A first taste of Non-Linear Beam Dynamics

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Purpose of the lecture

- Introducing aspects of non-linear dynamics
  - Mathematical tools for modelling nonlinear dynamics
    - Power series (Taylor) maps and symplectic maps.
  - Effects of nonlinear perturbations
    - Resonances, tune shifts, dynamic aperture.
  - Analysis methods:
    - Normal forms, frequency map analysis.
- Employ two types of accelerator systems for illustrating the methods and tools
  - Bunch compressor (a single-pass system)
  - A storage ring (a multi-turn system).

Aim of the 2nd Lecture

- Describe some of the **phenomena** associated with nonlinearities in periodic beamlines (such as storage rings)
- Explain significance of **symplectic maps**, and describe some of the challenges in calculating and applying symplectic maps
- Outline some of the **analysis methods** that can be used to characterise nonlinear beam dynamics in periodic beamlines.

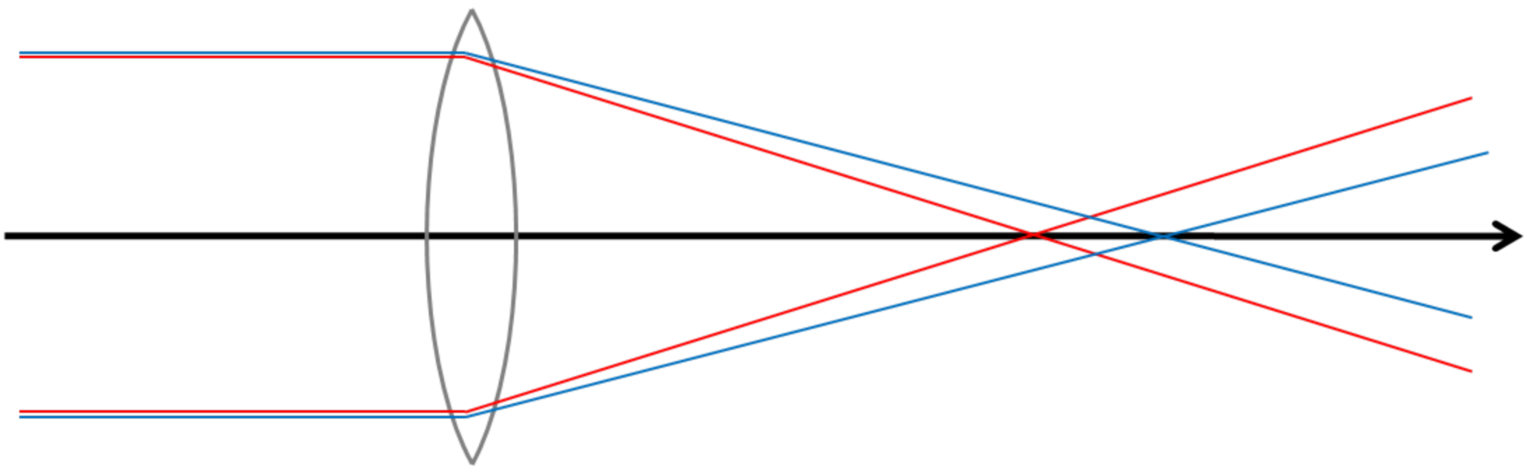
Example of a periodic system: a simple storage ring

A simple storage ring

- As example, consider the transverse dynamics in a **simple storage ring**, assuming:
  - The storage ring is constructed from some number of **identical cells** consisting of dipoles, quadrupoles and sextupoles.
  - The **phase advance** per cell can be tuned from close to zero, up to about $0.5 \times 2\pi$.
  - There is **one sextupole per cell**, which is located at a point where the horizontal beta function is 1 m, and the alpha function is zero.
- Usually, storage rings will contain (at least) two sextupoles per cell, to correct horizontal and vertical chromaticity. To keep things simple, we will use only one sextupole per cell.

Reminder: chromaticity correction

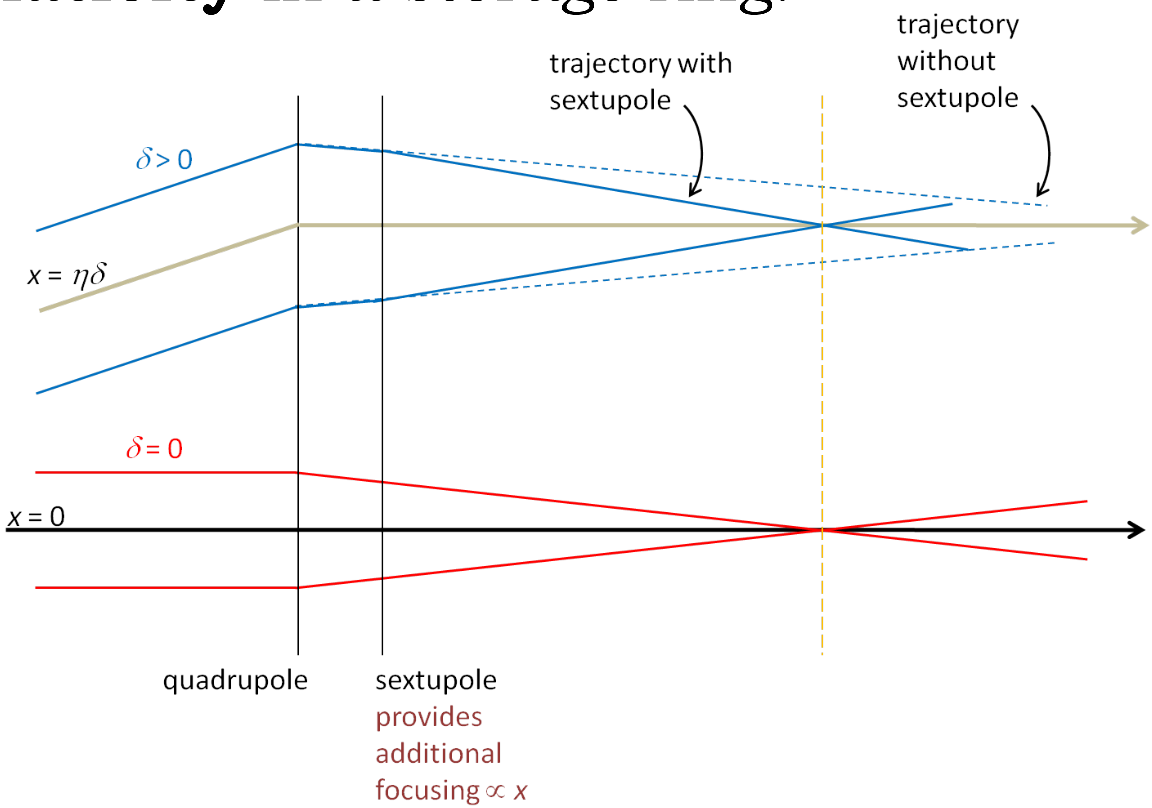
- Sextupoles are needed in a storage ring to compensate for the fact that quadrupoles have **lower focusing** strength for particles of **higher energy**:



- The change in focusing strength with particle energy has undesirable consequences, especially in storage rings: it can lead to **particle motion** becoming **unstable** because of **resonances**

Reminder: chromaticity correction

- A **sextupole** can be regarded as a **quadrupole** with **focusing strength** that **increases** with **horizontal offset** from the axis.
- If **sextupoles** are located where there is **non-zero dispersion**, they can be used to control the **chromaticity** in a storage ring.



Linear dynamics in a storage ring

- The **chromaticity**, and hence the sextupole strength, will normally be a **function** of the **phase advance**
- However, just to investigate the nonlinear effects of the sextupoles, we shall keep the **sextupole strength** $k_2 L$ **fixed**, and **change** only the **phase advance**
- We can assume that the **map** from **one sextupole** to the **next** is **linear**, and corresponds to a **rotation** in phase space through an angle equal to the phase advance:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto \begin{pmatrix} \cos\mu_x & \sin\mu_x \\ -\sin\mu_x & \cos\mu_x \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}$$

- Again to keep things simple, we shall consider only **horizontal motion**, and assume that the vertical co-ordinate $y = 0$

Nonlinear transfer map: sextupole

- Recall that the **vertical field component** in a sextupole magnet is:

$$\frac{B_y}{B\rho} = \frac{1}{2} k_2 x^2$$

with $B\rho$ the beam rigidity and the normalized sextupole gradient is

$$k2 = \frac{1}{B\rho} \frac{\partial^2 B_y}{\partial x^2}$$

- In the "thin lens" approximation, the **deflection** of a particle passing through the sextupole of length L is

$$\Delta p_x = -\int \frac{B_y}{B\rho} ds = -\frac{1}{2} k_2 L x^2$$

Nonlinear transfer map: sextupole

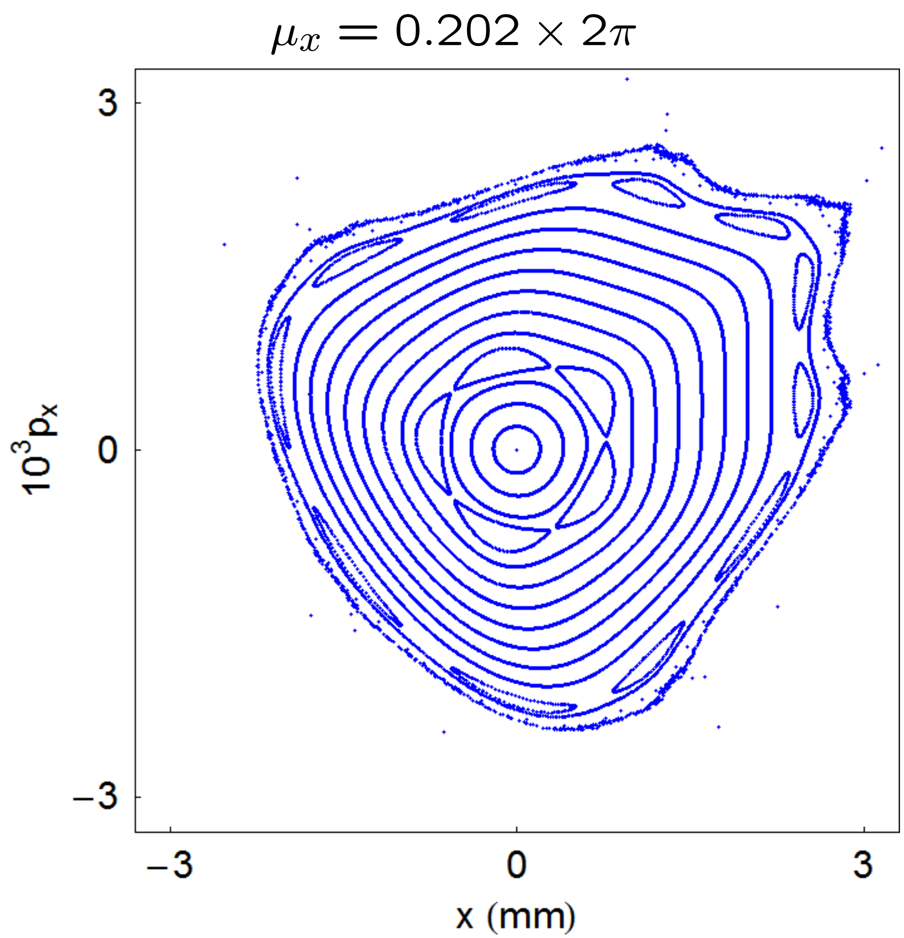
- The map for a particle moving through a short **sextupole** can be represented by a **“kick”** in the horizontal momentum:

$$
\begin{aligned}
x &\mapsto x \,, \\
p_x &\mapsto p_x - \frac{1}{2} k_2 L x^2
\end{aligned}
$$

- Let us choose a fixed value $k_2 L = -600 \text{ m}^{-2}$, and look at the effects of the maps for different phase advances.
- For each case, we construct a **phase space portrait** by plotting the values of the dynamical variables after repeated application of the map (rotation + sextupole) for a range of initial conditions.
- First, let us look at the phase space portraits for a range of phase advances from **0.2 × 2π** to **0.5 × 2π**

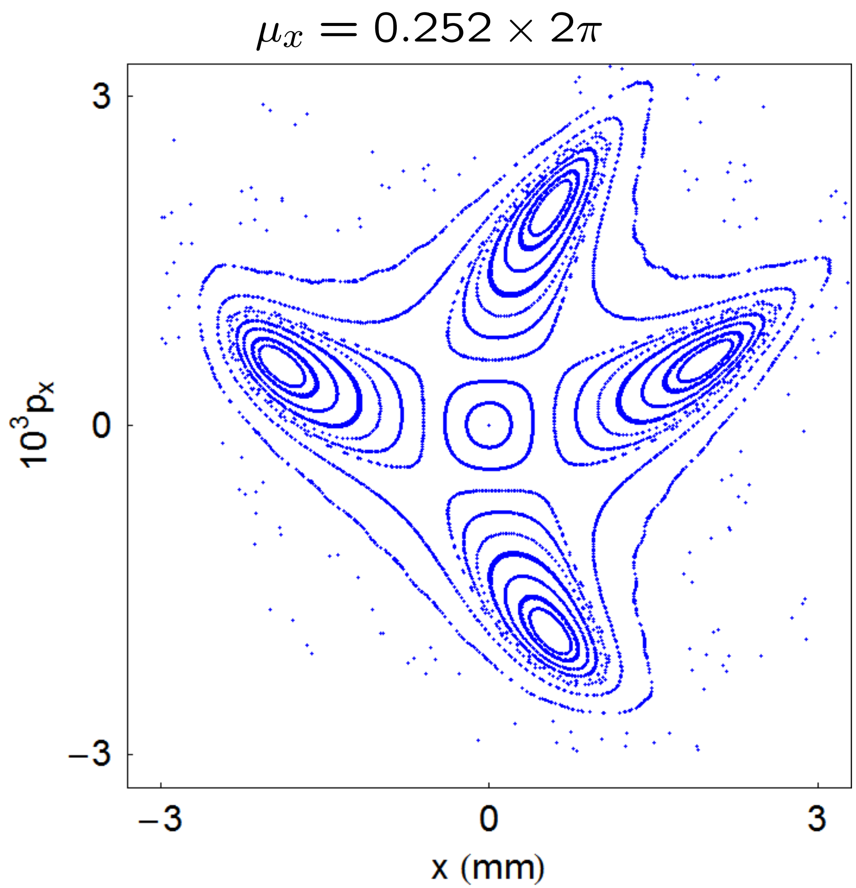
Phase space: single sextupole

- For **small amplitudes**, particles trace out **closed loops** around the origin: this is what we expect for a purely **linear map**
- As the **amplitude** is **increased**, there appear "**islands**" in phase space. The **phase advance** (for the linear map) is often close to m/p, where m is an integer and p is the number of islands
- **Larger number** of islands appears at **larger amplitude**



Phase space: single sextupole

- Usually, there is a **closed curve** that divides a region of **stable** from a region of **chaotic** motion.
- **Outside** that curve, the **amplitude** of particles **increases without limit**
- The area of the stable region **depends** strongly on the **phase advance**



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- **Outside** that curve, the **amplitude** of particles **increases without limit**
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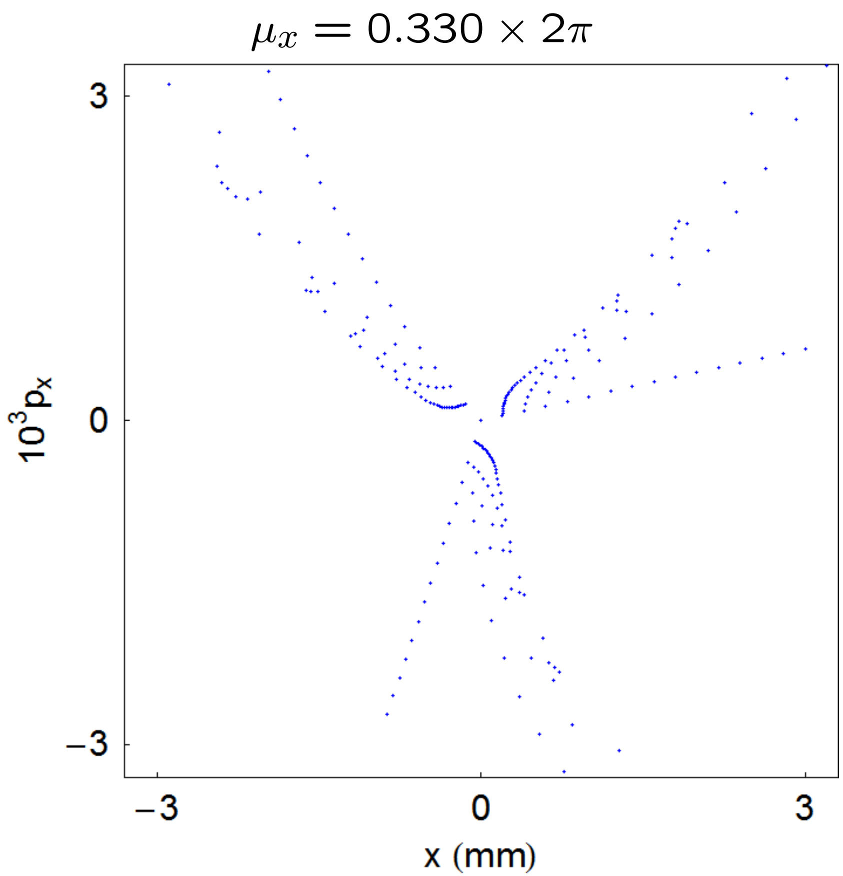
- For **phase advance** close to **2π/3**, it appears that the **stable region** almost **vanishes** altogether



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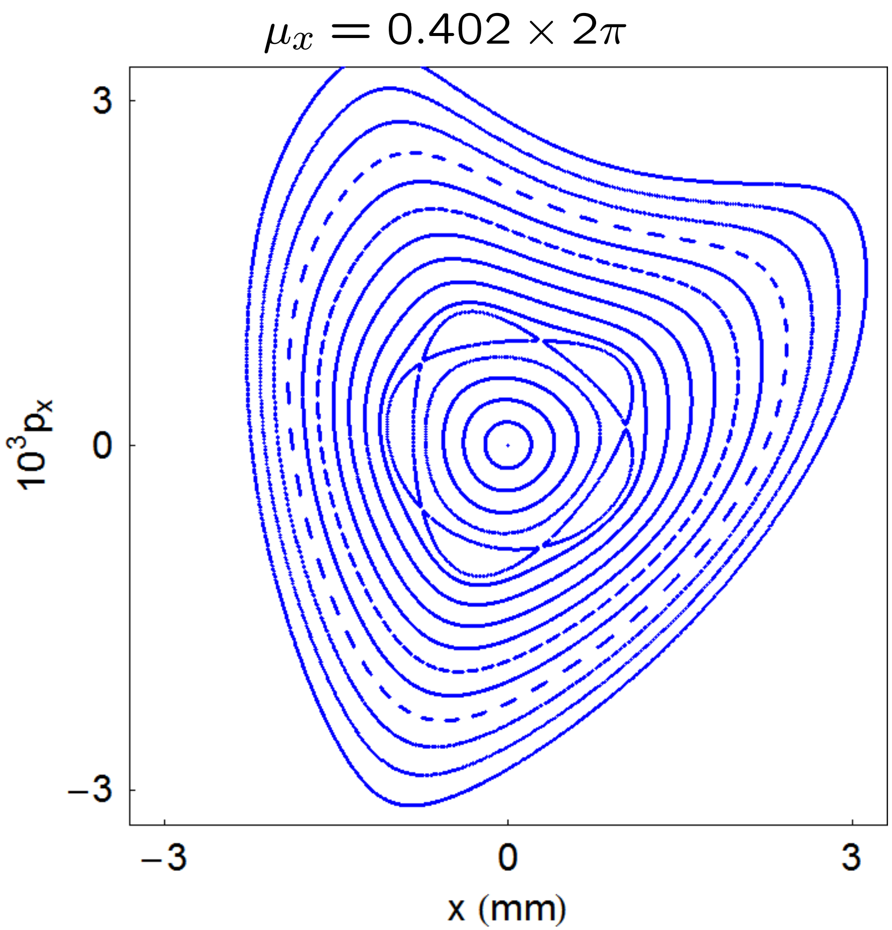
- When **phase advance** approaches **π**, the **stable area** becomes **large**, and distortions from the linear ellipse become less evident



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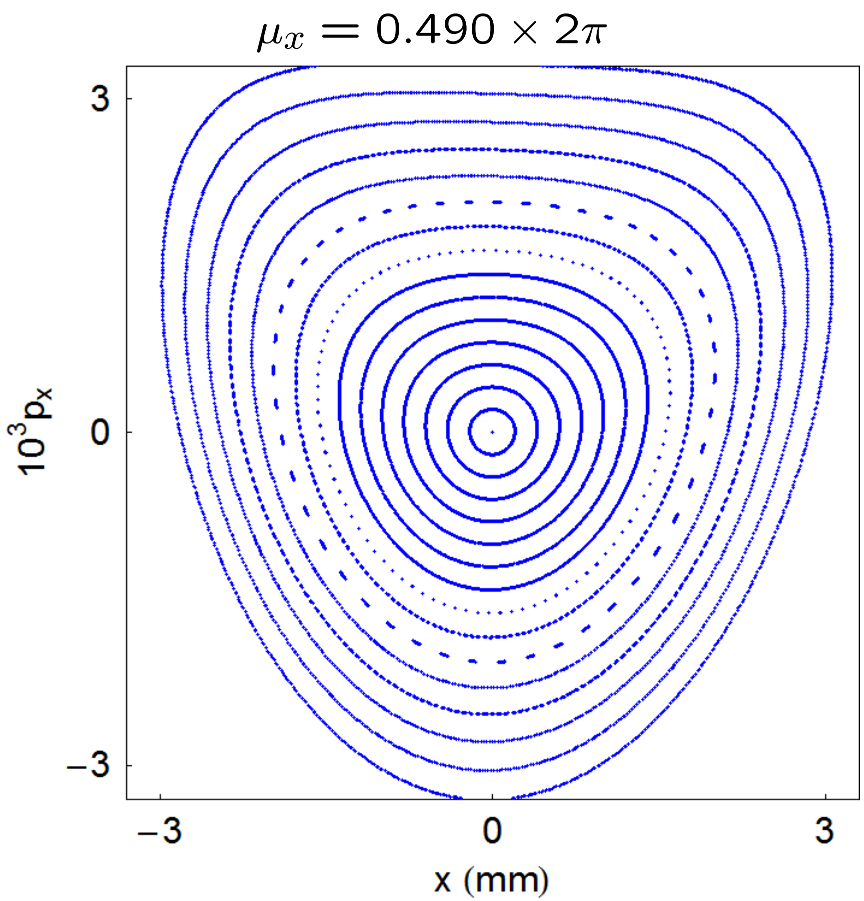


Effect of phase advance on nonlinear dynamics

Effect of phase advance

- An important observation is that the **effect** of the sextupole in the periodic cell **depends strongly** on the **phase advance** across the cell
- We can start to understand the significance of the phase advance by considering **two special cases**:
  - Phase advance equal to an **integer** times 2π
  - Phase advance equal to a **half integer** times 2π

Integer phase advance

- Let us consider first a **phase advance** equal to an **integer**. In that case, the linear part of the map is just the identity

$$\begin{aligned} x &\mapsto x \,, \\ p_x &\mapsto p_x \end{aligned}$$

- So the **combined effect** of the **linear map** and the **sextupole kick** is:

$$\begin{aligned} x &\mapsto x \,, \\ p_x &\mapsto p_x - \frac{1}{2} k_2 L x^2 \end{aligned}$$

- Clearly, the **horizontal momentum** will **increase** without limit
- There are **no stable regions** of phase space, apart from the line $x = 0$

Half- Integer phase advance

- Now consider what happens if the phase advance of a cell is a **half integer** times 2π, so the linear part of the map is just a rotation through π.
- If a **particle** starts at the entrance of a sextupole with $x = x_0$ and $p_x = p_{x0}$, then at the **exit** of that sextupole:

$$\begin{aligned} x_1 &= x_0 \, , \\ p_{x1} &= p_{x0} - \frac{1}{2} k_2 L x_0^2 \end{aligned}$$

- Then, after passing to the **entrance** of the **next sextupole**, the co-ordinates will be:

$$\begin{aligned} x_2 &= \cos(\pi) x_1 = -x_1 = -x_0 \, , \\ p_{x2} &= \cos(\pi) p_{x1} = -p_{x1} = -p_{x0} + \frac{1}{2} k_2 L x_0^2 \end{aligned}$$

Half- Integer phase advance

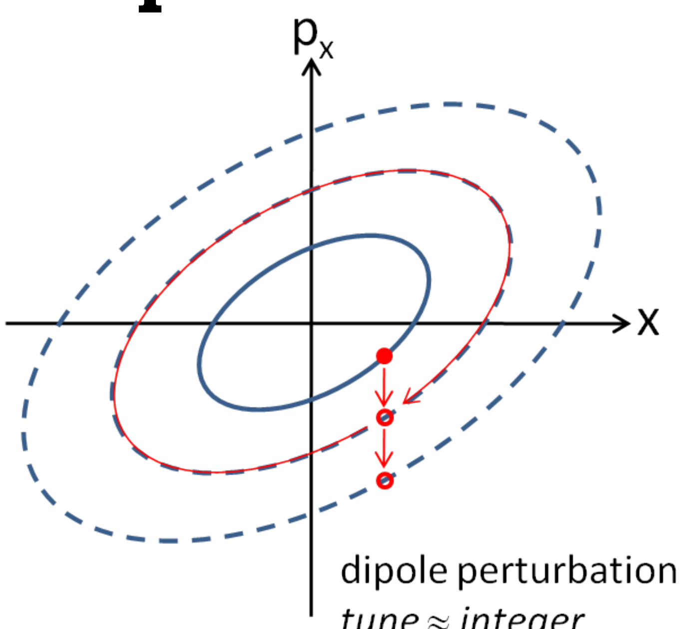
- Finally, on **passing** through the **second sextupole**:

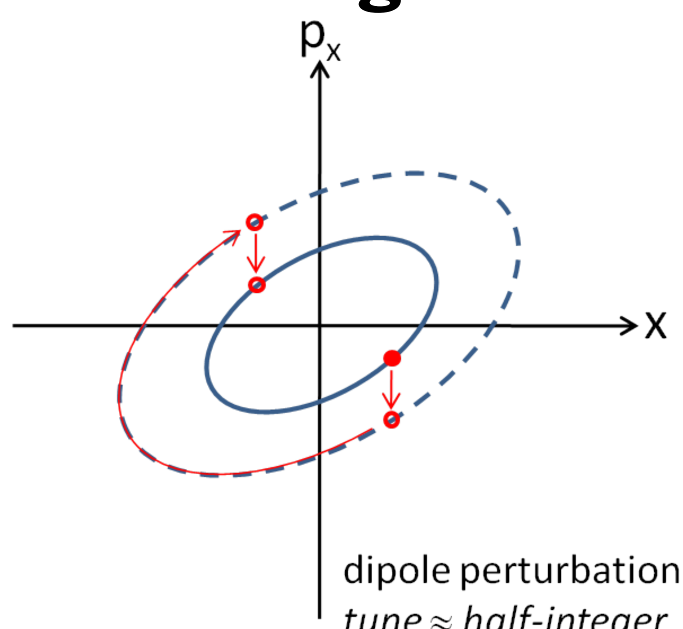
$$\begin{aligned} x_3 &= x_2 = -x_0 \ , \\ p_{x3} &= p_{x2} - \frac{1}{2} k_2 L x_2^2 = -p_{x0} \end{aligned}$$

- In other words, the **momentum kicks** from the two sextupoles **cancel** each other exactly
- The resulting map is a purely **linear phase space rotation** by π.
- In this situation, we expect the **motion** to be **stable** (and periodic), no matter what the amplitude

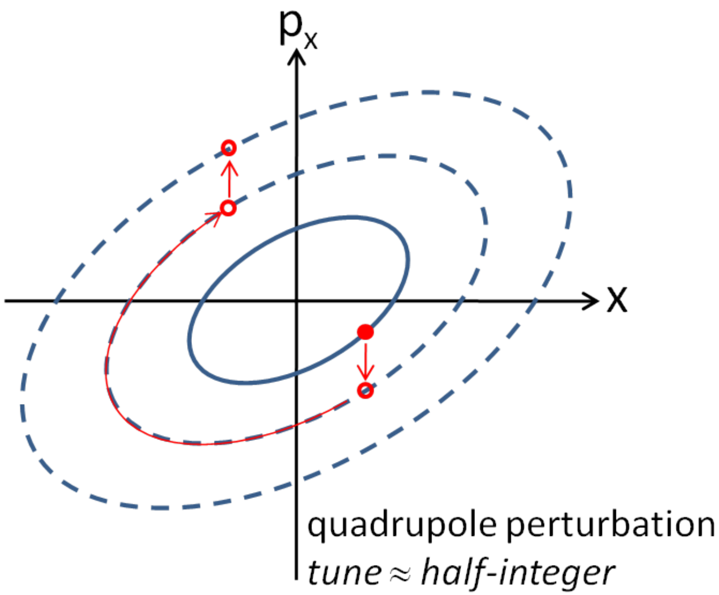
Impact of phase advance: dipoles

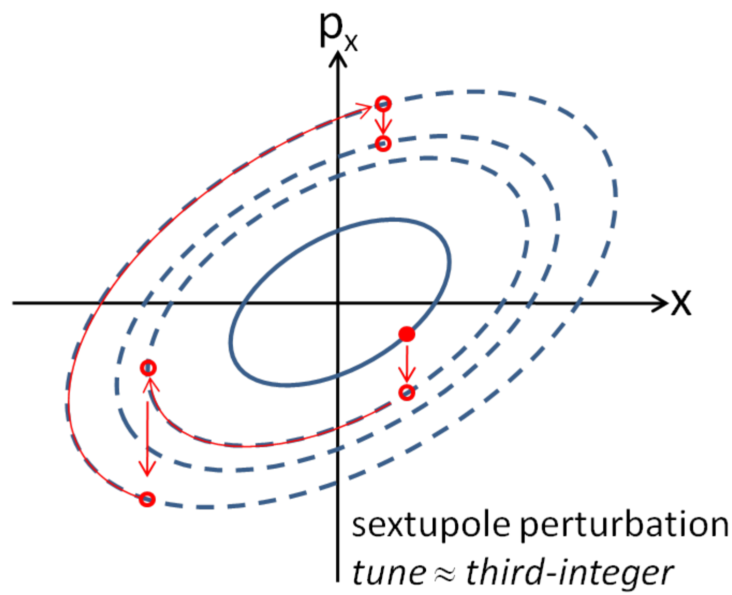
- The effect of the phase advance on the sextupole "kicks" is **similar** to the effect on **perturbations** arising from **dipole** and **quadrupole errors** in a storage ring
- In the case of dipole errors, the **kicks add up** if the phase advance is an **integer**, and **cancel** if the **phase advance** is a **half integer**





- In the case of **quadrupole errors**, the **kicks add up** if the phase advance is a **half integer**.
- **Higher-order multipoles** drive **higher-order resonances** but the effects are less easily illustrated on a phase space diagram.





Resonances

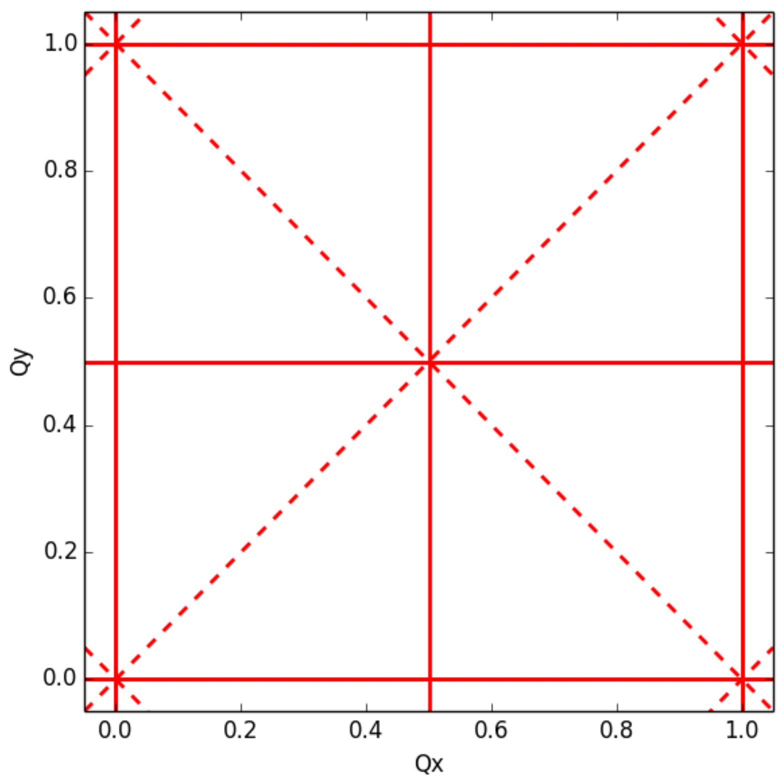
Resonances

- If we include vertical as well as horizontal motion, then we find that **resonances** occur when the tunes satisfy

$$\boxed{n_x Q_x + n_y Q_y = r}$$

where n_x, n_y and r are integers and resonance is of **order** $|n_x| + |n_y|$

Resonances up to order 2



normal resonances (= even n_y)

skew resonances (= odd n_y)

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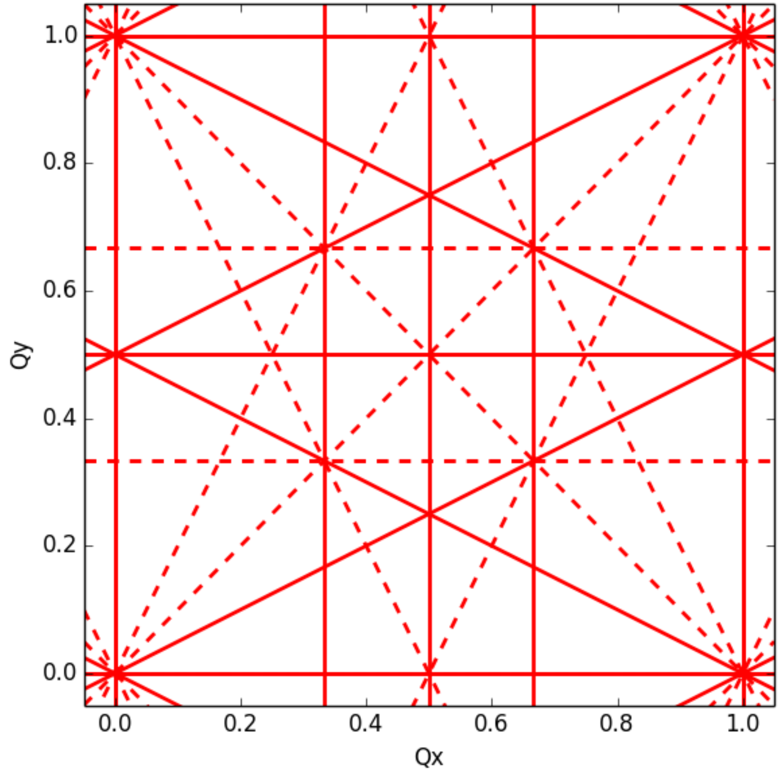
Resonances up to order 3



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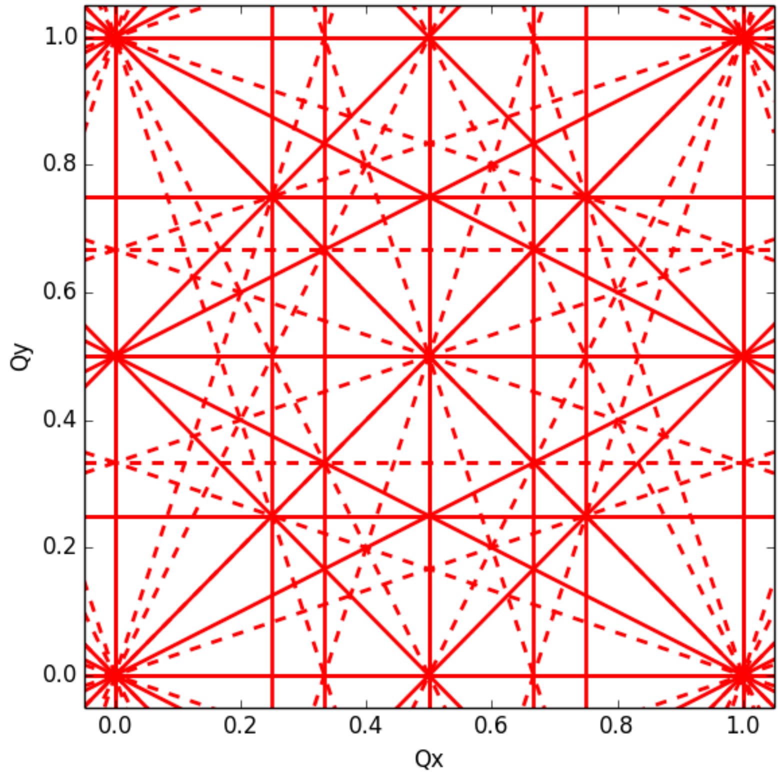
Resonances up to order 4



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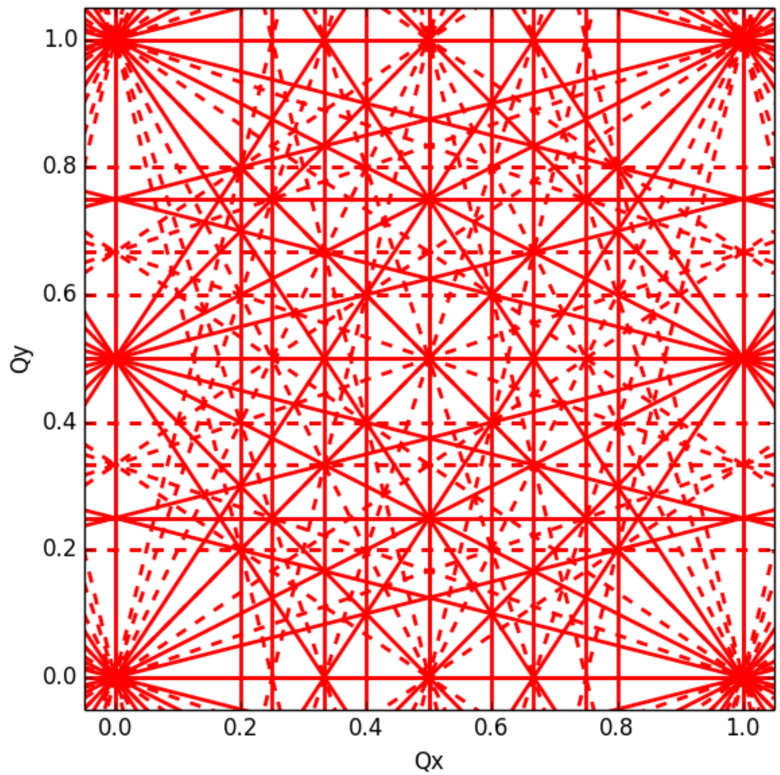
Resonances up to order 5



normal resonances (= even n_y)

skew resonances (= odd n_y)

Resonances in storage ring

- **Resonances** are associated with **chaotic motion** for particles in storage rings.
- However, the number of **resonance lines** in tune space is **infinite**: any point in tune space will be close to a resonance of some order.
- This observation raises two questions:
  - How do we know what the real effect of any given resonance line will be?
  - How can we design a storage ring to minimise the adverse effects of resonances?
- These are not easy questions to answer and usually necessitate numerical integration of the equations of motion
- We shall discuss some of the issues in the remaining parts of this lecture.

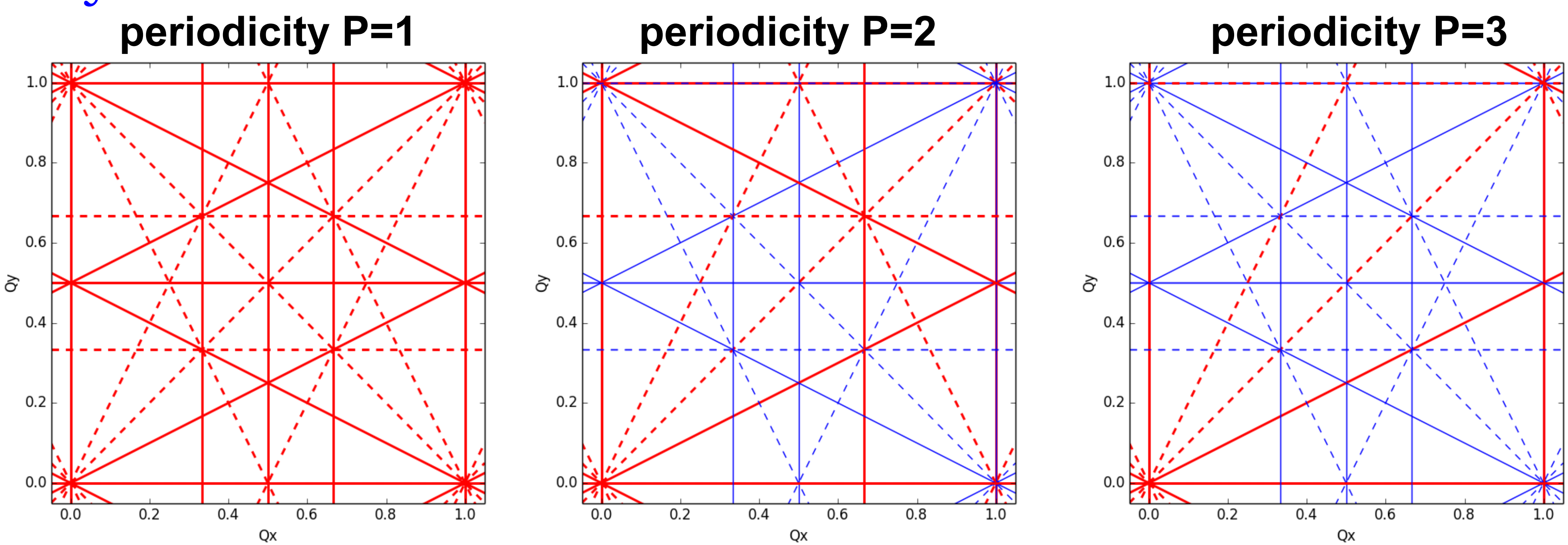
Resonance cancellation by periodicity

- By imposing a **periodicity *P*** in the lattice (i.e. building a machine from *P* identical cells) the resonance condition becomes

$$\boxed{n_x Q_x + n_y Q_y = rP}$$

- Resonances for which $r \times P$ is integer → systematic

- If $r \times P$ is NOT integer the resonance cancels → non-systematic



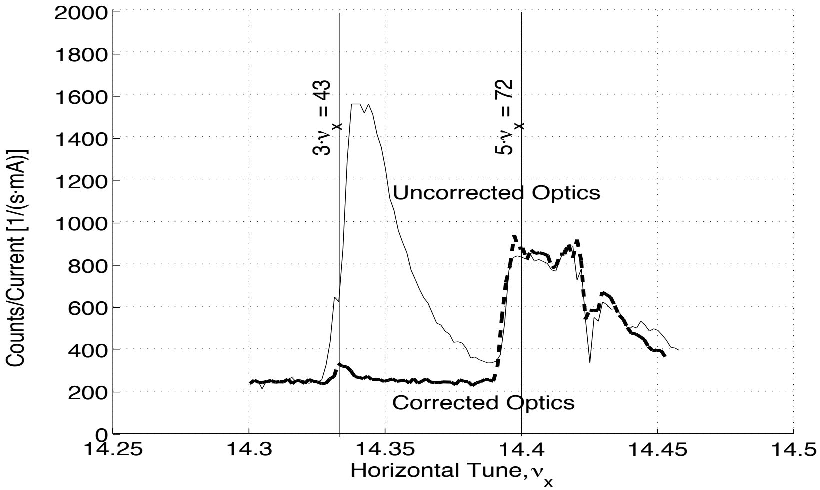
solid lines: normal resonances

dashed lines: skew resonances

Real life example for periodicity: ALS

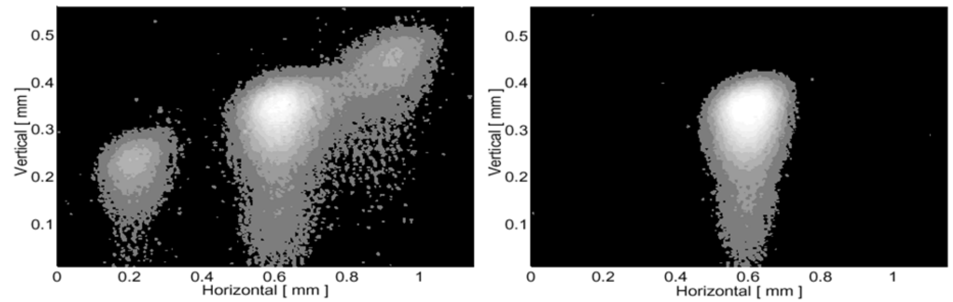
Advanced Light Source design lattice periodicity: 12

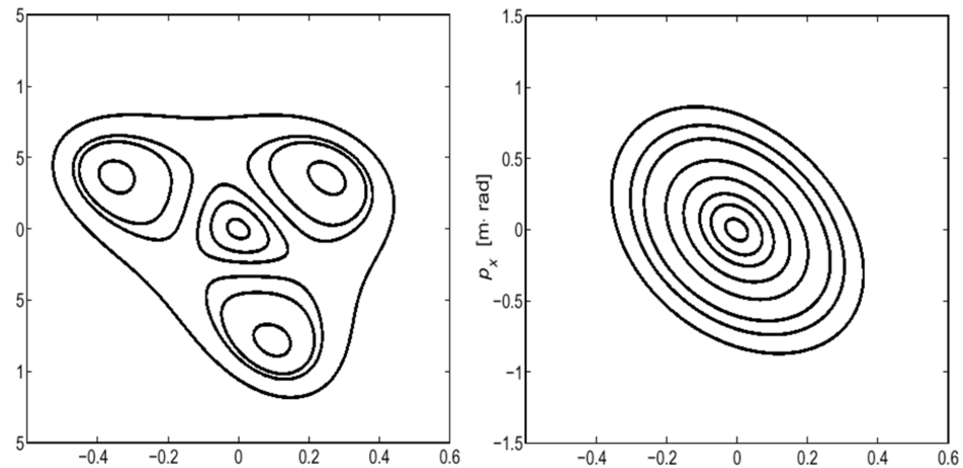
Measurement of beam loss as function of tune



Synchrotron light beam spot

Uncorrected optics **Corrected optics**





Simulated phase space

Beta beating

Before optics correction: ~30%

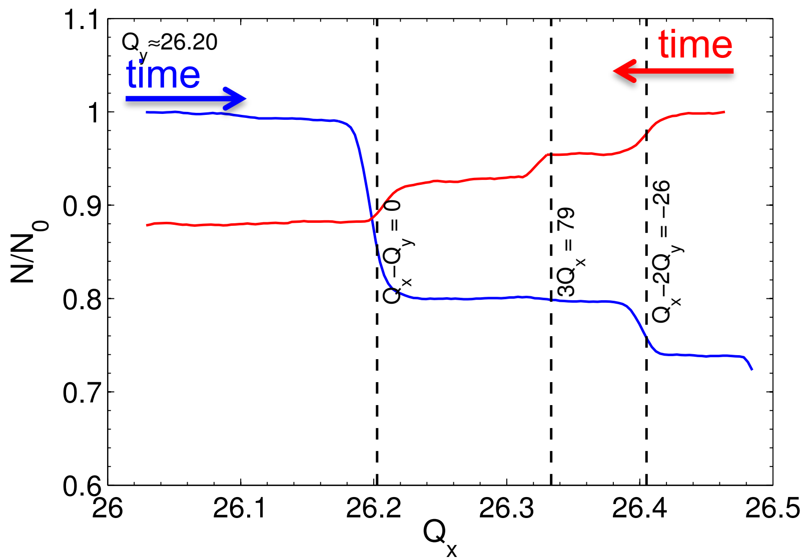
After optics correction: <1%

D. Robin, C. Steier, J. Safranek, W. Decking, "Enhanced performance of the ALS through periodicity restoration of the lattice," proc. EPAC 2000.

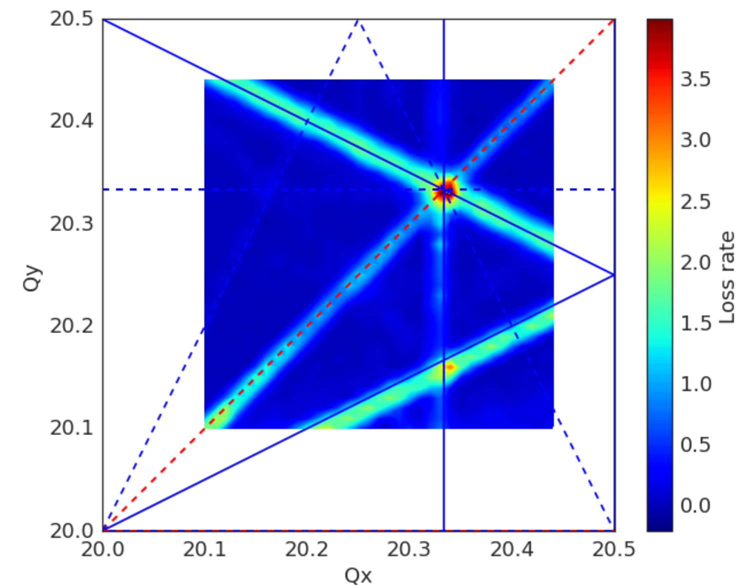
Real life example for periodicity: SPS

- SPS (hadron machine) has design lattice **periodicity** of **6**
- Some indication for the strength of individual resonance lines can be inferred from the **beam loss rate** during dynamic tune scans, i.e. the derivative of the beam intensity at the moment of resonance crossing
- **Sextupole resonances** can be clearly identified although they should be suppressed by lattice periodicity … but **SPS has no individual quadrupoles** to restore optics functions distortions

Measured losses during tune scan



Measured loss rate in 2D scan



Analytical methods for nonlinear dynamics

Analytical methods for NLD

- There are two approaches widely used in accelerator physics: **perturbation theory** and **normal form** analysis.
- In both these techniques, the goal is to construct a **quantity** that is **invariant** under application of the single-turn transfer map. Unfortunately, in both cases the mathematics is complicated and fairly cumbersome
- In the case of a **single sextupole** in a storage ring, we find from normal form analysis the following expression for the betatron action as a function of the betatron phase (angle variable):

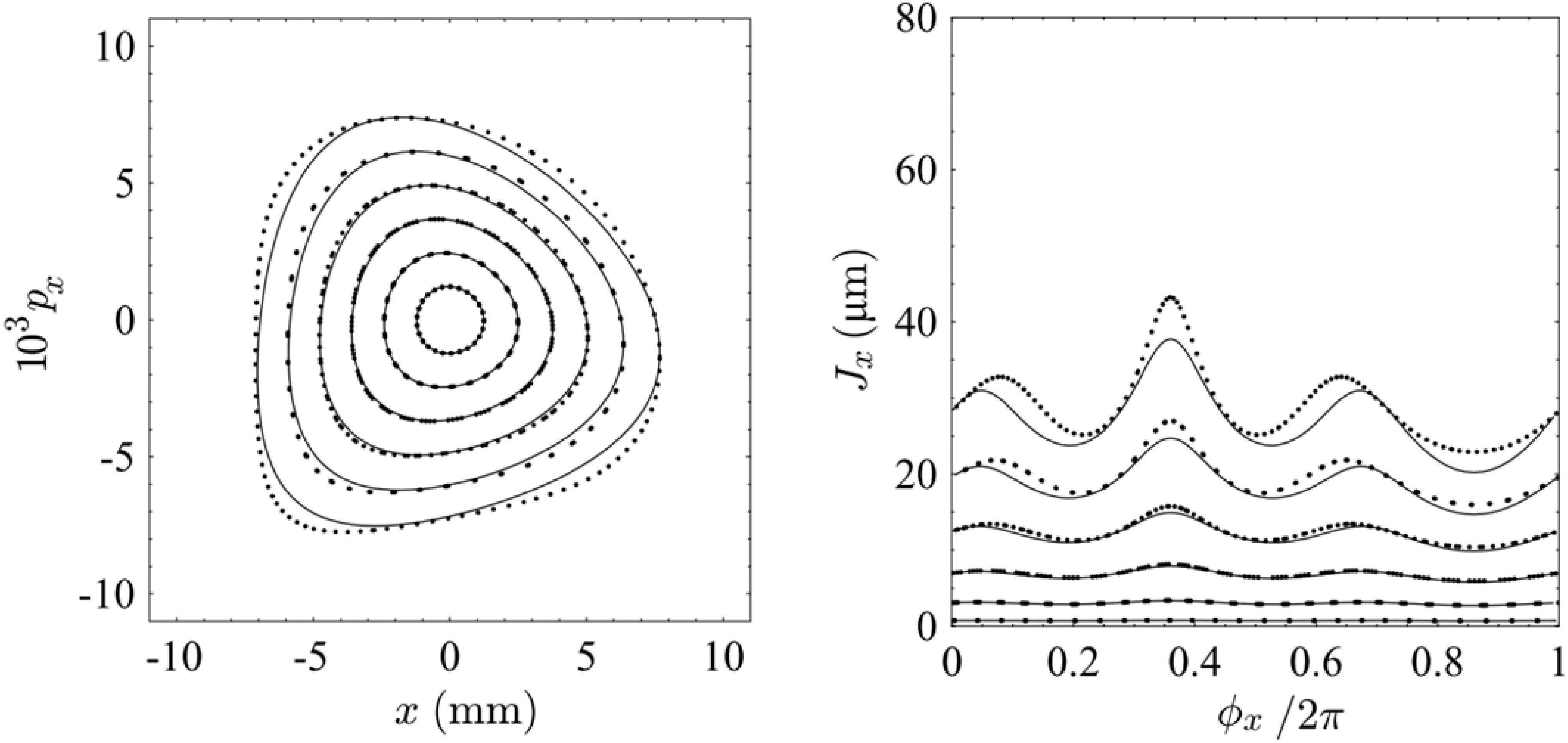
$$J_x \approx I_0 - \frac{k2L}{8}(2\beta_x I_0)^{3/2}\frac{\cos(3\mu_x/2 + 2\phi_x) + \cos(\mu_x/2)}{\sin(3\mu_x/2)} + O(I_0^2)$$

where I_0 is a constant (an invariant of the motion), ϕ_x is the angle variable, and μ_x is the phase advance per cell.

- The second term becomes **very large** when μ_x is close to **third integer**

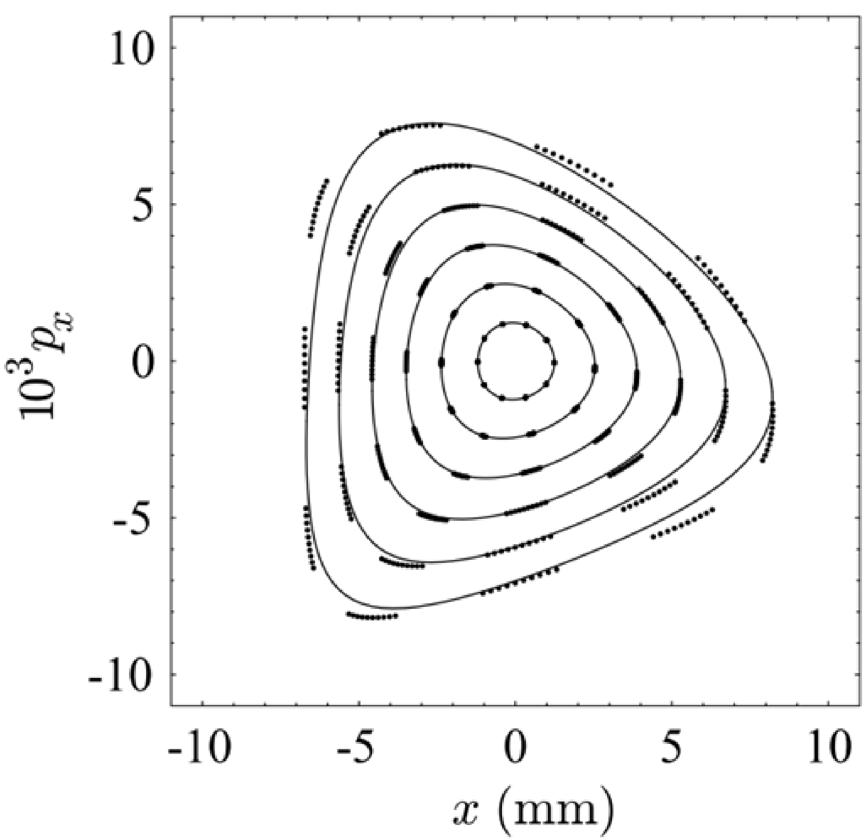
Normal form for sextupole

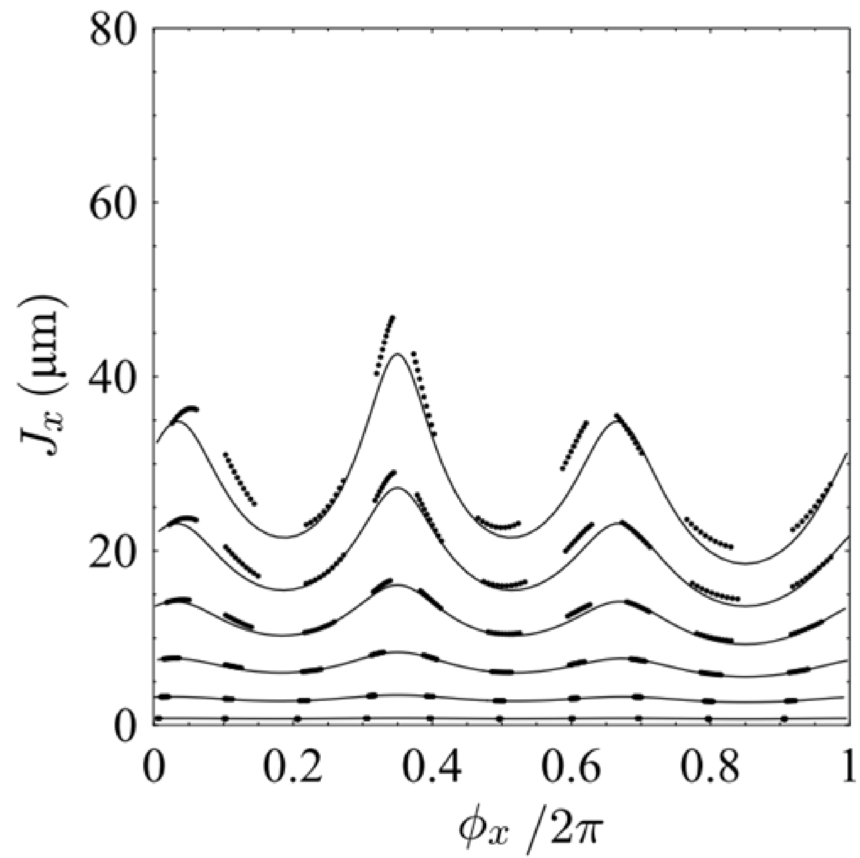
phase advance $\mu_x = 0.28 \times 2\pi$



Normal form for sextupole

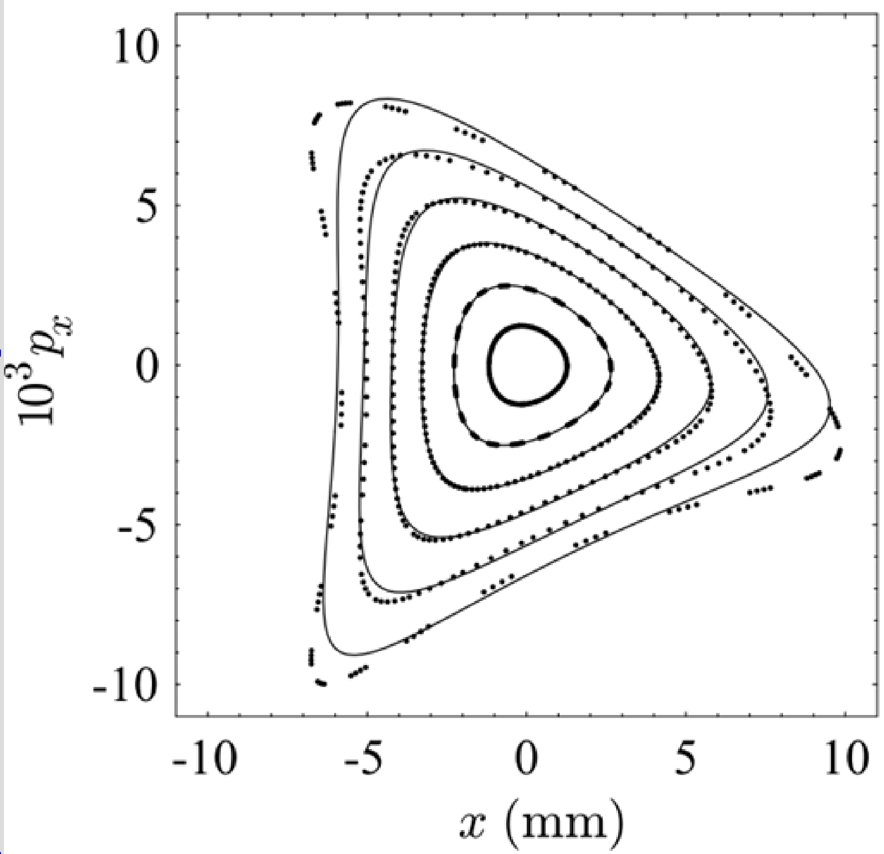
phase advance $\mu_x = 0.30 \times 2\pi$

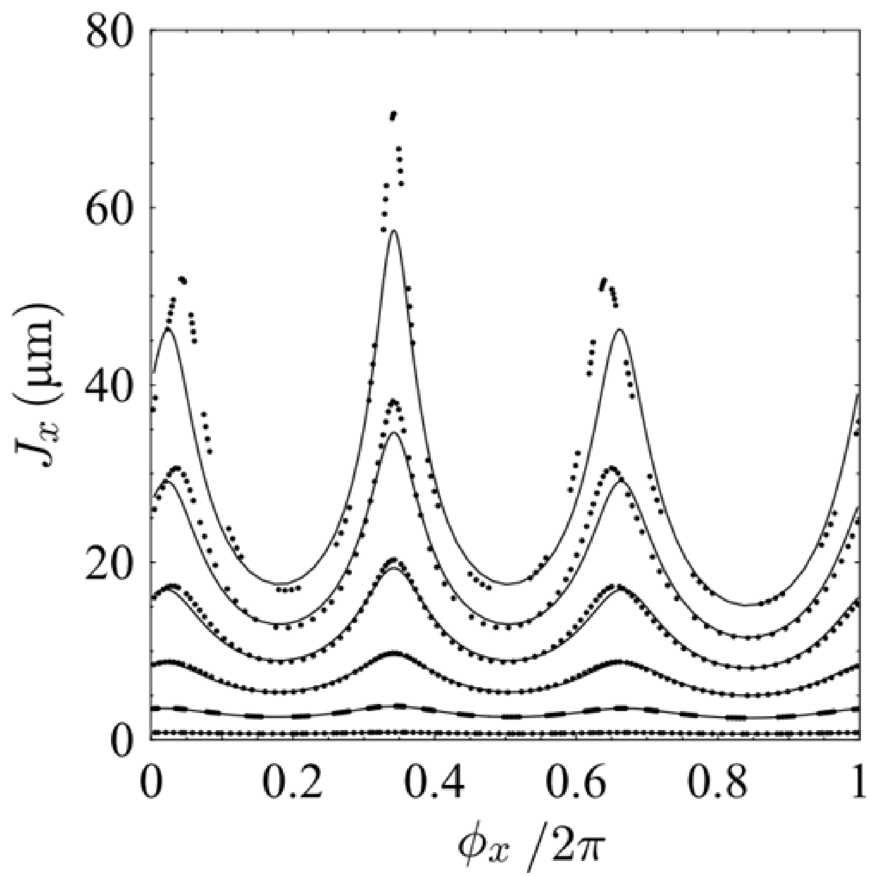




Normal form for sextupole

phase advance $\mu_x = 0.315 \times 2\pi$





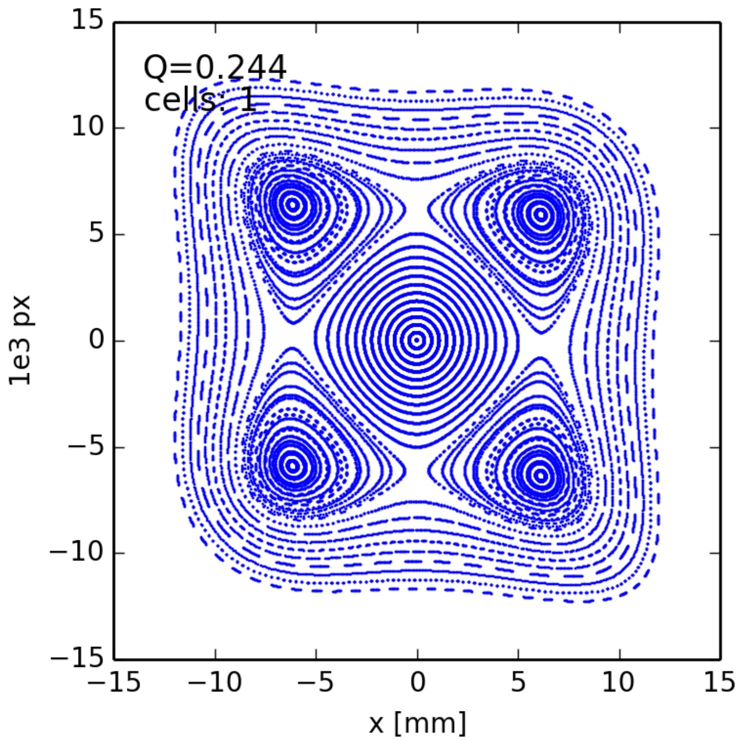
Tune-shift with amplitude

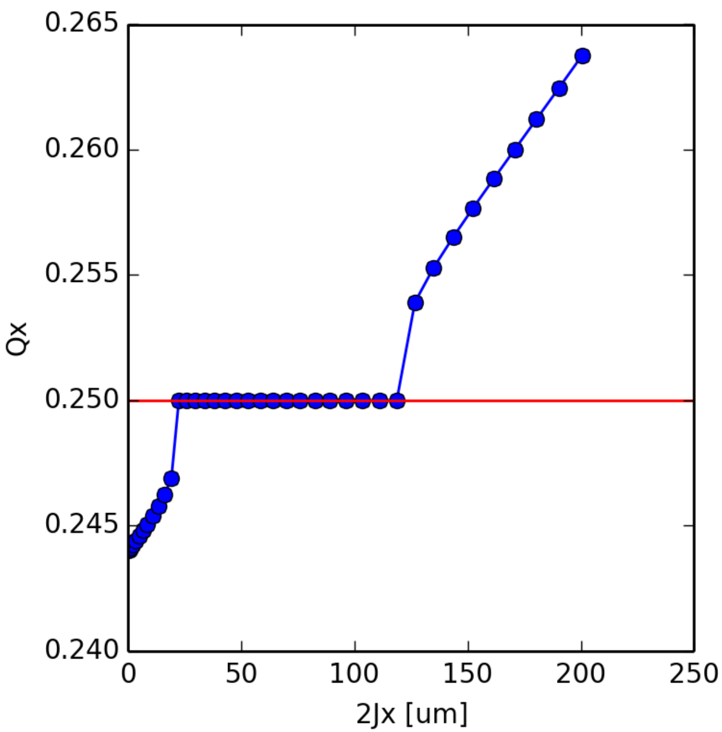
- Close inspection of the plots on the previous slides reveals another effect, in addition to the obvious distortion of the phase space ellipses: the **phase advance** per turn (i.e. the tune) **varies** with increasing **betatron amplitude**
- Normal form analysis (and perturbation theory) can be used to obtain **estimates** for the **tune shift** with amplitude
- In the case of a sextupole, the **tune shift** is **higher-order** in the **sextupole strength**
- An octupole, however, does have a **first-order** in the octupole strength **tune shift** with amplitude, given by:

$$\nu_x \approx \nu_{x0} + \frac{k3L\beta_x^2}{16\pi} J_x + O(J_x^2)$$

Particle trapped in 4th order resonance

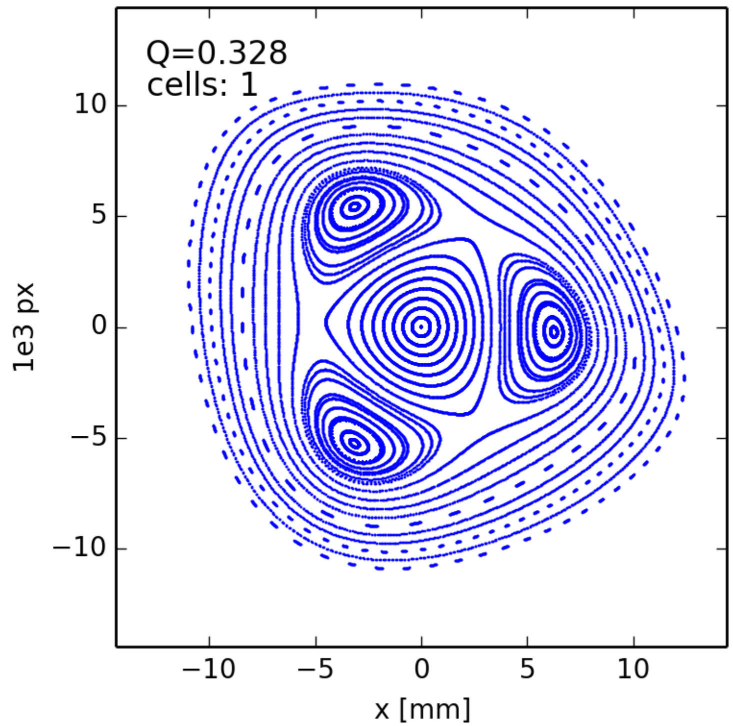
- Simulation of simple storage ring with a **single octupole** close to 4th order resonance
- **Detuning with amplitude** (linear in action)
- Particles in the stable islands have tune **locked** to **resonance**

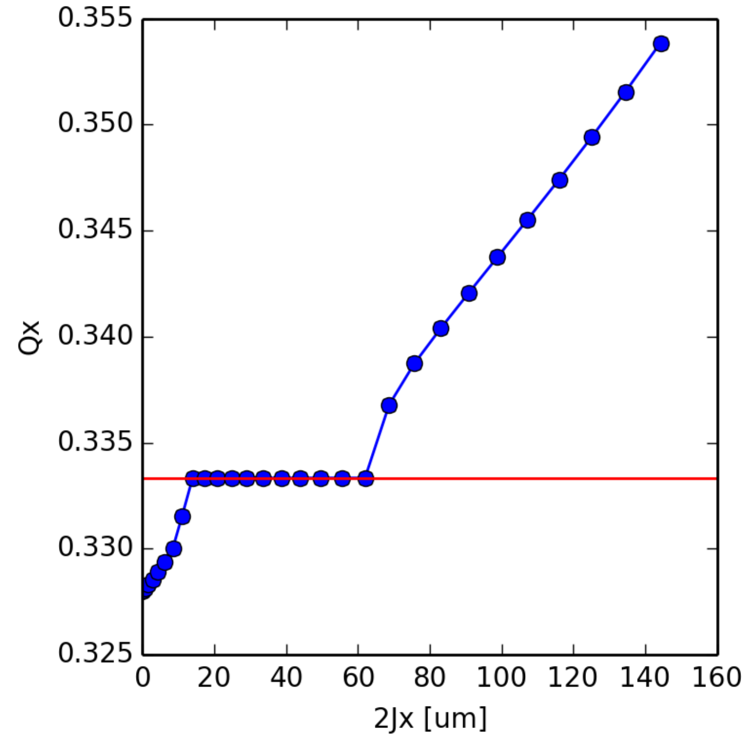




Particle trapped in 3rd order resonance

- Simulation of simple storage ring with a **sextupole** and an **octupole** close to **3rd order resonance**
- The **amplitude detuning** induced by the octupole can create **stable islands** even for the **3rd order resonance**
- The tune of particles in islands is locked to the resonance





Non-linear map representation

Taylor maps

- For any dynamical variable z_j the **Taylor map** up to **3rd order** can be written as

$$z_j^{new} = \sum_{k=1}^{6} R_{jk} z_k + \sum_{k=1}^{6}\sum_{l=1}^{6} T_{jkl} z_k z_l + \sum_{k=1}^{6}\sum_{l=1}^{6}\sum_{m=1}^{6} U_{jklm} z_k z_l z_m$$

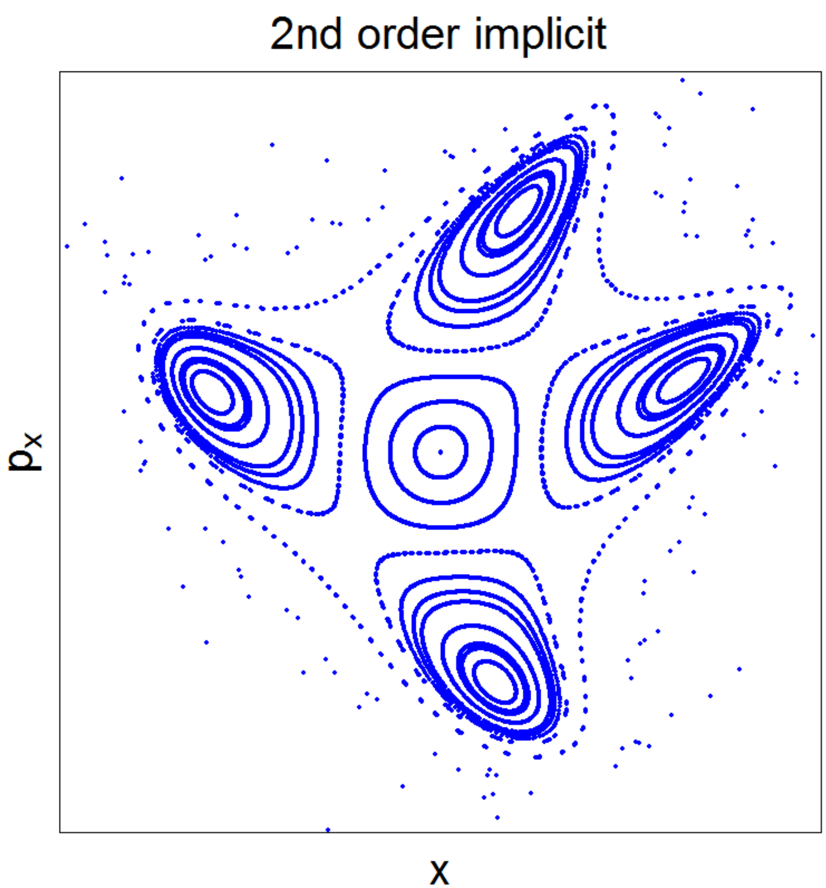
- **Taylor series** provide a convenient way of systematically representing **transfer maps** for **beamline components**, or sections of beamline
- The **main drawback** of Taylor series is that in general, transfer maps can only be represented exactly by series with an **infinite number** of **terms**
- In practice, we have to **truncate** a **Taylor map** at some order, and we then lose certain desirable properties of the map
- In particular, a **truncated map** will be usually be **non-symplectic**.

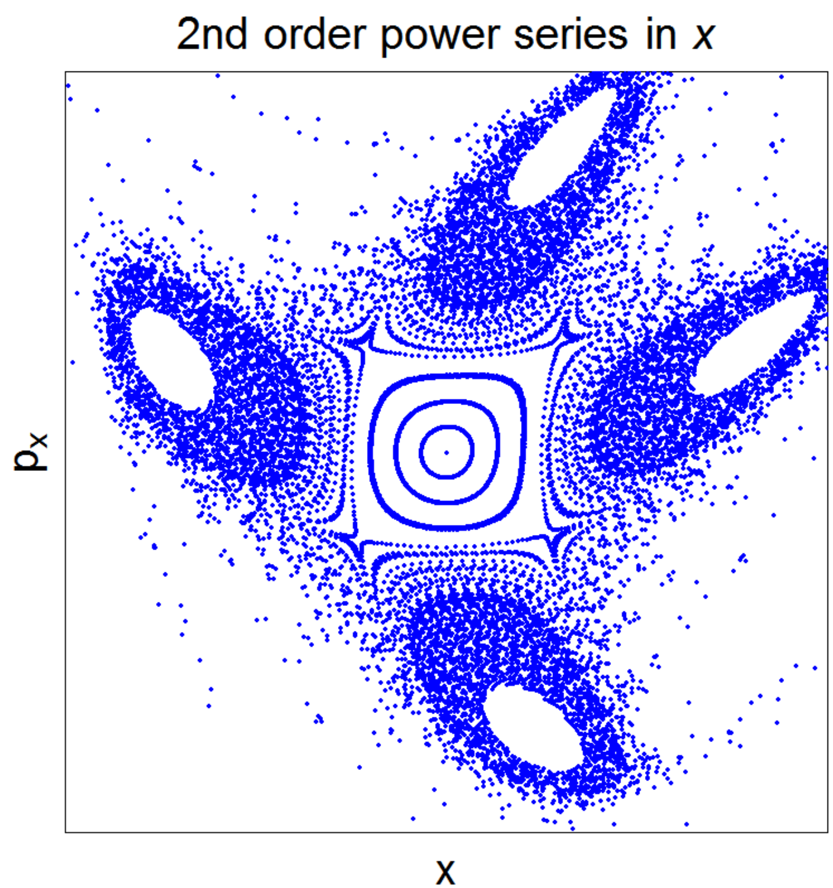
Symplectic maps

- Consider two sets of canonical variables $\mathbf{z}$, $\bar{\mathbf{z}}$ which may be even considered as the evolution of the system between two points in phase space
- A transformation from the one to the other set can be constructed through a **map** $\mathcal{M} : \mathbf{z} \mapsto \bar{\mathbf{z}}$
- The **Jacobian matrix** of the map $M = M(\mathbf{z}, t)$ is composed by the elements $M_{ij} \equiv \dfrac{\partial \bar{z}_i}{\partial z_j}$
- The map is **symplectic** if $M^T J M = J$, with $J = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$
- It can be shown that $\det(M) = 1$
- Physically, a **symplectic transfer map** conserves **phase space volumes** when the map is applied
- This is **Liouville's theorem**, and is a property of charged particles moving in electromagnetic fields, in the absence of radiation.

Symplectic maps

- The effect of **losing symplecticity** becomes apparent if we compare phase space portraits constructed using symplectic (below, left) and non-symplectic (below, right) transfer maps.





Symplectic integration

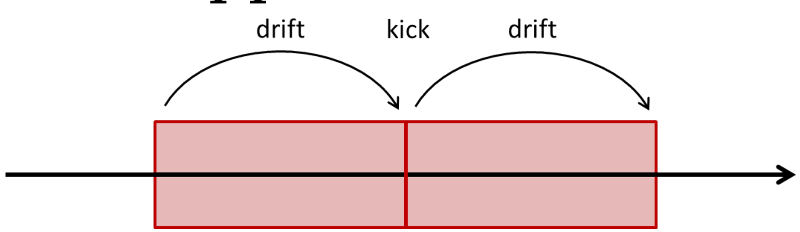
- Consider a sextupole with equations of motion:

$$\frac{dx}{ds} = p_x, \qquad \frac{dp_x}{ds} = -\frac{1}{2}k_2 x^2.$$

- Exact solutions using some elementary functions do not exist
- By splitting integration into three steps, it is possible to write an **explicit** and **symplectic** approximate solution

$$\begin{aligned}
0 \le s < L/2: &\qquad x_1 = x_0 + p_{x0}, \quad p_{x1} = p_{x0}, \\
s = L/2: &\qquad x_2 = x_1, \quad p_{x2} = p_{x1} - \frac{1}{2}k_2 L x_1^2, \\
L/2 < s \le L: &\qquad x_3 = x_2 + p_{x2}, \quad p_{x3} = p_{x2}.
\end{aligned}$$

- This an example of a symplectic integrator known as a **“drift–kick–drift”** approximation.



Numerical methods: Dynamic aperture

Dynamic aperture

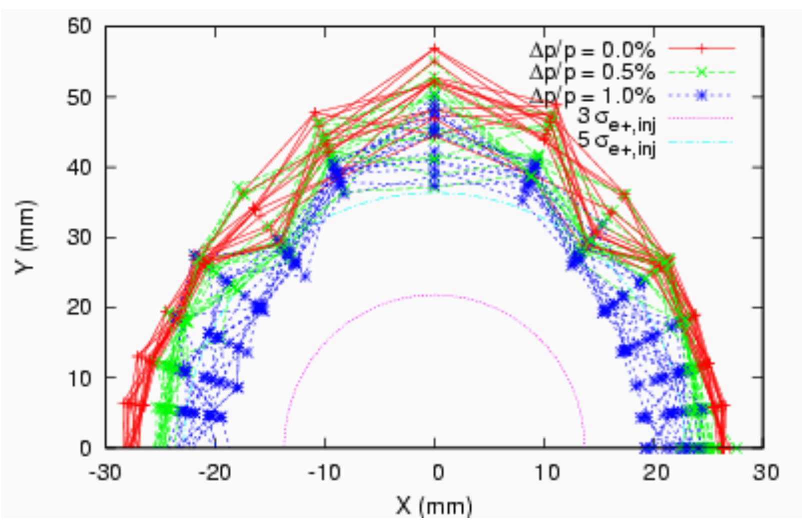
- The most direct way to evaluate the non-linear dynamics performance of a ring is the computation of **Dynamic Aperture** (short: DA)
- Particle motion due to multi-pole errors is generally **non-bounded**, so chaotic particles can escape to infinity. This is not true for all non-linearities (e.g. the beam-beam force)
- Need a **symplectic tracking code** to follow particle trajectories (a lot of initial conditions) for a number of turns (depending on the given problem) until the particles start getting lost. This boundary defines the **Dynamic aperture**
- As multi-pole errors may not be completely known, one has to track through several machine models built by **random distribution** of these errors
- One could start with **4D (only transverse)** tracking but certainly needs to simulate **5D (constant energy deviation)** and finally **6D (synchrotron motion included)**

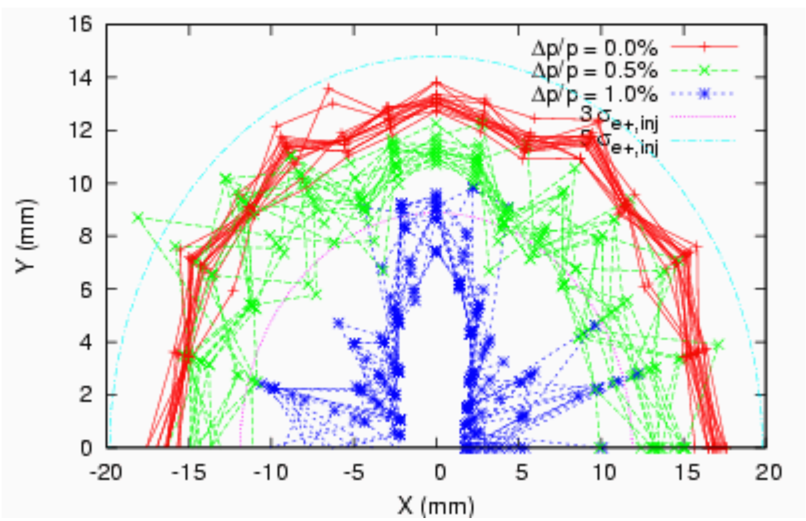
Dynamic Aperture plots

- Dynamic aperture plots show the **maximum initial values** of **stable trajectories** in x-y coordinate space at a particular point in the lattice, for a range of energy errors

- The **beam size** can be shown on the same plot

- Generally, the goal is to allow some **significant margin** in the design – the measured dynamic aperture is often smaller than the predicted dynamic aperture

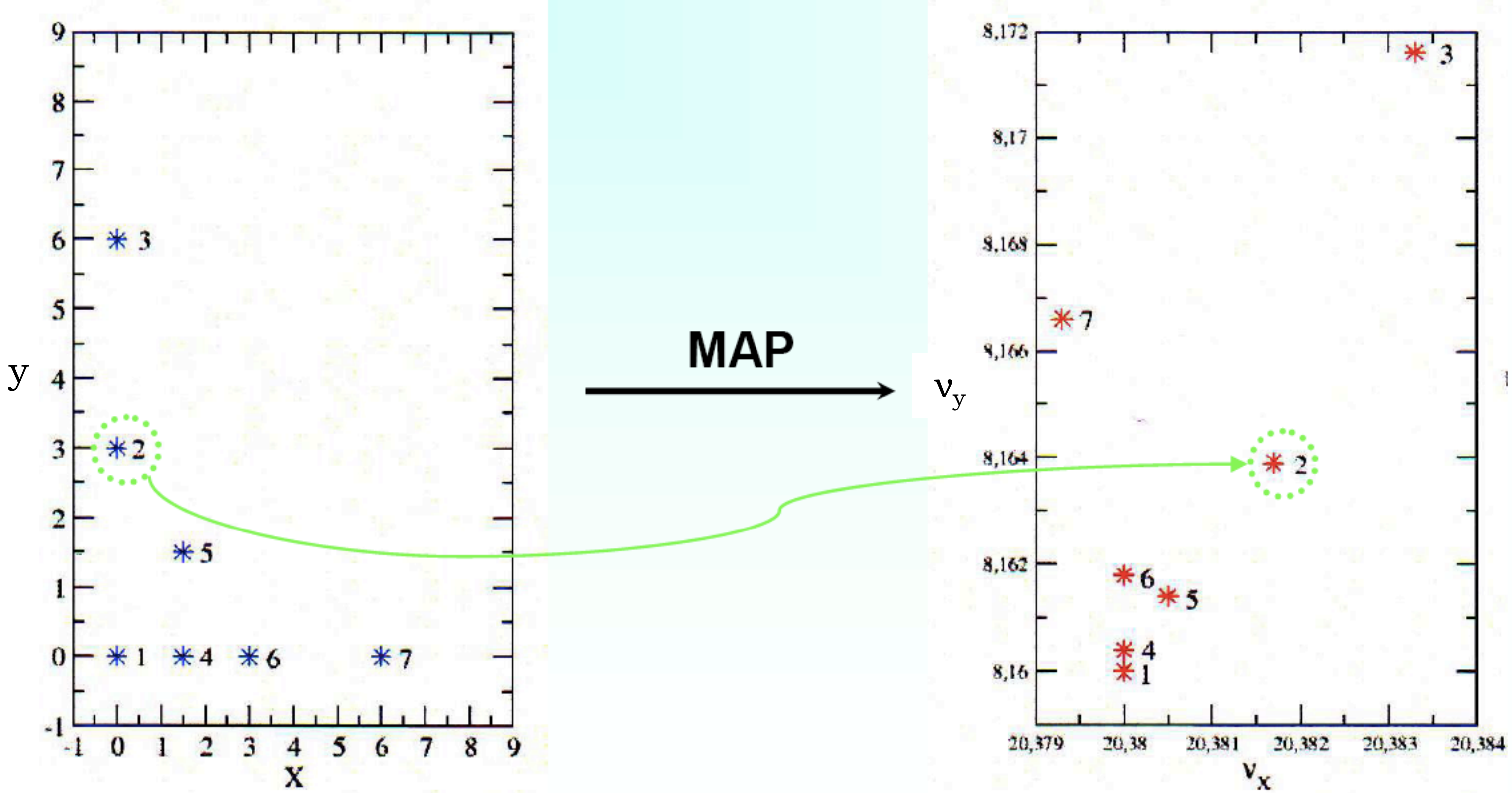




Numerical methods: Frequency map analysis

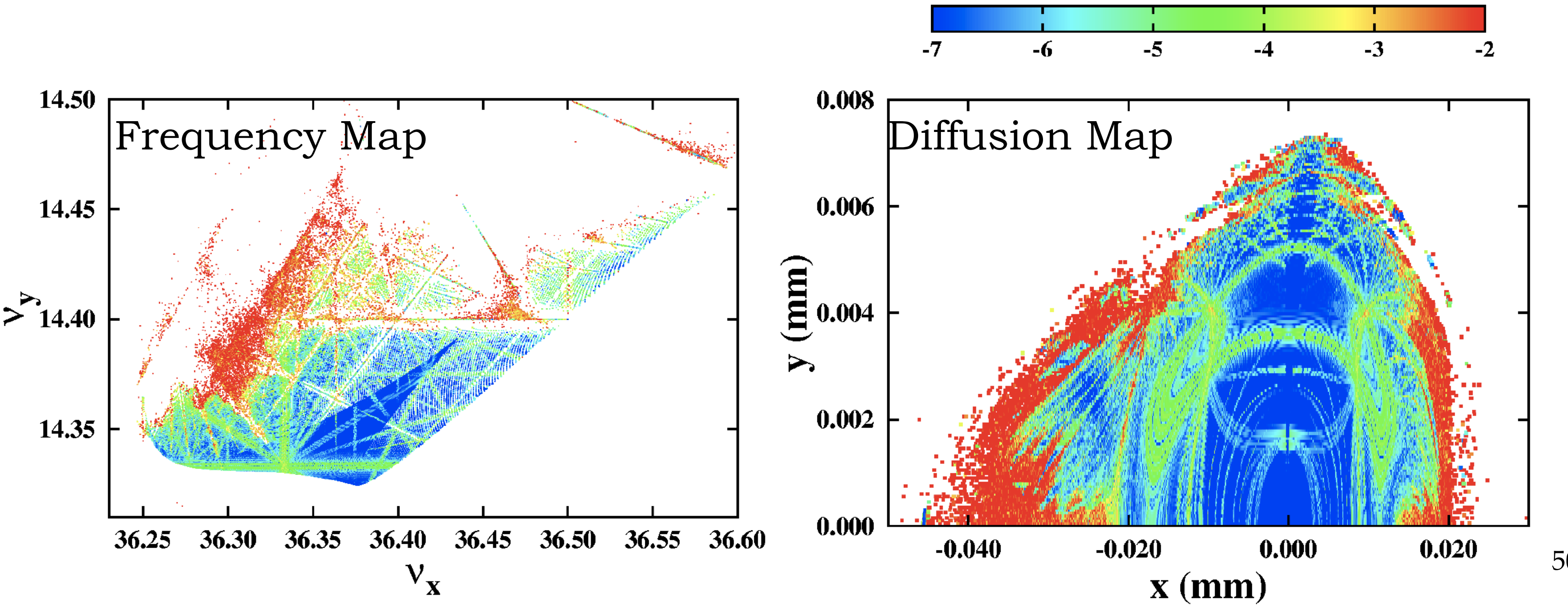
Building the frequency map

- Choose coordinates (x_i, y_i) with $p_x=p_y=0$
- Numerically integrate the phase trajectories through the lattice for sufficient number of turns
- Compute through advanced Fourier methods (**NAFF** algorithm) Q_x and Q_y after sufficient number of turns
- Plot them in the tune diagram



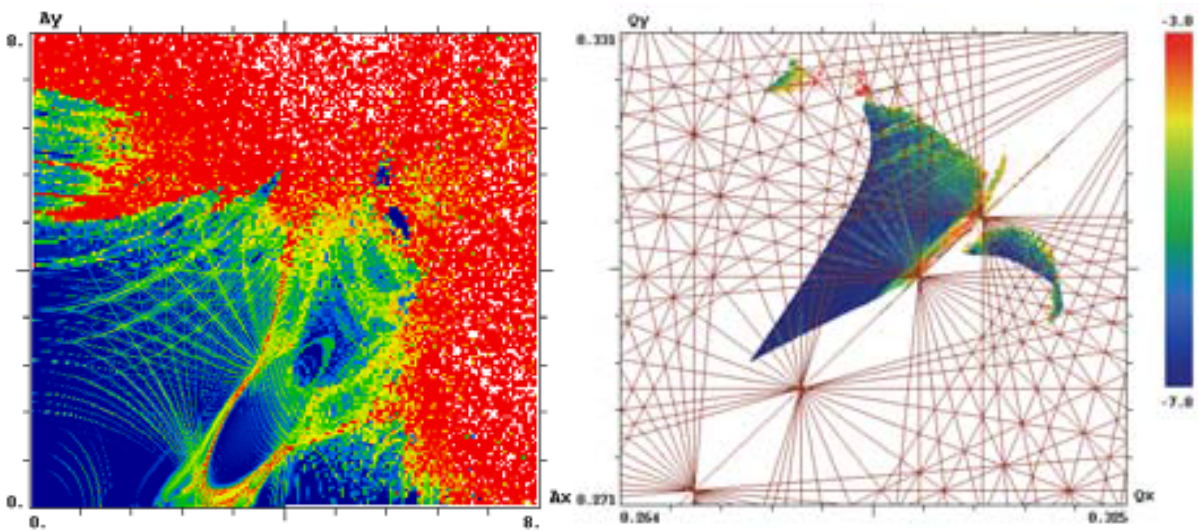
Frequency Map for the ESRF

- All dynamics represented in two plots (Frequency Map / Diffusion Map)
- **Regular motion** represented by blue colors
- **Resonances** appear as distorted lines in frequency space (or curves in initial condition space)
- **Chaotic motion** is represented by red scattered particles and defines dynamic aperture of the machine
- FMA shows also nicely the detuning with amplitude

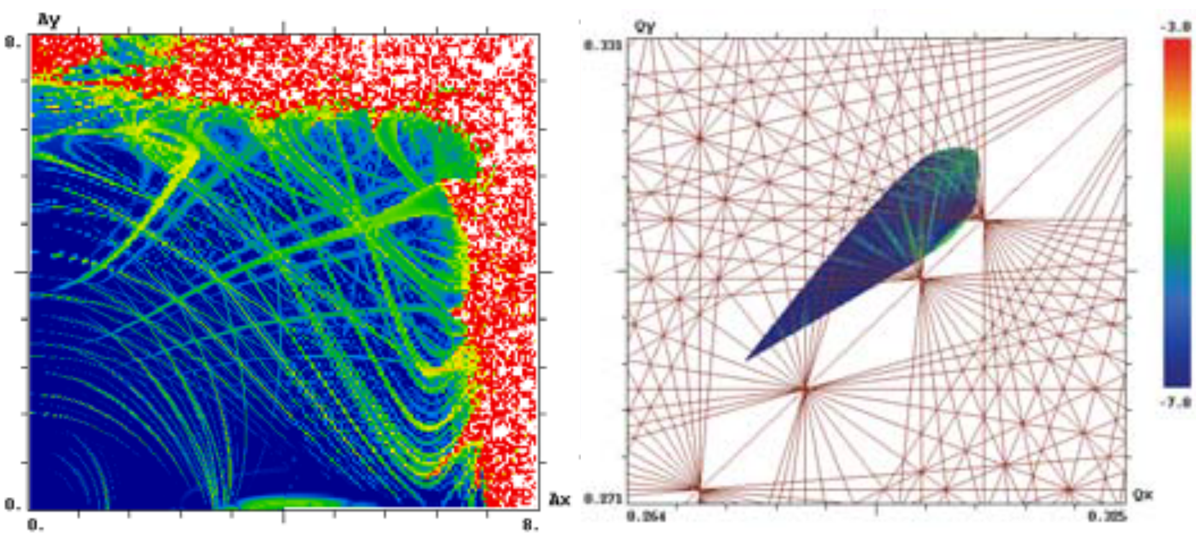


Frequency Map for LHC in collision

- Frequency map analysis for HL-LHC in collision



Large tune footprint and DA reduction due to "long range beam-beam" forces (electromagnetic field of other beam in interaction region)



DA clearly improved when compensating long range beam-beam with a wire

S. Fartoukh et al., PRSTAB, 2015

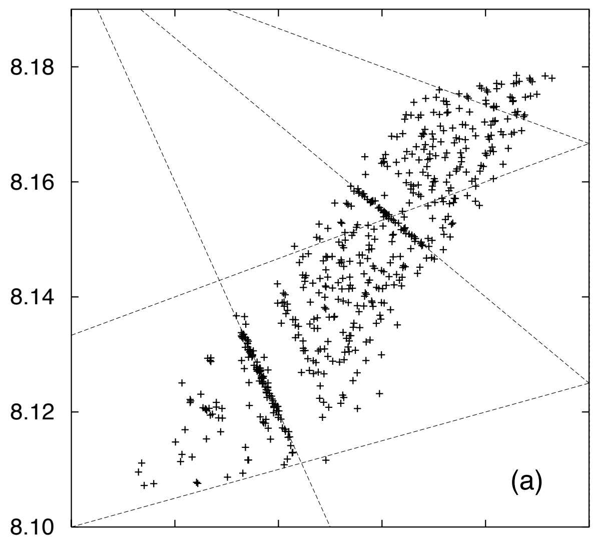
Experimental frequency maps

- Frequency analysis of **turn-by-turn data** of beam oscillations produced by a **fast kicker magnet** and recorded on a **Beam Position Monitor**

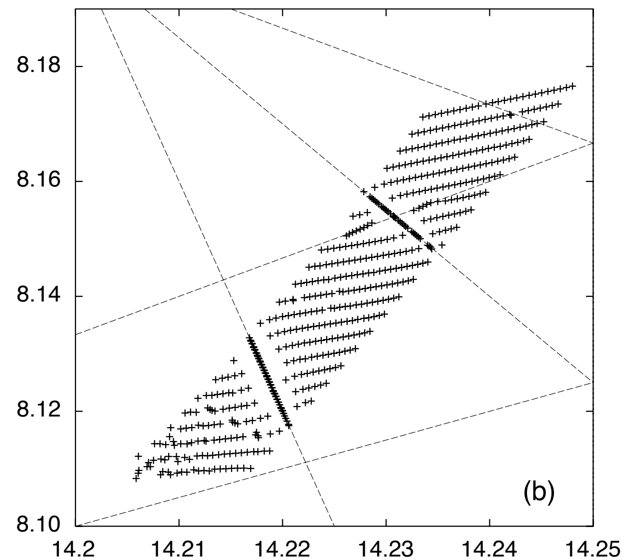
- Reproduction of the **non-linear model** of the **Advanced Light Source** storage ring and working point optimization for increasing beam lifetime

D. Robin, C. Steier, J. Laskar, and L. Nadolski, PRL 2000

experiment



simulation



Conclusions and Summary

Summary

- **Nonlinear dynamics** appear in a wide variety of accelerator systems, including **single-pass** systems (such as bunch compressors) and **multi-turn** systems (such as storage rings)
- It is possible to model nonlinear dynamics in a given component or section of beamline by representing the **transfer map** as a **power series**
- **Conservation** of **phase space volumes** is an important feature of the beam dynamics in many systems. To conserve phase space volumes, transfer maps must be **symplectic**
- In general, (**truncated**) **power series** maps are **not symplectic**.
- To construct a **symplectic transfer map**, the equations of motion in a given accelerator component must be solved using a **symplectic integrator** (e.g. the “drift–kick–drift” approximation for a multipole magnet).

Summary

- **Common features** of nonlinear dynamics in accelerators include **phase space distortion**, **tune shifts with amplitude**, **resonances**, and **chaotic** particle trajectories at large amplitudes (**dynamic aperture** limits)
- Analytical methods such as **perturbation theory** and **normal form** analysis can be used to estimate the impact of nonlinear perturbations in terms of quantities such as **resonance strengths** and **tune shifts with amplitude**
- **Frequency map analysis** provides a useful numerical tool for characterising tune shifts and resonance strengths from tracking data.
- This can give some **insight** into **limitations** on the **dynamic aperture**