Particle Motion in Hamiltonian Formalism
Lecture 1

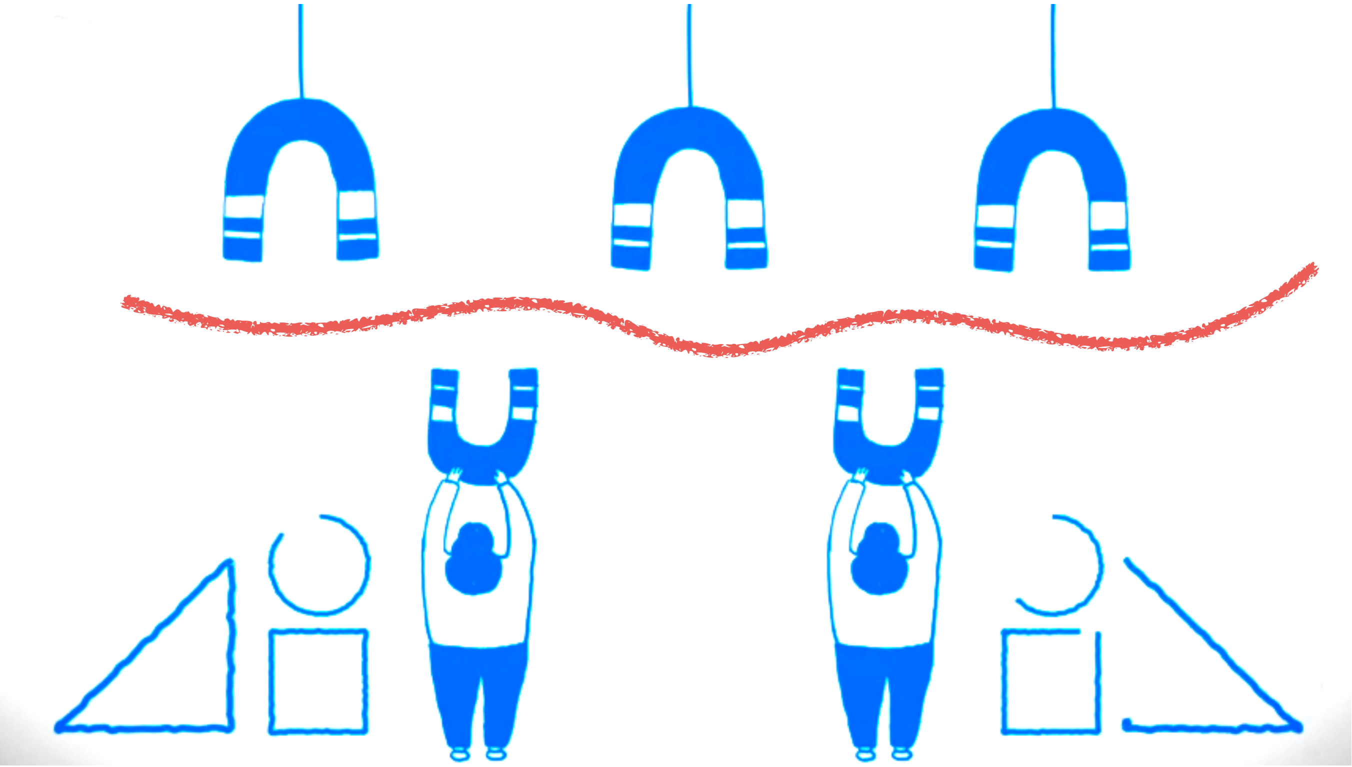

Image: Andrew Khosravani, 2016

http://richannel.org/collections/2016/particle-accelerators-for-humanity

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Where are we now?

So far, you should already have learned about:

- Some types of accelerators
- Intro to types of magnets and basic beam parameters
- Electromagnetic fields
- Maxwell's equations
- The Lorentz force
- Introduction to linear optics

We have started looking at how to describe particle motion in accelerators, now we will look at this from a different angle.

Aims for this lecture

These lectures cover the question "WHY? Do we design, study and understand particle accelerators in the way we do?"

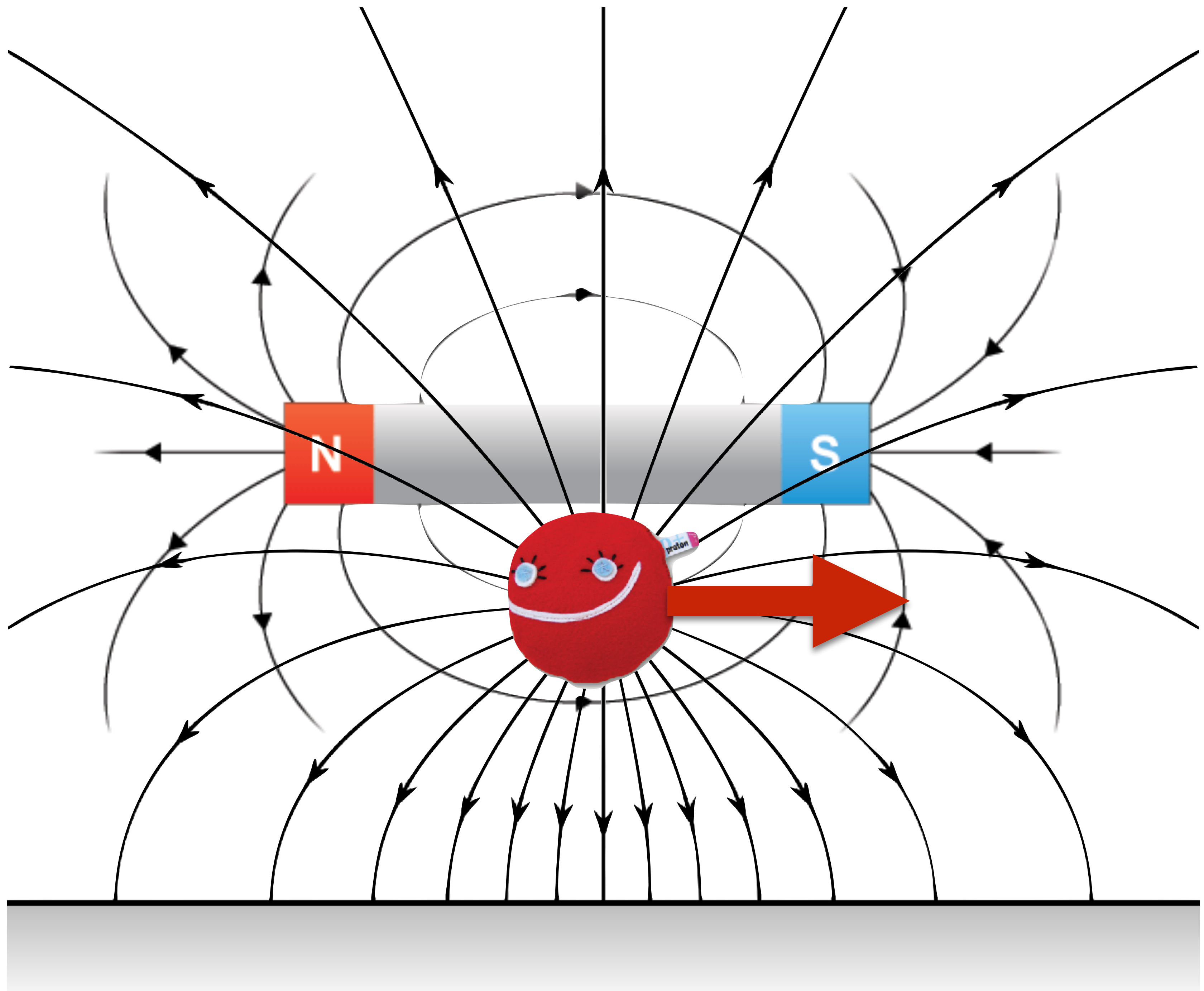
With a conceptual understanding
(i.e. without the full mathematics):

1. Understand the framework used to design or study an accelerator
2. Develop general description of particle motion in Hamiltonian framework
2. Understand what "maps" are, and how they relate to particle motion and simulation
3. Derive some basic maps from the equations of motion

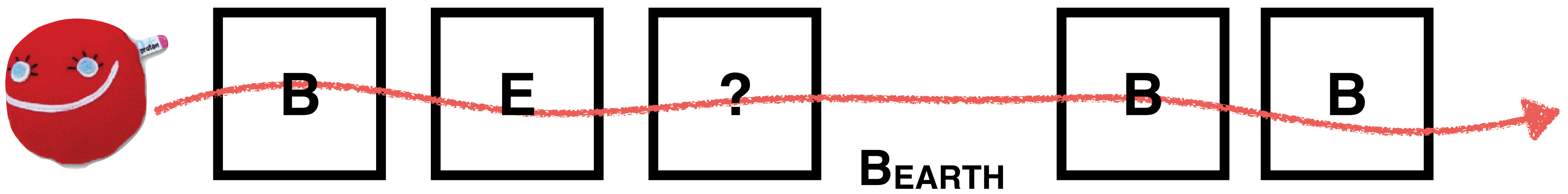
Our trajectory…

- **Lecture 1: Where do we begin?**
  - How do we describe the motion?
  - Approach: Hamiltonian dynamics
  - Setting up: Reference trajectory & variables and reference frame
  - Which EM fields do we need to consider (at first)?
- **Lecture 2: We now have the equations & the fields… what next?**
  - “Maps” and “matrices” transporting co-ordinates in EM fields
  - Derive maps for drift, dipole, quadrupole etc…
  - Adding the pieces together: getting to Hills equation of motion
  - Hints of non-linear dynamics and harder stuff to come later…

Particle in an EM field
How can we describe the motion?

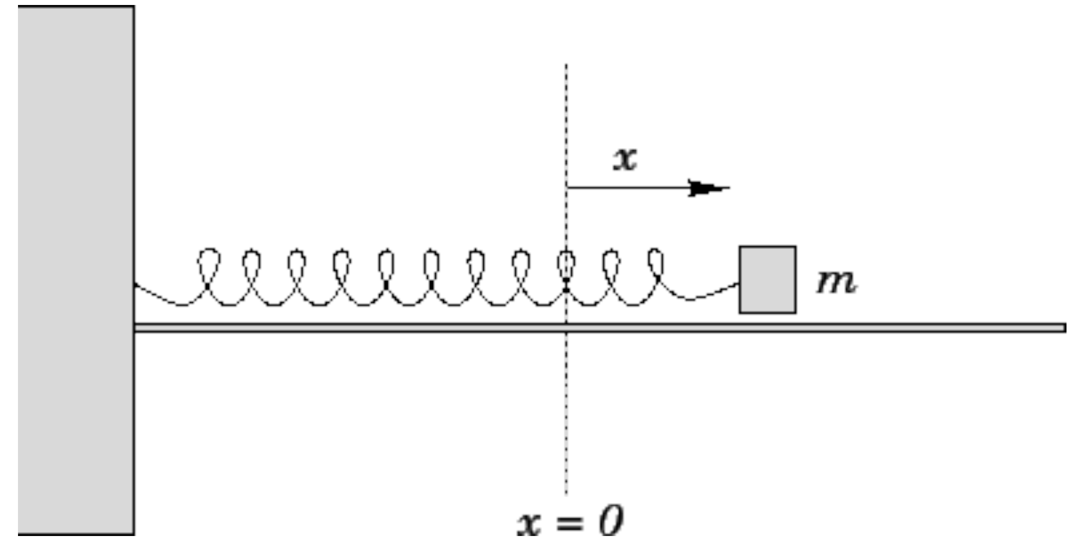


How can we describe the motion?



A simple example, how to solve?

- Take a spring, with a mass on it.
- Set it oscillating.
- Describe the subsequent motion.



Easy, right? We can start with $\mathbf{F} = ma$

We know 'Hooke's law' gives the restoring force from the spring: $\mathbf{F} = -kx$

And thus we can equate the two: $m\ddot{x} = -kx$

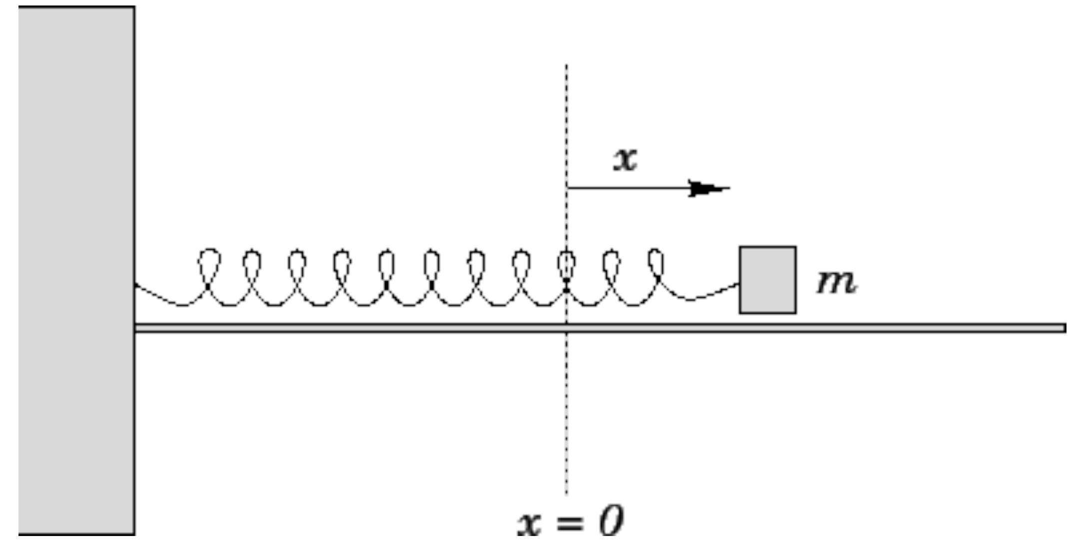
The solution (known for ages) is: $x = a\cos(\omega t - \phi)$

$\omega = \sqrt{\frac{k}{m}}$

Image: http://farside.ph.utexas.edu/teaching/315/Waves/node3.html

Another option… the Lagrangian

$$L = T - V$$



Kinetic energy: $T = \frac{1}{2}m\dot{x}^2$

Potential energy: $V = \frac{1}{2}kx^2$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

How to proceed? Use the Euler-Lagrange equation (don't worry if you don't remember or haven't done this…)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow \frac{d}{dt}(m\dot{x}) = -kx \rightarrow \boxed{m\ddot{x} = -kx}$$

same as before!

Third option… the Hamiltonian

First, we need to make sure we are using 'canonically conjugate co-ordinates'. Bit tricky, but we can find them using L from before:

$$H = T + V$$

often, but not always…

$$p = \frac{\partial L}{\partial \dot{x}}$$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad \rightarrow \quad p = m\dot{x}$$

from before…

$$H = \sum_i \dot{q}_i p_i - L \quad \rightarrow \quad H = \dot{x}p - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Write in terms of x, p (always for H!)

$$\rightarrow \quad \boxed{H = \frac{p^2}{2m} + \frac{1}{2}kx^2}$$

So in this case, the Hamiltonian is just the sum of kinetic & potential energies

Third option… the Hamiltonian (2/2)

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Now instead of Euler-Lagrange equations, we use Hamilton's equations to get the equations of motion:

$$\frac{\partial x}{\partial t} = \frac{\partial H}{\partial p} \qquad \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial x}$$

$$\rightarrow \quad \frac{\partial x}{\partial t} = \frac{p}{m} \qquad \frac{\partial p}{\partial t} = -kx$$

And we get 2, first order equations of motion.

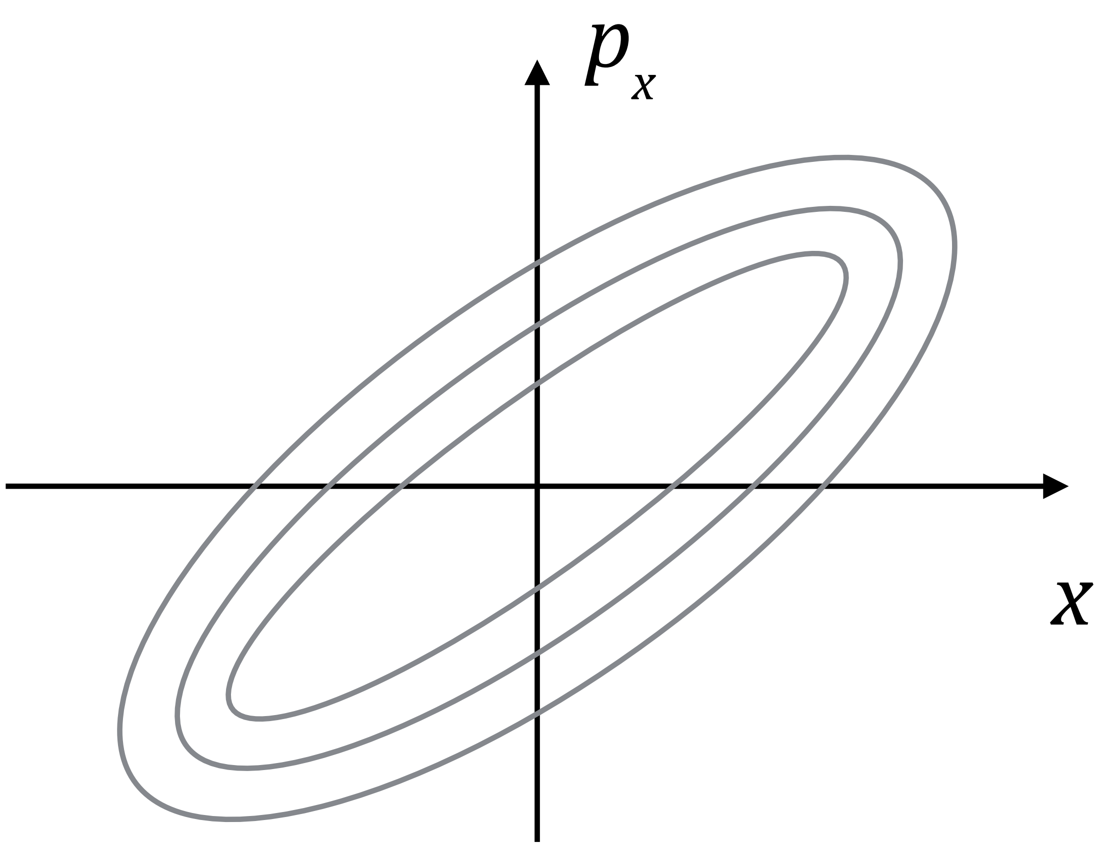
- The right is equivalent to Newton's laws for this system.
- But we started from thinking about the ENERGY of the system, without knowing the Forces applied.
- Note that H does not depend explicitly on t, so it is conserved
- Looks like it takes more effort? But provides rigorous framework!

Phase space

The most relevant ‘canonically conjugate’ variables for us are:

x (space) and p (momentum)

E (energy) and t (time)

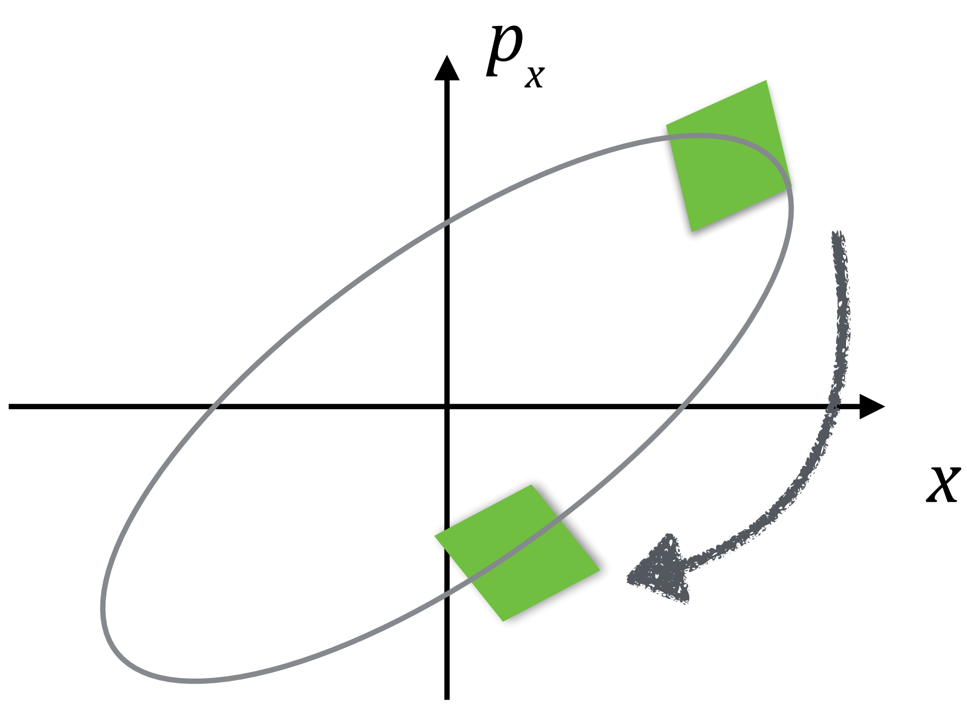


- As time progresses, particles trace out ellipses in phase space.
- Particles with larger ‘amplitude’ trace out a larger ellipse

In accelerator physics, we often hear the phrase ‘phase space’ where someone really means ‘trace space’ (x,x’) where x’=dx/ds is an angle. The two are subtly different. Technically, we should use ‘phase space’ (x, p_x).

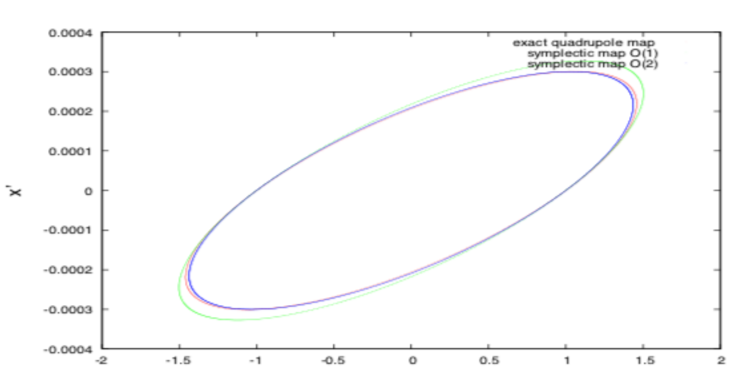
Symplecticity is guaranteed

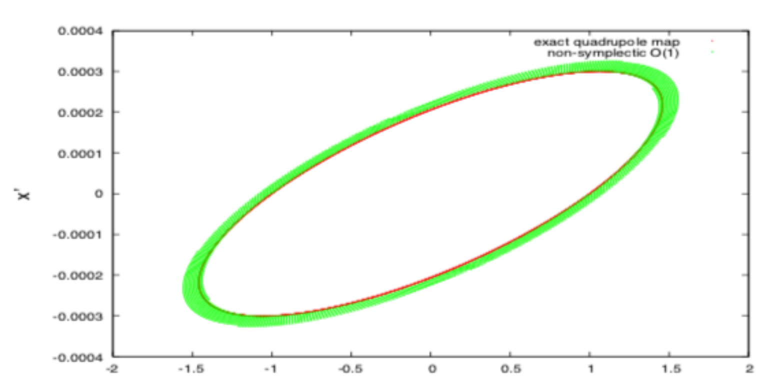
Hamiltonian mechanics (as opposed to Newton's laws/Lorentz force) is particularly powerful because we can write down a conservative system and solve it, ensuring symplecticity.



$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

cf. Werner Herr advanced school:





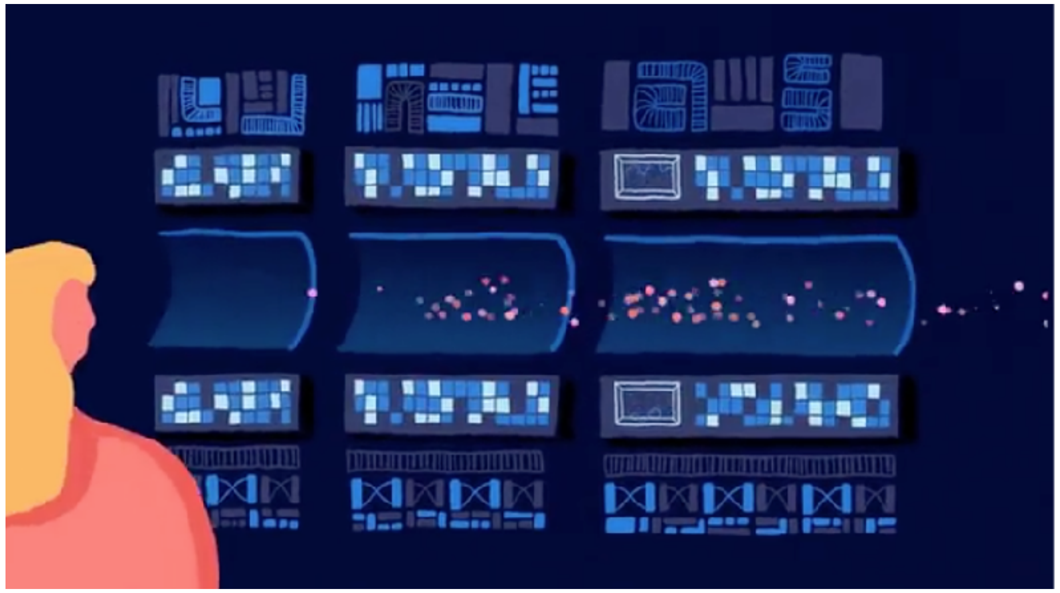
- Formally defined:

$$M^T(t) \cdot S \cdot M(t) = S$$

- In practise/physics:
  - Phase space areas are conserved (even in non-linear dynamics case).
  - i.e. Emittance is conserved.
  - e.g. A simulation code that is 'non-symplectic' will lead to particles eventually undergoing amplitude growth due to computational errors instead of a real effect!

Back to the Motion of Charged Particles

- Let’s start by looking at the motion of single particles.



- “*…in principle, there are only two steps in the analysis of any dynamical system. The first step is to write down the equations of motion; and the second step is to solve them*” [A. Wolski, pp. 59]

The approach: what options?

- We could just use Newton's laws & Lorentz force?

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \qquad (1.1) \qquad\qquad \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (1.2)$$

  - Newton's laws took astronauts to the moon. So maybe they are good enough for us? Well, it depends on the problem…

  - In Newtonian mechanics, to get time evolution, we:
    - Compute the total force exerted on each particle
    - Use Newtons 2nd law to calculate time evolution of position & velocity

Classical mechanics - our options:

- We could start with the Lagrangian:
  - Describes system in generalised co-ordinates (q) and velocities (dq/dt) and the “time” (t).
  - The Lagrange equations then consist of second-order differential equations describing the motion of the system.
  - Uses q and dq/dt and t (p and q are independent, the others aren’t)
- In Hamiltonian mechanics, we describe the motion of a particle (or a ball, or a planet) by:
  - First compute the “Hamiltonian” in “generalised co-ordinates” (q, p)
  - Then plug that into “Hamilton’s equations”
  - Get 2 (n) first order differential equations to solve

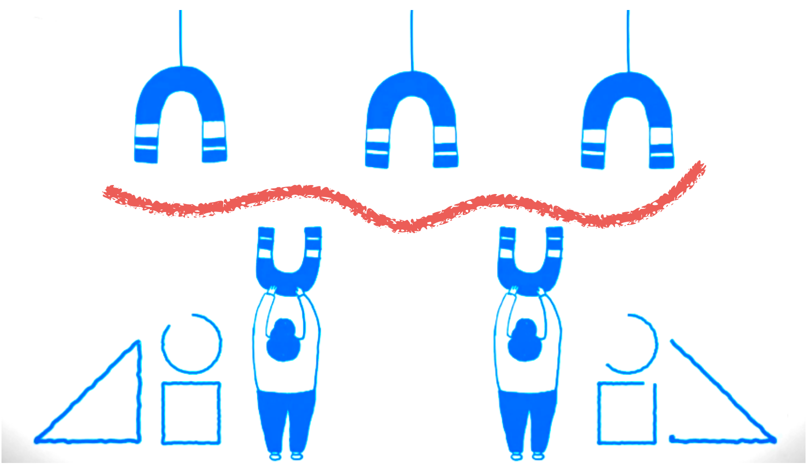
What’s the big deal of using p vs v? Isn’t p=mv? Well yes in rectangular co-ordinates, but in generalised co-ordinates that’s no longer true.

Which method?

	Lagrangian	Hamiltonian	Newton
Function of	Position, velocity, time L(q,v,t)	Position, momenta (generalised) H(q,p,t)	
Need to know	T-V	T+V	Forces
Results in	n 2nd order differential equations	2n x 1st order differential equation	
Conservative system	Yes	Yes	

Why use the Hamiltonian approach?

- In an accelerator, magnets and rf cavities are generally defined along a trajectory (i.e. we know where they are in distance, not time). The Hamiltonian approach is lets us use this fact.
- The motion of particles in electromagnetic fields is conservative, and similar to a harmonic oscillator with perturbations. We have lots of mathematical tools to treat this in the *Hamiltonian formalism*. (Stability analysis, linear methods etc…)
- Ultimately, it makes our lives easier.



If we know the Hamiltonian and Hamilton's equations, we can find the equations of motion for a dynamical system.

Our first goal: find out the Hamiltonian!

Our trajectory…

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Hamiltonian (straight beam line)

- The Hamiltonian represents the total energy of the particle
- We need a Hamiltonian that gives (1.1) and (1.2) when substituted into (1.3) and (1.4)

$$H = H(x_i, p_i; t)$$

Hamilton's equations

$$\frac{dx_i}{dt} = \frac{\delta H}{\delta p_i} \qquad (1.3)$$

$$\frac{dp_i}{dt} = -\frac{\delta H}{\delta x_i} \qquad (1.4)$$

We propose the following Hamiltonian for a relativistic charged particle moving in an electromagnetic field:

$$H = c\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2} + q\phi \qquad (1.5)$$

But this is still defined in terms of time… so we want to change it.

Where does this come from??

$$L = \frac{1}{2}mv^2 - q\phi(\vec{r},t) + q\vec{v}.\vec{A}(\vec{r},t)$$

$$H = \vec{v}.\vec{p} - L \qquad p_x = \frac{\partial L}{\partial v_x} = mv_x + qA_x$$

https://lampx.tugraz.at/~hadley/ss1/IQHE/cpimf.php

Hamiltonian (straight beam line)

(1.5) $$H = c\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2} + q\phi$$

We don't need to go through every step here, but here's what we do.

STEP 1: We take (1.5) and change the independent variable to z, the distance along the beam line

$$(x, p_x) \quad (y, p_y) \quad (t, -E)$$

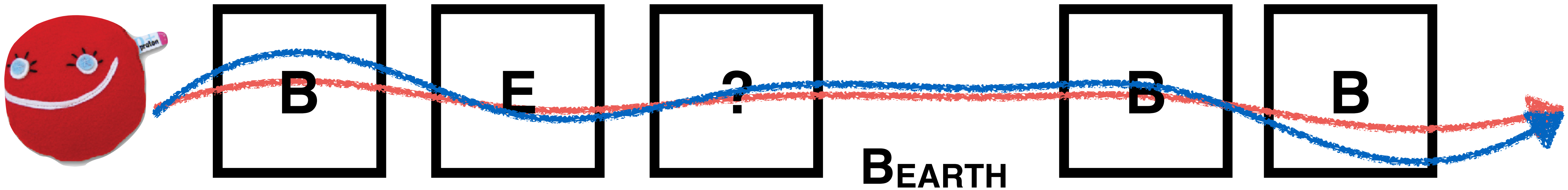
We get (try this at home…), use H= total energy E

$$H_{new} = -p_z = -\sqrt{\frac{(E - q\phi)^2}{c^2} - (p_x - qA_x)^2 - (p_y - qA_y)^2 - m^2c^2} - qA_z \quad (1.6)$$

Note that if the E and B fields are static, the Hamiltonian is independent of time and the total energy of the particle is constant. Happy days.

Straight Beamline Hamiltonian

We're not going to go through all the steps…



Blue: "reference" trajectory
Red: actual particle trajectory

STEP 2: We choose new (canonical) variables for the position & momentum that stay small as the particle moves along the beam line and scale by reference momentum P_0 (subscript '0' denotes reference)

STEP 3: We define new (canonical) longitudinal variables.

Variables for Beam Dynamics

Eventually, our Hamiltonian with independent variable 's' along the beamline, becomes:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\delta + \frac{1}{\beta_0} - \frac{q\phi}{cP_0}\right)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{1}{\beta_0^2\gamma_0^2}} - a_z \qquad (1.7)$$

$$\mathbf{a} = \frac{q}{P_0}\mathbf{A}$$

Scaled vector potential

$$\delta = \frac{E}{cP_0} - \frac{1}{\beta_0}$$

Energy deviation

$$(x, p_x, y, p_y, s, \delta)$$

Co-ordinates & momenta

- Using the Hamiltonian, we can get the equations of motion for a particle in a (straight) beam line.

- Usually these equations are *too complex to solve exactly,* so we have to *make some approximations.*

Hamiltonian (curved beam line)

- Now we complicate things slightly with curved reference frame
- We want to measure s along a path which curves with the trajectory of a particle on a curved orbit, which we call the *reference trajectory*
- Spoiler alert: simply turns out as a factor in front of the straight line Hamiltonian

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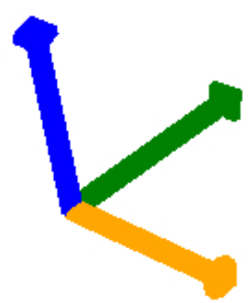
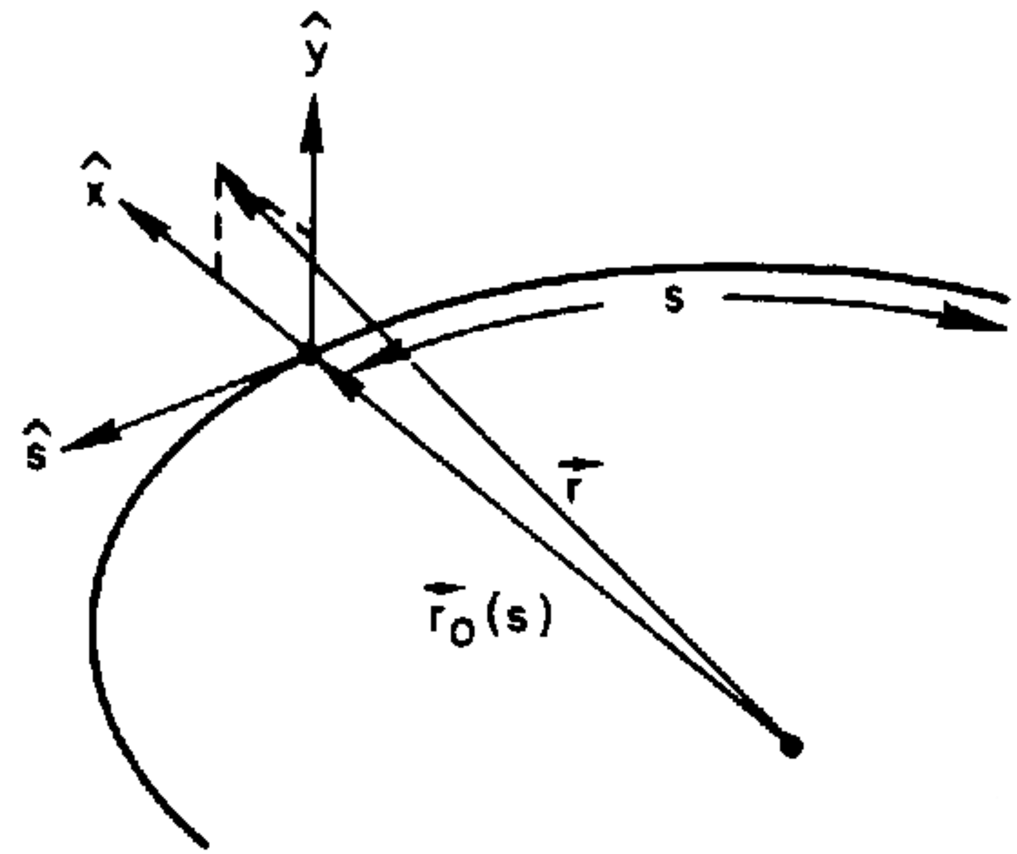
In accelerator physics we ask: "What are the particles' generalized coordinates when they reach a certain point in space?"

First, we convert to a non-inertial reference frame.
We use the 'Frenet-Serret' co-ordinate system





Particle motion is described with respect to a **reference orbit** in the non-inertial frame (x, y, s). This co-ordinate system is known as *Frenet-Serret*

- *First, we convert to ‘Frenet-Serret’ co-ordinate system*

$$\hat{s}(s) = \frac{d\vec{r}_0(s)}{ds}$$ Tangent unit vector to closed orbit

$$\hat{x}(s) = -\rho(s)\frac{d\hat{s}(s)}{ds}$$ Unit vector perpendicular to tangent vector

$$\hat{y}(s) = \hat{x}(s) \times \hat{s}(s)$$ Third unit vector…

Particle trajectory: $\vec{r}(s) = \vec{r}_0(s) + x\hat{x}(s) + y\hat{y}(s)$

nb. the reference frame moves WITH the particle

$$F_3(\vec{P}; x, s, y) = -\vec{P}.[\vec{r}_0(s) + x\hat{x}(s) + y\hat{y}(s)]$$ Generating function for canonical transformation

And we follow pretty much the same procedure as before…

$$H = e\phi + c\sqrt{m^2c^2 + \frac{(p_s - eA_s)^2}{(1 + x/\rho)^2} + (p_x - eA_x)^2 + (p_y - eA_y)^2}$$

Hamiltonian looks a little different and we can see that now the factor 1/ρ starts to appear. Remarkably, the addition of the one simple term x/ρ(s) gives all the new non-inertial dynamics

- *As before, we change the independent variable from t to s*

The new conjugate phase space variables are $(x, p_x, y, p_y, t, -H)$

And the new Hamiltonian (s-dependent) is $\tilde{H} = -p_s$

$$\tilde{H} = -(1 + x/\rho)\left[\frac{(H - e\phi)^2}{c^2} - m^2c^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2\right]^{1/2} - eA_s$$

Which is time-independent (if also ϕ, A are time-independent)

Expanding the Hamiltonian to second order in p_x, p_y

$$\tilde{H} \approx -p(1 + x/\rho) + \frac{1 + x/\rho}{2p}\left[(p_x - eA_x)^2 + (p_y - eA_y)^2\right]^{1/2} - eA_s$$

$H - e\phi = E$ is the total particle energy

$p = \sqrt{E^2/c^2 - m^2c^2}$ is the total particle momentum

Where are we now?

So far, we have been looking in general for any

Vector potential $\mathbf{A}$

Scalar potential ϕ

But in reality, we (usually) use electric fields to accelerate particles and magnetic fields to bend, focus and manipulate the beams.

So we need to know

"which magnetic fields can we really create?"

(We'll come back to the Hamiltonian later…)

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Magnetic Fields

- Maxwell's equations, time independent, no sources, so: $\vec{J} = 0$

$$\nabla \times \vec{B} = 0 \qquad \vec{B} = \mu_0 \vec{H}$$

$$\nabla \cdot \vec{B} = 0$$

- We'll "guess" that the following obeys these equations:
- A constant longitudinal field B_z, and

$$B_y + iB_x = C_n (x + iy)^{n-1}$$

- n is an integer > 0, C is a complex number
- (real part understood)

Does this obey Maxwell in free space?

Now apply $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ to each side of $B_y + iB_x = C_n(x+iy)^{n-1}$

LHS: $$= \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} + i\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)$$

$$= \left[\nabla \times \vec{B}\right]_z + i\nabla \cdot \vec{B}$$ Where we know B_z is constant.

RHS: $$= (n-1)(x+iy)^{n-2} + i^2(n-1)(x+iy)^{n-2} = 0$$

$\therefore \quad \nabla \times \vec{B} = 0$ and $\nabla \cdot \vec{B} = 0$

So we find that as expected, the field $B_y + iB_x = C_n(x+iy)^{n-1}$ satisfies Maxwell’s equations in free space

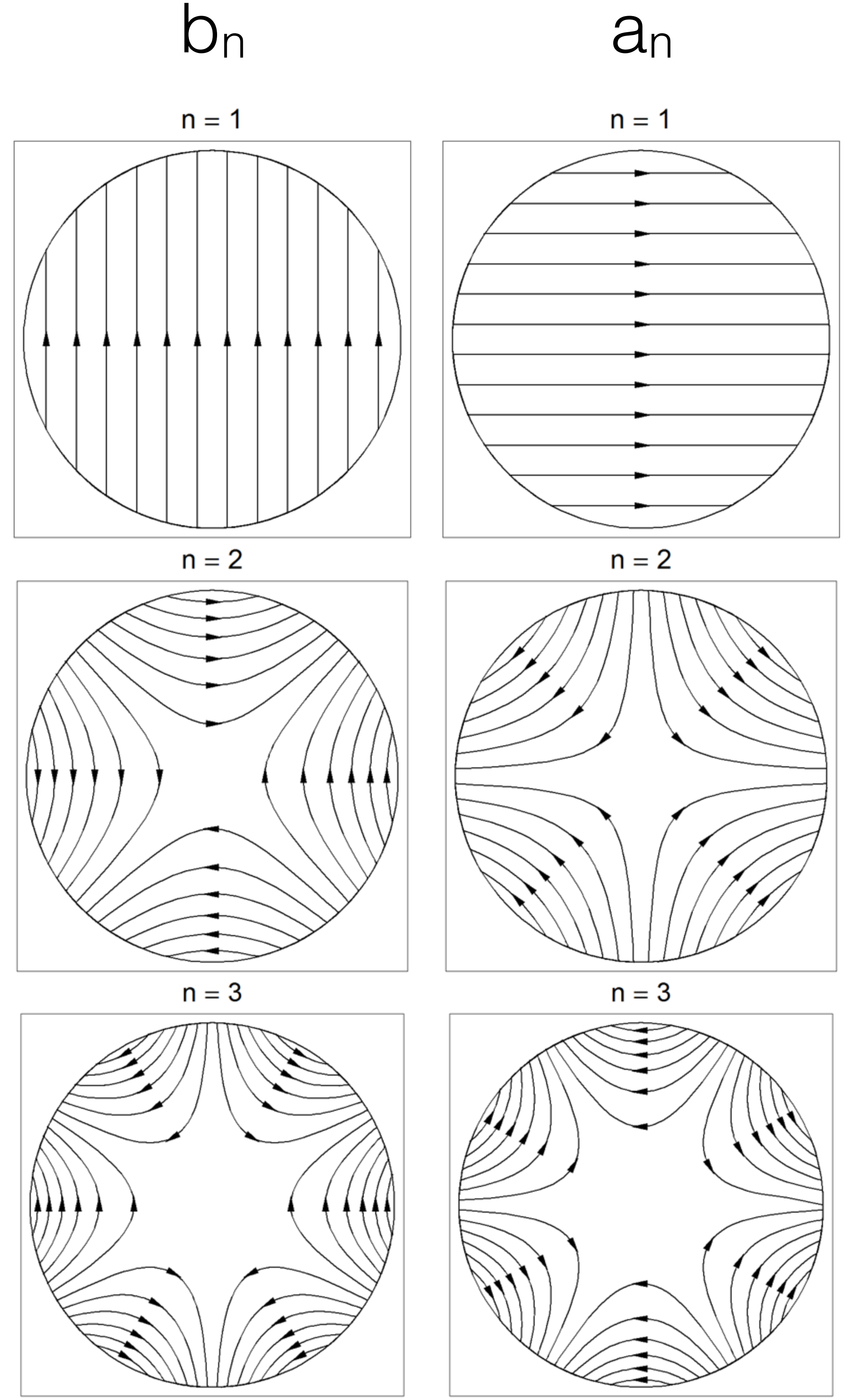
Multipole fields

In the usual notation:

$$B_y + iB_x = B_{ref} \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x+iy}{R_{ref}} \right)^{n-1}$$

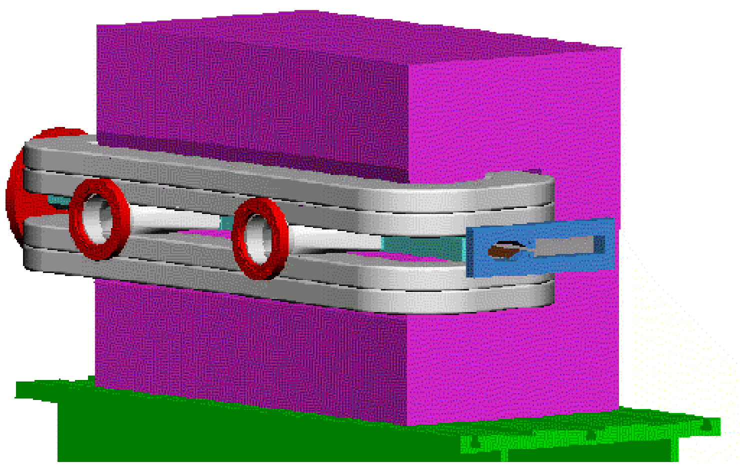
b_n are "normal multipole coefficients" (LEFT)
and a_n are "skew multipole coefficients" (RIGHT)
'ref' means some reference value

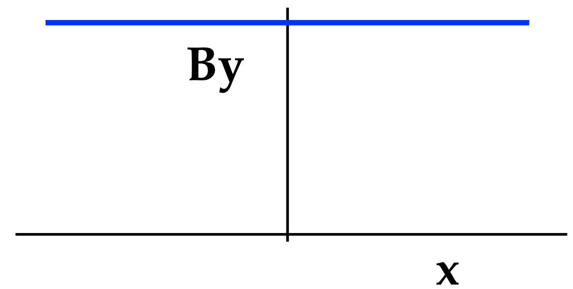
n=1, dipole field
n=2, quadrupole field
n=3, sextupole field

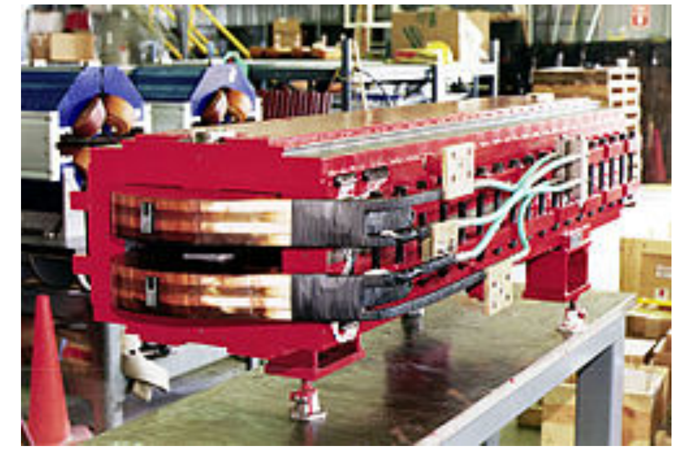


Images: A. Wolski, https://cds.cern.ch/record/1333874

Multipole Magnets







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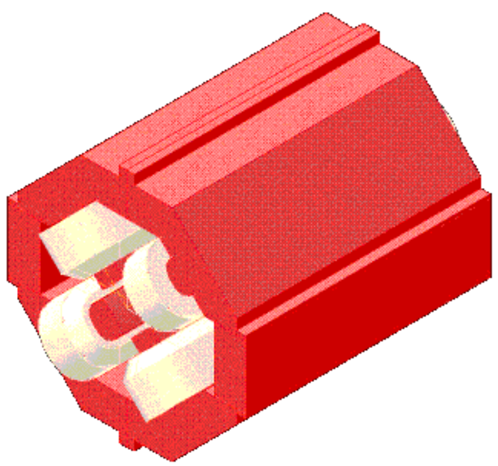

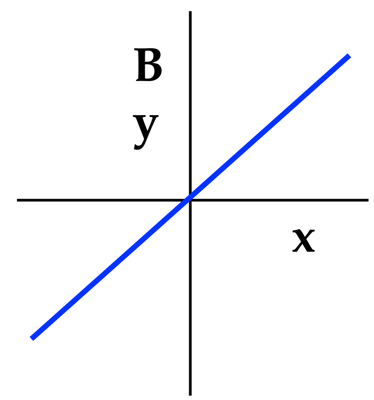

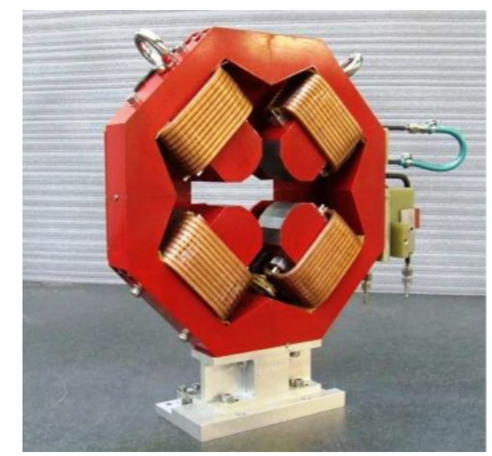

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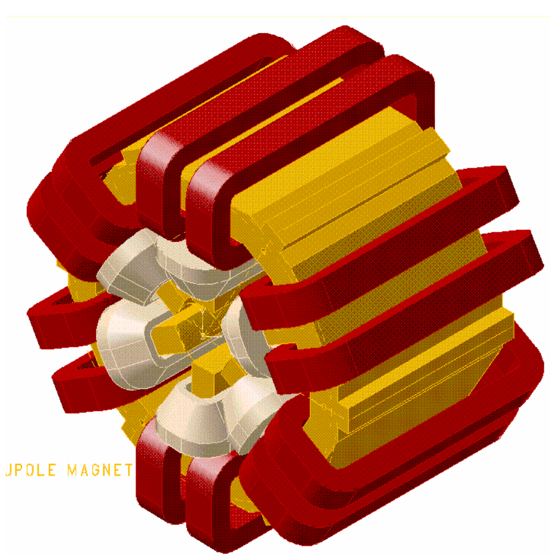
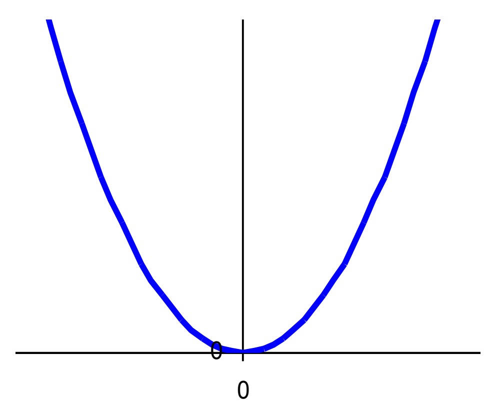




Images: Ted Wilson, JAI Course 2012

Image: Danfysik

Put these back into the Hamiltonian…
(We'll do more of this next lecture)

Hamiltonians of some machine elements (3D)

In general for multipole n:

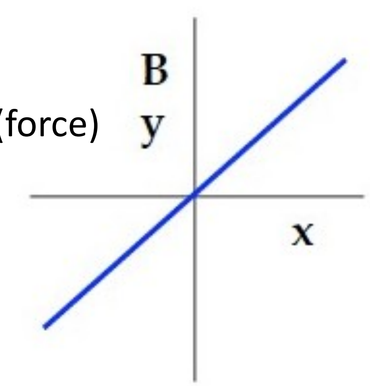
$$H_n = \frac{1}{1+n}\mathcal{R}e\left[(k_n + \mathrm{i}k_n^{(s)})(x+\mathrm{i}y)^{n+1}\right] + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

We get for some important types (normal components k_n only):

dipole: $$H = -\frac{-x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

quadrupole: $$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

Such a field (force) we need for focusing



sextupole: $$H = \frac{1}{3}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

So we now have:

An idea of how to get the Hamiltonian

and

An idea of which types of magnetic fields we might encounter

Lecture 2

FYI… in the next lecture we will:

4. Understand the approach to compute linear and non-linear maps

5. Derive and look at transfer matrices for main types of magnets used in accelerators

5. Get a glimpse (only) of non-linear dynamics (covered further later & at the Advanced CAS school)

Our trajectory…

- **Lecture 1: Where do we begin?**
  - How do we describe the motion?
  - Approach: Hamiltonian dynamics
  - Setting up: Reference trajectory & variables and reference frame
  - Which EM fields do we need to consider (at first)?
- **Lecture 2: We now have the equations & the fields… what next?**
  - “Maps” and “matrices” transporting co-ordinates in EM fields
  - Derive maps for drift, dipole, quadrupole etc…
  - Adding the pieces together: getting to Hills equation of motion
  - Hints of non-linear dynamics and harder stuff to come later…

References

Beam Dynamics:

- A. Wolski, “Beam Dynamics in High Energy Particle Accelerators”, Imperial College Press, 2014.
- S. Y. Lee, “Accelerator Physics”, 3rd Edition, World Scientific, 2011.
- K. Brown, SLAC-75-rev-4 (1982); SLAC-91-rev-2 (1977)

Electromagnetism:

- J. D. Jackson, Classical Electrodynamics, 3rd Ed, Wiley & sons (1999).

Hamiltonian Mechanics:

- H. Goldstein et al., Classical Mechanics (3rd ed.). Addison-wesley (2001).

Most of the images/animations I’ve used here:
http://richannel.org/collections/2016/particle-accelerators-for-humanity

Particle Motion in Hamiltonian Formalism
Lecture 2

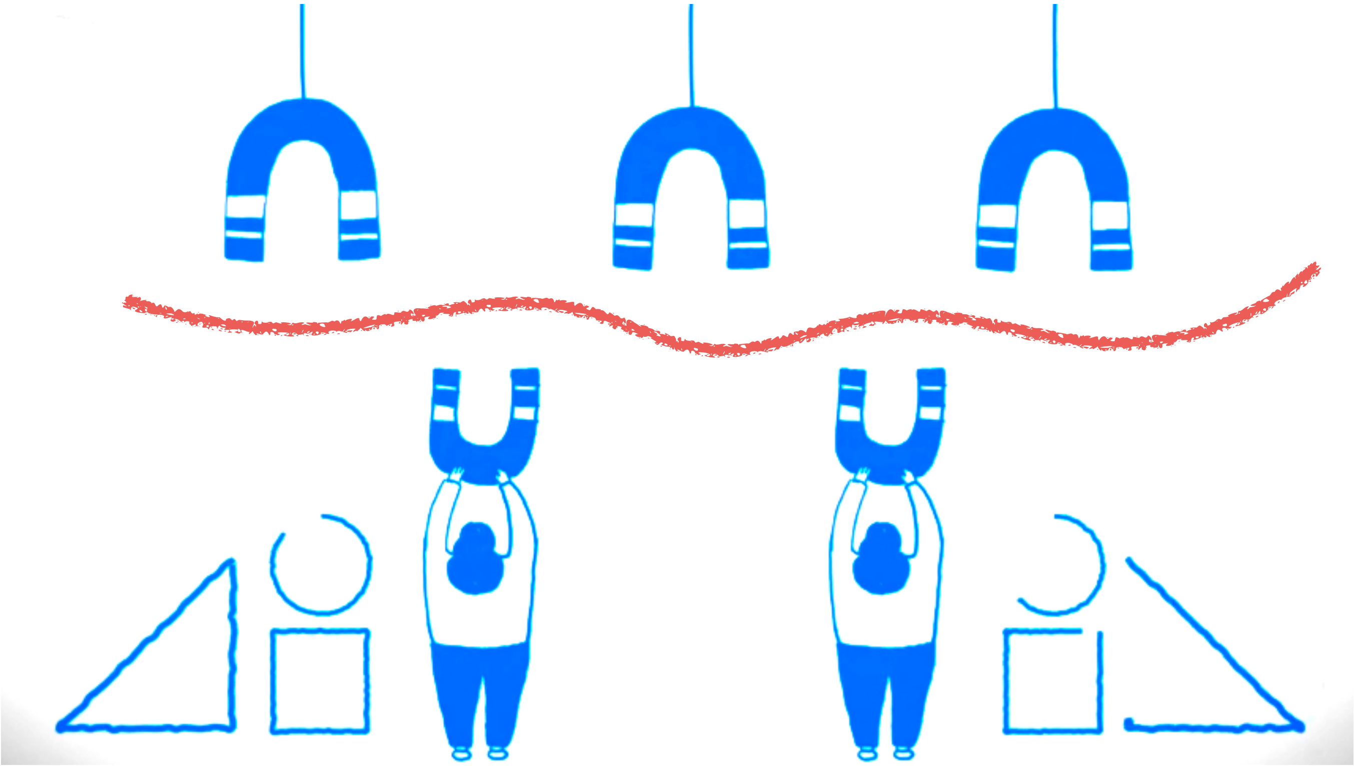
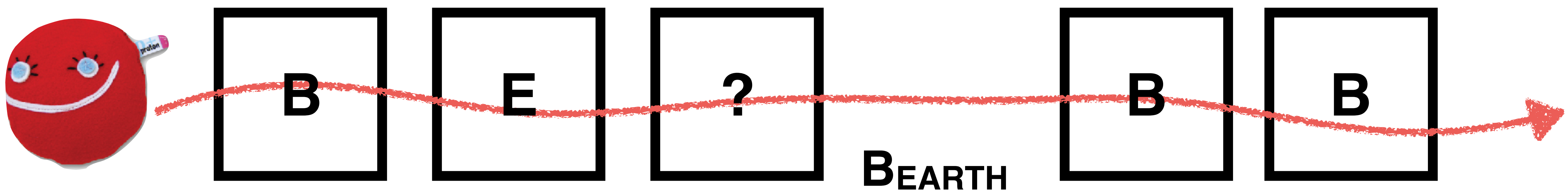


Image: Andrew Khosravani, 2016

http://richannel.org/collections/2016/particle-accelerators-for-humanity

Dr. Suzie Sheehy
John Adams Institute for Accelerator Science
University of Oxford

From Lecture 1:
How can we describe the motion?



What are maps?

- Maps are the basic mathematical instruments to transport particles through (arbitrary) EM fields.

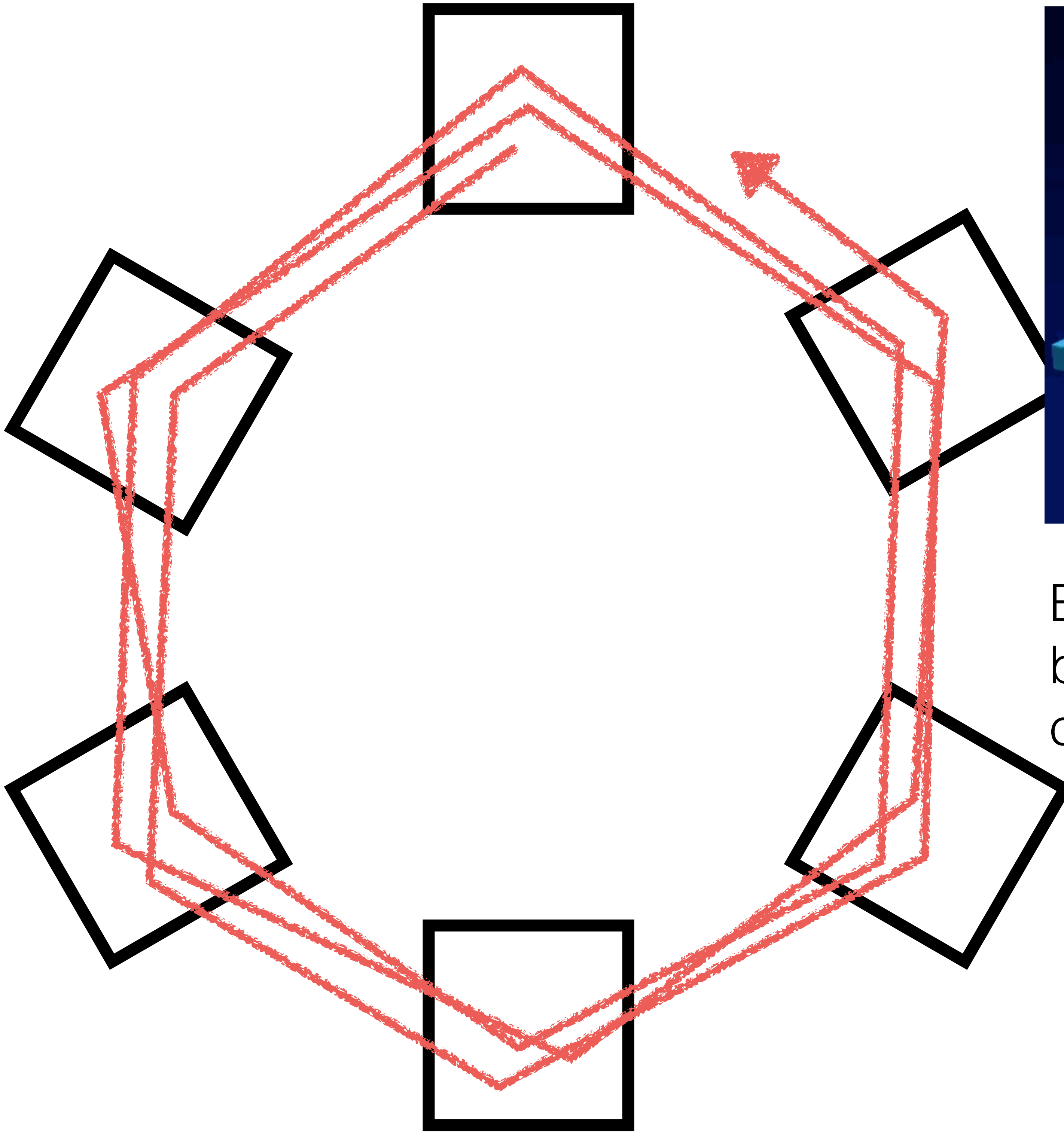
- They require some input and give an output.

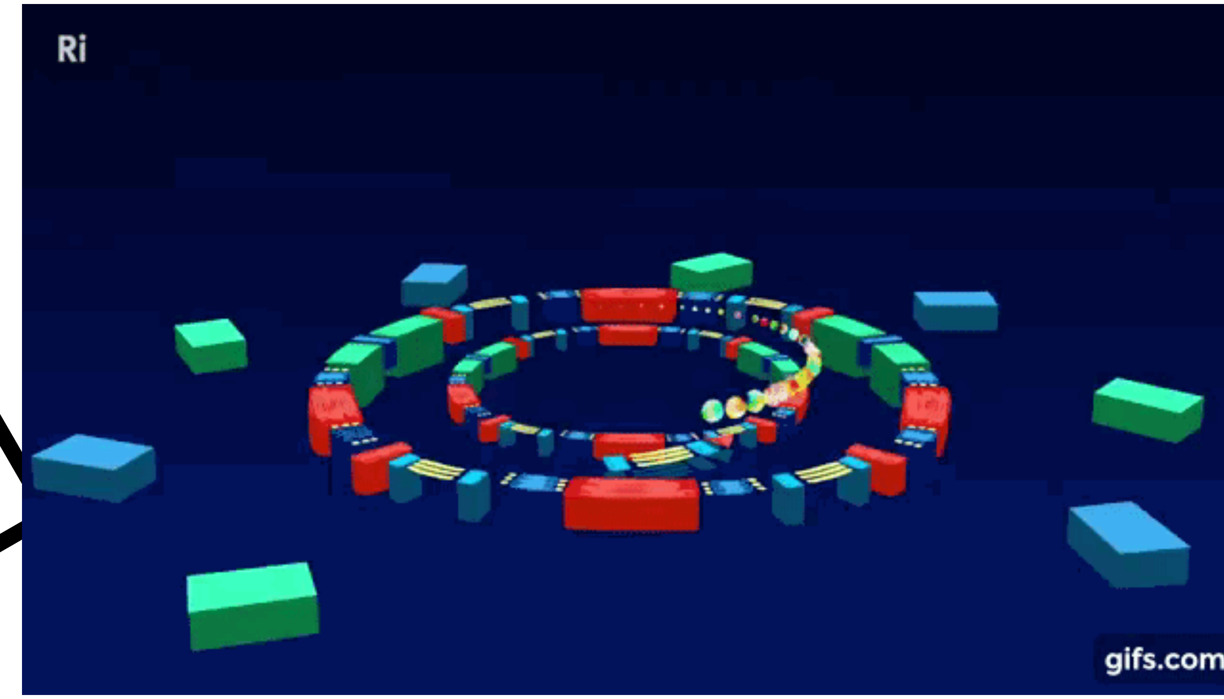
- A *transfer map* is a statement of how *dynamical variables* change at different points. They are *abstract objects*, and can be matrices, Taylor series, Lie Transforms or other objects.

- In this case we talk about particle co-ordinates but we could propagate any mathematical object (beam sizes, spin, etc…) in similar way

$$\vec{x} = (x, p_x, y, p_y, z, \delta)$$
$$\vec{x}(s_1) = \overrightarrow{M}(\vec{x}(s_0))$$

In the Introductory school, we will focus on matrices (1st order maps), but you should be aware that the concept is more general. Maps can be based on symplectic integrators (usually done in modern tracking programs), and other objects.





Each of these 'elements' (and drifts) can be thought of as transforming the particle co-ordinates

$$\vec{x} = (x, p_x, y, p_y, z, \delta)$$

$$\vec{x}(s_1) = \overrightarrow{M}(\vec{x}(s_0))$$

Mathematically, they are 'maps'

Maps for Circular Machines

Accelerators are made up of beamline elements, each with their own linear and nonlinear fields, they might be mis-aligned, mis-powered etc…

If one tried to write down the entire Hamiltonian for that system, it would be pretty darn complicated!

So instead, we take a piecewise approach:

1. First compute the maps for individual beamline elements using a local coordinate system that is appropriate to the element.

2. Then the maps are combined to produce a one-turn map, and then we can do our analysis on that map.

.

Map Definition and Conventions

- If the equations of motion are linear, we can write the map as a matrix.
- We can understand many of the effects of dynamics of particles by using linear equations.
- (Even though, in fact, the equations of motion are non-linear in reality, even in a drift space!)

Transfer matrices

$$\vec{x}(s) = M(s \mid s_0)\vec{x}(s_0) \qquad \vec{x}(s) = \begin{pmatrix} x(s) \\ p_x(s) \end{pmatrix}$$

Where M is the ‘transfer matrix’.

The effect of a succession of drifts & lenses can be found by multiplying their transfer matrices…

$$\vec{x}(s_n) = M_n(s_n \mid s_{n-1})\ldots M_3(s_3 \mid s_2)M_2(s_2 \mid s_1)M_1(s_1 \mid s_0)\vec{x}(s_0)$$

We could do this for a whole ring, but usually can exploit some symmetry (superperiod or cell)

Drift space

- Let's consider the simplest case, a drift. There are no electric or magnetic fields and we can set potentials to zero. $\phi = 0, \mathbf{A} = 0$

- The Hamiltonian is then: $H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2\gamma_0^2}$

- Where we've expanded to second order, and dropped terms of 3rd and higher order.

- WHY? Because we want to construct LINEAR maps.

- It is also possible to have NON-LINEAR maps (see later)

Wait, let's go back a step…

The actual Hamiltonian in a drift (no potentials) is:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\delta + \frac{1}{\beta_0}\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}}$$

Expanding to second order gives:

$$H = \cancel{-1} + \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2 \gamma_0^2} + \cancel{O(3)}$$

- And we can drop the 3rd and higher order terms (and zeroth order as this doesn’t contribute to the dynamics)
- This is called the *paraxial approximation* and you will hear people talking about it. It’s worth knowing that we do this, and (for instance) whether the simulation code you use takes this approximation or not.
- We COULD solve the equations of motion and then approximate, but it is useful to start with a Hamiltonian (even an approximate one) as it has conserved quantities

Back to our drift space…

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2\gamma_0^2}$$

$$\frac{dx_i}{ds} = \frac{\delta H}{\delta p_i} \qquad \frac{dp_i}{ds} = -\frac{\delta H}{\delta x_i}$$

- Using Hamilton's equations (above) to find equations of motion:

$$\frac{dx}{ds} = \frac{\delta H}{\delta p_x} = p_x$$

$$\frac{dp_x}{ds} = -\frac{\delta H}{\delta x} = 0$$

(for y are the same format…)

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = \frac{\delta}{\beta_0^2\gamma_0^2}$$

$$\frac{d\delta}{ds} = -\frac{\delta H}{\delta z} = 0$$

Solve exactly to get

$$x_1 = x_0 + L p_{x0}$$

$$p_{x1} = p_{x0}$$

$$y_1 = y_0 + L p_{y0}$$

$$p_{y1} = p_{y0}$$

$$z_1 = z_0 + \frac{L}{\beta_0^2\gamma_0^2}\delta_0$$

$$\delta_1 = \delta_0$$

In matrix form

- We can express this *linear map* as a *matrix*

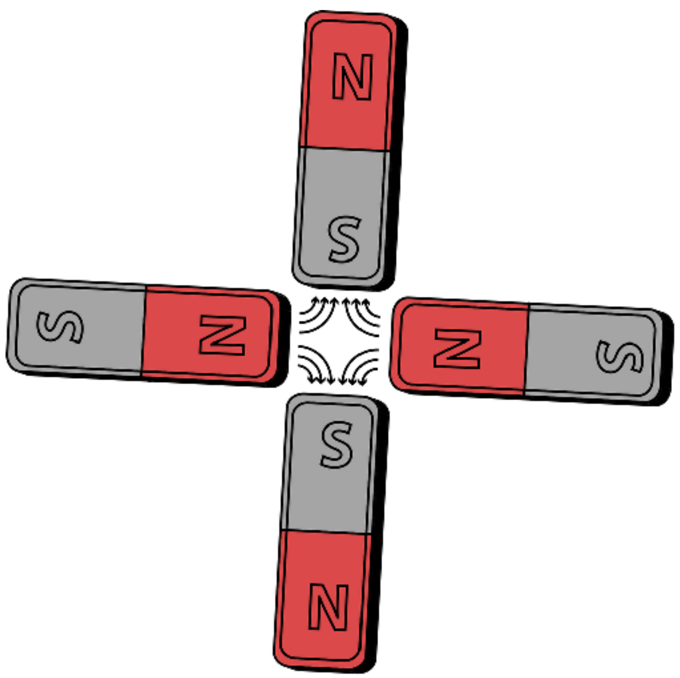
$$\vec{x}_1 = R_{drift}\,\vec{x}_0$$

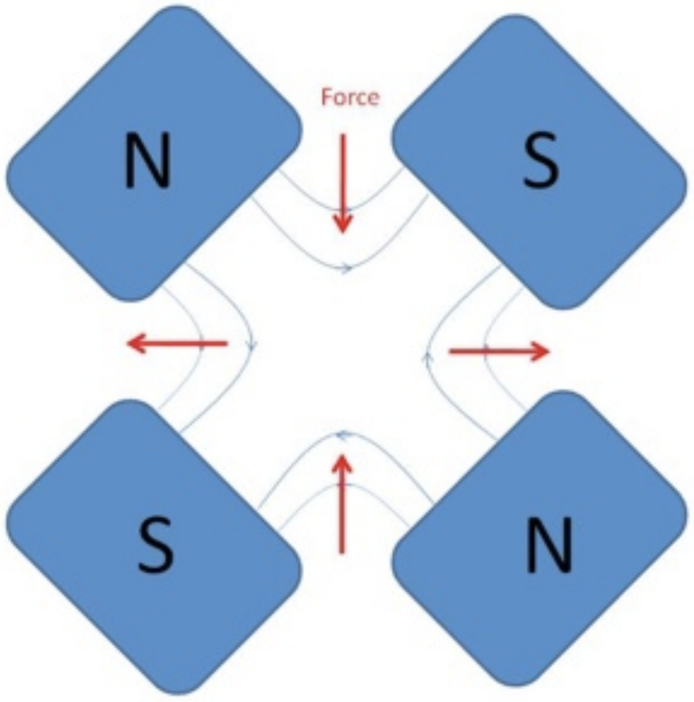
$$\vec{x} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \qquad R_{drift} = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

So given particle co-ordinates, we can transform them to the next ‘step’

Quadrupole magnet

- Let's try a non-trivial example, and one that is commonly used in accelerators: the quadrupole
- Quadrupoles provide transverse focusing





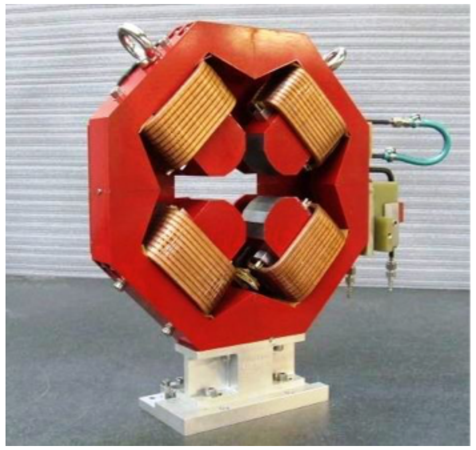
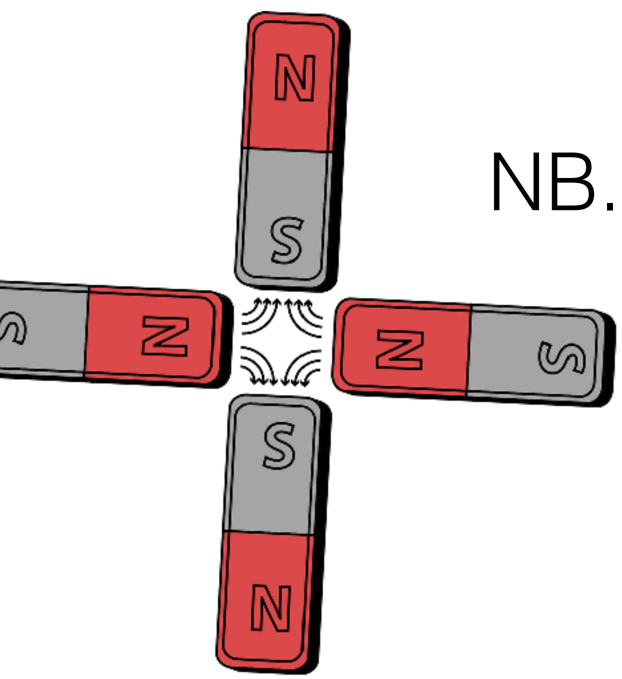


Image: STFC

Quadrupole magnet

- Remember our equation for magnetic multipoles?

$$B_y + iB_x = B_{ref} \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x+iy}{R_{ref}} \right)^{n-1}$$



NB. When working with the Hamiltonian, we need the vector potential, which can be written:

$$\mathbf{A} = (0,0,A_z) \qquad A_z = -B_{ref} \operatorname{Re} \sum_{n=1}^{\infty} (b_n + ia_n) \frac{(x+iy)^n}{nR_{ref}^{n-1}}$$

This becomes… for n=2 (quadrupole):

$$A_z = -\frac{B_{ref} b_2}{2 R_{ref}} (x^2 - y^2)$$

NB. For a pure multipole field, we normally use the multipole fields normalised by q / P_0

So, we can define the normalised multipole strength, k:

$$k_{n-1} = \frac{q}{P_0} \frac{\partial^{n-1} B_y}{\partial x^{n-1}} = (n-1)! \frac{B_{ref}}{R_{ref}^n} b_n$$

So for our quadrupole, n=2, we have: $k_1 = \frac{q}{P_0} \frac{\partial B_y}{\partial x} = \frac{B_{ref}}{R_{ref}} b_2$

And we get the magnetic field and vector potentials:

$$\vec{B} = (k_1 y, k_1 x, 0)$$

$$\mathbf{a} = (0, 0, -\frac{k_1}{2}(x^2 - y^2))$$

this is the bit we need next…

- Now we go back down the Hamiltonian rabbit hole…

$$\mathbf{a} = (0,0,-\frac{k_1}{2}(x^2 - y^2))$$

Oh wait, it's not so bad… it's just like the drift, but with the new potential!

$$H_{quad} = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2\gamma_0^2} + \frac{k_1}{2}(x^2 - y^2)$$

nb. we've made the paraxial approximation again

- Now we can solve the equations of motion again…

$$H_{quad} = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{\delta^2}{2\beta_0^2\gamma_0^2} + \frac{k_1}{2}(x^2 - y^2)$$

$$\frac{dx}{ds} = \frac{\delta H}{\delta p_x} = p_x \qquad \frac{dy}{ds} = \frac{\delta H}{\delta p_y} = p_y \qquad \frac{dz}{ds} = \frac{\partial H}{\partial \delta} = \frac{\delta}{\beta_0^2\gamma_0^2}$$

$$\frac{dp_x}{ds} = -\frac{\delta H}{\delta x} = k_1 x \qquad \frac{dp_y}{ds} = -\frac{\delta H}{\delta y} = -k_1 y \qquad \frac{d\delta}{ds} = -\frac{\delta H}{\delta z} = 0$$

etc… so that…

$$\frac{d^2x}{ds^2} - k_1 x = 0 \qquad \frac{d^2y}{ds^2} + k_1 y = 0$$

Quadrupole transfer matrix

Through a quad of length L, strength k_1

$$\vec{x}_1 = R_{quad}\vec{x}_0 \qquad \omega = \sqrt{k_1}$$

$$\vec{x} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \qquad R_{quad} = \begin{pmatrix} \cos(\omega L) & \frac{\sin(\omega L)}{\omega} & 0 & 0 & 0 & 0 \\ -\omega\sin(\omega L) & \cos(\omega L) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(\omega L) & \frac{\sinh(\omega L)}{\omega} & 0 & 0 \\ 0 & 0 & \omega\sinh(\omega L) & \cosh(\omega L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let's look at the x, px part

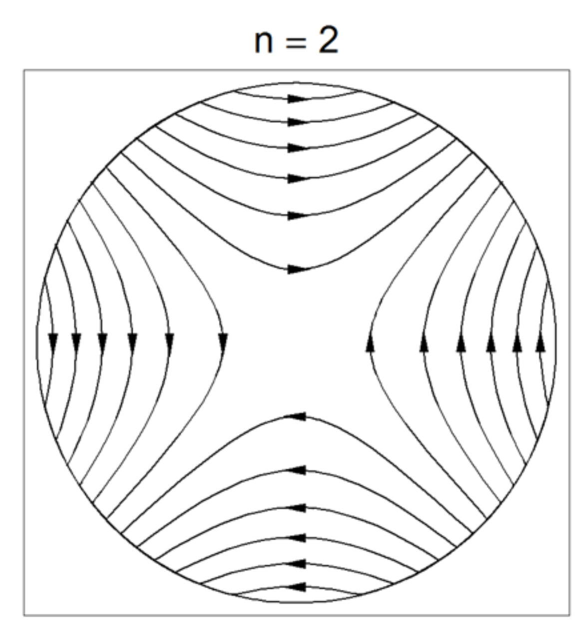
$$R_{quad} = \begin{pmatrix} \cos(\omega L) & \frac{\sin \omega L}{\omega} \\ -\omega \sin(\omega L) & \cos(\omega L) \end{pmatrix}$$

If $\omega = \sqrt{k_1}$ is real so $k_1>0$, the motion is a harmonic oscillator

(but in accelerators, the particle won't undergo a whole oscillation just in a single quadrupole)

But note that when $k_1<0$ the motion is defocusing (while in the vertical it becomes focusing)

i.e. we can't focus in x and y at the same time with a single quadrupole, the other plane is always defocusing.



But we can get around that (later)…

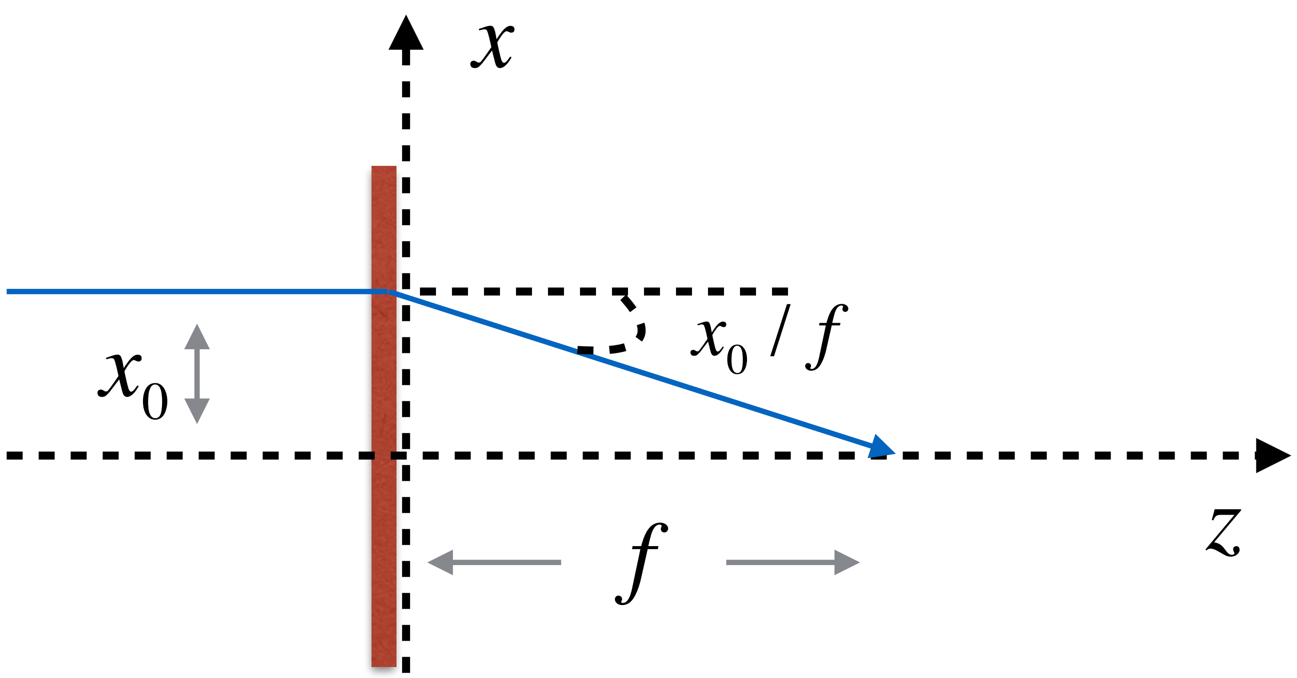
Thin lens approximation

- If we view a quadrupole as having length $L_q \to 0$
- f is a constant. $k_1 L_q \to 1/f$

$$k_1 = \frac{q}{P_0}\frac{\partial B_y}{\partial x}$$

$$R_F^x = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \qquad R_F^y = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

focusing in x-plane reversed (defocusing) in y-plane



thin lens focusing “F” quadrupole

Sneak peak: AG focusing - thin lens

For infinitesimally short lenses, we can recover most of the physics

$K(s) = \pm\delta(s)/f$ where f is the focal length.

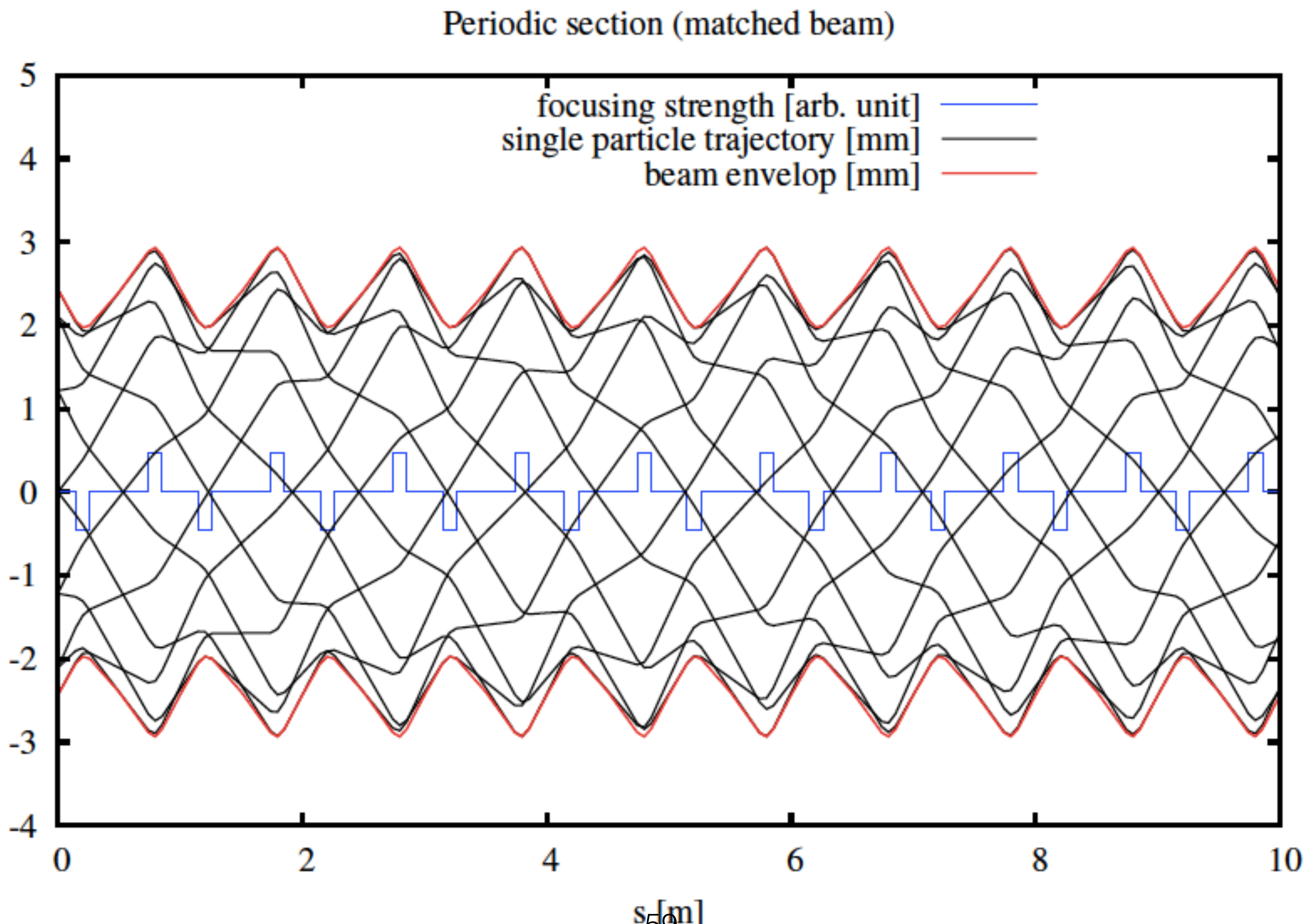
In the 'thin lens' approximation:

$$M = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{d}{f} - \frac{d^2}{f^2} & 2d + \frac{d^2}{f} \\ -\frac{d}{f^2} & 1 + d/f \end{pmatrix}$$

Focusing & defocusing with a drift between doesn't cancel out.
This is what gives us 'alternating gradient' focusing

Particle in AG focusing



Dipole magnet

- In a dipole magnet, the field is (ideally) uniform
- But it's not so simple as the reference trajectory is curved - so that the dynamical variables stay small

The vector potential in *curvilinear co-ordinates* is

$$A_x = 0 \qquad A_y = 0 \qquad A_s = -B_0\left(x - \frac{x^2}{(x+\rho)}\right)$$

let's look at the resulting linear transfer matrix
(after using the 2nd order Hamiltonian as usual)

Transfer Matrix for a Dipole

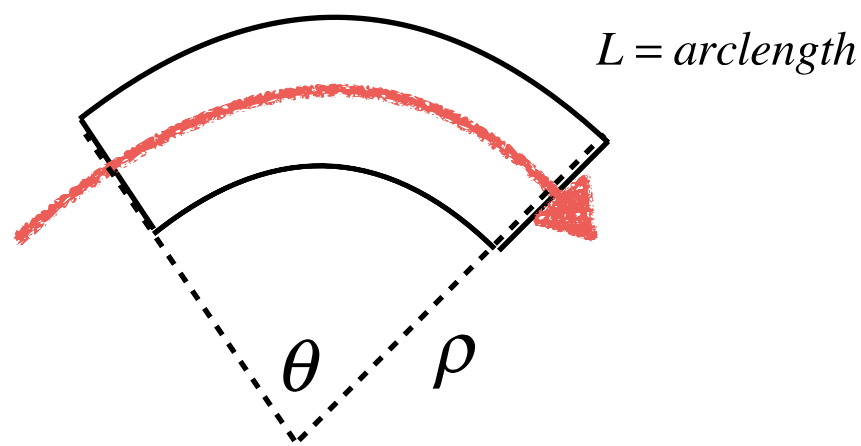
Pure sector dipole

$$R_{dipole} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -(1/\rho)\sin\theta & \cos\theta \end{pmatrix}$$

$$k_x = 1/\rho^2$$

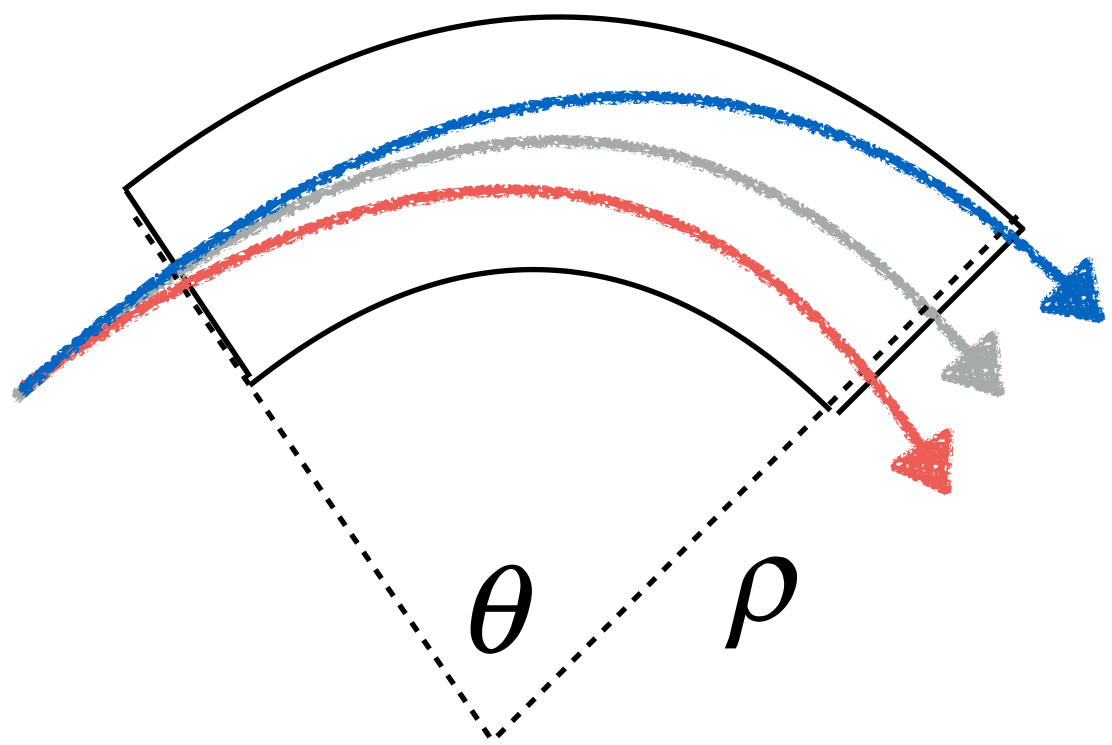
$$\theta = L/\rho$$

in the deflecting plane…



A few things to note:

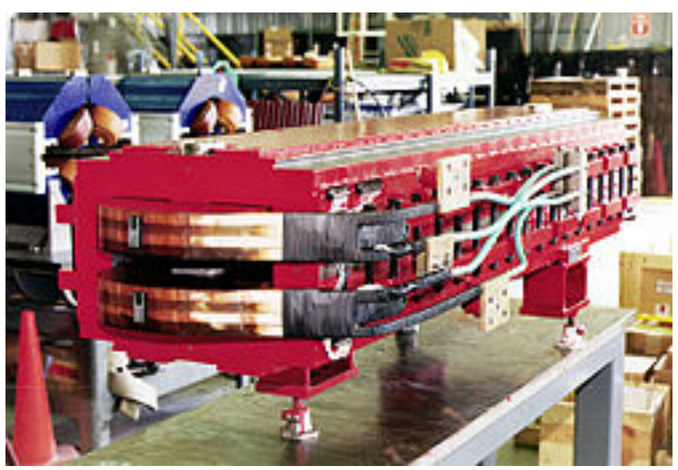
Because of curvilinear co-ordinate system it should look like a drift… but it looks like there is focusing (weak focusing)

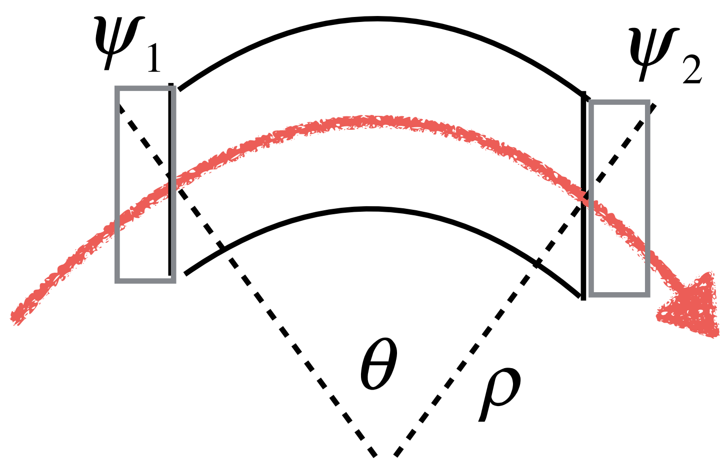


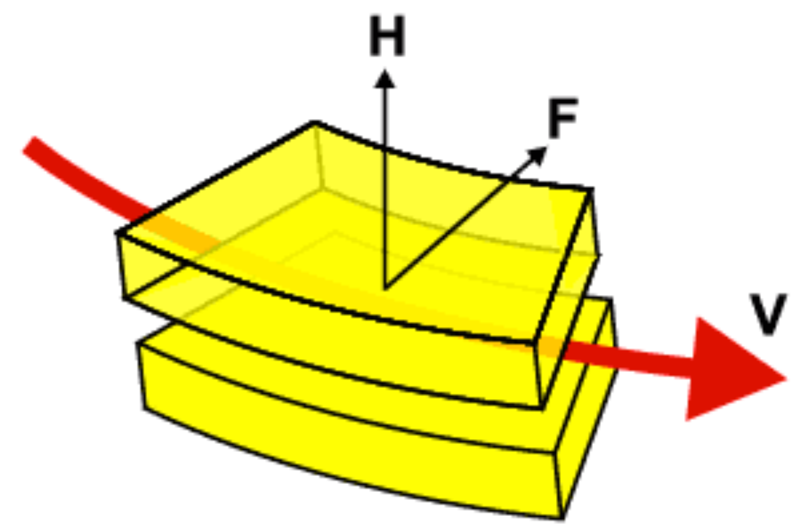
Dispersion - bending of a dipole changes with momentum
(Remember this for later lectures…)

A note on dipoles…

- Note that the map we've seen is for a particle inside the dipole, but there are also *end effects* (*fringe field effects*).
- We cannot neglect these!
- Results in 'edge focusing' if the beam trajectory is not perpendicular to the magnet entrance/exit face



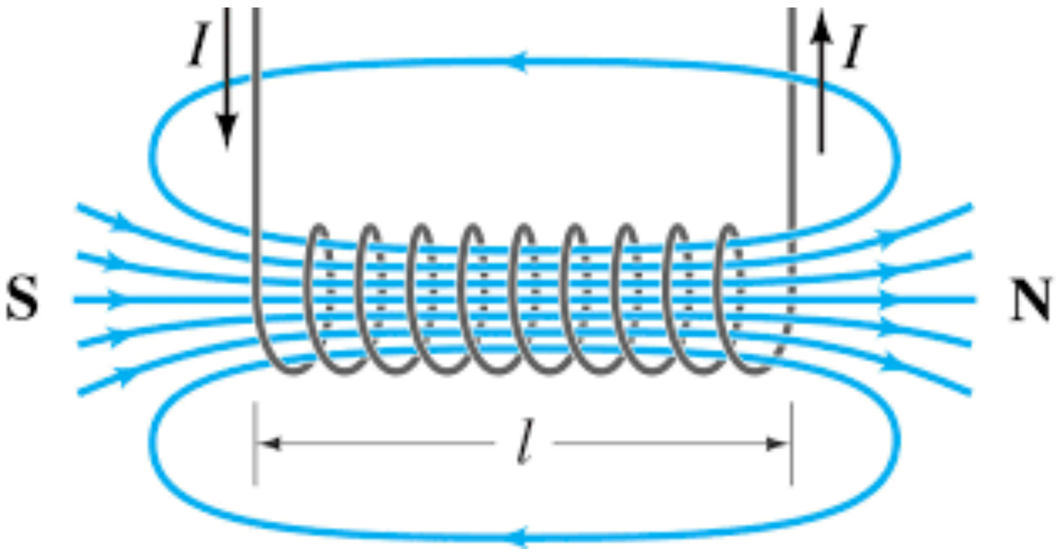


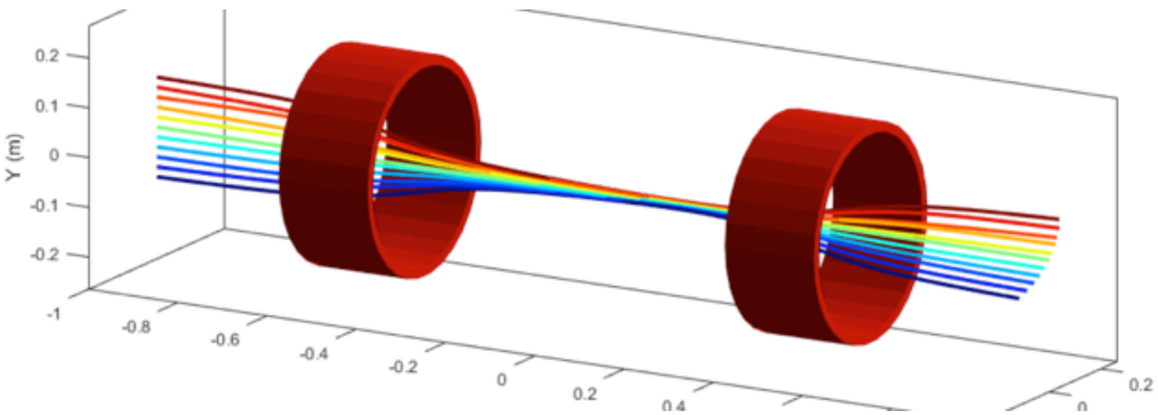


In the limit that the fringe field extent is zero…

$$R_{fringe,x} = \begin{pmatrix} 1 & 0 \\ k_0 \tan\psi_1 & 1 \end{pmatrix}$$

Solenoid

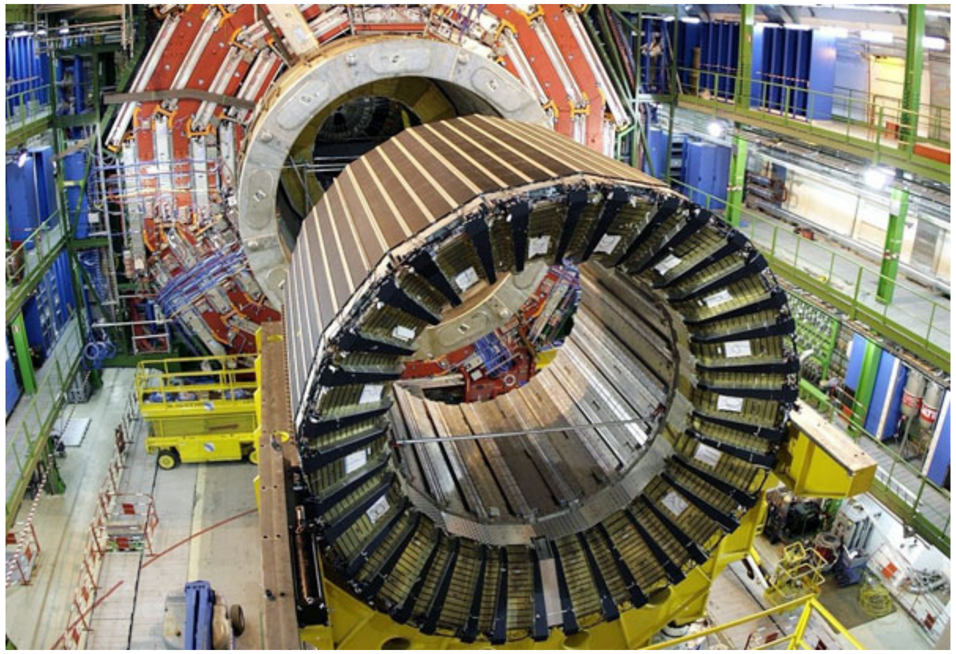




- Produces a uniform field parallel to beam direction
- Can provide transverse focusing, particle capture etc and
- Also used in detectors for PP experiments

$$\mathbf{B} = (0, 0, B_0)$$

$$\mathbf{A} = \left(\frac{-B_0 y}{2}, \frac{B_0 x}{2}, 0\right)$$



Transfer Matrix for a Solenoid

$$R_{sol} = \begin{pmatrix} \cos^2(\omega L) & \frac{1}{2\omega}\sin(2\omega L) & \frac{1}{2}\sin(2\omega L) & \frac{1}{\omega}\sin^2(2\omega L) & 0 & 0 \\ -\frac{\omega}{2}\sin(2\omega L) & \cos^2(\omega L) & -\omega\sin^2(\omega L) & \frac{1}{2}\sin(2\omega L) & 0 & 0 \\ -\frac{1}{2}\sin(2\omega L) & -\frac{1}{\omega}\sin^2(2\omega L) & \cos^2(\omega L) & \frac{1}{2\omega}\sin(2\omega L) & 0 & 0 \\ \omega\sin^2(\omega L) & -\frac{1}{2}\sin(2\omega L) & -\frac{\omega}{2}\sin(2\omega L) & \cos^2(\omega L) & 0 & 0 \end{pmatrix}$$

There is coupling between horizontal & vertical motion

(And we have lost some higher order effects (chromaticity)
by using 2nd order Hamiltonian)

$$\omega_s = k_s = \frac{q}{P_0}\frac{B_0}{2}$$

“Knowing is not enough; we must apply.
Willing is not enough; we must do.”

– Johann Wolfgang von Goethe

Now let's look at the equation of motion

- In a circular accelerator… with linear magnetic field components.

Hamilton's equations of motion* are:

$$\frac{dx}{ds} = \frac{\partial H}{\partial p_x}, \quad \frac{dp_x}{ds} = -\frac{\partial H}{\partial x} \qquad \frac{dy}{ds} = \frac{\partial H}{\partial p_y}, \quad \frac{dp_y}{ds} = -\frac{\partial H}{\partial y}$$

With transverse magnetic fields we can show (scaled & in (x,y,s)) :

$$\vec{B} = B_x(x,y)\hat{x} + B_y(x,y)\hat{y}$$

$$B_x = -\frac{1}{(1+x/\rho)}\frac{\partial A_s}{\partial y} \qquad B_y = -\frac{1}{(1+x/\rho)}\frac{\partial A_s}{\partial x}$$

Betatron equations of motion become: (neglect higher order terms)

$$x'' - \frac{\rho + x}{\rho^2} = \frac{B_y}{B\rho}\frac{p_0}{p}\left(1+\frac{x}{\rho}\right)^2 \qquad y'' = -\frac{B_x}{B\rho}\frac{p_0}{p}\left(1+\frac{x}{\rho}\right)^2$$

*neglecting synchrotron motion

Getting to Hill's equation (2)

So we have these equations:

$$x'' - \frac{\rho + x}{\rho^2} = \frac{B_y}{B\rho}\frac{p_0}{p}\left(1 + \frac{x}{\rho}\right)^2 \qquad y'' = -\frac{B_x}{B\rho}\frac{p_0}{p}\left(1 + \frac{x}{\rho}\right)^2$$

Expand the B field to first order in x,y:

$$B_y = -B_0 + \frac{\partial B_y}{\partial x}x \qquad B_x = \frac{\partial B_y}{\partial x}y$$

$$\frac{B_0}{B\rho} = \frac{1}{\rho}$$ ie. dipole field defines the closed orbit

$$x'' + K_x(s)x = 0 \qquad K_x = 1/\rho^2 - K_1(s)$$

$$y'' + K_y(s)y = 0 \qquad K_y = K_1(s)$$

$$K_1(s) = \frac{1}{B\rho}\frac{\partial B_1}{\partial x}$$

nb. in a quadrupole $K_x = -K_y$

Hill's Equation

Hill's equation is a linearised equation of motion describing particle oscillations:

$$\frac{d^2x}{ds^2} + k_x(s)x = 0 \qquad \frac{d^2y}{ds^2} + k_y(s)y = 0$$

Where k changes along the path, and $B_1(s) = \partial B_y / \partial x$

$$k_x(s) = \frac{1}{\rho^2} - \frac{B_1(s)}{B\rho} \qquad k_y(s) = \frac{B_1(s)}{B\rho}$$

evaluated at the closed orbit

Focusing functions are periodic over length L , ie. $K_{x,y}(s+L) = K_{x,y}(s)$

nb. In a quadrupole: $k_x(s) = -\frac{B_1(s)}{B\rho}$

Following similar notation to S. Y. Lee, Accelerator Physics, pp.41

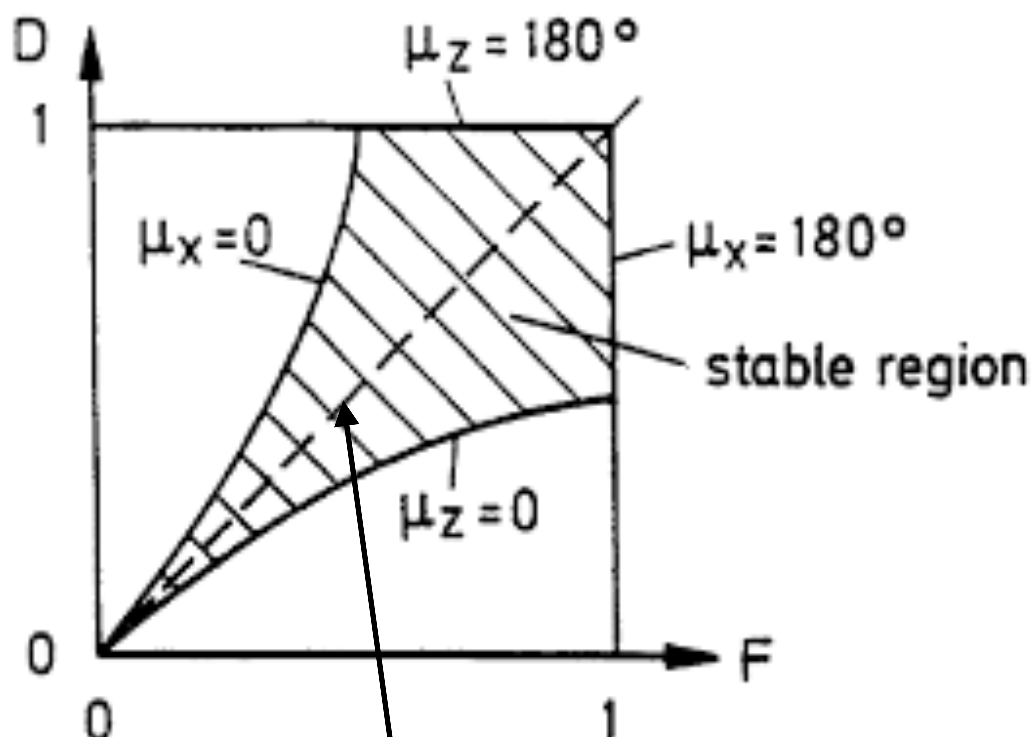
E. D. Courant and H. S. Snyder, "Theory of the alternating-gradient synchrotron," Annals of Physics, vol. 3, no. 1, pp. 1–48, 1958.

Stability: an example

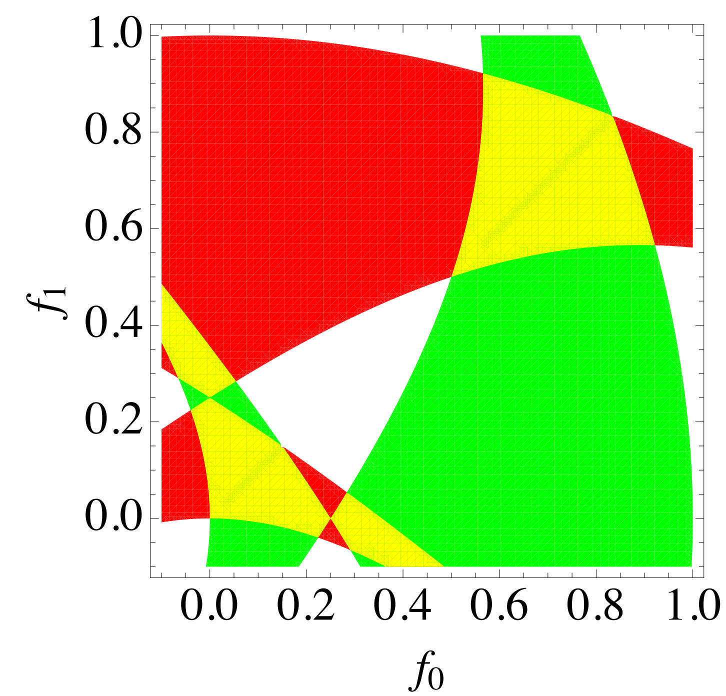
This solution is 'stable' in a periodic system when there is a real betatron phase advance or tune, such that:

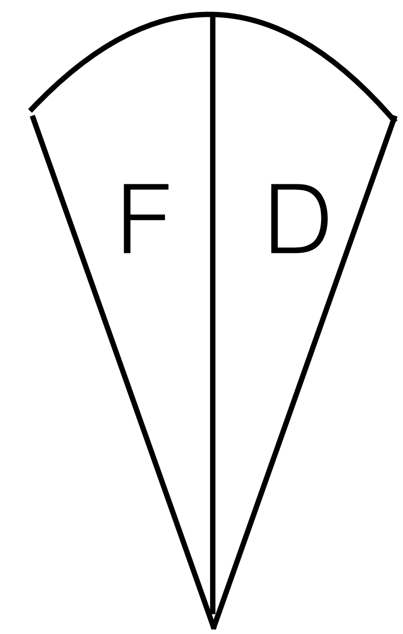
So let's test this out…

$$|Tr(M)| \leq 2$$



'stability region'





$$f(\theta) = \begin{cases} f_0 + f_1 = const., & 0 < \theta < \frac{1}{2}\theta_0 \\ f_0 - f_1 = const., & \frac{1}{2}\theta_0 < \theta < \theta_0. \end{cases}$$

I parameterise it in terms of f, think of f as focal length

Non-Linear Maps

Note that *the equations of motion are, in reality, nonlinear*

But we can often make linear approximations

$$x_2 = \mathbf{M}x_1 \qquad\qquad x_{a,2} = \sum R_{ab}x_{b,1} + \sum T_{abc}x_{b,1}x_{c,1} + \cdots$$

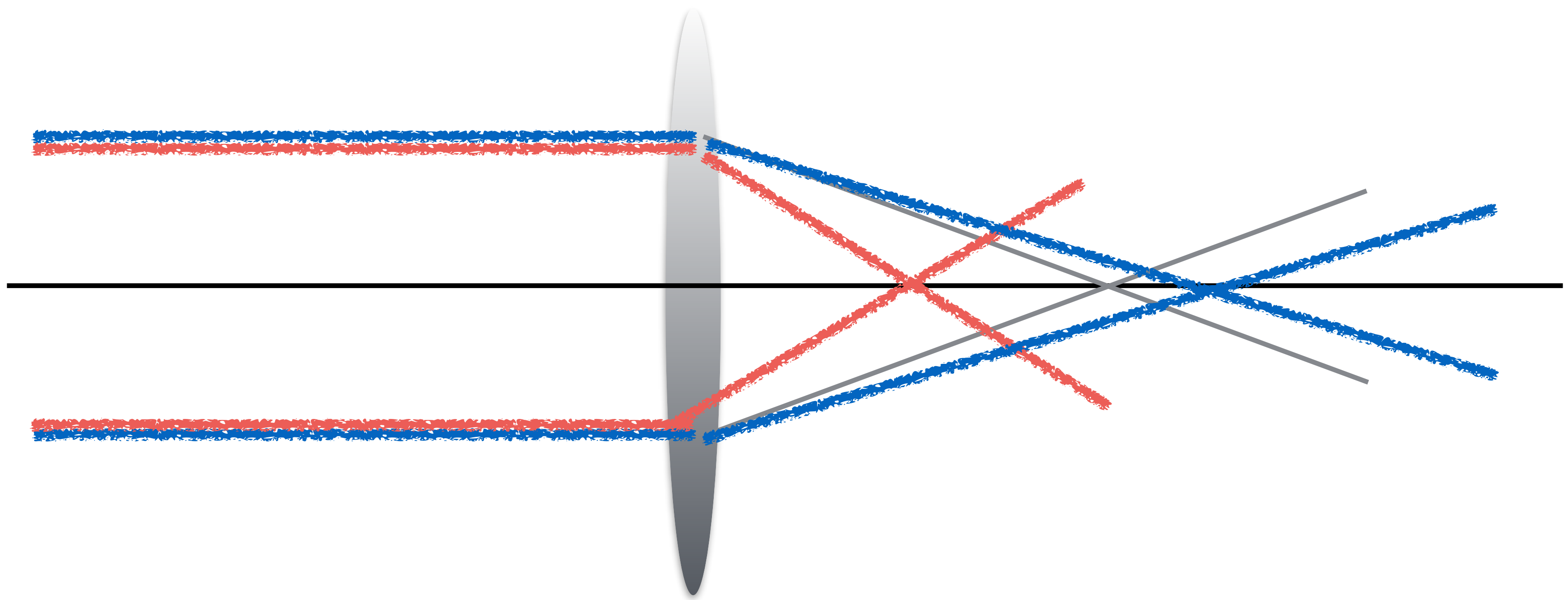
Taylor expansion of the map, where
R is the linear transfer matrix
T is the second order transfer matrix

But other 'map' objects can be used!

See for example:
A. Dragt, IEEE Trans. Nucl. Sci. Vol. NS-26, No.3, 1979.
http://accelconf.web.cern.ch/AccelConf/p79/PDF/PAC1979_3601.PDF

A glimpse of non-linear effects

- In fact, there are many things we've glanced over that are actually non-linear…



e.g. Chromaticity - focusing of a quadrupole changes with momentum

Particle Motion in EM Fields

Hopefully, we have done the following:

1. Arrived at a general description of particle motion in EM fields

2. Understood what “maps” are, and how they relate to particle motion and simulation

3. Derived some basic maps from the equations of motion

4. Understood the approach to compute linear and non-linear maps

5. Derived and looked at transfer matrices for main types of magnets used in accelerators

5. Got a glimpse of non-linear dynamics

References

Beam Dynamics:

- A. Wolski, "Beam Dynamics in High Energy Particle Accelerators", Imperial College Press, 2014.
- S. Y. Lee, "Accelerator Physics", 3rd Edition, World Scientific, 2011.
- K. Brown, SLAC-75-rev-4 (1982); SLAC-91-rev-2 (1977)

Electromagnetism:

- J. D. Jackson, Classical Electrodynamics, 3rd Ed, Wiley & sons (1999).

Hamiltonian Mechanics:

- H. Goldstein et al., Classical Mechanics (3rd ed.). Addison-wesley (2001).

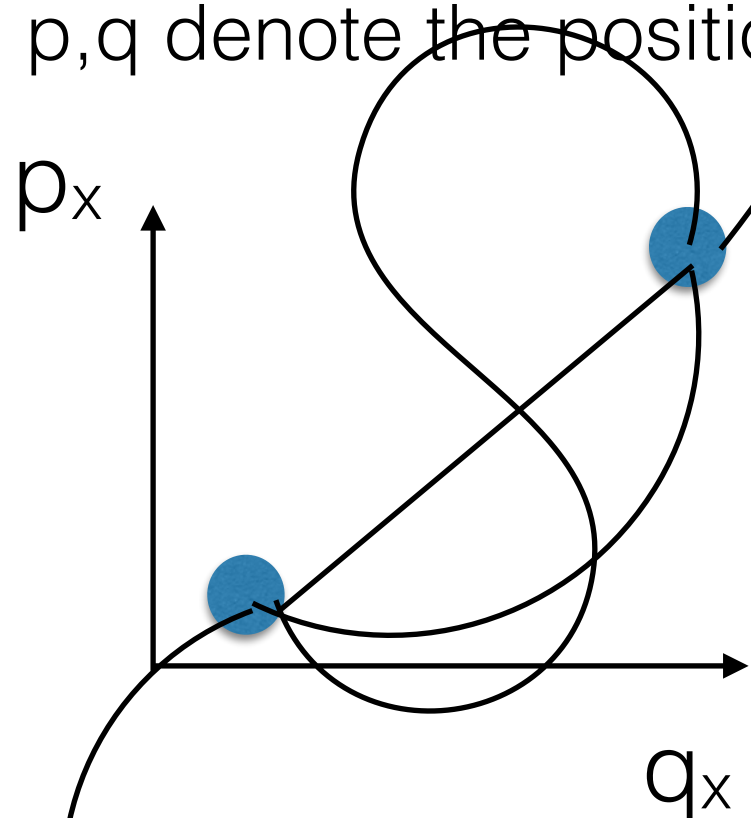
Most of the images/animations I've used here:
http://richannel.org/collections/2016/particle-accelerators-for-humanity

Additional Material

Aside (if needed): Let's recap some classical dynamics

- "Canonically conjugate variables" (q,p) =
  - q = 'generalised co-ordinate'
  - p = 'generalised momentum'

- p,q denote the position of a particle in *phase space*



$$\partial S = \partial\left[\int_{t1}^{t2} L.dt\right] = 0$$

Action = 'S'
Principle of least action: nature takes the path between t1 & t2 which minimises S.

Let's check if the following solves Hill's equation… $x'' + kx = 0$

$$\boxed{x = \sqrt{\beta(s)\varepsilon}\cos(\phi(s) + \phi_0)}$$

Substitute $w = \sqrt{\beta}$ $\phi = \phi(s) + \phi_0$

& differentiate…

$$x' = \sqrt{\varepsilon}\left\{ w'(s)\cos\phi - \frac{d\phi}{ds} w(s)\sin\phi \right\}$$

nb. we need: $\dfrac{d\phi}{ds} = \dfrac{1}{\beta(s)} = \dfrac{1}{w^2(s)}$

Differentiate again…

$$x'' = \sqrt{\varepsilon}\left\{ w''(s)\cos\phi \underbrace{- \frac{w'(s)}{w^2(s)}\sin\phi + \frac{w'(s)}{w^2(s)}\sin\phi}_{=0} - \frac{1}{w^3}\cos\phi \right\}$$

Sub into Hill's…

$$\sqrt{\varepsilon}\left\{ w''(s)\cos\phi - \frac{1}{w^3}\cos\phi \right\} + kw\sqrt{\varepsilon}\cos\phi = 0$$

gives… $\boxed{w''(s) - \dfrac{1}{w^3} + kw = 0}$

'envelope equation' $\boxed{\dfrac{1}{2}\beta\beta'' - \dfrac{1}{4}\beta'^2 + k\beta^2 = 1}$

The usual approach - transfer matrices

Express solution in matrix form…

$$\vec{x}(s) = M(s \mid s_0)\vec{x}(s_0) \qquad \vec{x}(s) = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

Where M is the ‘transfer matrix’.

We already know (because we showed)

$$\frac{d\phi(s)}{ds} = \frac{1}{w^2}$$

$$x = w\sqrt{\varepsilon}\cos(\phi(s) + \phi_0)$$

reminder…

$$\frac{d(\cos(f(x))}{dx} = -\sin(f(x))\frac{df(x)}{dx}$$

Take derivative for x’…

$$x' = w'\sqrt{\varepsilon}\cos(\phi(s) + \phi_0) - \frac{\sqrt{\varepsilon}}{w}\sin(\phi(s) + \phi_0)$$

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

Trace two rays…

‘cosine like’ $\phi = 0$ ‘sine like’ $\phi = \pi / 2$

$$x = w\sqrt{\varepsilon}\cos(\phi(s) + \phi_0)$$

$$x' = w'\sqrt{\varepsilon}\cos(\phi(s) + \phi_0) - \frac{\sqrt{\varepsilon}}{w}\sin(\phi(s) + \phi_0)$$

Yields 4 simultaneous equations so we can solve for a,b,c,d…

$$\mu = \phi_2 - \phi_1$$

$$M_{12} = \begin{pmatrix} \frac{w_2}{w_1}\cos\mu - w_2 w_1{}' \sin\mu & w_1 w_2 \sin\mu \\ -\frac{1 + w_1 w_1{}' w_2 w_2{}'}{w_1 w_2}\sin\mu - \left(\frac{w_1{}'}{w_2} - \frac{w_2{}'}{w_1}\right)\cos\mu & \frac{w_1}{w_2}\cos\mu + w_1 w_2{}' \sin\mu \end{pmatrix}$$

You will see this later…

Simplify by considering a period or ‘turn’, and w’s are equal.

$$M_{period} = \begin{pmatrix} \cos\mu - ww' \sin\mu & w^2 \sin\mu \\ -\frac{1 + w^2 w'^2}{w^2} \sin\mu & \cos\mu + ww' \sin\mu \end{pmatrix}$$

If we define the so-called ‘Twiss’ or ‘Courant-Snyder’ parameters:

$$\beta = w^2 \qquad \alpha = -\frac{1}{2}\beta' \qquad \gamma = \frac{1+\alpha}{\beta}$$

$$M_{period} = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

(sorry that we are reusing symbols again… these are NOT the relativistic parameters)