Lattice Design

G. Sterbini, BE-ABP-HSI, CERN

9-21 June 2019, CAS, Slangerup, Denmark

guido.sterbini@cern.ch

Lattice Design

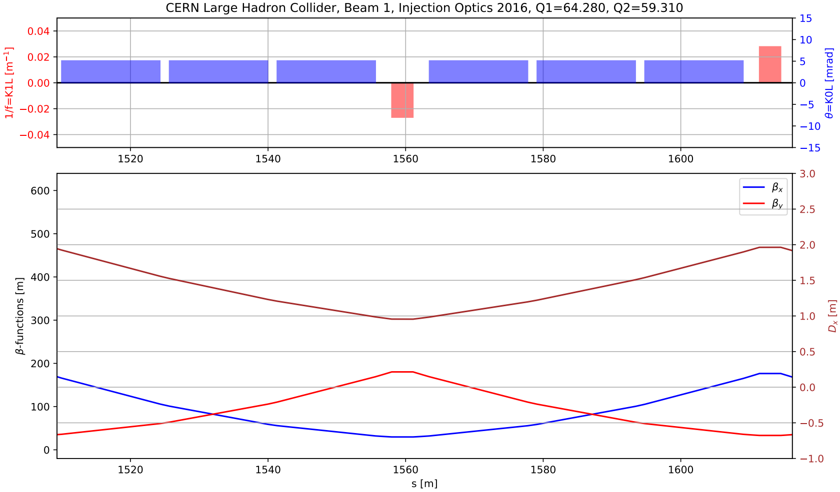
The analysis of the cell stability and betatron functions can be done via an algorithmic approach[1] using the method presented yesterday:

1. compute *symbolically* the M_{OTM},
2. diagonalize it $M_{OTM} = PDP^{-1}$, with $det(P) = -i$ and $P_{11} = P_{12}$,
3. impose that all the eigenvalues amplitude is 1 to get the stability condition,
4. study P to get the periodic solution for β and α at the start of the cell,
5. propagate the solution from the start of the cell along the different lattice element.

We will start considering a **FODO cell**.

[1] http://cern.ch/go/J8TP

The CERN Large Hadron Collider FODO cell



The FODO cell description

Let's consider a FODO cell of length L_{cell} in **thin lens approximation**, where

1. the space of the focusing (F) and defocusing (D) quadrupoles is equal to $L_{cell}/2$ and
2. the focal length of the F and D quadrupoles equal in module, that is $f_D = -f_F$ with $f_F > 0$.

For convenience we will start and end the FODO cell with half of an F quadrupole (i.e., with focal lenght $2 \times f_F$) and we will consider, as first step, the horizontal plane.

The FODO M_{OTM} diagonalization

Using symbolic tools (e.g., *sympy*) one can compute[2]

$$M_{OTM} = \begin{bmatrix} -\frac{L_{cell}^2}{8f^2} + 1 & \frac{L_{cell}^2}{4f} + L_{cell} \\ \frac{L_{cell}(L_{cell}-4f)}{16f^3} & -\frac{L_{cell}^2}{8f^2} + 1 \end{bmatrix}$$

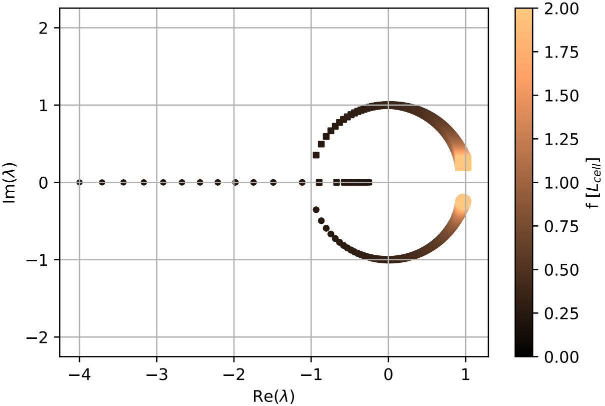
$$D = \begin{bmatrix} \frac{-L_{cell}^2 + L_{cell}\sqrt{L_{cell}^2 - 16f^2} + 8f^2}{8f^2} & 0 \\ 0 & \frac{-L_{cell}^2 - L_{cell}\sqrt{L_{cell}^2 - 16f^2} + 8f^2}{8f^2} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{\sqrt{f}}{\sqrt{-\frac{i}{(L_{cell}-4f)\sqrt{L_{cell}^2-16f^2}}}}(-L_{cell}+4f) & \frac{\sqrt{f}}{\sqrt{-\frac{i}{(L_{cell}-4f)\sqrt{L_{cell}^2-16f^2}}}}(-L_{cell}+4f) \\ -\frac{1}{2\sqrt{f}\sqrt{-\frac{i}{(L_{cell}-4f)\sqrt{L_{cell}^2-16f^2}}}\sqrt{L_{cell}^2-16f^2}} & \frac{1}{2\sqrt{f}\sqrt{-\frac{i}{(L_{cell}-4f)\sqrt{L_{cell}^2-16f^2}}}\sqrt{L_{cell}^2-16f^2}} \end{bmatrix}$$

[2] http://cern.ch/go/J8TP

The FODO stability I

The stability on the horizontal plane is achieved if λ_1 and λ_2 have unitary module.



The FODO stability II

This implies $-1 < \frac{\lambda_1+\lambda_2}{2} = \cos\mu < 1$, that is

$$\boxed{\frac{L_{cell}}{4} < f}.$$

The stability condition in the vertical plane is exactly equivalent, since D(f)=D(-f).

The stability condition of a FODO lattice (thin lens approximation and no dipoles) imposes an F quadrupole with f larger than $L_{cell}/4$.

The FODO phase advance

Remembering that

$$\mu = \arccos \frac{\lambda_1 + \lambda_2}{2},$$

one gets

$$\boxed{\mu = \arccos \left(1 - \frac{L_{cell}^2}{8f^2}\right)},$$

or, equivalently, from[3]

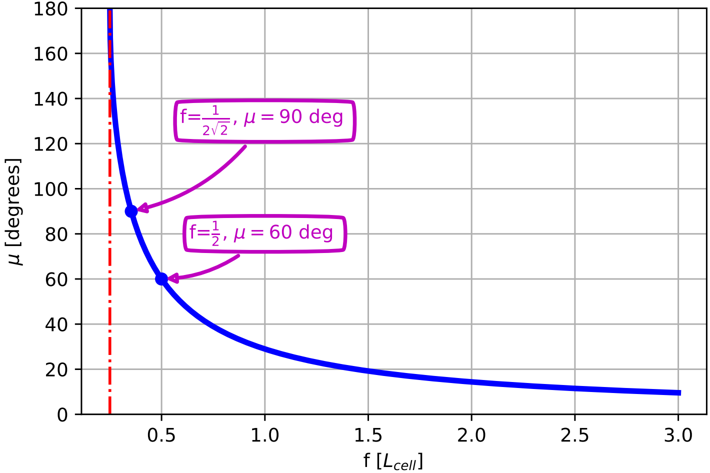
$$\sin\left(\frac{\arccos(1-x)}{2}\right) = \sqrt{\frac{x}{2}}$$

we get

$$\boxed{\sin\left(\frac{\mu}{2}\right) = \frac{L_{cell}}{4f}}.$$

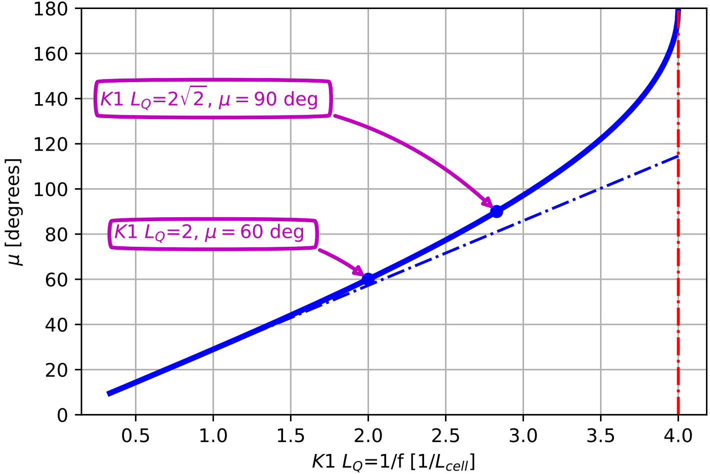
[3] http://cern.ch/go/8qcf

μ vs f and 1/f



f=$\frac{1}{2\sqrt{2}}$, $\mu = 90$ deg
f=$\frac{1}{2}$, $\mu = 60$ deg
μ [degrees]
f [L_{cell}]
0
20
40
60
80
100
120
140
160
180
0.5
1.0
1.5
2.0
2.5
3.0

μ vs f and 1/f



$K1\ L_Q$=$2\sqrt{2}$, $\mu = 90$ deg
$K1\ L_Q$=2, $\mu = 60$ deg
μ [degrees]
$K1\ L_Q$=1/f [$1/L_{cell}$]
0
20
40
60
80
100
120
140
160
180
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

FODO Optics Functions I

Remembering that

$$P = \begin{pmatrix} \sqrt{\dfrac{\beta}{2}} & \sqrt{\dfrac{\beta}{2}} \\ \dfrac{-\alpha + i}{\sqrt{2\beta}} & \dfrac{-\alpha - i}{\sqrt{2\beta}} \end{pmatrix}$$

we have

$$\boxed{\beta(0) = 2\ P_{11}^2} \text{ and } \boxed{\alpha(0) = -P_{11}(P_{21} + P_{22})}.$$

FODO Optics Functions II

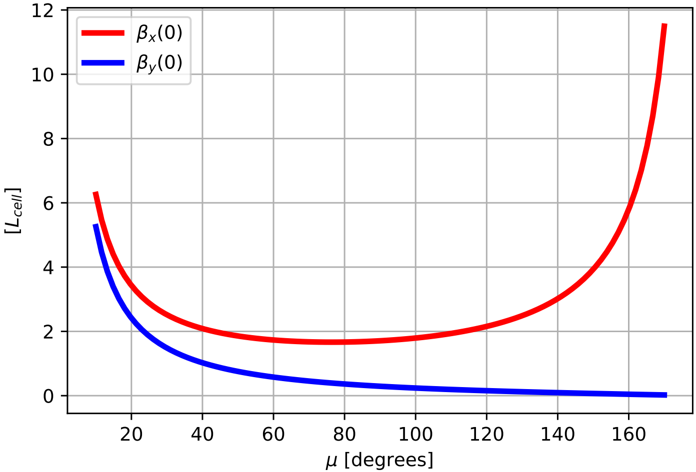
This yields

$$\begin{aligned}\beta_x(0) &= \frac{2f\sqrt{4f+L_{cell}}}{\sqrt{4f-L_{cell}}} = L_{cell}\frac{1+\sin(\mu/2)}{\sin(\mu)}\\ \alpha_x(0) &= 0.\end{aligned}$$

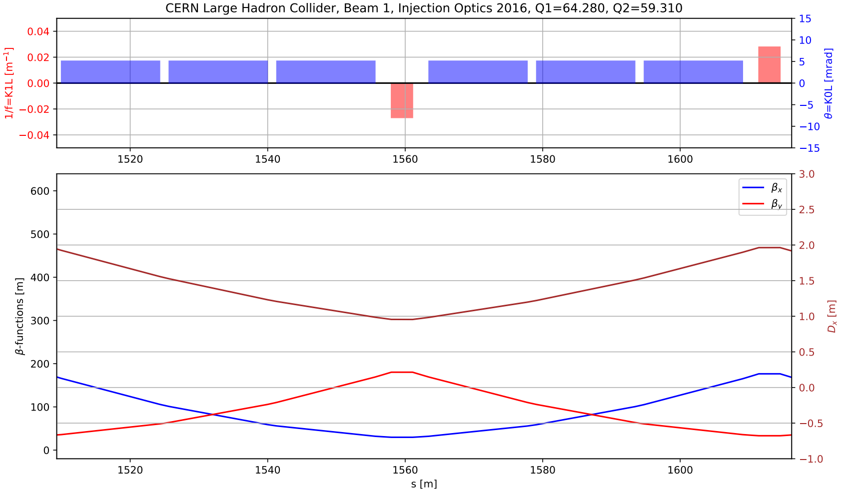
With a similar approach, we can compute the y-plane optical functions by considering P(-f), getting

$$\begin{aligned}\beta_y(0) &= \frac{2f\sqrt{4f-L_{cell}}}{\sqrt{4f+L_{cell}}} = L_{cell}\frac{1-\sin(\mu/2)}{\sin(\mu)}\\ \alpha_y(0) &= 0.\end{aligned}$$

β-function vs μ



β-function vs μ



Chromaticity of a FODO I

The definition of the linear chromaticity is

$$\xi = \frac{\Delta Q}{\frac{\Delta p}{p_0}} = \frac{1}{2\pi}\frac{\Delta \mu}{\frac{\Delta p}{p_0}}. \tag{1}$$

From the relation

$$f\left(\frac{\Delta p}{p_0}\right) = f \times \left(1 + \frac{\Delta p}{p_0}\right) \tag{2}$$

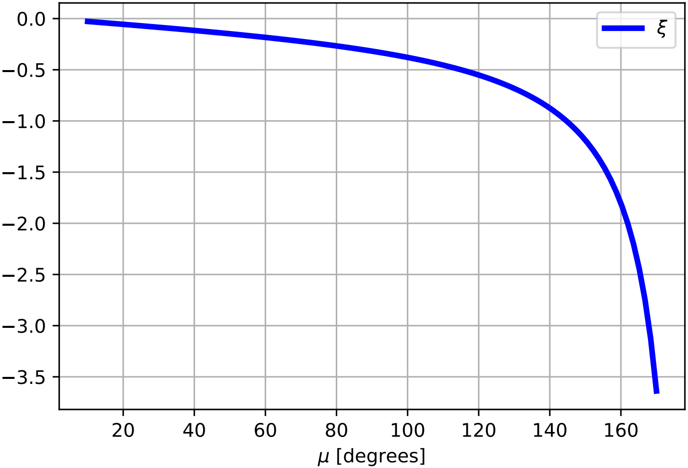
and from

$$\sin\left(\frac{\mu}{2}\right) = \frac{L_{cell}}{4f}, \tag{3}$$

one can compute the FODO lattice chromaticity

$$\xi = -\frac{1}{4\pi}\frac{L_{cell}}{f}\frac{1}{\cos(\mu/2)} = \boxed{-\frac{1}{\pi}\tan\left(\frac{\mu}{2}\right)} \tag{4}$$

Chromaticity of a FODO II



FODO flavours I

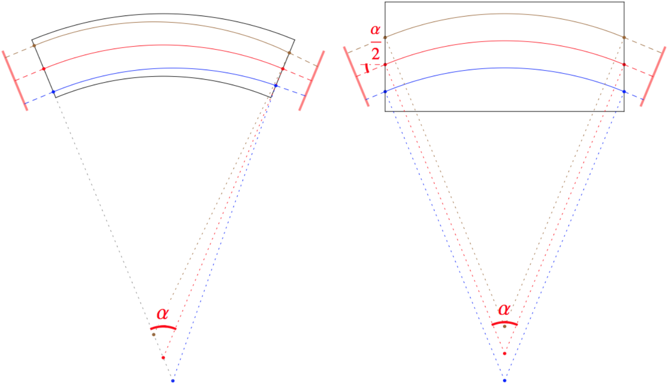
From the FODO lattice we can define at least two additional "flavours":

1. different focal length in the F and D quadrupoles,
2. uneven distance between quadrupoles.

The stability of the two cases is discussed in http://cern.ch/go/J8TP.

FODO flavours II

In addition an example on the effect of the dipoles (sector and rectangular bends) and thick quadrupoles is given in http://cern.ch/go/J8TP using MAD-X.

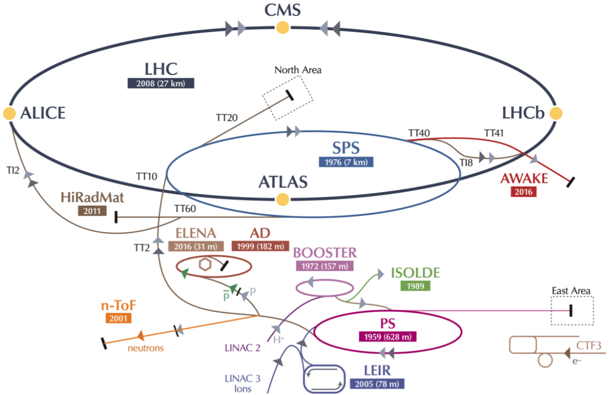


Triplet cell

Starting from the FODO we can consider other lattice cells. As an example, by putting back-to-back two OFOD, we have a triplet cell (OFODDOFO).

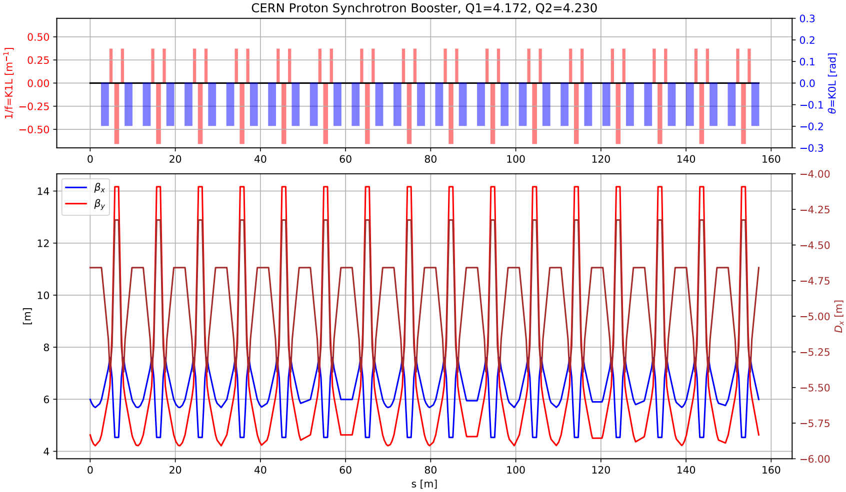
An example of triplet lattice is presented in http://cern.ch/go/J8TP where the stability condition is discussed.

In the following we present few of the CERN Accelerator Complex optics[4].

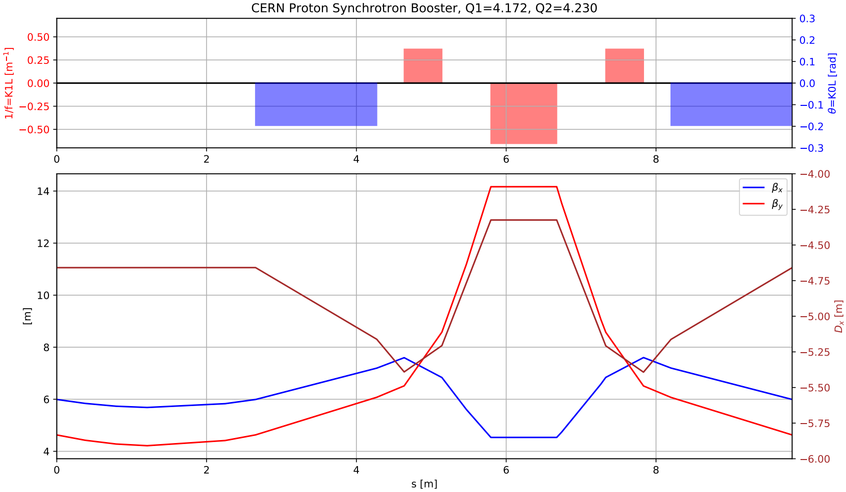


[4] http://cern.ch/go/MKp7

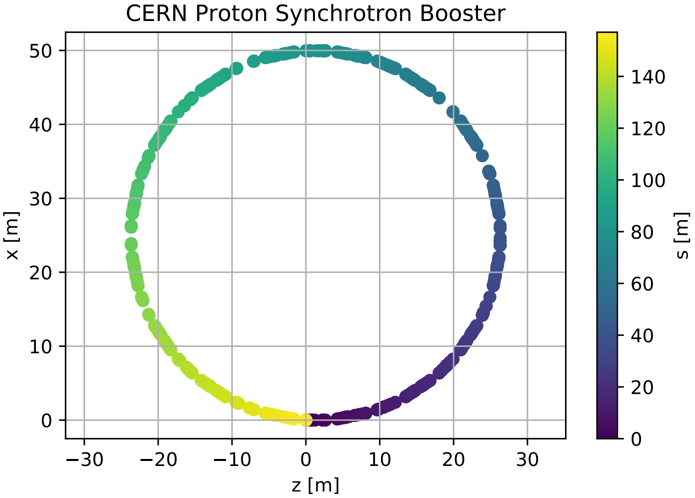
CERN Proton Synchrotron Booster



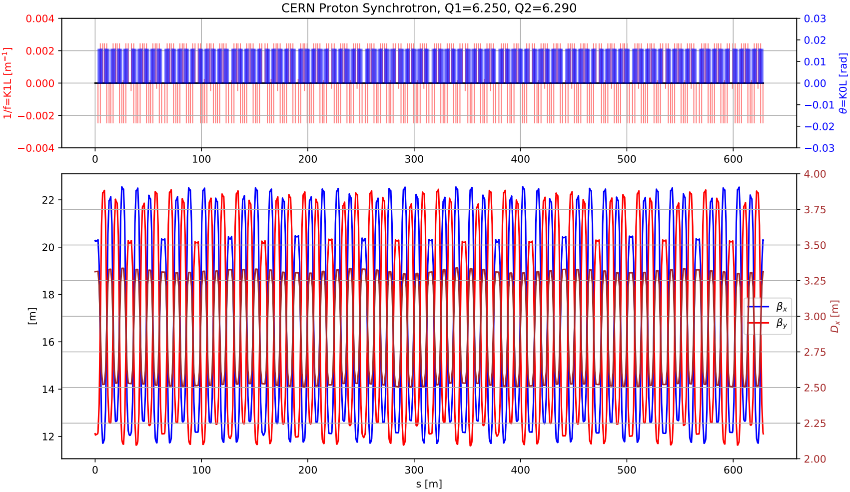
CERN Proton Synchrotron Booster



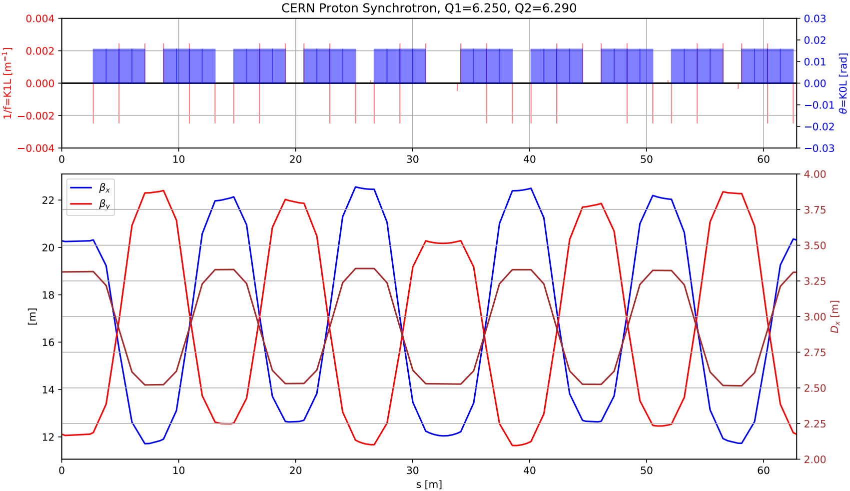
CERN Proton Synchrotron Booster



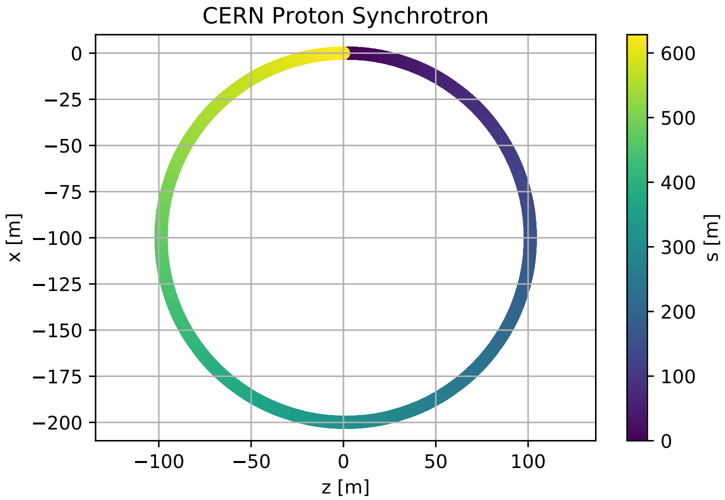
CERN Proton Synchrotron



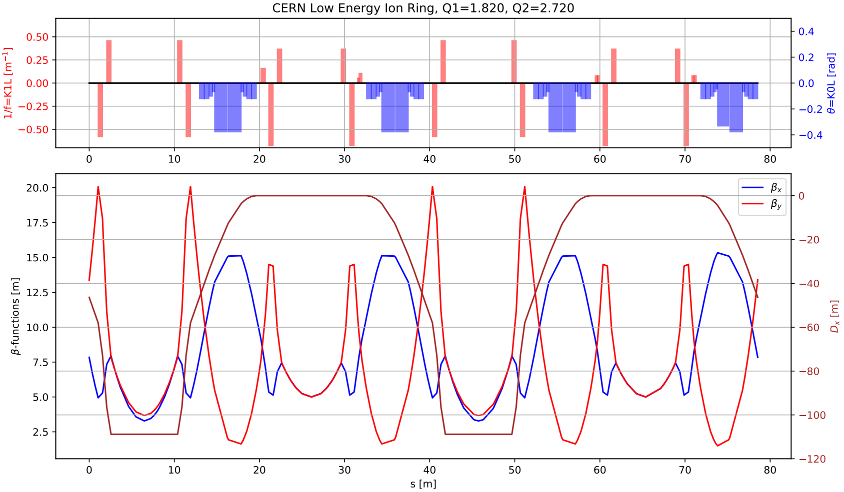
CERN Proton Synchrotron



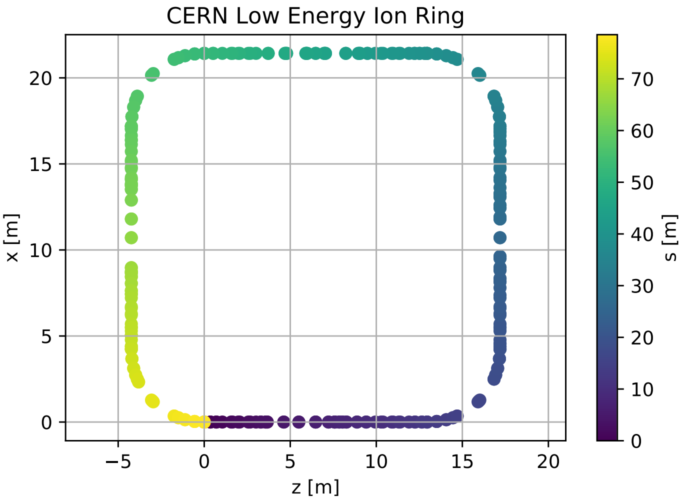
CERN Proton Synchrotron



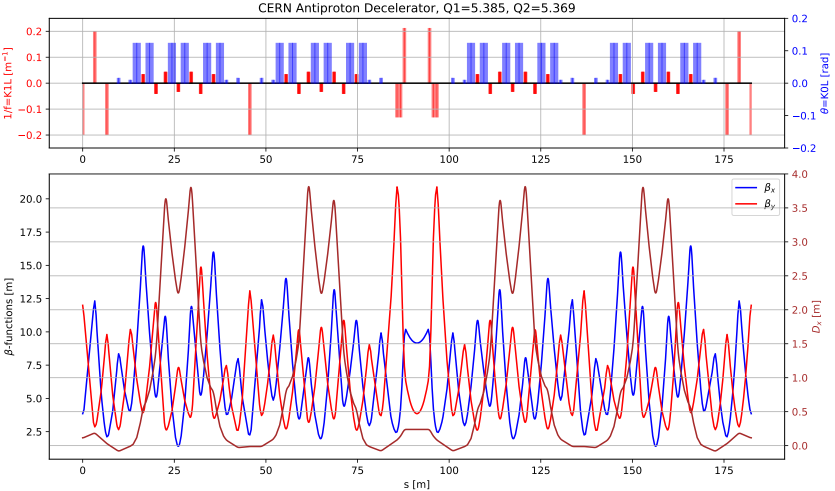
CERN Low Energy Ion Ring

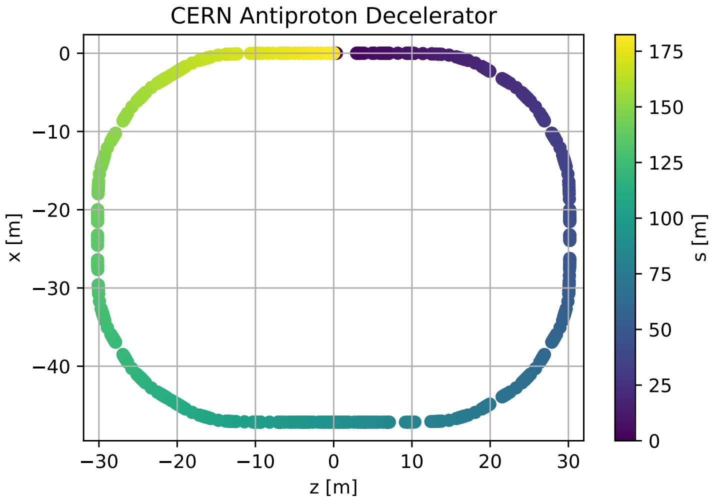


CERN Low Energy Ion Ring

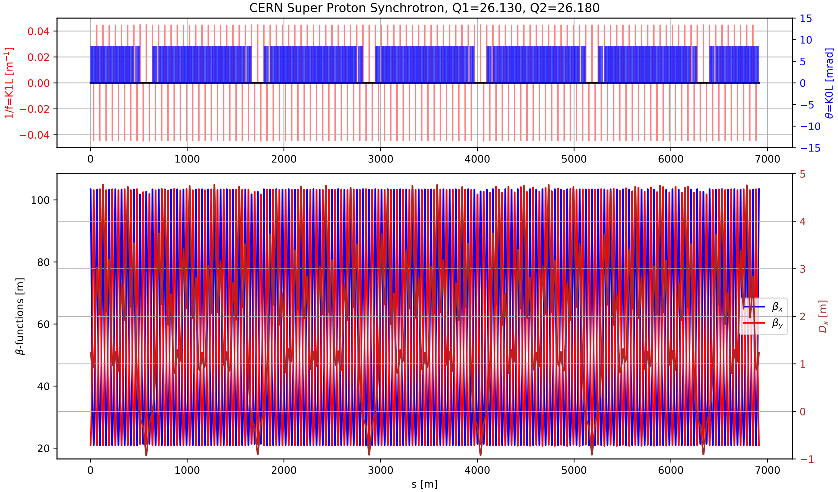


CERN Antiproton Deceleration

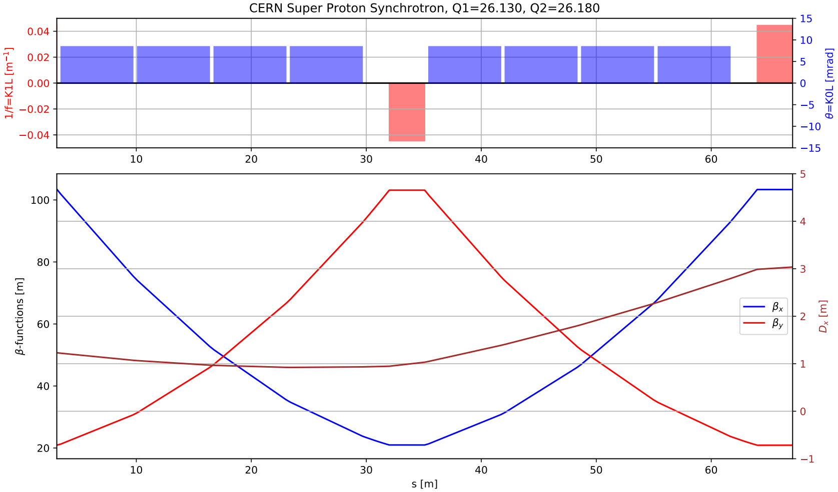



CERN Antiproton Decelerator
0
−10
−20
−30
−40
x [m]
−30
−20
−10
0
10
20
30
z [m]
175
150
125
100
75
50
25
0
s [m]

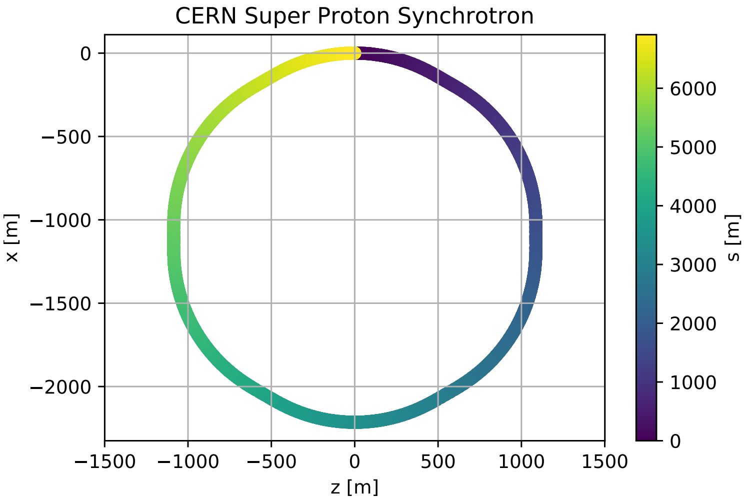
CERN Super Proton Synchrotron



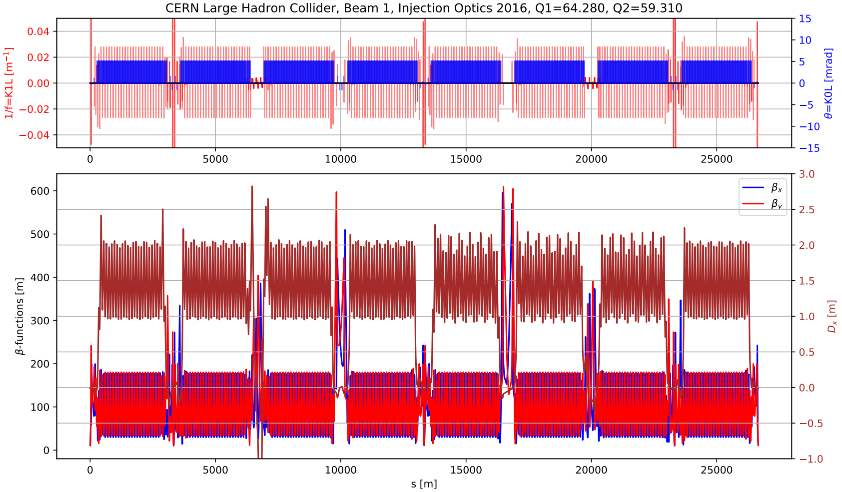
CERN Super Proton Synchrotron



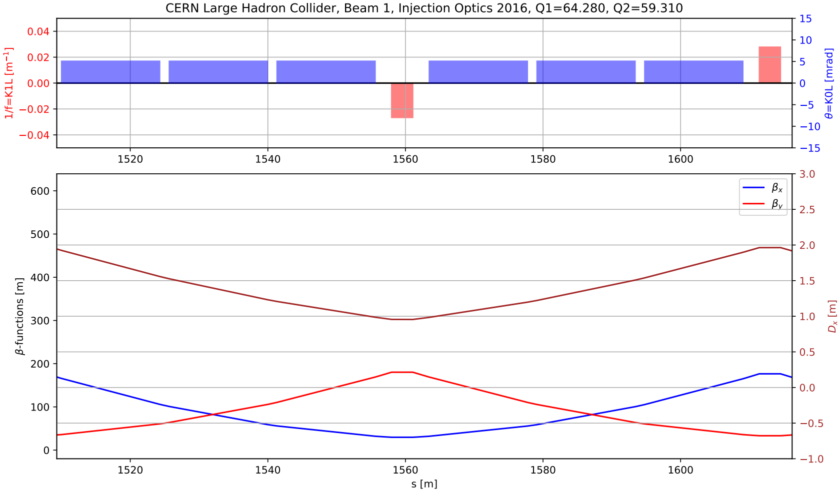
CERN Super Proton Synchrotron



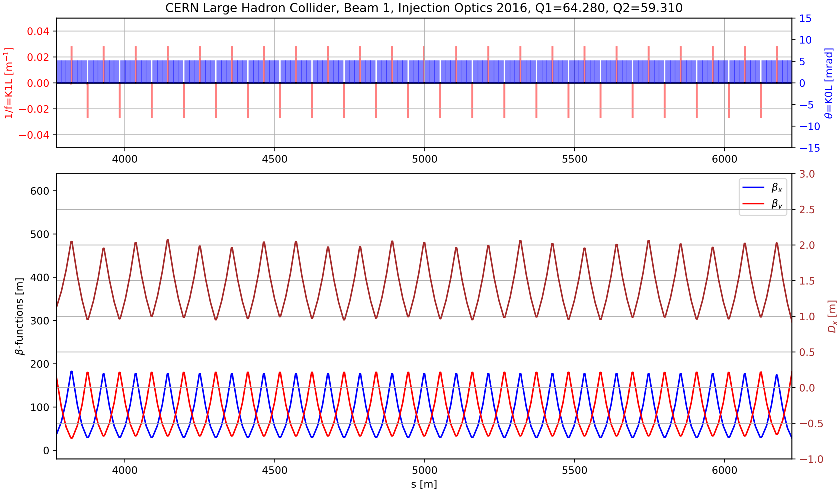
CERN Large Hadron Collider Beam 1



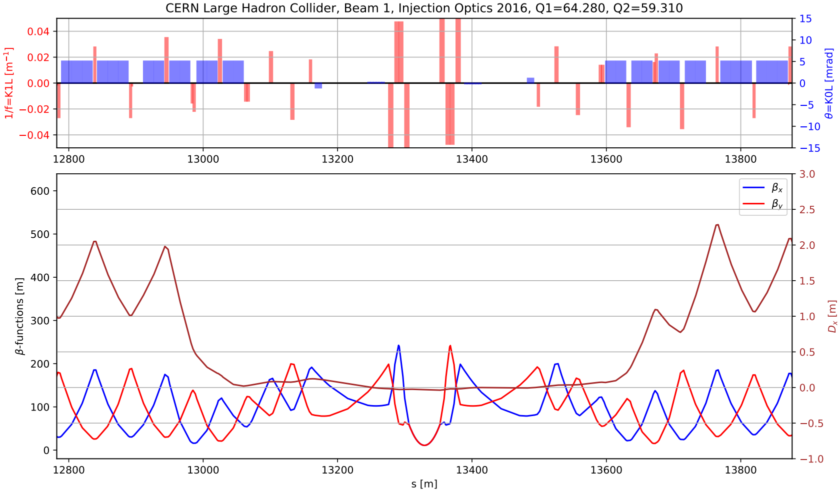
CERN Large Hadron Collider Beam 1



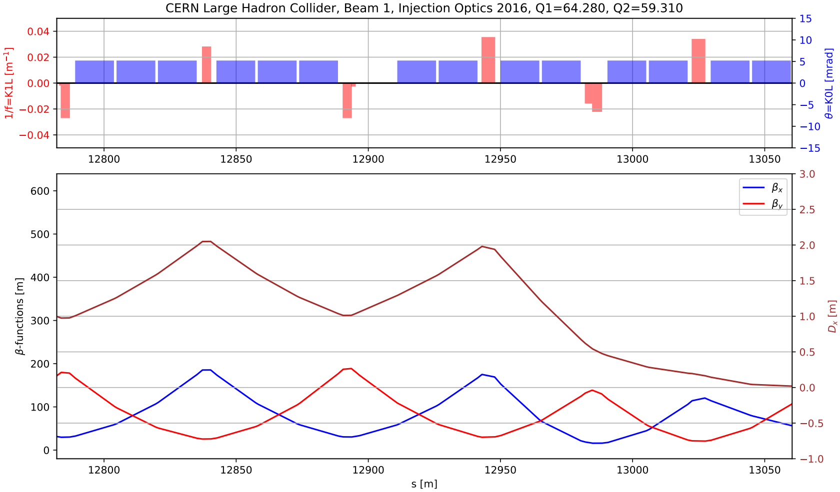
CERN Large Hadron Collider Beam 1



CERN Large Hadron Collider Beam 1



CERN Large Hadron Collider Beam 1



CERN Large Hadron Collider Beam 1