Introduction to Special Relativity

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Basics of Accelerator Physics and Technology

7-11 October 2019, Archamps, France

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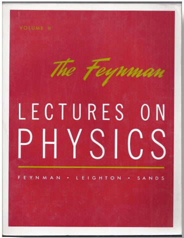
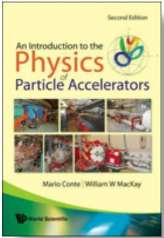
"Particle accelerator physics is the realm of applied special relativity"

Some references

1. Richard P. Feynman, Lectures on Physics, 1963, on-line
2. Mario Conte, William W. MacKay, *An Introduction to the Physics of Particle Accelerators*, Second Edition, World Scientific, 2008





Part 1.

Introduction:

Principle of Special Relativity

Einstein's principle of relativity

In Albert Einstein's original treatment, in 1905, the principle of relativity is based on two postulates:

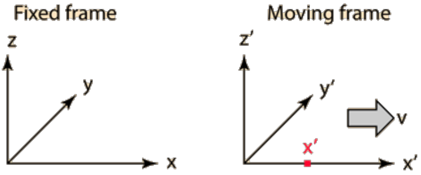
1. the laws of physics are invariant (i.e. identical) in all inertial frames of reference (i.e. non-accelerating frames of reference); and
2. the speed of light in a vacuum is the same for all observers, regardless of the motion of the light source or observer.

Newton's principle of relativity and Galilean transformation

The principle of relativity was first stated by Galileo Galilei in 1632, and later by Newton in one of his corollaries to the laws of motion:

- *"The motions of bodies included in a given space are the same among themselves, whether that space is at rest or moves uniformly forward in a straight line."*

At the time of Newton the relation of the coordinates between two systems in motion with relative velocity v, was defined by the Galilean transformation of motion:



$$\begin{aligned} x' &= x - v\,t \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{r}' &= \mathbf{r} - \mathbf{v}t \\ t' &= t \end{aligned}$$

with $\mathbf{r} = (x, \quad y, \quad z)$.

Galilean transformation and Maxwell's equations

- The principle of relativity has been used in mechanics for a long time. It was employed by various people, in particular Huygens, to obtain the rules for the collision of billiard balls, based on the conservation of momentum.

- In the 19th century interest in it was heightened as the result of investigations into the phenomena of electricity, magnetism, and light. A long series of careful studies of these phenomena by many people culminated in Maxwell's equations of the electromagnetic field, which describe electricity, magnetism, and light in one uniform system.

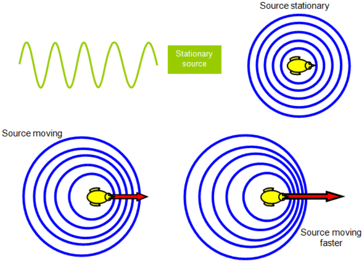
- However, the Maxwell equations did not seem to obey the principle of relativity. That is, if we transform Maxwell's equations by the substitution of equations (of the Galilean transformation, their form does not remain the same:
  - Therefore, it seemed like if in a moving space ship the electrical and optical phenomena should be different from those in a stationary ship.

Maxwell's equations and principle of relativity

- One of the consequences of Maxwell's equations is that if there is a disturbance in the field such that light is generated, these electromagnetic waves go out in all directions equally and at the same speed

$$c = 2.99792458 \times 10^8 \text{ m/s}$$

- Another consequence of the equations is that if the source of the disturbance is moving, the light emitted goes through space at the same speed c.



This is analogous to the case of sound, the speed of sound waves being likewise independent of the motion of the source.

These considerations lead to the idea of a "ether". Attempts were made to determine the absolute velocity of the earth through the hypothetical "ether" that was supposed to pervade all space.

The problem with Galilean transformation

Maxwell describes light as waves, wave equation reads (see previous lecture):

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\Psi \quad = \quad 0$$

With Galilei transformation $x = x' - vt,\ y' = y,\ z' = z,\ t' = t$ **:**

$$\left(\left[1 - \frac{v^2}{c^2}\right]\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x \partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi \quad = \quad 0$$

... not quite the same form, other physics laws ??

<u>Reason:</u> Waves are required to move in a medium (ether !) which travels along in a fixed reference frame, observed from another frame the speed is different ...

Incompatible with experiments:

- **Speed of light <u>in vacuum</u> is independent of the motion of the source, i.e.** $v_{tot} = c + v' = c$
- **Speed of light <u>in vacuum</u>** c **is the maximum speed and cannot be exceeded** **c = 299792458.000 m/s**

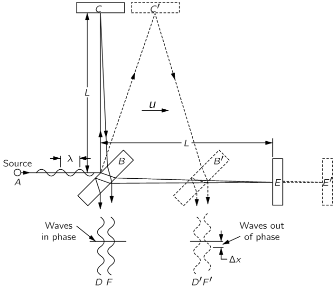
  (Michelson, Morely, 1897)
- **There is no ether, light is not a wave, but many - though not all - properties can be well described by them**

Note (origin of confusing and obscure arguments) :

The definition : $\frac{\Delta s}{\Delta t}$ is not the correct definition of speed !

The Michelson-Morley experiment (1887)

The goal was to determine the absolute velocity of the earth through this hypothetical "ether":



A: light source; B: partially silvered glass plate; C and E: mirrors; D and F: superimposed light beams

- Time from B to E and back:

$$
\begin{aligned}
B \to E: &\quad ct_1 = L + ut_1 \Rightarrow t_1 = L/(c-u)\\
E \to B: &\quad ct_2 = L - ut_2 \Rightarrow t_2 = L/(c+u)
\end{aligned}
$$

total time:

$$t_1 + t_2 = \frac{2L}{(1 - u^2/c^2)}$$

- Time from B to C and back:

$$
\begin{aligned}
B \to C: &\quad (ct_3)^2 = L^2 + (ut_3)^2 \Rightarrow t_3 = L/\sqrt{c^2 - u^2}\\
C \to B: &\quad t_4 = t_3
\end{aligned}
$$

total time:

$$t_3 + t_4 = \frac{2L}{\sqrt{c^2 - u^2}}$$

If there is an "ether drift" then

$$\boxed{t_1 + t_2 \neq t_3 + t_4}$$

The apparatus was amply sensitive to observe such an effect, <u>but no time difference was found</u>—the velocity of the earth through the ether could not be detected. *The result of the experiment was null*.

Lorentz transformation

- To solve the Michelson-Morley puzzle, Lorentz suggested that material bodies contract when they are moving, and that this shortening is only in the direction of the motion, and also, that if the length is L_0 when a body is at rest, then when it moves with speed v parallel to its length, the new length, $L_{\|}$ is given by:

$$L_{\|} = L_0 \sqrt{1 - v^2/c^2}$$

- He also noticed that the remarkable and curious thing that the Maxwell equations were invariant under the following substitutions:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

These equations are known as *Lorentz transformation*.

- In their 1905 papers on electrodynamics, Henri Poincaré and Albert Einstein explained that with the Lorentz transformation the relativity principle holds perfectly.

Rewrite of the Lorentz transformation

In relativity, it is common practice to write the velocity v as β

$$\beta = \frac{v}{c} \qquad \in [0, 1]$$

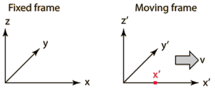
and to define, γ, the Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \in [1, \infty)$$

Therefore

$$\boxed{\begin{aligned} x' &= \gamma\left(x - \beta ct\right) \\ y' &= y \\ z' &= z \\ ct' &= \gamma\left(ct - \beta x\right) \end{aligned}} \quad \Rightarrow \quad \boxed{\begin{aligned} \mathbf{r}'_{\|} &= \gamma\left(\mathbf{r}_{\|} - \boldsymbol{\beta} ct\right) \\ \mathbf{r}'_{\perp} &= \mathbf{r}_{\perp} \\ ct' &= \gamma\left(ct - \mathbf{r}_{\|} \cdot \boldsymbol{\beta}\right) \end{aligned}}$$

where $\mathbf{r}_{\|}$ and $\mathbf{r}_{\perp}$ are the components of $\mathbf{r}$ w.r.t. $\boldsymbol{\beta}$ (or $\mathbf{v}$).



where $\mathbf{v}$ is the relative velocity of the two frames.

Time dilation and length contraction

- Length contraction, or Lorentz contraction, is the solution that Lorentz proposed to solve the Michelson-Morley experiment:
  - is the phenomenon that a moving object's length is measured to be shorter than its proper length, which is the length as measured in the object's own rest frame

$$\Delta x' = \frac{\Delta x}{\gamma}$$

- Time dilation:
  - is a difference in the elapsed time measured by two clocks, either due to them having a velocity relative to each other (or by there being a gravitational potential difference between their locations)

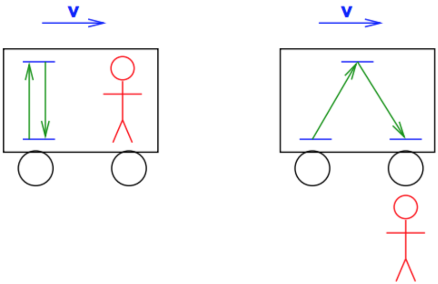
$$\Delta t' = \gamma \Delta t$$

Definitions:

- proper mass: mass of a body at rest
- proper time: time as measured in its own frame
- proper length: length as measured in its own frame

Time dilation made intuitive

Reflection of light between 2 mirrors seen inside moving frame and from outside

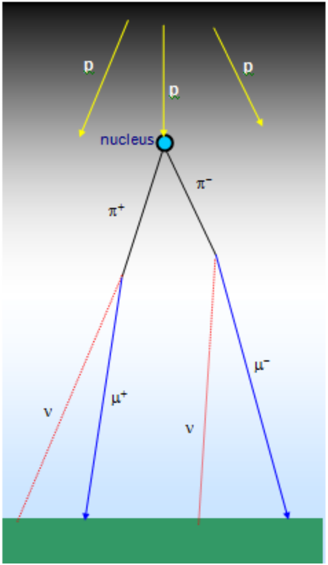


Frame moving with velocity v

Seen from outside the path is longer, but c must be the same ..

Example of time dilation: muon decay in the atmosphere

Muons are formed in collisions of cosmic rays with nuclei of atmosphere's atoms, at heights of about 12000 m



- The half-life of a muon is 2.2 microseconds and so even moving at 0.994 c they would only expect to travel about 660 m before half of them decayed.
- As they are formed at 12000 m altitude it would take 40 μs, or about 20 half lives, to reach the ground.
- This would mean that only $1/2^{20}$ of the original number would be detected.
- The fact that the proportion is much higher than this means that the muons are living longer.
- Their relativistic factor is:

$$\gamma = \frac{1}{\sqrt{1 - 0.994^2}} = 9.1424$$

Their time slows down, and 2.2 μs become about γ times longer, or
Lengths contract and the 12000 m become $12000/\gamma$ m.

Relativistic velocity-addition formula

In Galilean relativity, the velocity of an object $\mathbf{u}$, in motion over a moving reference with velocity $\mathbf{v}$, is seen from an observer at rest as moving with a velocity $\mathbf{u}'$:

$$\mathbf{u}' = \mathbf{u} + \mathbf{v}$$

The equation of Einstein's special relativity is different: for collinear motions, the velocity u' is

$$u' = \frac{u + v}{1 + vu/c^2}$$

For arbitrary velocities the 3d vector formula is more complex:

$$\mathbf{u}'_{\parallel} = \frac{\mathbf{u}_{\parallel} + \mathbf{v}}{1 + \frac{\mathbf{v}\cdot\mathbf{u}_{\parallel}}{c^2}}; \qquad \mathbf{u}'_{\perp} = \frac{\sqrt{1 - \frac{v^2}{c^2}}\mathbf{u}_{\perp}}{1 + \frac{\mathbf{v}\cdot\mathbf{u}_{\parallel}}{c^2}}$$

Question: what happens if $u = v = c$?

Einstein postulates

- **Space and time are NOT independent quantities**
- **Relativistic phenomena (with relevance for accelerators):**
  - **No speed of moving objects can exceed speed of light**
  - **(Non-) Simultaneity of events in independent frames**
  - **Lorentz Contraction and Time Dilation**
  - **Relativistic Doppler effect**
- **There are no absolute time and space, no absolute motion**

Inertial system: It is not possible to know whether one is moving or not

There is not simultaneity

To be clear

- **Lorentz contraction:**
  - **It is not the matter that is compressed (what Lorentz thought)**
  - **It is the space that is modified (Einstein)**
- **Time dilation:**
  - **It is not the clock that is changed (what Lorentz and others thought)**
  - **It is the time that is modified (Einstein)**
- **Einstein's main contribution: to believe it !**

About mass

Interpretation: $E = mc^2 = m_0c^2 + T$

- **Kinetic energy of particle is** T **(old concept)**
- **Energy of particle at rest is** $E_0 = m_0c^2$ **(new concept)**
- **Total energy** E **of a particle "in motion" is** $E = mc^2$ **(brand new concept)**

Generally true: $E = m \cdot c^2 = \gamma m_0 \cdot c^2$

using the definition of relativistic mass again: $m = \gamma m_0$

Part 2.

Relativistic Kinematics

Definitions

$$\beta_r = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} = \frac{1}{\sqrt{(1 - \beta_r^2)}}$$

β_r **relativistic speed:** β_r **= [0, 1]**

LHC: β_r $\approx$ **0.999999991**

γ **Lorentz factor:** γ **= [1,** ∞**)**

LHC: γ $\approx$ **7461**

Definitions and practical units

m	rest mass	MeV/c^2
$E_0 = mc^2$	rest energy	MeV
$E = \gamma mc^2$	total energy	MeV
$K = E - mc^2$	kinetic energy	MeV
$\mathbf{v}$	velocity	m/s
$\boldsymbol{\beta} = \mathbf{v}/c$	relativistic velocity	–
$\gamma = 1/\sqrt{1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta}}$	lorentz factor	–
$\mathbf{P} = \boldsymbol{\beta}\gamma mc$	momentum	MeV/c
$E^2 = (Pc)^2 + \left(mc^2\right)^2$	total energy	MeV

Useful relations and quantities

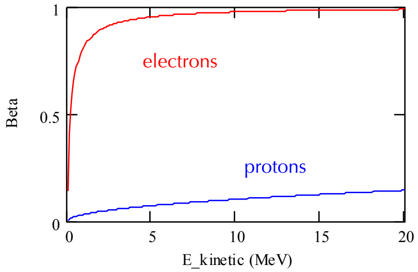
$E^2 = P^2c^2 + m^2c^4$	total energy	MeV
$\mathbf{P}c = E\boldsymbol{\beta}$	total momentum times c	
$m_e = 0.510999$	rest mass of the electron	MeV$/c^2$
$m_p = 938.272$	rest mass of the proton	MeV$/c^2$
$m_\mu = 105.66$	rest mass of the muon	MeV$/c^2$

Frequent subdivisions

$\gamma \simeq 1$	non-relativistic
$\gamma > 1$	relativistic
$\gamma \gg 1$	ultra-relativistic

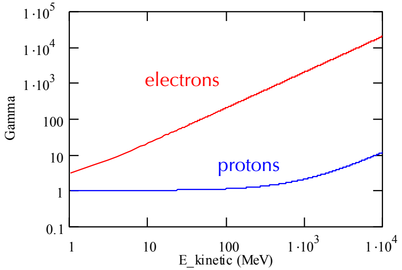
normalized velocity

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$



total energy / rest energy

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$



Forces and momentum

In presence of a force, for example, the Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Newton's second law of motion is

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}$$

In presence of a constant collinear force, e.g. a constant acceleration through an appropriate field E, the momentum of a particle grows linearly, and so does the particle's mass. Recall: $P = (\gamma m)\,\beta c$

Transformations of electromagnetic fields

Unprimed quantities are in the lab frame, primed quantities in the a frame moving with velocity v along the z axis

$$\begin{cases} E_x = \gamma\left(E'_x + vB'_y\right) \\ E_y = \gamma\left(E'_y - vB'_x\right) \\ E_z = E'_z \end{cases} \qquad \begin{cases} B_x = \gamma\left(B'_x - vE'_y/c\right) \\ B_y = \gamma\left(B'_y + vE'_x/c\right) \\ B_z = B'_z \end{cases}$$

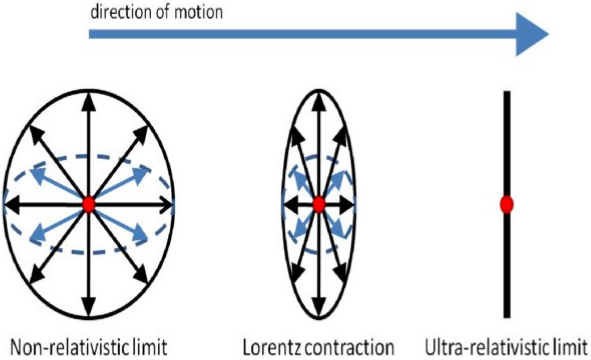
In compact 3d vector form, with $\gamma = 1/\sqrt{1 - v^2/c^2}$, for a frame moving with arbitrary velocity $\vec{v}$:

$$\mathbf{E} = \gamma\left(\mathbf{E}' - \mathbf{v} \times \mathbf{B}'\right) - \frac{\gamma^2}{1+\gamma}\left(\mathbf{v} \cdot \mathbf{E}'\right)\mathbf{v}$$

$$\mathbf{B} = \gamma\left(\mathbf{B}' + \mathbf{v} \times \mathbf{E}'\right) - \frac{\gamma^2}{1+\gamma}\left(\mathbf{v} \cdot \mathbf{B}'\right)\mathbf{v}$$

Quote Einstein: For a charge moving in an electromagnetic field, the force experienced by the charge is equal to the electric force, transformed into the rest frame of the charge.

Example of the electric field



At high energies, electric fields get compressed in the direction of propagation

Every day example

GPS satellite:

- 20'000 km above ground, (unlike popular believe: not on geostationary orbits)
- Orbital speed 14'000 km/h (i.e. relative to observer on earth)
- On-board clock accuracy 1 ns
- Relative precision of satellite orbit $\leq 10^{-8}$
- At GPS receiver, for 5 m need clock accuracy $\approx$ 10 ns
- Do we correct for relativistic effects ?

If you do the math...

Orbital speed 14000 km/h $\approx$ 3.9 km/s

→ $\beta \approx 1.3 \cdot 10^{-5}$, $\gamma \approx 1.000000000084$

Small, but accumulates 7 μs during one day compared to reference time on earth !

After one day: your position wrong by $\approx$ 2 km !!

(including general relativity error is 10 km per day)

→ **Countermeasures:**

(1) Minimum 4 satellites (avoid reference time on earth)

(2) Detune data transmission frequency from 1.023 MHz to 1.022999999543 MHz prior to launch

...The End!

Thank you

for your attention!