CERN

Basic Mathematics and Units

Rende Steerenberg – BE/OP

Contents

- **Vectors & Matrices**
- Differential Equations
- Some Units we use

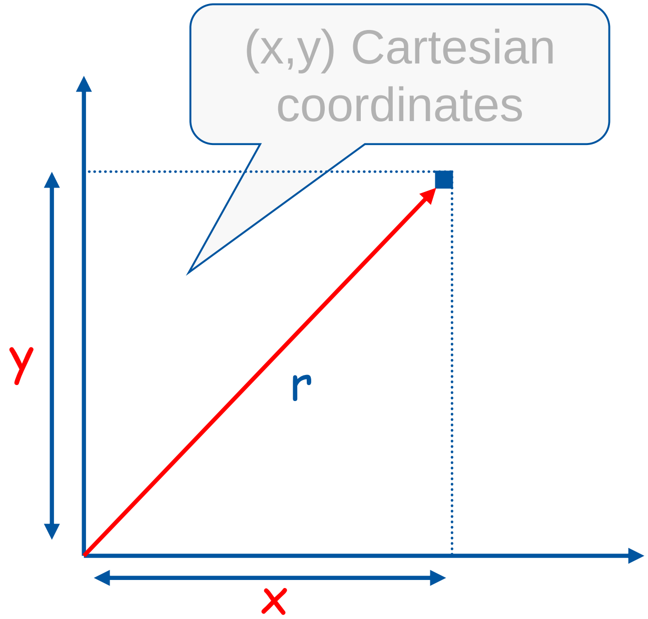
Scalars & Vectors

Scalar, a single quantity or value

Vector, (origin,) length, direction

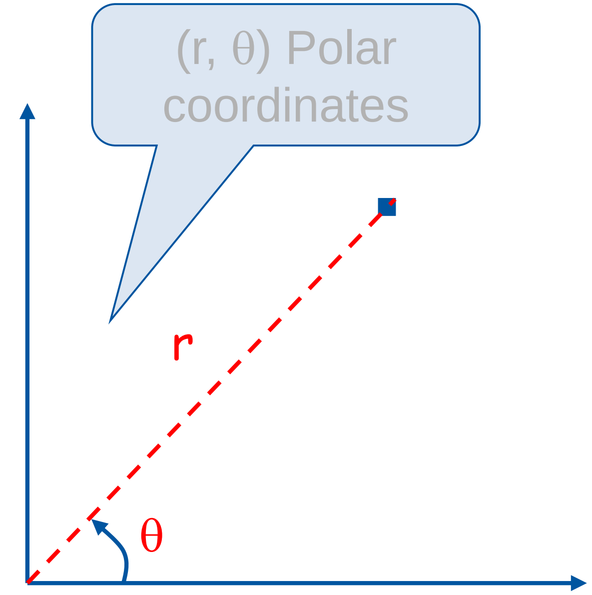
Coordinate systems

A **vector** has 2 or more quantities associated with it



r is the length of the vector

$$r = \sqrt{x^2 + y^2}$$

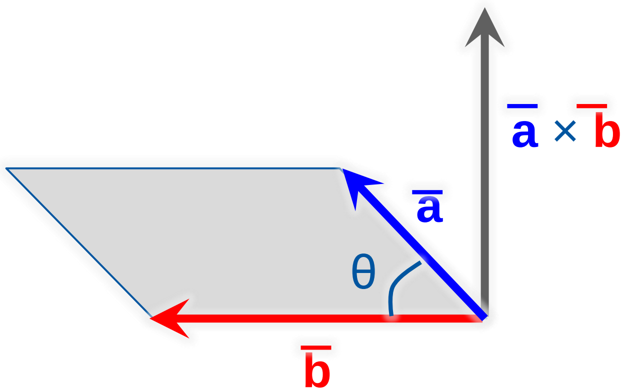


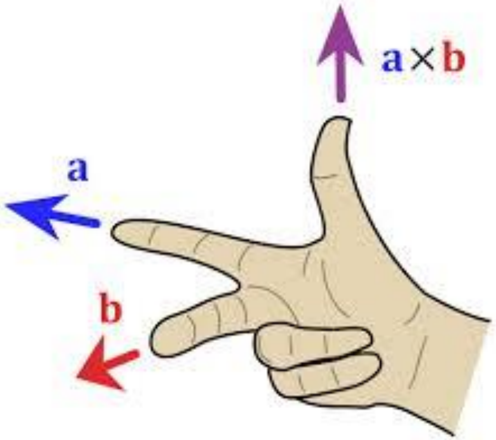
θ gives the direction of the vector

$$\tan(\theta) = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

Vector Cross Product

$\bar{a}$ and $\bar{b}$ are two vectors in the in a plane separated by angle θ





The cross product $\bar{a} \times \bar{b}$ is defined by:

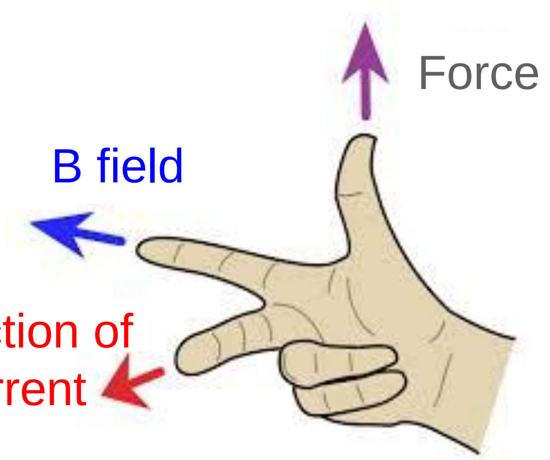
- **Direction**: $\bar{a} \times \bar{b}$ is perpendicular (normal) on the plane through $\bar{a}$ and $\bar{b}$
- The **length** of $\bar{a} \times \bar{b}$ is the surface of the parallelogram formed by $\bar{a}$ and $\bar{b}$

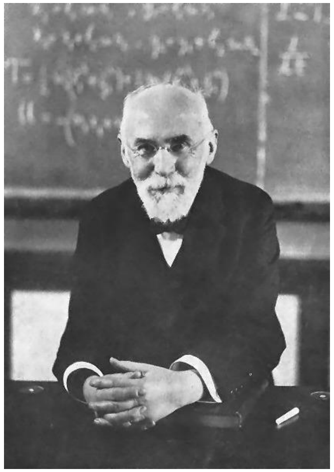
$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin(\theta)$$

Cross Product & Magnetic Field

The Lorentz force in a pure magnetic field expression

$$F = e(\vec{v} \times \vec{B})$$





The reason why our particles move around our “circular” machines under the influence of the magnetic fields

Tuesday

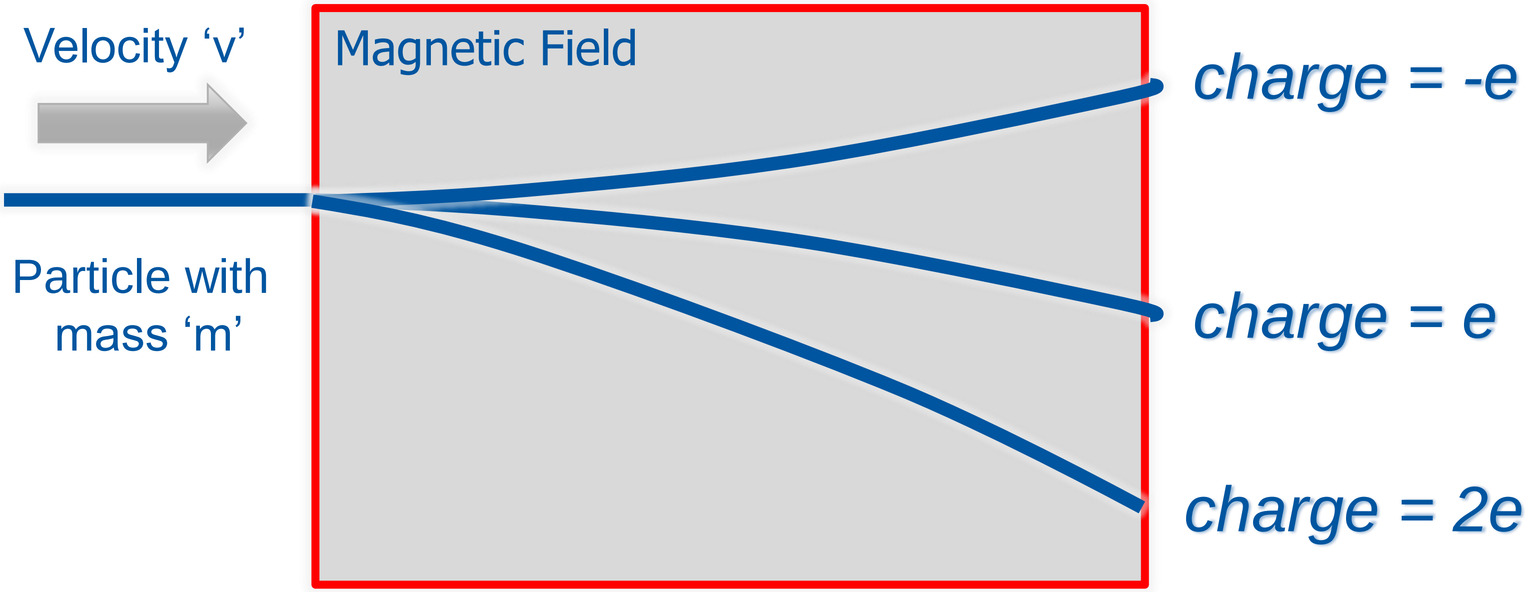
“Electro-Magnetic Theory” by Andrea Latina

After this

“Transverse Beam Dynamics” by Bernhard Holzer

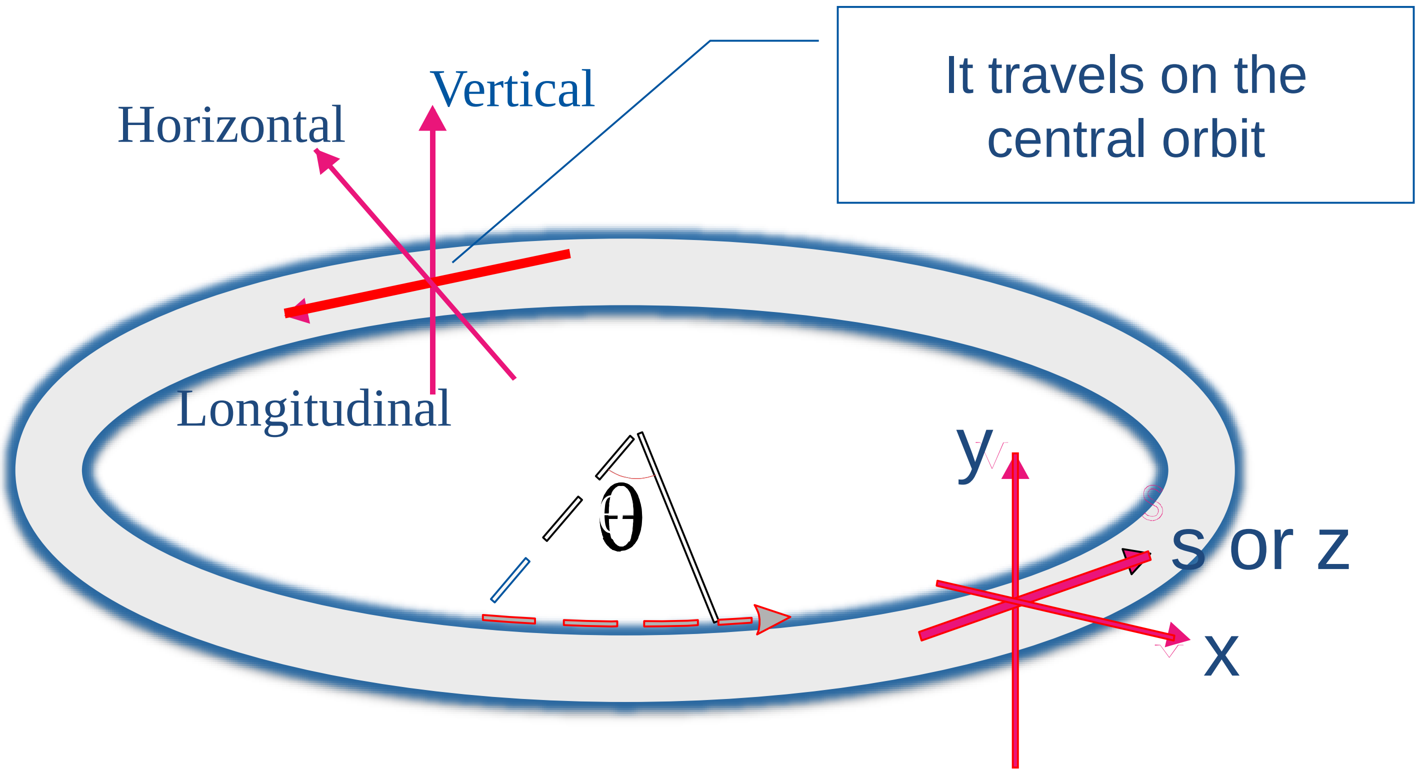
Lorentz Force in Action

$$F = e(\vec{v} \times \vec{B})$$



The larger the energy of the beam the larger the radius of curvature

A Rotating Coordinate System



Magnetic Rigidity

$$F = e(\vec{v} \times \vec{B})$$

- As a formula this is:

$$F = evB = \frac{mv^2}{\rho}$$

Radius of curvature

Like for a stone attached to a rotating rope

- Which can be written as:

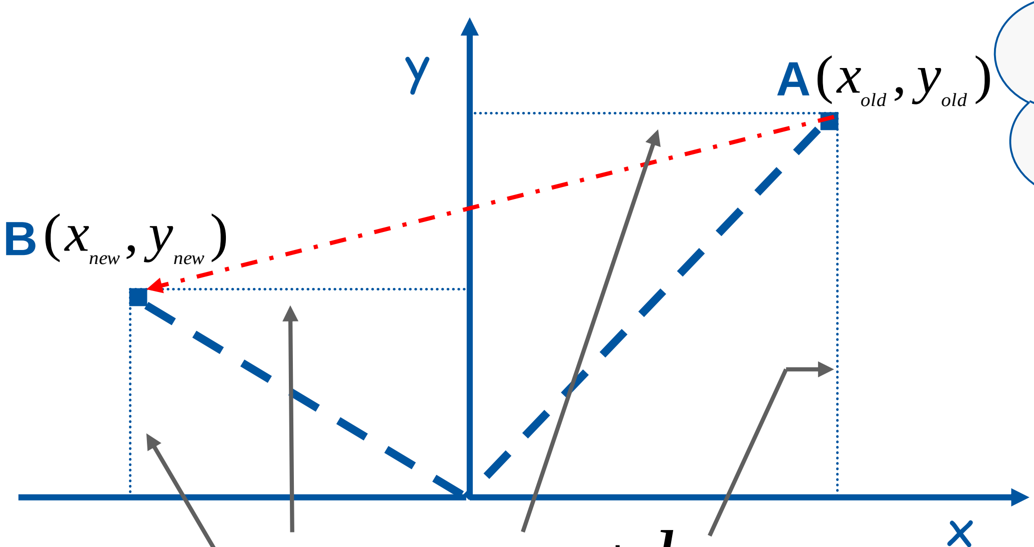
$$B\rho = \frac{mv}{e} = \frac{p}{e}$$

Momentum p=mv

- Bρ is called the magnetic rigidity, and if we put in all the correct units we get:
- Bρ = 33.356·p [kG·m] = 3.3356·p [T·m] (if p is in [GeV/c])

Moving a Point in a Coordinate System

To move from one point (A) to any other point (B) one needs control of both **Length** and **Direction**

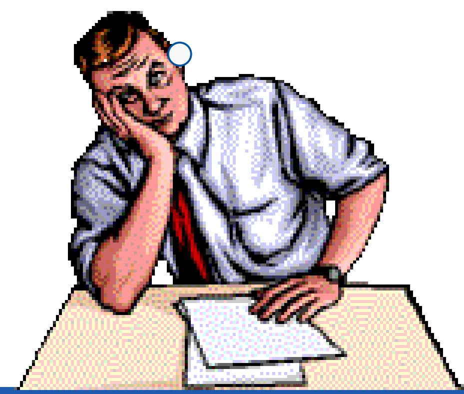


$$x_{new} = ax_{old} + by_{old}$$

$$y_{new} = cx_{old} + dy_{old}$$

2 equations needed !!!

Rather clumsy !
Is there a more efficient way of doing this ?



Matrices & Vectors

So, we have:

$$\begin{cases} x_{new} = a x_{old} + b y_{old} \\ y_{new} = c x_{old} + d y_{old} \end{cases}$$

Lets write this as **<u>one</u>** equation:

$$\bar{B} = M\bar{A}$$

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

Rows

Columns

- $\bar{A}$ and $\bar{B}$ are **<u>Vectors</u>** or **<u>Matrices</u>**
- $\bar{A}$ and $\bar{B}$ have 2 rows and 1 column
- **M** is a **<u>Matrix</u>** and has 2 rows and 2 columns

Matrix Multiplications

This implies:

$$\left.\begin{aligned} x_{new} &= a x_{old} + b y_{old} \\ y_{new} &= c x_{old} + d y_{old} \end{aligned}\right\} \text{Equals} \left\{ \begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix} \right.$$

This defines the rules for **<u>matrix multiplicatio</u>**

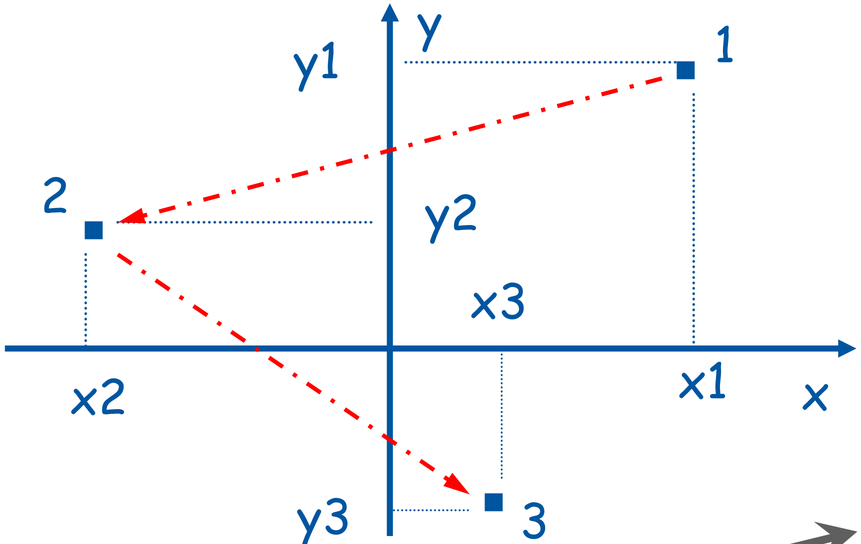
$$\begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

This matrix multiplication results in:

$$i = ae + bg,\ j = af + bh,\ k = ce + dg,\ l = cf + dh$$

Moving a Point & Matrices

Lets apply what we just learned and move a point around:



- M1 transforms 1 to 2
- M2 transforms 2 to 3
- This defines M3=M2M1

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = M1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = M2.M1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M3 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Matrices & Accelerators

- We use matrices to describe the various magnetic elements in our accelerator.
  - The **x** and **y** co-ordinates are the **position** and **angle** of each individual particle.
  - If we know the position and angle of any particle at one point, then to calculate its position and angle at another point we **multiply all the matrices** describing the magnetic elements between the two points to give a single matrix

- Now we are able to calculate the final co-ordinates for any initial pair of particle co-ordinates, provided all the element matrices are known.

The Unit Matrix

There is a special matrix that when multiplied with an initial point will result in the same final point.

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

The result is : $\begin{cases} X_{new} = X_{old} \\ Y_{new} = Y_{old} \end{cases}$

The **Unit matrix** has **no effect** on x and y

Going Backwards

What about **going back** from a **final** point to the corresponding **initial** point ?

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$ or $$\overline{B} = M\overline{A}$$

For the reverse we need another matrix M^{-1}

$$\overline{A} = M^{-1}\overline{B}$$ such that $$\overline{B} = MM^{-1}\overline{B}$$

The combination of M and M^{-1} does have no effect

$$MM^{-1} = Unit\ Matrix$$

M^{-1} is the "**inverse**" or "**reciprocal**" matrix of **M.**

Calculating the Inverse Matrix

If we have a 2 x 2 matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

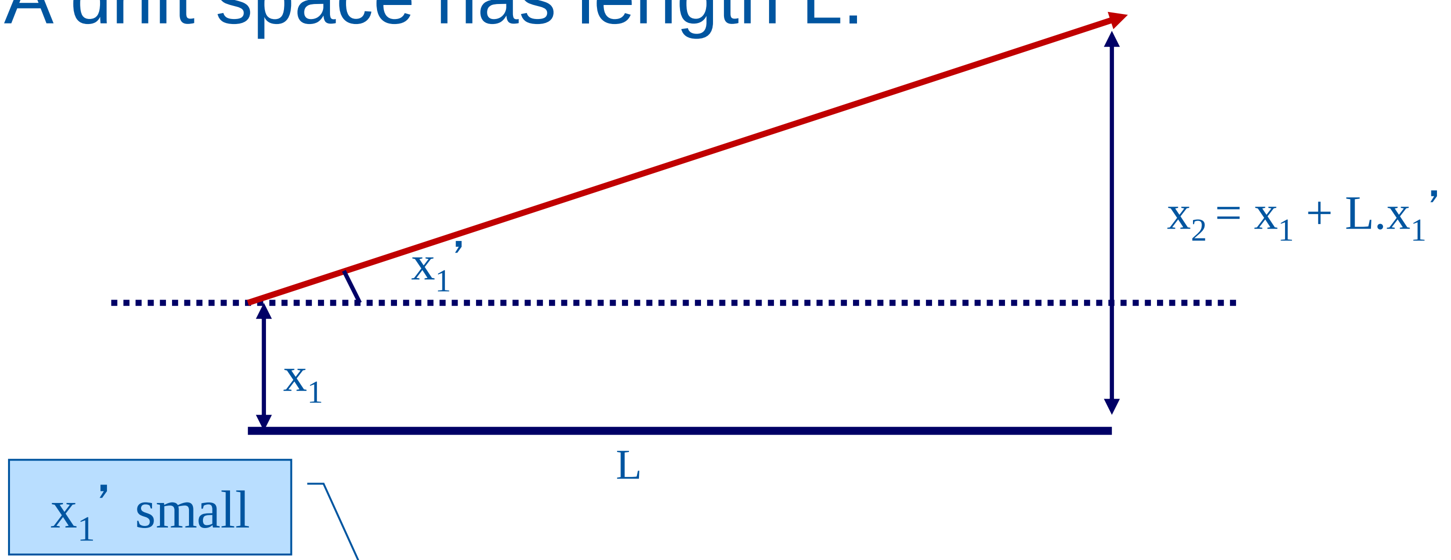
Then the **inverse matrix** is calculated by:

$$M^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The term **(ad – bc)** is called the **determinate**, which is just a **number** (scalar).

Example: Drift Space Matrix

- A drift space contains no magnetic field.
- A drift space has length L.



$$\left.\begin{aligned} x_2 &= x_1 + L x_1' \\ x_2' &= 0 + x_1' \end{aligned}\right\} \rightarrow \begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

A Practical Example

- Changing the current in two sets of quadrupole magnets (F & D) changes the horizontal and vertical tunes (Q_h & Q_v).
- This can be expressed by the following matrix relationship:

$$\begin{pmatrix} \Delta Q_h \\ \Delta Q_v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Delta I_F \\ \Delta I_D \end{pmatrix} \quad \text{or} \quad \overline{\Delta Q} = M \, \overline{\Delta I}$$

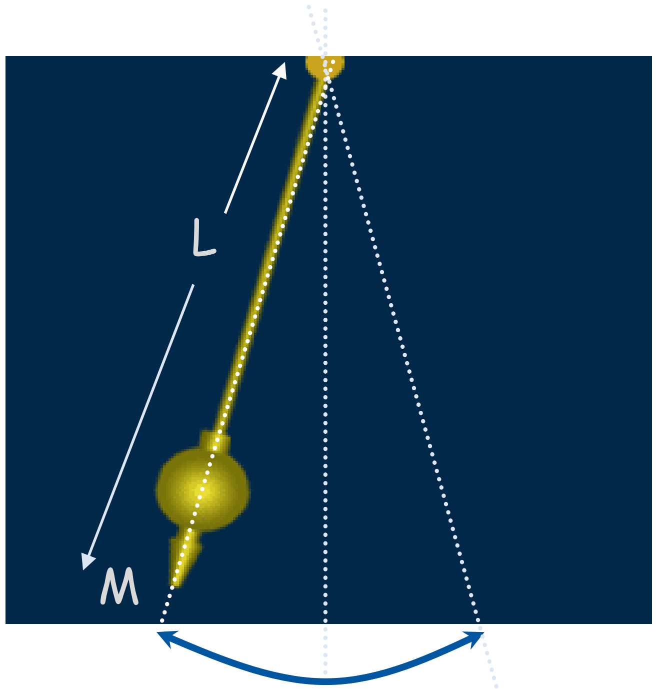
- Change I_F then I_D and measure the changes in Q_h and Q_v
- Calculate the matrix M
- Calculate the inverse matrix M^{-1}
- Use now M^{-1} to calculate the current changes (ΔI_F and ΔI_D) needed for any required change in tune (ΔQ_h and ΔQ_v).

$$\overline{\Delta I} = M^{-1} \, \overline{\Delta Q}$$

- Vectors & Matrices
- **Differential Equations**
- Some Units we use

The Pendulum

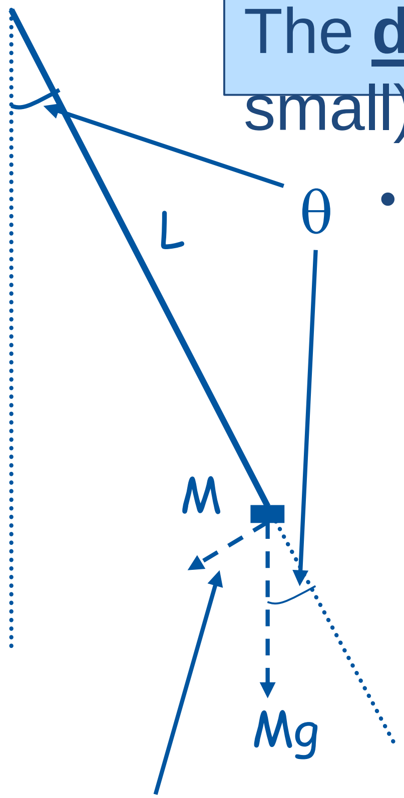
- Lets use a pendulum as example
- The **length** of the pendulum is **L**
- It has a **mass m** attached to it
- It moves back and forth under the **influence of gravity**



- Lets try to find an **equation** that **describes** the **motion** of the mass **m** makes.
- This will result in a **Differential Equation**

Differential Equation

The **distance** from the centre = **Lθ** (since θ is small)



- The **velocity** of mass M is: $v = \frac{d(L\theta)}{dt}$
- The **acceleration** of mass M is: $a = \frac{d^2(L\theta)}{dt^2}$
- Newton: **F**orce = **m**ass x **a**cceleration

$$-Mg\sin\theta = M\frac{d^2(L\theta)}{dt^2}$$

Restoring force due to gravity is

-M g sinθ

(force opposes motion)

$$\frac{d^2(\theta)}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

θ is small

L is constant

Solving a Differential Equation

$$\frac{d^2(\theta)}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

Differential equation describing the motion of a pendulum at small amplitudes.

Find a solution……Try a good "guess"……

$$\theta = A\cos(\omega t)$$

Differentiate our guess (twice)

$$\frac{d(\theta)}{dt} = -A\omega\sin(\omega t)$$ and $$\frac{d^2(\theta)}{dt^2} = -A\omega^2\cos(\omega t)$$

Put this and our "guess" back in the original Differential equation.

$$-\omega^2\cos(\omega t) + \left(\frac{g}{L}\right)\cos(\omega t) = 0$$

Solving a Differential Equation

Now we have to find the solution for the following equation:

$$-\omega^2 \cos(\omega t) + \left(\frac{g}{L}\right)\cos(\omega t) = 0$$

Solving this equation gives: $\omega = \sqrt{\frac{g}{L}}$

The final solution of our differential equation, describing the motion of a pendulum is as we expected :

$$\theta = A\cos\sqrt{\left(\frac{g}{L}\right)}\,t$$

Oscillation amplitude

Oscillation frequency

Position & Velocity

The differential equation that describes the transverse motion of the particles as they move around our accelerator.

$$\frac{d^2(x)}{dt^2} + (K)x = 0$$

The solution of this second order describes **oscillatory motion**

For any system, where the restoring force is proportional to the displacement, the solution for the displacement will be of the form:

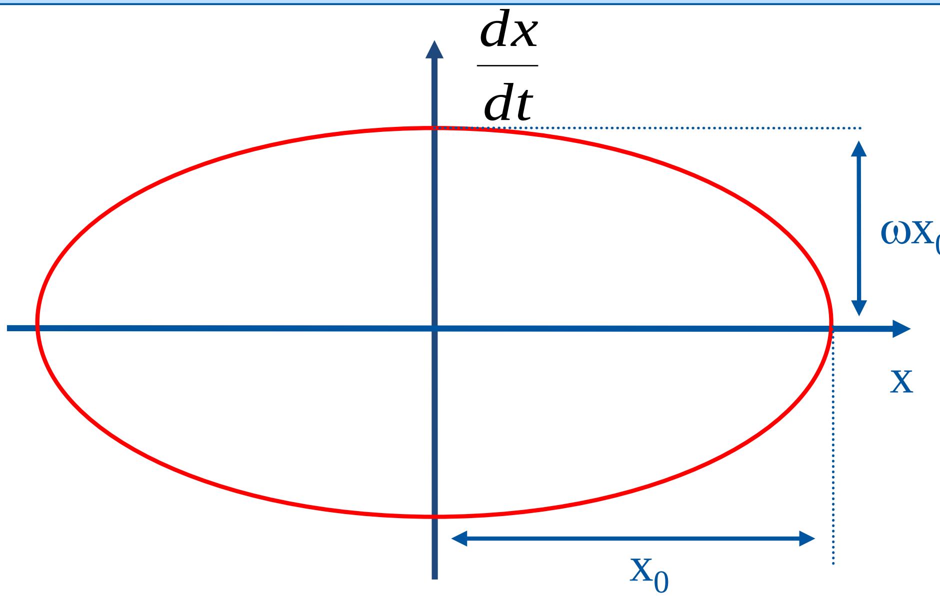
$$x = x_0 \cos(\omega t) \qquad \frac{dx}{dt} = -x_0 \omega \sin(\omega t)$$

Phase Space Plot

Plot the **velocity** as a function of **displacement**:

$$x = x_0 \cos(\omega t)$$

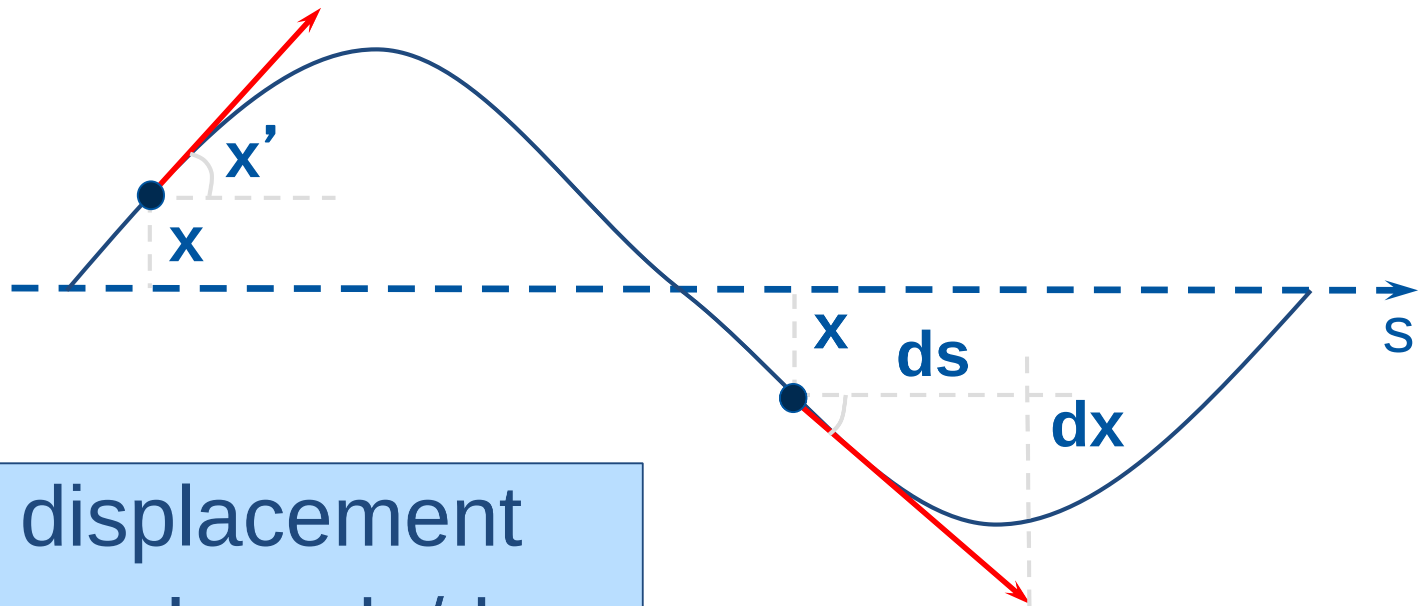
$$\frac{dx}{dt} = -x_0 \omega \sin(\omega t)$$



- It is an ellipse.
- As ωt advances by 2 π it repeats itself.
- This continues for (ω t + k 2π), with k=0,±1, ±2,..,..etc

Oscillations in Accelerators

Under the influence of the **magnetic fields** the **particle oscillate**



x = displacement

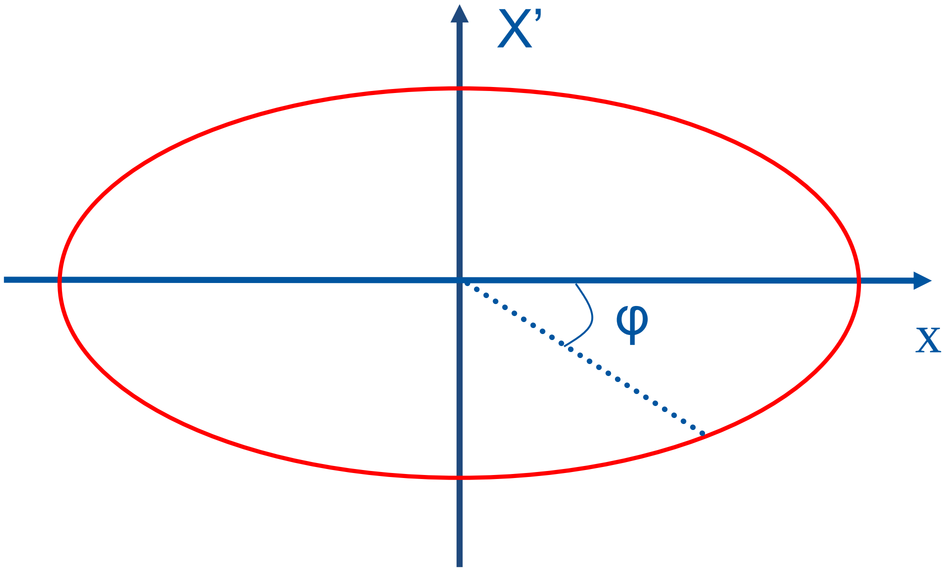
x' = angle = dx/ds

Transverse Phase Space Plot

This changes slightly the Phase Space plot

Position x

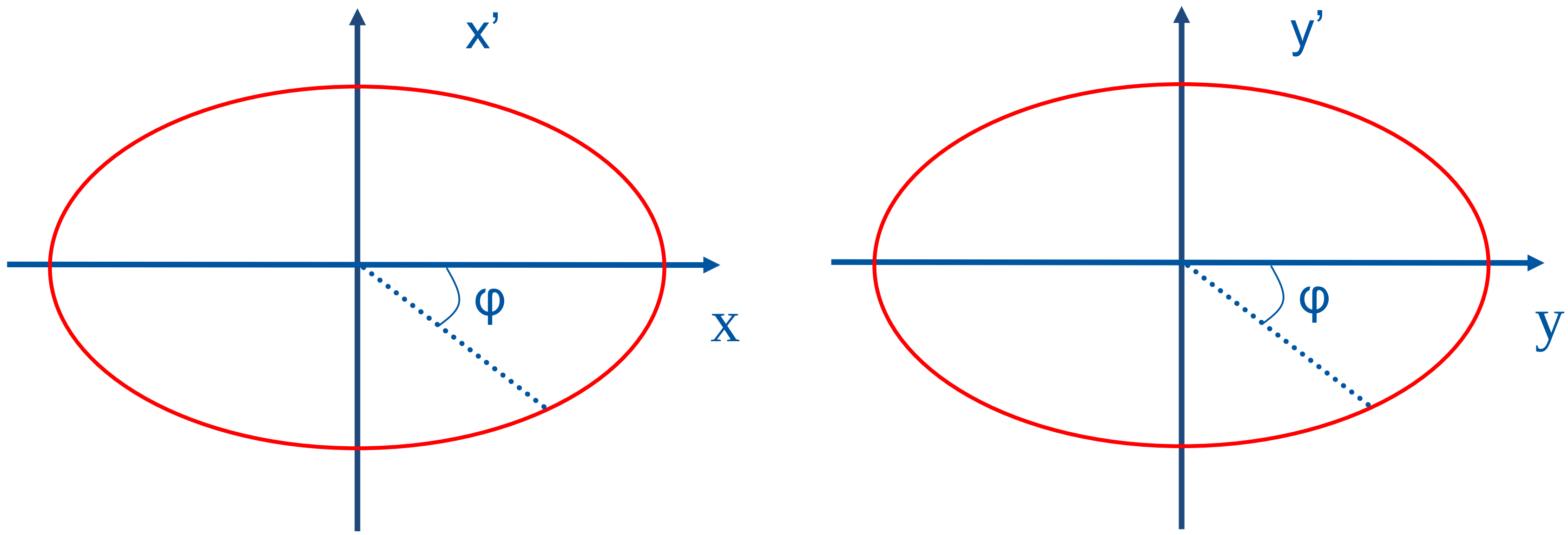
Angle $x' = \frac{dx}{ds}$



- φ = ωt is called the **phase angle**
- X-axis is the horizontal or vertical position (or time).
- Y-axis is the horizontal or vertical phase angle (or energy).

Transverse Phase Space Plot

We distinguish motion in the Horizontal & Vertical Plane

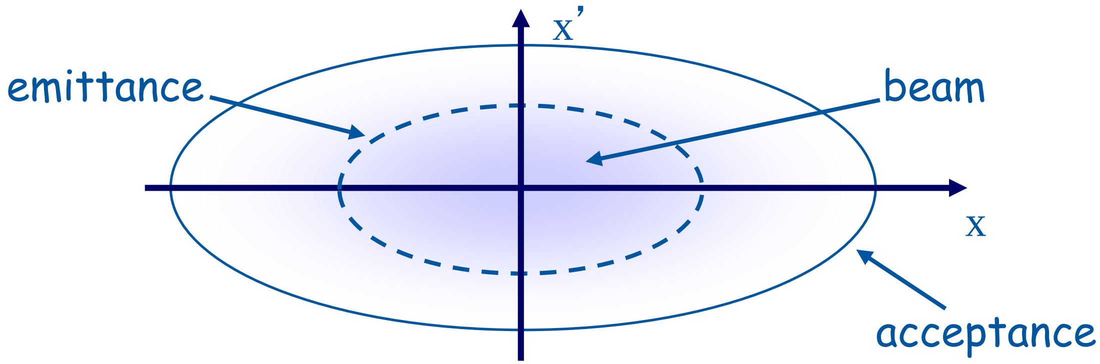


Horizontal Phase Space

Vertical Phase Space

Transverse Emittance

- To be rigorous we should define the emittance slightly differently.
- Observe all the particles at a single position on one turn and measure both their position and angle.
- This will give a large number of points in our phase space plot, each point representing a particle with its co-ordinates x, x'.

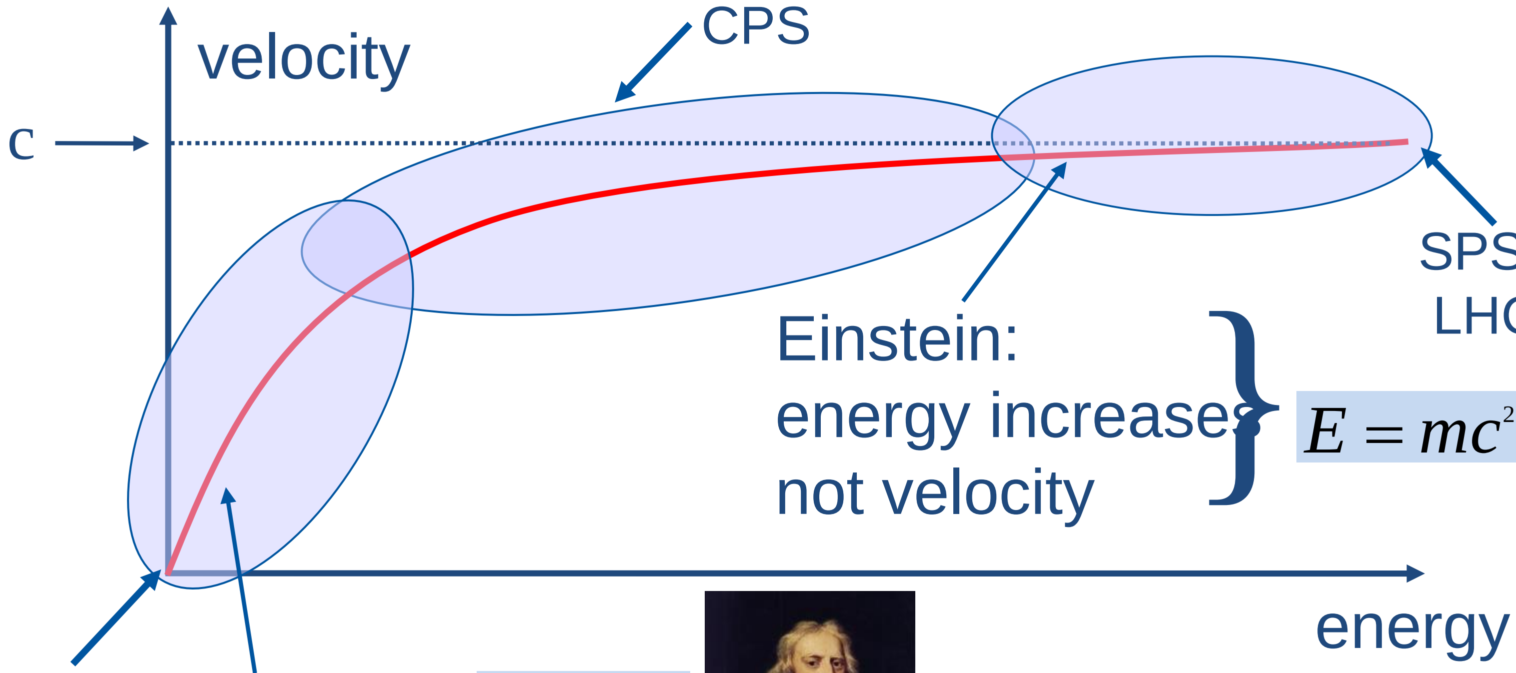


Symbol: ε
Expressed in 1σ, 2σ,..
Units: mm mrad

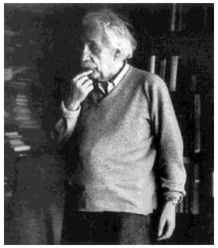
- The **emittance** is the **area** of the ellipse, which contains all, or a defined percentage, of the particles.
- The **acceptance** is the maximum **area** of the ellipse, which the emittance can attain without losing particles

- Vectors & Matrices
- Differential Equations
- **Some Units we use**

Relativity



Einstein: energy increases not velocity $\}$ $E = mc^2$



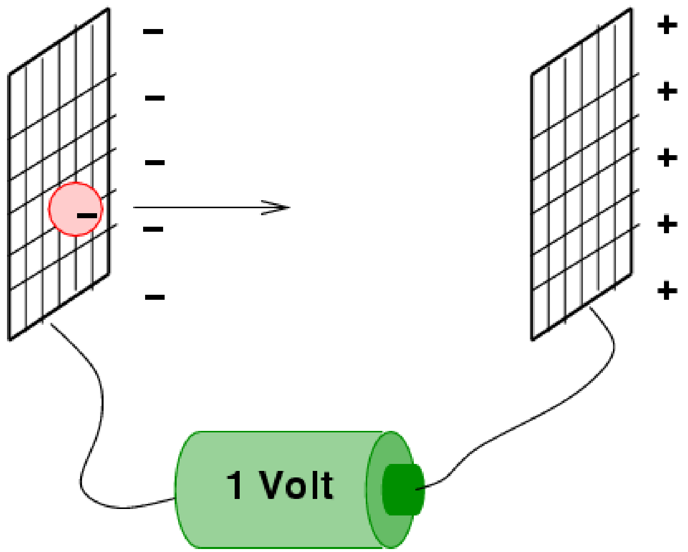
Newton: $E = \frac{1}{2}mv^2$

More about "Relativity" by Andrea Latina

This afternoon

The Units we use for Energy

- The energy acquired by an electron in a potential of 1 Volts is defined as being 1 eV



- The unit eV is too small to be used today, we use:

  1 KeV = 10^3, MeV = 10^6, GeV = 10^9, TeV = 10^{12}

Energy: eV versus Joules

- The unit most commonly used for **Energy** is **Joules [J]**
- In accelerator and particle physics we talk about **eV**….!?
- The **energy** acquired by an **electron** in a potential of **1 Volt** is defined as being **1 eV**
- **1 eV** is **1 elementary charge** ‘pushed’ by **1 Volt**.

$$\mathbf{1\ eV = 1.6 \times 10^{-19}\ Joules}$$

The Energy in the LHC beam

- The energy in one LHC beam at high energy is about 320 Million Joules
- This corresponds to the energy of a TGV engine going at 150 km/h

...... but then concentrated in the size of a needle

Energy versus Momentum

Einstein's formula:

$E = mc^2$ which for a mass at rest is: $E_0 = m_0 c^2$

The ratio between the total energy and the rest energy is

$$\gamma = \frac{E}{E_0}$$

The ratio between the real velocity and the velocity of light is

$$\beta = \frac{v}{c}$$

Then the mass of a moving particle is: $m = \gamma m_0$

We can write: $\beta = \frac{mvc}{mc^2}$

Momentum is: $p = mv$

$$\beta = \frac{pc}{E} \quad or \quad p = \frac{E\beta}{c}$$

Energy versus Momentum

$$p = \frac{E\beta}{c}$$

Momentum — Energy

- Therefore the **units** for
  - **momentum** are: MeV/c, GeV/c, …etc.
  - **Energy** are: MeV, GeV, …etc.

Attention:

when **β=1** **energy** and **momentum** are **equal**

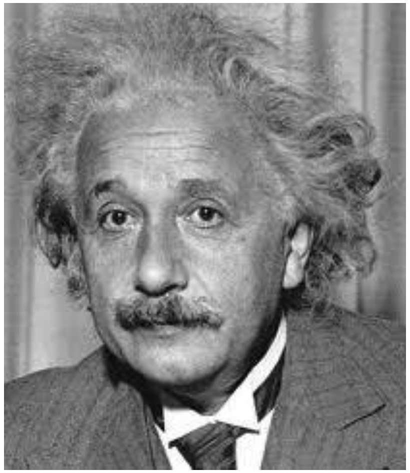
when **β<1** the **energy** and **momentum** are **not equal**

A Practical Example (PSB- PS)

- Kinetic energy at injection $E_{kinetic}$ = 1.4 GeV
- Proton rest energy E_0=938.27 MeV
- The total energy is then: $E = E_{kinetic} + E_0$ = **2.34 GeV**
- We know that $\gamma = \frac{E}{E_0}$, which gives γ = 2.4921
- We can derive $\beta = \sqrt{1 - \frac{1}{\gamma^2}}$, which gives **β = 0.91597**
- Using $p = \frac{E\beta}{c}$ we get p = **2.14 GeV/c**

In this case: **Energy ≠ Momentum**

Pure mathematics is, in its way, the poetry of logical ideas.



Albert Einstein