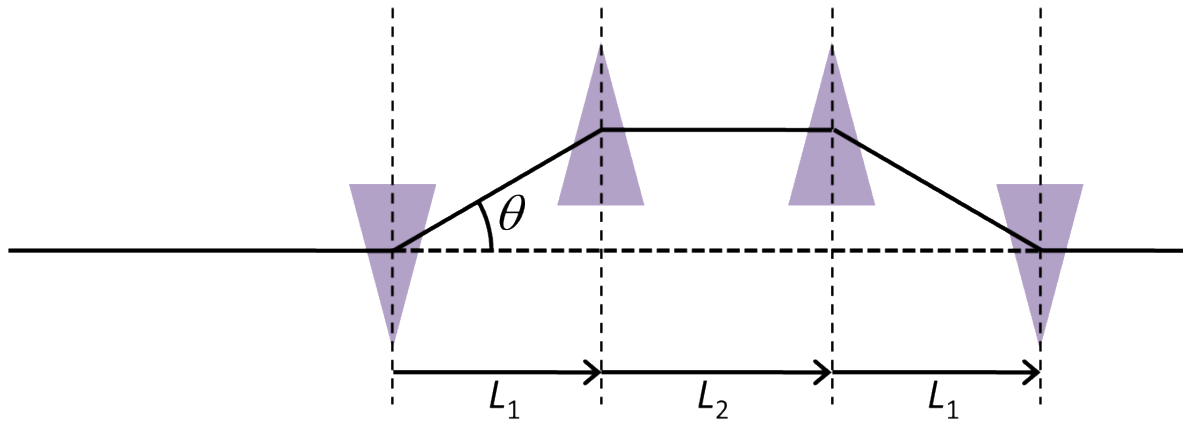
- The **RF cavity** is designed to "chirp" the bunch, i.e. to provide a **change** in **energy deviation** as a function of **longitudinal position** z within the bunch
- The **energy deviation** δ of a particle with energy E from a reference energy E_0 is defined as:

$$\delta = \frac{E - E_0}{E_0}$$

- The **transfer map** for the **RF cavity** in the bunch compressor with voltage V and frequency $\frac{\omega}{2\pi}$ is:

$$\begin{aligned} z_1 &= z_0 \ , \\ \delta_1 &= \delta_0 - \frac{eV}{E_0} \sin\left(\frac{\omega z_0}{c}\right) \end{aligned}$$

- Neglecting synchrotron radiation, the chicane does not change the energy of the particles. However, the **path length** L **depends** on the **energy** of the particle.



- If we assume that the bending angle in a dipole is small:

$$L = \frac{2L_1}{\cos\theta} + L_2$$

- The **bending angle** is a function of the **energy** of the particle:

$$\theta = \frac{\theta_0}{1+\delta}$$

- The **change** in the co-ordinate z is the **difference** between the **nominal** path length, and the length of the path actually taken by the particle
- Hence, the **chicane transfer map** can be written:

$$
\begin{aligned}
z_2 &= z_1 + 2L_1 \left(\frac{1}{\cos\theta_0} - \frac{1}{\cos(\theta(\delta_1))} \right) , \\
\delta_2 &= \delta_1
\end{aligned}
$$

where θ_0 is the nominal bending angle of each dipole in the chicane, and $\theta(\delta)$ is given by

$$\theta(\delta) = \frac{\theta_0}{1+\delta}$$

- Clearly, the **complete transfer map** for the bunch compressor is **nonlinear**, but how important are the nonlinear terms?

Bunch compressors linear dynamics

- To understand the effects of the nonlinear part of the map, we will study a **specific example**
- First, we will "**design**" a bunch compressor using only the **linear part** of the map
- The linear part of a transfer map can be obtained by **expanding** the map as a **Taylor series** in the dynamical variables, and keeping only the **first-order** terms
- After finding appropriate values for the various **parameters** using the **linear transfer map**, we shall see how our **design** has to be **modified** to take account of the **nonlinearities**

- To **first order** in the dynamical variables, the **map** for the **RF cavity** can be written:

$$\begin{aligned} z_1 &= z_0 \ , \\ \delta_1 &= \delta_0 + R_{65} z_0 \end{aligned} \qquad \text{with} \qquad R_{65} = -\frac{eV}{E_0}\frac{\omega}{c}$$

- The **map** for the **chicane** is

$$\begin{aligned} z_2 &= z_1 + R_{56}\delta_1 \ , \\ \delta_2 &= \delta_1 \end{aligned} \qquad \text{with} \qquad R_{56} = 2L_1 \frac{\theta_0 \sin\theta_0}{\cos^2\theta_0}$$

Bunch compressors linear dynamics

- As a specific example, consider a bunch compressor for the International Linear Collider (ILC)

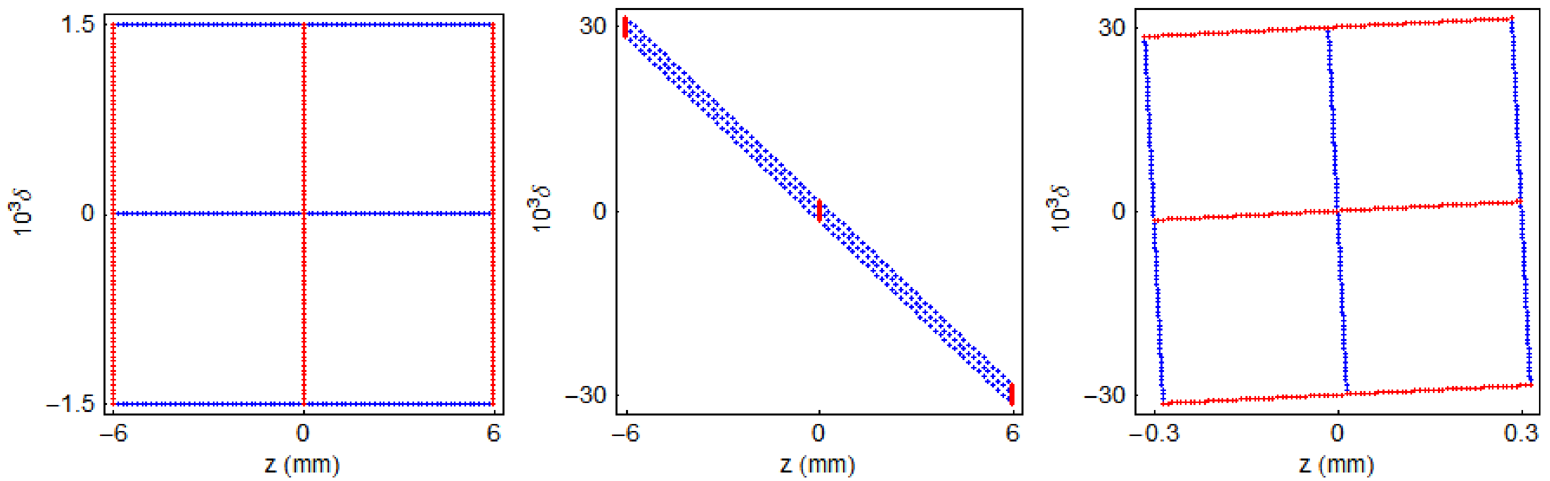
Initial rms bunch length	$\sqrt{\langle z_0^2 \rangle}$	6 mm
Initial rms energy spread	$\sqrt{\langle \delta_0^2 \rangle}$	0.15%
Final rms bunch length	$\sqrt{\langle z_2^2 \rangle}$	0.3 mm

- **Two constraints** determine the values of R_{65} and R_{56}
  - The **bunch length** should be **reduced** by a **factor 20**
  - There should be **no "chirp"** on the bunch at the exit of the bunch compressor,
- With these constraints, we find (see Appendix):

$$R_{65} = -4.9937 \text{ m}^{-1} \qquad R_{56} = 0.19975 \text{ m}$$

Bunch compressors linear dynamics

- We can illustrate the effect of the linearised bunch compressor map on **phase space** using an artificial **"window frame" distribution**:



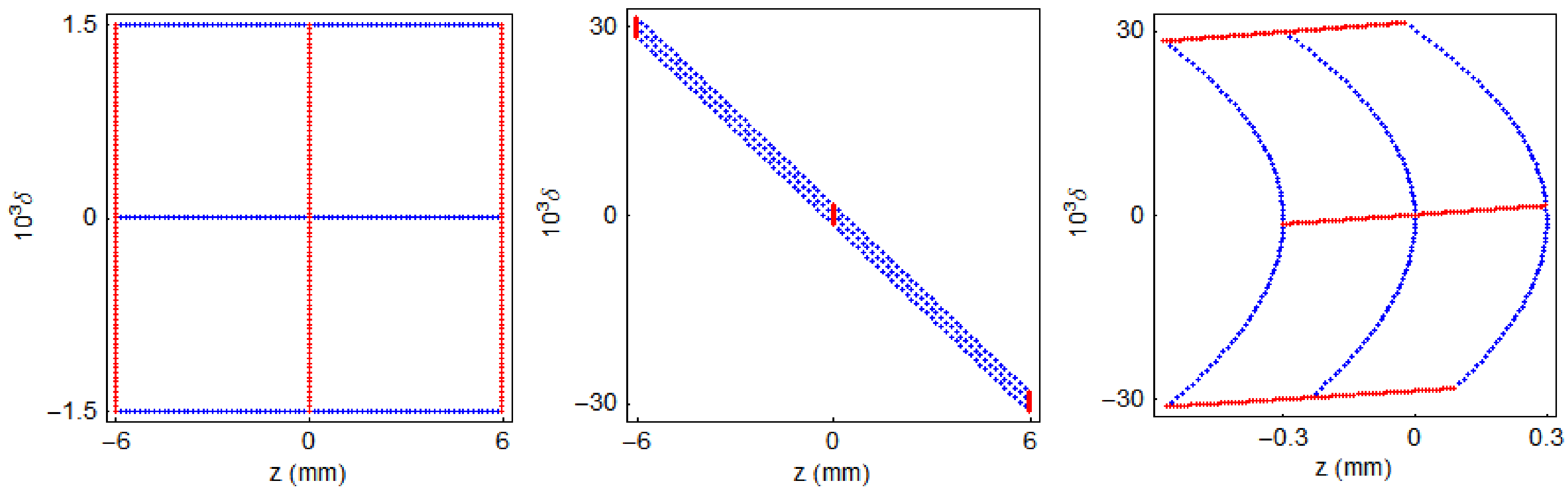
- The rms **bunch length** is **reduced** by a factor of 20 as required, but the **rms energy spread** is **increased** by the same factor, because the transfer map is **symplectic**, so phase space areas are conserved under the transformation

- Let's apply now the **full nonlinear map** for the bunch compressor.
- We need first to make some **assumptions** for the **RF voltage** and **frequency**, and the dipole **bending angle** and chicane **length** in order for the coefficient R_{65} and R_{56} to have the appropriate values

Beam (reference) energy	E_0	5 GeV
RF frequency	f_{rf}	1.3 GHz
RF voltage	V_{rf}	916 MV
Dipole bending angle	θ_0	3°
Dipole spacing	L_1	36.3 m

Bunch compressors non-linear dynamics

- As before, we illustrate the effect of the bunch compressor map on phase space using a "window frame" distribution:



- Although the **bunch length** has been **reduced**, there is significant **distortion** of the distribution: the **rms bunch length** will be **significantly longer** than we are aiming for
- To reduce the distortion, we need to understand where it comes from, by looking at the map more closely

Bunch compressors non-linear dynamics

- Consider a particle entering the bunch compressor with initial phase space co-ordinates z_0 and δ_0. We can write the co-ordinates z_1 and δ_1 of the particle after the **RF cavity** to **2nd order** in z_0 and δ_0:

$$\begin{aligned} z_1 &= z_0 \;, \\ \delta_1 &= \delta_0 + R_{65} z_0 + T_{655} z_0^2 \end{aligned}$$

- The co-ordinates of the particle after the **chicane** are (to **2nd order**):

$$\begin{aligned} z_2 &= z_1 + R_{56}\delta_1 + T_{566}\delta_1^2 \;, \\ \delta_2 &= \delta_1 \end{aligned}$$

- If we **combine** the **maps** for the RF and the chicane, we get:

$$\begin{aligned} z_2 = {} & (1 + R_{56}R_{65}) z_0 + R_{56}\delta_0 \\ & + (R_{56}T_{655} + R_{65}^2 T_{566}) z_0^2 \\ & + 2R_{65}T_{566} z_0 \delta_0 + T_{566}\delta_0^2, \\ \delta_2 = {} & \delta_0 + R_{65} z_0 + T_{655} z_0^2 \end{aligned}$$

- In order to eliminate the strong **non-linear distortion**, we have to **eliminate** the **second term**, i.e.

$$R_{56}T_{655} + R_{65}^2 T_{566} = 0$$

- By expanding the original map,

$$z_2 = z_1 + 2L_1 \left(\frac{1}{\cos\theta_0} - \frac{1}{\cos(\theta(\delta_1))} \right)$$

as a Taylor series in δ, we find that for small angles:

$$T_{566} \approx -3L_1\theta_0^2$$

- Now, it remains to determine T_{655}, i.e. the **coefficient** for the **second-order** dependence of the **energy deviation** on **longitudinal position**

Bunch compressors non-linear dynamics

- The **map** of the **energy deviation**

$$\delta_1 = \delta_0 - \frac{eV}{E_0} \sin\left(\frac{\omega z_0}{c}\right)$$

contains only **odd order terms** unless the RF cavity is operated **out of phase**, i.e.

$$\delta_1 = \delta_0 - \frac{eV}{E_0} \sin\left(\frac{\omega z_0}{c} + \phi_0\right)$$

- The **first** and **second order** coefficients in the transfer map for the **energy deviation** are:

$$R_{65} = -\frac{eV}{E_0}\frac{\omega}{c}\cos\phi_0 \quad \text{and} \quad T_{655} = -\frac{1}{2}\frac{eV}{E_0}\left(\frac{\omega}{c}\right)^2 \sin\phi_0$$

- Recall that $R_{65} = -4.9937 \text{ m}^{-1}$ and $R_{56} = 0.19975 \text{ m}$
- We also obtain

$$T_{566} \approx -3L_1\theta_0^2 = -0.29963 \text{ m}$$

- By imposing $R_{56}T_{655} + R_{65}^2 T_{566} = 0$, we have that

$$T_{655} = 37.406 \text{ m}^{-2}$$

- Using the expressions

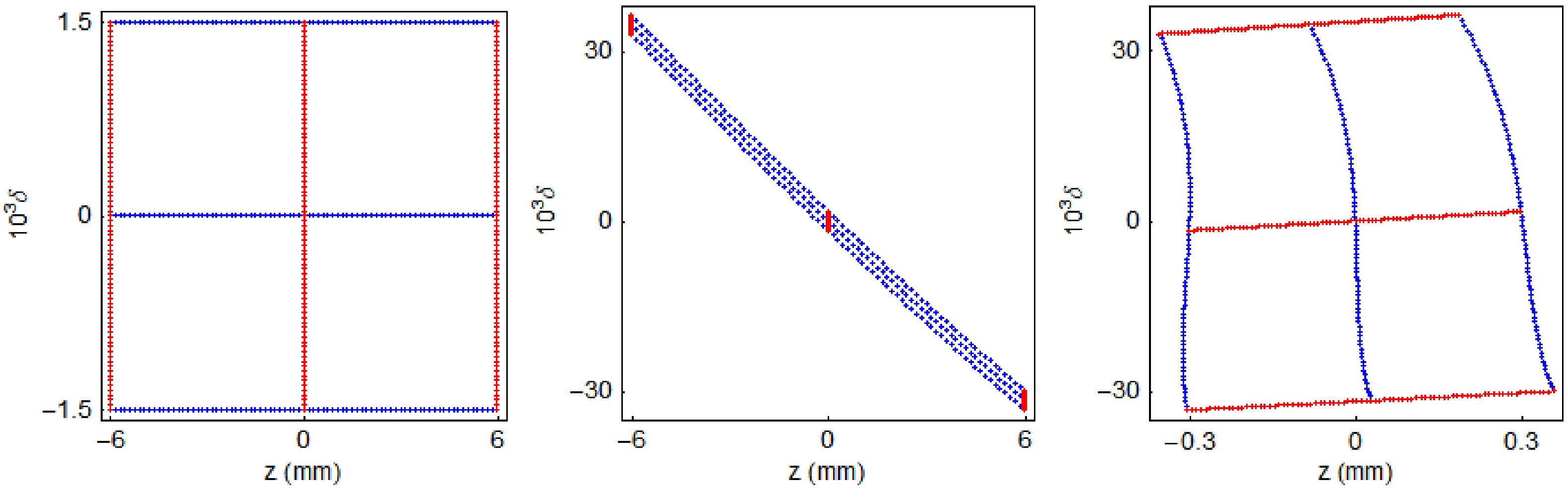
$$R_{65} = -\frac{eV}{E_0}\frac{\omega}{c}\cos\phi_0 \quad \text{and} \quad T_{655} = -\frac{1}{2}\frac{eV}{E_0}\left(\frac{\omega}{c}\right)^2 \sin\phi_0$$

the **voltage** and **phase** are determined as

$$V = 1046 \text{ MV} \quad \text{and} \quad \phi_0 = 28.8^\circ$$

Bunch compressors non-linear dynamics

- As before, we illustrate the effect of the bunch compressor on phase space using a "window frame" distribution, using the parameters determined above, to try to compress by a factor 20, while minimising the second-order distortion:



- The **dominant distortion** now appears to be **3rd order**, and looks **small enough** that it should not significantly affect the performance

Conclusions and Summary

Some conclusions

- **Nonlinear effects** can **limit** the **performance** of an accelerator system
- Sometimes the **effects** are **small enough** that they can be **ignored**
- In many cases, a **system designed without** taking account of **nonlinearities** will **not achieve** the specified **performance**
- If we analyse and understand the **nonlinear behaviour** of a system, then, we may be able to devise means of **compensating** any adverse effects

Summary

- **Nonlinear effects** can arise from a number of **sources** in accelerators, including stray fields, higher-order multipole components in magnets, space-charge...
- The **transfer map** for a nonlinear element (such as a sextupole) may be represented as a **power series** in the initial values of the phase space variables
- The effects of **nonlinearities** in accelerator system vary widely, depending on the **type of system** in which they occur (e.g. single-pass, or periodic)
- In some cases, **nonlinear effects** can limit the **performance** of an accelerator system. In such cases, it is important to take nonlinearities into account in the **design** of the system

Appendix

In a linear approximation, the maps for the rf cavity and the chicane in a bunch compressor may be represented as matrices:

$$M_{\mathrm{rf}} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}, \qquad M_{\mathrm{ch}} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \tag{45}$$

where:

$$a = \frac{eV}{E_0}\frac{\omega}{c}, \quad \text{and} \quad b = 2L_1 \frac{\theta_0 \sin\theta_0}{\cos^2\theta_0}. \tag{46}$$

The matrix representing the total map for the bunch compressor, M_{bc}, is then:

$$M_{\mathrm{bc}} = M_{\mathrm{ch}} M_{\mathrm{rf}} = \begin{pmatrix} 1 - ab & b \\ -a & 1 \end{pmatrix} = \begin{pmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{pmatrix}. \tag{47}$$

The effect of the map is written:

$$\vec{z} \mapsto M_{\mathrm{bc}}\vec{z}, \quad \text{where} \quad \vec{z} = \begin{pmatrix} z \\ \delta \end{pmatrix}. \tag{48}$$

Bunch compressor long. dynamics

Now we proceed to derive expressions for the required values of the parameters a and b, in terms of the desired initial and final bunch length and energy spread.

We construct the beam distribution *sigma* matrix by taking the outer product of the phase space vector for each particle, then averaging over all particles in the bunch:

$$\Sigma = \langle \vec{z}\,\vec{z}^{\mathsf{T}} \rangle = \begin{pmatrix} \langle z^2 \rangle & \langle z\delta \rangle \\ \langle z\delta \rangle & \langle \delta^2 \rangle \end{pmatrix}. \tag{49}$$

The transformation of Σ under a linear map represented by a matrix M is given by:

$$\Sigma \mapsto M \cdot \Sigma \cdot M^{\mathsf{T}}. \tag{50}$$

Usually, a bunch compressor is designed so that the correlation $\langle z\delta \rangle = 0$ at the start and end of the compressor. In that case, using (47) for the linear map M, and (50) for the transformation of the sigma matrix, we find that the parameters a and b must satisfy:

$$(1 - ab)\frac{a}{b} = \frac{\langle \delta_0^2 \rangle}{\langle z_0^2 \rangle} \tag{51}$$

where the subscript 0 indicates that the average is taken over the *initial* values of the dynamical variables.

Given the constraint (51), the compression factor r is given by:

$$r^2 \equiv \frac{\langle z_1^2 \rangle}{\langle z_0^2 \rangle} = 1 - ab, \tag{52}$$

where the subscript 1 indicates that the average is taken over the final values of the dynamical variables.

Bunch compressor long. dynamics

We note in passing that the linear part of the map is *symplectic*. A linear map is symplectic if the matrix M representing the map is symplectic, i.e. satisfies:

$$M^{\mathsf{T}} \cdot S \cdot M = S, \tag{53}$$

where, in one degree of freedom (i.e. two dynamical variables), S is the matrix:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{54}$$

In more degrees of freedom, S is constructed by repeating the 2×2 matrix above on the block diagonal, as often as necessary.

In one degree of freedom, it is a necessary and sufficient condition for a matrix to be symplectic, that it has unit determinant: but this condition does *not* generalise to more degrees of freedom.

As a specific example, consider a bunch compressor for the International Linear Collider:

Initial rms bunch length	$\sqrt{\langle z_0^2 \rangle}$	6 mm
Initial rms energy spread	$\sqrt{\langle \delta_0^2 \rangle}$	0.15%
Final rms bunch length	$\sqrt{\langle z_1^2 \rangle}$	0.3 mm

Solving equations (51) and (52) with the above values for rms bunch lengths and energy spread, we find:

$$a = 4.9937\,\mathrm{m}^{-1}, \quad \text{and} \quad b = 0.19975\,\mathrm{m}. \tag{55}$$