Space Charge in Linear Machines

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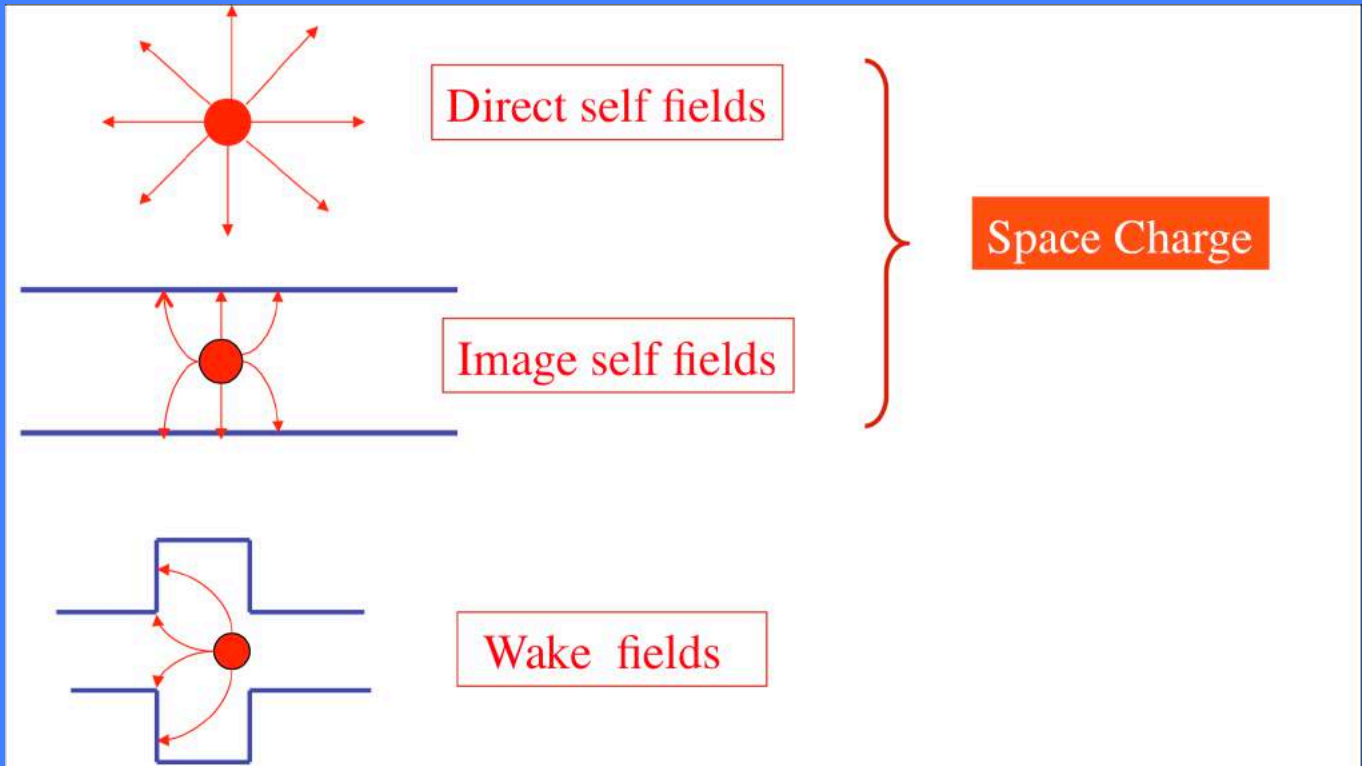


The CERN Accelerator School

Slangerup – June 12 - 2019

SPARC LAB

OUTLINE

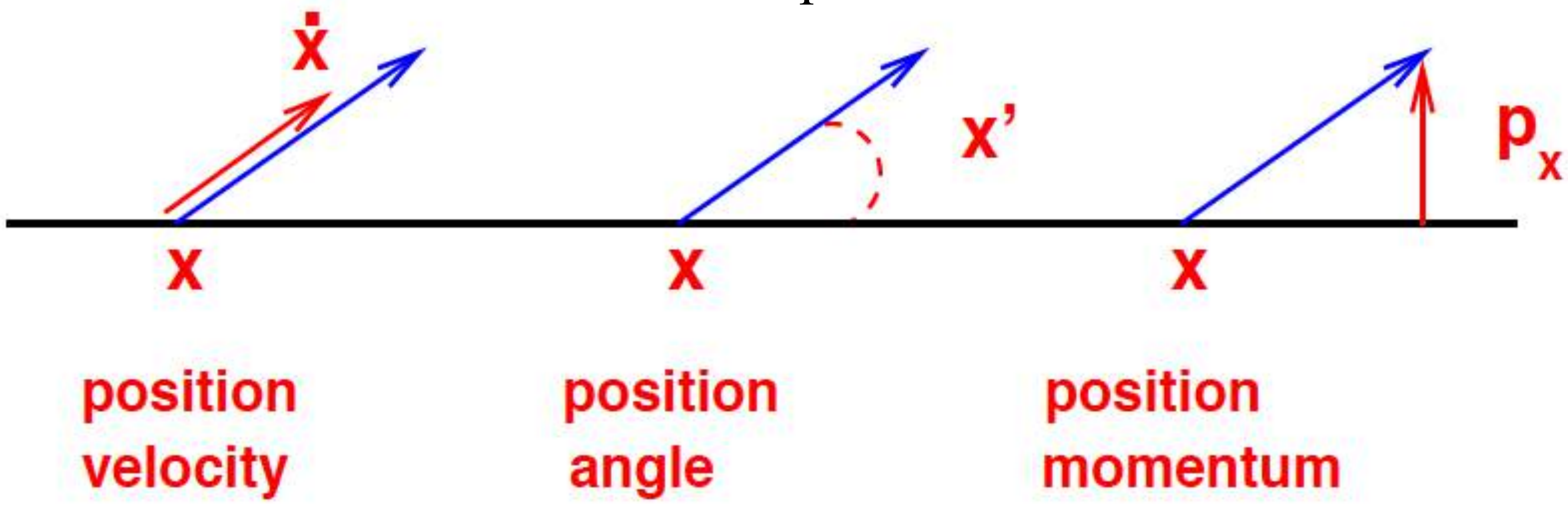
- The rms emittance concept
- rms envelope equation
- Space charge forces
- Space charge induced emittance oscillations
- Matching conditions and emittance compensation

Typical coordinates to describe the particle motion
(6 per particle)

Configuration Space

Trace Space

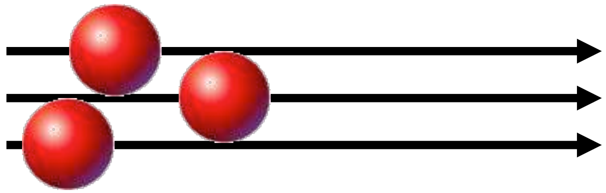
Phase Space



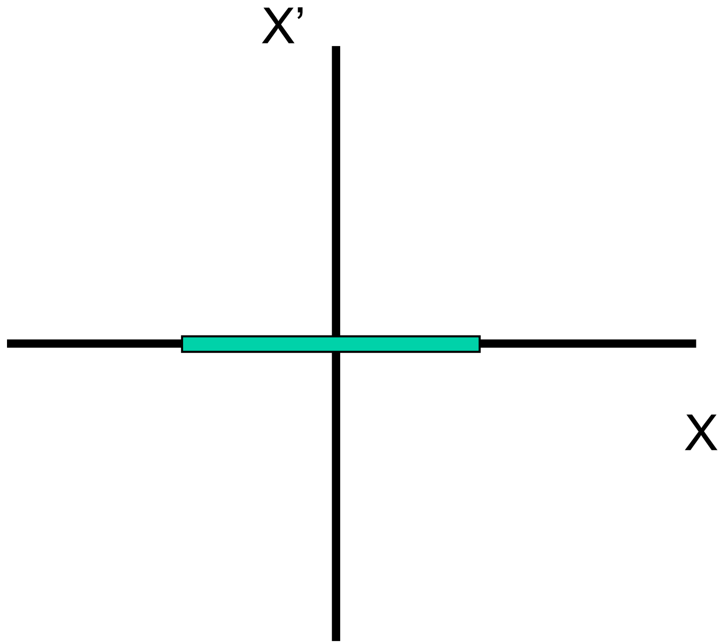
$$\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \end{cases}$$

$$p_z = \gamma m_o v_z$$
$$= \beta_z \gamma m_o c$$

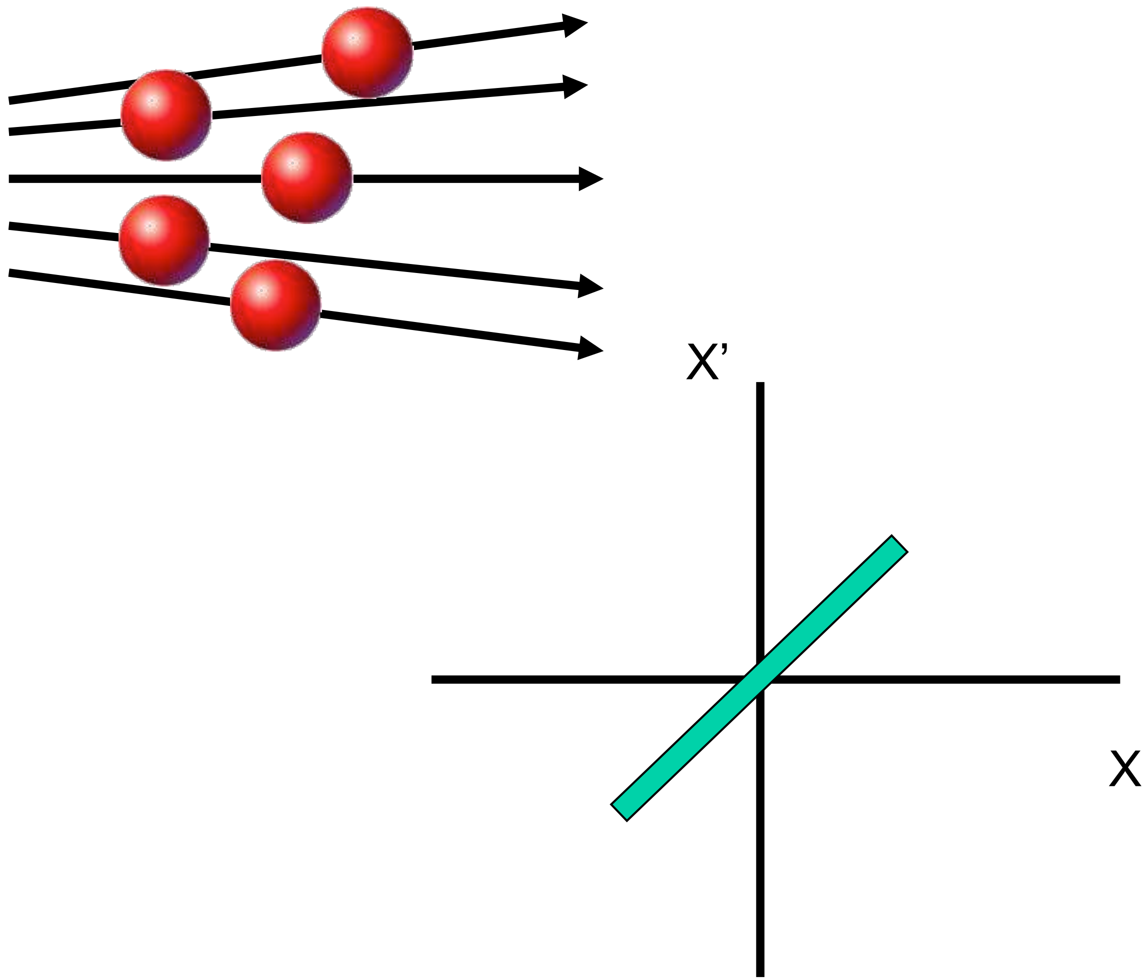
Trace space of an ideal laminar beam



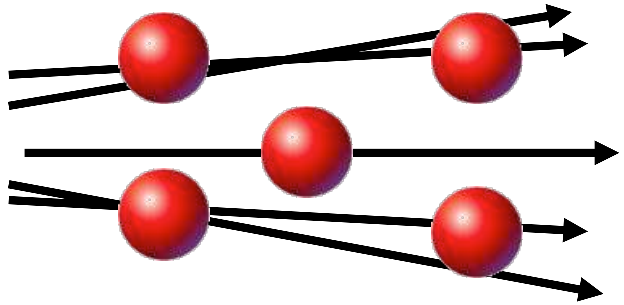
$$\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \end{cases} \qquad p_x << p_z$$

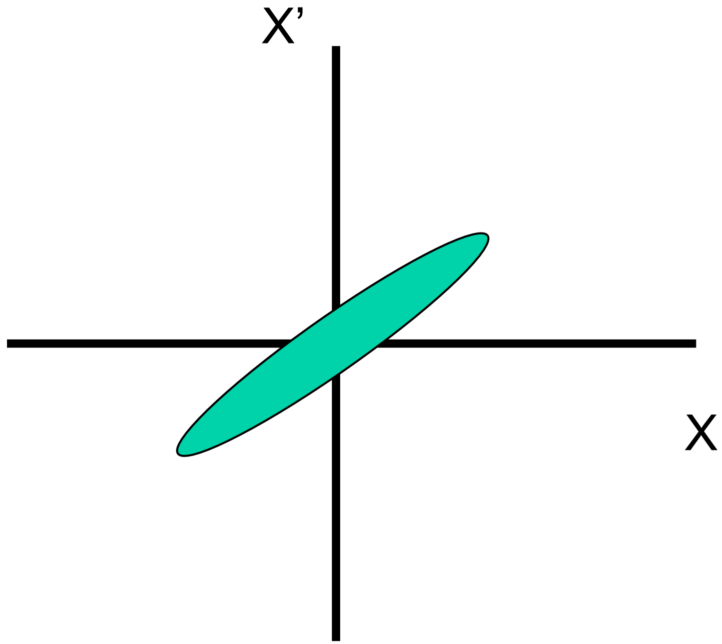


Trace space of a laminar beam



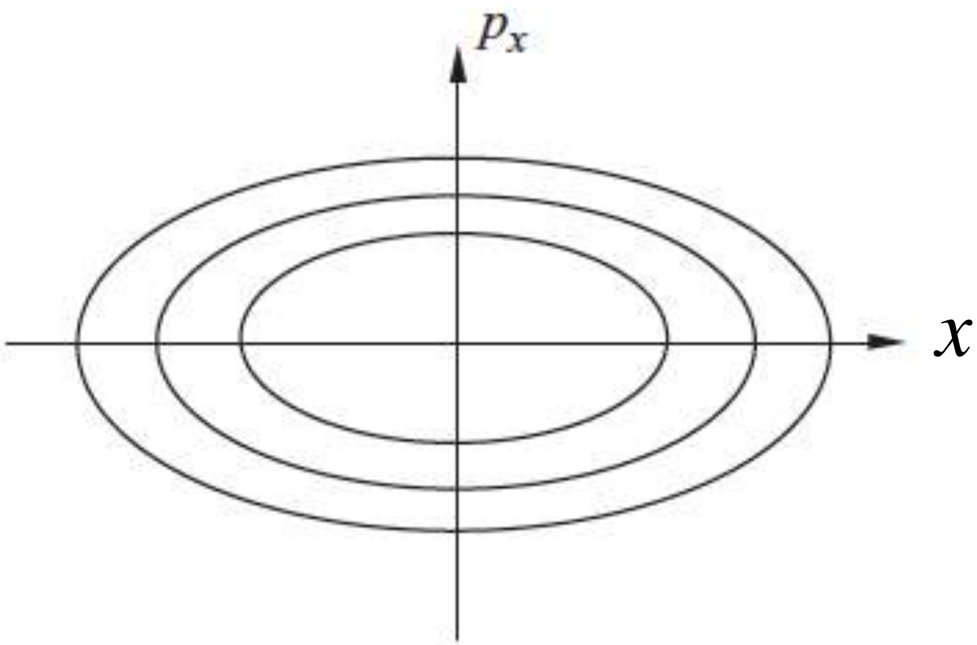
Trace space of non laminar beam





In a system where all the forces acting on the particles are linear (i.e., proportional to the particle's displacement x from the beam axis), it is useful to assume an elliptical shape for the area occupied by the beam in x-x‘ trace space or x-p_x phase space.

$$\ddot{x} + k^2 x = 0$$



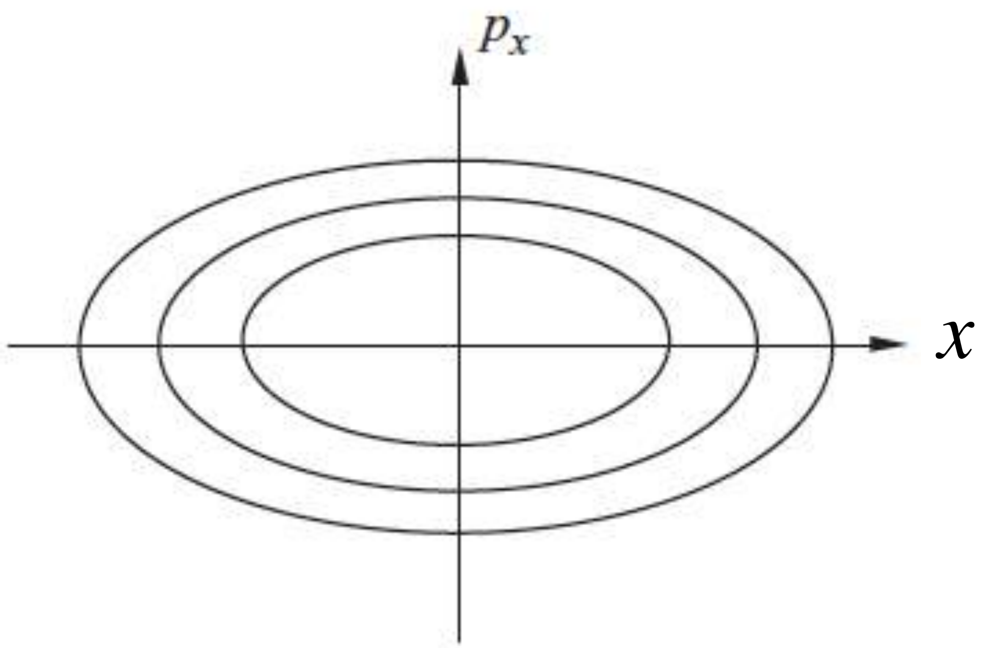
$$H = \frac{1}{2m}[p_x^2 + m^2\omega^2 x^2]$$

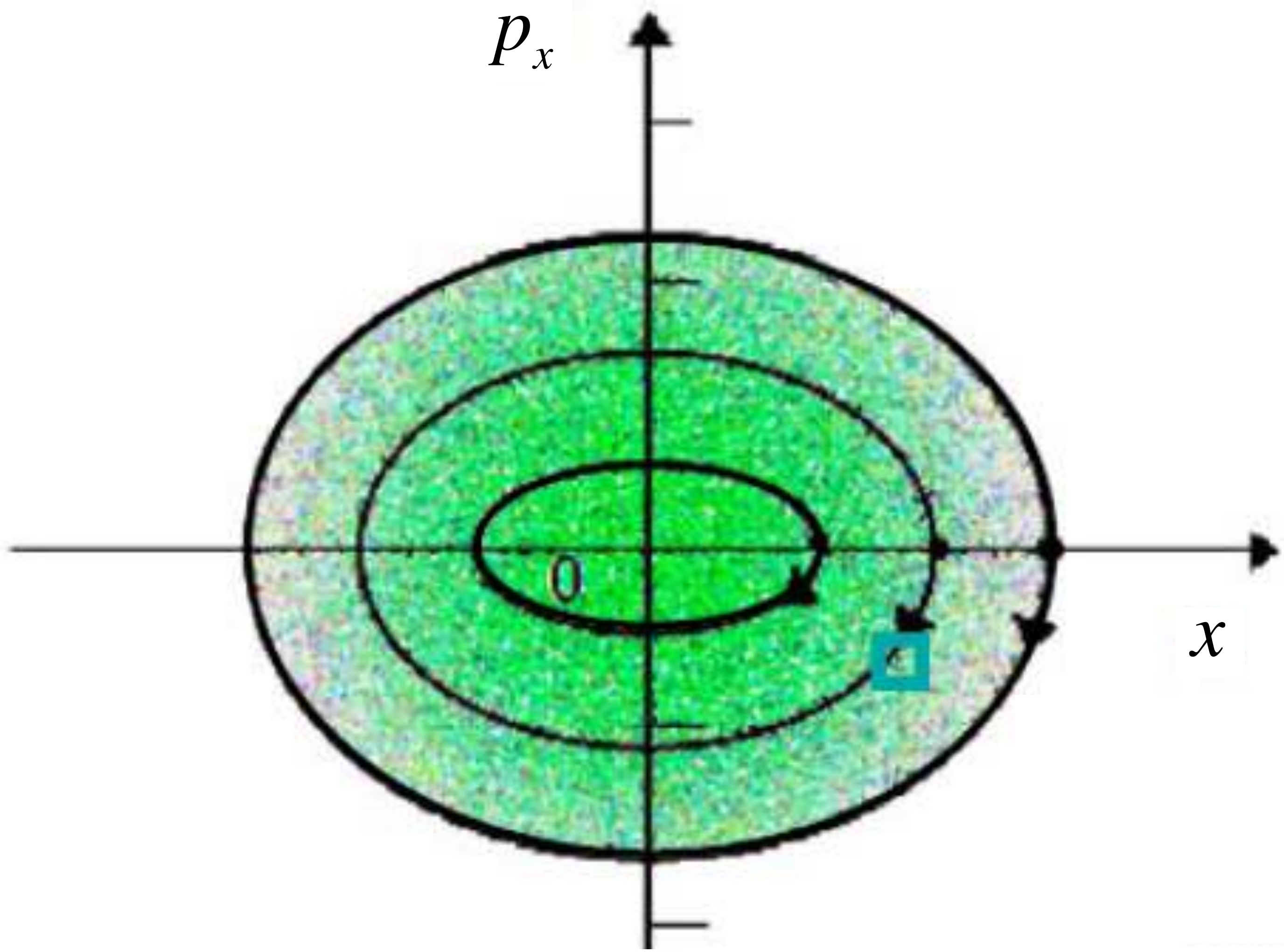
$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i},$$

The action is related to the area enclosed by the phase space trajectory.

$$J = \frac{1}{2\pi} \oint p_x \, \mathrm{d}x.$$

The action is also generally known to be an *adiabatic invariant*, in that when the parameters of an oscillatory system are changed slowly, the action remains a constant.





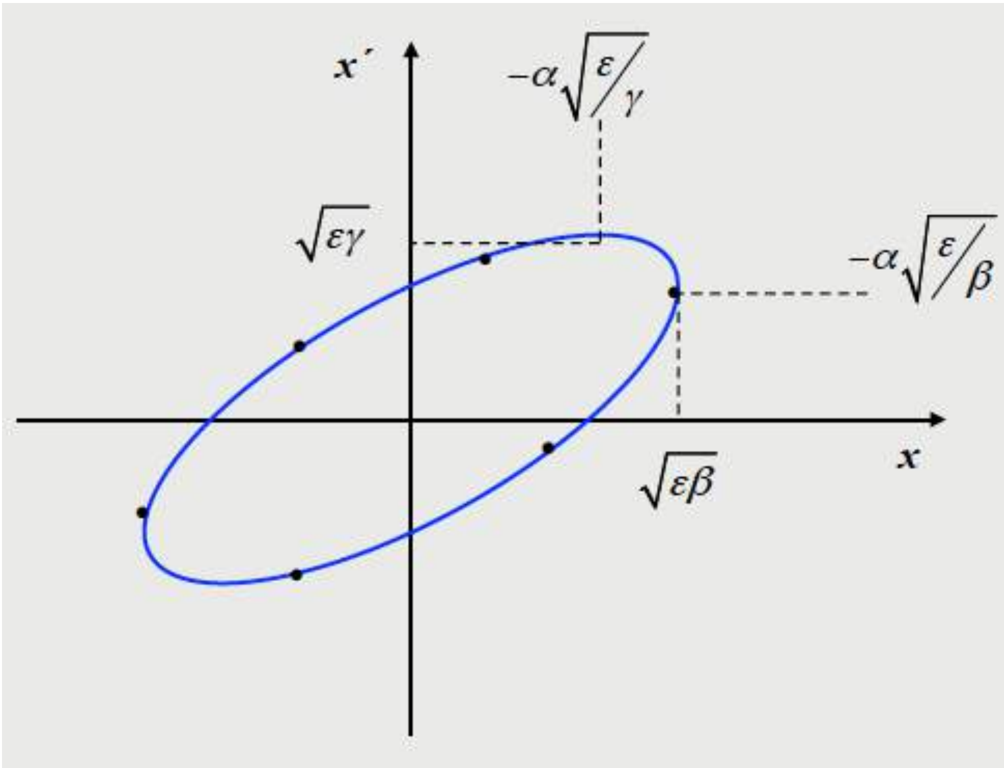
p_x
0
x

Geometric emittance: ε_g

Ellipse equation: $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_g$

Twiss parameters: $\beta\gamma - \alpha^2 = 1$ $\beta' = -2\alpha$

Ellipse area: $A = \pi\varepsilon_g$



Analytical Geometry: Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ Canonical Ellipse equation $$Area = \pi ab$$

$$Ax^2 + Bxy + Cy^2 = 1$$ Rotated Ellipse $$Area = \frac{2\pi}{\sqrt{4AC - B^2}}$$

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon$$ Emittance Ellipse

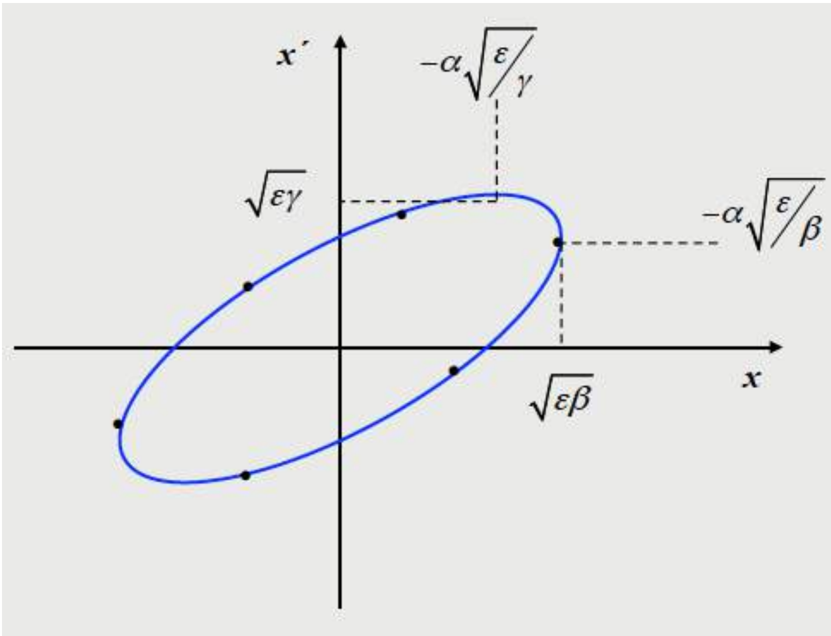
$$Area = \frac{\pi\varepsilon}{\sqrt{\gamma\beta - \alpha^2}} = \pi\varepsilon \Leftrightarrow \gamma\beta - \alpha^2 = 1$$

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon$$

$$\gamma\beta - \alpha^2 = 1$$

Solving for x=f(x') and computing df(x')/dx'=0 =>

$$\begin{cases} x'_{x_{\max}} = -\alpha\sqrt{\dfrac{\varepsilon}{\beta}} \\ x_{\max} = \sqrt{\varepsilon\beta} \end{cases}$$



From:

$$\frac{dx_{\max}}{dz} = x'_{x_{\max}} \qquad \frac{\beta'}{2}\sqrt{\frac{\varepsilon}{\beta}} = -\alpha\sqrt{\frac{\varepsilon}{\beta}}$$

$$\beta' = -2\alpha$$

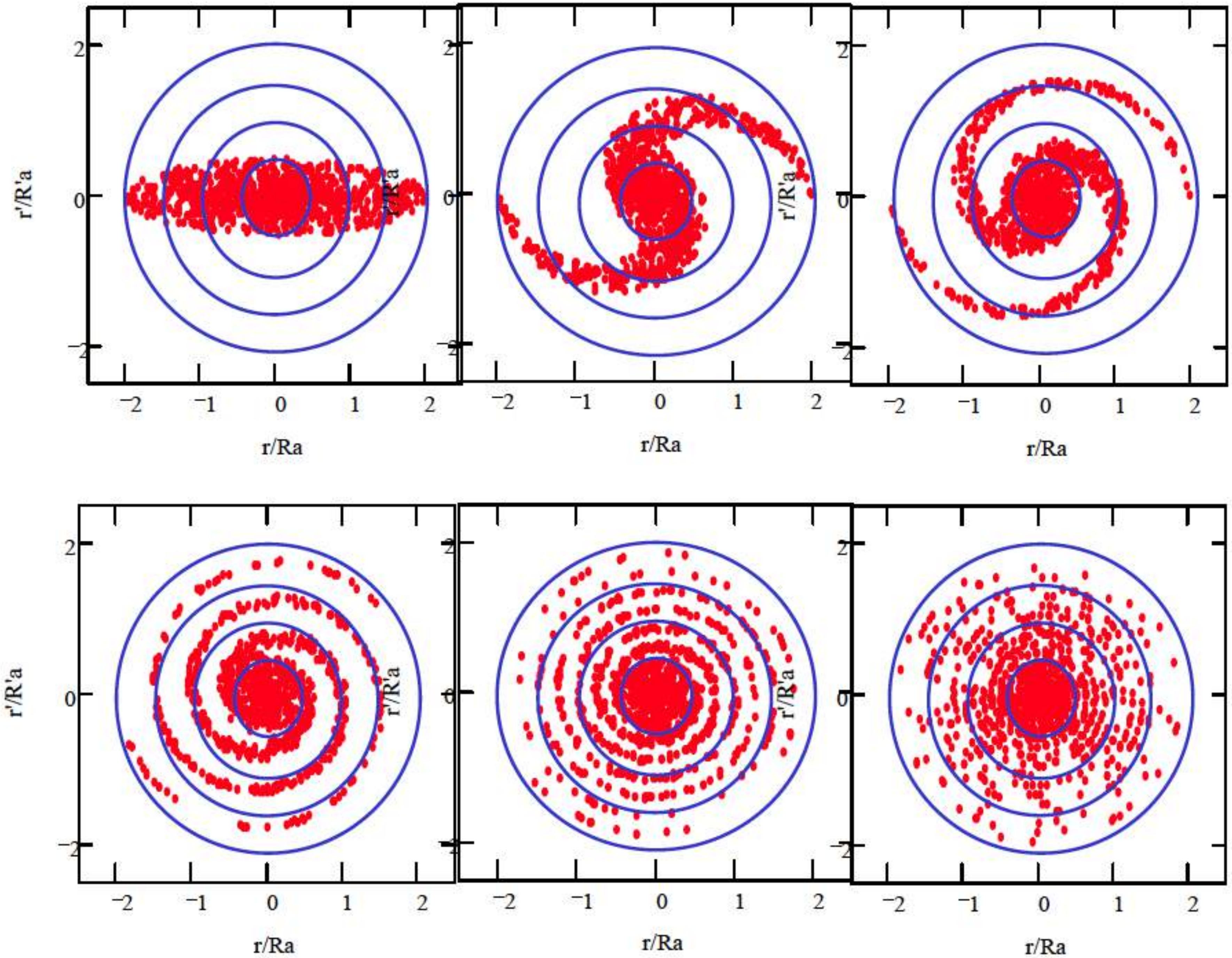
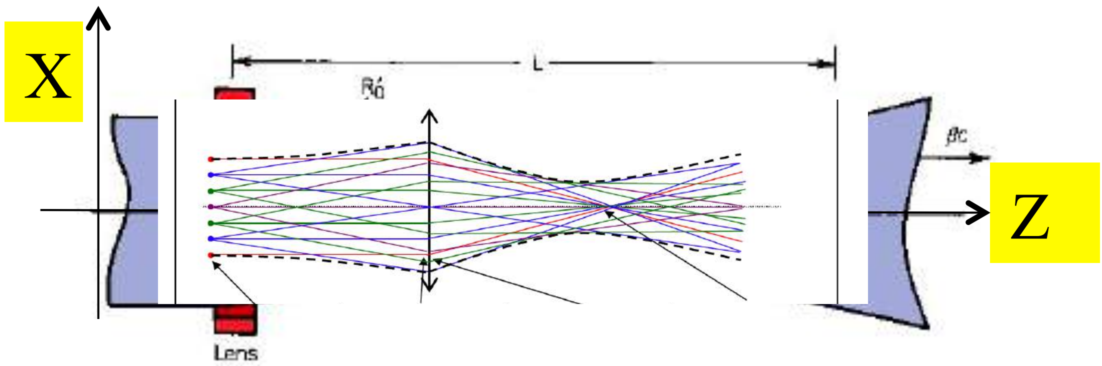
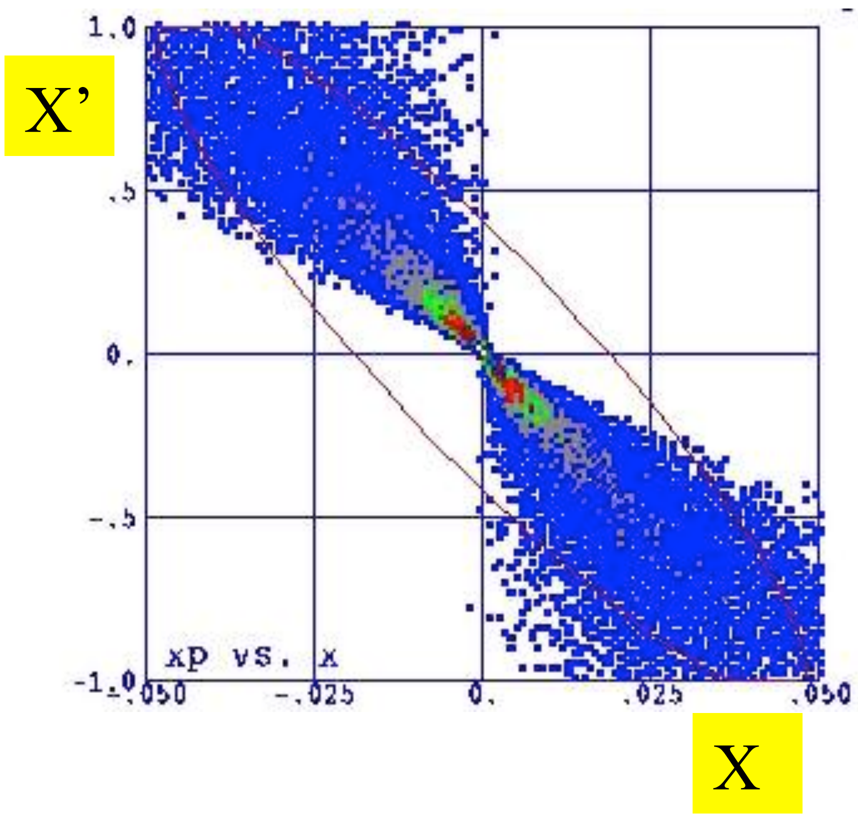


Fig. 17: Filamentation of mismatched beam in non-linear force

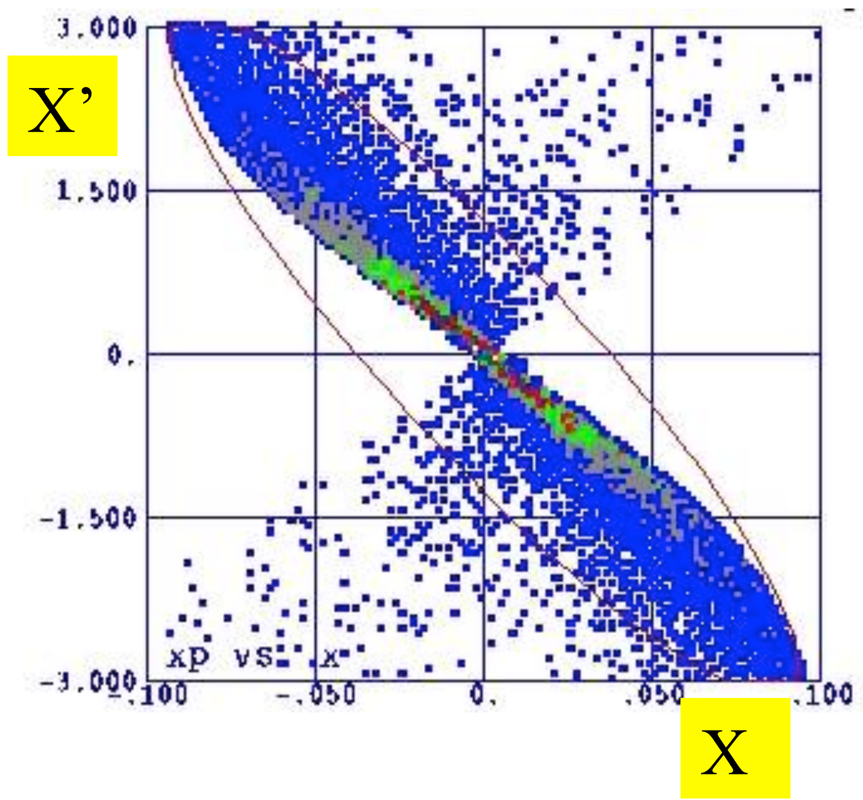
Phase space evolution



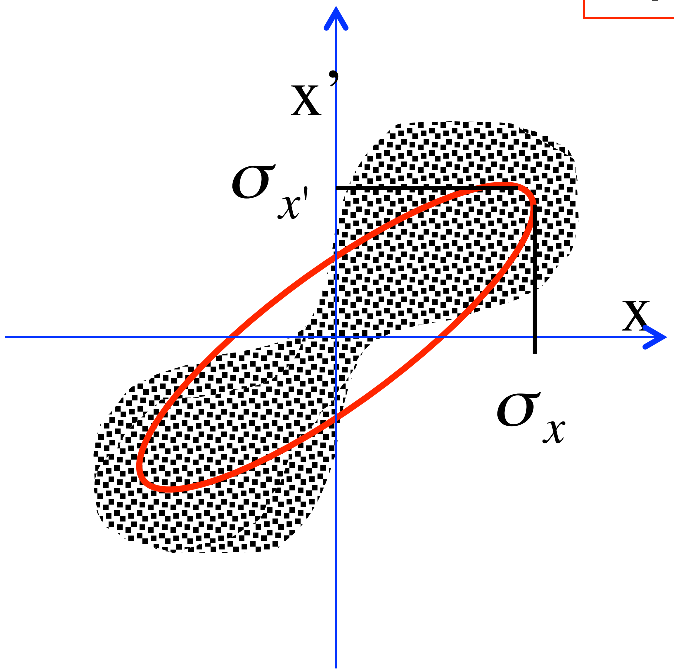
No space charge => cross over



With space charge => no cross over



rms emittance ε_{rms}



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,x')\,dx\,dx' = 1$$

$$f'(x,x') = 0$$

rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,x')\,dx\,dx'$$

Define rms emittance:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}$$

such that:

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

Since: $\beta' = -2\alpha$

it follows:

$$\alpha = -\frac{1}{2\varepsilon_{rms}} \frac{d}{dz} \langle x^2 \rangle = -\frac{\langle x x' \rangle}{\varepsilon_{rms}} = -\frac{\sigma_{xx'}}{\varepsilon_{rms}}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma'_x = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

$$\sigma_{xx'} = \langle xx' \rangle = -\alpha \varepsilon_{rms}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

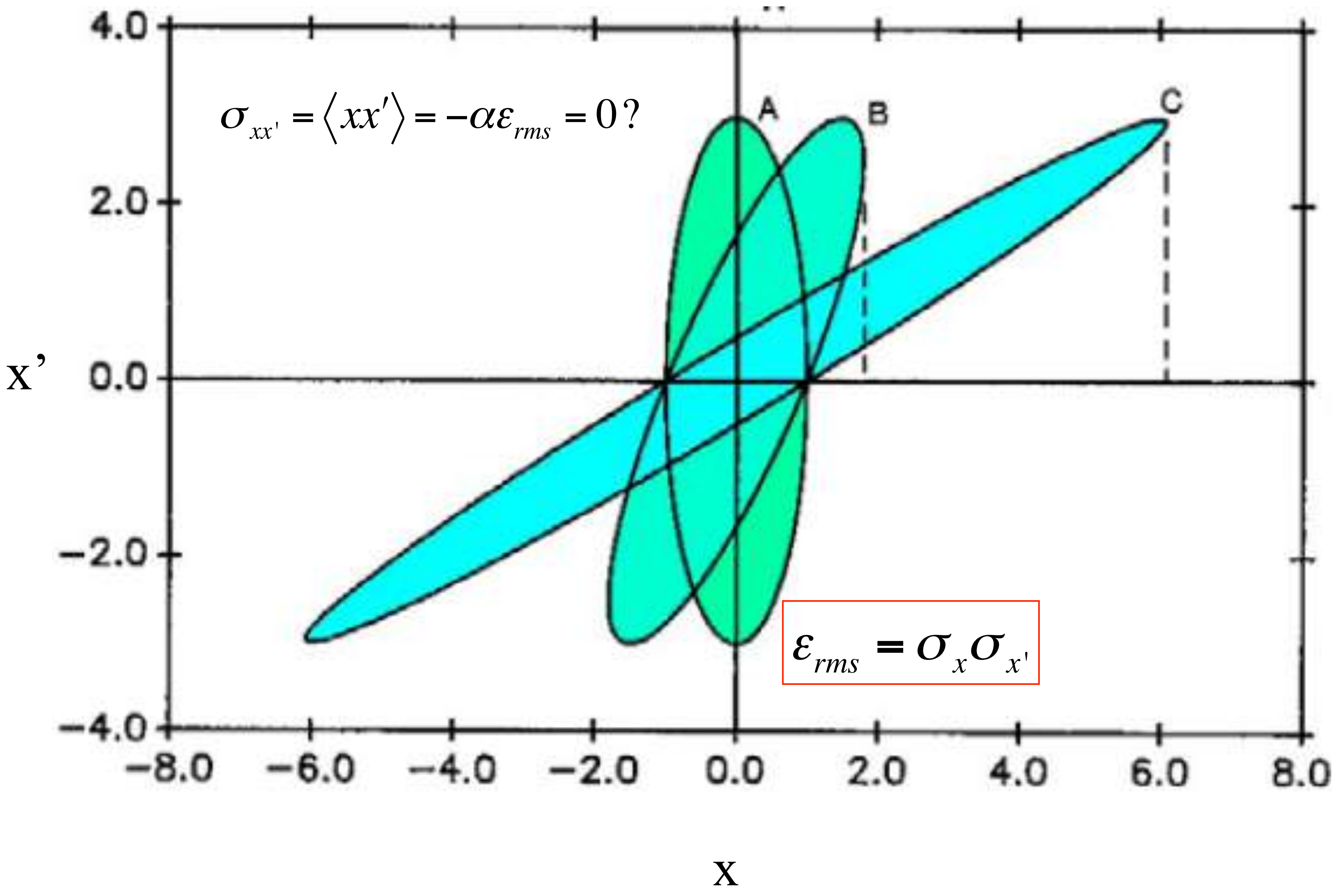
Substituting α, β, γ we get $\frac{\sigma_{x'}^2}{\varepsilon_{rms}} \frac{\sigma_x^2}{\varepsilon_{rms}} - \left(\frac{\sigma_{xx'}}{\varepsilon_{rms}} \right)^2 = 1$

We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \right)}$$

$$x' = \frac{p_x}{p_z}$$

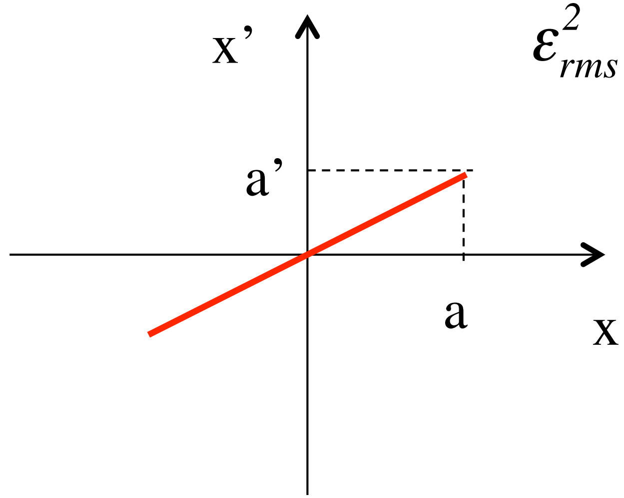
Which distribution has no correlations?



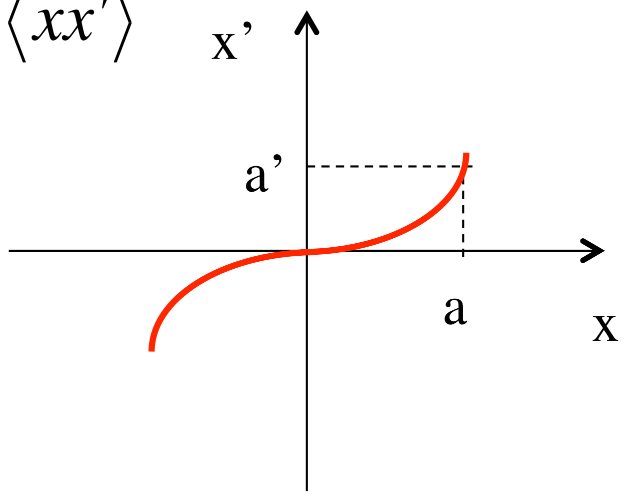
$$\sigma_{xx'} = \langle xx' \rangle = -\alpha\varepsilon_{rms} = 0\,?$$

$$\varepsilon_{rms} = \sigma_x \sigma_{x'}$$

What does rms emittance tell us about phase space distributions under linear or non-linear forces acting on the beam?



$$\varepsilon_{rms}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2$$



Assuming a generic x, x' correlation of the type: $x' = Cx^n$

$$\varepsilon_{rms}^2 = C^2 \left(\langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

When n = 1 ==> $\varepsilon_{rms} = 0$

When n ≠ 1 ==> $\varepsilon_{rms} \neq 0$

Constant under linear transformation only

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right] = 2\langle xx'\rangle\langle x'^2\rangle + 2\langle x^2\rangle\langle x'\rangle\langle x''\rangle - 2\langle xx''\rangle\langle xx'\rangle = 0$$

For linear transformations, $x'' = -k_x^2 x$, and the right-hand side of the equation is

$$2k_x^2\langle x^2\rangle\langle xx'\rangle - 2\langle x^2\rangle\langle xx'\rangle k_x^2 = 0,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right] = 0$$

And without acceleration:

$$x' = \frac{p_x}{p_z}$$

Normalized rms emittance: $\varepsilon_{n,rms}$

Canonical transverse momentum: $p_x = p_z x' = m_o c\beta\gamma x'$ $\quad p_z \approx p$

$$\varepsilon_{n,rms} = \frac{1}{m_o c}\sqrt{\sigma_x^2\sigma_{p_x}^2 - \sigma_{xp_x}^2} = \frac{1}{m_o c}\sqrt{\left(\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2\right)} \approx \langle\beta\gamma\rangle\varepsilon_{rms}$$

Liouville theorem: the density of particles *n*, or the volume V occupied by a given number of particles in phase space (x,p_x,y,p_y,z,p_z) remains invariant under conservative forces.

$$\frac{dn}{dt} = 0$$

It hold also in the projected phase spaces (x,p_x),(y,p_y)(,z,p_z) provided that there are no couplings.

But rms emittance is not Liouvillian!

Limit of single particle emittance

Limits are set by Quantum Mechanics on the knowledge of the two conjugate variables (x,p_x). According to Heisenberg:

$$\sigma_x \sigma_{p_x} \geq \frac{\hbar}{2}$$

This limitation can be expressed by saying that the state of a particle is not exactly represented by a point, but by a small uncertainty volume of the order of $\hbar^3$ in the 6D phase space.

In particular for a single electron in 2D phase space it holds:

$$\varepsilon_{n,rms} = \frac{1}{m_o c}\sqrt{\sigma_x^2 \sigma_{p_x}^2 - \sigma_{xp_x}^2} \Rightarrow \begin{cases} = 0 & \text{classical limit} \\ \geq \dfrac{1}{2}\dfrac{\hbar}{m_o c} = \dfrac{\lambda\!\!\!^{-}_c}{2} = 1.9 \times 10^{-13}\, m & \text{quantum limit} \end{cases}$$

Where $\lambda\!\!\!^{-}_c$ is the reduced Compton wavelength.

OUTLINE

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Envelope Equation without Acceleration

Now take the derivatives:

$$\frac{d\sigma_x}{dz} = \frac{d}{dz}\sqrt{\langle x^2 \rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\langle x^2 \rangle = \frac{1}{2\sigma_x}2\langle xx' \rangle = \frac{\sigma_{xx'}}{\sigma_x}$$

$$\frac{d^2\sigma_x}{dz^2} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\langle x'^2 \rangle + \langle xx' \rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{\sigma_{x'}^2 + \langle xx'' \rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

And simplify:

$$\sigma_x'' = \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x}$$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term.

$$\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

$$\frac{\varepsilon_{rms}^2}{\sigma_x^3} \approx \frac{T}{V} \approx P$$

Beam Thermodynamics

Kinetic theory of gases defines temperatures in each directions and global as:

$$k_B T_x = m\left\langle v_x^2 \right\rangle \qquad T = \frac{1}{3}\left(T_x + T_y + T_z\right) \qquad E_k = \frac{1}{2} m \left\langle v^2 \right\rangle = \frac{3}{2} k_B T$$

Definition of beam temperature in analogy:

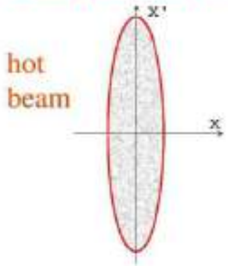
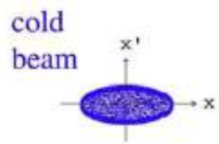
$$k_B T_{beam,x} = \gamma m_o \left\langle v_x^2 \right\rangle \qquad \left\langle v_x^2 \right\rangle = \beta^2 c^2 \left\langle x'^2 \right\rangle = \beta^2 c^2 \sigma_{x'}^2 = \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}$$

We get:

$$k_B T_{beam,x} = \gamma m_o \left\langle v_x^2 \right\rangle = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}$$

$$P_{beam,x} = n k_B T_{beam,x} = n \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = N_T \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_L \sigma_x^2}$$

$$k_B T_{beam,x} = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}$$

Property	Hot beam	Cold beam
ion mass (m_o)	heavy ion	light ion
ion energy ($\beta\gamma$)	high energy	low energy
beam emittance (ε)	large emittance	small emittance
lattice properties ($\gamma_{x,y} \approx 1/\beta_{x,y}$)	strong focus (low β)	high β
phase space portrait		

Electron Cooling: Temperature relaxation by mixing a hot ion beam with co-moving cold (light) electron beam.

Particle Accelerators
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EMITTANCE, ENTROPY AND INFORMATION

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$$S = kN \log(\pi\varepsilon)$$

Envelope Equation with Linear Focusing

$$\sigma_x'' - \frac{\langle x x'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

Assuming that each particle is subject only to a linear focusing force, without acceleration: $x'' + k_x^2 x = 0$

take the average over the entire particle ensemble $\langle x x'' \rangle = -k_x^2 \langle x^2 \rangle$

$$\sigma_x'' + k_x^2 \sigma_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

We obtain the rms envelope equation with a linear focusing force in which, unlike in the single particle equation of motion, the rms emittance enters as defocusing pressure like term.

OUTLINE

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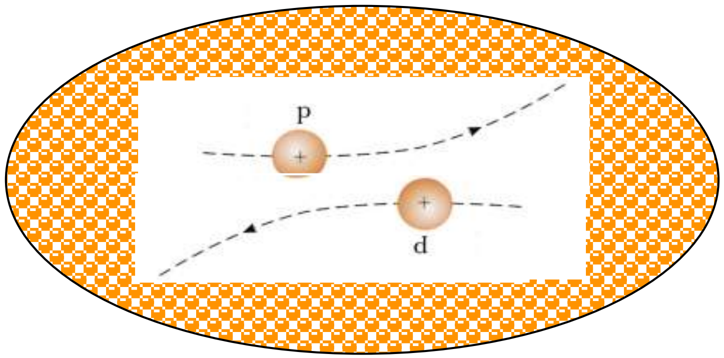
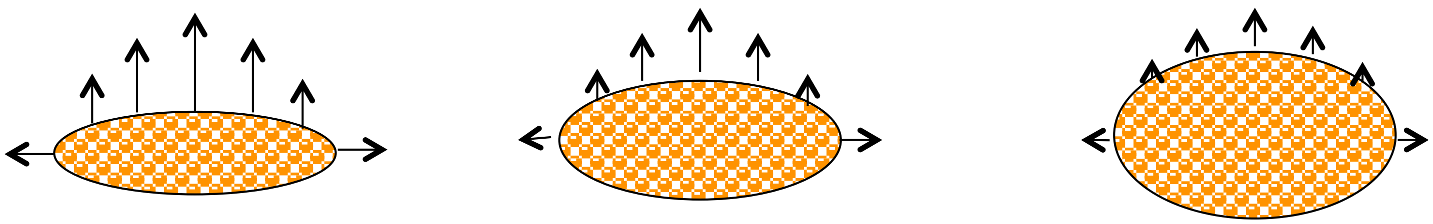
Space Charge: what does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**



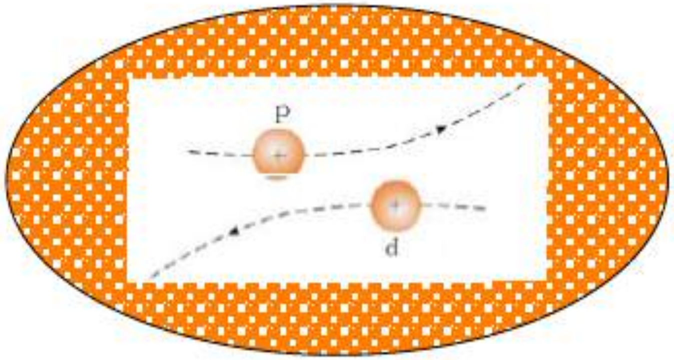
2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects**



The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**

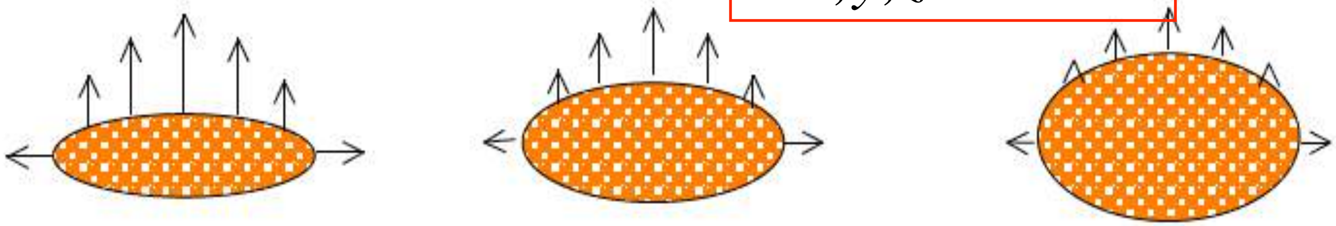
$$\lambda_D = \gamma \sqrt{\frac{\varepsilon_o k_B T}{e^2 n}}$$



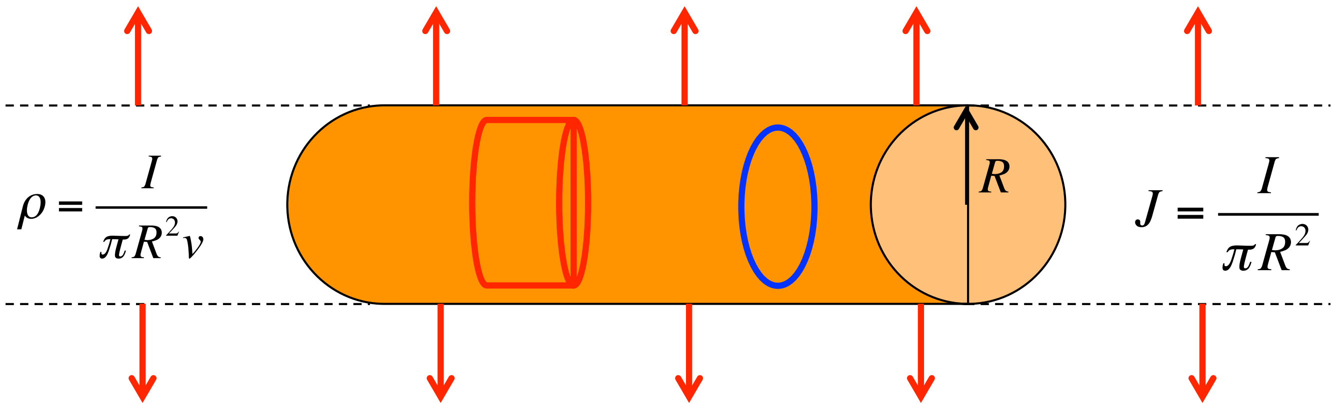
$$\sigma_{x,y,z} << \lambda_D$$

2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects, Single Component Cold Plasma**

$$\sigma_{x,y,z} >> \lambda_D$$



Continuous Uniform Cylindrical Beam Model



Gauss' s law

$$\int \varepsilon_o E \cdot dS = \int \rho dV$$

$$E_r = \frac{I}{2\pi\varepsilon_o R^2 v} r \quad \text{for} \quad r \le R$$

$$E_r = \frac{I}{2\pi\varepsilon_o v} \frac{1}{r} \quad \text{for} \quad r > R$$

$$B_\vartheta = \frac{\beta}{c} E_r$$

Ampere' s law

$$\int B \cdot dl = \mu_o \int J \cdot dS$$

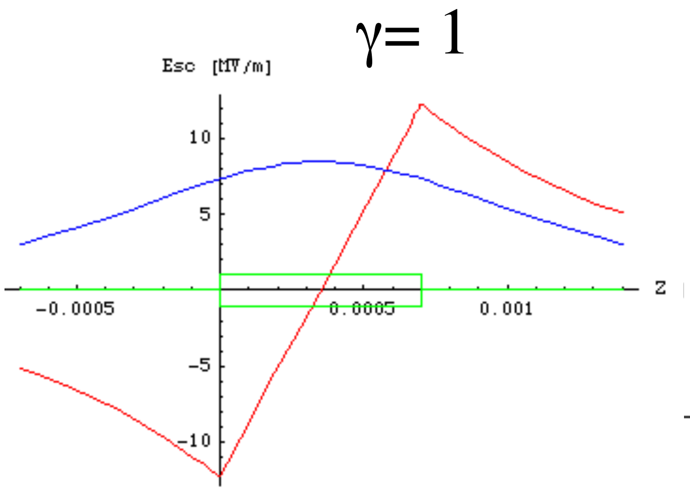
$$B_\vartheta = \mu_o \frac{Ir}{2\pi R^2} \quad \text{for} \quad r \le R$$

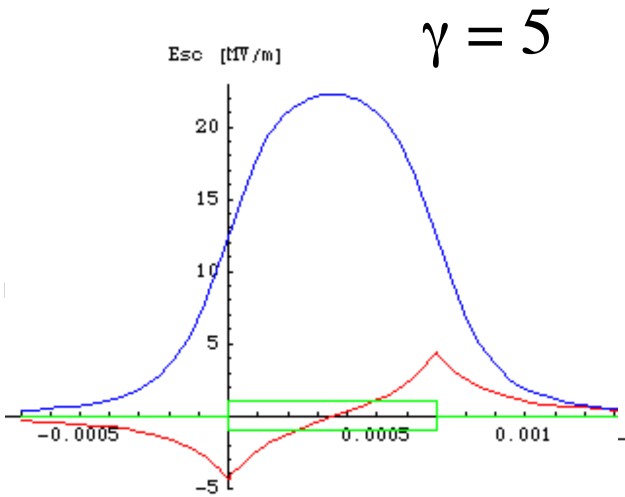
$$B_\vartheta = \mu_o \frac{I}{2\pi r} \quad \text{for} \quad r > R$$

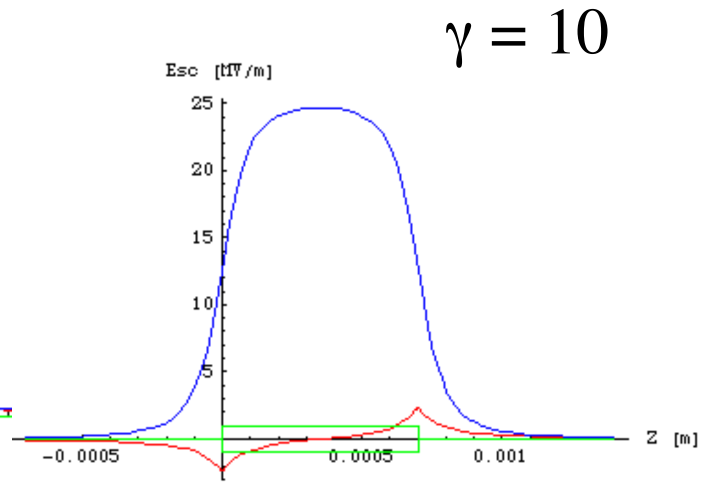
Bunched Uniform Cylindrical Beam Model

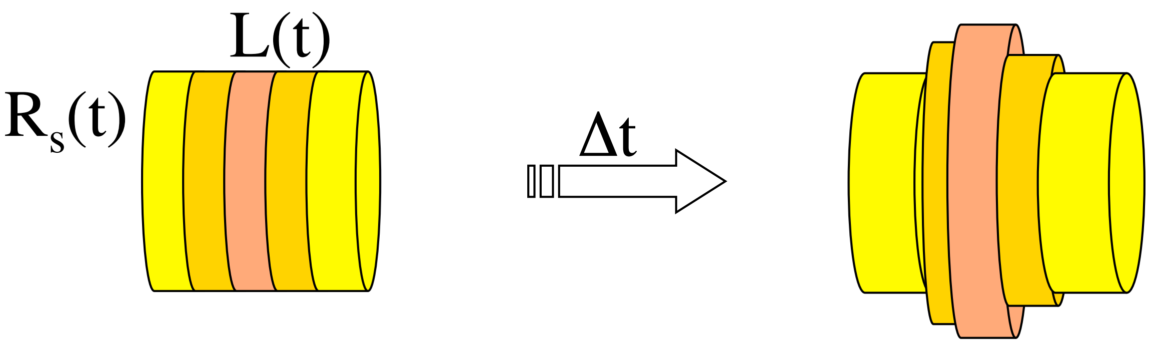
$$E_z(0,s,\gamma) = \frac{I}{2\pi\gamma\varepsilon_0 R^2 \beta c} h(s,\gamma)$$

$$E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$









$$E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$

Lorentz Force

$$F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1-\beta^2\right)E_r = \frac{eE_r}{\gamma^2}$$

$$B_\vartheta = \frac{\beta}{c} E_r$$

is a **linear** function of the transverse coordinate

$$\frac{dp_r}{dt} = F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g(s,\gamma)$$

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore space charge defocusing is primarily a non-relativistic effect. Using $R=2\sigma_x$ for a uniform distribution:

$$F_x = \frac{eIx}{8\pi\gamma^2\varepsilon_0\sigma_x^2\beta c} g(s,\gamma)$$

Envelope Equation with Space Charge

Single particle transverse motion:

$$\frac{dp_x}{dt} = F_x \qquad p_x = p\,x' = \beta\gamma m_o c x' \qquad p = const.$$

$$\frac{d}{dt}\left(px'\right) = \beta c \frac{d}{dz}\left(p\,x'\right) = F_x$$

$$x'' = \frac{F_x}{\beta c p}$$

$$F_x = \frac{eIx}{8\pi\gamma^2\varepsilon_0\sigma_x^2\beta c} g\left(s,\gamma\right)$$

$$x'' = \frac{k_{sc}\left(s,\gamma\right)}{\sigma_x^2} x$$

$$k_{sc} = \frac{2I}{I_A} g\left(s,\gamma\right)$$

$$I_A = \frac{4\pi\varepsilon_o m_o c^3}{e}$$

Now we can calculate the term $\langle xx'' \rangle$ that enters in the envelope equation

$$\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\langle xx'' \rangle}{\sigma_x}$$

$$\langle xx'' \rangle = \frac{k_{sc}}{\sigma_x^2} \langle x^2 \rangle = k_{sc}$$

Including all the other terms the envelope equation reads:

Space Charge De-focusing Force

$$\sigma_x'' + k^2 \sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

Emittance Pressure

External Focusing Forces

Laminarity Parameter: $\rho = \dfrac{(\beta\gamma)^2 k_{sc} \sigma_x^2}{\varepsilon_n^2}$

The beam undergoes two regimes along the accelerator

$$\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

ρ>>1 — Laminar Beam

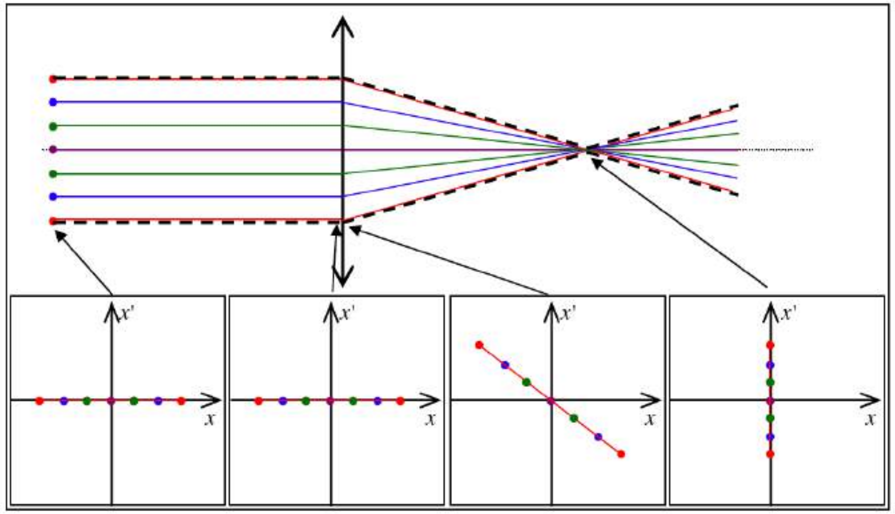


Fig. 10: Particle trajectories in laminar beam

$$\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

ρ<<1 — Thermal Beam

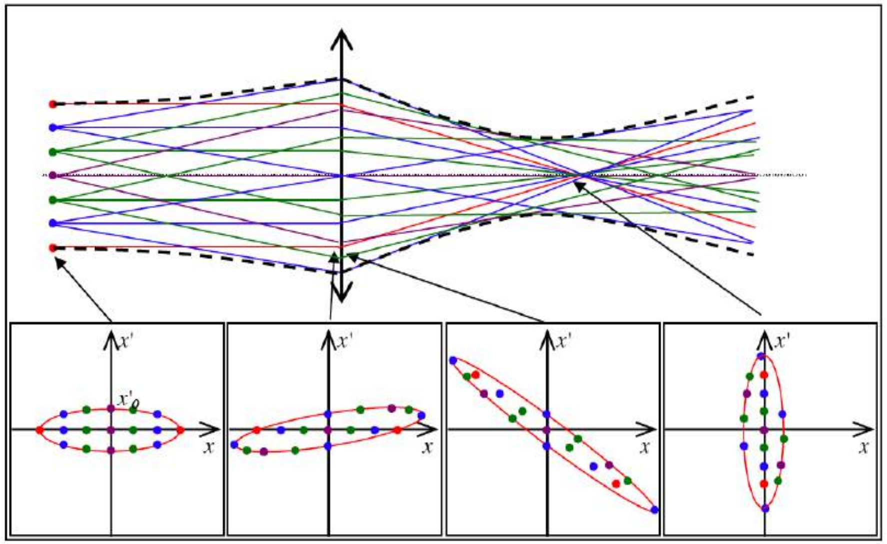
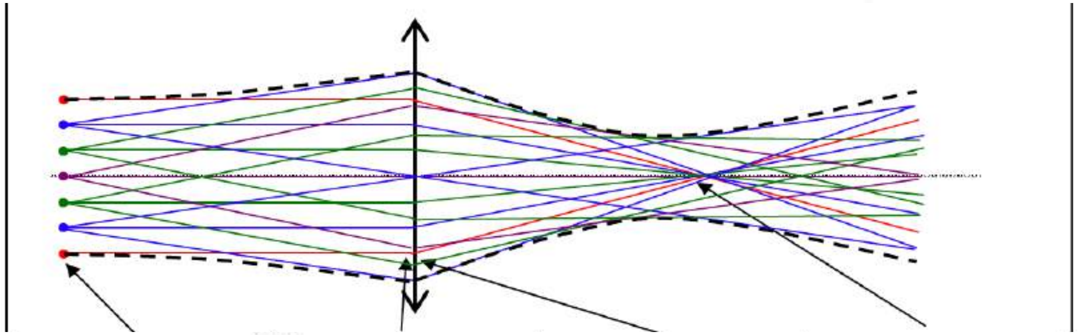
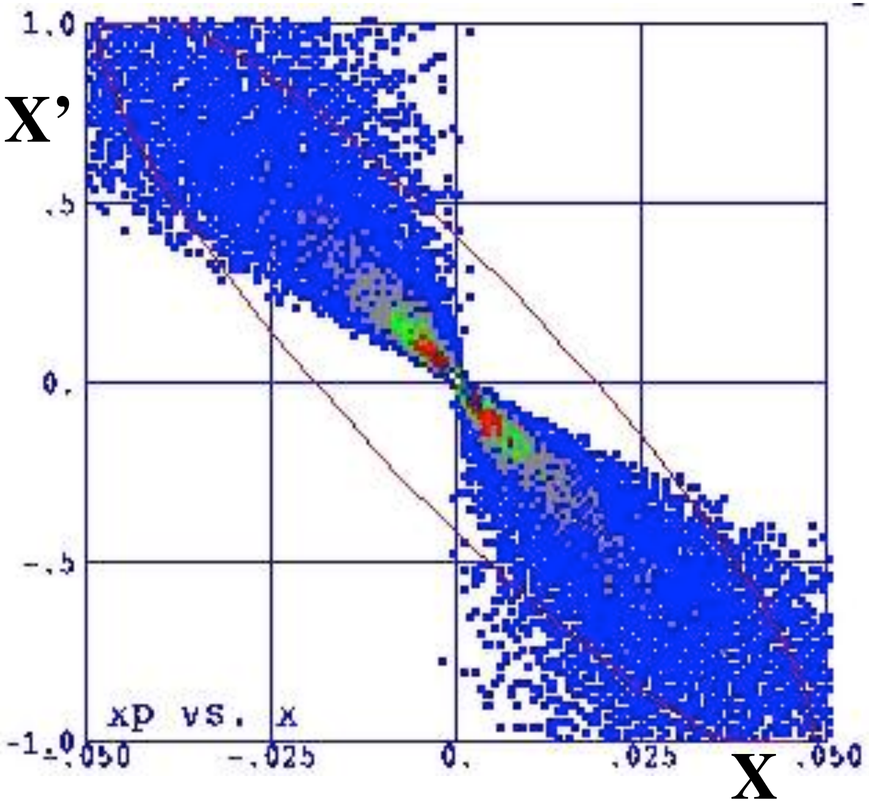


Fig. 11: Particle trajectories in non-zero emittance beam

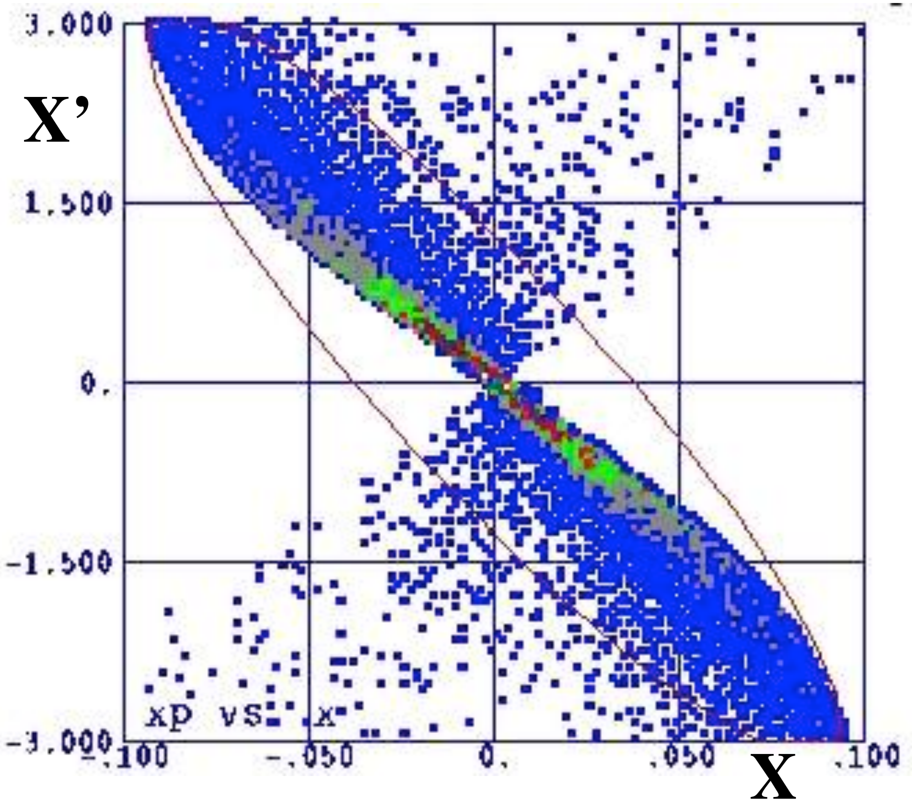
Trace space evolution



No space charge => cross over



With space charge => no cross over

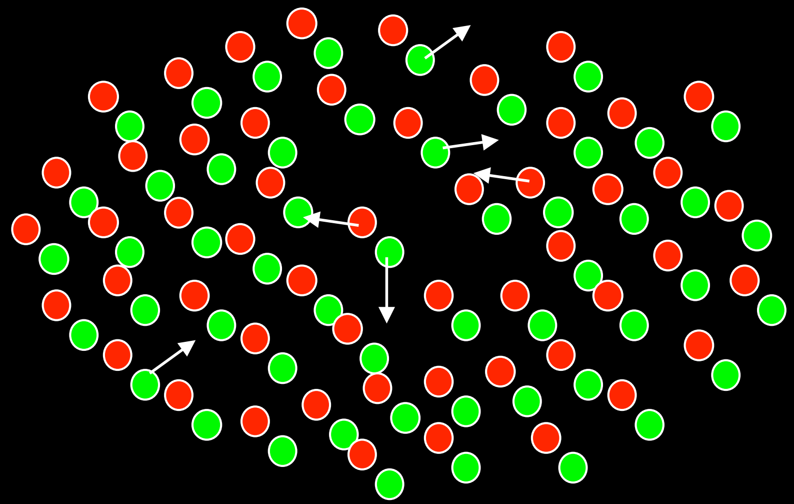


OUTLINE

- The rms emittance concept
- rms envelope equation
- Space charge forces
- Space charge induced emittance oscillations
- Matching conditions and emittance compensation

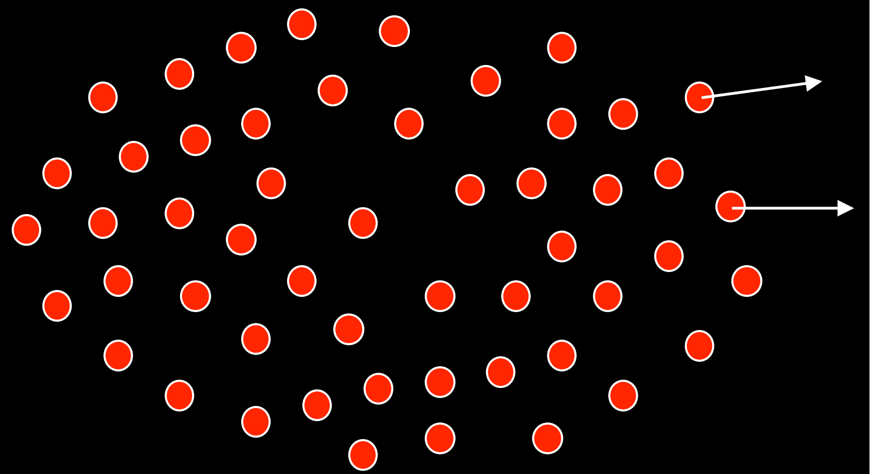
Neutral Plasma

- Oscillations
- Instabilities
- EM Wave propagation



Single Component Cold Relativistic Plasma

Magnetic focusing

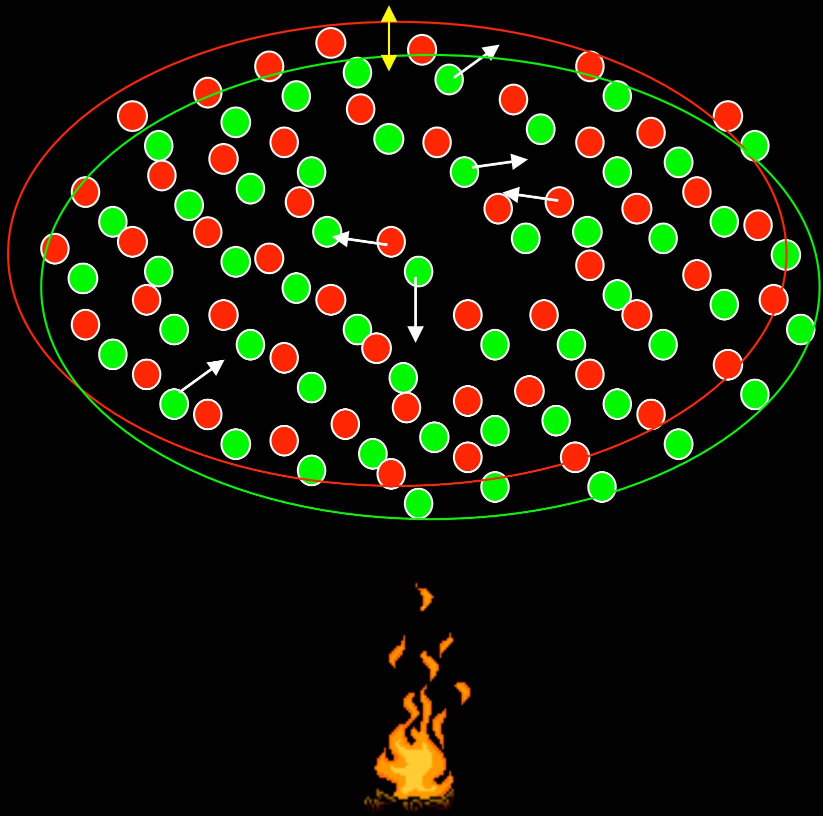


Magnetic focusing

Surface charge density

$$\sigma = e\, n\, \delta x$$

δx



Surface electric field

$$E_x = -\sigma/\epsilon_0 = -e\, n\, \delta x/\epsilon_0$$

Restoring force

$$m \frac{d^2 \delta x}{dt^2} = e\, E_x = -m\, {\omega_p}^2\, \delta x$$

Plasma frequency

$${\omega_p}^2 = \frac{n\, e^2}{\epsilon_0\, m}$$

Plasma oscillations

$$\delta x = (\delta x)_0 \cos(\omega_p t)$$

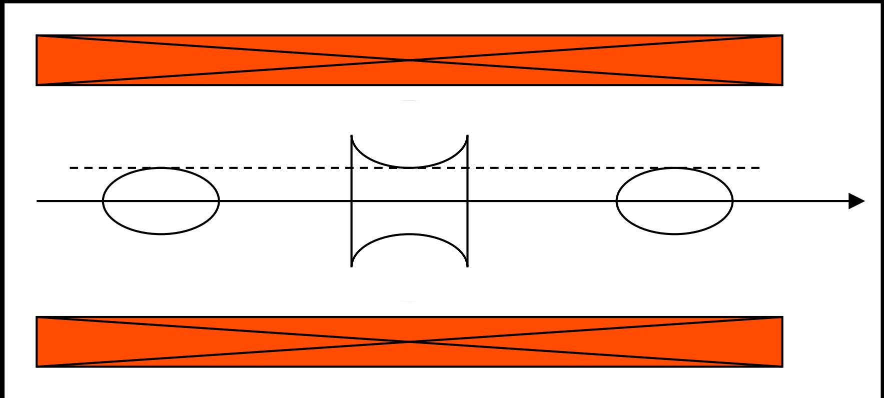
Single Component Relativistic Plasma

$$\sigma'' + k_s^2 \sigma = \frac{k_{sc}(s,\gamma)}{\sigma}$$

$$k_s = \frac{qB}{2mc\beta\gamma}$$

Equilibrium solution:

$$\sigma_{eq}(s,\gamma) = \frac{\sqrt{k_{sc}(s,\gamma)}}{k_s}$$



Small perturbation:

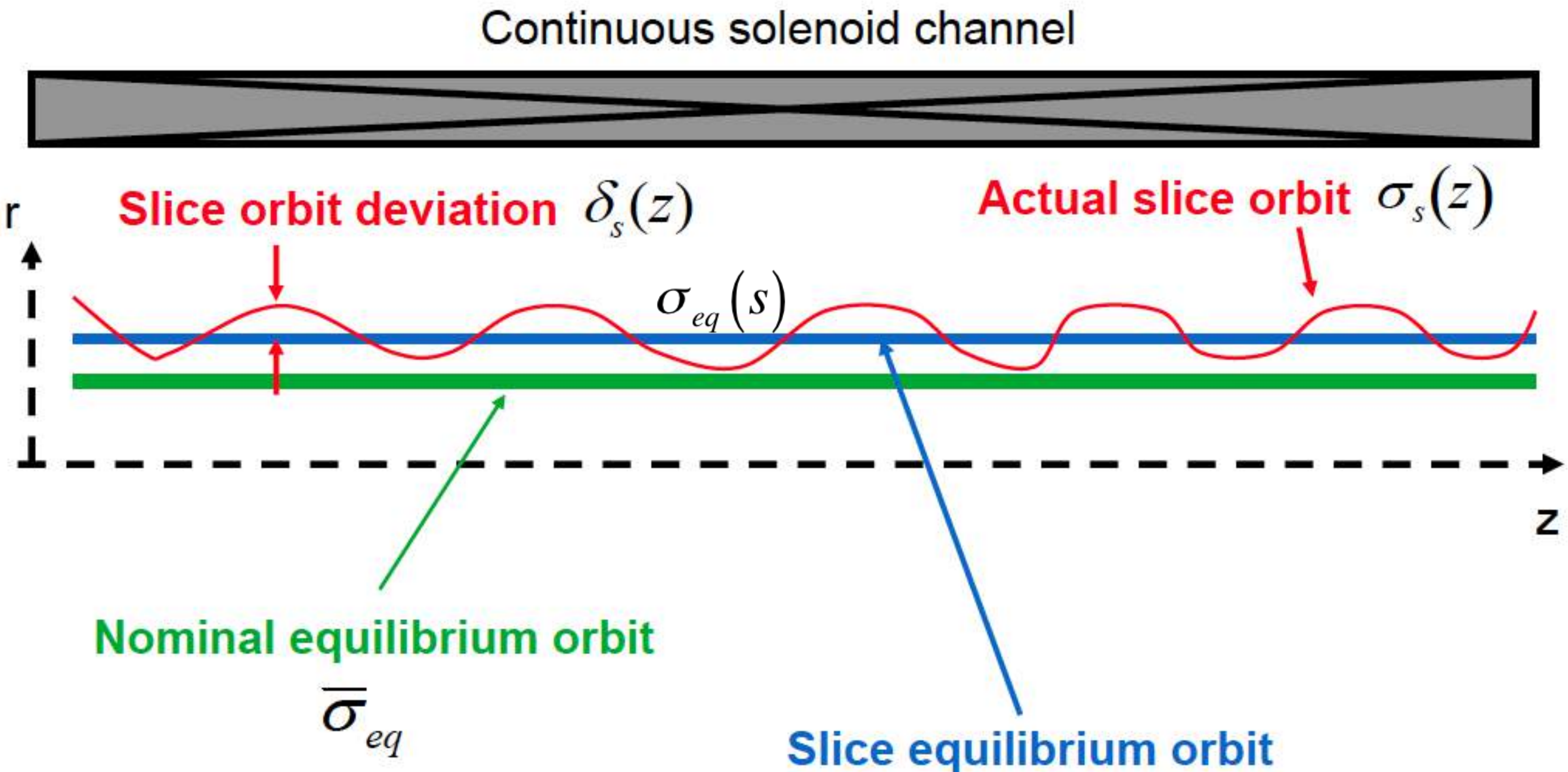
$$\sigma(\zeta) = \sigma_{eq}(s) + \delta\sigma(s)$$

$$\delta\sigma''(s) + 2k_s^2\delta\sigma(s) = 0$$

$$\delta\sigma(s) = \delta\sigma_o(s)\,cos\left(\sqrt{2}k_s z\right)$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma_o(s)\,cos\left(\sqrt{2}k_s z\right)$$

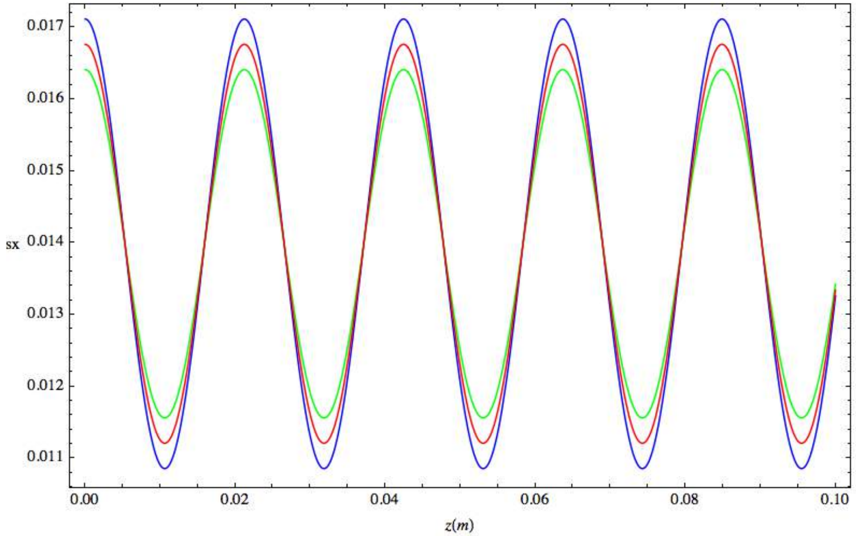


Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

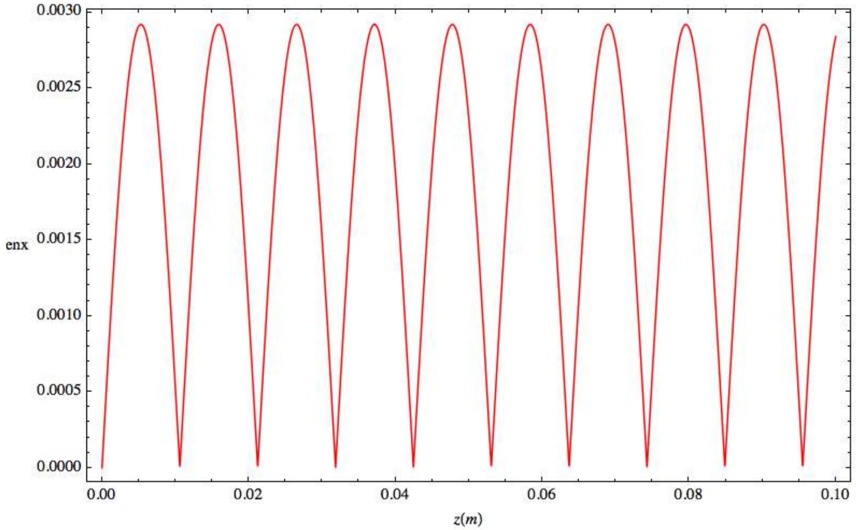
$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma_o(s)\cos\left(\sqrt{2}k_s z\right)$$

Envelope oscillations drive Emittance oscillations

σ(z)



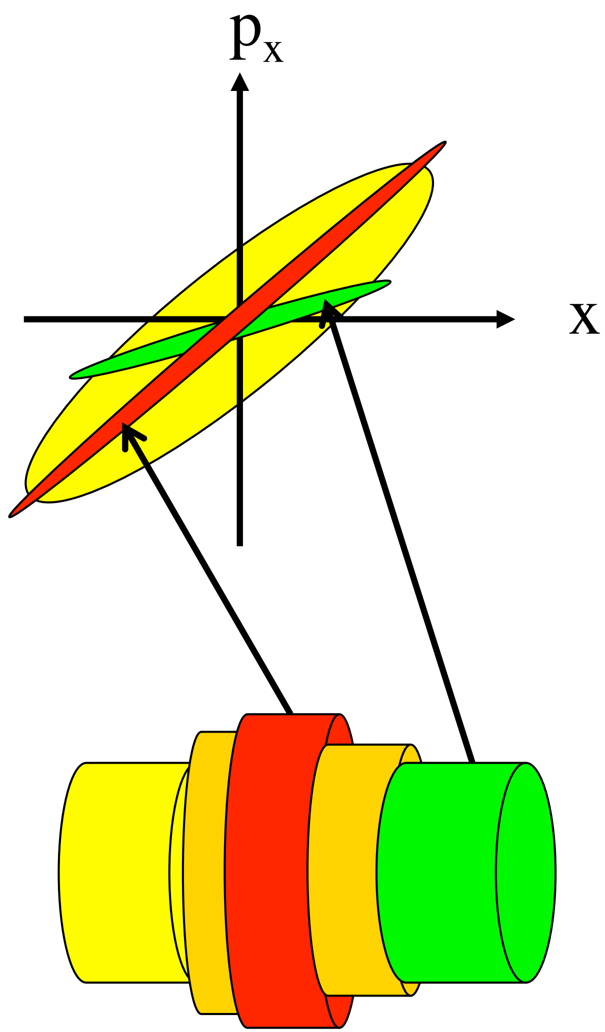
ε(z)



$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}{}^2} = \sqrt{\left(\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2 \right)} \approx \left| sin\left(\sqrt{2} k_s z \right) \right|$$

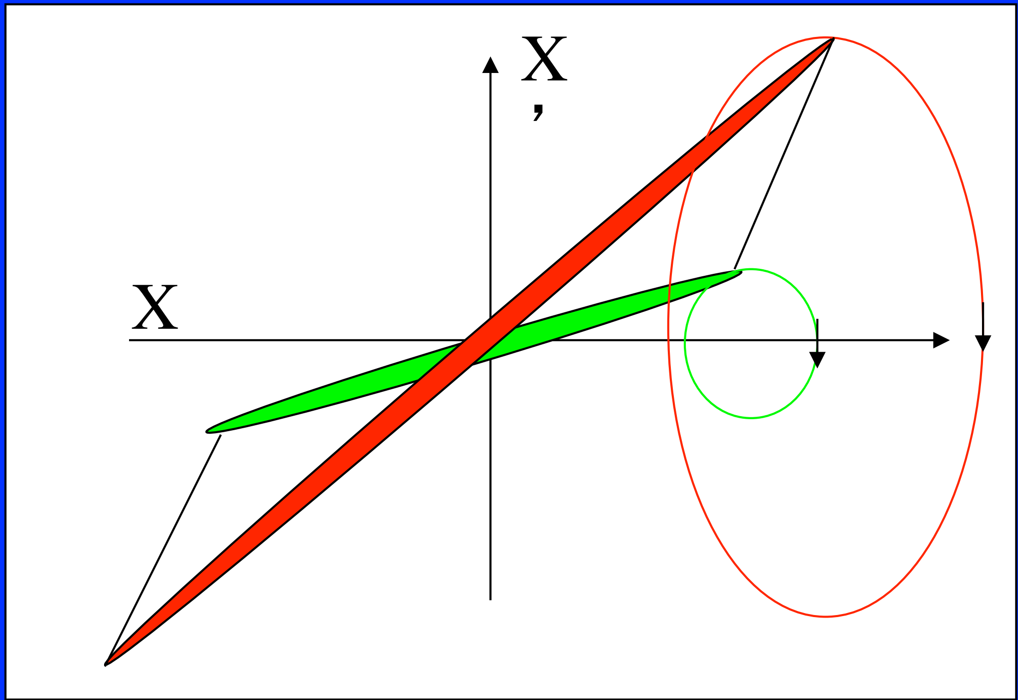
Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam

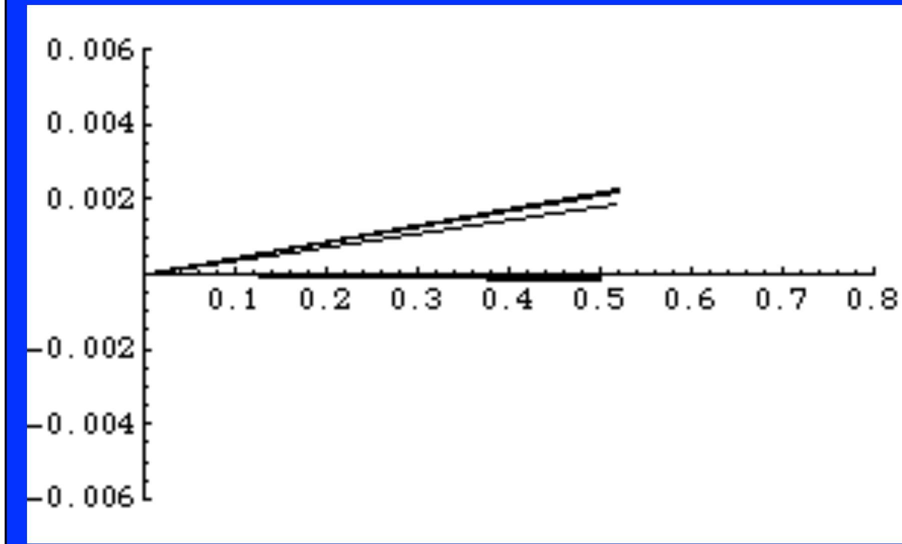
Projected Phase Space



Slice Phase Spaces

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes





OUTLINE

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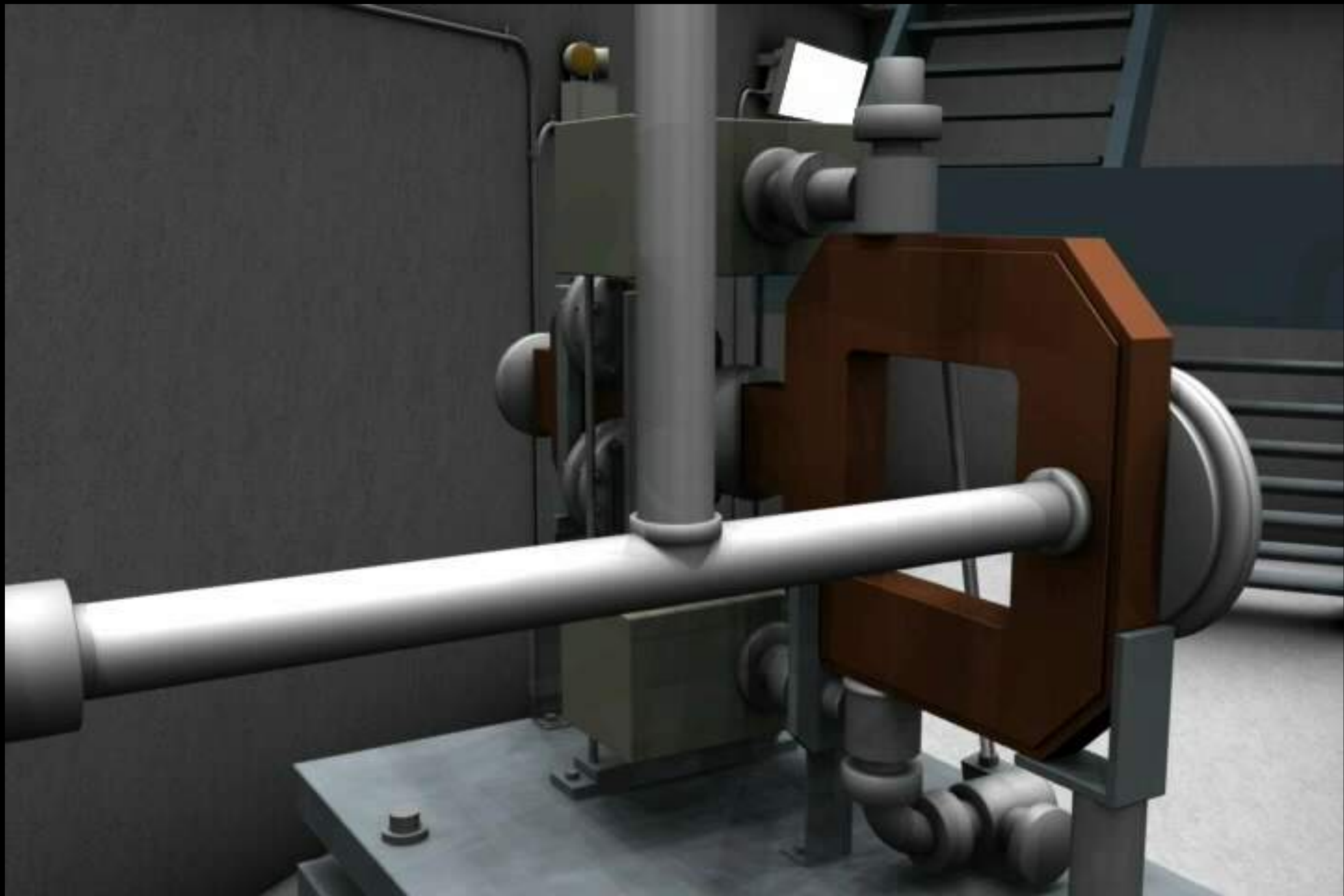
High Brightness Photo-Injector



Envelope Equation with Acceleration

$$\frac{dp_x}{dt} = \frac{d}{dt}\left(px'\right) = \beta c \frac{d}{dz}\left(px'\right) = 0$$

$$p = \beta\gamma m_o c$$

$$x'' + \frac{p'}{p}x' = 0$$

$$x'' = -\frac{\left(\beta\gamma\right)'}{\beta\gamma}x'$$

$$\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\left\langle xx''\right\rangle}{\sigma_x}$$

$$\left\langle xx''\right\rangle = -\frac{\left(\beta\gamma\right)'}{\beta\gamma}\left\langle xx'\right\rangle = -\frac{\left(\beta\gamma\right)'}{\beta\gamma}\sigma_{xx'} = -\frac{\left(\beta\gamma\right)'}{\beta\gamma}\sigma_x\sigma_x'$$

Space Charge De-focusing Force

$$\sigma_x'' + \frac{\left(\beta\gamma\right)'}{\beta\gamma}\sigma_x' + k^2\sigma_x = \frac{\varepsilon_n^2}{\left(\beta\gamma\right)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

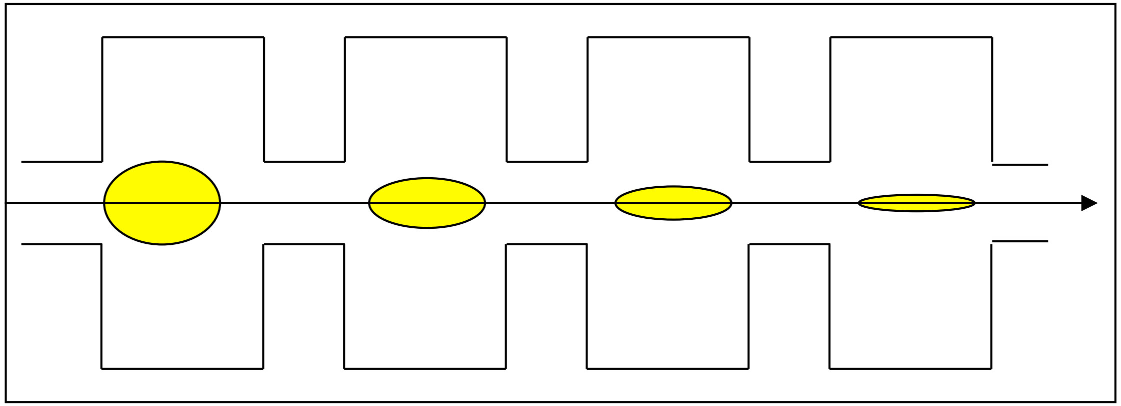
Adiabatic Damping

Emittance Pressure

Other External Focusing Forces

$$\varepsilon_n = \beta\gamma\varepsilon_{rms}$$

Beam subject to strong acceleration



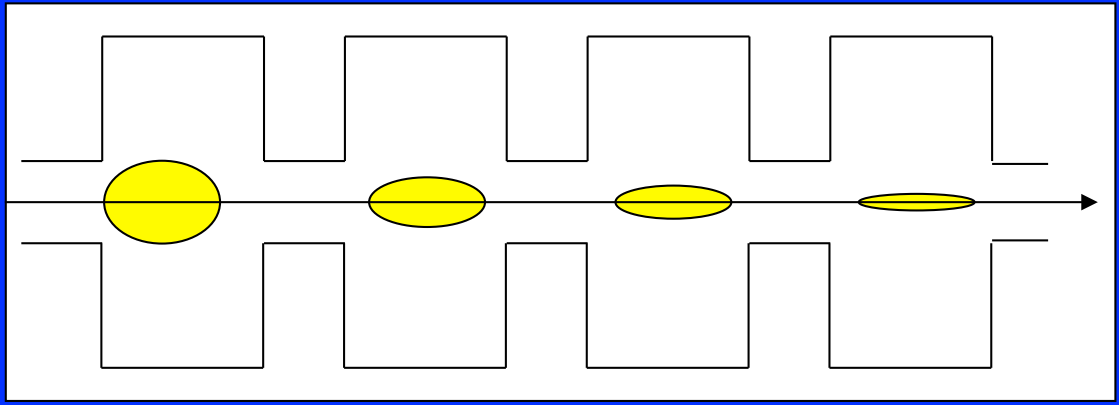
$$\sigma_x'' + \frac{\gamma'}{\gamma}\sigma_x' + \frac{k_{RF}^2}{\gamma^2}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2\sigma_x^3} + \frac{k_{sc}^o}{\gamma^3\sigma_x}$$

We must include also the RF focusing force: $k_{RF}^2 = \frac{\gamma'^2}{2}$

$$k_{sc}^o = \frac{2I}{I_A} g(s,\gamma)$$

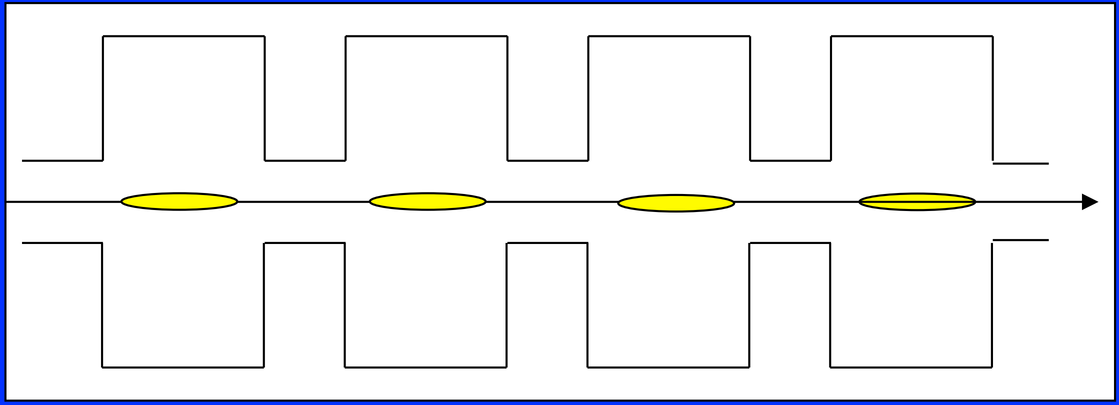
Space charge dominated beam (Laminar)

$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A\gamma}}$$

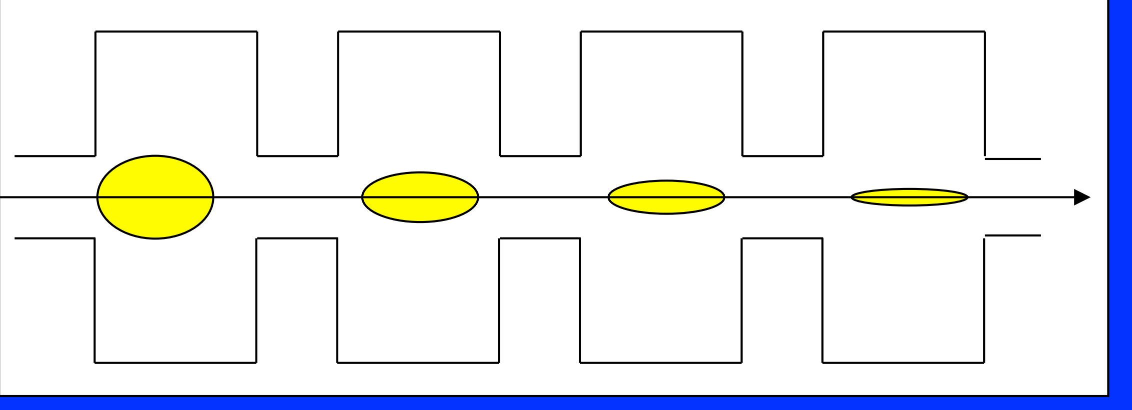


Emittance dominated beam (Thermal)

$$\sigma_\varepsilon = \sqrt{\frac{2\varepsilon_n}{\gamma'}}$$



$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A\gamma}}$$



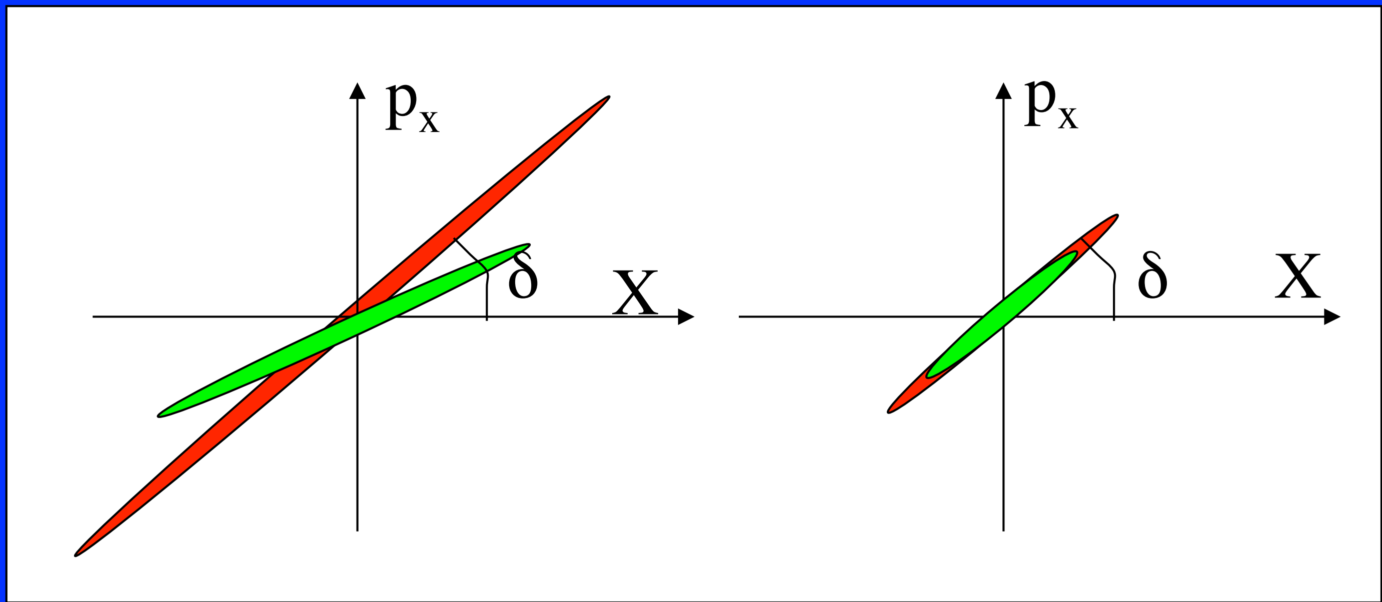
This solution represents a **beam equilibrium mode** that turns out to be the transport mode for achieving minimum emittance at the end of the **emittance correction process**

An important property of the laminar beam

$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A\gamma}}$$

$$\sigma'_q = -\sqrt{\frac{2I}{I_A\gamma^3}}$$

Constant phase space angle: $\delta = \frac{\gamma\sigma'_q}{\sigma_q} = -\frac{\gamma'}{2}$

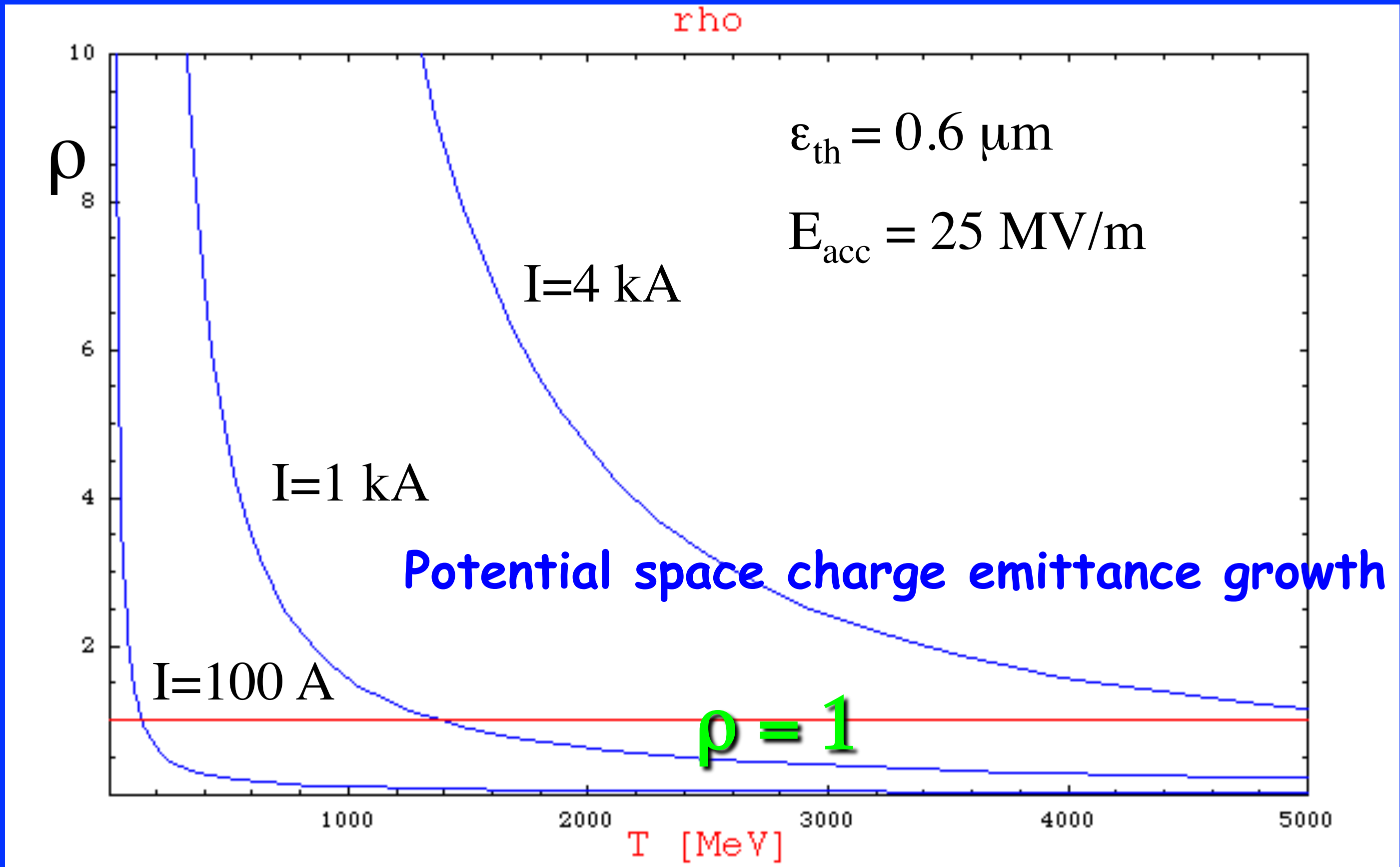


Laminarity parameter

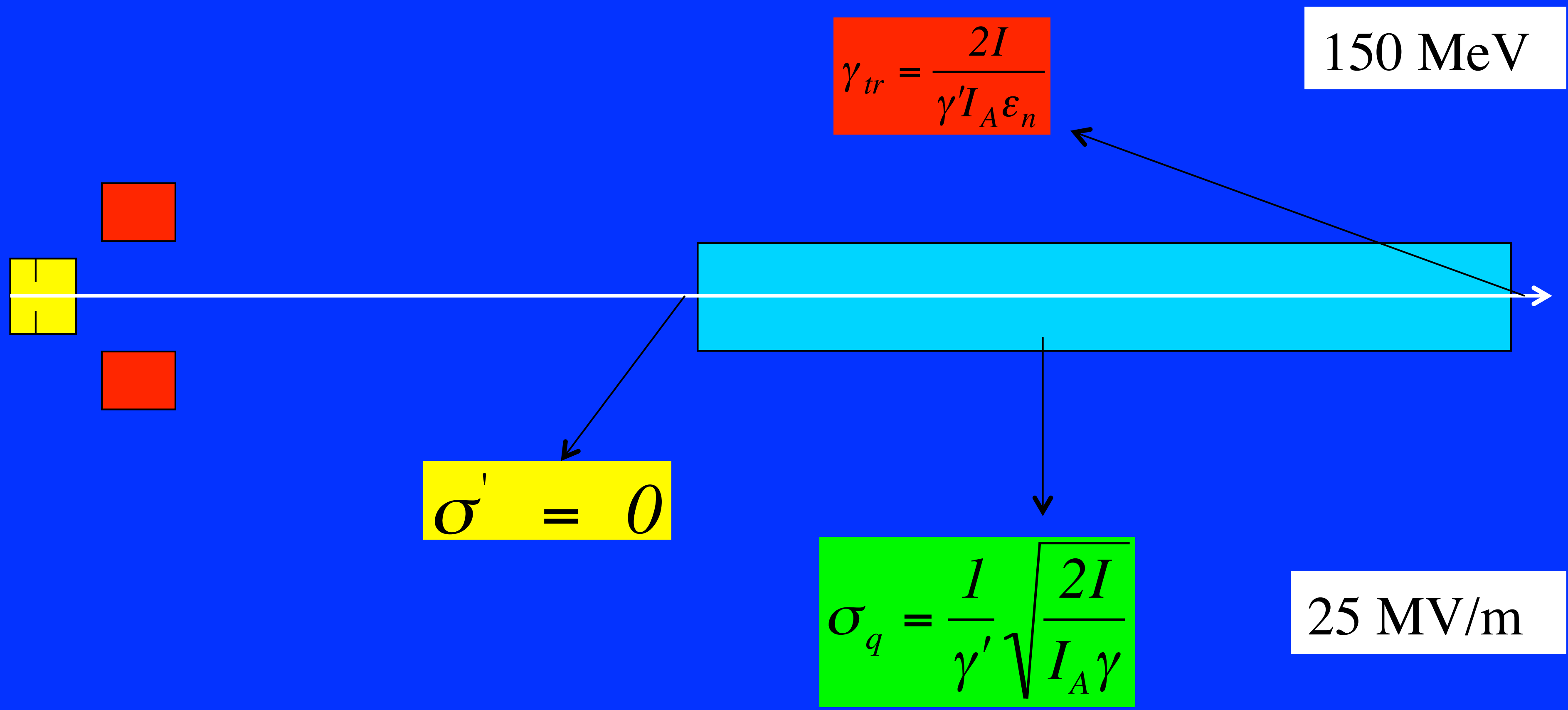
$$\rho = \frac{2I\sigma^2}{\gamma I_A \varepsilon_n^2} \equiv \frac{2I\sigma_q^2}{\gamma I_A \varepsilon_n^2} = \frac{4I^2}{\gamma'^2 I_A^2 \varepsilon_n^2 \gamma^2}$$

Transition Energy (ρ=1)

$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$



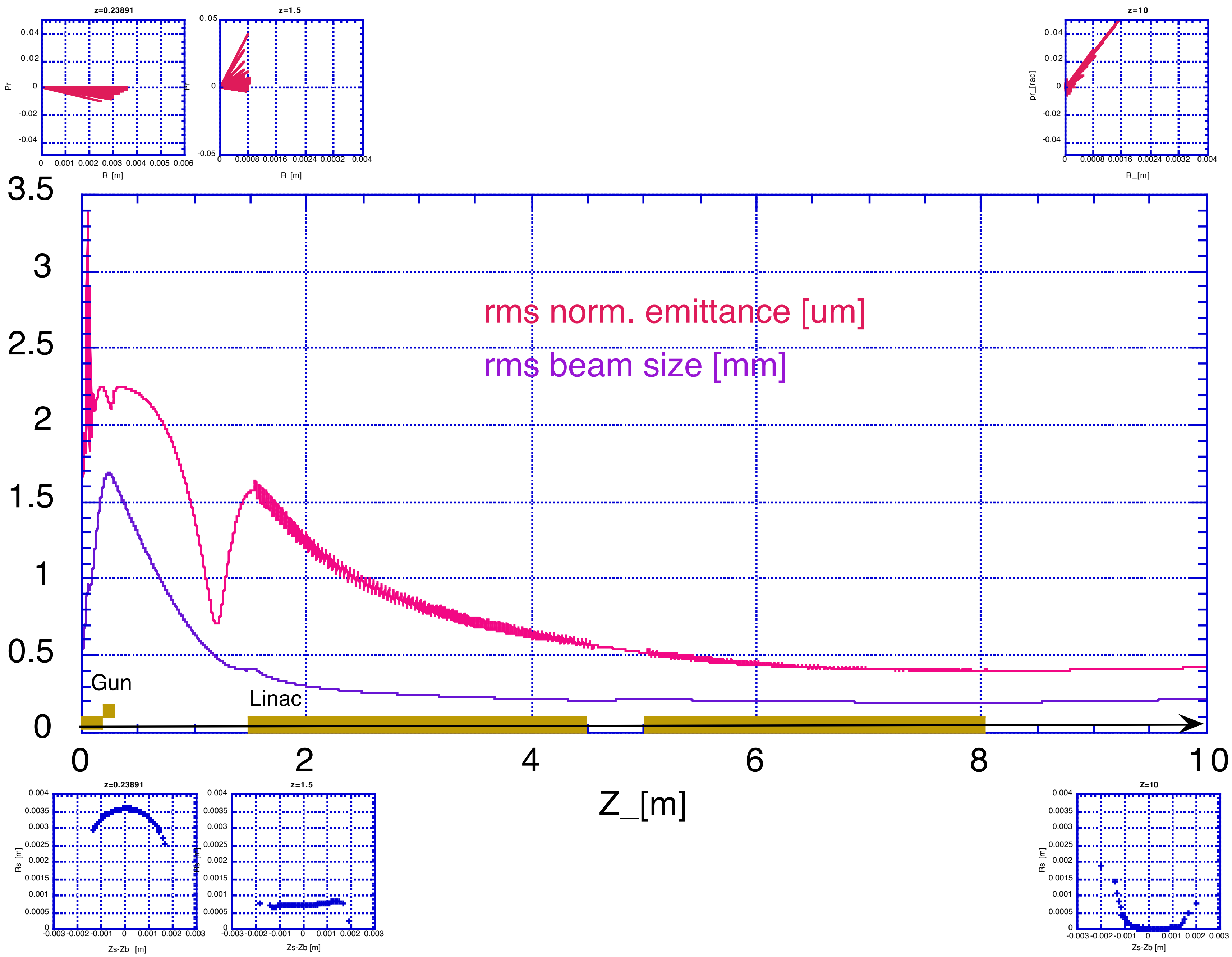
Matching Conditions with a TW Linac



$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$

$$\sigma' = 0$$

$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A \gamma}}$$



z=0.23891
Pr
R [m]
z=1.5
R [m]
z=10
pr_[rad]
R_[m]
rms norm. emittance [um]
rms beam size [mm]
Gun
Linac
Z_[m]
z=0.23891
Rs [m]
Zs-Zb [m]
z=1.5
Zs-Zb [m]
Z=10
Rs [m]
Zs-Zb [m]

Emittance Compensation for a SC dominated beam: Controlled Damping of Plasma Oscillations

- ε_n oscillations are driven by Space Charge

- propagation close to the laminar solution allows control of ε_n oscillation "phase"

- ε_n sensitive to SC up to the transition energy

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