CAS SCHOOL, SEP 2021 BEGINNERS

# INTRODUCTION TO **ELECTROMAGNETISM**

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# COPIRIGHTS



National Research Tomsk State University

MATERIALS

## CAS Website

These slides and the video will be available the  CAS school website

## Proceedings

There will be the electronic version of the proceedings for the school

## Books

1.. J. David Jackson,"Classical Electrodynamics"
2. Chabay, Sherwood "Matter & Interactions"

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# Variables and Units

| | | |
|---|---|---|
| $\mathbf{E}$ | | electric field [V/m] |
| $\mathbf{B}$ | | magnetic field [T] |

| | | |
|---|---|---|
| $q$ | | electric charge [C] |
| $\rho$ | | electric charge density [C/m$^3$] |
| $\mathbf{j}$ | $= \rho \mathbf{v}$ | current density [A/m$^2$] |

| | | |
|---|---|---|
| $\epsilon_0$ | | permittivity of vacuum, $8.854 \cdot 10^{-12}$ [F/m] |
| $\mu_0$ | $= \frac{1}{\epsilon_0 c^2}$ | permeability of vacuum, $4\pi \cdot 10^{-7}$ [H/m or N/A$^2$] |
| $c$ | | speed of light in vacuum, $2.99792458 \cdot 10^8$ [m/s] |

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# Differentiation with vectors

We define operator "nabla" which we treat as a special vector

$$\nabla \stackrel{\text{def}}{=} \left( \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} \right)$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$ Divergence

$$\nabla \times \mathbf{F} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \quad \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \quad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$ Curl

$$\nabla \phi = \left( \frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial z} \right)$$ Gradient

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# WHY EM?

## EM is our first example of a field theory

To work in the accelerator physics field you really should understand field theory and understand that well

## EM teaches us about special relativity

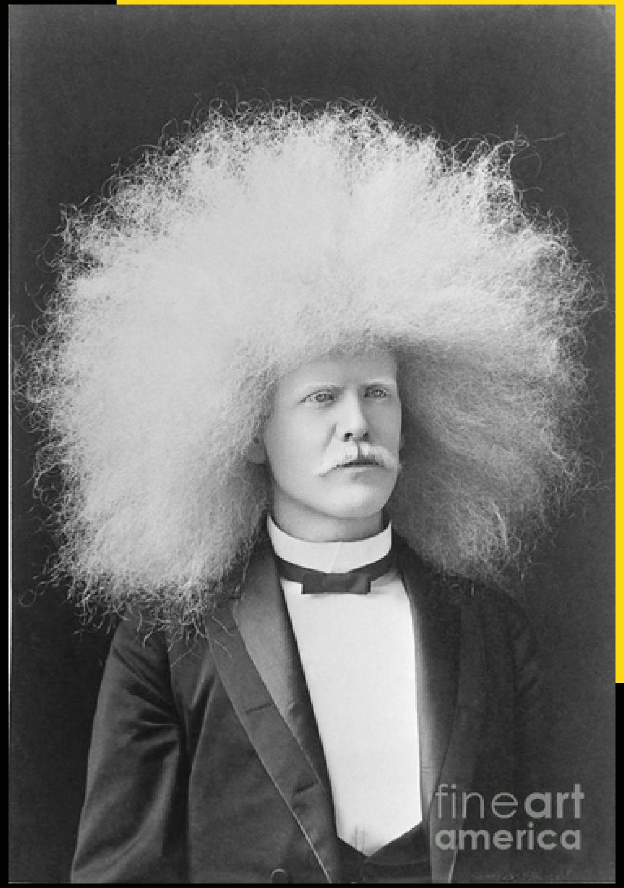
See Special Relativity lecture

## Modern physics

Electromagnetism is the first example of  using theories unification

# Examples

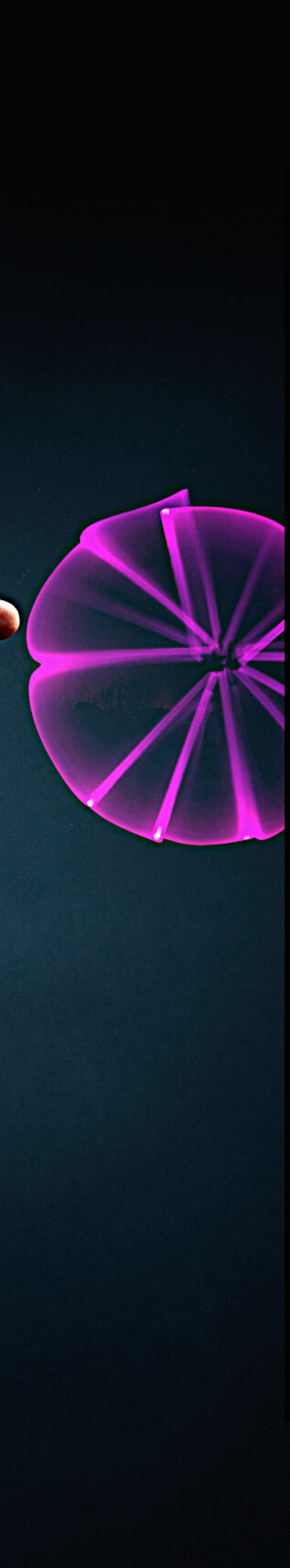
**Electric Force**




**Magnetic Force**

# CONTENT OF THE COURSE

- INTRODUCTION
- ELECTROSTATICS
- MAGNETOSTATICS
- ELECTROMAGNETISM



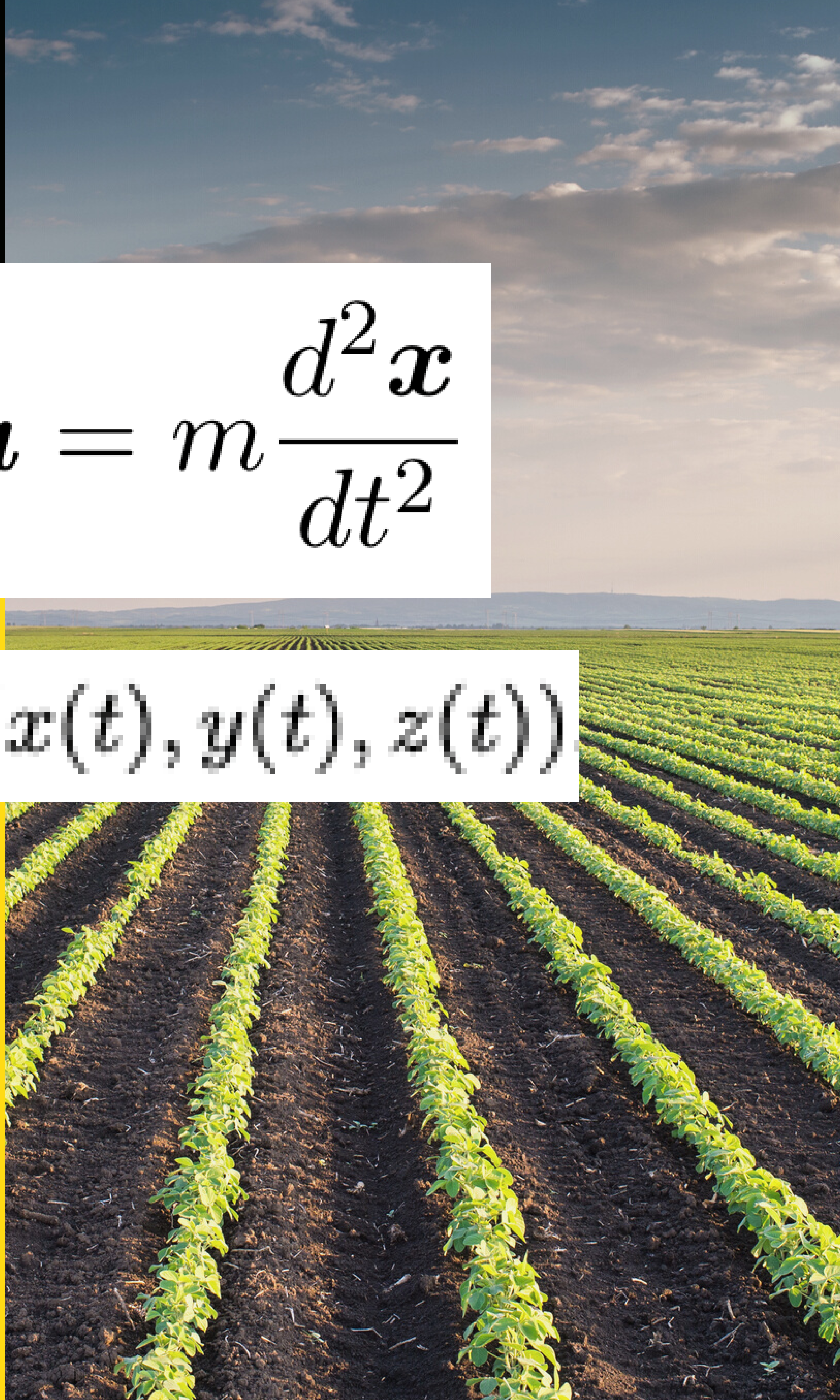
# INTRODUCTION

- **Introduction to Fields**
- Charge and Current
- Conservation Law
- Lorentz Force
- Maxwell Equations

# INTRODUCTION TO FIELDS

$$\boldsymbol{F} = m\boldsymbol{a} = m\frac{d^2\boldsymbol{x}}{dt^2}$$

$$(x(t), y(t), z(t))$$



## GRAVITATIONAL FORCE

The force exerted by the earth on a particle.

## GRAVITATIONAL FIELD

Instead of saying that the earth exerts a force on a falling object, it is more useful to say that the earth sets up **a gravitational force field**.

**Any object near the earth is acted upon by the gravitational force field at that location.**

# INTRODUCTION TO FIELDS

**F** is the force acting on a particle of mass m and **g** – the acceleration due to gravity.

- **F** and **g** are fields;
- the mass of the particle m is not a field

$$F = mg$$

## GRAVITATIONAL FORCE

1. We can split the system into a **source** which produces the field and an **object** which reacts to the field
2. We treat both pieces separately

# INTRODUCTION TO FIELDS

## ELECTRIC FORCE

The force between charged particles. Charged particles exert forces on each other

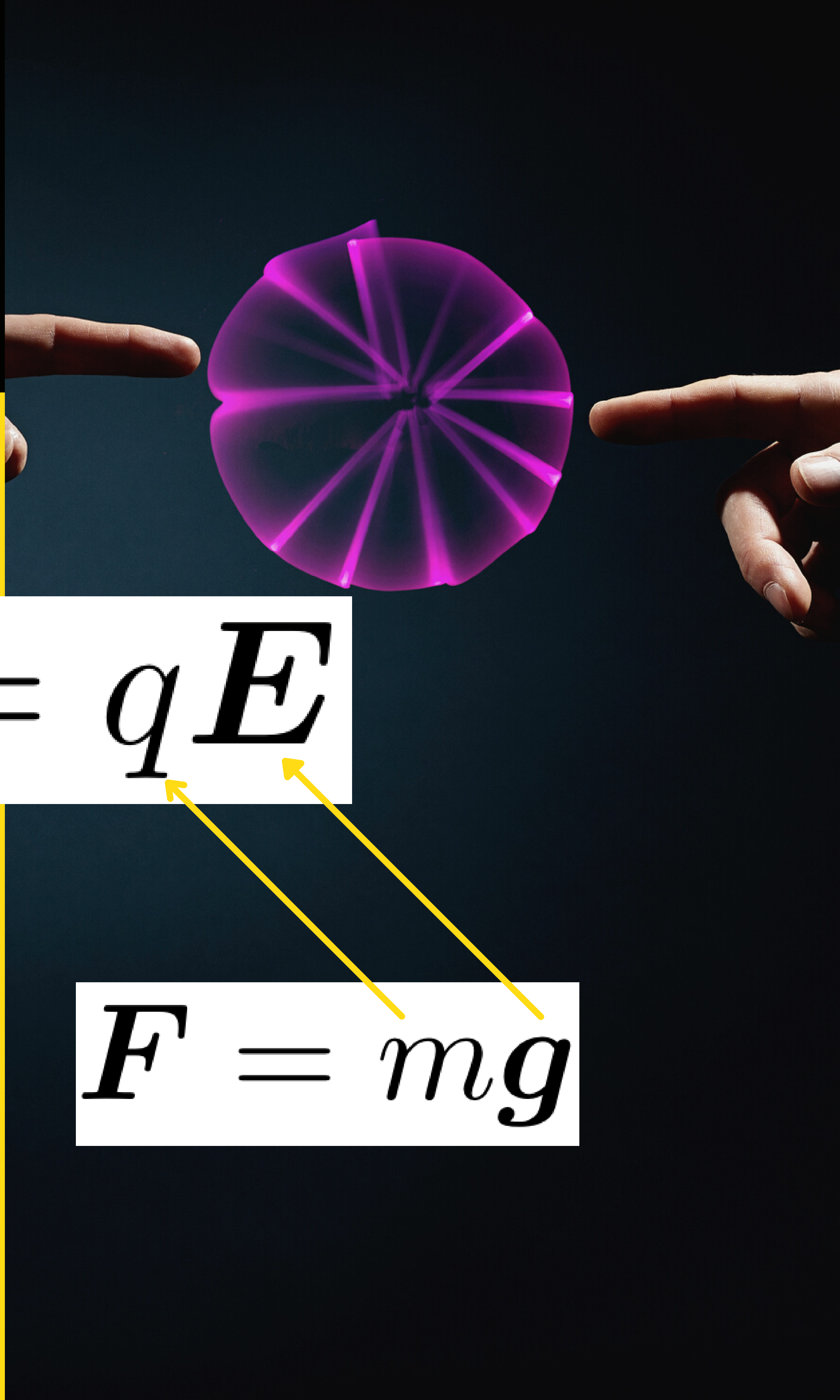
$$F = qE$$

## ELECTRIC FIELD

- The charge q of our particle replaces the mass m of our particle. q is a single number associated with the object that experiences the field.
- The electric field **E** replaces the gravitational field **g**

$$F = mg$$

**We are splitting things up into a source that produces a field and an object that experiences the field**



# INTRODUCTION TO FIELDS

## ELECTROMAGNETIC FORCE

To describe the **force of electromagnetism**, we need to introduce **two fields**, each of which is a three-dimensional vector. They are called

**ELECTRIC** FIELD , **E**

$$\mathbf{E}(\mathbf{x}, t)$$

AND

$$\mathbf{B}(\mathbf{x}, t)$$

**MAGNETIC** FIELD **, B**

# INTRODUCTION

- Introduction to Fields
- **Charge and Current**
- Conservation Law
- Lorentz Force
- Maxwell Equations

# CHARGE AND CURRENT

$$e = 1.602176634 \times 10^{-19}\ C$$

$$q = ne \qquad n \in \mathbf{Z}$$

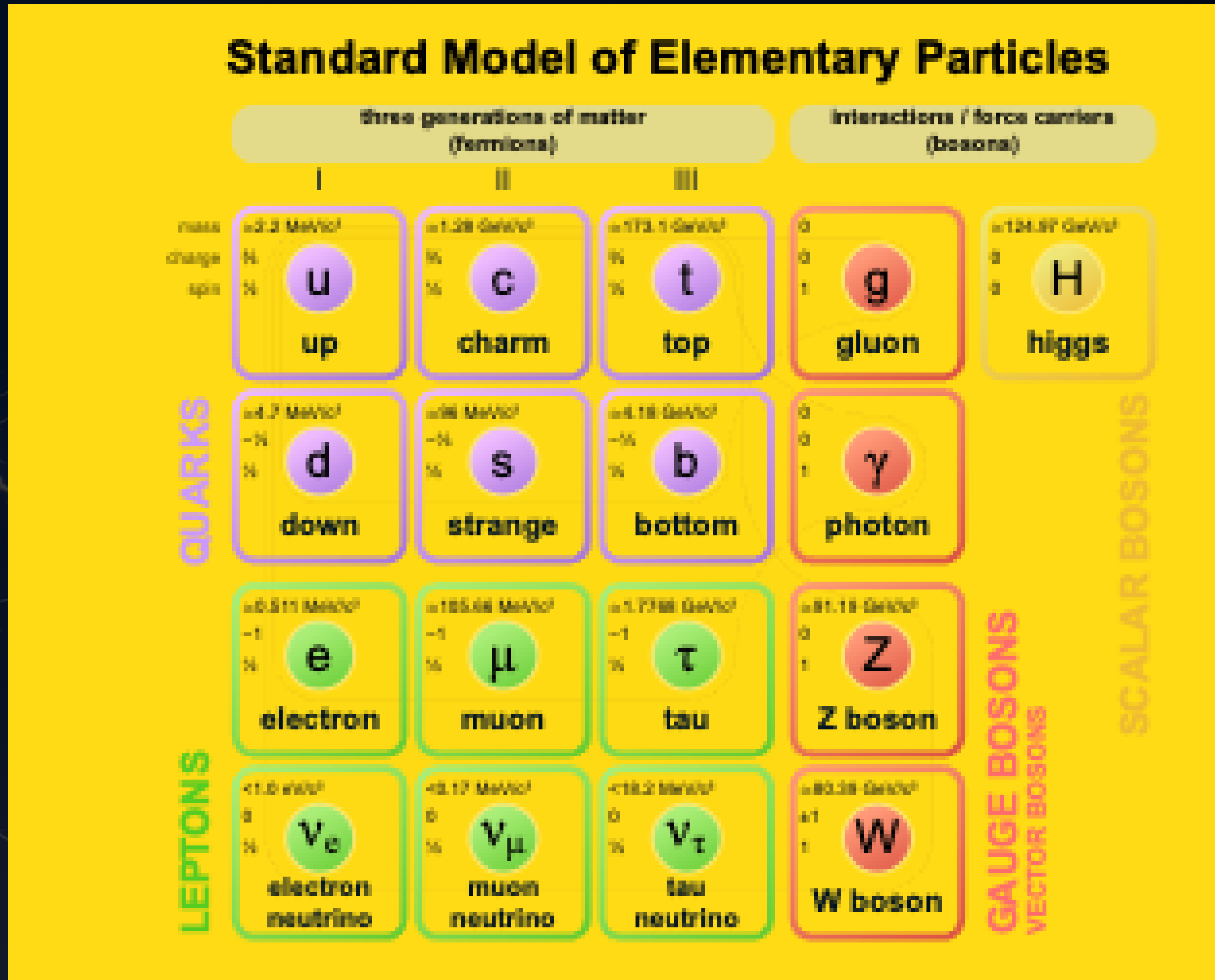
The SI unit of charge is the **Coulomb**, denoted by **C**

A much more natural unit . Then, proton/electron: n = ±1

$$q = -e/3$$

$$q = 2e/3$$

the charge of quarks



# CHARGE AND CURRENT

$$\rho(\mathbf{x}, t)$$

$$Q = \int_V d^3x \; \rho(\mathbf{x}, t)$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

the charge density – charge per unit volume

the total charge **Q** in a given region **V**

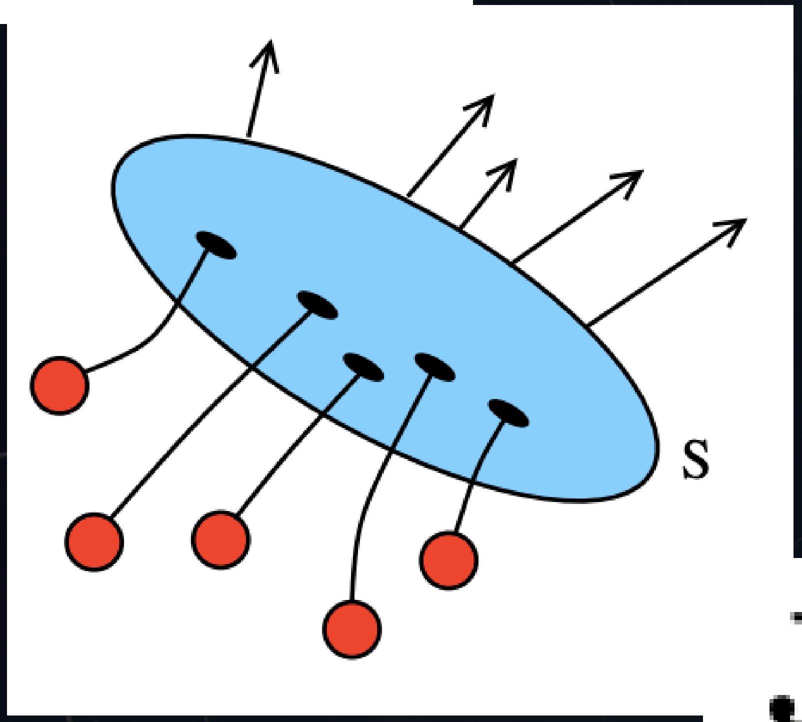
the movement of charge from one place to another is captured by the **current density J.**

**I** is called the **current**.

**The current density is the current-per-unit-area**

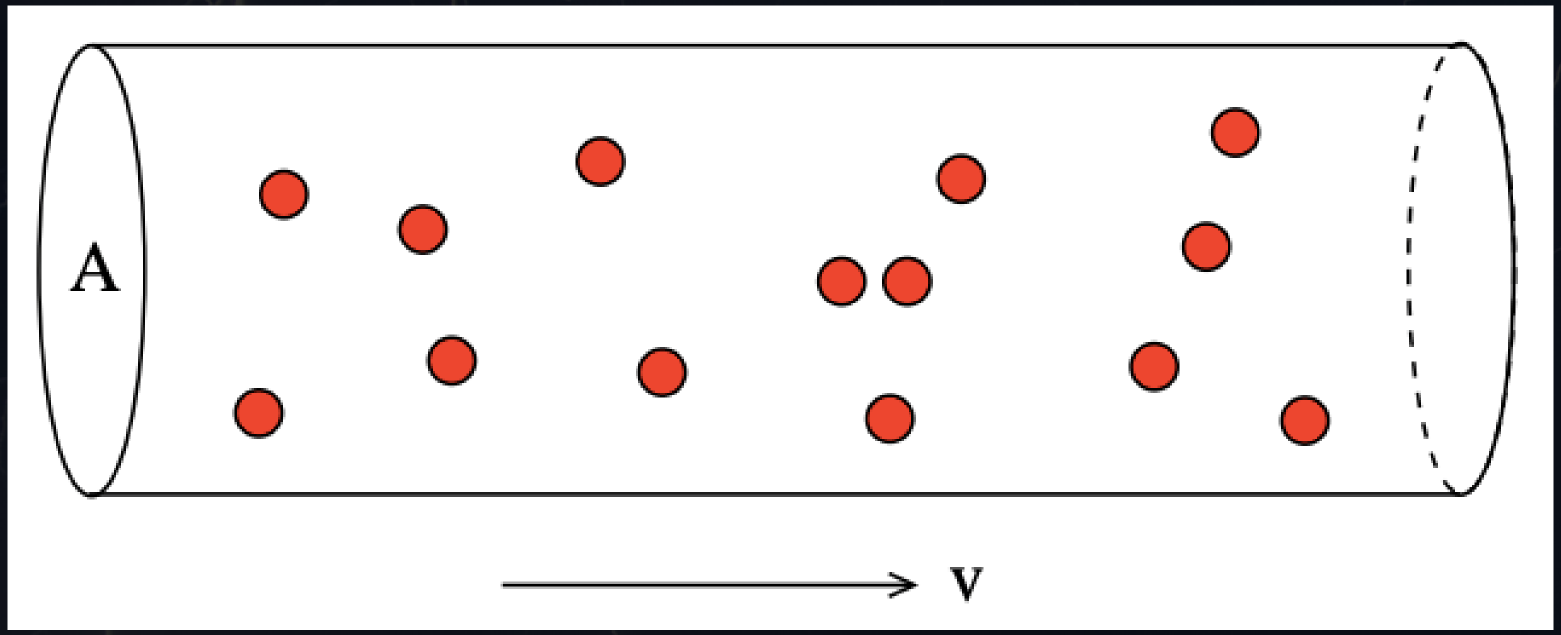
# CHARGE AND CURRENT

Current flux




$$\mathbf{J} = \rho \mathbf{v}$$

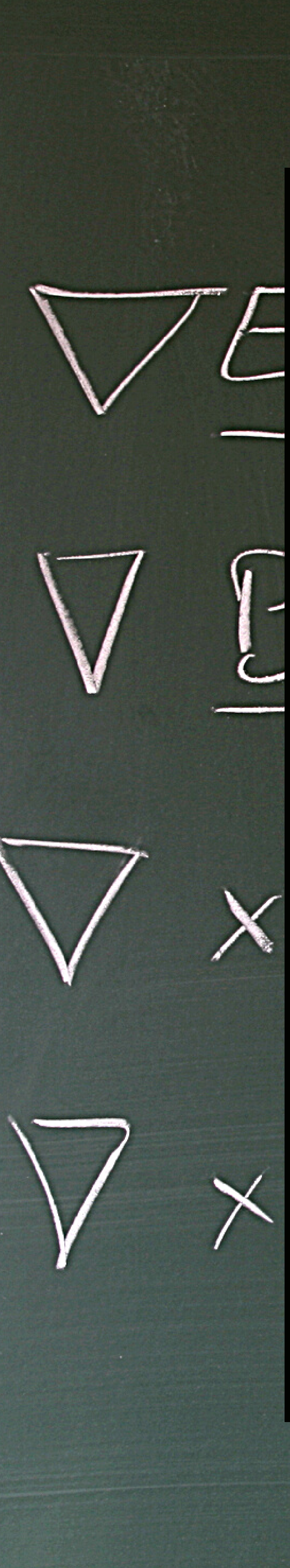
Move intuitive way:
A continuous charge distribution in which the velocity of a small volume, at point **x**, is given by **v(x, t)**




Electrons moving along a wire

$$\mathbf{J} = nq\mathbf{v}$$

$$I = |\mathbf{J}|A$$



# INTRODUCTION

- Introduction to Fields
- Charge and Current
- **Conservation Law**
- Lorentz Force
- Maxwell Equations

**CONSERVATION LAW**

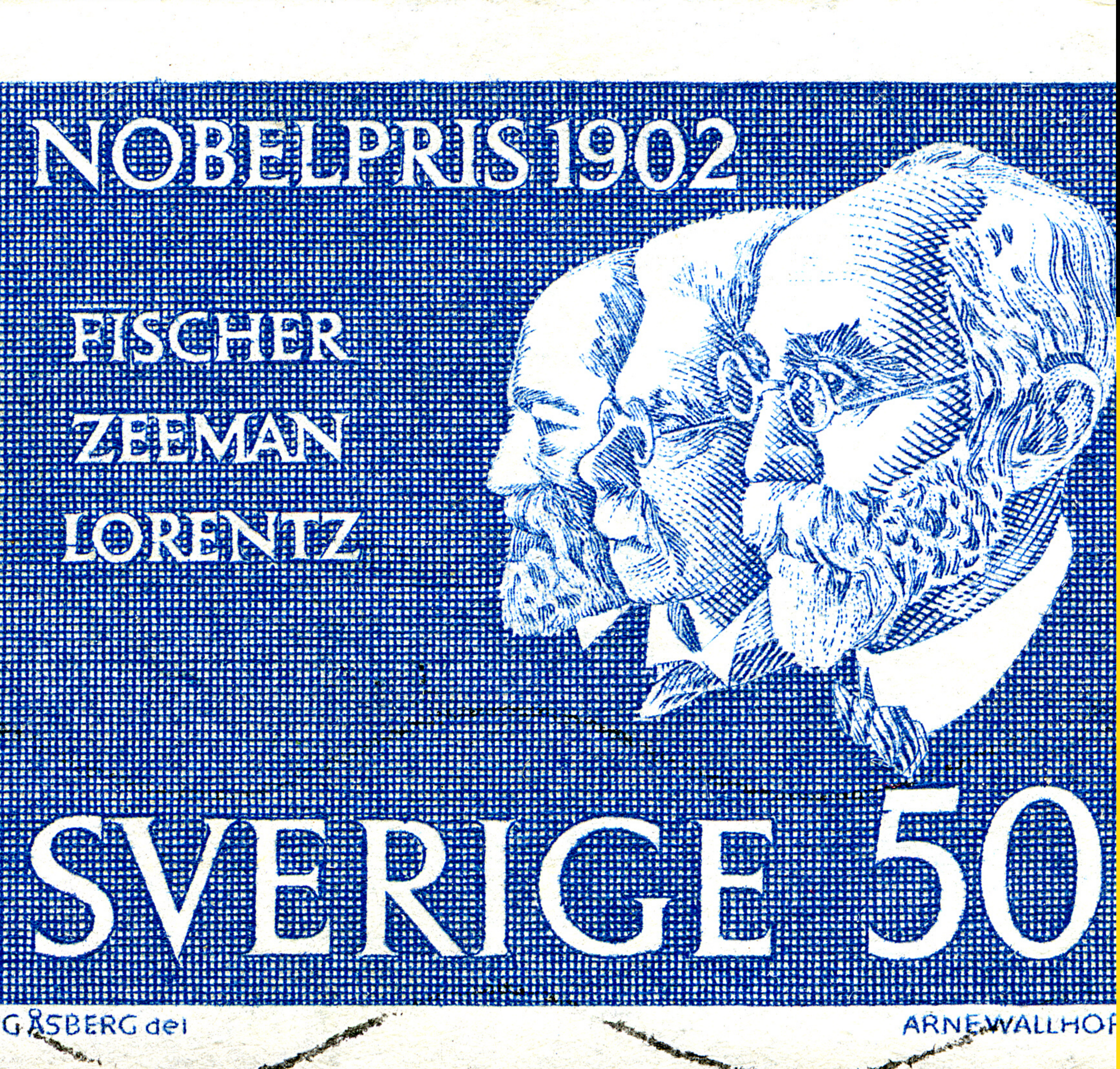
**Continuity equation:**
charge density can change in time only if there is a compensating current flowing into or out of that region

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

$$\frac{dQ}{dt} = \int_V d^3x \, \frac{\partial \rho}{\partial t} = -\int_V d^3x \, \nabla \cdot \mathbf{J} = -\int_S \mathbf{J} \cdot d\mathbf{S}$$

the change in the total charge **Q** contained in some region **V.** The minus sign is to ensure that if the net flow of current is outwards, then the total charge decreases.

If there is no current flowing out of the region, then $dQ/dt = 0$




# INTRODUCTION

- Introduction to Fields
- Charge and Current
- Conservation Law
- **LORENTZ FORCE**
- Maxwell Equations

# LORENTZ FORCE

$$F = mg \rightarrow F = qE \rightarrow \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

**Lorentz Force**

an electric field accelerates a particle in the direction **E**, while a magnetic field causes a particle to move in circles in the plane perpendicular to **B.**

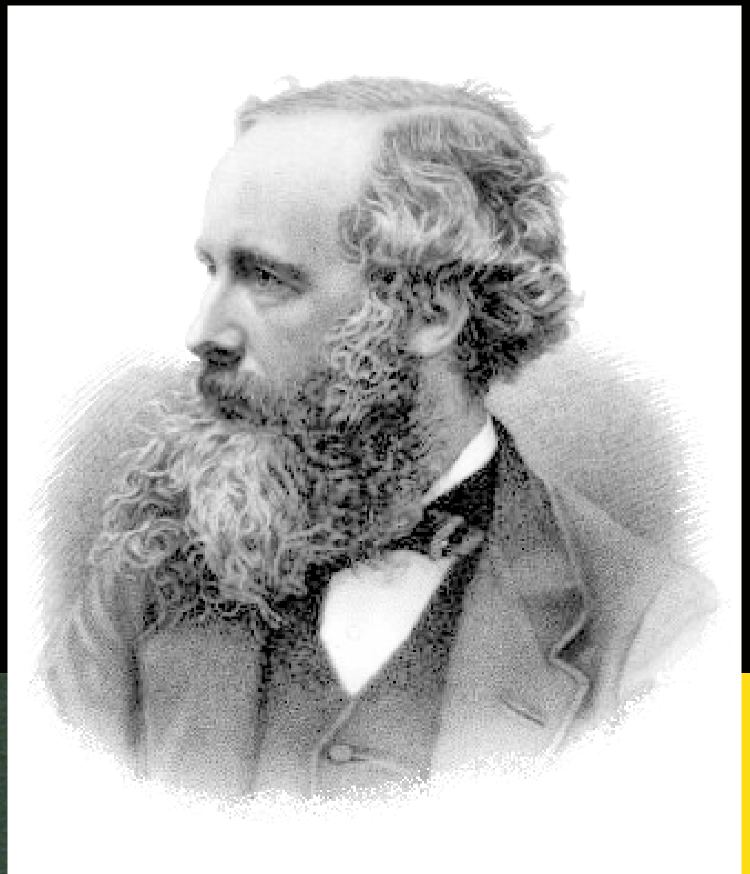
**Lorentz Force Law**

in terms of the charge distribution

Now we talk in terms of the force density **f**(**x**, t), which is the force acting on a small volume at point **x**

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$$

# INTRODUCTION



- Introduction to Fields
- Charge and Current
- Conservation Law
- Lorentz Force
- **MAXWELL EQUATIONS**

# MAXWELL EQUATIONS

## DIFFERENTIAL FORM

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

**GAUSS'S LAW** FOR **E**

$$\nabla \cdot \mathbf{B} = 0$$

**GAUSS'S LAW** FOR **B**

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

**FARADAY'S LAW**
for time-varying magnetic fields

$$\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

**AMPERE(-MAXWELL) LAW**
for time-varying electric fields

# MAXWELL EQUATIONS

## INTEGRAL FORM

$$\int_{S=\partial V} \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{Q}{\epsilon_0}$$

GAUSS'S LAW FOR E

$$\int_{\partial V} \boldsymbol{B} \cdot d\boldsymbol{S} = 0$$

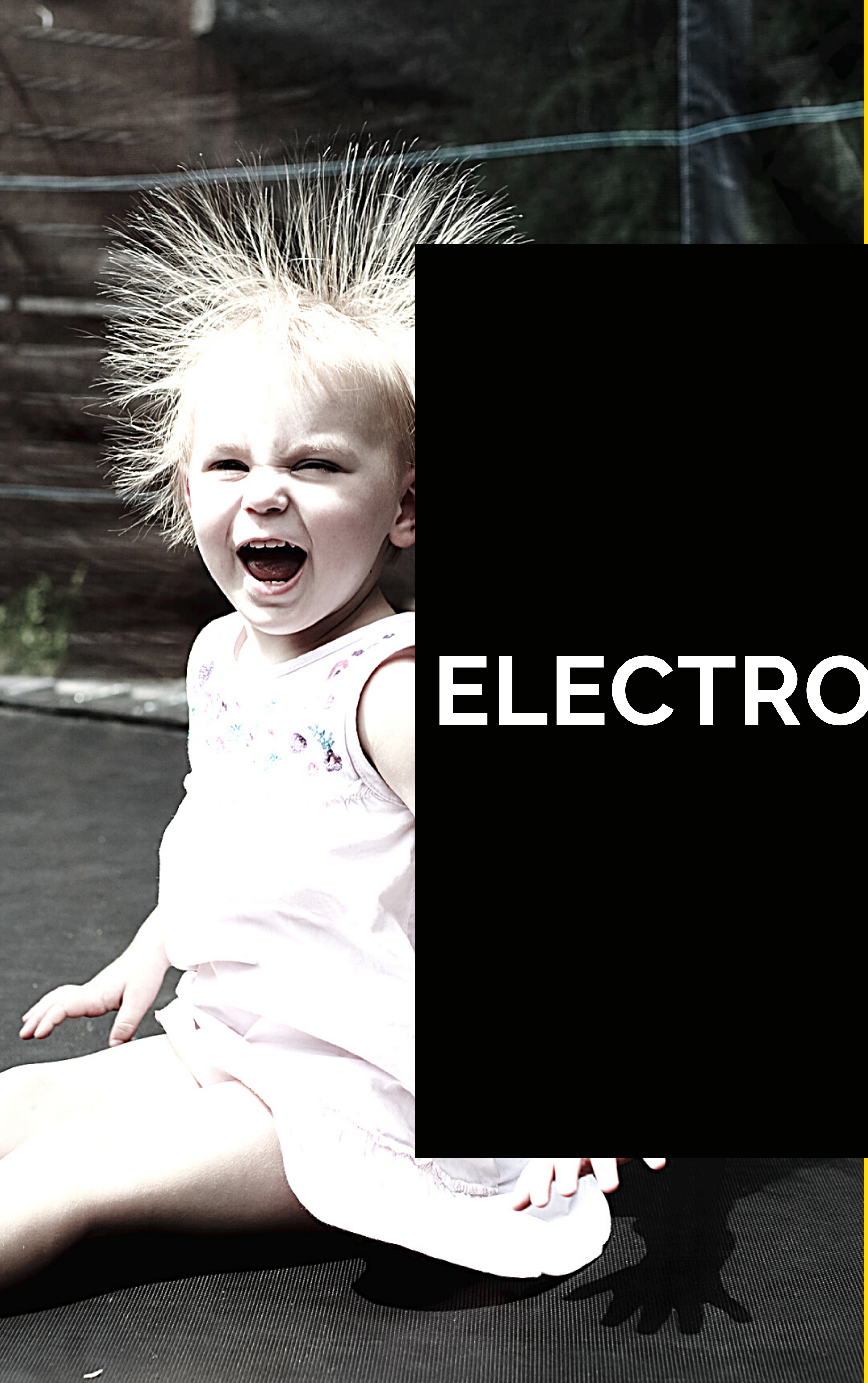
GAUSS'S LAW FOR B

$$\int_C \boldsymbol{E} \cdot d\boldsymbol{r} = \int_S \nabla \times \mathbf{E} \cdot d\boldsymbol{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\boldsymbol{S} = -\frac{d}{dt} \int_S \boldsymbol{B} \cdot d\boldsymbol{S}$$

FARADAY'S LAW

$$\int_S \nabla \times \mathbf{B} \cdot d\boldsymbol{S} = \mu_0 \int_S \boldsymbol{J} \cdot d\boldsymbol{S} + \mu_0 \epsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\boldsymbol{S}$$

AMPERE (-MAXWELL) LAW



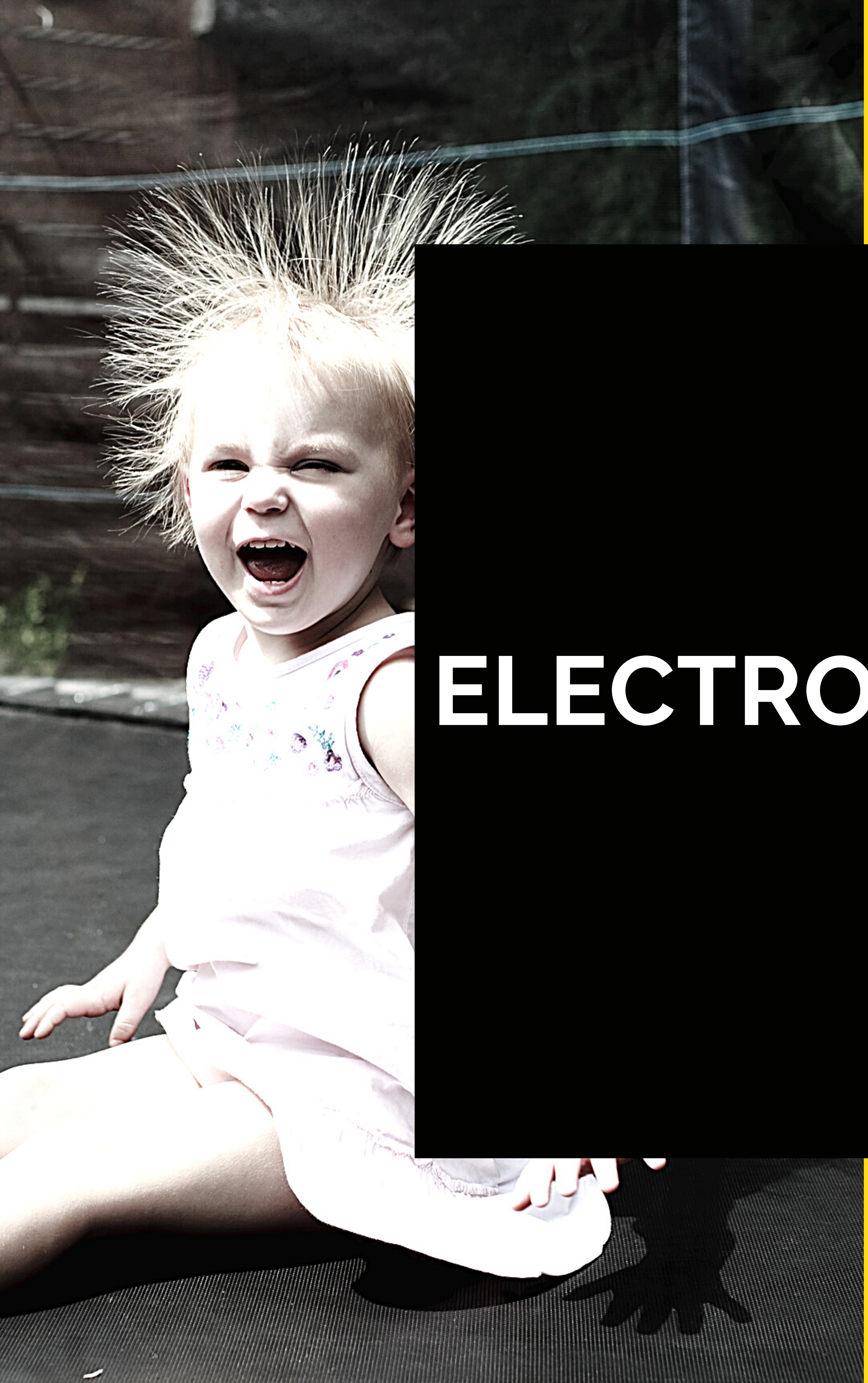
# ELECTROSTATICS

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad 0$$

$$\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$



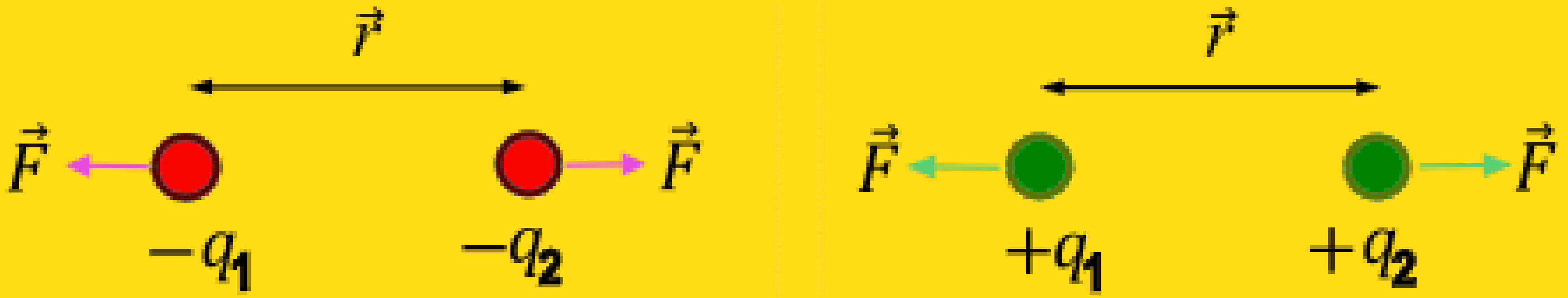
# ELECTROSTATICS

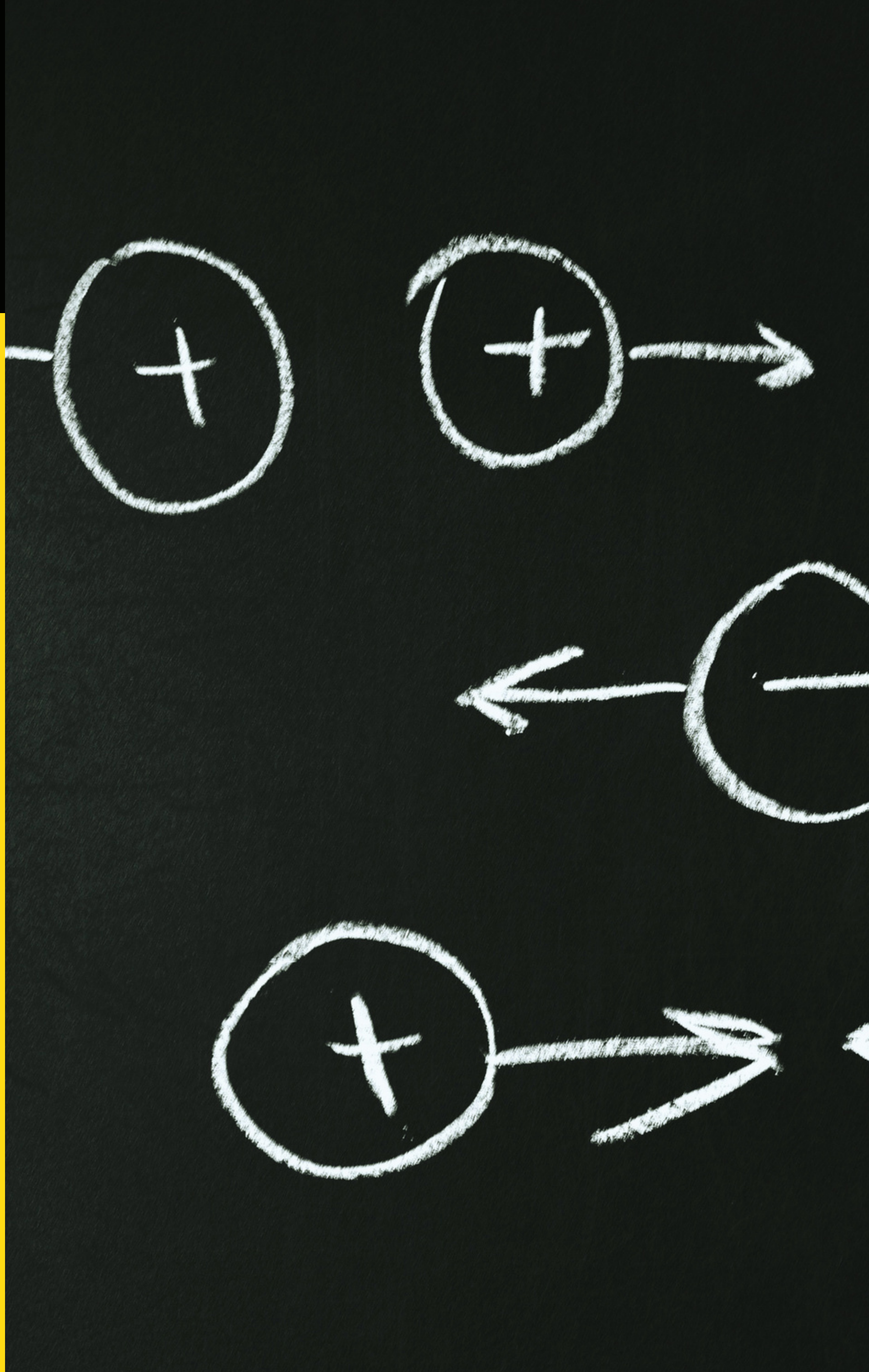
- **COULOMB FORCE**
- Electrostatic Potential
- Principle of Superposition
- Continuous distribution of charges

# COULOMB FORCE

$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2}$$

- Like charges repel and unlike charges attract;
- The force acts along the line joining the two point charges



# COULOMB FORCE

## ELECTROSTATIC FORCE

$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2}$$

- Proportional to electric charge of each of the two interacting objects
- Inversely proportional to square of the distance
- Proportional to Coulomb constant **K,** which depends on medium type (vacuum, air, water, etc)

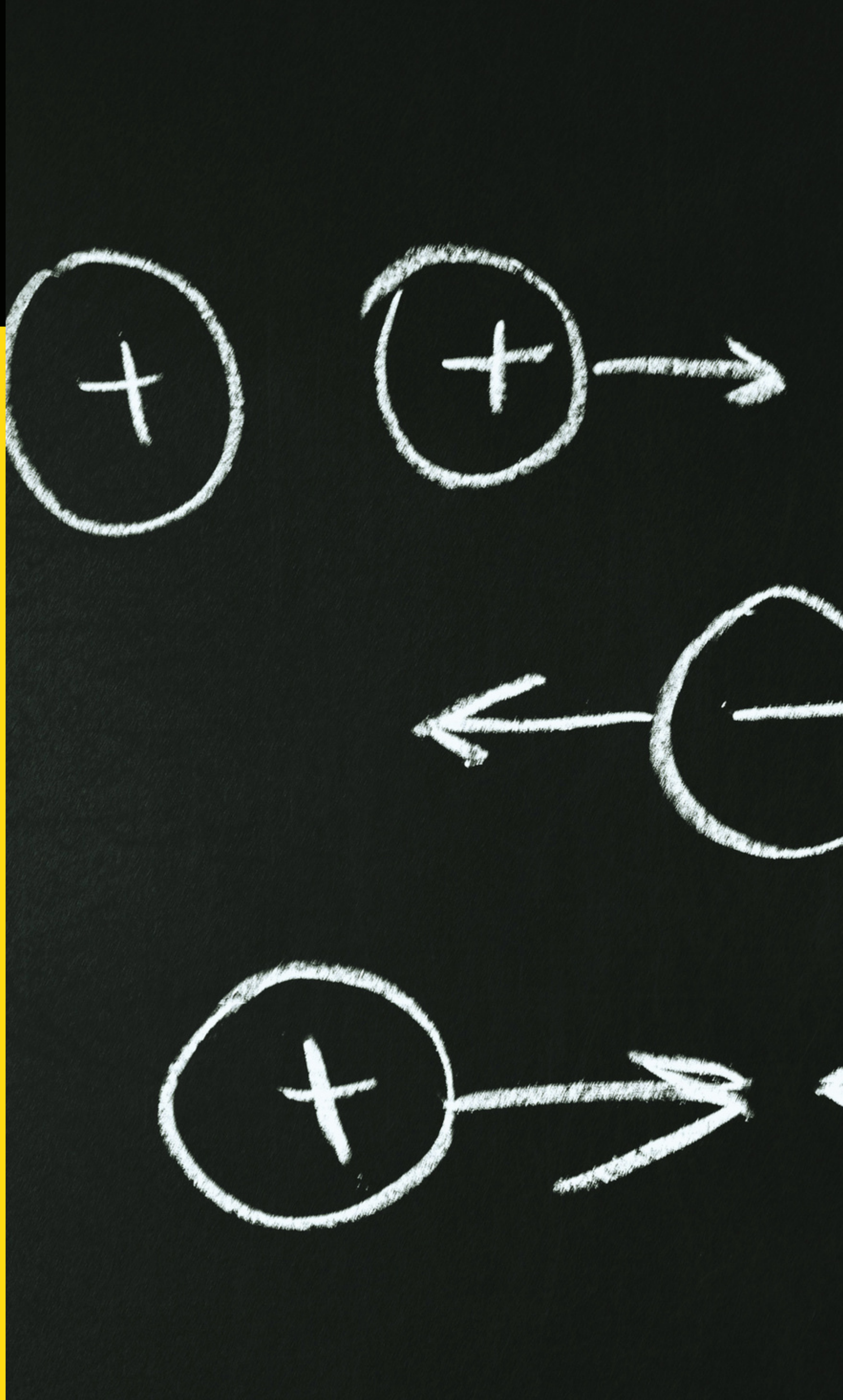
$$K = \frac{1}{4\pi\varepsilon_0} = 9 \cdot 10^9 \frac{C}{N \cdot m}$$

$$K = \frac{1}{4\pi\varepsilon}$$

material permittivity of dielectric

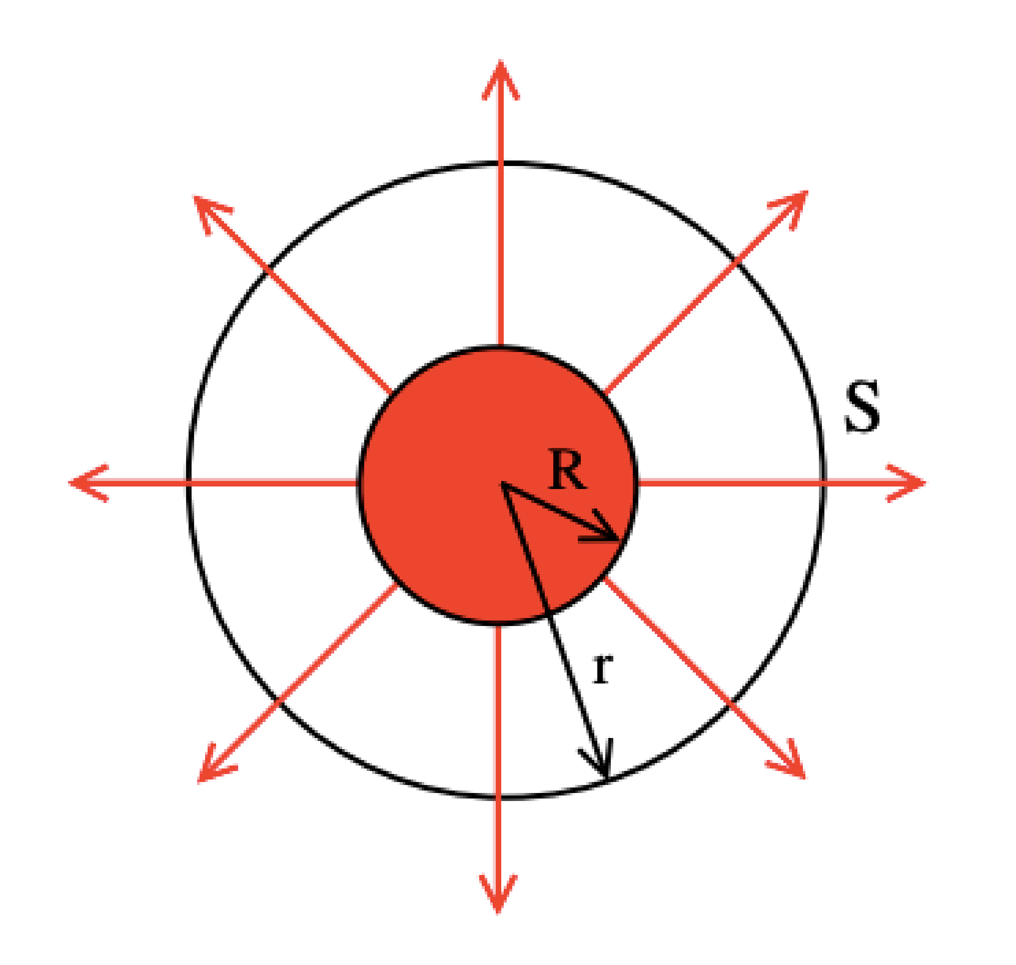
$$\varepsilon = \varepsilon_r \varepsilon_0 = (1 + \chi)\varepsilon_0$$

**ε_r** - relative permittivity
**χ** - susceptibility of the material



# COULOMB FORCE VS GAUSS LAW

- Take a particle of charge **Q** and radius **R** and Gaussian surface **S** to be a sphere of radius **r**
- We want to know the electric field at some radius **r > R**




$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}$$

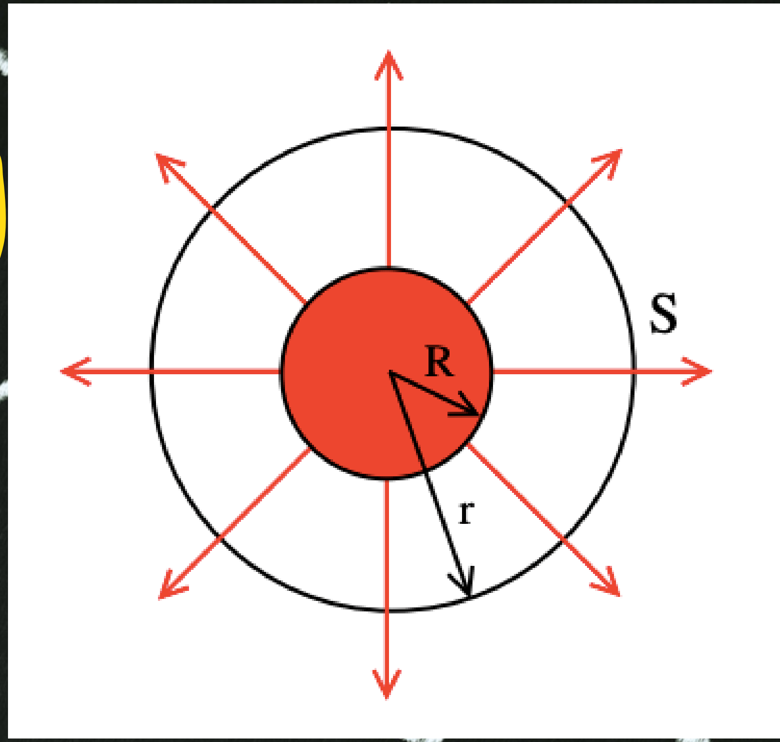
$$\mathbf{E}(\mathbf{x}) = E(r)\hat{\mathbf{r}}$$

$$\int_S \mathbf{E} \cdot d\mathbf{S} = E(r) \int_S \hat{\mathbf{r}} \cdot d\mathbf{S} = E(r)\, 4\pi r^2 = \frac{Q}{\epsilon_0}$$

# COULOMB FORCE VS GAUSS LAW

$$\int_S \mathbf{E}\cdot d\mathbf{S} = E(r)\int_S \hat{\mathbf{r}}\cdot d\mathbf{S} = E(r)\,4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\mathbf{E}(\mathbf{x}) = E(r)\hat{\mathbf{r}}$$




$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

electric field outside a sherically symmetric distribution of charge **Q**

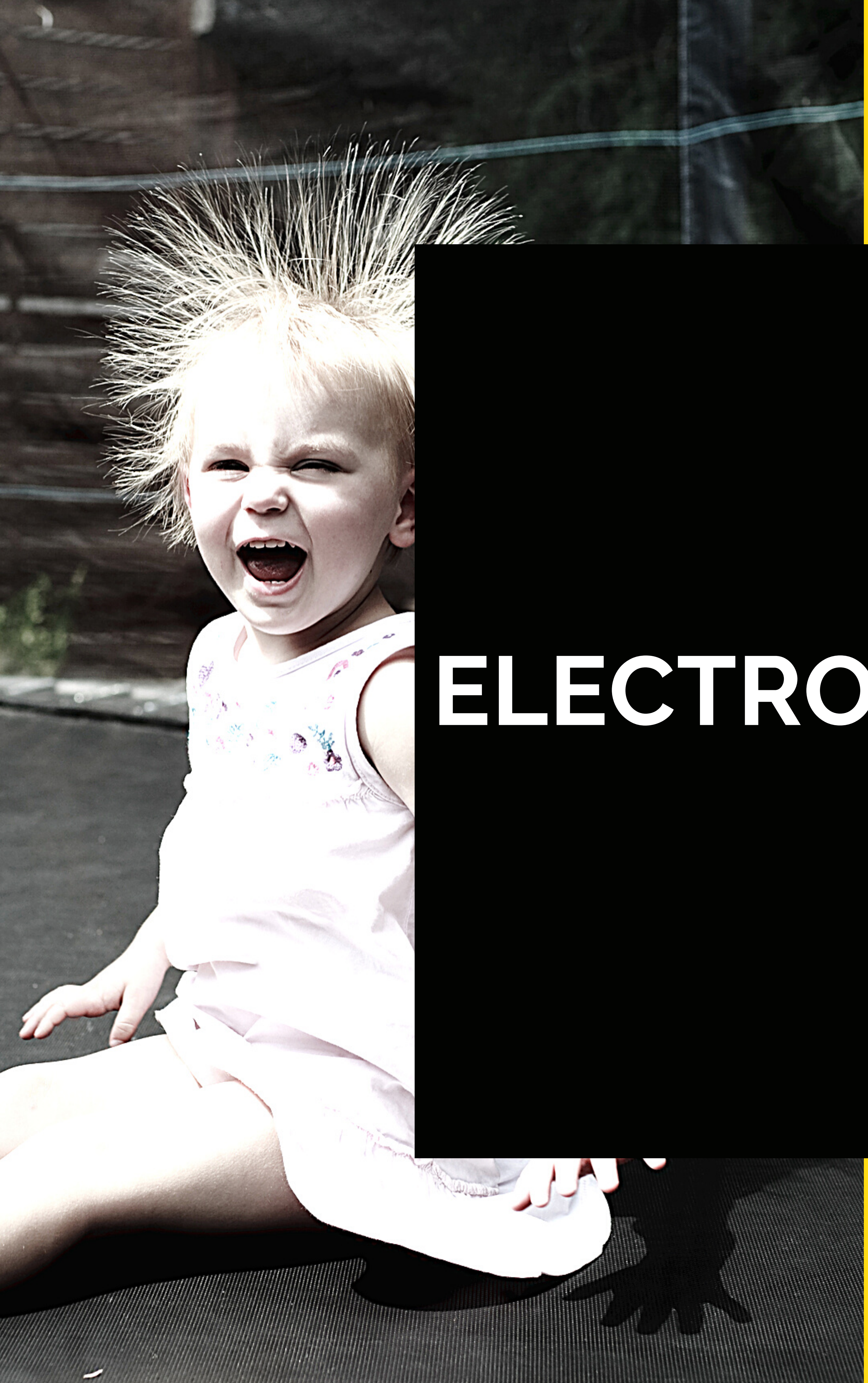
$$\mathbf{F} = \frac{Qq}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

$$F_E = K\cdot\frac{q_1\cdot q_2}{r^2}$$

By the **Lorentz force law:** $\boldsymbol{F} = q\boldsymbol{E}$

force experienced by a test charge **q**

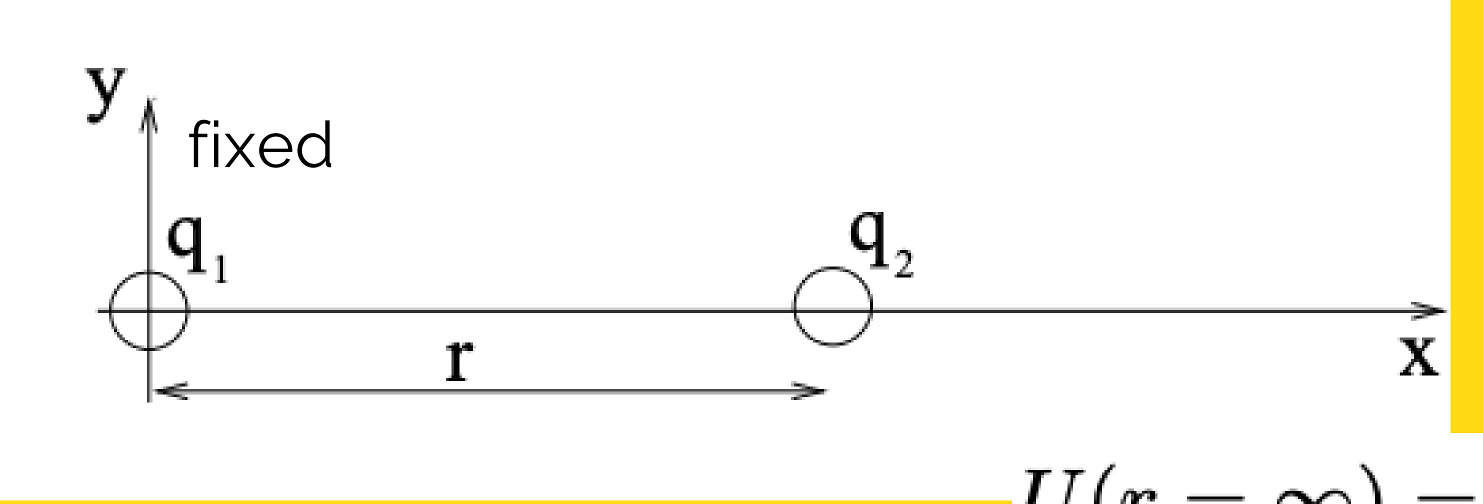
moving in the region **r>R**



# ELECTROSTATICS

- Coulomb Force
- **ELECTROSTATIC POTENTIAL**
- Principle of Superposition
- Continuous distribution of charges

# Electrostatic Potential energy




$$U(r=\infty)=0$$

## Energy

If we let the charge **q2** move upon electrostatic force, then it starts accelerating and gain kinetic energy. Consequently it will lose potential energy.

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

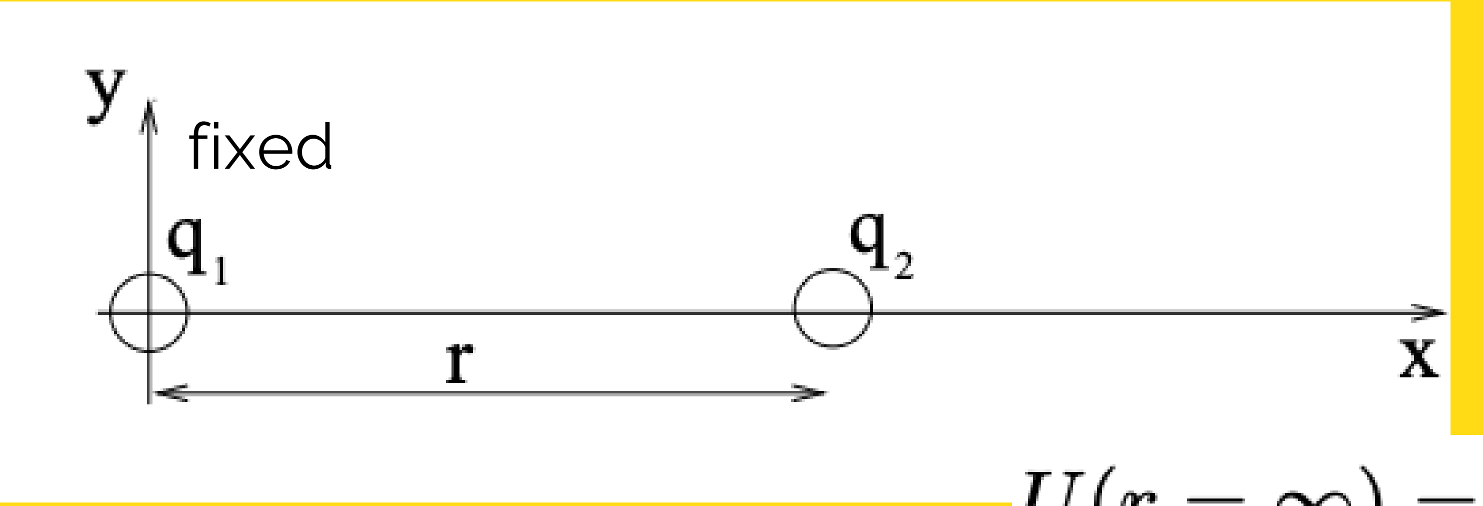
## Potential Energy

Work needed to bring 2 point-like charges together (or to a distance **r**) .

$$W = \int_\infty^r \boldsymbol{F}\cdot d\boldsymbol{r} = q_1\int_\infty^r \boldsymbol{E}\cdot d\boldsymbol{r} = Kq_1q_2\int_\infty^r \frac{dr}{r^2} = Kq_1\, q_2\frac{1}{r}$$

$$U_E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1\cdot q_2}{r}$$

This work **W** is stored as potential energy **U**




$$U(r=\infty)=0$$

## Potential Energy

The potential energy for a collection of point charges is. the **sum of contributions** for each pair of particles.

## Energy

If we let the charge **q2** move upon

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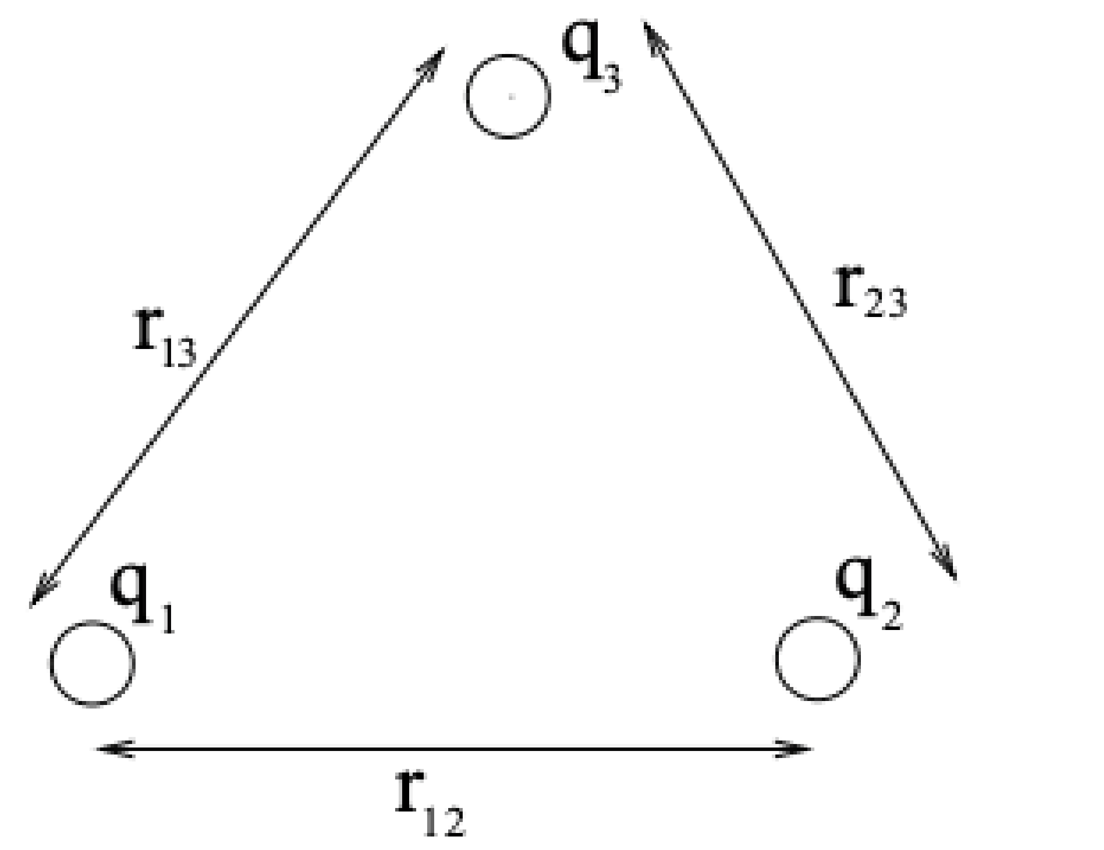



$$U_E=\frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1\cdot q_2}{r}$$

$$U=\frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r_{12}}+\frac{1}{4\pi\epsilon_0}\frac{q_1q_3}{r_{13}}+\frac{1}{4\pi\epsilon_0}\frac{q_2q_3}{r_{23}}$$

# Electrostatic Potential

## Electric Potential

$$U = q\phi$$

volts

the **electrical potential energy per charge** is the electric potential. The scalar is called the **electrostatic potential** or **scalar potential** (or, sometimes, just the **potential)**.

## Maxwell Equations: Electrostatics.

The two can be combined into the **Poisson equation**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = 0$$

$$\mathbf{E} = -\nabla\phi$$

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

# Electrostatic Potential

The two can be combined into the **Poisson equation**

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

**Laplace equation**

$$\nabla^2 \phi = 0$$

Solutions to the Laplace equation are said to be **harmonic functions.**

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi$$

Laplacian

# ELECTROSTATICS

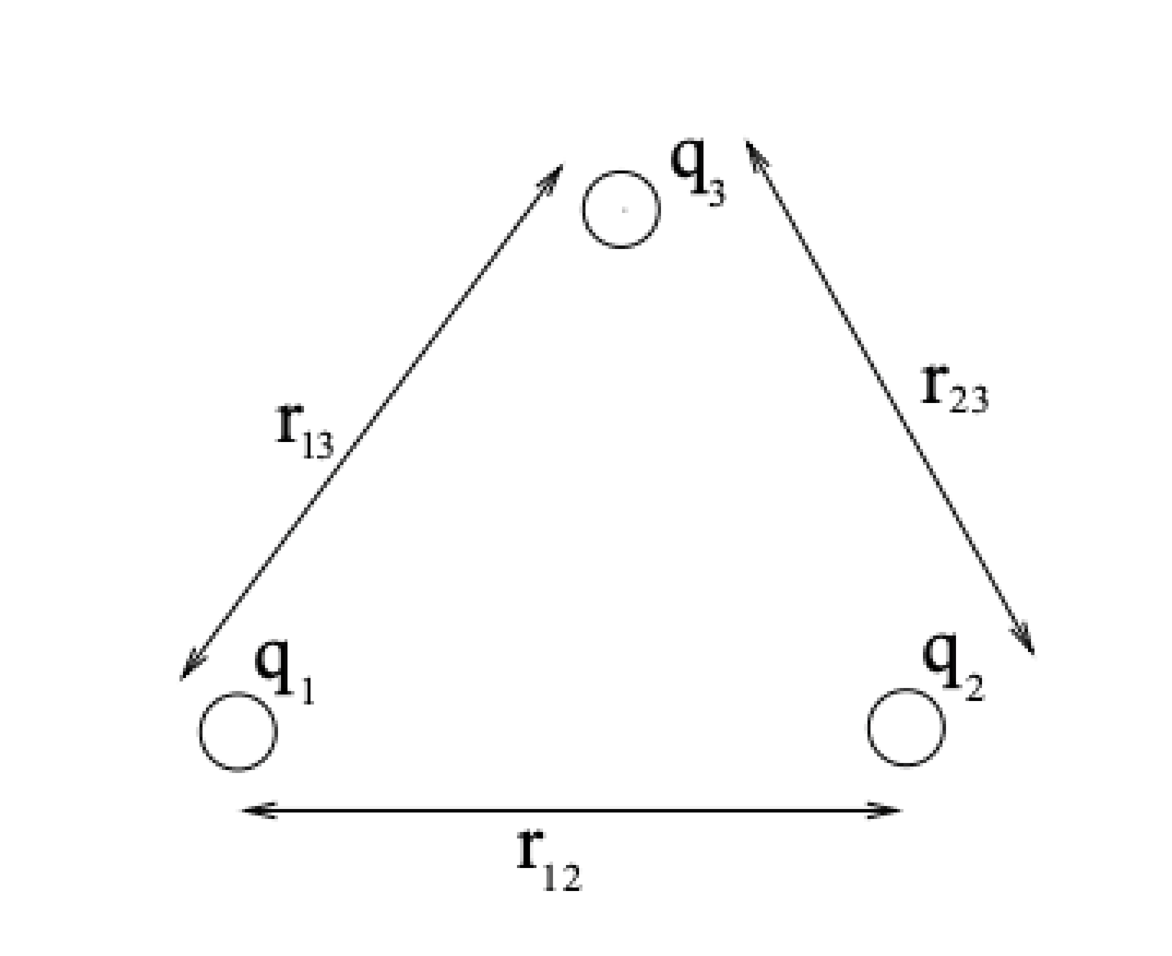
- Coulomb Force
- Electrostatic Potential
- **PRINCIPLE OF SUPERPOSITION**
- Continuous distribution of charges

# PRINCIPLE OF SUPERPOSITION

**The net electric field at a location in space is equal to the vector sum of individual electric fields contributed by all charged particles located elsewhere.**

Thus, the electric field contributed by a charged particle is unaffected by the presence of other charged particles.

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}}$$



# ELECTROSTATICS

- Coulomb Force
- Electrostatic Potential
- Principle of Superposition
- **CONTINUOUS DISTRIBUTION OF CHARGES**

# CONTINUOUS DISTRIBUTION OF CHARGE

The region in which charges are closely spaced is said to have

**CONTINUOUS DISTRIBUTION OF CHARGE.**

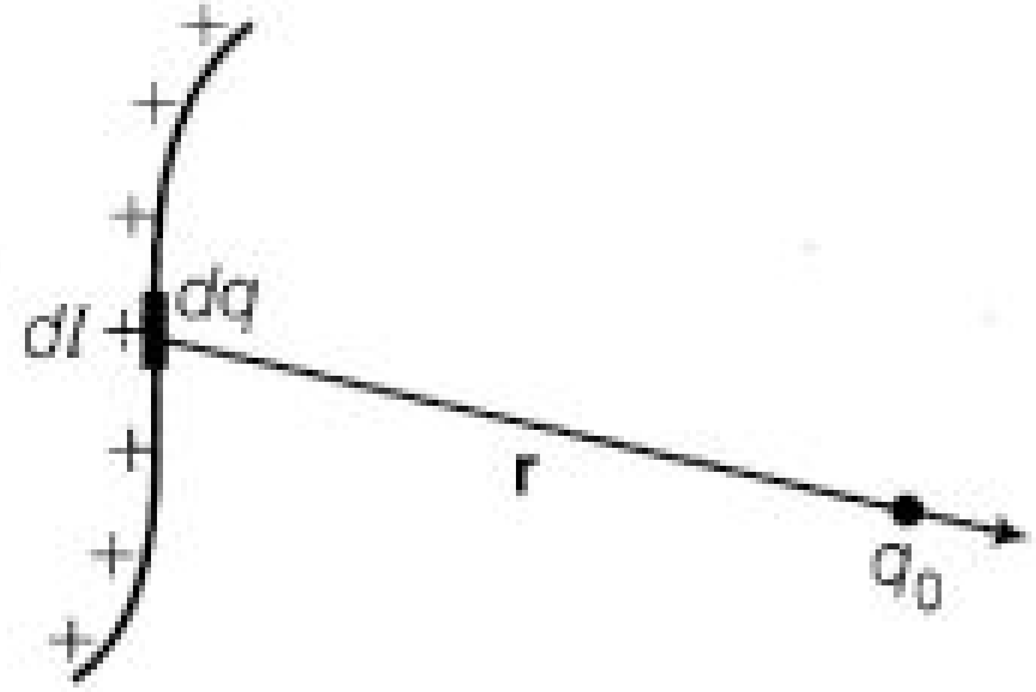

**(i) Linear Charge Distribution**

$$dq = \lambda\, dl$$

where, $\lambda$ = linear charge density

$$d\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_0\,(dq)}{|\mathbf{r}|^2}\,\hat{\mathbf{r}} \Rightarrow d\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_0\,(\lambda\, dl)}{|\mathbf{r}|^2}\,\hat{\mathbf{r}}$$

Net force on charge $q_0$, $\mathbf{F} = \frac{q_0}{4\pi\varepsilon_0} \int_l \frac{\lambda\, dl}{|\mathbf{r}|^2}\,\hat{\mathbf{r}}$



# CONTINUOUS DISTRIBUTION OF CHARGE

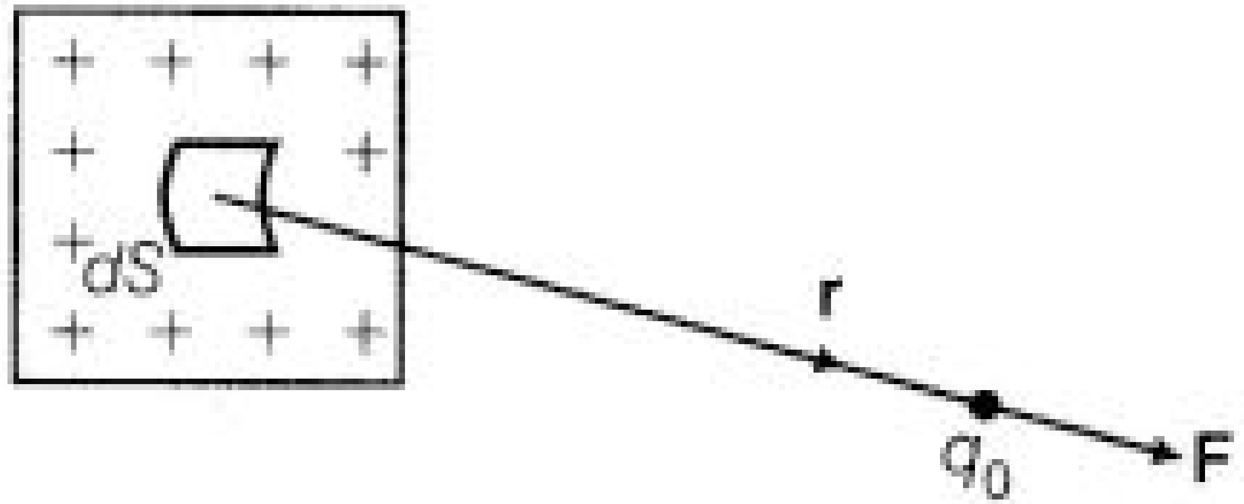

The region in which charges are closely spaced is said to have **CONTINUOUS DISTRIBUTION OF CHARGE.**

**(ii) Surface Charge Distribution**

$$dq = \sigma \, dS$$

where, $\sigma$ = surface charge density

Net force on charge $q_0$, $\mathbf{F} = \dfrac{q_0}{4\pi\varepsilon_0} \int_S \dfrac{\sigma \, dS}{|\mathbf{r}|^2} \hat{\mathbf{r}}$



# CONTINUOUS DISTRIBUTION OF CHARGE

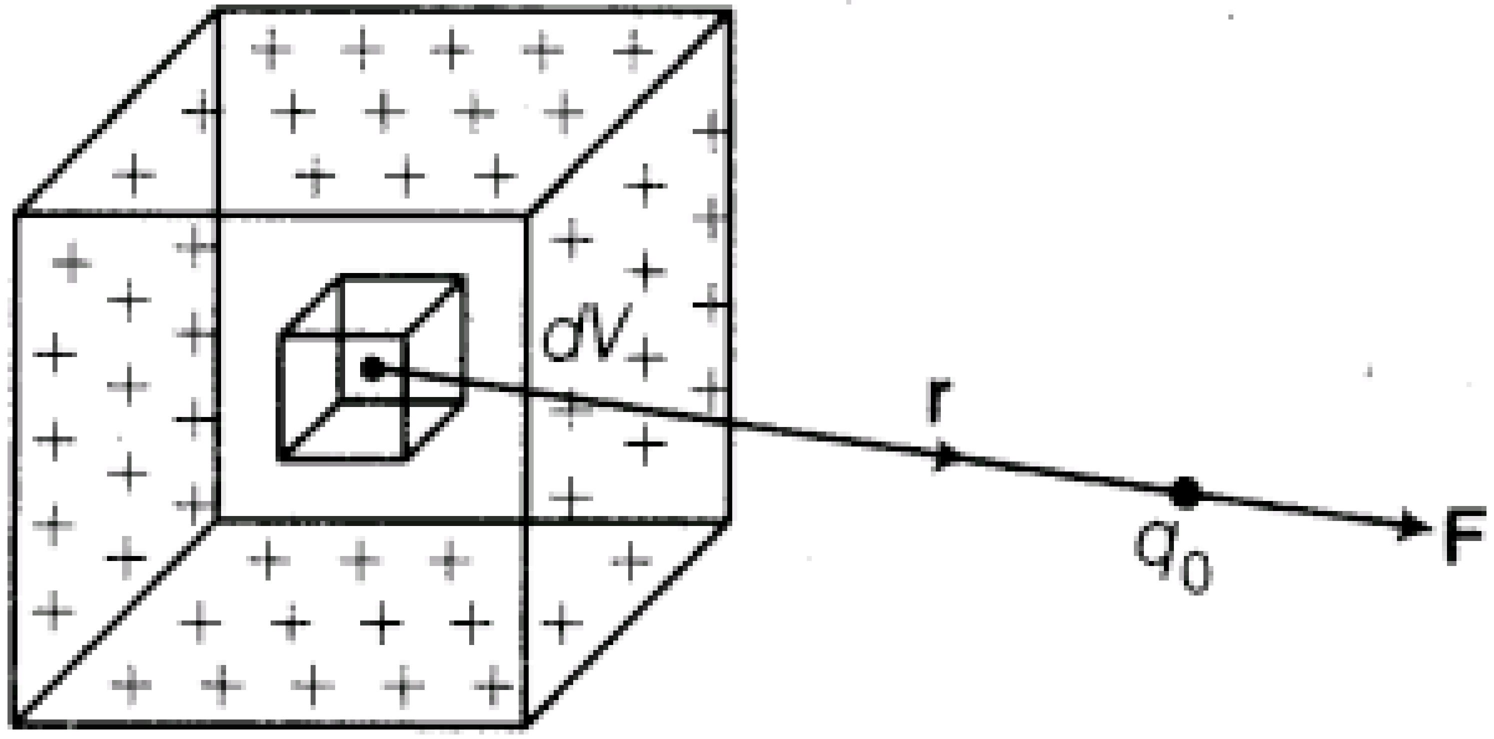
The region in which charges are closely spaced is said to have **CONTINUOUS DISTRIBUTION OF CHARGE.**


## (*iii*) Volume Charge Distribution

$$dq = \rho\, dV$$

where, $\rho$ = volume charge density

Net force on charge $q_0$, $\mathbf{F} = \dfrac{q_0}{4\pi\varepsilon_0}\displaystyle\int_V \dfrac{\rho\, dV}{|\mathbf{r}|^2}\,\hat{\mathbf{r}}$




$$\frac{1}{4\pi\epsilon_0}\int_V d^3r'\ \rho(\mathbf{r}')\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

## MAGNETOSTATICS

- Charges give rise to electric fields.
- Current give rise to magnetic fields.
- Moving charge particles make a magnetic field which is different from the electric field
- The magnetic field is induced by steady currents - continuous flow of charge

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

# MAGNETOSTATICS

$$\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

- **STEADY CURRENT**
- Ampère's Law
- Vector Potential
- Biot-Savart Law
- Motion of a charged particle

# Steady Current

**Continuity equation, which captures the conservation of electric charge:**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

charge density can change in time only if there is a compensating current flowing into or out of that region

**Since the charge density is unchanging (and, indeed, vanishing)...**

$$\cancel{\frac{\partial \rho}{\partial t}} + \nabla \cdot \mathbf{J} = 0$$

# Steady Current

MATHEMATICALLY:

**IF A CURRENT FLOWS INTO SOME REGION OF SPACE, AN EQUAL CURRENT MUST FLOW OUT TO AVOID THE BUILD UP OF CHARGE.**

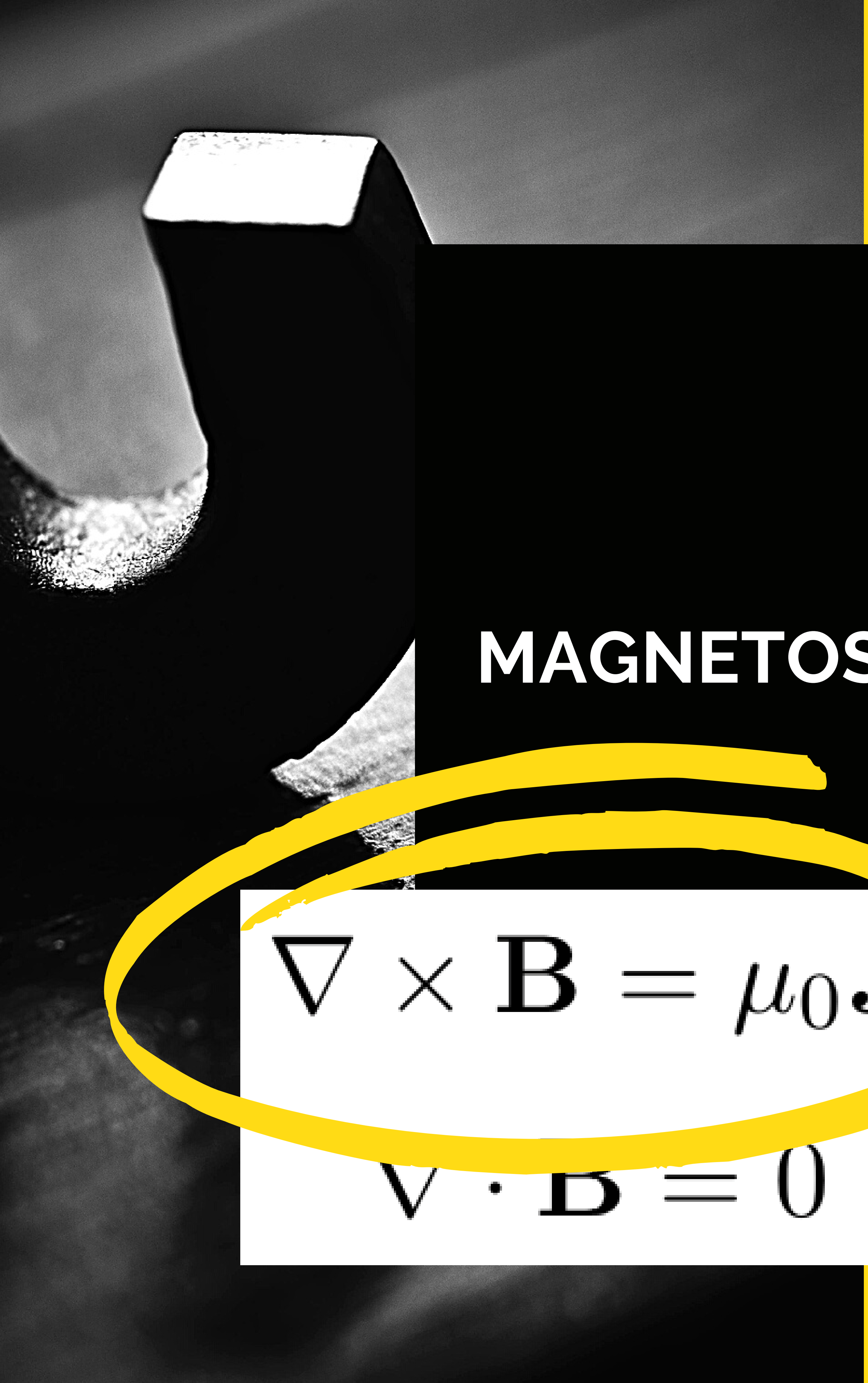
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

This is consistent with Maxwell Equations for magnetostatics

$$\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0$$



## MAGNETOSTATICS

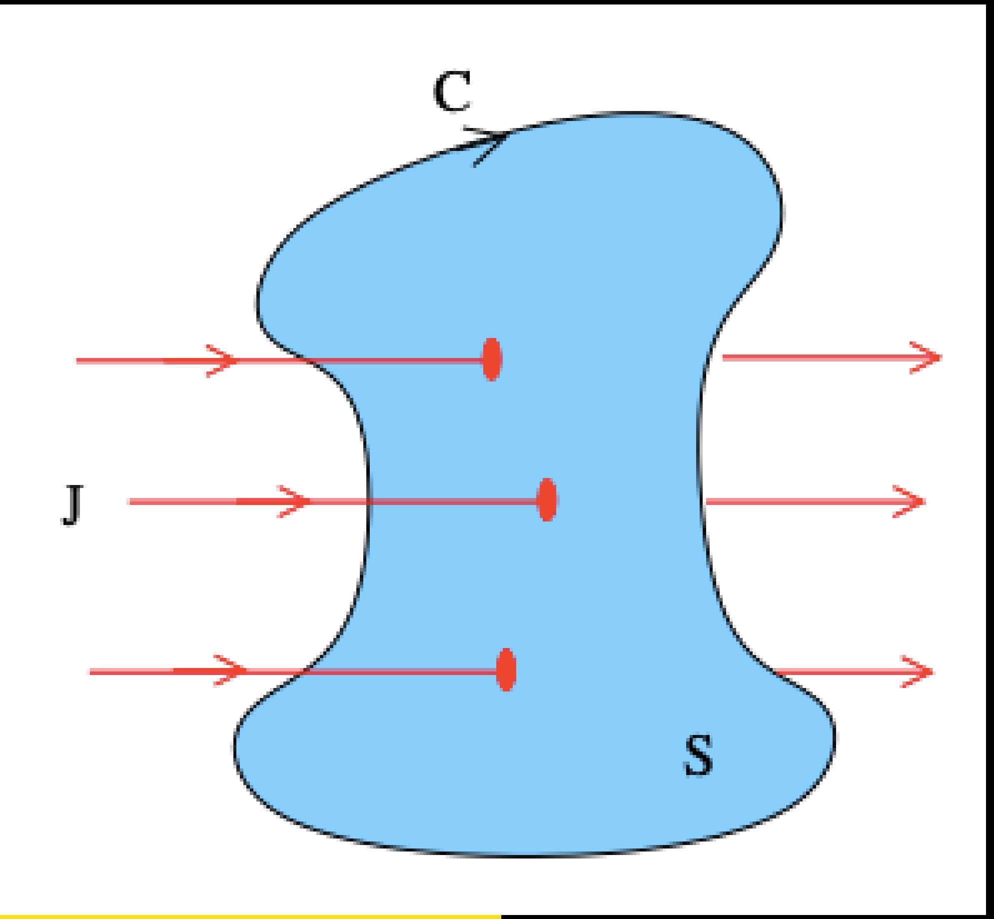
$$\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

- Steady Current
- **AMPÈRE LAW**
- Vector Potential
- Biot-Savart Law
- Motion of a charged particle

# AMPÈRE LAW

$$\nabla \times \mathbf{B} = \mu_0 \boldsymbol{J}$$



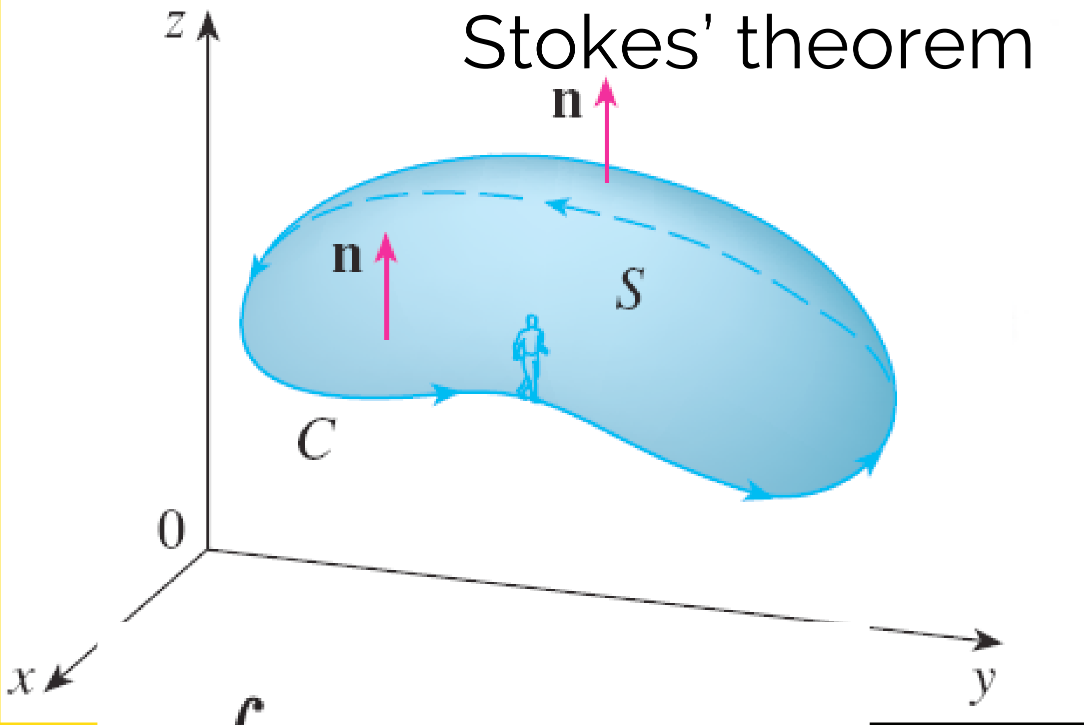

## RELATIONSHIP BETWEEN A CURRENT AND THE MAGNETIC FIELD IT GENERATES

Ampere's law states that the integral of the magnetic field around the contour **C** equals

$$\int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}$$

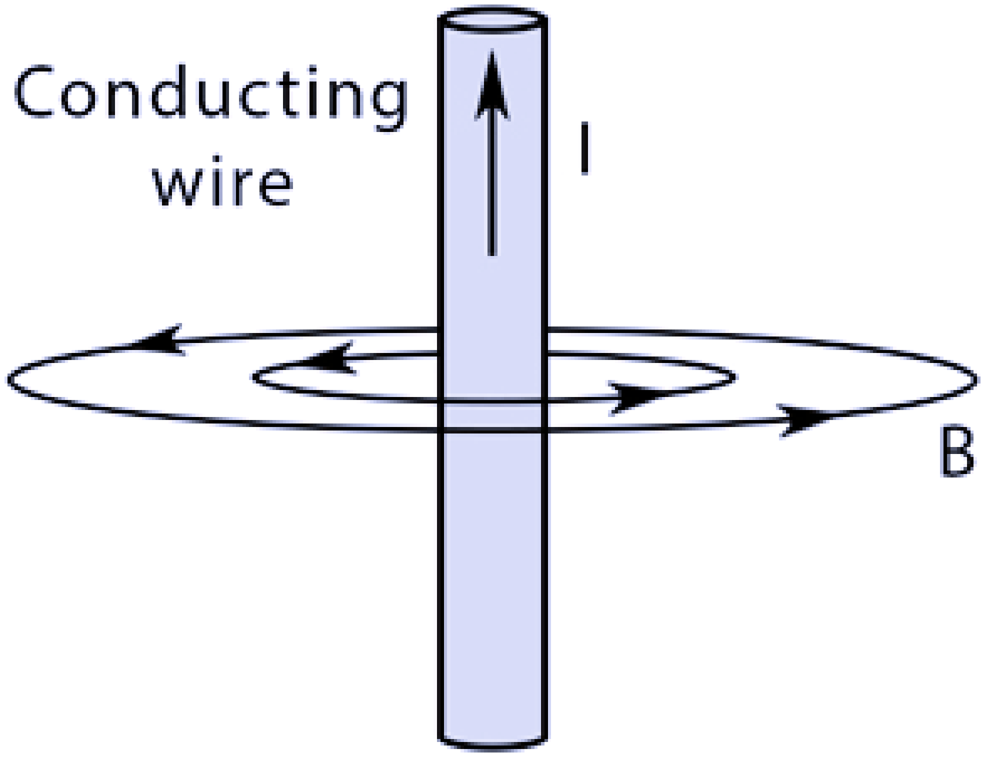
= I

Stokes' theorem

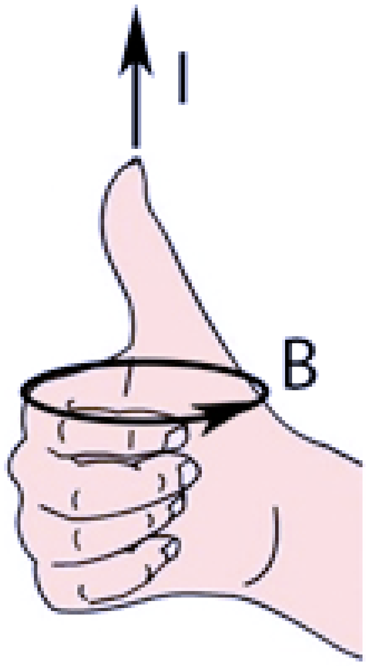



$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

# Ampere's Law




Right hand thumb rule




Integral form: $\oint \vec{B}.\vec{dl} = \mu_o I$

Differential form: $\vec{\nabla} X \vec{B} = \mu_o \vec{J}$

I : Electric curent

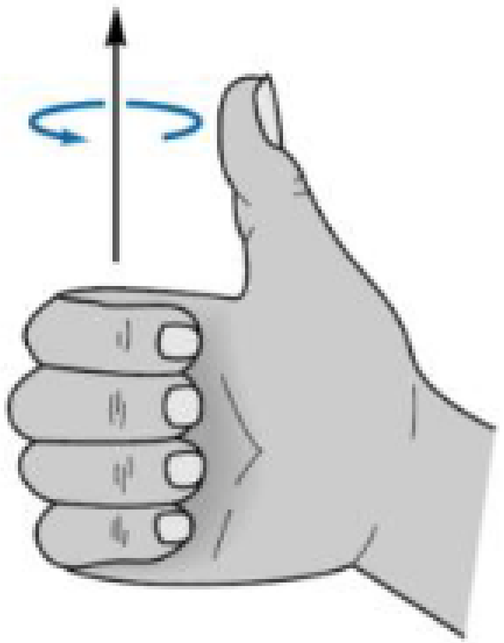
B : Magnetic field

$\mu_o$: Permeability of free space

J : Current density

Thumb points in the direction of the electric current and fingers curl around the current indicating the direction of the magnetic field

Stokes' theorem



When the thumb points in the direction of $\hat{\mathbf{n}}$, the fingers curl in the forward direction around $C$

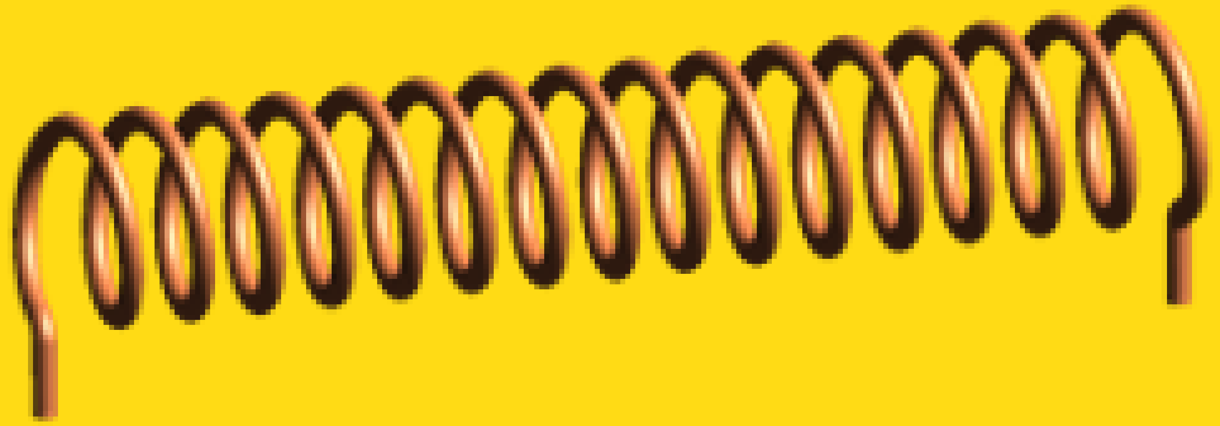
For positive current direction of magnetic field is determined with rule of right hand

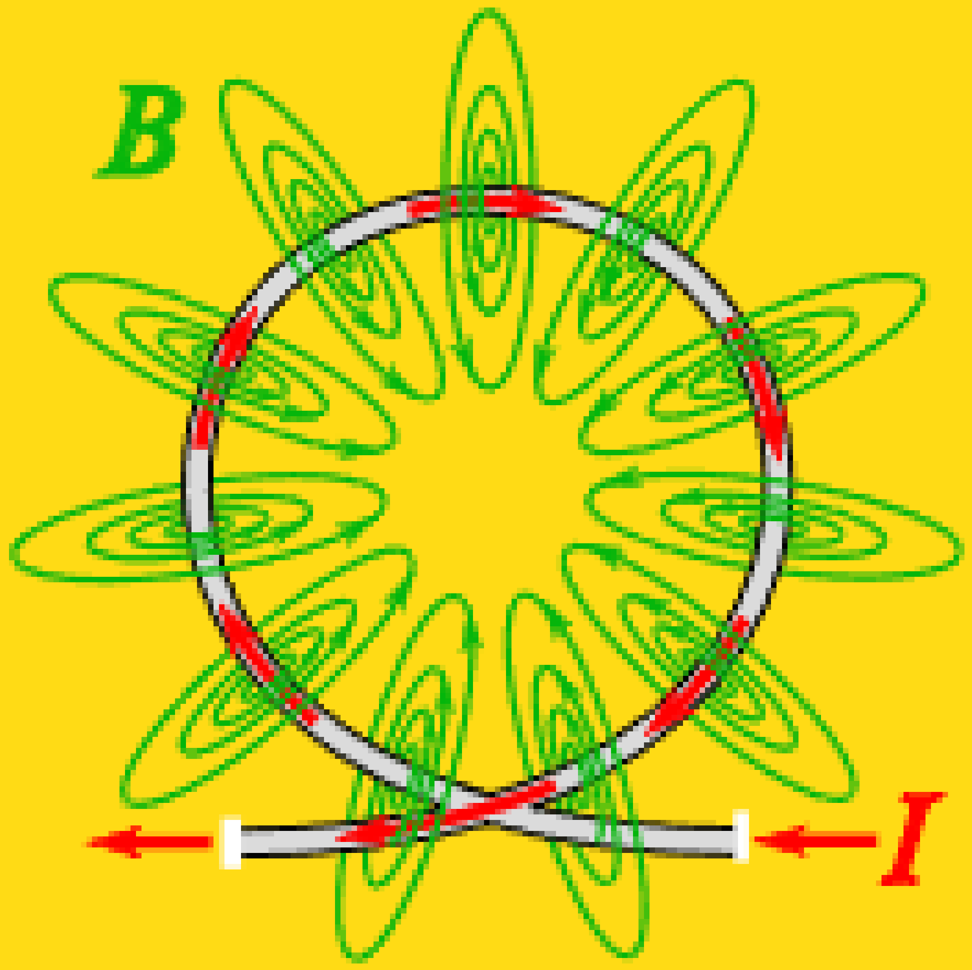
# AMPÈRE LAW

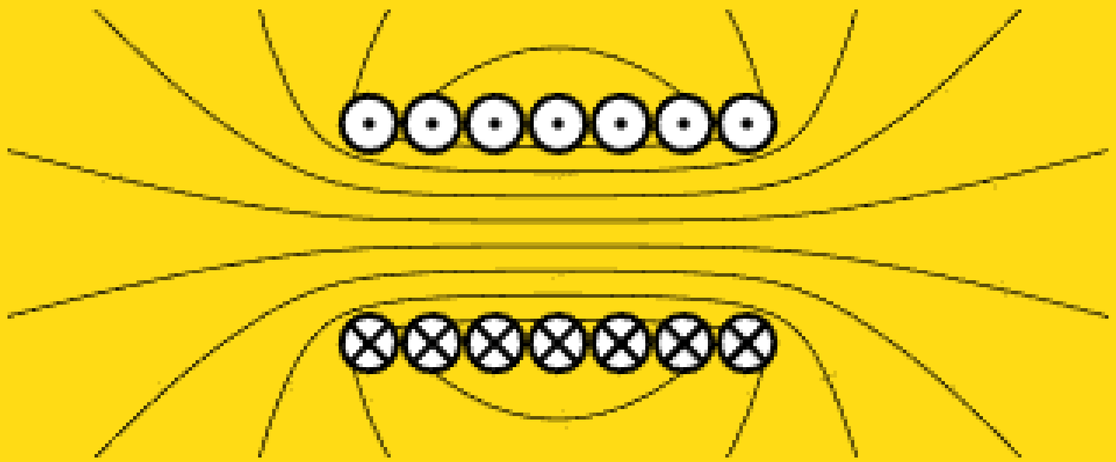
THE PRIMARY USAGE OF THE AMPERE LAW IS

**CALCULATING THE MAGNETIC FIELD GENERATED BY AN ELECTRIC CURRENT**

Ex: a long straight conducting wire, coaxial cable, cylindrical conductor, solenoid, and toroid





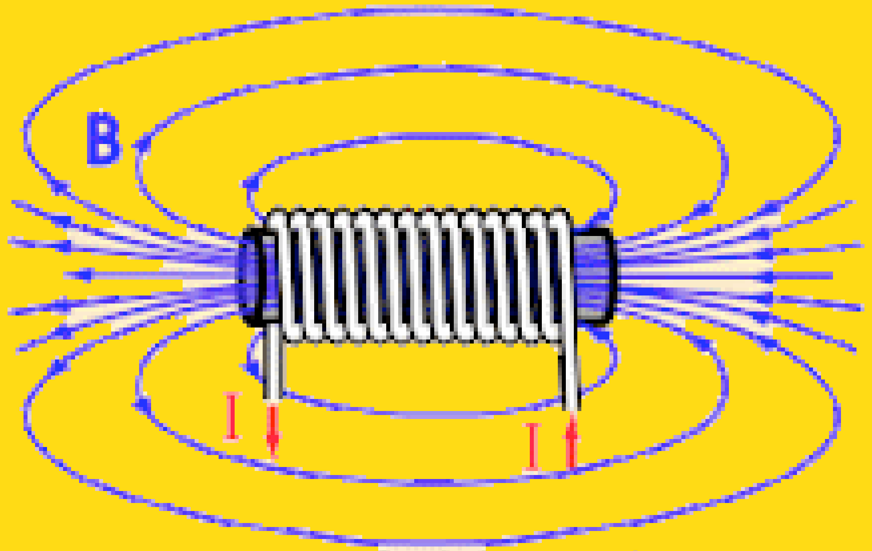
# AMPÈRE LAW

$$\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

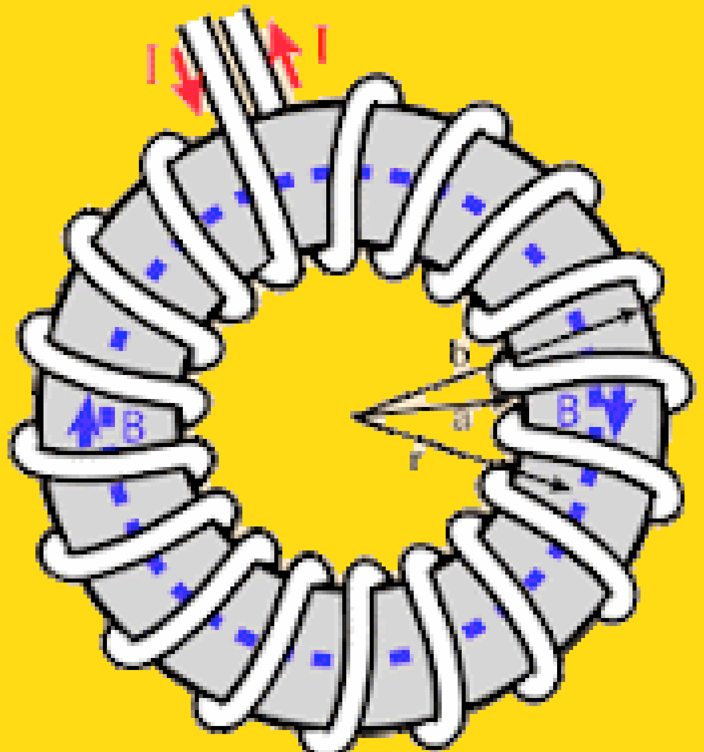
$BL = \mu NI$

$B = \mu \frac{N}{L} I$

$B = \mu n I$



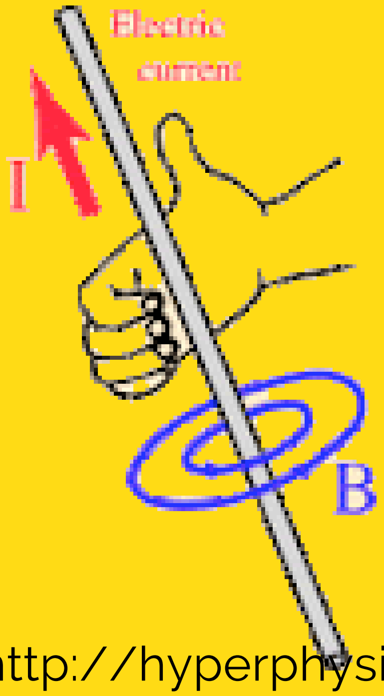

Magnetic field inside a long solenoic.



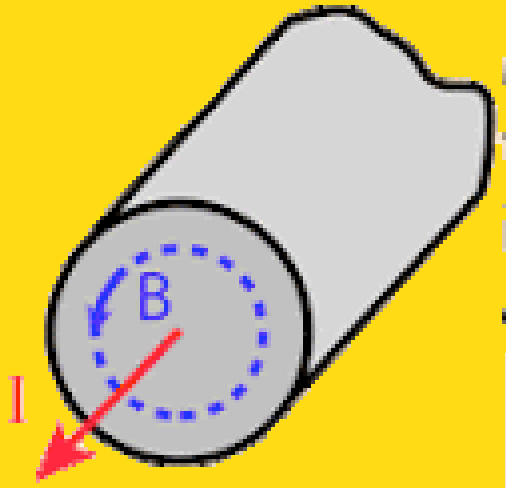

$$B = \frac{\mu NI}{2\pi r}$$

Magnetic field inside a toroidal coil.

$$B = \frac{\mu_0 I}{2\pi r}$$




Magnetic field from a long straight wire.




Magnetic field inside a conductor.

$$B = \frac{\mu J r}{2} = \frac{\mu r I}{2\pi R^2}$$

which at the surface approaches:

$$B_{surface} = \frac{\mu I}{2\pi R}$$

Outside the surface,

$$B 2\pi r' = \mu_0 I \quad \text{or} \quad B = \frac{\mu_0 I}{2\pi r'}$$

http://hyperphysics.phy-astr.gsu.edu

# MAGNETOSTATICS

$$\nabla \cdot \mathbf{B} = 0$$

- Steady Current
- Ampère Law
- **VECTOR POTENTIAL**
- Biot-Savart Law
- Motion of a charged particle

# VECTOR POTENTIAL

**To guaranteed a solution to** $\nabla \cdot \mathbf{B} = 0$

we write the magnetic field as the curl of some vector field

$$\mathbf{B} = \nabla \times \mathbf{A}$$

**A** – is called the **vector potential**

While magnetic fields that can be written in this form certainly satisfy the given condition, the converse is also true

**Ampère law becomes**

This is the equation that we have to solve to determine **A** and, through that, **B**

$$\nabla \times \mathbf{B} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mu_0 \mathbf{J}$$

MAGNETIC MONOPOLE

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \longrightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

**It says that there are no magnetic charges.**

A point-like magnetic charge **g** would source the magnetic field, giving rise a $1/r^2$ fall-off

$$\mathbf{B} = \frac{g\hat{\mathbf{r}}}{4\pi r^2}$$

object with this behaviour **– magnetic monopole**

**Maxwell's equations says that they don't exist**

# MAGNETOSTATICS

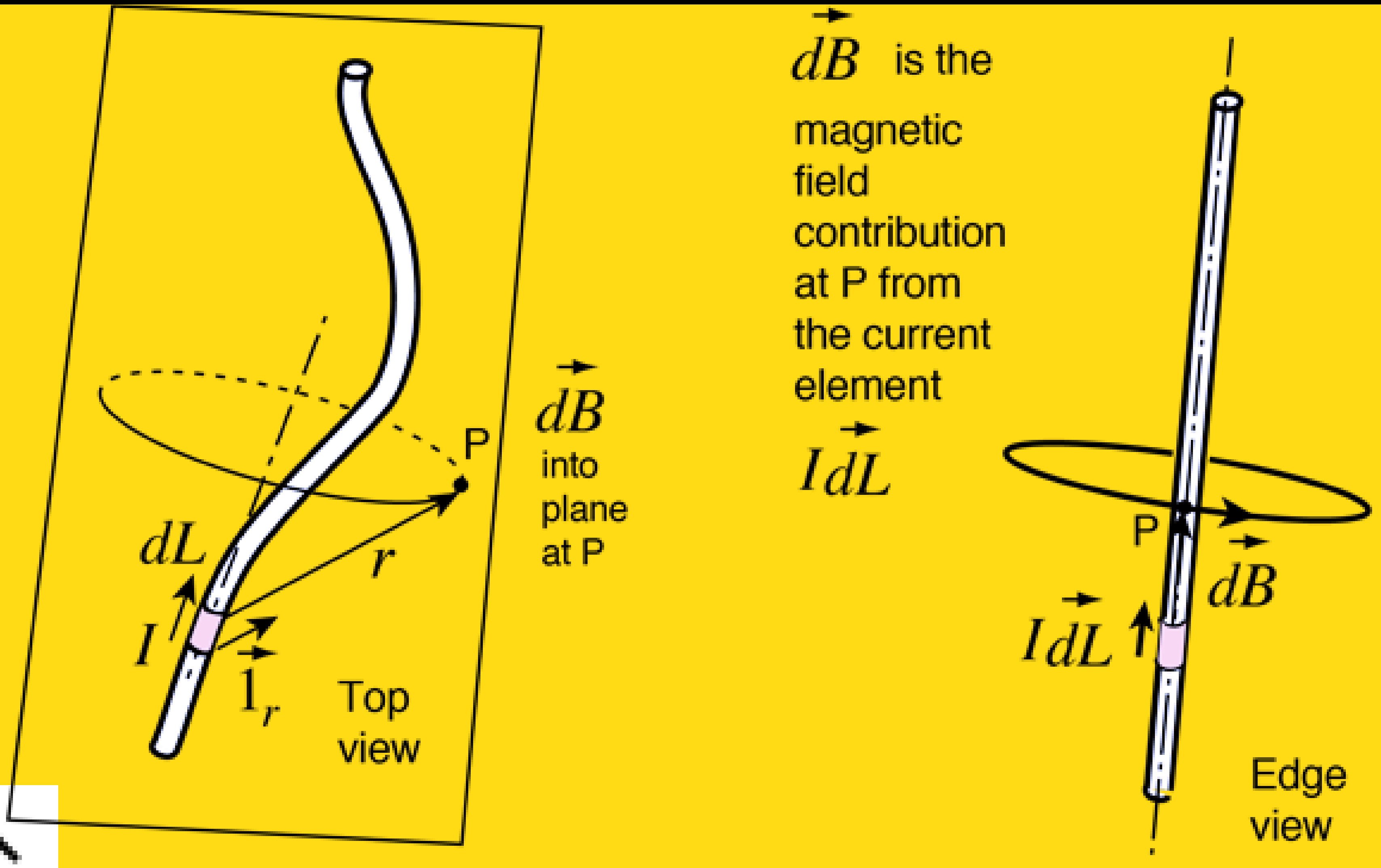
- Steady Current
- Ampère Law
- Vector Potential
- **BIOT-SAVART LAW**
- Motion of a charged particle

# BIOT-SAVART LAW

## THE ANALOGOUS OF COULOMB LAW

A segment of wire of length dl, carrying a current **I** sets up a magnetic field

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2}$$




Biot-Savart law for currents



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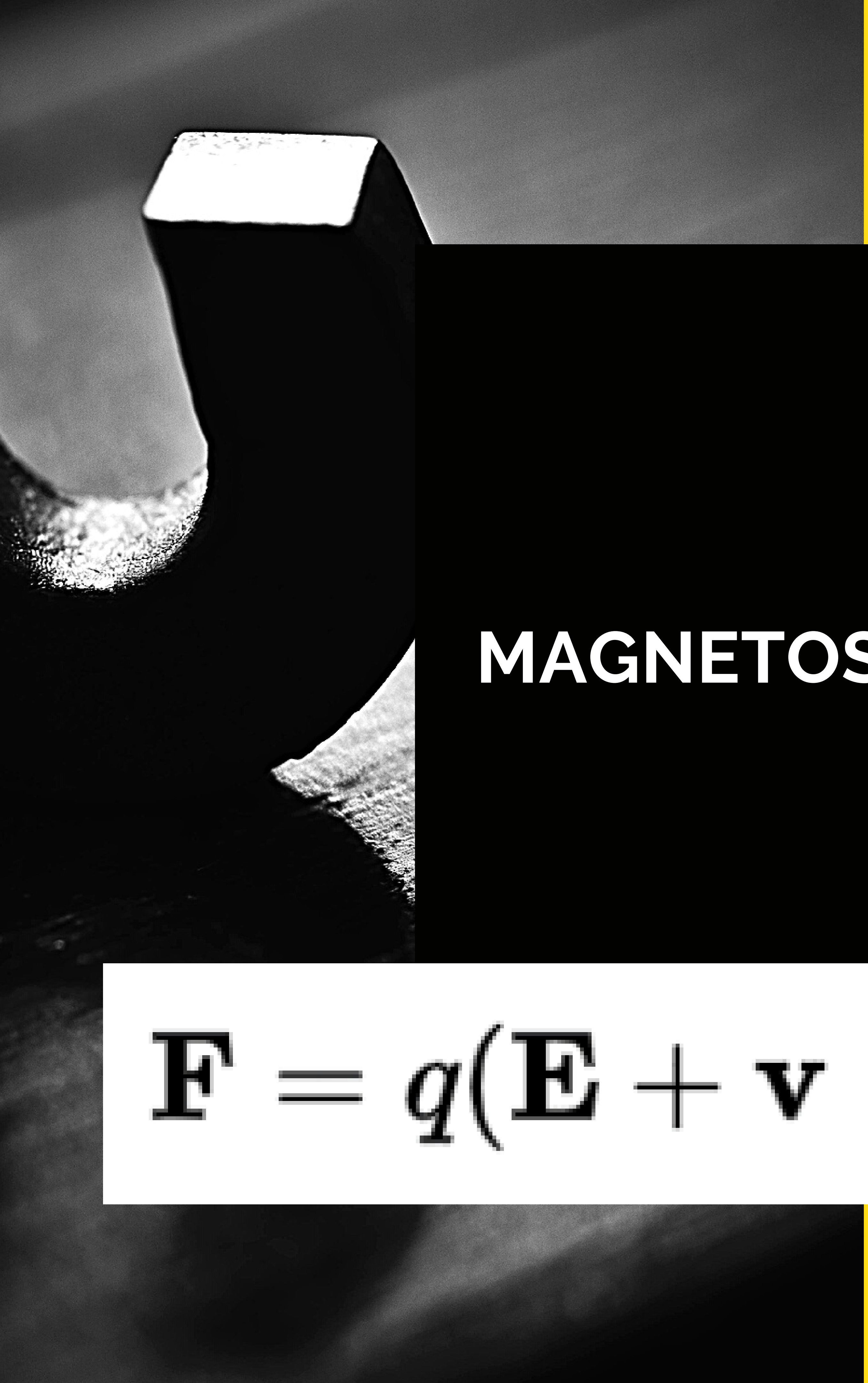


# MAGNETOSTATICS

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

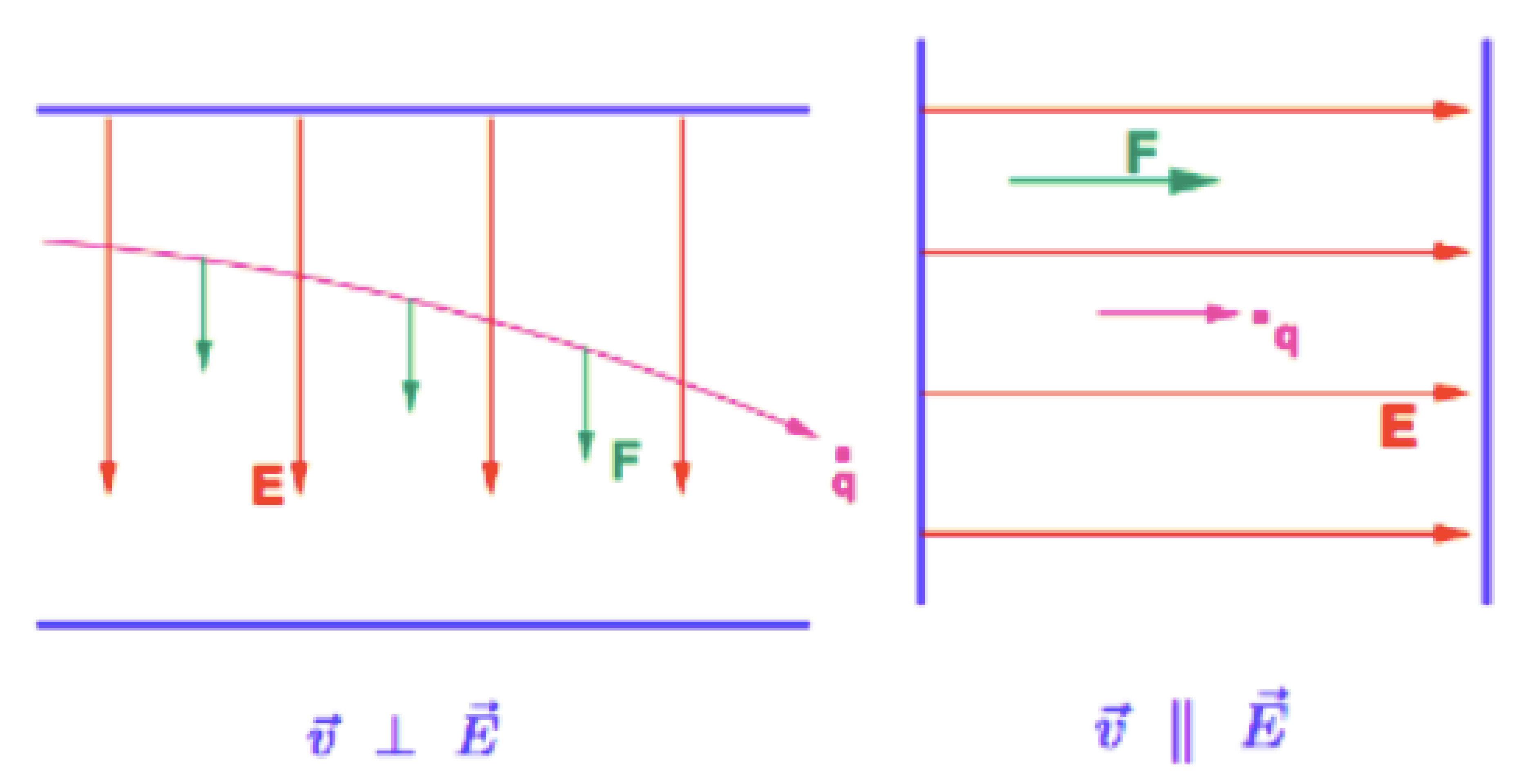
- Steady Current
- Ampère Law
- Vector Potential
- Biot-Savart Law
- **MOTION OF A CHARGED PARTICLE**

# LORENTZ FORCE

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{F} = q(\mathbf{E} + \cancel{\mathbf{v} \times \mathbf{B}})$$

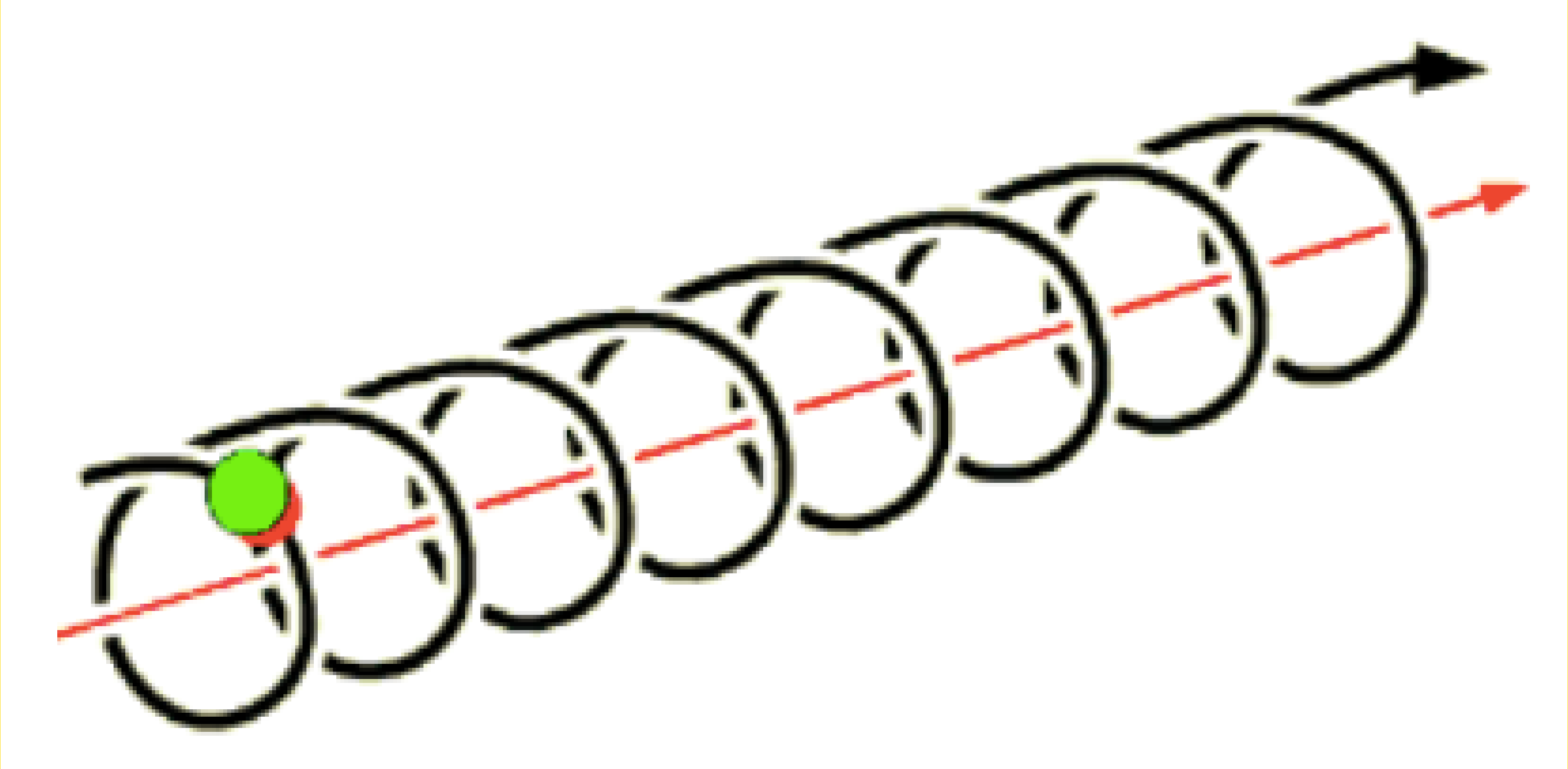
In case of an electric field, the force is always in the direction of the field, also for particles in rest.



MOTION OF A CHARGED PARTICLE

# LORENTZ FORCE

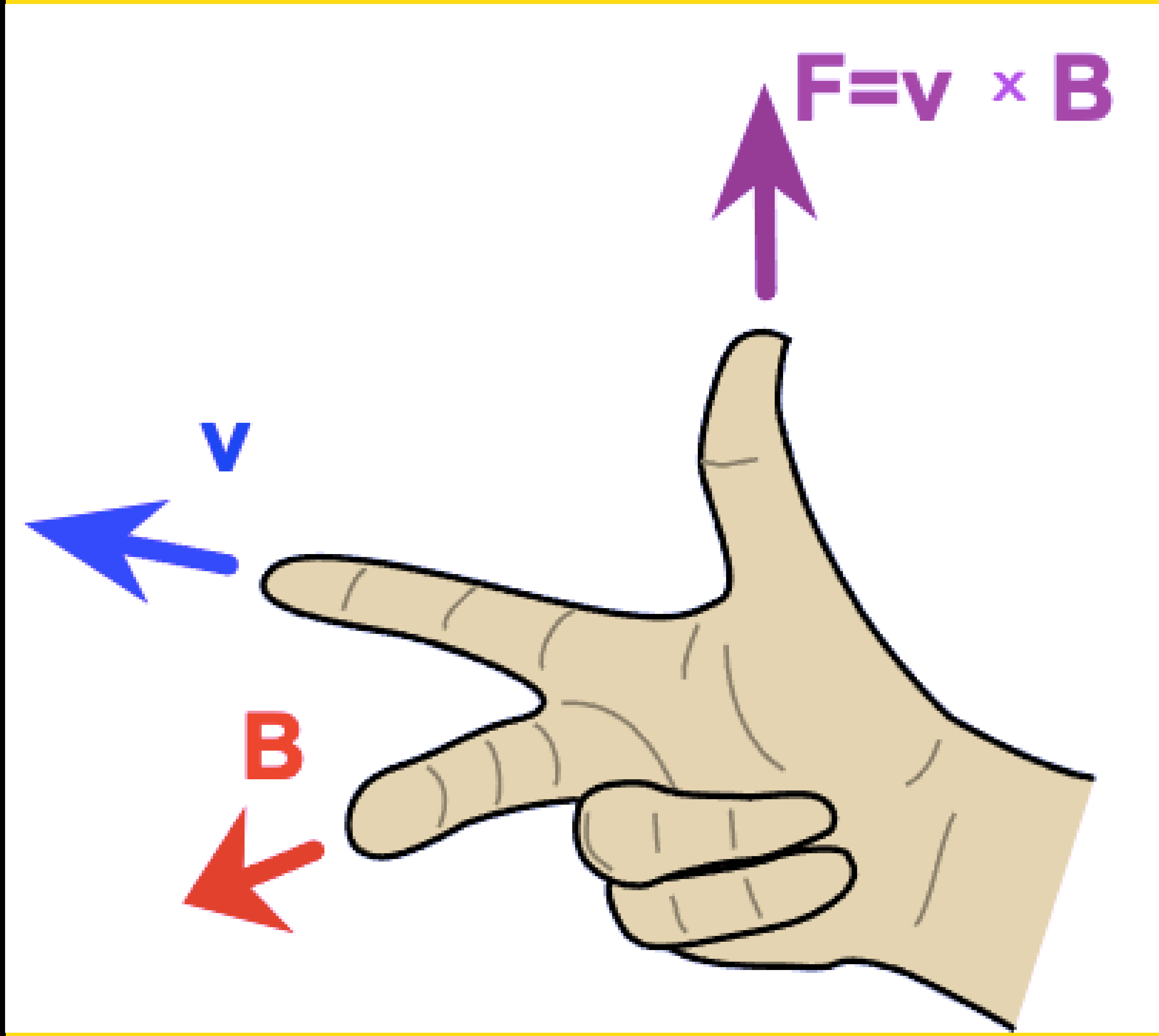
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

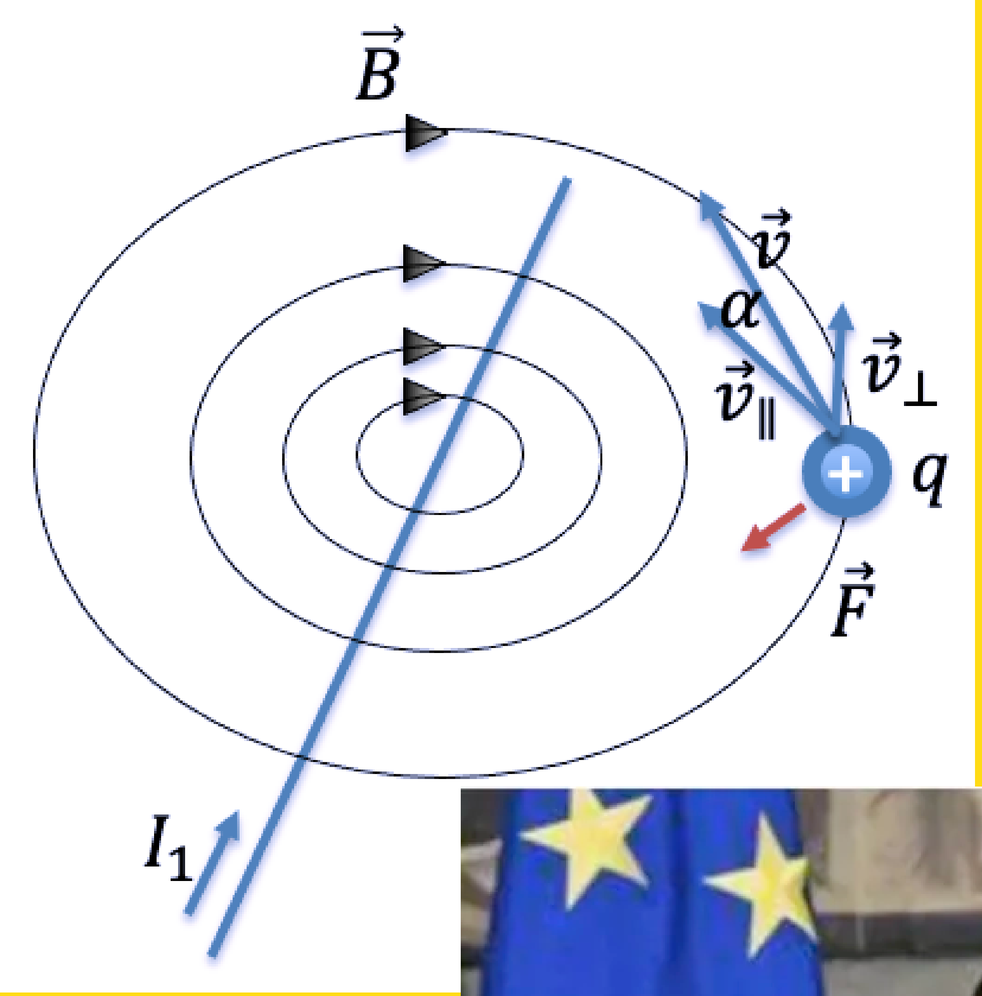


$$\mathbf{F} = q(\cancel{\mathbf{E}} + \mathbf{v} \times \mathbf{B})$$

In this case the force is perpendicular to both, **v** and **B**

# MOTION OF A CHARGED PARTICLE




# **ELECTROMAGNETISM:** NON-STATIC CASE

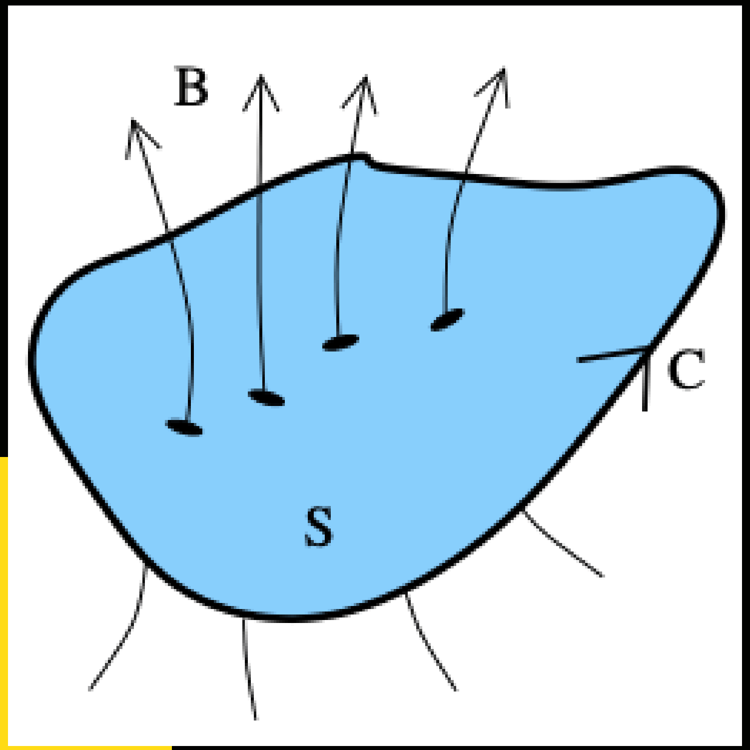
- **FARADAY'S LAW OF INDUCTION**
- Wave Function
- Propagation of electromagnetic waves in a conductor
- Propagation of electromagnetic waves in a highly conductive materials

WORDS OF WISDOM

“I was at first almost frightened when I saw such mathematical force made to bear upon the subject, and then wondered to see that the subject stood it so well.”

Faraday to Maxwell, 1857

# FARADAY'S LAW OF INDUCTION




The process of creating a current through changing magnetic fields is called **INDUCTION.**

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Stokes theorem

$$\int_C \mathbf{E} \cdot d\mathbf{r} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

$$\mathcal{E} = \int_C \mathbf{E} \cdot d\mathbf{r}$$

electromotive force

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

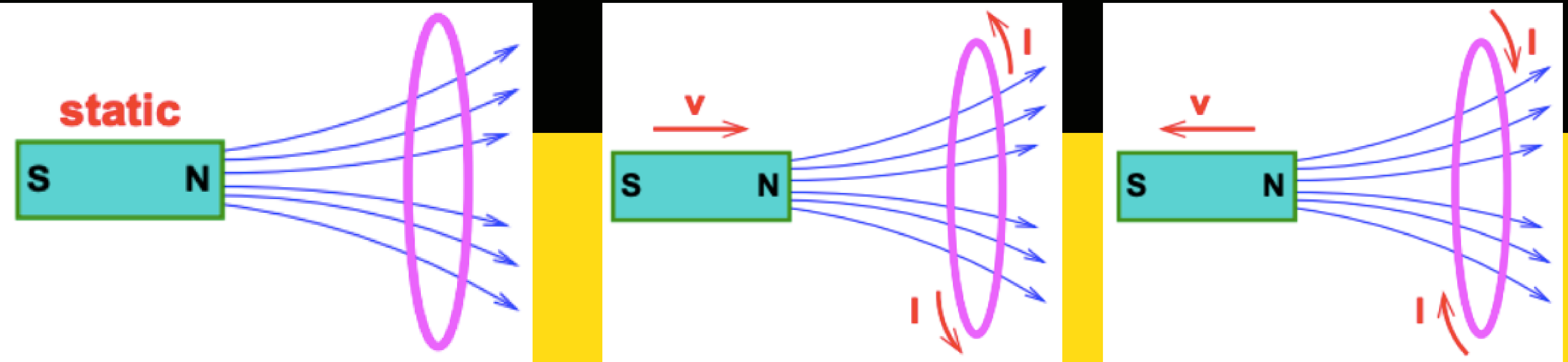
magnetic flux

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

**Faraday's Law**

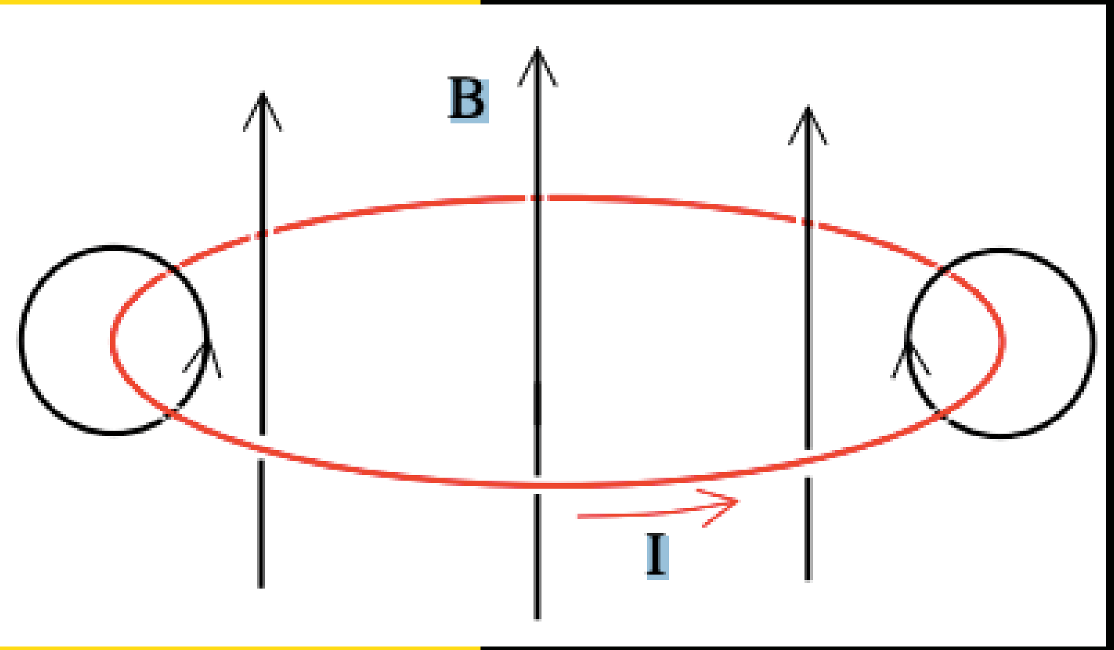
# FARADAY'S LAW OF INDUCTION

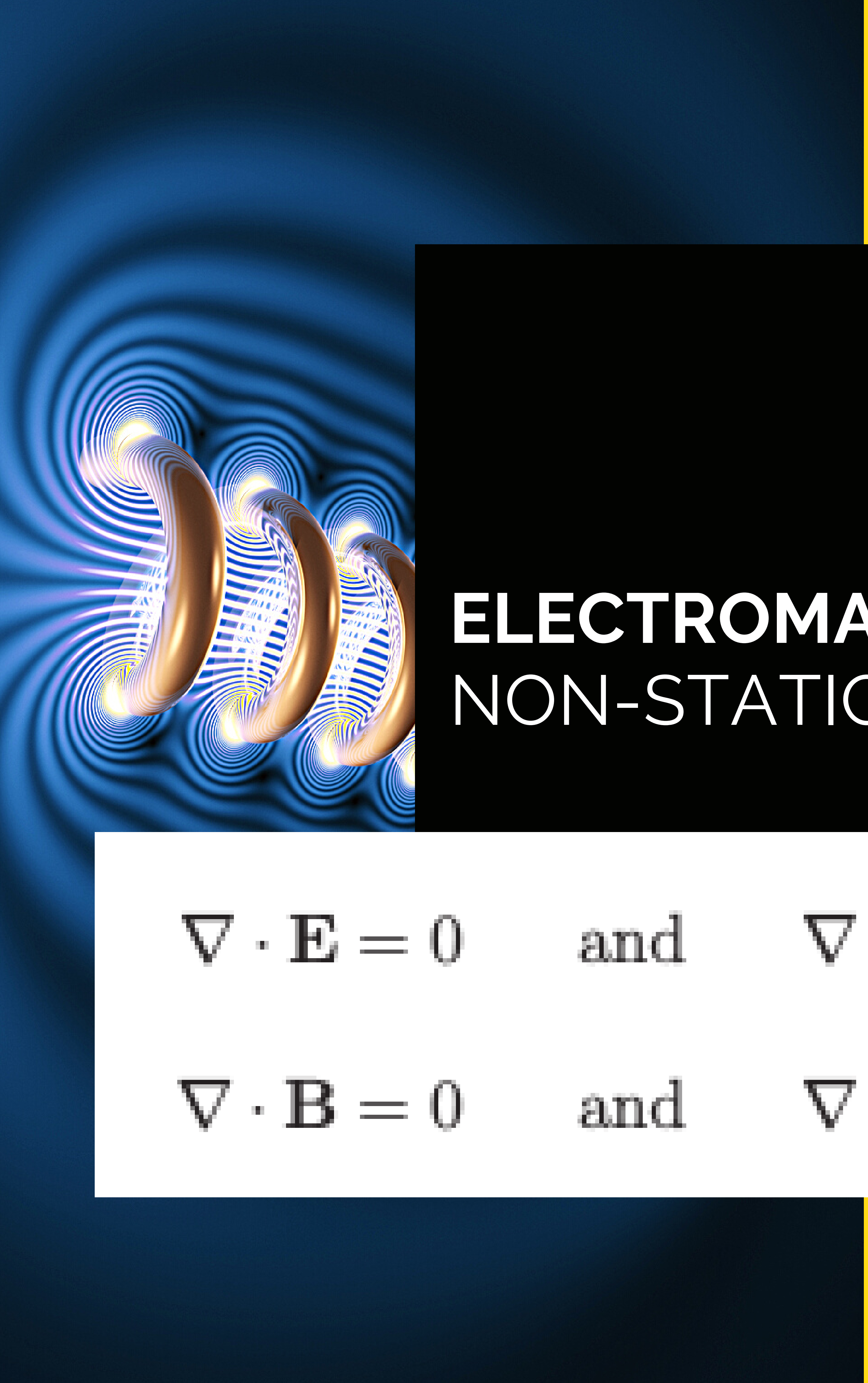
$$\mathcal{E} = -\frac{d\Phi}{dt}$$




**The electromotive force around a closed path is equal to the negative of the time rate of change of the magnetic flux enclosed by the path.**

Secondary effect: When a current flows in C, it will create its own magnetic field. This induced magnetic field will always be in the direction that opposes the change. This is called **Lenz's law.**





# **ELECTROMAGNETISM:** NON-STATIC CASE

$$\nabla \cdot \mathbf{E} = 0 \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- Faraday's Law of Induction
- **WAVE FUNCTION**
- Propagation of electromagnetic waves in a conductor
- Propagation of electromagnetic waves in a highly conductive materials

# WAVE FUNCTION

$$\nabla \cdot \mathbf{E} = 0 \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

ELECTRIC FIELD

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$$

**The wave equation**

MAGNETIC FIELD

$$\frac{1}{c^2} \frac{\partial^2 \mathrm{B}}{\partial t^2} - \nabla^2 \mathrm{B} = 0$$

$$c = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$$

SPEED OF LIGHT

AND THERE WAS LIGHT

“

**The velocity of transverse undulations in our hypothetical medium, calculated from the electro-magnetic experiments of MM. Kohlrausch and Weber, agrees so exactly with the velocity of light calculated from the optical experiments of M. Fizeau, that we can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena**

James Clerk Maxwell

# WAVE FUNCTION

| | |
|---|---|
| $\left\|\vec{k}\right\| = \frac{2\pi}{\lambda} = \frac{\omega}{c}$ | wave-number vector |
| $\lambda = \frac{c}{f}$ | wave length |
| $f$ | frequency |
| $\omega = 2\pi f$ | angular frequency |

$$\mathbf{E} = \mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \quad \text{and} \quad \mathbf{B} = \mathbf{B}_0\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

**k –** the wave-number vector with **|k|** = **k**, which gives the direction of propagation of the wave.

**ω** is more properly called the angular frequency **(f –** frequency)

$$\omega^2 = c^2 k^2$$

dispersion relation

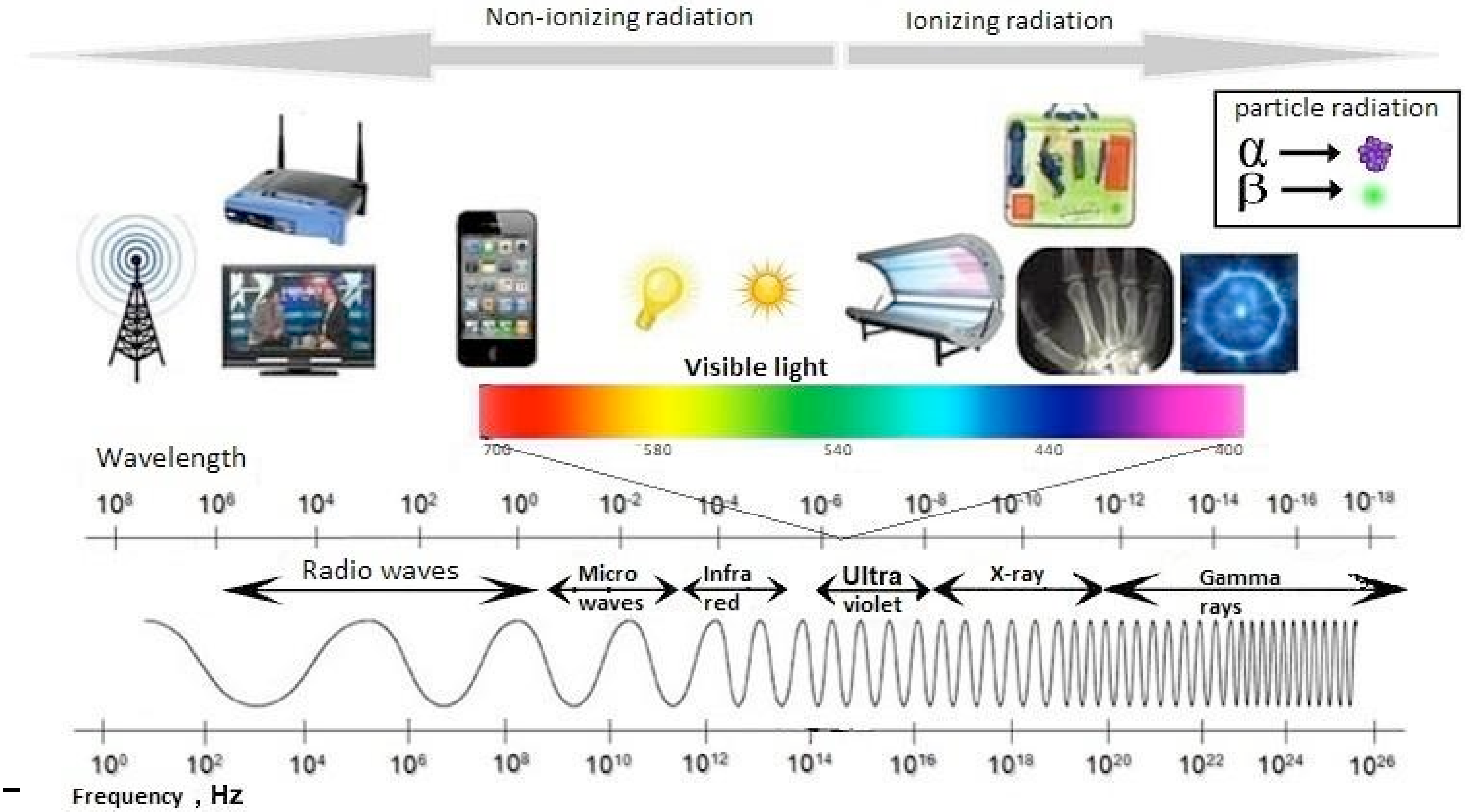
$$c = \frac{\omega}{|k|} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

**Eo, Bo** – constant vectors, the amplitude of the wave

**λ = 2π/k** - the wavelength of the wave

Short wave length →high frequency → high energy

# The electromagnetic spectrum



# WAVE FUNCTION. CONSTRAINS.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

→

$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0$$

$$\vec{k} \times \vec{B}_0 = -\frac{\omega}{c^2}\vec{E}_0$$

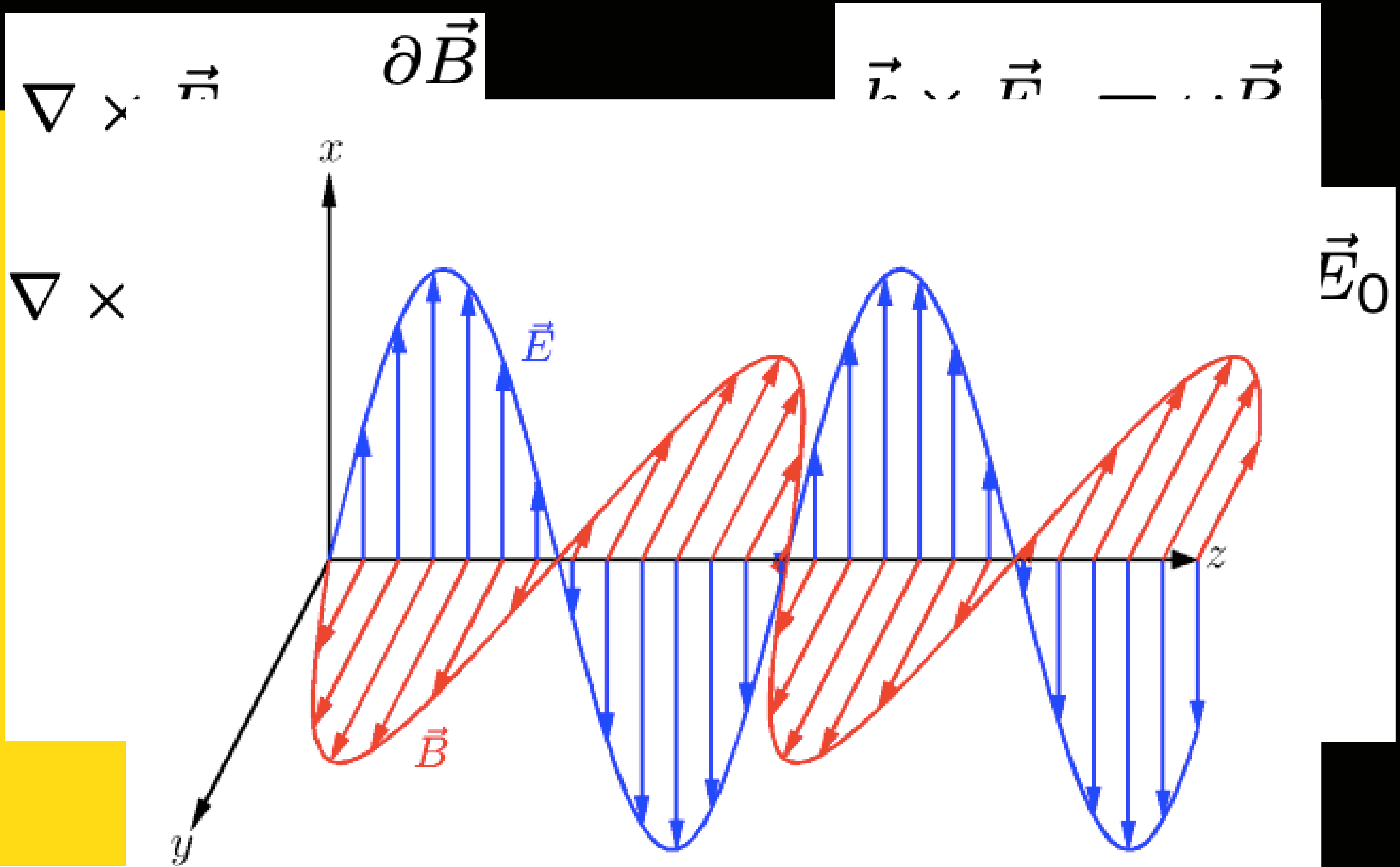
$$\vec{k} \cdot \vec{E}_0 = 0$$

$$\vec{k} \cdot \vec{B}_0 = 0$$

# WAVE FUNCTION. CONSTRAINS.

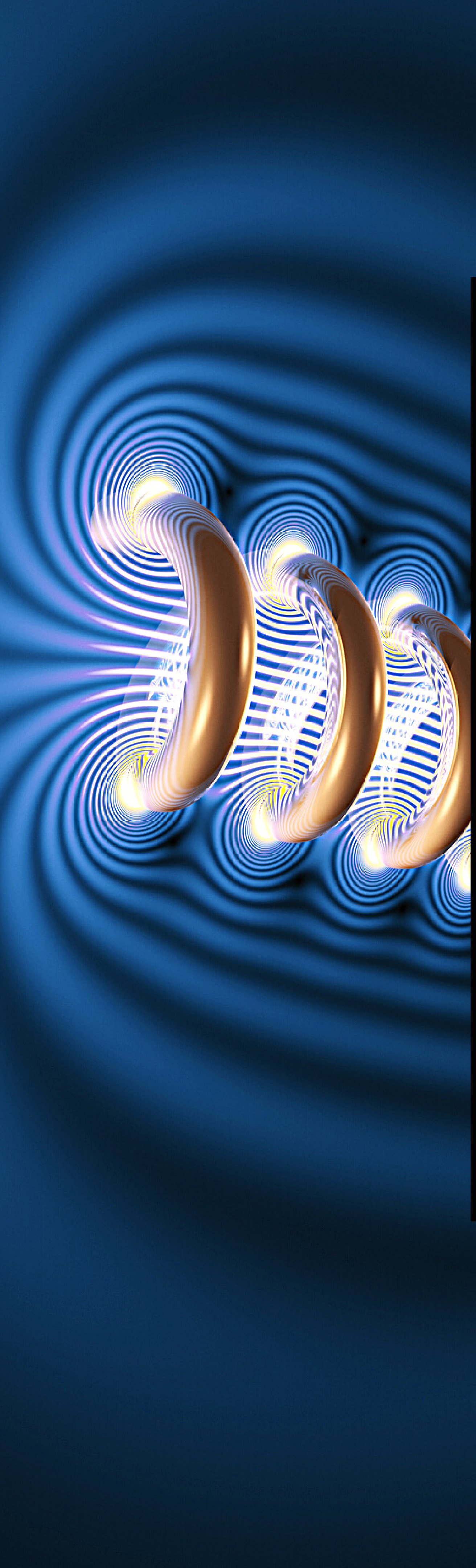
- **Eo, Bo, and k** are mutually perpendicular;
- The field amplitudes are related by

$$\frac{E_0}{B_0} = c$$




Magnetic and electric fields are transverse to direction of propagation:

$$\vec{E} \perp \vec{B} \perp \vec{k}$$



# **ELECTROMAGNETISM:** NON-STATIC CASE

- Faraday's Law of Induction
- Wave Function
- **PROPAGATION OF ELECTROMAGNETIC WAVES IN A CONDUCTOR**
- Propagation of electromagnetic waves in a  highly conductive materials

# Propagation of electromagnetic waves in a conductor

## OHMIC CONDUCTOR

One significant difference is that the electric field in the wave drives a flow of electric current in the conductor: this leads to ohmic energy losses

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

$$\nabla \cdot \boldsymbol{J} = \sigma \nabla \cdot \boldsymbol{E} = \frac{\sigma}{\epsilon_0}\rho$$

The constant **σ** is the conductivity of the material

## THE CONTINUITY EQUATION,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon_0}\rho = 0$$

$$\rho(t) = \rho_0 \exp -\frac{t}{\tau}$$

$$\tau = \frac{\epsilon_0}{\sigma}$$

relaxation time

# Propagation of electromagnetic waves in a conductor

$$\tau = \frac{\epsilon_0}{\sigma}$$

relaxation time

## PERFECT CONDUCTOR

$$\sigma \to \infty$$

Relaxation time is vanishing

## GOOD, BUT NOT PERFECT CONDUCTOR

charges move almost instantly to the surface of the conductor

$$\frac{\sigma}{\epsilon} \approx 10^{14}\mathrm{sec}^{-1}$$

## ISOLATOR

$$\sigma = 0$$

the solution of the wave equation is reduces to an ordinary plane wave

## SKIN DEPTH

inside a good conductor the field is attenuated in the direction of the propagation and its magnitude decreases exponentially as it penetrates into the conductor

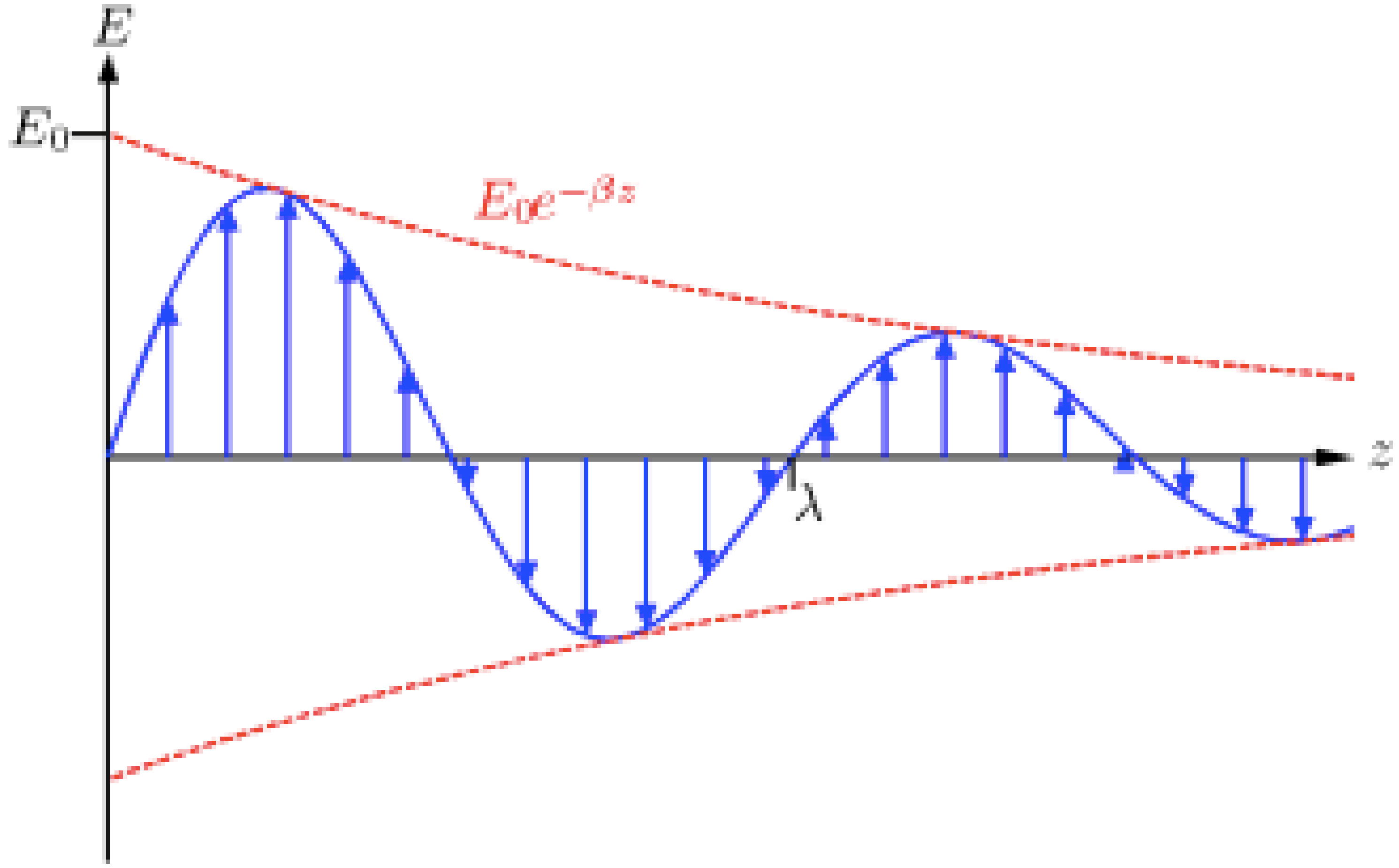
$$\delta = \sqrt{\frac{2}{\mu_r \sigma \omega}}$$

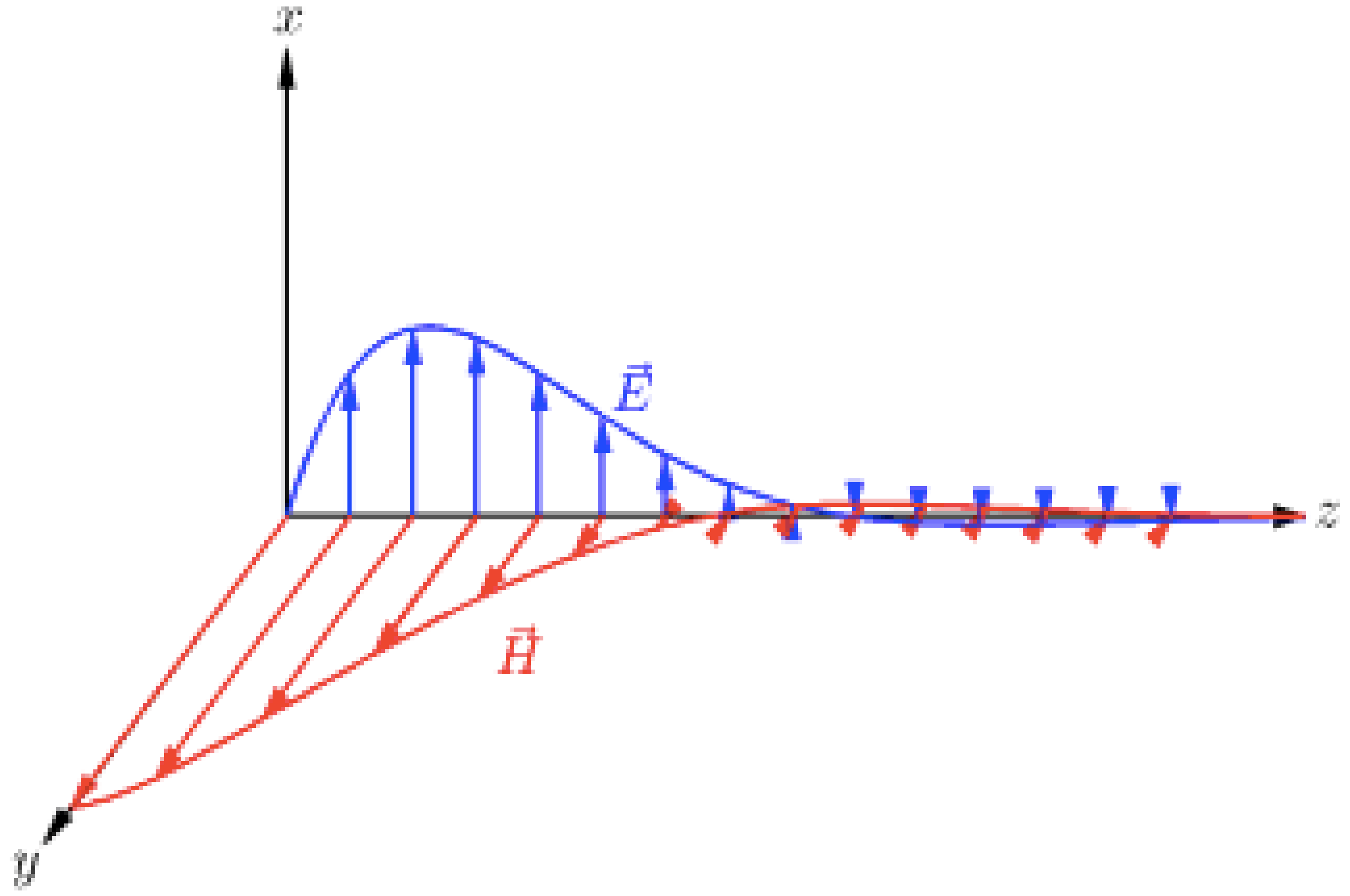
The amplitude of the wave falls by a factor **1/e** in a distance

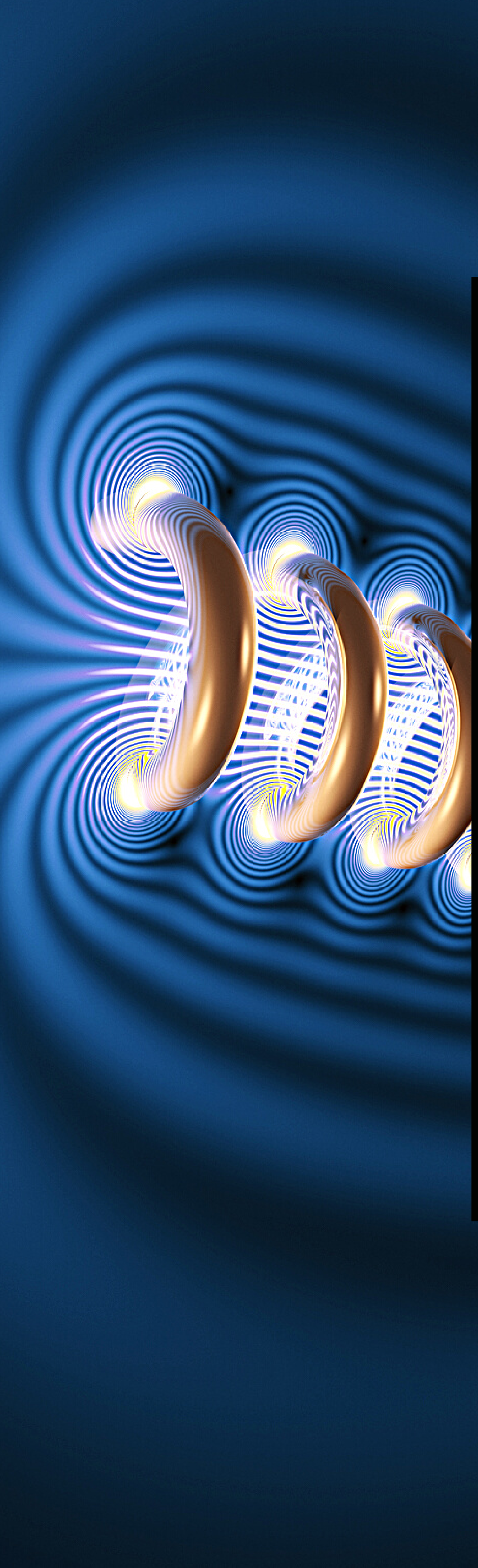
**δ = 1/β.**

**δ** is known as **the skin depth.**

The skin depth is **smaller for larger conductivity.** The better the conductivity of a material, **the less well an electromagnetic wave can penetrate the material.**





# **ELECTROMAGNETISM:** NON-STATIC CASE

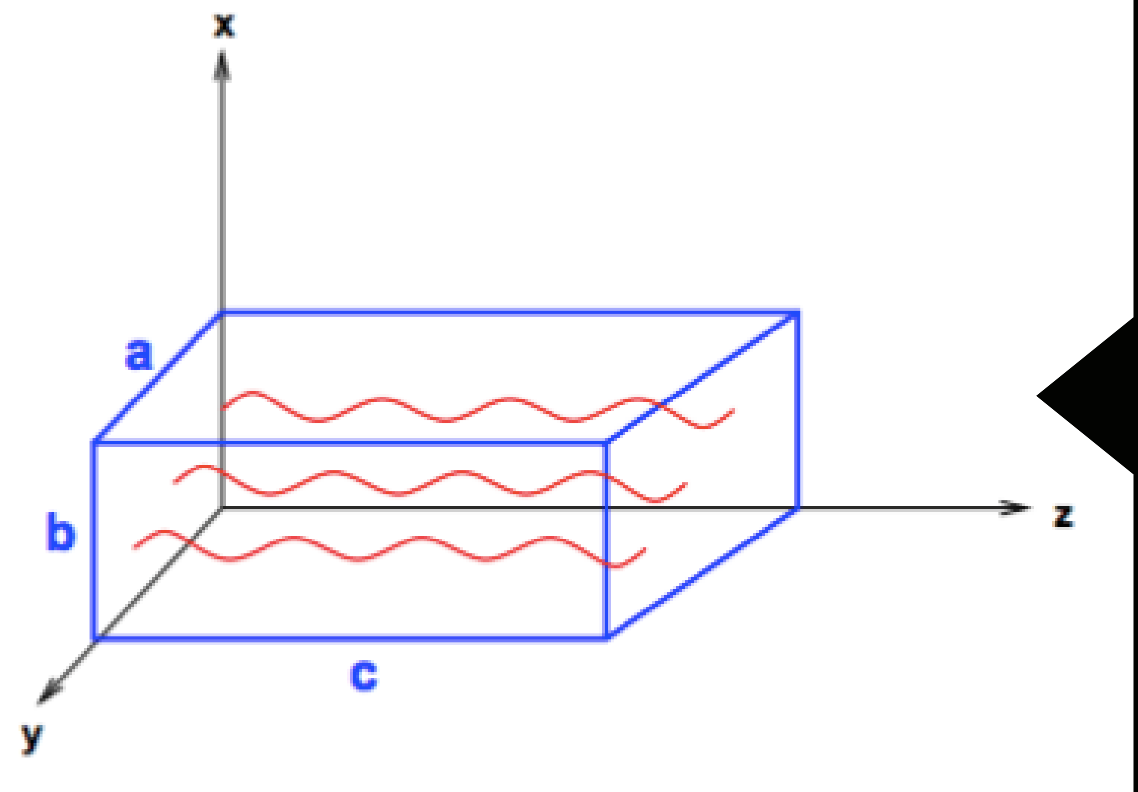
- Faraday's Law of Induction
- Wave Function
- Propagation of electromagnetic waves in a conductor
- **PROPAGATION OF ELECTROMAGNETIC WAVES IN A HIGHLY CONDUCTIVE MATERIALS**

## HISTORY OF WAVEGUIDES

1884 Sir Oliver Lodge detected electromagnetic waves from a spark at the end of a cylinder, and found that the amplitude did not fall off as $1/r^2$.

1897 Lord Rayleigh showed that two classes of waves are possible, "transverse electric" (TE) and "transverse magnetic" (TM). For each class, there is a minimum frequency for propagation.

1936 Barrow-Southworth showed that for practical guides, the attenuation in waveguides was much less than in wires or coaxial cables.

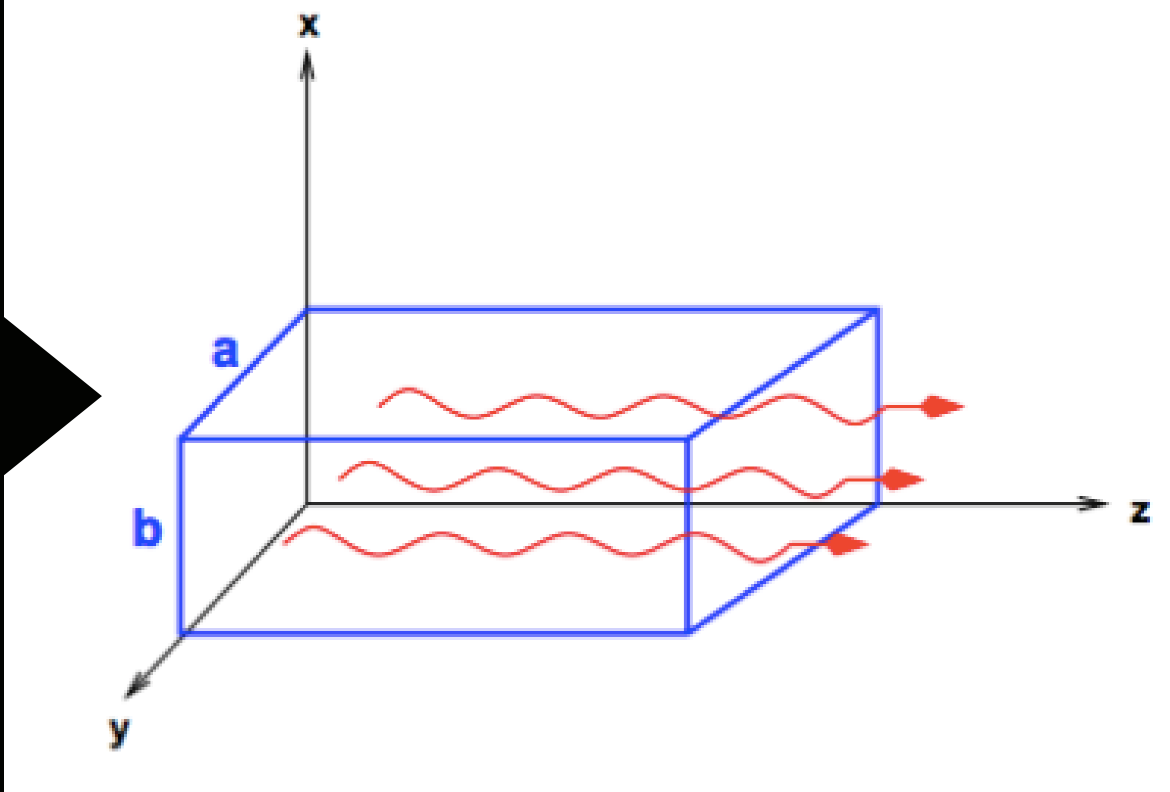



## RF CAVITIES

Field can persist and be stored

## WAVEGUIDES

Plane waves can propagate along waveguides



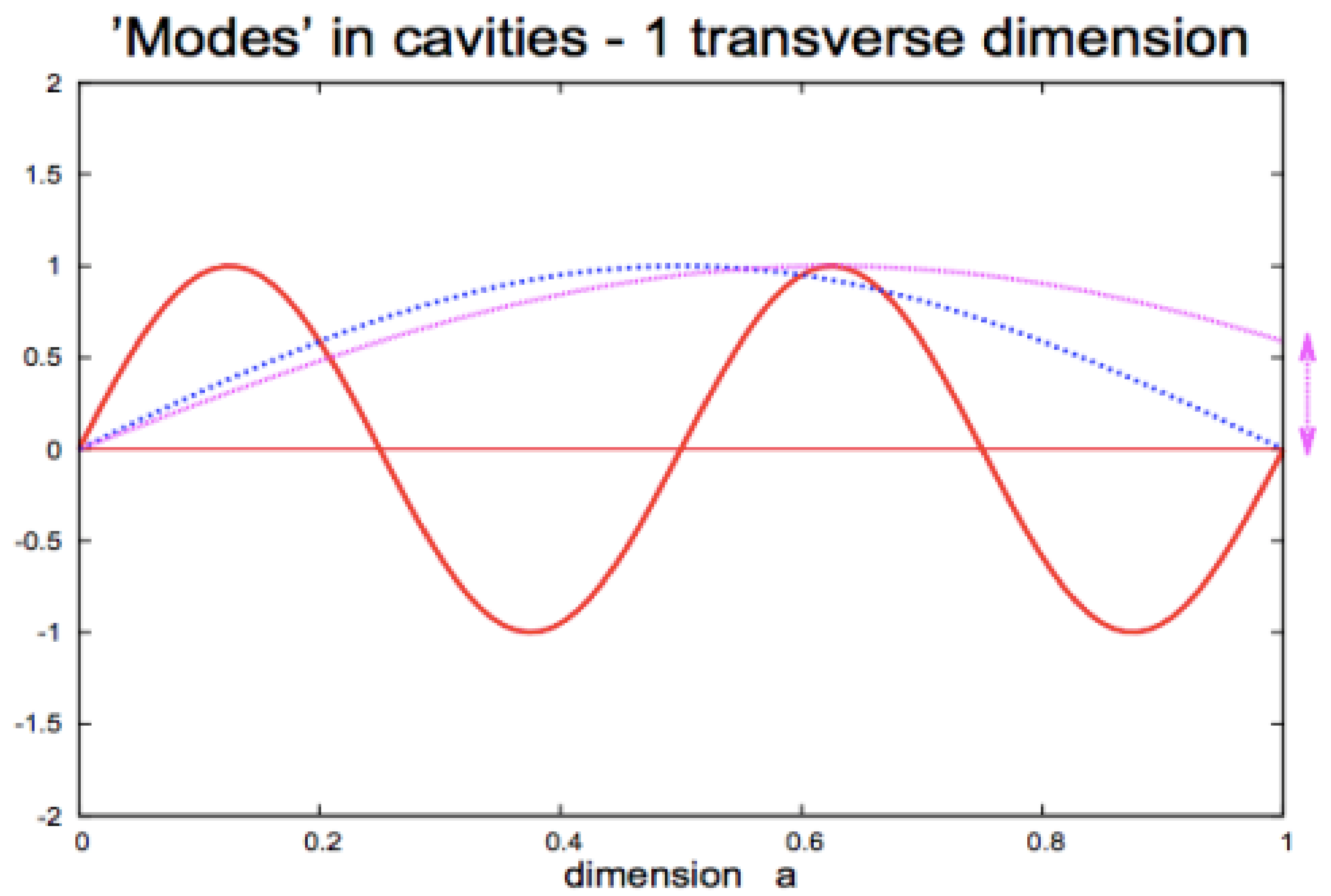
# **Example:** Fields in RF cavities

Rectangular RF cavity, an ideal conductor

$$E_x = E_{x0} \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$
$$E_y = E_{y0} \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$
$$E_z = E_{z0} \cdot \sin(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$

$$B_x = \frac{i}{\omega}(E_{y0}k_z - E_{z0}k_y) \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$
$$B_y = \frac{i}{\omega}(E_{z0}k_x - E_{x0}k_z) \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$
$$B_z = \frac{i}{\omega}(E_{x0}k_y - E_{y0}k_x) \cdot \cos(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$

# **Example:** Fields in RF cavities




**No electric field at boundaries, wave must have "nodes" = zero fields at the boundaries**

**Only modes which 'fit' into the cavity are allowed**

**In the example:** $\frac{\lambda}{2} = \frac{a}{4}$, $\frac{\lambda}{2} = \frac{a}{1}$, $\frac{\lambda}{2} = \frac{a}{0.8}$

**(then either "sin" or "cos" is 0)**

# **Consequences:** RF cavities

Field must be zero at conductor boundary, only possible if:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

and for $k_x, k_y, k_z$ we can write, (then they all fit):

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \frac{m_z \pi}{c},$$

The integer numbers $m_x, m_y, m_z$ are called mode numbers, important for design of cavity !

→ half wave length $\lambda/2$ must always fit exactly the size of the cavity.

(For cylindrical cavities: use cylindrical coordinates )

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# **Consequences:** wave guides

Similar considerations as for cavities, no field at boundary.
We must satisfy again the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

This leads to modes like (no boundaries in direction of propagation $z$):

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b},$$

The numbers $m_x, m_y$ are called mode numbers for planar waves in wave guides !

In z direction: No Boundary - No Boundary Condition ...



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# **Consequences:** wave guides

Re-writing the condition as:

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2 \quad \rightarrow \quad k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

Propagation without losses requires $k_z$ to be real, i.e.:

$$\frac{\omega^2}{c^2} > k_x^2 + k_y^2 = (\frac{m_x \pi}{a})^2 + (\frac{m_y \pi}{b})^2$$

which defines a cut-off frequency $\omega_c$. For lowest order mode:

$$\omega_c = \frac{\pi \cdot c}{a}$$

- Above cut-off frequency: propagation without loss
- At cut-off frequency: standing wave
- Below cut-off frequency: attenuated wave (it does not "fit in").

There is a very easy way to show that very high frequencies easily propagate

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# CLASSIFICATION OF MODES

## Transverse Electric (TE)

there is no longitudinal component of the electric field

$$E_z = 0 \quad \text{everywhere}; \quad B_z \neq 0$$

## Transverse Magnetic (TM)

there is no longitudinal component of the magnetic field

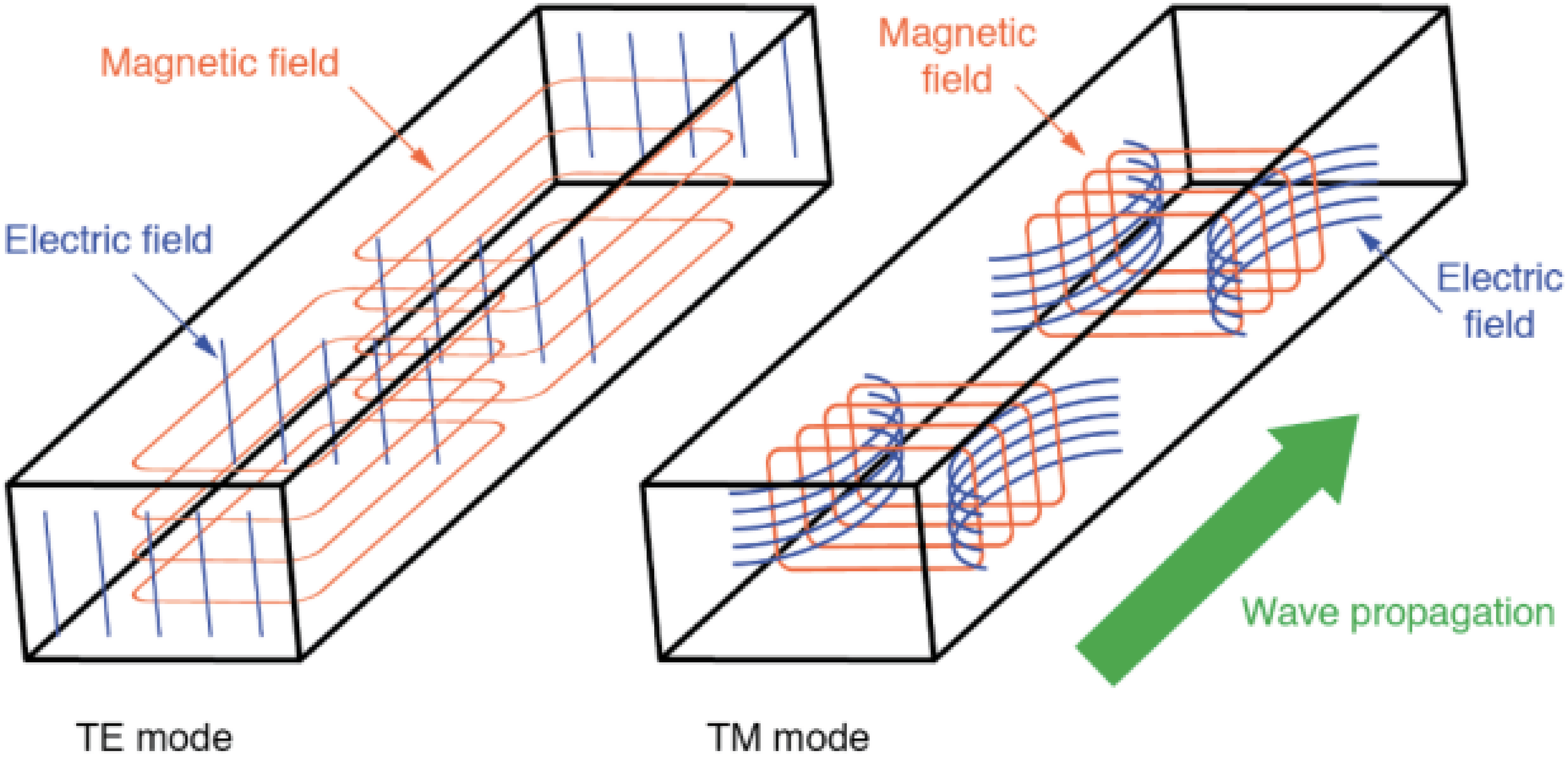
$$B_z = 0 \quad \text{everywhere}; \quad E_z \neq 0$$

## Transverse ElectroMagnetic (TEM)

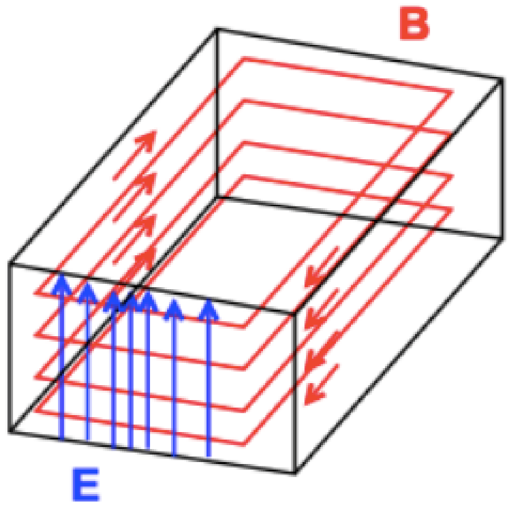
both electric and magnetic components are transverse to the wave guide axis

$$E_z, B_z = 0 \quad \text{everywhere}$$

# CLASSIFICATION OF MODES




*Magnetic flux lines appear as continuous loops*
*Electric flux lines appear with beginning and end points*




Note (here a TE mode) :

Electric field lines end at boundaries

Magnetic field lines appear as "loops"

# THE END

**Thank you for your attention!**

I would like to thank as well my colleagues  who gave the EM course previously (and I could profit from it while preparing the lecture (A. Latina, A. Wolski, P. Skowronski, W. Herr)

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