Classical Electrodynamics (2 hrs) and Special Relativity (1 hr)

CERN Accelerator School, 8-21 September 2019, Vysoke-Tatry, Slovakia

Recommended Reading Material (in this order)

[1] **R.P. Feynman,** *Feynman lectures on Physics*, **Vol2.**

[2] **Proceedings of CAS:** *RF for accelerators*, **Ebeltoft, Denmark, 8-17 June 2010, Edited by R. Bailey, CERN-2011-007.**

[3] **J.D. Jackson,** *Classical Electrodynamics* **(Wiley, 1998 ..)**

[4] **L. Landau, E. Lifschitz,** *The Classical Theory of Fields*, **Vol2. (Butterworth-Heinemann, 1975)**

[5] **J. Slater, N. Frank,** *Electromagnetism*, **(McGraw-Hill, 1947, and Dover Books, 1970)**

OUTLINE - following this strategy

This does not replace a full course (i.e. $\approx$ 60 hours, some additional material in backup slides, details in bibliography)

Also, it cannot be treated systematically without special relativity.

The main topics discussed:

- **Basic electromagnetic phenomena, to arrive at:**

  Maxwell's equations

- **Lorentz force, charged particles in electromagnetic fields**
- **Electromagnetic waves in vacuum**
- **Electromagnetic waves in conducting media, waves in RF cavities and wave guides**

→ **Special Relativity should come in here to sort out some issues**

Variables, notations and units used in this lecture

Formulae use SI units throughout.

$\vec{E}(\vec{r},t)$	=	**electric field [V/m]**
$\vec{H}(\vec{r},t)$	=	**magnetic field [A/m]**
$\vec{D}(\vec{r},t)$	=	**electric displacement [C/m^2]**
$\vec{B}(\vec{r},t)$	=	**magnetic flux density [T]**
q	=	**electric charge [C]**
$\rho(\vec{r},t)$	=	**electric charge density [C/m^3]**
$\vec{I}, \ \vec{j}(\vec{r},t)$	=	**current [A], current density [A/m^2]**
μ_0	=	**permeability of vacuum, 4** $\pi \cdot 10^{-7}$ **[H/m or N/A^2]**
ϵ_0	=	**permittivity of vacuum, 8.854** $\cdot 10^{-12}$ **[F/m]**
c	=	**speed of light in free space, 299792458.0 [m/s]**
h	=	**Planck constant, 6.62607** $\cdot 10^{-34}$ **[J s]**

FAQ: do we really need Maxwell's equations (instead of explaining the phenomena) ?

- They are fairly abstract and use some advanced mathematics, but they do not really say what is happening to fields and particles

- However they provide a (very successful) framework to model the observations, but cannot really call it a "theory").

- Some early attempts tried to explain fields as some kind of "gear wheels" or some "stress" between some kind of material (would still be compatible with Maxwell)

What is needed is a set of concepts to conveniently "describe" the effects in mathematical terms to arrive at:

> **A formulation, to figure out the characteristics of a solution of a problem, without actually solving it.**

Maxwell's equations do exactly that job !

- ELECTROSTATICS -

Electrostatics deals with:

- **Charges Q**
- **(Static) Electric fields $\vec{E}$ generated by the charges**

Recap: vectors and vector calculus → **define a special vector** ∇

called the "gradient": $$\nabla \stackrel{def}{=} (\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z})$$

Can be used like any other vector (e.g. in vector and scalar products), for example on a vector $\vec{F}(x, y, z)$ or a scalar function $\phi(x, y, z)$:

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3 = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\nabla \phi = (\frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial z})$$

A very versatile object ...

Consider it an <u>operator-in-waiting</u> (for something to work on ..)

Depending how it is used in products the results are <u>very</u> different:

$\nabla \cdot \vec{F}$ is a scalar (e.g. "density" of a source, see later)

$\nabla \phi$ is a "honest" vector (e.g. electric field $\vec{E}$, force)

$\nabla \times \vec{F}$ is a pseudo-vector

Works also on matrices (of course) and on itself (e.g.):

$\nabla \cdot \nabla$ (also written as Δ)

$$\nabla \times (\nabla \times \vec{F}) = \nabla \cdot (\nabla \cdot \vec{F}) - \Delta \vec{F}$$

etc. all kind of contraptions ...

Two operations with ∇ acting on vectors have special names:

DIVERGENCE (scalar product of ∇ with a vector):

$$\text{div}(\vec{F}) \quad \stackrel{def}{=} \quad \nabla \cdot \vec{F} \; = \; \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Physical significance: measure of something "coming out"

CURL (vector product of ∇ with a vector):

$$\text{curl}(\vec{F}) \quad \stackrel{def}{=} \quad \nabla \times \vec{F} \; = \; \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \;\; \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \;\; \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Physical significance: measure of something "circulating"

Example: Coulomb field of an isolated charge Q

A local charge Q (e.g. a particle in a beam) generates an electric field $\vec{E}$ according to:

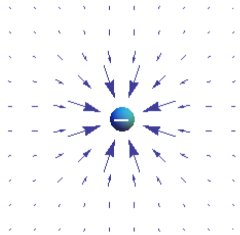
$$\boxed{\vec{E}(x,y,z) \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{\vec{r}}{r^3}} \qquad \vec{r} \;=\; (x,y,z)$$

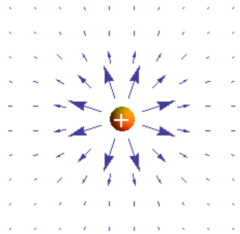
Absolute value depends on : $\left|\frac{\vec{r}}{r^3}\right| \;=\; \frac{1}{r^2}$

Field lines pointing <u>away</u> or <u>towards</u> the charge : $\vec{r}$

Charges are <u>pushed</u> or <u>attracted</u> along the field lines

Expect that div $\vec{E}$ should be relevant →





We can do the (non-trivial*) computation of the divergence:

$$\text{div } \vec{E} \;=\; \nabla \; \vec{E} \;=\; \frac{dE_x}{dx} \;+\; \frac{dE_y}{dy} + \; \frac{dE_z}{dz} \;=\; \frac{\rho}{\epsilon_0}$$

(negative charges) $\nabla \cdot \vec{E} < 0$

(positive charges) $\nabla \cdot \vec{E} > 0$

Divergence related to charge density ρ generating the field $\vec{E}$**

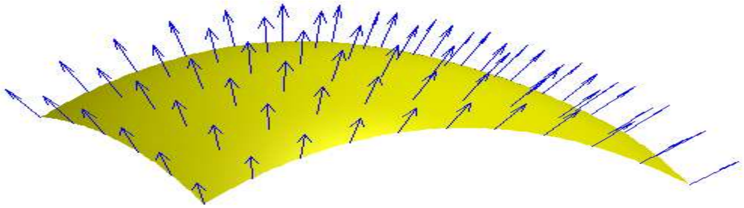
Charge density ρ is charge per volume*:** $\rho \;=\; \dfrac{Q}{V} \quad \Longrightarrow \quad \iiint \rho \; dV \;=\; Q$

* **see later**

** **sometime called "source density"**

*** **becomes important later**

Counting charges and field lines (a simple pictorial form)

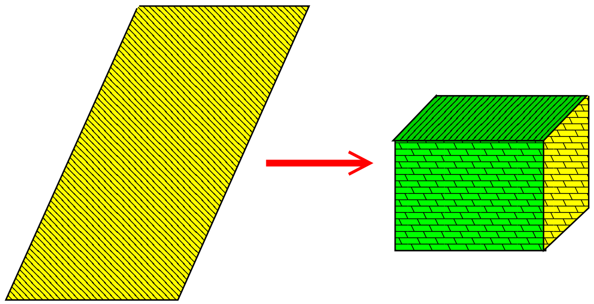


Surface integrals: integrate (adding up) field vectors passing through a surface S (or area A), we obtain the **Flux**

$$\Phi = \sum_i^N \vec{E} = \frac{Q}{\epsilon_0}$$

Intuitively: count "arrows", how many (N) and how long Q_i (strong fields)

The surface can also be wrapped up (closed) to encompass a volume, e.g.:



Replace individual charges by charge density:

$q_i \quad \Longrightarrow \rho(x,y,z)$ **i.e. charge per unit volume** dV

For a charge <u>density</u> the sum is replaced by an <u>integral</u> of $\vec{E}$ **through the surface element** $d\vec{A}$**:**

$$\Phi \;=\; \sum_i^N \vec{E} \quad \longrightarrow \quad \int_S \vec{E}\, d\vec{A} \qquad \left[\text{and} \quad \int_V \rho\, dV \;=\; Q \right]$$

The volume V **is the one enclosed by the surface** S

This holds for <u>any</u> arbitrary (closed) surface S**, and:**

- **Does not matter <u>how</u> the particles are distributed inside the volume**
- **Does not matter <u>whether</u> the particles are moving (for this part)**
- **Does not matter <u>whether</u> the particles are in vacuum or material**

Using this vector calculus (here you have to believe me) :

$$\underbrace{\int_S \vec{E} \cdot d\vec{A} = \int_V \nabla \vec{E} \cdot dV}_{\text{Gauss' formula}}$$

(relates surface and volume integrals)

$\Rightarrow$ $\nabla \vec{E} = \frac{\rho}{\epsilon_0}$ written as divergence : div $\vec{E}$

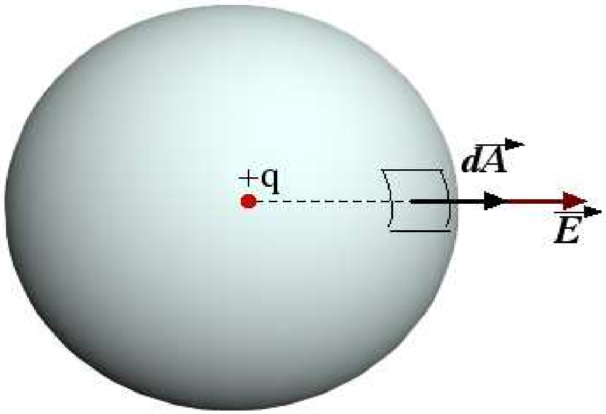
$$\Phi_E = \int_S \vec{E} \cdot d\vec{A} = \int_V \frac{\rho}{\epsilon_0} \cdot dV = \frac{Q}{\epsilon_0} = \Phi_E$$

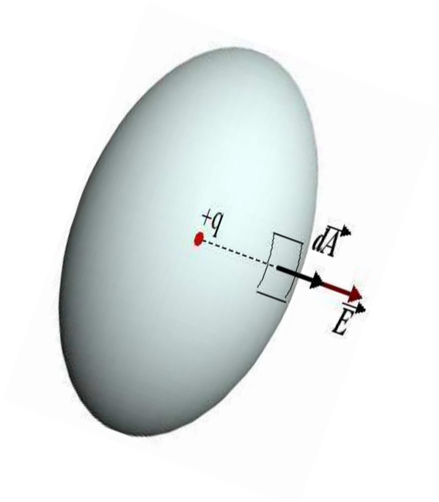
Flux of electric field $\vec{E}$ through any closed surface is proportional to net electric charge Q enclosed in the region (Gauss' Theorem).

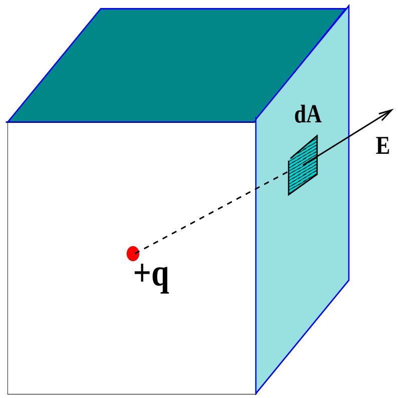
Written with charge density ρ we have:

$$\text{div}\vec{E} = \nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}$$

Simplest possible example: flux from a charge q







A charge q generates a field $\vec{E}$ according to (Coulomb):

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

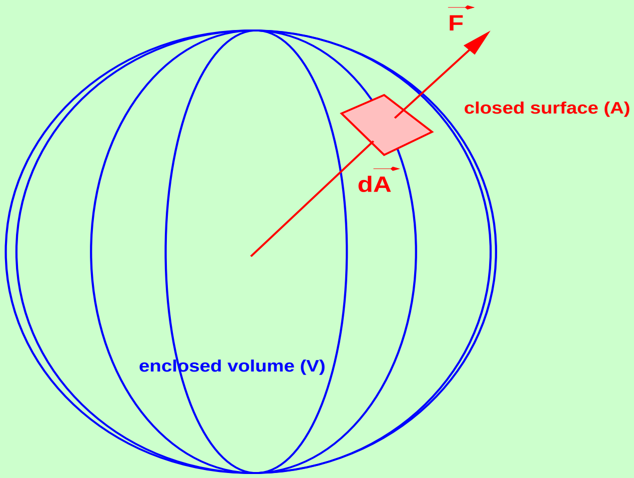
Enclose it by a sphere: $\vec{E} = const.$ **on a sphere (area is $4\pi \cdot r^2$):**

$$\int_{\text{sphere}} \vec{E} \cdot d\vec{A} = \frac{q}{4\pi\epsilon_0}\int_{\text{sphere}} \frac{dA}{r^2} = \frac{q}{\epsilon_0}$$

Surface integral through sphere A is charge inside the sphere (<u>any</u> radius, <u>any</u> shape)

<u>Exercise:</u> compute the surface integral for the cube, the result will be interesting

Once more to remember: Gauss' theorem to evaluate flux integral:



$$\iint_A \vec{E} \cdot d\vec{A} = \iiint_V \nabla \cdot \vec{E} \cdot dV \quad \text{or}$$

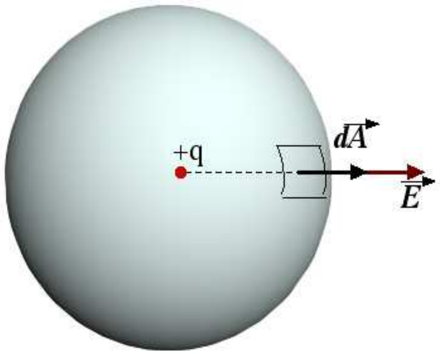
$$\iint_A \vec{E} \cdot d\vec{A} = \iiint_V \text{div } \vec{E} \cdot dV$$

Integral through closed surface (flux) is integral of divergence in the enclosed volume

Surface integral related to the divergence from the enclosed volume

Sum (integral) of all sources inside the volume gives the flux out of this region

Arrive at: Maxwell's first equation using Gauss's



density ρ and charge Q : $\displaystyle\int_V \frac{\rho}{\epsilon_0} \cdot dV = \frac{Q}{\epsilon_0}$

$$\underbrace{\int_A \vec{E} \cdot d\vec{A} = \int_V \nabla \cdot \vec{E} \cdot dV}_{\text{Gauss theorem}} = \frac{Q}{\epsilon_0} = \left[\int_V \frac{\rho}{\epsilon_0} \cdot dV \right]$$

Written with <u>charge density</u> ρ we get Maxwell's <u>first</u> equation:

Maxwell (I): $\boxed{\text{div } \vec{E} = \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$

The higher the charge density: The larger the divergence of the field

Some calculations of DIV and CURL: simplest possible charge distribution - point charge

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \cdot \frac{\vec{r}}{r^3}$$

What are div $\vec{E}$ **and** curl $\vec{E}$ **for a point charge ?**

First step: compute all derivatives (used for DIV and CURL)

$$\Rightarrow \quad \frac{\partial E_{(x,y,z)}}{\partial_{(x,y,z)}}$$

$$\frac{\partial E_x}{\partial x} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3x^2}{R^5}\right) \qquad \frac{\partial E_x}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \qquad \frac{\partial E_x}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5}$$

$$\frac{\partial E_y}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \qquad \frac{\partial E_y}{\partial y} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3y^2}{R^5}\right) \qquad \frac{\partial E_y}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5}$$

$$\frac{\partial E_z}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5} \qquad \frac{\partial E_z}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5} \qquad \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3z^2}{R^5}\right)$$

$$\frac{\partial E_x}{\partial x} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3x^2}{R^5}\right) \qquad \frac{\partial E_x}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \qquad \frac{\partial E_x}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5}$$

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$$\frac{\partial E_z}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5} \qquad \frac{\partial E_z}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5} \qquad \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3z^2}{R^5}\right)$$

$$\text{div } \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\epsilon_0}\left(\frac{3}{R^3} - \frac{3}{R^5}(x^2 + y^2 + z^2)\right)$$

$$\frac{\partial E_x}{\partial x} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3x^2}{R^5}\right) \qquad \frac{\partial E_x}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \qquad \frac{\partial E_x}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5}$$

$$\frac{\partial E_y}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \qquad \frac{\partial E_y}{\partial y} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3y^2}{R^5}\right) \qquad \frac{\partial E_y}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5}$$

$$\frac{\partial E_z}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5} \qquad \frac{\partial E_z}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5} \qquad \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3z^2}{R^5}\right)$$

$$\text{curl } \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial z}\right) = (0, \quad 0, \quad 0)$$

(there is nothing circulating around a point charge)

In general: Because there is nothing circulating (curl $\vec{E} = 0$) one can derive the field $\vec{E}$ from a scalar electrostatic potential $\phi(x, y, z)$, i.e.:

$$\vec{E} = -\,\text{grad}\ \phi = -\nabla\phi = -(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z})$$

then we have

$$\boxed{\nabla\vec{E} = -\nabla^2\phi = -(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}) = \frac{\rho(x, y, z)}{\epsilon_0}}$$

This is called <u>Poisson's equation</u>

All we need is ϕ to get the fields Example →

A very important example: 3D Gaussian distribution

$$\rho(x,y,z) = \frac{Q}{\sigma_x\sigma_y\sigma_z\sqrt{2\pi}^3}\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}-\frac{z^2}{2\sigma_z^2}\right)$$

($\sigma_x, \sigma_y, \sigma_z$ r.m.s. sizes)

$$\phi(x,y,z,\sigma_x,\sigma_y,\sigma_z) = \frac{Q}{4\pi\epsilon_0}\int_0^\infty \frac{\exp(-\frac{x^2}{2\sigma_x^2+t}-\frac{y^2}{2\sigma_y^2+t}-\frac{z^2}{2\sigma_z^2+t})}{\sqrt{(2\sigma_x^2+t)(2\sigma_y^2+t)(2\sigma_z^2+t)}}dt$$

For a derivation, see e.g. W. Herr, *Beam-Beam Effects*,
in Proceedings CAS Zeuthen, 2003, CERN-2006-002, and references therein.

This gives the compact formulae for the fields:

$$E_x = \frac{ne}{2\epsilon_0\sqrt{2\pi(\sigma_x^2 - \sigma_y^2)}} \mathrm{Im}\left[\mathrm{erf}\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{\left(-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right)}\mathrm{erf}\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

$$E_y = \frac{ne}{2\epsilon_0\sqrt{2\pi(\sigma_x^2 - \sigma_y^2)}} \mathrm{Re}\left[\mathrm{erf}\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{\left(-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right)}\mathrm{erf}\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

wherer erf **is the complex error function**

These formulae are often used to evaluate the forces due to beam-beam effects

Very important in practice:

Poisson's equation in Polar coordinates, i.e. 2D (r, φ)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) \;+\; \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} \;=\; -\frac{\rho}{\epsilon_0}$$

Poisson's equation in Cylindrical coordinates (r, φ, z)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) \;+\; \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} \;+\; \frac{\partial^2 \phi}{\partial z^2} \;=\; -\frac{\rho}{\epsilon_0}$$

Poisson's equation in Spherical coordinates (r, θ, φ)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) \;+\; \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right) \;+\; \frac{1}{r^2 \sin\theta}\frac{\partial^2 \phi}{\partial \varphi^2} \;=\; -\frac{\rho}{\epsilon_0}$$

Spherical coordinates sound like a good choice for local charges, e.g. point charges

What are div **and** curl **??**

$$\text{div } E(r,\vec{\theta},\varphi) \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) \;+\; \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\;\theta E_\theta) \;+\; \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}$$

(very useful !)

$$\text{curl } E(r,\vec{\theta},\varphi) \;=\; \frac{\hat{e_r}}{r\sin\theta}\left(\frac{\partial(E_\varphi\sin\theta)}{\partial\theta} \;-\; \frac{\partial E_\theta}{\partial\varphi}\right) \;+$$

$$\frac{\hat{e_\theta}}{r}\left(\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\varphi} \;-\; \frac{E_\varphi}{\partial r}(rE_\varphi\right) \;+\; \frac{\hat{e_\varphi}}{r}\left(\frac{\partial(rE_\theta)}{\partial r} \;-\; \frac{\partial E_r}{\partial\theta}\right)$$

Note: $\hat{e_\theta},\ \hat{e_r},\ \hat{e_\varphi}$ **are the corresponding orthogonal unit vectors**

(useful, but no so much used !)

Try it on a point charge

$$\vec{E} = -\nabla\Psi(r) = \frac{Q}{4\pi\epsilon_0}\cdot\frac{\vec{r}}{r^3}$$

$$\text{div}\ \vec{E}(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \underbrace{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\ \theta E_\theta)}_{\equiv\ 0} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}}_{\equiv\ 0}$$

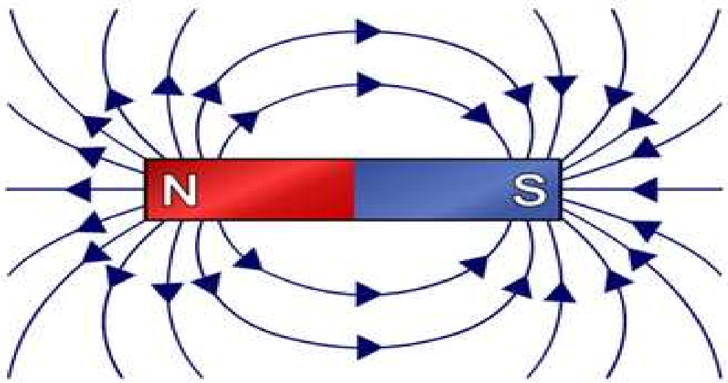
Only the radial component is non-zero:

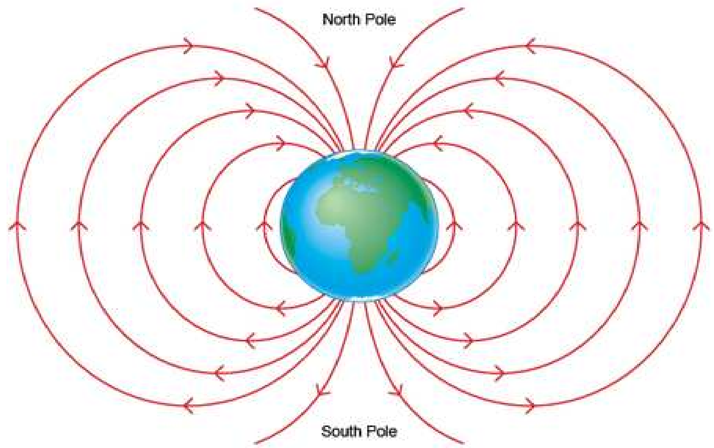
then : $\quad \text{div}\ \vec{E}(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r)$

➡ $\text{div}\ \vec{E}(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2\cdot Q}{4\pi\epsilon_0 r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\epsilon_0}\right) = \ ???$

(non-trivial indeed ...)

What about magnetic fields ? ...

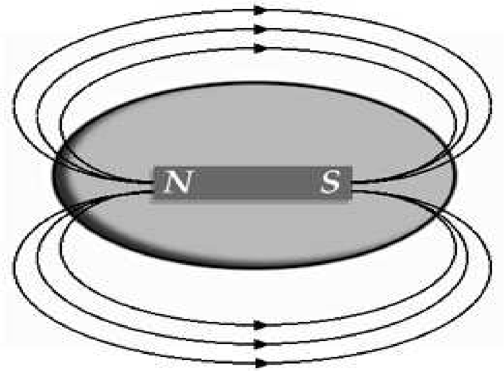




- **They have the direction (by definition): magnetic field lines from North to South**
- **Field lines of $\vec{B}$ are <u>always</u> closed**
- **Magnets (e.g. compass) are pushed or attracted along the field lines**

What about divergence of magnetic fields ?

Enclose it again by a surface - the result is found immediately:



$$\int\int_A \vec{B}\ \mathrm{d}\vec{A} = \int\int\int_V \nabla\vec{B}\ \mathrm{d}V = 0$$

Volume (thus dV) is <u>never</u> $= 0$

➡ $\nabla\vec{B} = \text{div } \vec{B} = 0$

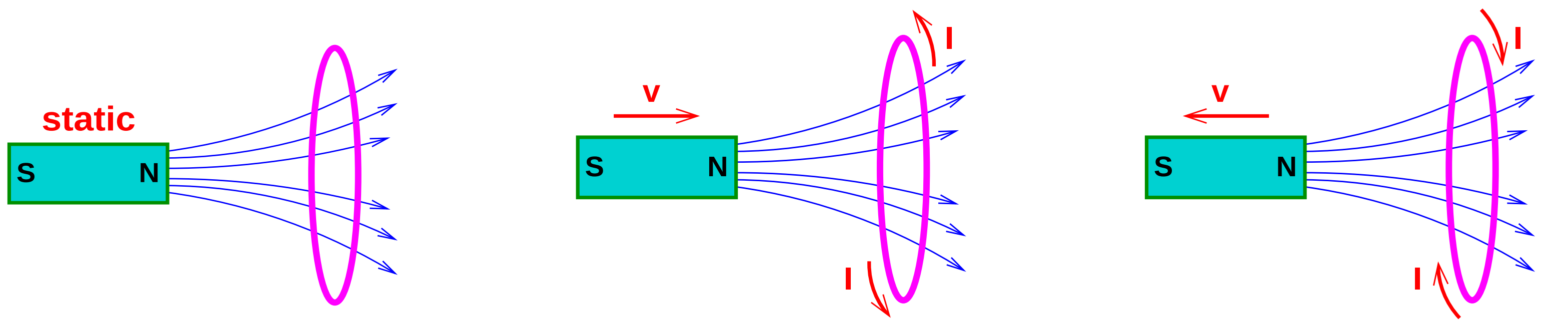
What goes into the closed surface also goes out

➡ **<u>Maxwell's second equation:</u>** $\nabla\vec{B} = \text{div } \vec{B} = 0$

➡ **Physical significance: (probably) no Magnetic Monopoles**

Enter Faraday: allow changing flux through an area

$$\text{static flux}: \quad \Omega = \int_A \vec{B} \cdot \mathrm{d}\vec{A} \qquad\qquad \text{changing flux}: \quad \frac{\partial \Omega}{\partial t} = \int_A \frac{\partial(\vec{B})}{\partial t} \cdot \mathrm{d}\vec{A}$$



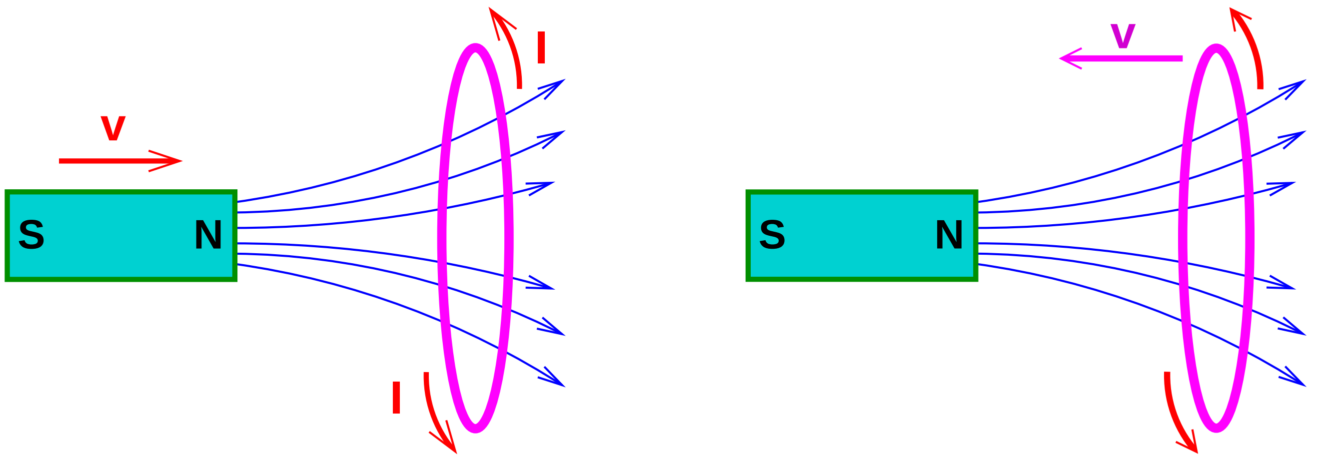
Moving the magnet changes the flux: more or fewer passing through the area (use a conducting coil, e.g. wedding ring) $\Longrightarrow$

Induces a circulating (curling) electric field $\vec{E}$ in the coil which "pushes" charges around the coil $\Longrightarrow$

Moving charges: Current I in the coil (observe its direction ..)

Experimental evidence:

It does not matter whether the magnet or the coil is moved (same direction of induced current):



Experimental evidence:

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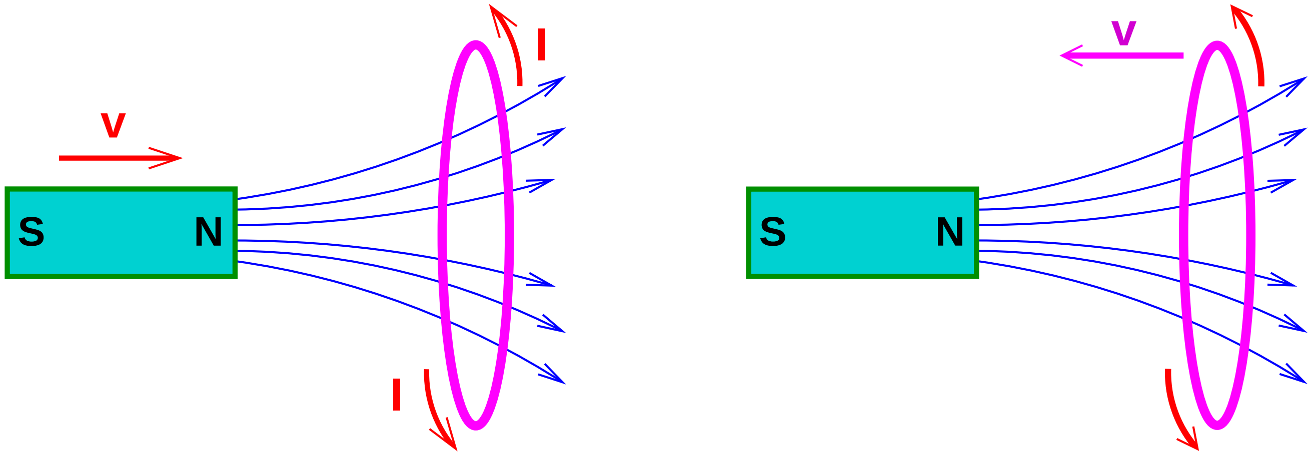


If you think it is obvious - no :

This was the reason for Einstein to develope special relativity !!!

Again: Maxwell different in the two systems (see lecture on Relativity)

Formally: A <u>changing</u> flux Ω through an area A produces circular electric field $\vec{E}$, "pushing" charges $\Longrightarrow$ a current I

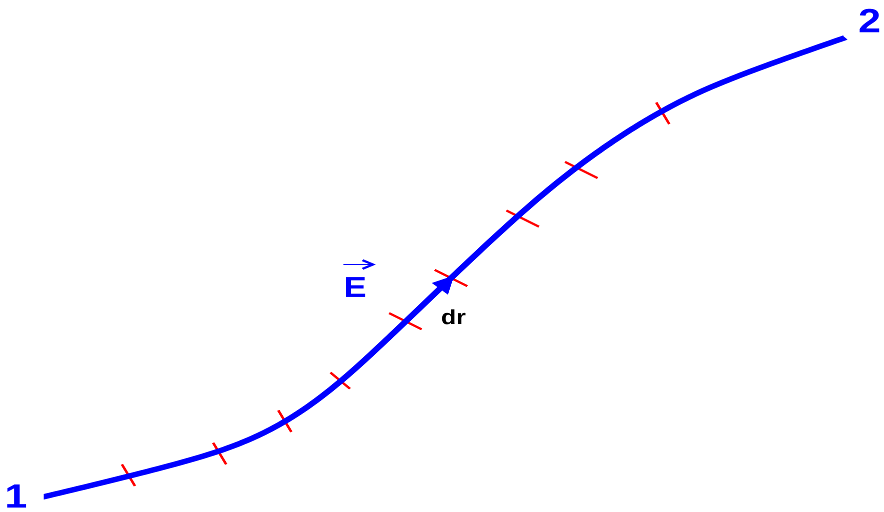
$$-\frac{\partial\Omega}{\partial t} = \frac{\partial}{\partial t}\overbrace{\int_A \vec{B}\cdot d\vec{A}}^{\text{flux }\Omega} = \underbrace{\oint_C \vec{E}\cdot d\vec{r}}_{\text{pushed charges}}$$

Flux can be changed by:

- **Change of magnetic field $\vec{B}$ with time t (e.g. transformers)**
- **Change of area A with time t (e.g. dynamos)**

How to count "pushed charges" $\left[\int_C \vec{E}\cdot d\vec{r}\right.$ is a <u>line integral</u>$\left.\right]$

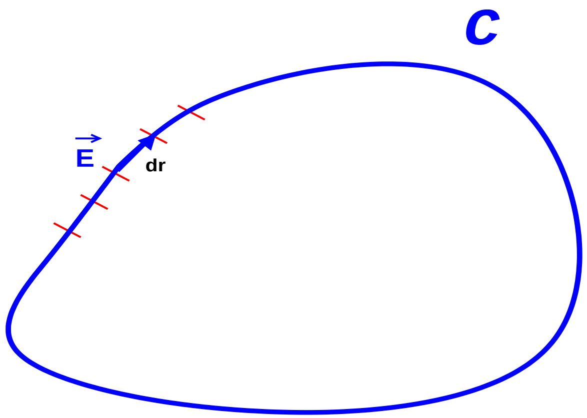
Line integrals sum up "pushes" along lines or curves:



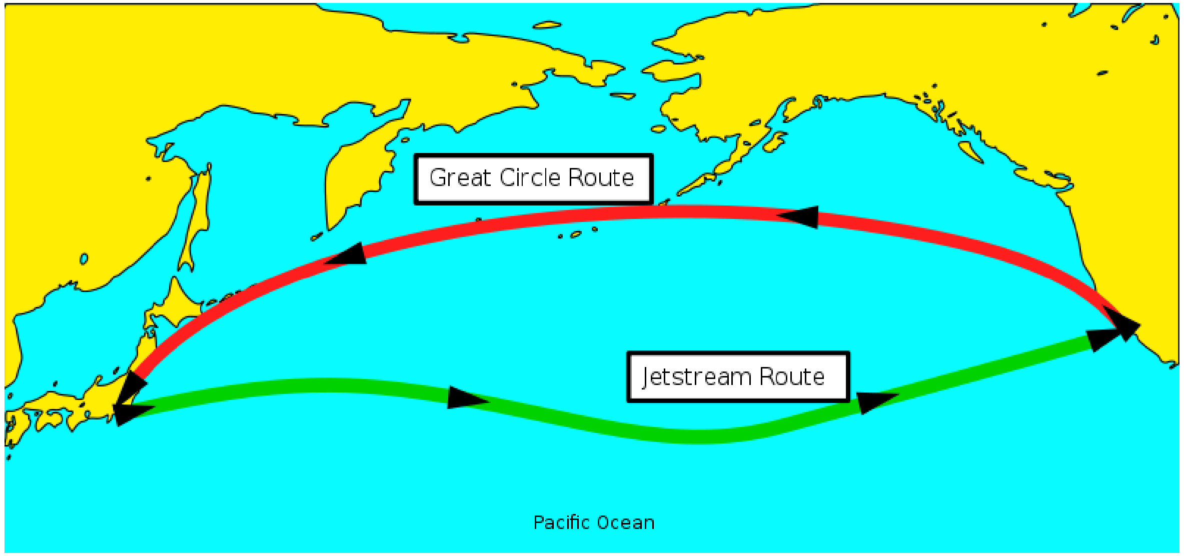
"lines" can be open or closed

Sum along all line elements $d\vec{r}$

$\int_1^2 \vec{E} \cdot d\vec{r}$ or $\int_C \vec{E} \cdot d\vec{r}$



Everyday example ..

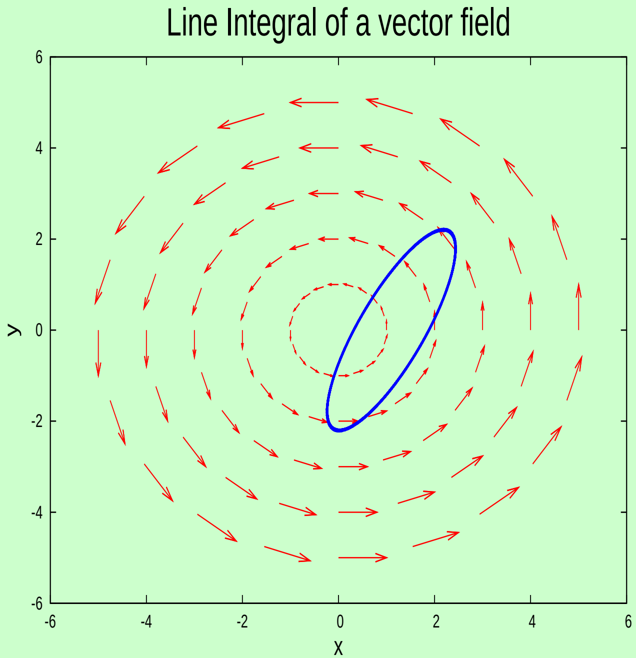


Line integrals: sum up "pushes" along the two Lines/Routes

Optimize: e.g. fuel consumption, time of flight (saves 1 hour !)

Like surface integral, the <u>closed</u> line integral $\int_C \vec{E} \cdot d\vec{r}$ **can be re-written:**

Used in the following: Stoke's theorem



$$\oint_C \vec{E} \cdot d\vec{r} = \iint_A \nabla \times \vec{E} \cdot d\vec{A} \quad \text{or}$$

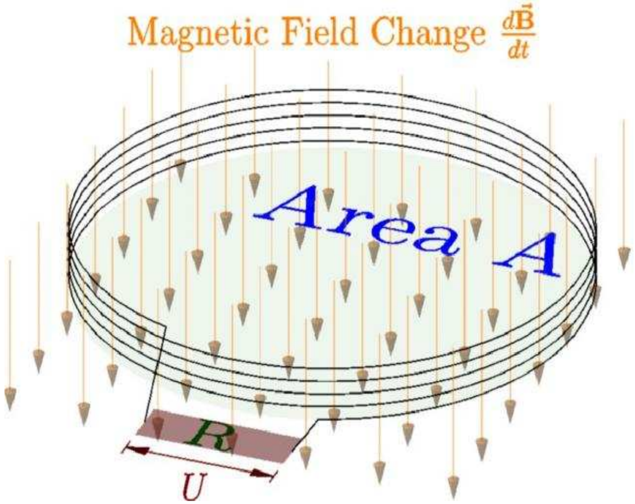
$$\oint_C \vec{E} \cdot d\vec{r} = \iint_A \text{curl } \vec{E} \cdot d\vec{A}$$

obviously : div $\vec{E} = 0$

Summing up all vectors inside the area: net effect is the sum along the closed curve

→ measures something that is circulating ("curling") inside and how strongly

Use this theorem for a coil enclosing a closed area



$$\underbrace{\int_A -\frac{\partial \vec{B}}{\partial t} d\vec{A} = \int_A \nabla \times \vec{E}\, d\vec{A}} = \oint_C \vec{E} \cdot d\vec{r}$$

$$\int_A -\frac{\partial \vec{B}}{\partial t} d\vec{A} = \underbrace{\int_A \nabla \times \vec{E}\, d\vec{A} = \oint_C \vec{E} \cdot d\vec{r}}_{\text{Stoke's formula}}$$

$$\underbrace{\int_A -\frac{\partial \vec{B}}{\partial t} d\vec{A} = \int_A \nabla \times \vec{E}\, d\vec{A}}_{\text{same Integration}}$$

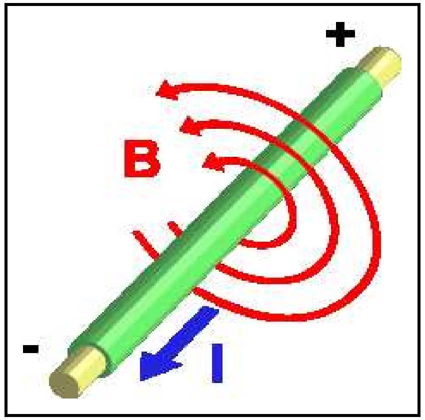
Re-written: changing magnetic field through an area induces circulating electric field around the area (Faraday)

Maxwell' 3rd equation

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \text{curl } \vec{E}$$

Next: Maxwell's fourth equation (part 1) ...

From Ampere's law, for example current density $\vec{j}$:



$$\int_A \nabla \times \vec{B} \;\; d\vec{A} \;=\; \oint_C \vec{B} \cdot d\vec{r} \;=\; \int_A \mu_0 \vec{j} \;\; d\vec{A}$$

$\vec{j}$: "amount" of charges through area $\vec{A}$

$$\int_A \mu_0 \vec{j} \;\; d\vec{A} \;=\; \mu_0 \; I \qquad \text{(total current)}$$

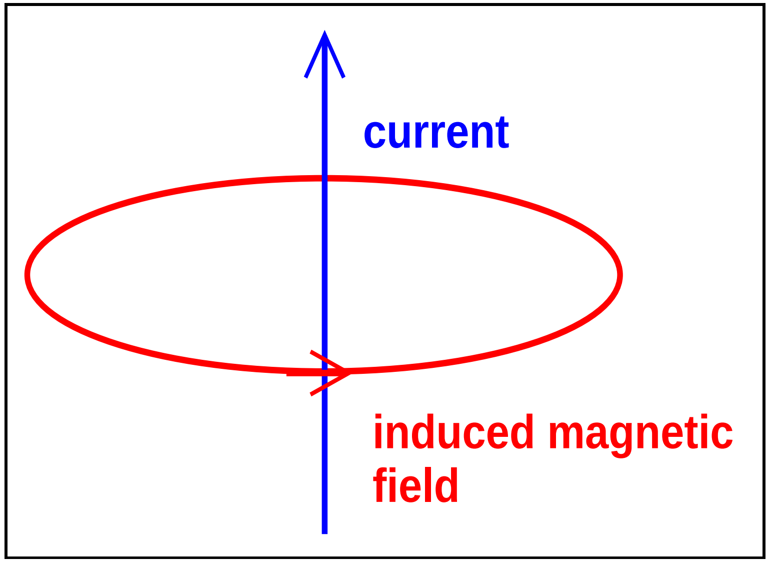
<u>Static</u> electric current induces circular magnetic field (e.g. in magnets)

Using the same argument as before (the same integral formula):

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{j}}$$

An important application:

For a static electric current I in a single (infinitely long) wire we get Biot-Savart law (using the area of a circle $A = r^2 \cdot \pi$, we can easily do the integral):



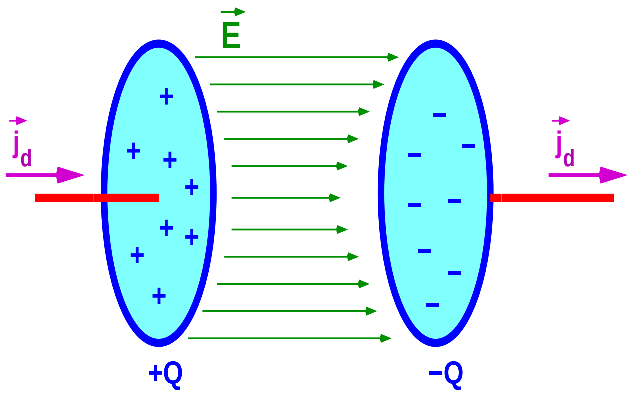
$$\vec{B} = \frac{\mu_0}{4\pi} \oint \vec{j} \cdot \frac{\vec{r} \cdot d\vec{r}}{r^3}$$

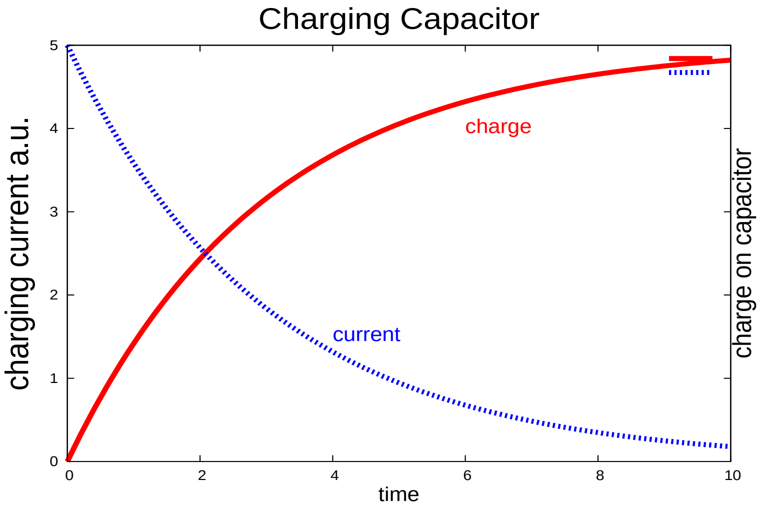
$$\vec{B} = \frac{\mu_0}{2\pi} \frac{\vec{j}}{r}$$

Application: magnetic field calculations in wires

Maxwell's fourth equation (part 2) ...

Charging capacitor: Current enters left plate - leaves from right plate, builds up an electric field between plates ➡ produces a "current" <u>during</u> the charging process

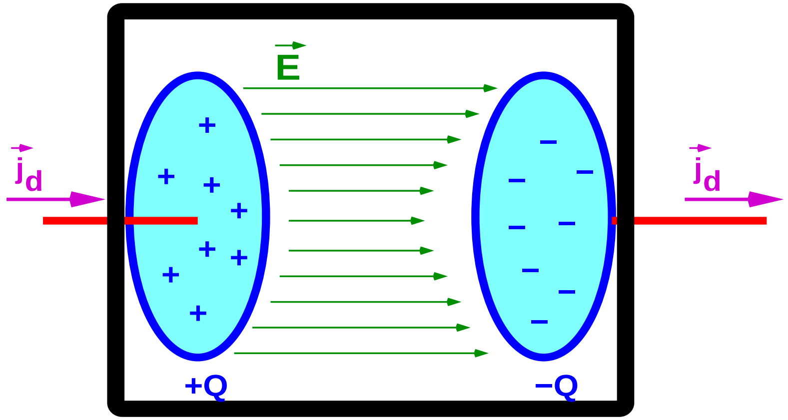




$$Q(t) \;=\; C \cdot V_b \cdot (1 \;-\; \exp(-t/RC)) \quad \textbf{and} \quad I(t) \;=\; \frac{V_b}{R}\;\exp(-t/RC)$$

Part 2: Maxwell's fourth equation

Charging capacitor: Current enters left plate - leaves from right plate, builds up an electric field between plates → produces a "current" during the charging process



Displacement Current :

$$\vec{j}_d = \frac{d\vec{E}}{dt}$$

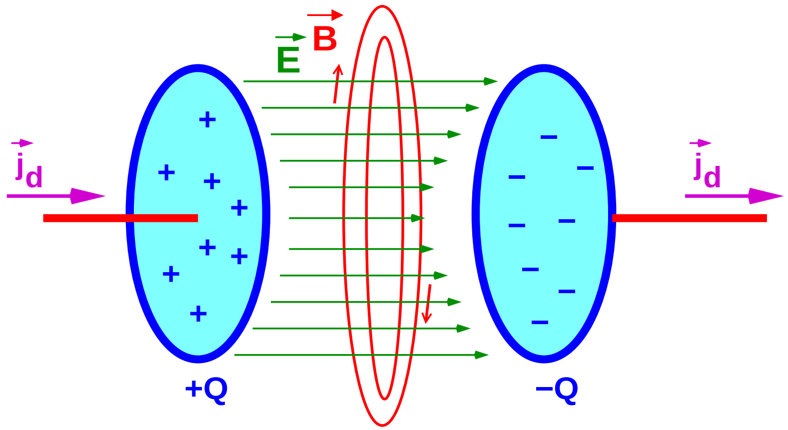
This is not a current from charges moving through a wire

This is a "current" from time varying electric fields

Once charged: fields are constant, (displacement) "current" stops

Cannot distinguish the origin of a current - apply Ampere's law to j_d

- **Displacement current** j_d **produces magnetic field, just like "real currents" do ...**



- **Time varying electric field** **induces time varying circular magnetic field (using the current density $\vec{j}_d$)**

$$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{d\vec{E}}{dt}$$

Bottom line:

Magnetic fields $\vec{B}$ can be generated by two different "currents":

$$\nabla \times \vec{B} = \mu_0 \vec{j} \qquad \textbf{(electrical current)}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \qquad \textbf{(displacement current)}$$

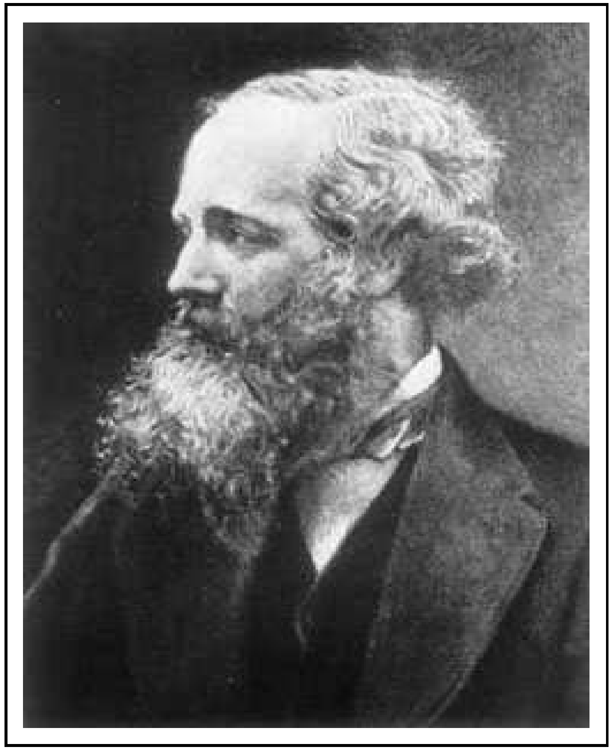
or putting them together to get Maxwell's fourth equation:

$$\nabla \times \vec{B} = \mu_0 (\vec{j} + \vec{j}_d) = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

or in integral form:

$$\int_A \nabla \times \vec{B} \cdot d\vec{A} = \int_A \left(\mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

SUMMARY: MAXWELL'S EQUATIONS



$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

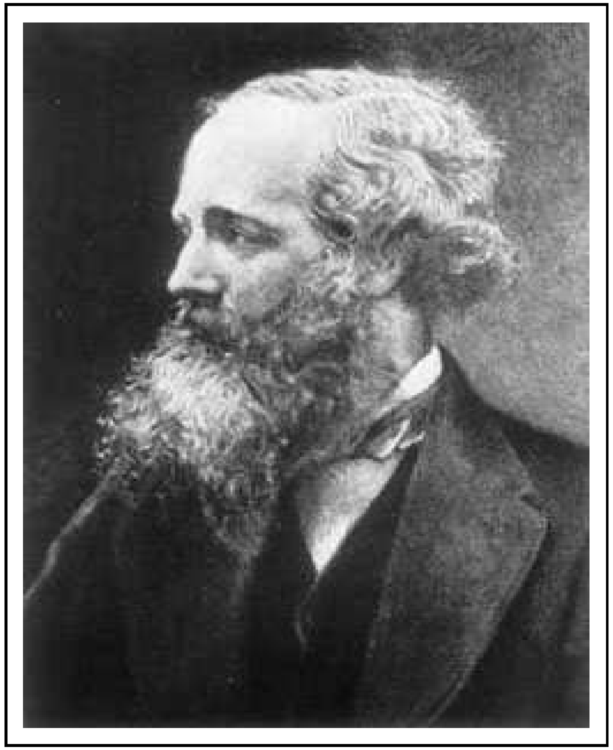
$$\int_A \vec{B} \cdot d\vec{A} = 0$$

$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt} \right) \cdot d\vec{A}$$

$$\oint_C \vec{B} \cdot d\vec{r} = \int_A \left(\mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} \right) \cdot d\vec{A}$$

Written in Integral form

SUMMARY: MAXWELL'S EQUATIONS



$$\nabla \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} \; + \; \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

Written in Differential form (my preference)

V.G.F.A.Q:

Why :

$$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\text{curl } \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\text{div } \vec{B} = 0$$

$$\text{curl } \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

Why Not :

$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt} \right) \cdot d\vec{A}$$

$$\int_A \vec{B} \cdot d\vec{A} = 0$$

$$\oint_C \vec{B} \cdot d\vec{r} = \int_A \left(\mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} \right) \cdot d\vec{A}$$

$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$ something $(\vec{E})$ flowing out

$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$???

$\text{curl } \vec{E} = -\frac{d\vec{B}}{dt}$ something $(\vec{E})$ circulating

$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt} \right) \cdot d\vec{A}$???

Maxwell's Equations - compact

1. **Electric fields $\vec{E}$ are generated by charges and proportional to total charge**

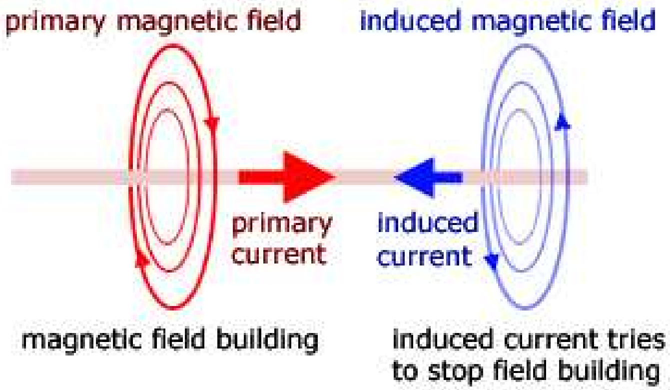
2. **Magnetic monopoles do (probably) not exist**

3. **Changing magnetic flux generates circular electric fields/currents**

4.1 **Changing electric flux generates circular magnetic fields**

4.2 **Static electric current generates circular magnetic fields**

Changing fields: Powering and self-induction



- **If the current is not static:**
- **Primary magnetic flux $\vec{B}$ changes with changing current**

➡ **Induces an electric field, resulting in a current and induced magnetic field $\vec{B}_i$**

➡ **Induced current will oppose a change of the primary current**

➡ **If we want to change a current to ramp a magnet ...**

Ramp rate defines required Voltage:

$$U = -L\frac{\partial I}{\partial t}$$

Inductance L in Henry (H)

Example:

- **Required ramp rate: 10 A/s**
- **With $L = 15.1\ H$ per powering sector**

➡ **Required Voltage is $\approx$ 150 V**

Surprise - as is always the case:

Units:	**Gauss law**	**Ampere/Maxwell**
SI	$\nabla\vec{E} = \frac{\rho}{\epsilon_0}$	$\nabla\times\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{d\vec{E}}{dt}$
Electro-static ($\epsilon_0 = 1$)	$\nabla\vec{E} = 4\pi\rho$	$\nabla\times\vec{B} = \frac{4\pi}{c^2}\vec{j} + \frac{1}{c^2}\frac{d\vec{E}}{dt}$
Electro-magnetic ($\mu_0 = 1$)	$\nabla\vec{E} = 4\pi c^2\rho$	$\nabla\times\vec{B} = 4\pi\vec{j} + \frac{1}{c^2}\frac{d\vec{E}}{dt}$
Gauss cgs	$\nabla\vec{E} = 4\pi\rho$	$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{d\vec{E}}{dt}$
Lorentz	$\nabla\vec{E} = \rho$	$\nabla\times\vec{B} = \frac{1}{c}\vec{j} + \frac{1}{c}\frac{d\vec{E}}{dt}$

Also: $\vec{B}^{Gauss} = \sqrt{\frac{4\pi}{\mu_0}}\vec{B}^{SI}$ $\rho^{Gauss} = \frac{\rho^{SI}}{\sqrt{4\pi\epsilon_0}}$ **and so on .....**

That's not all → Electromagnetic fields in material

In vacuum:

$$\vec{D} = \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_0 \cdot \vec{H}$$

In a material:

$$\vec{D} = \epsilon_r \cdot \epsilon_0 \cdot \vec{E} \;=\; \epsilon_0 \vec{E} \;+\; \vec{P}$$

$$\vec{B} = \mu_r \cdot \mu_0 \cdot \vec{H} \;=\; \mu_0 \vec{H} \;+\; \vec{M}$$

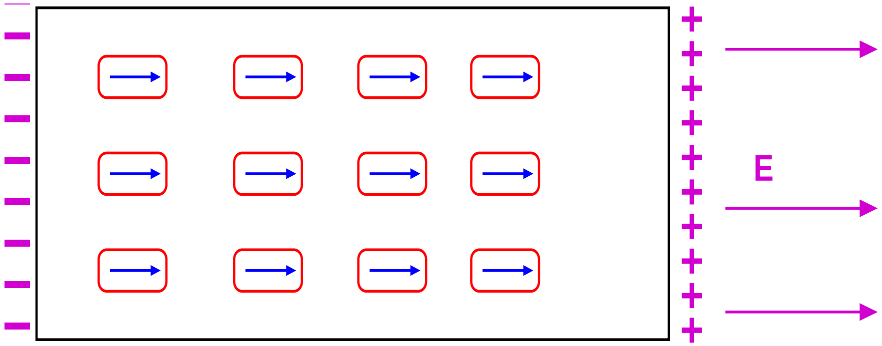
Origin: $\vec{P}$**olarization and** $\vec{M}$**agnetization**

$\epsilon_r(\vec{E}, \vec{r}, \omega)$ → ϵ_r **is relative permittivity** $\approx \quad [1 - 10^5]$

$\mu_r(\vec{H}, \vec{r}, \omega)$ → μ_r **is relative permeability** $\approx \quad [0(!) - 10^6]$

(i.e.: linear, isotropic, non-dispersive)

Polarization $\vec{P}$: displacement of charges in non-conducting material



Appears as electric dipole

$\vec{P} = \xi_e \cdot \vec{E}$ (ξ_e **is electric susceptibility)**

Dielectric displacement follows: $\vec{D} = (1 + 4\pi\xi_e)\vec{E} = \epsilon \cdot \vec{E}$

Magnetism: occurence of circular currents of atomic electrons

Classification of magnetic material properties:

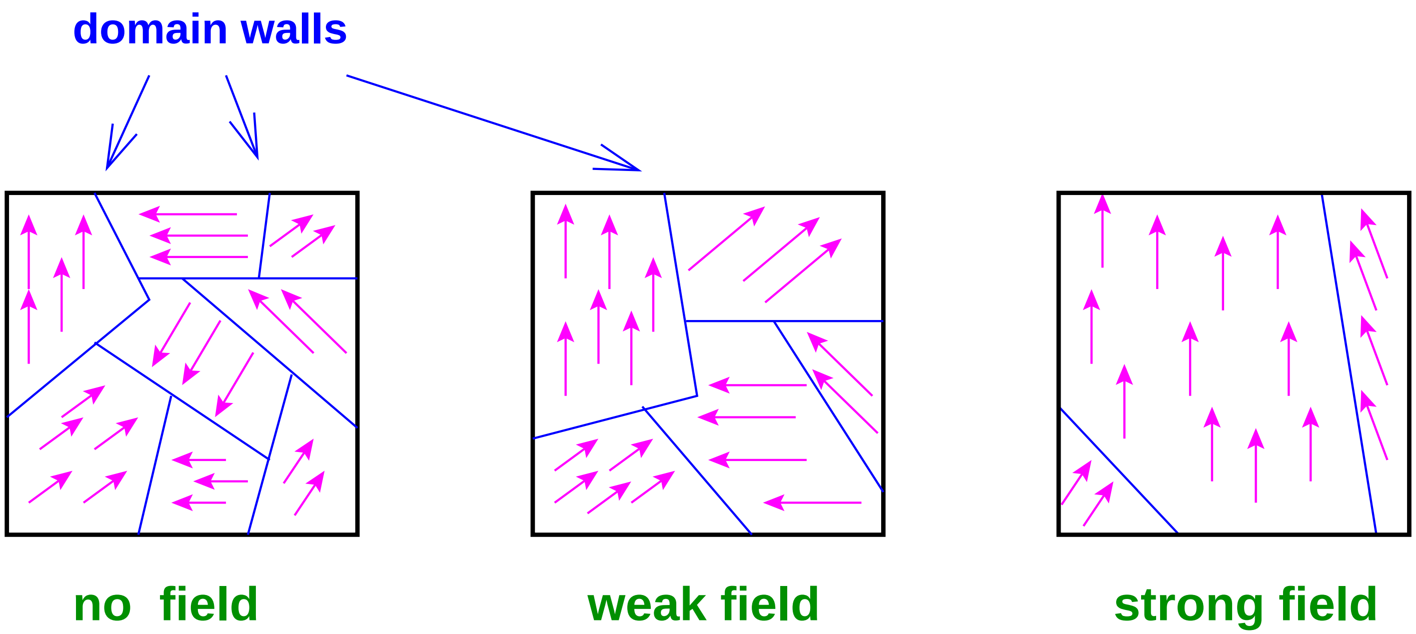
Diamagnetism	**repelled**	$\mu < 1$	$\xi_m < 0$	**typical:** $\xi_m \approx -10^{-7}$
Paramagnetism	**aligned**	$\mu > 1$	$\xi_m > 0$	**typical:** $\xi_m \approx +10^{-7}$
Ferromagnetism	**aligned**	$\mu \gg 1$	$\xi_m \gg 0$	**typical:** $\xi_m \approx +10^{6}$

Diamagnetism: Atoms and molecules without magnetic moment

Paramagnetism: Atoms and molecules with magnetic moment

Ferromagnetism: Saturation magnetization occur within microscopic domains (Weiss domains)

Magnetism in ferromagnetic: not rigorous - just to get an idea ...



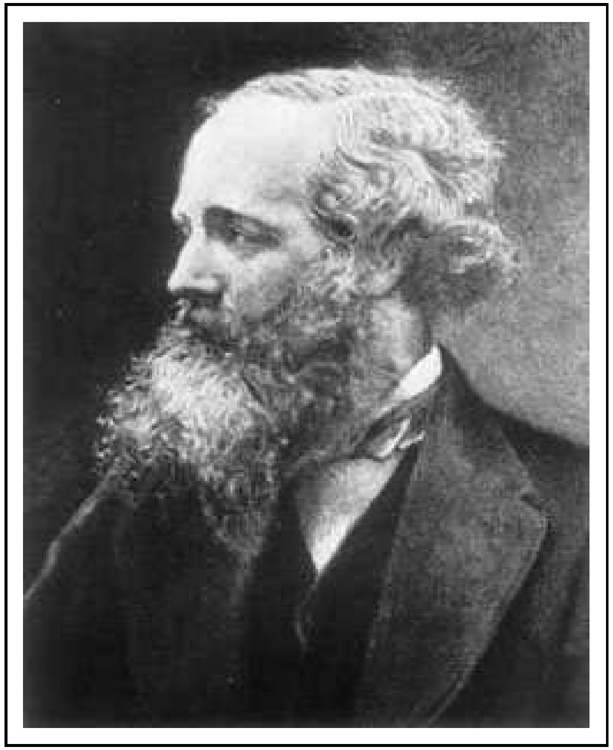
Unmagnetized Iron: spontaneous magnetization around seeds, leading to "Weiss domains"

Randomly oriented domains cancel out

With external fields: domain walls move - domains in direction of external field "grow"! They do not align (although this is sometimes said)!

Magnetic field follows : $\vec{B} = (1 + 4\pi\xi_m)\vec{H} = \mu \cdot \vec{H}$

Once more: Maxwell's Equations



$$\nabla \vec{D} = \rho$$
$$\nabla \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$
$$\nabla \times \vec{H} = \vec{j} + \frac{d\vec{D}}{dt}$$

(a.k.a. Macroscopic Maxwell equations)

Something on potentials (needed in lecture on Relativity):

In general: Scalar Potentials are related to static field conditions, Vector Potentials are related to dynamic field conditions

Electric fields can be derived from a (scalar) potential ϕ:

$$\vec{E} = -\vec{\nabla} \cdot \phi$$

Magnetic fields can be derived from a (vector) potential $\vec{A}$:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \text{curl } \vec{A}$$

Combining Maxwell(I) + Maxwell(III):

$$\vec{E} = -\vec{\nabla} \cdot \phi - \frac{\partial \vec{A}}{\partial t}$$

Fields can be written as derivatives of scalar and vector potentials $\Phi(x, y, z)$ and $\vec{A}(x, y, z)$

But watch out and remember for later: $\vec{\nabla} \cdot \ldots$ versus $\vec{\nabla} \times \ldots$!

(absolute values of potentials Φ and $\vec{A}$ can not be measured ..)

The Coulomb potential of a static charge q is written as:

$$\Psi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{|\vec{r} - \vec{r}_q|} \qquad \left[\text{ or } \int \frac{1}{4\pi\epsilon_0} \cdot \frac{\rho(\vec{r_q})}{|\vec{r} - \vec{r}_q|} \right]$$

where $\vec{r}$ is the observation point* and $\vec{r}_q$ the location of the charge

The vector potential is linked to the current $\vec{j}$:

$$\nabla^2 \vec{A} = \mu_0 \vec{j}$$

The knowledge of the potentials allows the computation of the fields ➡ see lecture on relativity (fields of moving charges)

* **a shameless lie: one cannot observe/measure a potential (only fields) !**

Accelerator magnets ➡ Multipole expansion

If Ψ is periodic*) in θ, the components in cylindrical coordinates are:

$$B_r(\theta, r) = -\left(\frac{\partial \Psi}{\partial r}\right) = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}}\right)^{n-1} \sin(n(\theta - \alpha_n)) \quad \text{and}$$

$$B_\theta(\theta, r) = -\left(\frac{1}{r}\right)\left(\frac{\partial \Psi}{\partial \theta}\right) = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}}\right)^{n-1} \cos(n(\theta - \alpha_n))$$

$C(n)$ **is the strength of the 2n-pole component of the total field**

R_{ref} **is a reference radius (LHC: 17 mm for 28 mm aperture, typical ratio)**

α_n **is a constant (related to the orientation of the 2n component)**

*) **a good assumption for accelerator magnets**

The n-th component in $B_r(r,\theta)$ has n South poles and n North poles as a function of the azimuthal angle θ

This implies (for any n):

$$\theta_S = \frac{\pi}{2n} + \alpha_n; \qquad \frac{5\pi}{2n} + \alpha_n; \qquad \frac{9\pi}{2n} + \alpha_n; \ldots \qquad \text{(South poles)}$$

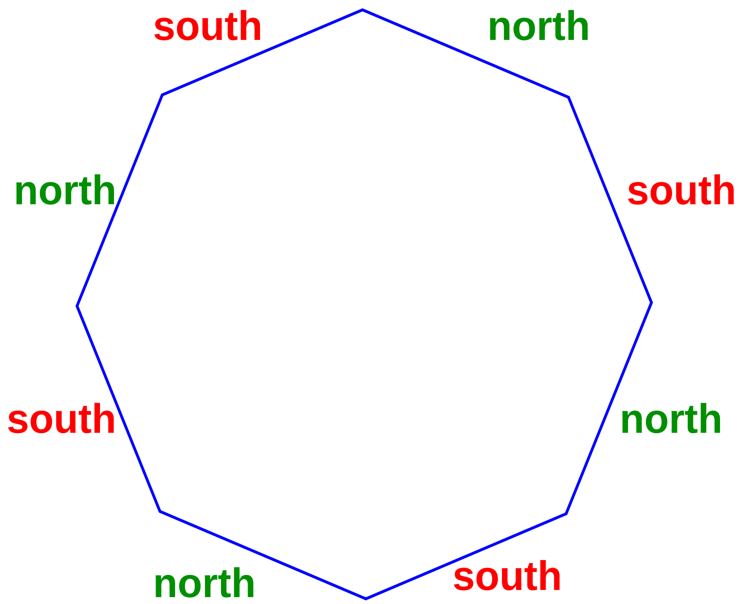
$$\theta_N = \frac{3\pi}{2n} + \alpha_n; \qquad \frac{7\pi}{2n} + \alpha_n; \qquad \frac{11\pi}{2n} + \alpha_n; \ldots \qquad \text{(North poles)}$$

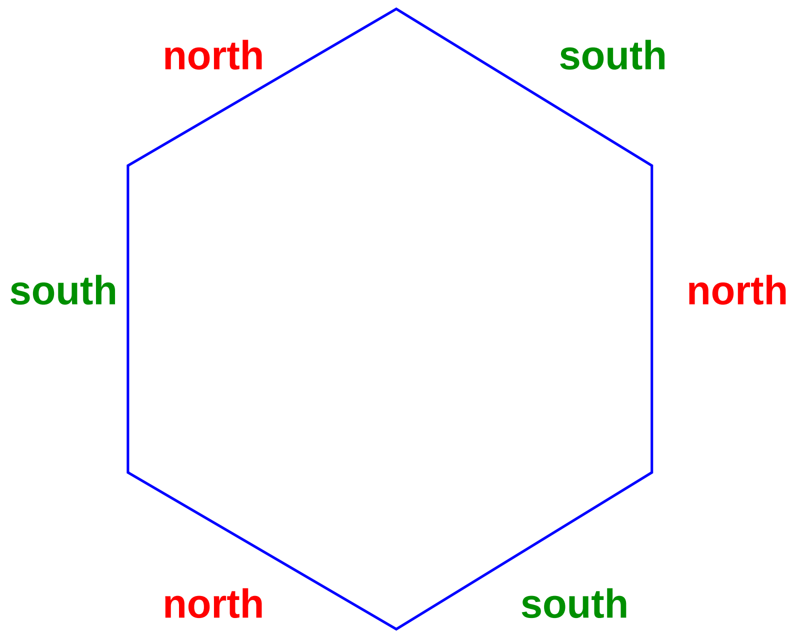
A focusing Quadrupole ($n = 2,\ \alpha_2 = 0$):

$$\theta_S = 45\,\mathrm{deg}; \qquad 225\,\mathrm{deg}; \qquad 405\,\mathrm{deg}; \ldots.. \qquad \text{(South poles)}$$

$$\theta_N = 135\,\mathrm{deg}; \qquad 315\,\mathrm{deg}; \qquad 495\,\mathrm{deg}; \ldots.. \qquad \text{(North poles)}$$

(What if: $\alpha_2 = \frac{\pi}{2}$ or $\alpha_2 = \frac{\pi}{4}$ **???)**





A little exercise: what are n**,** Θ**,** α_n **?**

If Cartesian coordinates are preferred, starting from:

$$B_x(\theta, r) = B_r \cos\theta - B_\theta \sin\theta = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}}\right)^{n-1} \sin[(n-1)\theta - n\alpha_n)]$$

$$B_y(\theta, r) = B_r \sin\theta + B_\theta \cos\theta = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}}\right)^{n-1} \cos[(n-1)\theta - n\alpha_n)]$$

or defined as <u>complex</u> field* :

$$\vec{B}(z) = B_y(x, y) + iB_x(x, y) = \sum_{n=1}^{\infty} [C(n) \exp(-in\alpha_n)] \left(\frac{z}{R_{ref}}\right)^{n-1}$$

then one gets a "physical" picture:

$$C(n) \exp(-in\alpha_n) = (2n \quad pole)_{\text{normal}} + i(2n \quad pole)_{\text{skew}} = b_n + ia_n$$

Example: **for** $\alpha_2 = \frac{\pi}{4} \neq 0$ → $b_2 = 0, \quad a_2 = 1$

* **Euler:** $z = x + iy = r \cdot \exp(i\theta) = r(\cos(\theta) + i \sin(\theta))$

Finally replacing $C(n)\exp(-in\alpha_n)$ by $(b_n + ia_n)$ (and some confusion):

$$B_y + iB_x = \sum_{n=0}^{\infty} (b_n + i \cdot a_n) \left(\frac{r}{R_{ref}}\right)^{n} \qquad \text{(U.S. convention)}$$

$$B_y + iB_x = \sum_{n=1}^{\infty} (b_n + i \cdot a_n) \left(\frac{r}{R_{ref}}\right)^{n-1} \qquad \text{(European and LHC convention)}$$

more physical significance (and more confusion):

$$b_{n+1} = \frac{R_{ref}^n}{n!} \left(\frac{\partial^n B_y}{\partial x^n}\right) \qquad [\ = b_n \ (U.S.)\ ]$$

$$a_{n+1} = \frac{R_{ref}^n}{n!} \left(\frac{\partial^n B_x}{\partial x^n}\right) \qquad [\ = a_n \ (U.S.)\ ]$$

Lorentz force on charged particles

So far: Lorentz force is added to Maxwell equations !

From experience and experiments, can not be derived/understood without Relativity (but then it comes out easily, see lecture !)

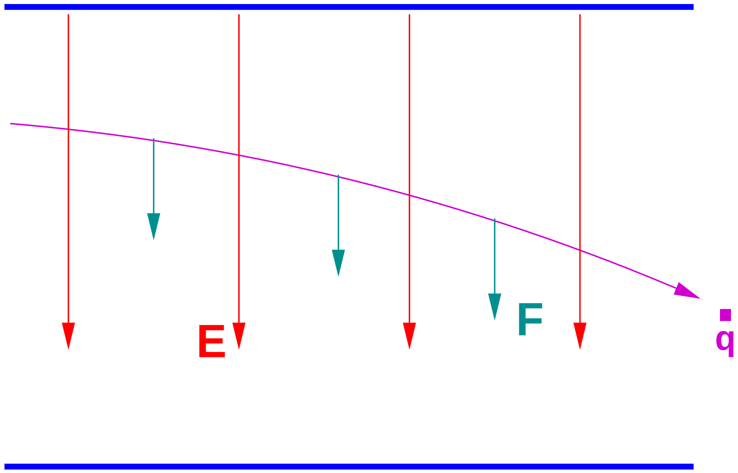
Moving ($\vec{v}$) charged (q) particles in electric ($\vec{E}$) and magnetic ($\vec{B}$) fields experience the Lorentz force $\vec{f}$:

$$\vec{f} \;=\; q \cdot (\vec{E} \;+\; \vec{v} \;\times\; \vec{B})$$

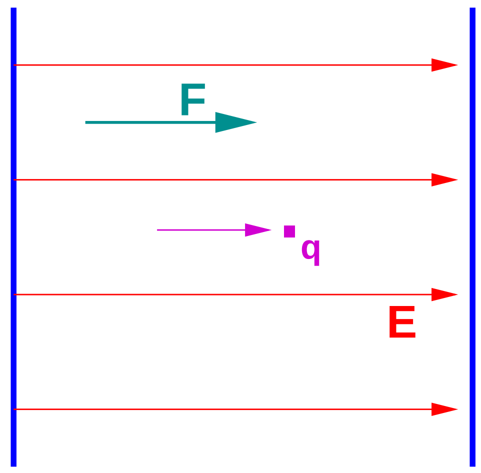
for the equation of motion we get (using Newton's law);

$$\frac{d}{dt}(m\vec{v}) \;=\; \vec{f} \;=\; q \cdot (\vec{E} \;+\; \vec{v} \;\times\; \vec{B})$$

Motion in electric fields



$\vec{v} \perp \vec{E}$



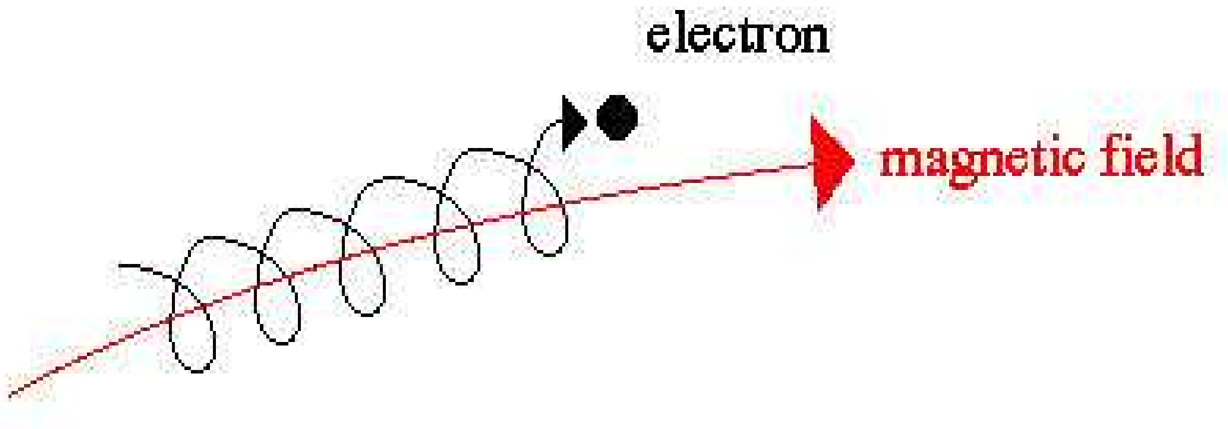
$\vec{v} \parallel \vec{E}$

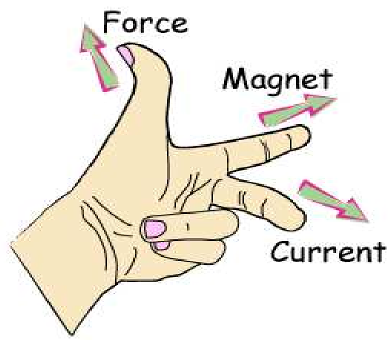
Assume no magnetic field:

$$\frac{d}{dt}(m\vec{v}) = \vec{f} = q \cdot \vec{E}$$

Force always in direction of field $\vec{E}$, also for particles at rest.

Motion in magnetic fields





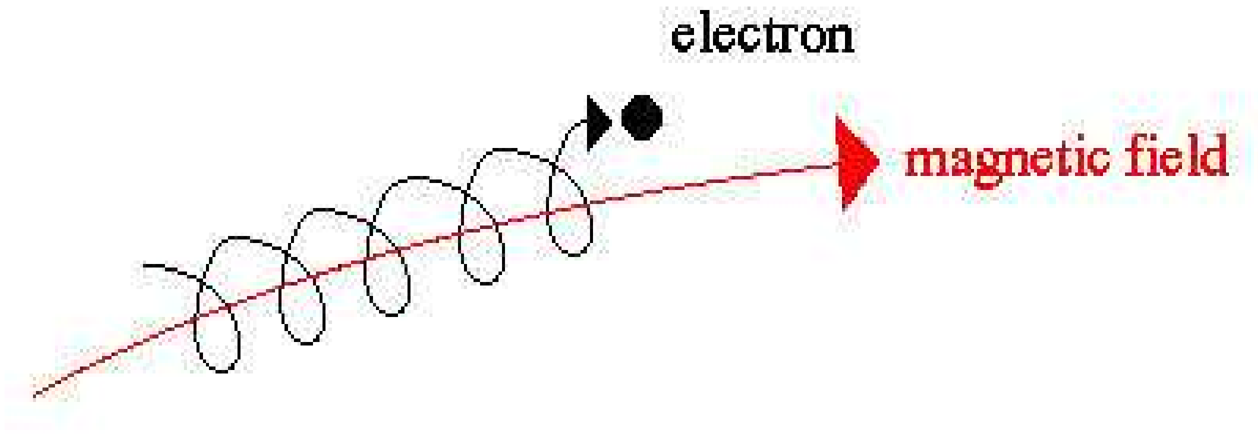
Without electric field : $$\frac{d}{dt}(m\vec{v}) = \vec{f} = q \cdot \vec{v} \times \vec{B}$$

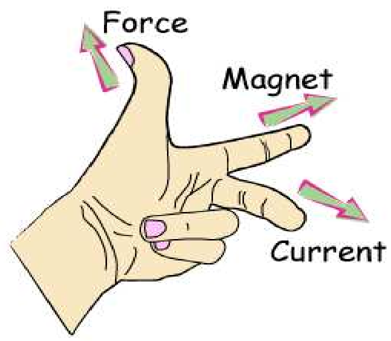
Force is perpendicular to both, $\vec{v}$ and $\vec{B}$

<u>No</u> force on particles at rest - do we understand that ?

Or is it just a empirical story to get the right answer ?

Motion in magnetic fields





Without electric field : $$\frac{d}{dt}(m\vec{v}) \;=\; \vec{f} \;=\; q \cdot \vec{v} \times \vec{B}$$

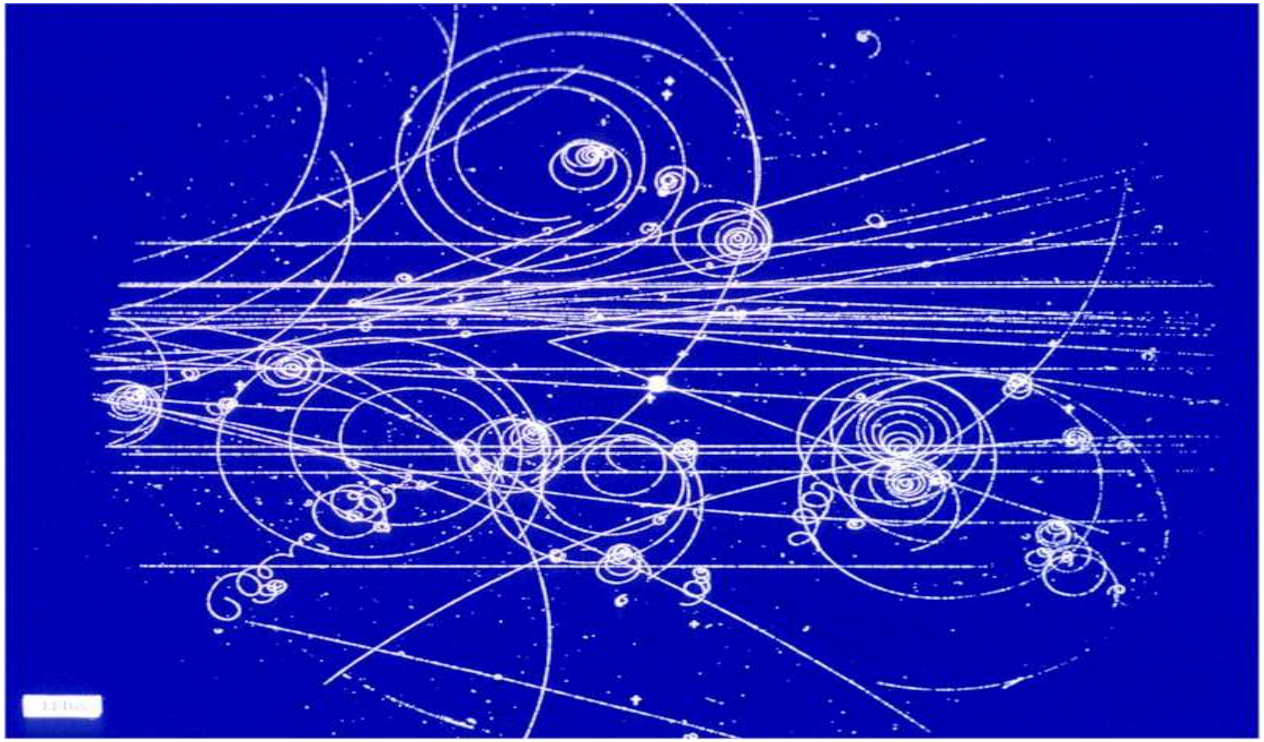
Force is perpendicular to both, $\vec{v}$ and $\vec{B}$

No force on particles at rest - do we understand that ?

Or is it just a empirical story to get the right answer ?

Yes, but see next lecture ...

Important application:



Tracks from particle collisions, lower energy particles have smaller bending radius, allows determination of momenta ..

Q1: what is the direction of the magnetic field ???

Q2: what is the charge of the incoming particle ???

Practical units:

$$B\ [T] \cdot \rho\ [m] \quad = \quad \frac{p\ [eV/c]}{c\ [m/s]}$$

Example LHC:

B = 8.33 T, p = 7^{12} eV/c ➡ **ρ = 2804 m**

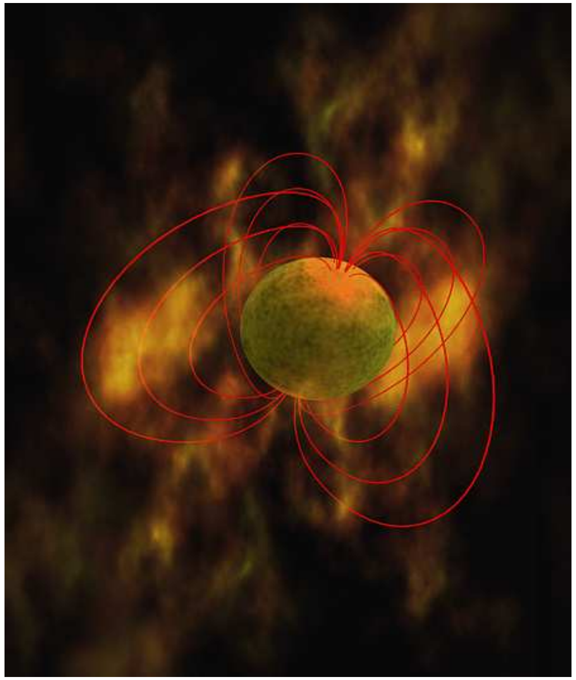
More - bending angle α of a dipole magnet of length L:

$$\alpha \ = \ \frac{B\ [T] \cdot L\ [m] \cdot 0.3}{p\ [GeV/c]}$$

Example LHC:

B = 8.33 T, p = 7000 GeV/c, L = 14.3 m ➡ **α = 5.11 mrad**

... and some really strong magnetic fields



Example : CXOUJ164710.2 – 45516

Diameter : 10 – 20 km

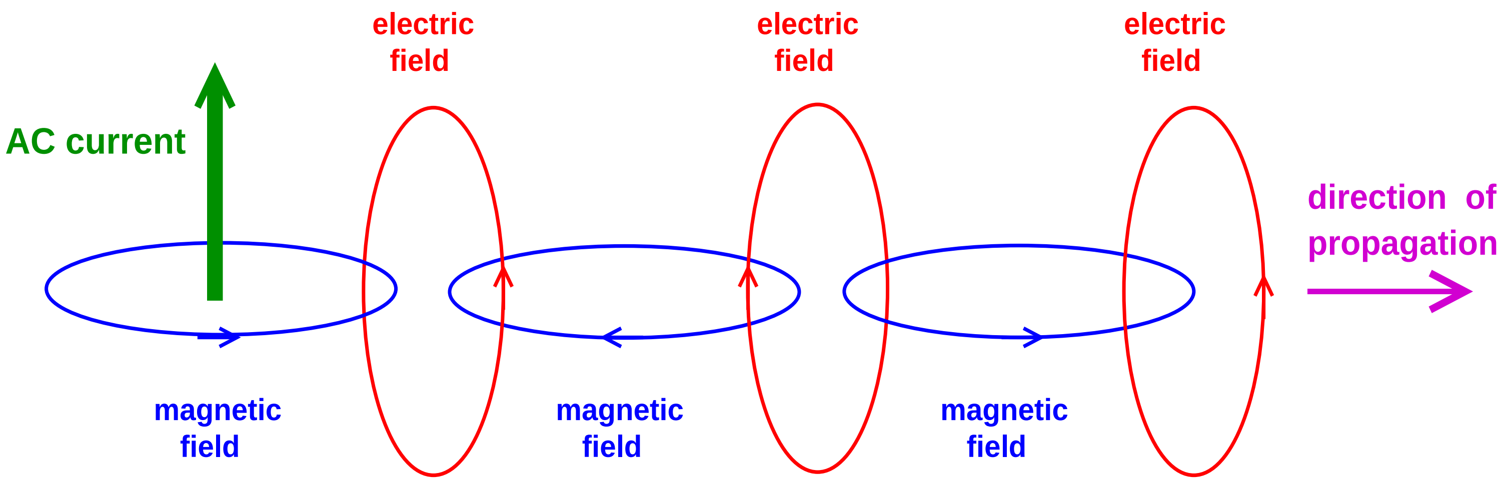
Field : $\approx 10^{12}$ Tesla

As accelerator : $\approx 10^{12}$ TeV

Very fast time varying electromagnetic fields - γ-ray bursts up to 10^{40} W (sun: $\approx 4 \cdot 10^{26}$ W)

Punchline:
Grand Unification (world formula) may show up near 10^{12} TeV

Time Varying Fields - (Maxwell 1864)



Time varying magnetic fields produce circular electric fields

Time varying electric fields produce circular magnetic fields

→ Can produce self-sustaining, propagating fields (i.e. waves)

→ Rather useful picture in the classical framework, but ...

In vacuum: only fields, no charges ($\rho = 0$), no current ($j = 0$) ...

From Maxwell's (III and IV) and some educated guess:

$$\nabla \times (\nabla \times \vec{B}) \;=\; \nabla^2 \vec{E} \;=\; -\nabla \times (\frac{\partial \vec{B}}{\partial t}) \;=\; -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\Longrightarrow \quad \nabla^2 \vec{E} \;=\; \frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \;=\; \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} \;=\; \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{E}}{\partial t^2} \;=\; \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

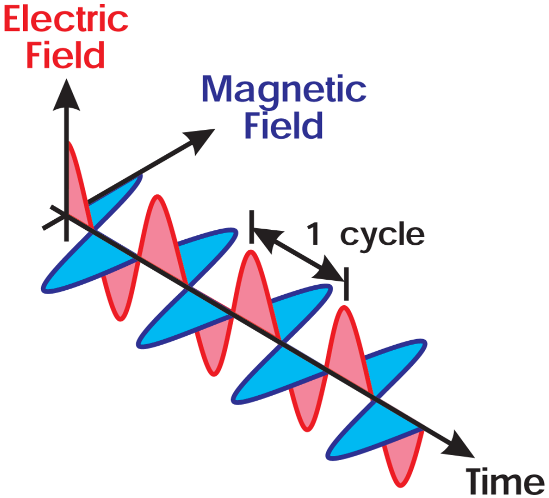
(correspondingly for $\vec{B}$)

→ **Equation for a wave with speed:** $c \;=\; \dfrac{1}{\sqrt{\mu_0 \cdot \epsilon_0}}$

It has a different form for a resting and a moving system !!!
see lecture on Special Relativity why and how it saves the day ...

Electromagnetic waves

$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$
$$\vec{B} = \vec{B}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$



Important quantities :

$\vec{k}$ = (propagation vector)

λ = (wave length, 1 cycle)

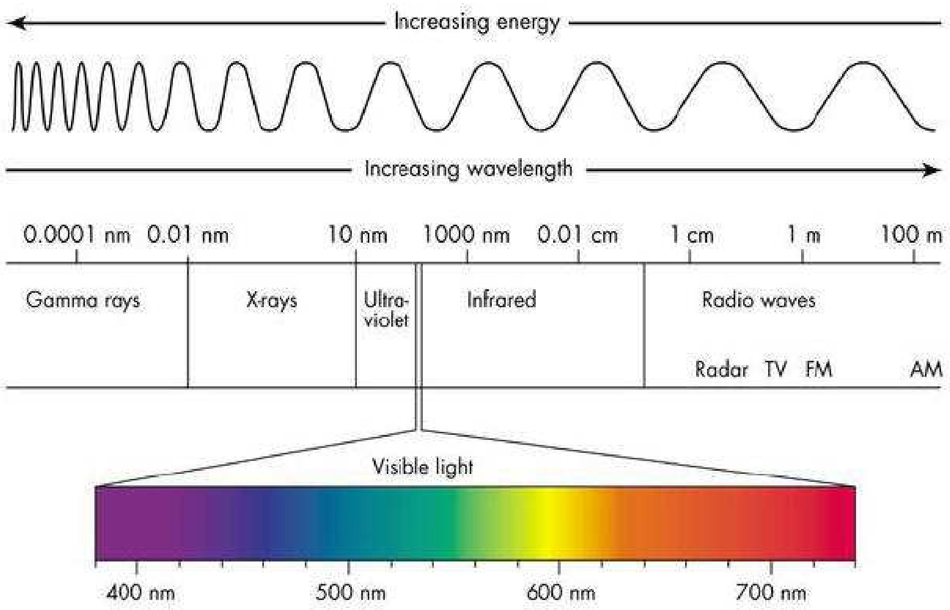
ω = (frequency $\cdot\, 2\pi$)

$\vec{k}\cdot\vec{k} = \dfrac{\omega^2}{c^2}$ (Dispersion Relation)

Magnetic and electric fields are transverse to direction of propagation:
$\vec{E} \perp \vec{B} \perp \vec{k}$

Short wave length $\rightarrow$ **high frequency** $\rightarrow$ **high energy**

Spectrum of Electromagnetic waves



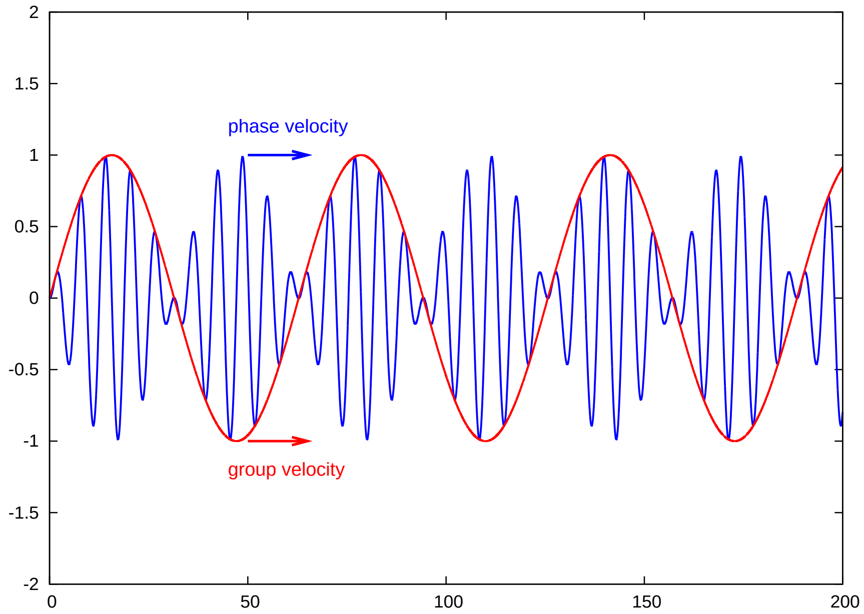
Example: yellow light ➡ $\approx\ 5 \cdot 10^{14}$ **Hz (i.e.** $\approx$ **2 eV !)**

LEP (SR) ➡ $\leq\ 2 \cdot 10^{20}$ **Hz (i.e.** $\approx$ **0.8 MeV !)**

gamma rays ➡ $\geq\ 3 \cdot 10^{20}$ **Hz (1 MeV to 10 TeV !)**

(For estimates using temperature: 2.71 K $\approx$ **0.00023 eV)**

Modulated wave: Group velocity and Phase velocity



carrier wave :

$$v_p = \frac{\omega}{k} \quad \text{(phase velocity)}$$

wave packet :

$$v_g = \frac{\partial \omega}{\partial k} \quad \text{(group velocity)}$$

Wave packet can be considered as superposition of a number of harmonic waves

- **Carrier wave moves at phase velocity, can be larger the c**
- **Wave packet moves at group velocity, carries information**

v_p **and** v_g **may or may not be equal depends on "dispersion relation"**

Energy in electromagnetic waves (in brief, details in [2, 3, 4]):

define Poynting vector : $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ (in direction of propagation)

"Energy flux": energy crossing a unit area, per second $[\frac{J}{m^2 s}]$

In free space: energy is shared between electric and magnetic field

The energy density U **would be:**

$$U_{EB} = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right)$$

Some example: $B = \mathbf{5 \cdot 10^{-5}}$ **T (= 0.5 Gauss)**

➡ $U_B \approx$ **1 mJ/m^3** **(corresponds to** $\underline{\approx 10^{16}\ \gamma s}$ **at visible light)**

➡ **Classical Electrodynamics is a <u>very</u> good approximation ..**

Waves interacting with material

Need to look at the behaviour of electromagnetic fields at boundaries between different materials (air-glass, air-water, vacuum-metal, ...).

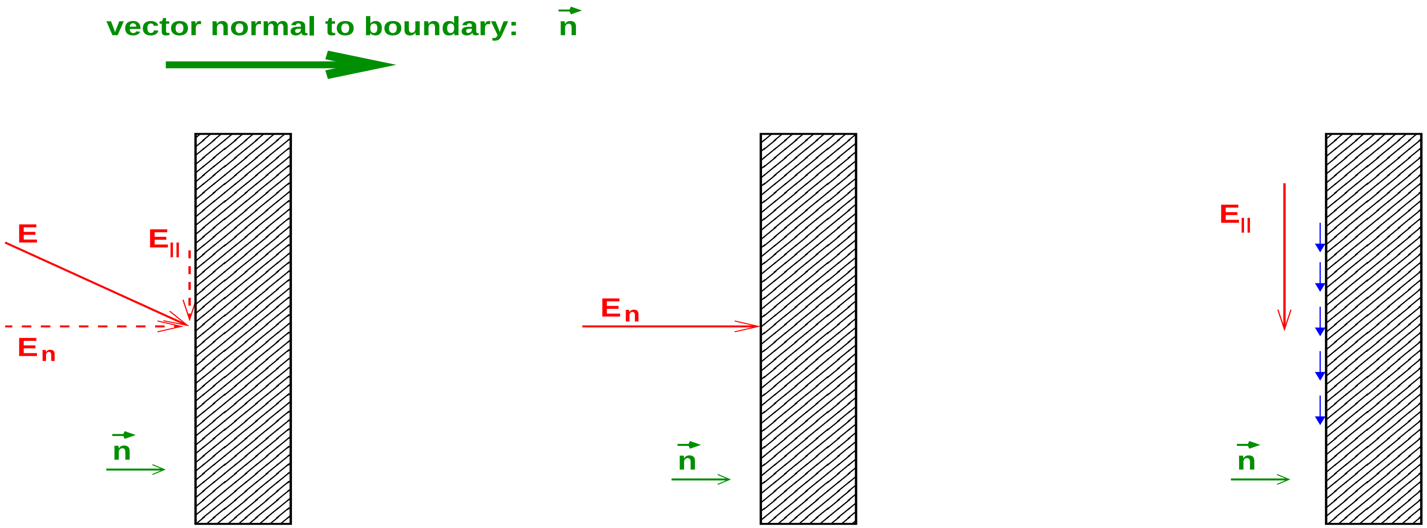
Have to consider two particular cases:

- **Ideal conductor (i.e. no resistance), apply to:**
  - RF cavities
  - Wave guides
- **Conductor with finite resistance, apply to:**
  - Penetration and attenuation of fields in material (skin depth)
  - Impedance calculations

Can be derived from Maxwell's equations, here only the results !

Boundary conditions: air/vacuum and conductor

A simple case ($\vec{E}$-fields on a conducting surface):



Field parallel to surface $E_{\parallel}$ cannot exist (it would move charges and we get a surface current): $E_{\parallel} \ = \ 0$

- **Only a field normal (orthogonal) to surface E_n is possible**

Extreme case: surface of ideal conductor

For an ideal conductor (i.e. no resistance) the tangential electric field must vanish Corresponding conditions for normal magnetic fields. We must have:

$$\vec{E}_t \;=\; 0, \qquad \vec{B}_n \;=\; 0$$

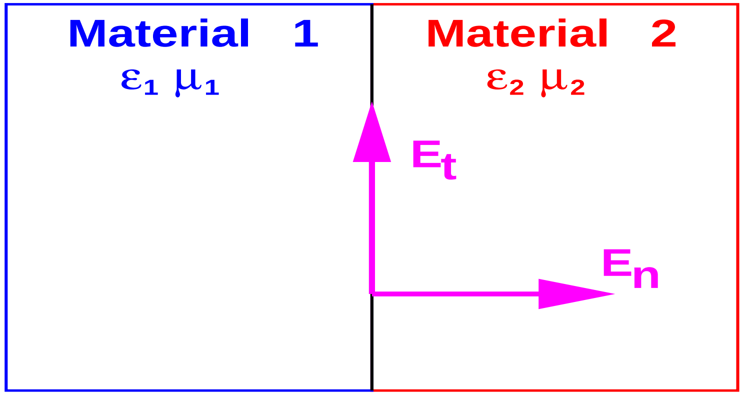
This implies:

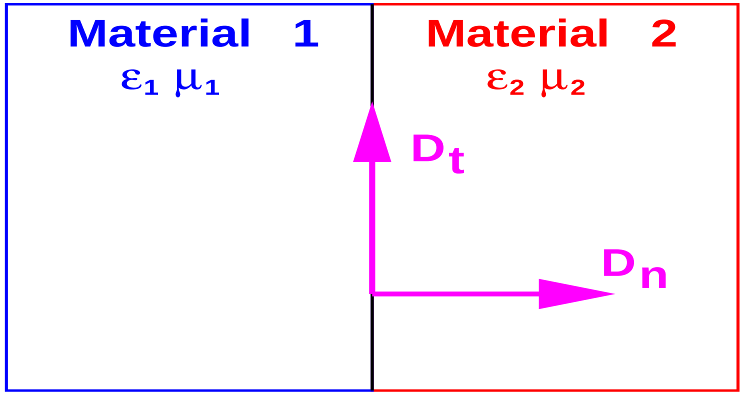
- Fields at any point in the conductor are zero.
- Only some field patterns are allowed in waveguides and RF cavities

A very nice lecture in R.P.Feynman, Vol. II

Now for Boundary Conditions between two different regions →

Boundary conditions for electric fields





Assuming no surface charges (proof e.g. [3, 5])*:

From curl $\vec{E} = 0$**:**

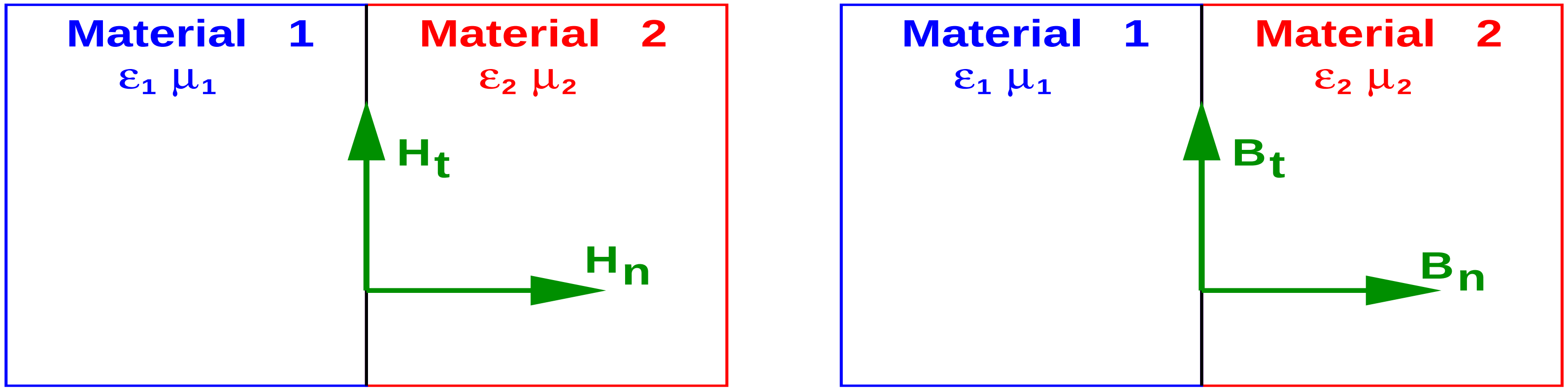
➡ **tangential $\vec{E}$-field continuous across boundary** $(E_t^1 \;=\; E_t^2)$

From div $\vec{D} = \rho$**:**

➡ **normal $\vec{D}$-field continuous across boundary** $(D_n^1 \;=\; D_n^2)$

* **with surface charges, see backup slides**

Boundary conditions for magnetic fields



Assuming no surface currents (proof e.g. [3, 5])*:

From curl $\vec{H} = \vec{j}$**:**

➡ **tangential $\vec{H}$-field continuous across boundary** $(H_t^1 = H_t^2)$

From div $\vec{B} = 0$**:**

➡ **normal $\vec{B}$-field continuous across boundary** $(B_n^1 = B_n^2)$

* with surface current, see backup slides

Summary: boundary conditions for fields

Electromagnetic fields at boundaries between different materials with different permittivity and permeability ($\epsilon_1, \epsilon_2, \mu_1, \mu_2$).

$$(E_t^1 \;=\; E_t^2), \quad (E_n^1 \;\neq\; E_n^2) \qquad \text{or :} \quad (\vec{E}_2 - \vec{E}_1) \;\times\; \vec{n} \;=\; 0$$

$$(D_t^1 \;\neq\; D_t^2), \quad (D_n^1 \;=\; D_n^2) \qquad \text{or :} \quad (\vec{D}_2 - \vec{D}_1) \;\cdot\; \vec{n} \;=\; \rho_{surface}$$

$$(H_t^1 \;=\; H_t^2), \quad (H_n^1 \;\neq\; H_n^2) \qquad \text{or :} \quad (\vec{H}_2 - \vec{H}_1) \;\times\; \vec{n} \;=\; \vec{J}_{surface}$$

$$(B_t^1 \;\neq\; B_t^2), \quad (B_n^1 \;=\; B_n^2) \qquad \text{or :} \quad (\vec{B}_2 - \vec{B}_1) \;\cdot\; \vec{n} \;=\; 0$$

where $\vec{n}$ is the unit vector pointing into medium 2
(derivation deserves its own lectures, just believe it)

They determine: reflection, refraction and refraction index n.

Waves in material → **Index of refraction: n**

Speed of electromagnetic waves in vacuum: $c = \frac{1}{\sqrt{\mu_0 \cdot \epsilon_0}}$

$$\text{n} = \frac{\text{Speed of light in vacuum}}{\text{Speed of light in material}} = \frac{c}{v_p}$$

For water $\text{n} \approx 1.33$

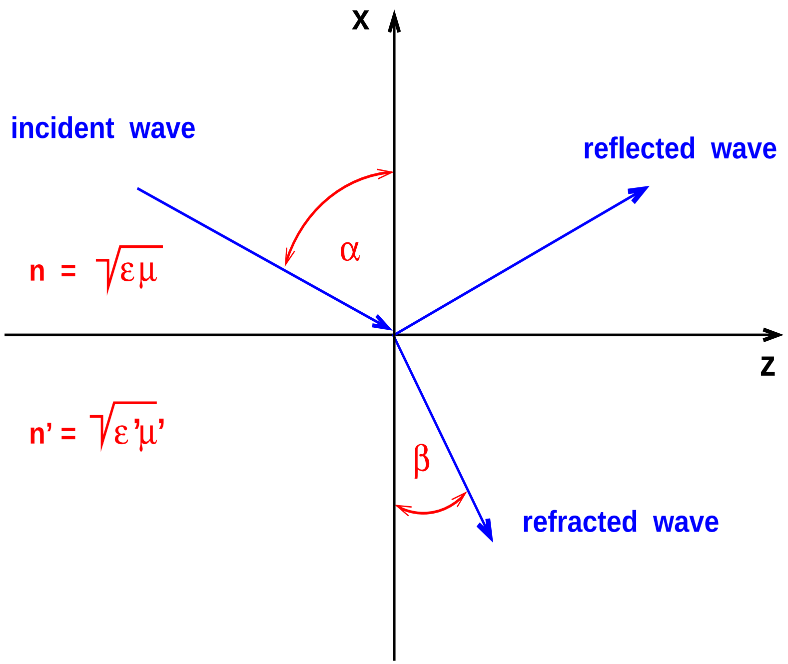
Depends on wavelength

$\text{n} \approx 1.32 - 1.39$

<u>some others</u>	**ice:**	**1.31**
	alcohol:	**1.36**
	sugar solution:	**1.49**
	glass:	**1.51**

(propose an experiment !)

Reflection and refraction angles related to the refraction index n and n':



$$\frac{\sin\alpha}{\sin\beta} = \frac{n'}{n} = \tan\alpha_B$$

If light is incident under angle α_B [3]:
Reflected light is linearly polarized perpendicular to plane of incidence

(Application: fishing → air-water gives $\alpha_B \approx 53^o$)

Polarization of EM waves (Classical Picture !):

The solutions of the wave equations imply monochromatic plane waves:

$$\vec{E} = \vec{E_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{B} = \vec{B_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

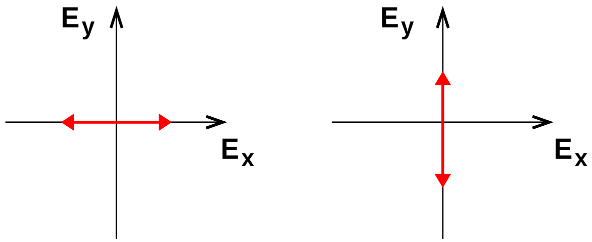
As defined, consider only electric field, re-written using unit vectors in the plane transverse to propagation: $\vec{\epsilon_1} \perp \vec{\epsilon_2}$

Two Components: $\vec{E_1} = \vec{\epsilon_1} E_1 \; e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{E_2} = \vec{\epsilon_2} E_2 \; e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

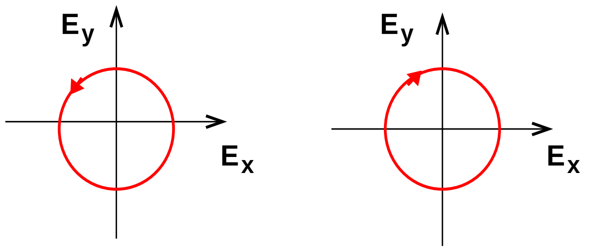
$$\Rightarrow \vec{E} = (\vec{E_1} + \vec{E_2}) = (\vec{\epsilon_1} E_1 + \vec{\epsilon_2} E_2) \; e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

With a relative phase ϕ between the two directions:

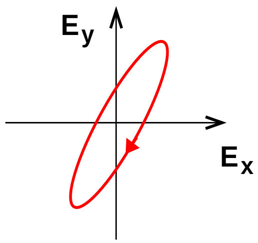
$$\vec{E} = \vec{\epsilon_1} E_1 \; e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \vec{\epsilon_2} E_2 \; e^{i(\vec{k}\cdot\vec{r}-\omega t+\phi)}$$



linear : $E_2 = 0$ or $E_1 = 0$



circular : $\phi \pm \frac{\pi}{2}$



elliptical : ϕ e.g. $\frac{\pi}{4}$

Polarized light - why is it interesting:

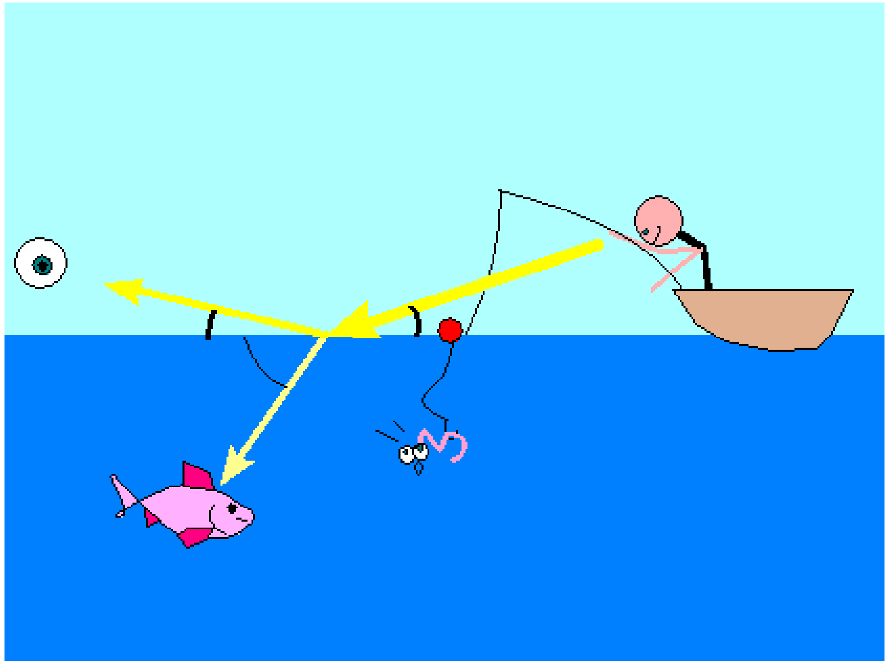
Produced (amongst others) in Synchrotron light machines (linearly and circularly polarized light, adjustable) blue sky (!), reflections (e.g. water surface)

Accelerator and other applications:

- **Polarized light reacts differently with charged particles**
- **Radio communication, capacity doubling**
- **Beam diagnostics, medical diagnostics (blood sugar, ..)**
- **Inverse FEL**
- **3-D motion pictures, LCD display, cameras (reduce glare)**
- **Outdoor activities (e.g. Fishing, driving a car through a rainy night, ...)**
- **...**

Less classical, in Quantum Electro Dynamics: photons with spin +1 or -1

Practical application:



- **Some of the light is reflected**
- **Some of the light is transmitted and refracted**

For α_B, can be largely supressed using polarised glasses

Extreme case: ideal conductor

For an ideal conductor (i.e. no resistance) we must have:

$$\vec{E}_{\|} \ = \ 0, \qquad \vec{B}_n \ = \ 0$$

otherwise the surface current becomes infinite

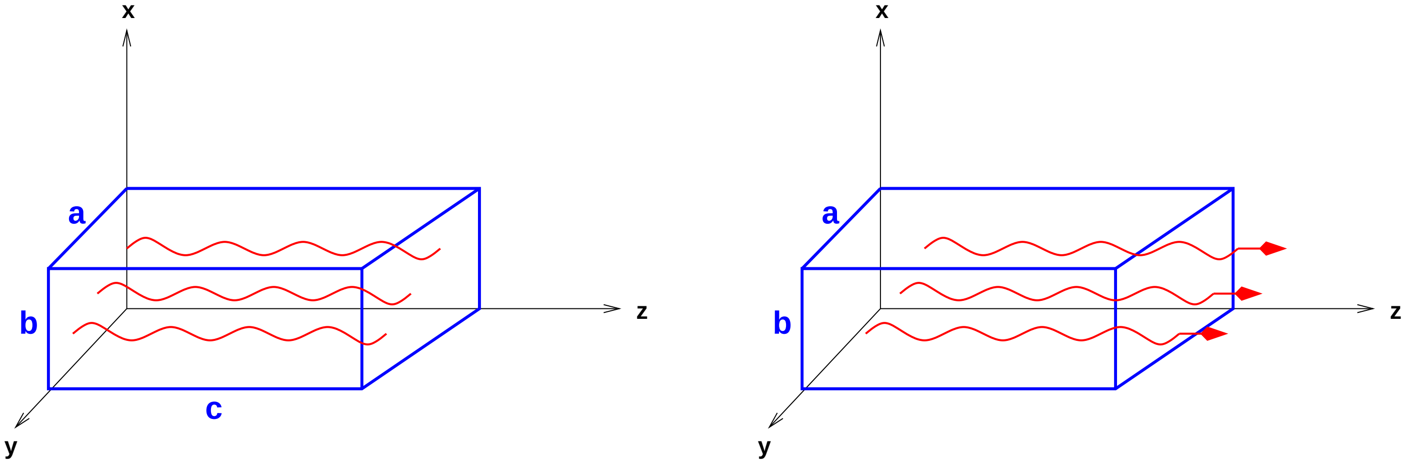
This implies:

- **All energy of an electromagnetic wave is reflected from the surface of an ideal conductor.**
- **Fields at any point in the ideal conductor are zero.**
- **Only some fieldpatterns are allowed in waveguides and RF cavities**

A very nice lecture in R.P.Feynman, Vol. II

Examples: cavities and wave guides

Rectangular, conducting cavities and wave guides (schematic) with dimensions $a \times b \times c$ and $a \times b$:



- **RF cavity, fields can persist and be stored (reflection !)**
- **Plane waves can propagate along wave guides, here in z-direction**

(here just the basics, many details in "RF Systems" by Frank Tecker)

Fields in RF cavities - as reference

Assume a rectangular RF cavity ($a,\ b,\ c$), ideal conductor.

Without derivations, the components of the fields are:

$$E_x = E_{x0} \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$

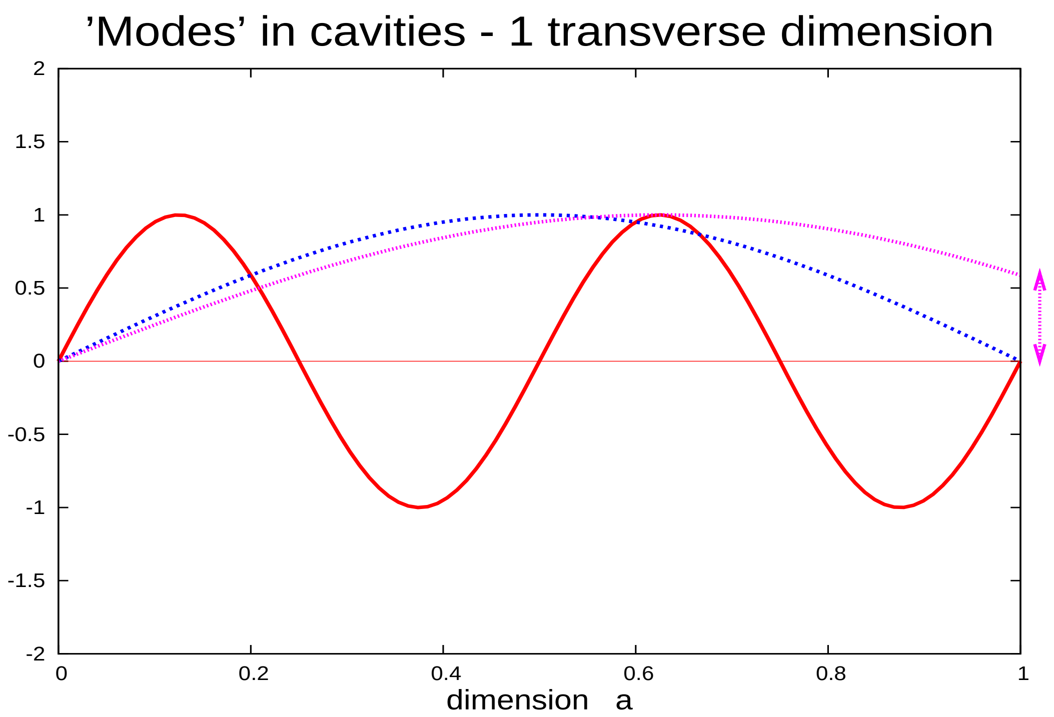
$$E_y = E_{y0} \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$

$$E_z = E_{z0} \cdot \sin(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$

$$B_x = \frac{i}{\omega}(E_{y0} k_z - E_{z0} k_y) \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$

$$B_y = \frac{i}{\omega}(E_{z0} k_x - E_{x0} k_z) \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$

$$B_z = \frac{i}{\omega}(E_{x0} k_y - E_{y0} k_x) \cdot \cos(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$



No electric field at boundaries, wave must have "nodes" = zero fields at the boundaries

Only modes which 'fit' into the cavity are allowed

In the example: $\frac{\lambda}{2} = \frac{a}{4}$, $\quad \frac{\lambda}{2} = \frac{a}{1}$, $\quad \frac{\lambda}{2} = \frac{a}{0.8}$

(then either "sin" or "cos" is 0)

Consequences for RF cavities

Field must be zero at conductor boundary, only possible if:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

and for k_x, k_y, k_z **we can write, (then they all fit):**

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \frac{m_z \pi}{c},$$

The integer numbers m_x, m_y, m_z **are called mode numbers, important for design of cavity !**

$\rightarrow$ **half wave length** $\lambda/2$ **must always fit exactly the size of the cavity.**

(For cylindrical cavities: use cylindrical coordinates)

Wave guides and cavities are more likely to be circular:

Derivation using the Laplace equation in cylindrical coordinates, for a derivation see e.g. [2, 3]:

$$
\begin{aligned}
E_r &= E_0 \, \frac{k_z}{k_r} J_n'(k_r) \cdot \cos(n\theta) \cdot \sin(k_z z) \cdot e^{-i\omega t} \\
E_\theta &= E_0 \, \frac{n k_z}{k_r^2 r} J_n(k_r) \cdot \sin(n\theta) \cdot \sin(k_z z) \cdot e^{-i\omega t} \\
E_z &= E_0 \, J_n(k_r r) \cdot cos(n\theta) \cdot sin(k_z z) \cdot e^{-i\omega t}
\end{aligned}
$$

$$
\begin{aligned}
B_r &= iE_0 \, \frac{\omega}{c^2 k_r^2 r} J_n(k_r r) \cdot \sin(n\theta) \cdot \cos(k_z z) \cdot e^{-i\omega t} \\
B_\theta &= iE_0 \, \frac{\omega}{c^2 k_r r} J_n'(k_r r) \cdot \cos(n\theta) \cdot \cos(k_z z) \cdot e^{-i\omega t} \\
B_z &= 0
\end{aligned}
$$

Homework: write it down for wave guides ..

Accelerating circular cavities

For accelerating cavities we need longitudinal electric field component $E_z \neq 0$ and magnetic field purely transverse.

$$
\begin{aligned}
E_r &= 0 \\
E_\theta &= 0 \\
E_z &= E_0 \; J_0(p_{01}\frac{r}{R}) \cdot e^{-i\omega t}
\end{aligned}
$$

$$
\begin{aligned}
B_r &= 0 \\
B_\theta &= -iE_0 \; J_1(p_{01}\frac{r}{R}) \cdot e^{-i\omega t} \\
B_z &= 0
\end{aligned}
$$

(p_{nm} is the mth zero of J_n, e.g. $p_{01} \approx$ 2.405)

This would be a cavity with a TM$_{010}$ mode: $\omega_{010} = p_{01} \cdot \frac{c}{R}$

Similar considerations lead to (propagating) solutions in (rectangular) wave guides:

$$E_x = E_{x0} \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$E_y = E_{y0} \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$E_z = i \cdot E_{z0} \cdot \sin(k_x x) \cdot \sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_x = \frac{1}{\omega}(E_{y0} k_z - E_{z0} k_y) \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_y = \frac{1}{\omega}(E_{z0} k_x - E_{x0} k_z) \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_z = \frac{1}{i \cdot \omega}(E_{x0} k_y - E_{y0} k_x) \cdot \cos(k_x x) \cdot \cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

This part is new: $e^{i(k_z z)}$ $\Longrightarrow$ **something moving in z direction**

In z direction: No Boundary - No Boundary Condition ...

Consequences for wave guides

Similar considerations as for cavities, no field at boundary.

We must satisfy again the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \qquad \Longrightarrow \qquad k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

This leads to modes like (no boundaries in direction of propagation, only k_x and k_y have to "fit"):

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{m_x \pi}{a}\right)^2 - \left(\frac{m_y \pi}{b}\right)^2}$$

The numbers m_x, m_y are called mode numbers for waves in wave guides !

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Should we worry about k_z ?? Argument of root can become negative .. !

Propagation without losses requires k_z to be real, i.e.:

$$\frac{\omega^2}{c^2} > \left(\frac{m_x \pi}{a}\right)^2 + \left(\frac{m_y \pi}{b}\right)^2 \quad \rightarrow \quad \boxed{\omega > \pi c \sqrt{\left(\frac{m_x}{a}\right)^2 + \left(\frac{m_y}{b}\right)^2}}$$

defines a (minimum) cut-off frequency ω_c:

$$\boxed{\omega_c = \frac{\pi \cdot c}{a}}$$

(with a the longest side length of the wave guide)

- **Above cut-off frequency: propagation without loss**
- **At cut-off frequency: standing wave**
- **Below cut-off frequency: attenuated wave (it does not "fit in").**

(if bored : try to compute v_p and v_g for the limits : $\omega \rightarrow \infty$ and $\omega \rightarrow \omega_c$)

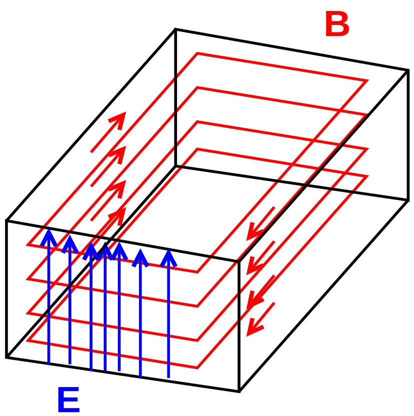
Classification of modes:

Transverse electric modes (TE): $E_z = 0 \quad H_z \neq 0$

Transverse magnetic modes (TM): $E_z \neq 0 \quad H_z = 0$

Transverse electric-magnetic modes (TEM): $E_z = 0 \quad H_z = 0$

(Not all of them can be used for acceleration ... !)



Note (here a TE mode) :

Electric field lines end at boundaries

Magnetic field lines appear as "loops"

Other case: finite conductivity

Starting from Maxwell equation:

$$\nabla \times \vec{B} = \mu \vec{j} \; + \; \mu\epsilon \frac{d\vec{E}}{dt} \; = \; \underbrace{\overbrace{\sigma \cdot \vec{E}}^{\vec{j}}}_{Ohm's\ law} + \mu\epsilon \frac{d\vec{E}}{dt}$$

Wave equations:

$$\vec{E} \; = \; \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}, \qquad \vec{B} \; = \; \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

We want to know k with this new contribution:

$$k^2 \; = \; \frac{\omega^2}{c^2} \; - \; \underbrace{i\omega\sigma\mu}_{new}$$

Consequence → Skin Depth

Electromagnetic waves can now penetrate <u>into</u> the conductor !

For a good conductor $\sigma \gg \omega\epsilon$:

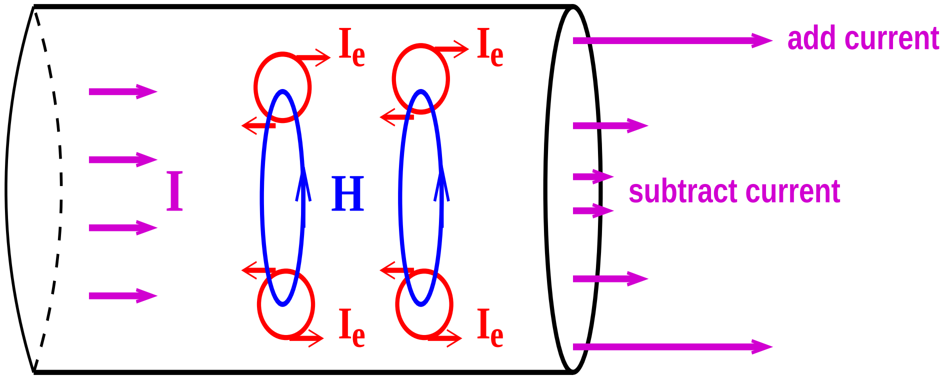
$$k^2 \approx -i\omega\mu\sigma \quad \rightarrow \quad k \approx \sqrt{\frac{\omega\mu\sigma}{2}}(1+i) = \frac{1}{\delta}(1+i)$$

$$\boxed{\delta = \sqrt{\frac{2}{\omega\mu\sigma}}}$$

is the <u>Skin Depth</u>

- **High frequency waves "avoid" penetrating into a conductor, flow near the surface**
- **Penetration depth small for large conductivity**

"Explanation" - inside a conductor (very schematic)



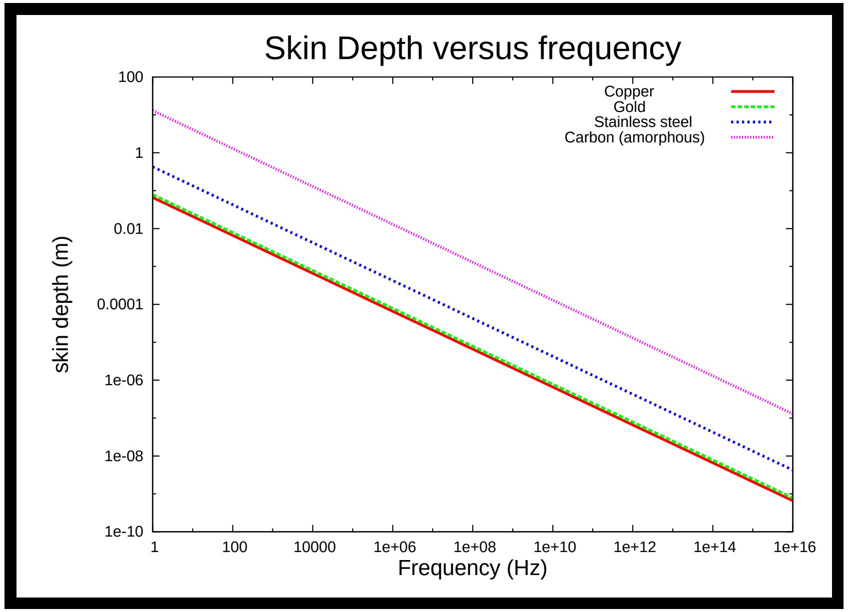
Eddy currents I_E **from changing** $\vec{H}$**-field:** $\nabla \times \vec{E} = \mu_0 \frac{d\vec{H}}{dt}$

Cancel current flow in the centre of the conductor $\mathbf{I} - \mathbf{I_e}$

Enforce current flow near the "skin" (surface) $\mathbf{I} + \mathbf{I_e}$

Q: Why are high frequency cables thin ??

Attenuated waves - penetration depth



Waves incident on conducting material are attenuated

Basically by the Skin depth :

(attenuation to 1/e)

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Wave form:

$$e^{i(kz-\omega t)} = e^{i((1+i)z/\delta-\omega t)} = e^{\frac{-z}{\delta}} \cdot e^{i(\frac{z}{\delta}-\omega t)}$$

Values of δ can have a very large range ..

- **Skin depth Copper ($\sigma \approx 6 \cdot 10^7$ S/m):**

  2.45 GHz: $\delta \approx$ 1.5 μm, 50 Hz: $\delta \approx$ 10 mm

- **Penetration depth Glass (strong variation, σ typically $6 \cdot 10^{-13}$ S/m):**

  2.45 GHz: $\delta >$ km

- **Penetration depth Sushi (strong variation, σ typically $3 \cdot 10^{-2}$ S/m):**

  2.45 GHz: $\delta \approx$ 6 cm

- **Penetration depth Seawater ($\sigma \approx 4$ S/m):**

  76 Hz: $\delta \approx$ 25 - 30 m (Design an antenna !!, very low bandwidth)

Done list:

1. **Review of basics and write down Maxwell's equations**
2. **Electromagnetic fields in vacuum and in material**
3. **Add Lorentz force and motion of particles in EM fields**
4. **Electromagnetic waves in vacuum**
5. **Electromagnetic waves in conducting media**
   - **Waves in RF cavities**
   - **Waves in wave guides**
   - **Important concepts: mode numbers, cut-off frequency, skin depth**

But still a few (important) problems to sort out →

What next:

- We have to deal with moving charges in accelerators
- Applied to moving charges Maxwell's equations are not compatible with observations of electromagnetic phenomena
- Electromagnetism and laws of classical mechanics are inconsistent
- Ad hoc introduction of Lorentz force
- Electromagnetic "wave" concept is fuzzy: no medium (more in lecture on Special Relativity)

Needed: sort in out in a systematic framework

The fix: Special Relativity (next lecture)

- Formulate Maxwell's equations relativistically invariant
- Problems solved (easily !) ...

- BACKUP SLIDES -

Relativity and electrodynamics

- Back to the start: electrodynamics and Maxwell equations
- Life made easy with four-vectors ..
- Strategy: one $+$ three

Write potentials and currents as four-vectors:

$$\Psi, \ \vec{A} \quad \Rightarrow \quad A^{\mu} \ = \ (\frac{\Psi}{c}, \ \vec{A})$$

$$\rho, \ \vec{j} \quad \Rightarrow \quad J^{\mu} \ = \ (\rho \cdot c, \ \vec{j})$$

What about the transformation of current and potentials ?

Transform the four-current like:

$$\begin{pmatrix} \rho' c \\ j'_x \\ j'_y \\ j'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho c \\ j_x \\ j_y \\ j_z \end{pmatrix}$$

It transforms via: $J'^{\mu} = \Lambda \; J^{\mu}$ **(always the same** Λ**)**

Ditto for: $A'^{\mu} = \Lambda \; A^{\mu}$ **(always the same** Λ**)**

Note: $\partial_\mu J^\mu = \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\vec{j} = 0$ **(charge conservation)**

Electromagnetic fields using potentials:

Magnetic field: $\vec{B} = \nabla \times \vec{A}$

e.g. the x-component:

$$B_x = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

Electric field: $\vec{E} = -\nabla \Psi - \frac{\partial \vec{A}}{\partial t}$

e.g. for the x-component:

$$E_x = -\frac{\partial A_0}{\partial x} - \frac{\partial A_1}{\partial t} = -\frac{\partial A_t}{\partial x} - \frac{\partial A_x}{\partial t}$$

→ **after getting all combinations ..**

Electromagnetic fields described by field-tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} = \begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

It transforms via: $F'^{\mu\nu} = \Lambda \; F^{\mu\nu} \; \Lambda^T$ **(same Λ as before)**

(Warning: There are different ways to write the field-tensor $F^{\mu\nu}$,
I use the convention from [1, 3, 5])

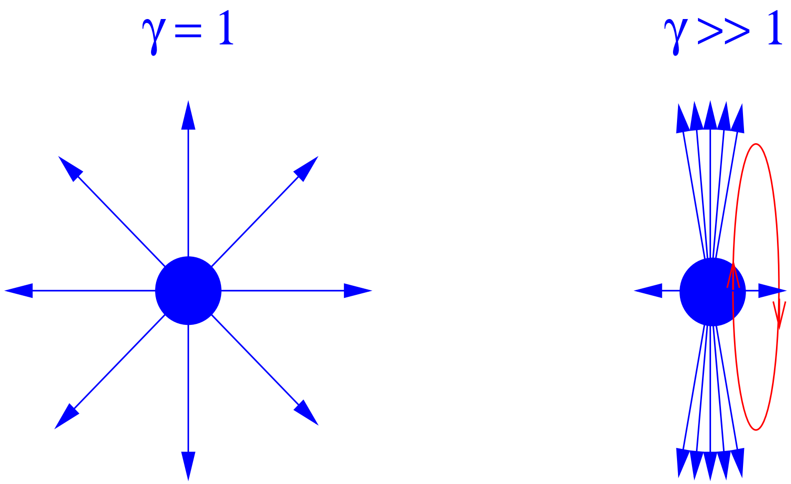
Transformation of fields into a moving frame (x-direction):

Use Lorentz transformation of $F^{\mu\nu}$ and write for components:

$$
\begin{aligned}
E'_x &= E_x & B'_x &= B_x \\
E'_y &= \gamma(E_y - v \cdot B_z) & B'_y &= \gamma(B_y + \tfrac{v}{c^2} \cdot E_z) \\
E'_z &= \gamma(E_z + v \cdot B_y) & B'_z &= \gamma(B_z - \tfrac{v}{c^2} \cdot E_y)
\end{aligned}
$$

Fields perpendicular to movement are transformed

Example Coulomb field: (a charge moving with constant speed)



- In rest frame purely electrostatic forces
- In moving frame $\vec{E}$ transformed and $\vec{B}$ appears

How do the fields look like ?

Needed to compute e.g. radiation of a moving charge, wake fields, ...

For the static charge we have the Coulomb potential (see lecture on Electrodynamics) and $\vec{A} = 0$

Transformation into the new frame (moving in x-direction) with our transformation of four-potentials:

$$\frac{\Psi'}{c} = \gamma(\frac{\Psi}{c} - A_x) = \gamma\frac{\Psi}{c}$$

$$A'_x = \gamma\left(A_x - \frac{v\Psi}{c^2}\right) = -\gamma\frac{v}{c^2}\Psi = -\frac{v}{c^2}\Psi'$$

i.e. all we need to know is Ψ' $\Rightarrow$

$$\Psi'(\vec{r}) = \gamma\Psi(\vec{r}) = \gamma \cdot \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{|\vec{r} - \vec{r}_q|}$$

After transformation of coordinates, e.g. $x = \gamma(x' - vt')$
The resulting potentials can be used to compute the fields.

Watch out !!

We have to take care of causality:
The field observed at a position $\vec{r}$ at time t was caused at an **earlier** **time $t_r < t$ at the location $\vec{r}_0(t_r)$**

$$\Psi(\vec{r},t) \;=\; \frac{qc}{|\vec{R}|c - \vec{R}\vec{v}} \qquad \vec{A}(\vec{r},t) \;=\; \frac{q\vec{v}}{|\vec{R}|c - \vec{R}\vec{v}}$$

The potentials $\Psi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ depend on the state at **retarted** **time t_r, not t**
$\vec{v}$ is the velocity at time t_r and $\vec{R} \;=\; \vec{r} - \vec{r}_0(t_r)$ relates the retarted position to the observation point.

Q: Can we also write a Four-Maxwell ?

Re-write Maxwell's equations using four-vectors and $F^{\mu\nu}$:

$$\nabla \vec{E} = \frac{\rho}{\epsilon_0} \quad \textbf{and} \quad \nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \quad \xrightarrow{1+3}$$

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \quad \text{(Inhomogeneous Maxwell equation)}$$

$$\nabla \vec{B} = 0 \quad \textbf{and} \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \xrightarrow{1+3}$$

$$\partial_\gamma F^{\mu\nu} + \partial_\mu F^{\nu\lambda} + \partial_\nu F^{\lambda\mu} = 0 \quad \text{(Homogeneous Maxwell equation)}$$

We have Maxwell's equation in a very compact form, transformation between moving systems very easy

How to use all that stuff ??? **Look at first equation:**

$$\partial_\mu F^{\mu\nu} \;=\; \mu_0 J^\nu$$

Written explicitly (Einstein convention, sum over μ):

$$\partial_\mu F^{\mu\nu} \;=\; \sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} \;=\; \partial_0 F^{0\nu} \;+\; \partial_1 F^{1\nu} \;+\; \partial_2 F^{2\nu} \;+\; \partial_3 F^{3\nu} \;=\; \mu_0 J^\nu$$

Choose e.g. $\nu = 0$ and replace $F^{\mu\nu}$ by corresponding elements:

$$\partial_0 F^{00} \quad + \partial_1 F^{10} \quad + \partial_2 F^{20} \quad + \partial_3 F^{30} \quad = \quad \mu_0 J^0$$

$$0 \quad + \partial_x \frac{E_x}{c} \quad + \partial_y \frac{E_y}{c} \quad + \partial_z \frac{E_z}{c} \quad = \quad \mu_0 J^0 \quad = \quad \mu_0 \; c\rho$$

This corresponds exactly to:

$$\vec{\nabla} \cdot \; \vec{E} \;=\; \frac{\rho}{\epsilon_0} \qquad\qquad (c^2 \;=\; \epsilon_0 \mu_0)$$

➡ **For $\nu = 1, 2, 3$ you get Ampere's law**

For example in the x-plane ($\nu = 1$) and the S frame:

$$\partial_y B_z - \partial_z B_y - \partial_t \frac{E_x}{c} = \mu_0 J^x$$

after transforming ∂^γ and $F^{\mu\nu}$ to the S' frame:

$$\partial'_y B'_z - \partial'_z B'_y - \partial'_t \frac{E'_x}{c} = \mu_0 J'^x$$

Now Maxwell's equation have the identical form in S and S'

(In matter: can be re-written with $\vec{D}$ and $\vec{H}$ using "magnetization tensor")

Finally: since $$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

$$\partial_\gamma F^{\mu\nu} + \partial_\mu F^{\nu\lambda} + \partial_\nu F^{\lambda\mu} = 0$$

We can re-write them two-in-one in a new form:

$$\boxed{\frac{\partial^2 A^\mu}{\partial x_\nu \partial x^\nu} = \mu_0 J^\mu}$$

This contains all four Maxwell's equations, and the only one which stays the same in all frames !!

There are <u>no</u> separate electric and magnetic fields, just a frame dependent manifestation of a single electromagnetic field

Quite obvious in Quantum ElectroDynamics !