Injection, Extraction & Matching

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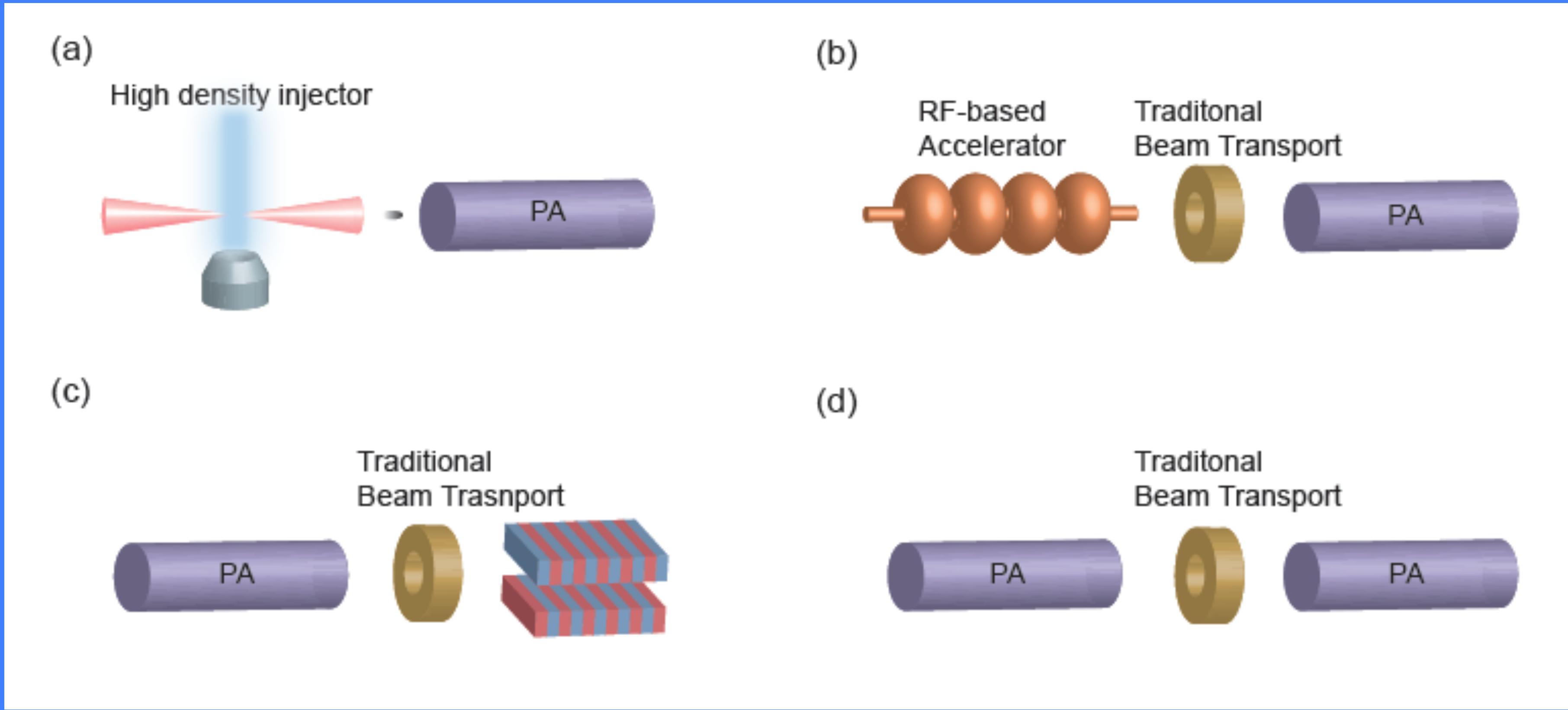
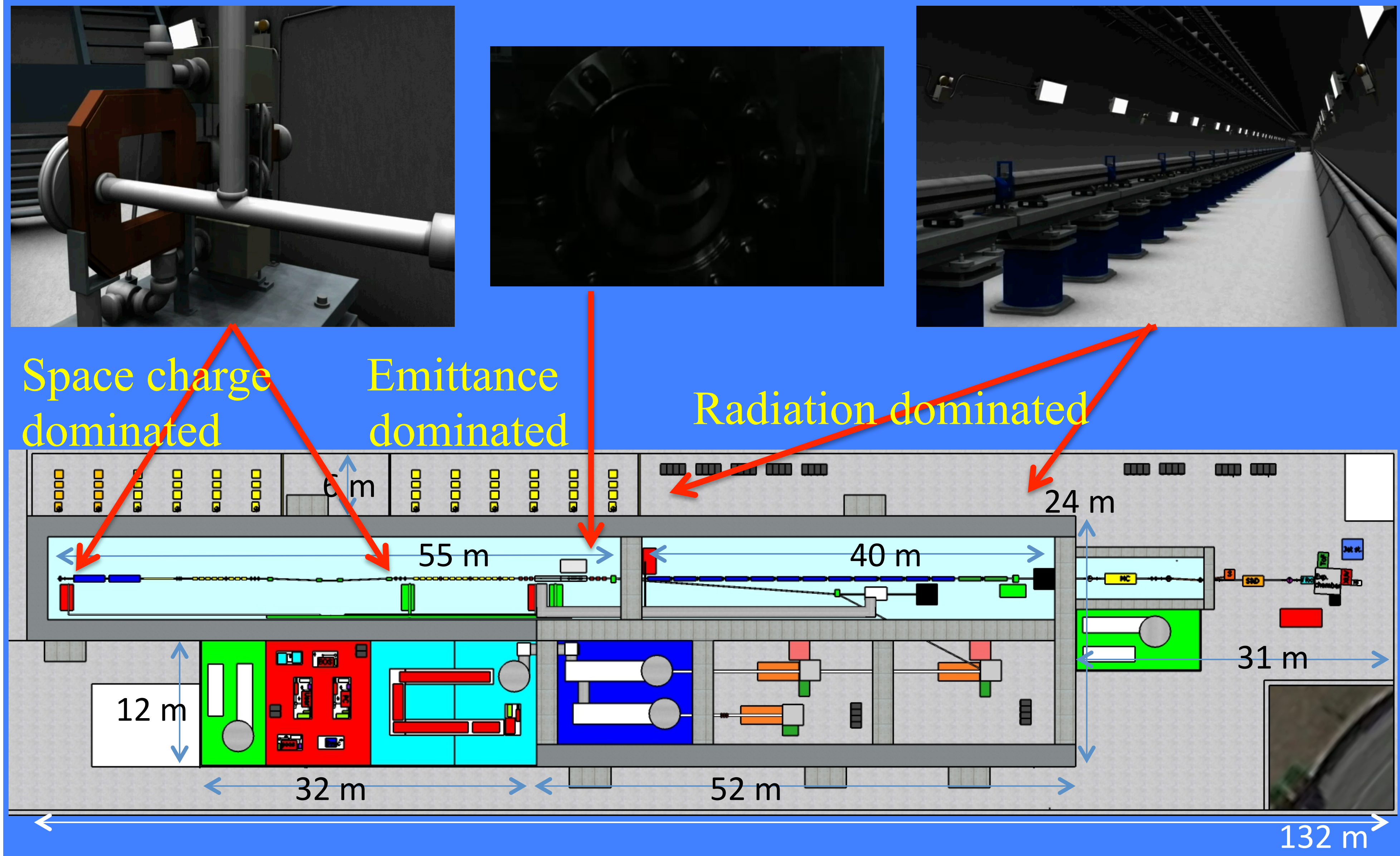


- EuPRAXIA@SPARC_LAB



Future accelerators require high quality beams:

==> High Luminosity & High Brightness

==> High Energy & Low Energy Spread

↑ $$L = \frac{N_{e+} N_{e-} f_r}{4\pi \sigma_x^* \sigma_y^*}$$

- N of particles per pulse => 10^9
- High rep. rate f_r => bunch trains

- Small spot size => low emittance

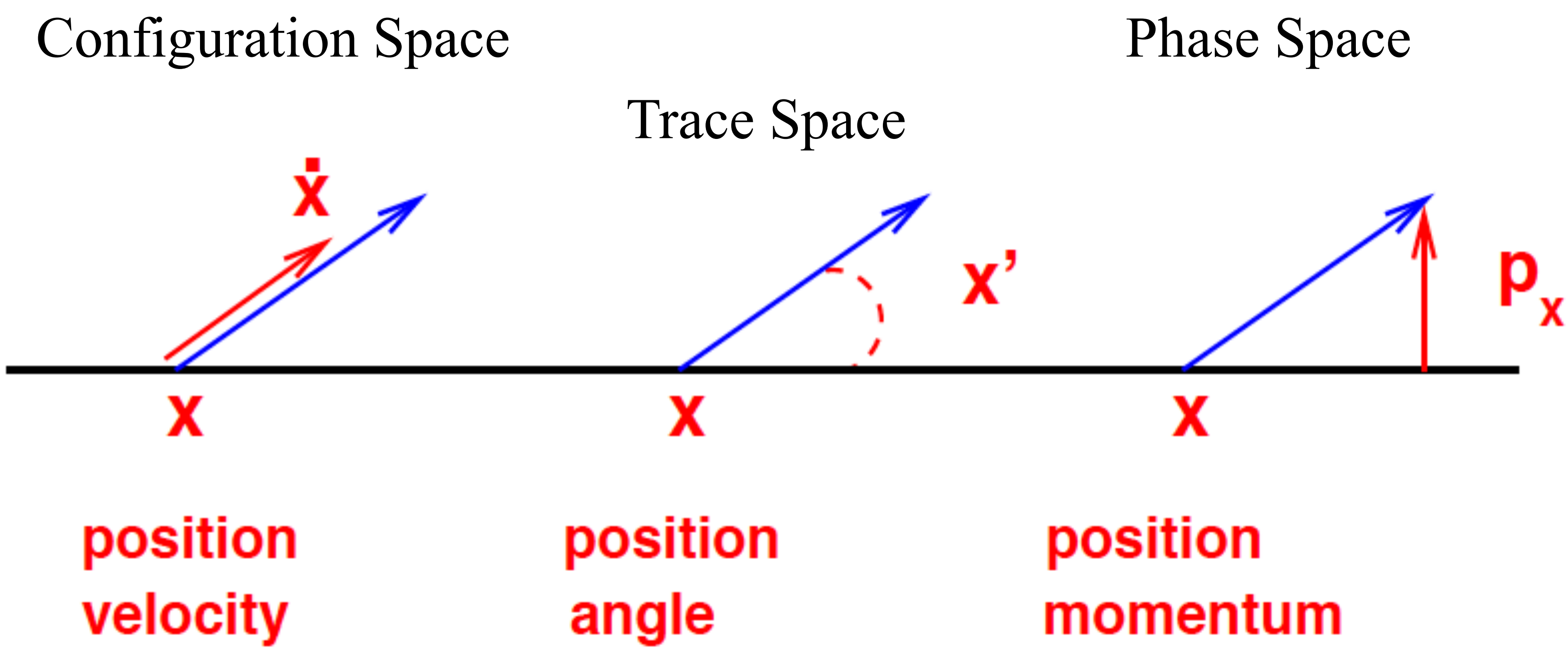
↑ $$B_n \approx \frac{2I}{\varepsilon_n^2}$$

- Short pulse (ps to fs)

- Little spread in transverse momentum and angle => low emittance

- The rms emittance concept

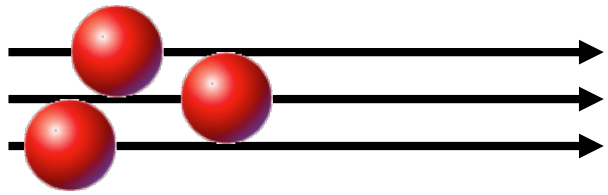
Typical coordinates to describe the particle motion
(6 per particle)



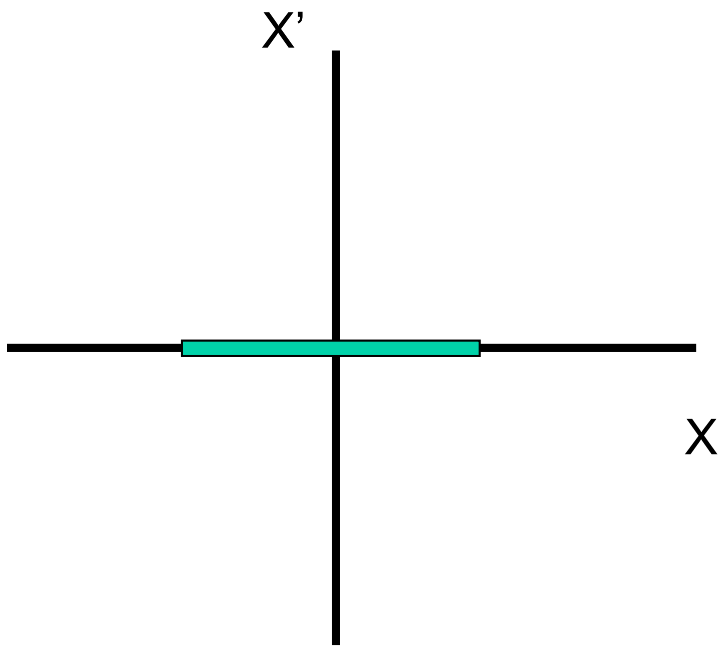
$$\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \qquad p_x << p_z \end{cases}$$

$$p_z = \gamma m_o v_z$$
$$= \beta_z \gamma m_o c$$

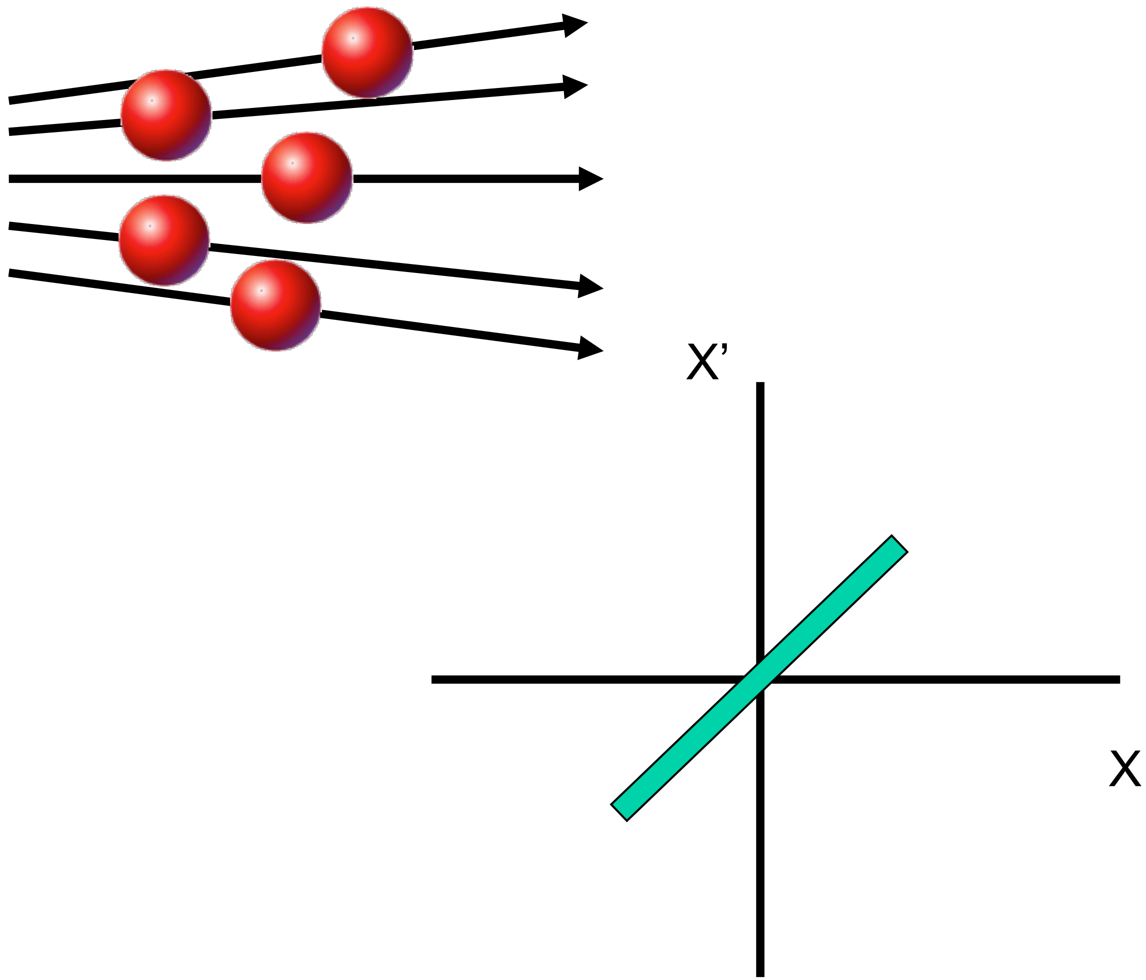
Trace space of an ideal laminar beam



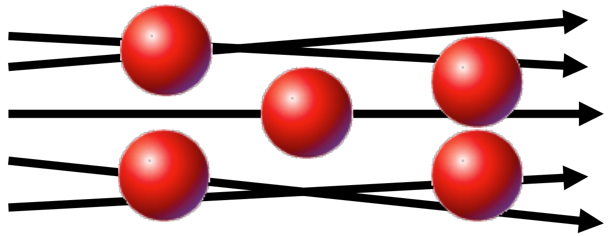
$$\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \end{cases} \qquad p_x << p_z$$

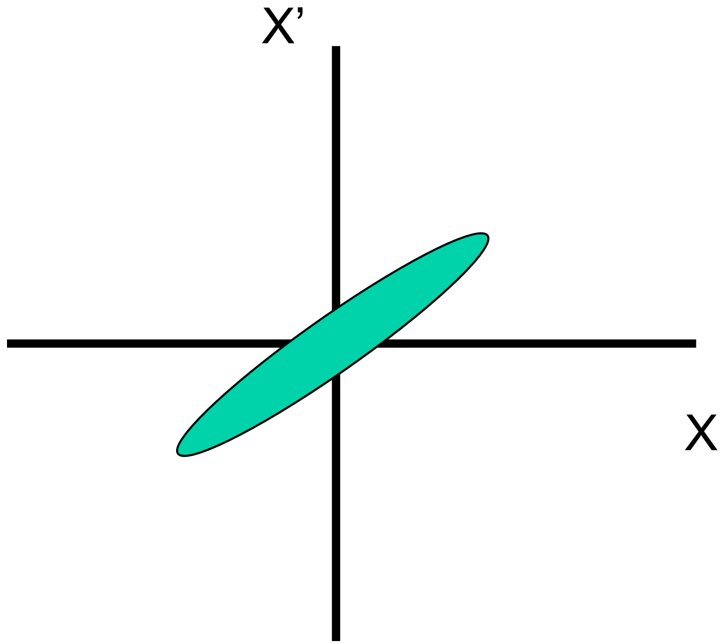


Trace space laminar beam



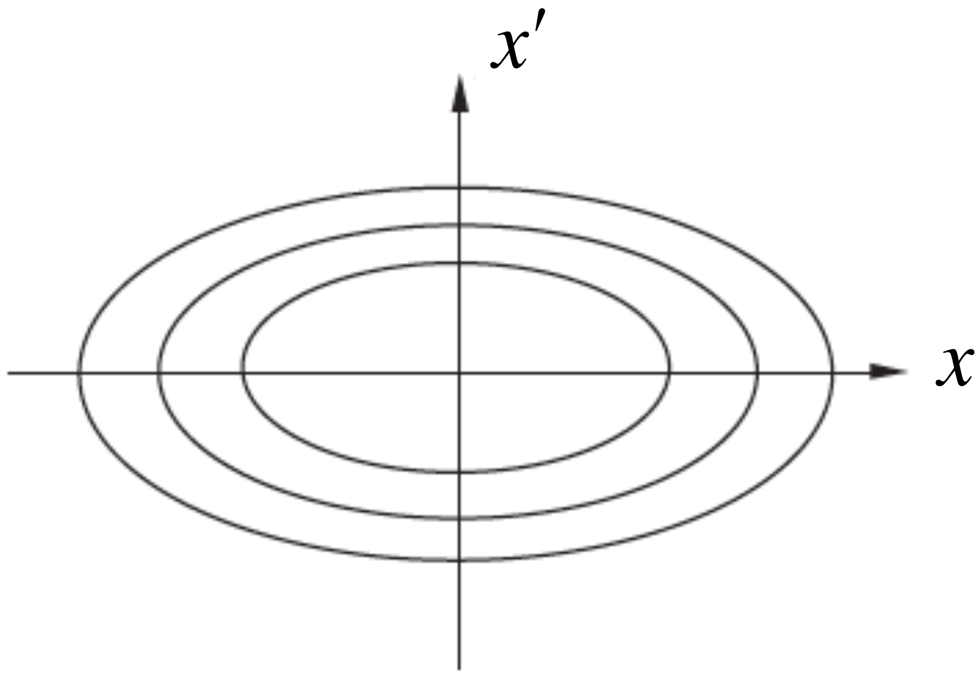
Trace space of non laminar beam





In a system where all the forces acting on the particles are linear (i.e., proportional to the particle's displacement x from the beam axis), it is useful to assume an elliptical shape for the area occupied by the beam in x-x‘ trace space.

$$\ddot{x} + k^2 x = 0$$



$$H = \frac{1}{2m}[p_x^2 + m^2\omega^2 x^2]$$

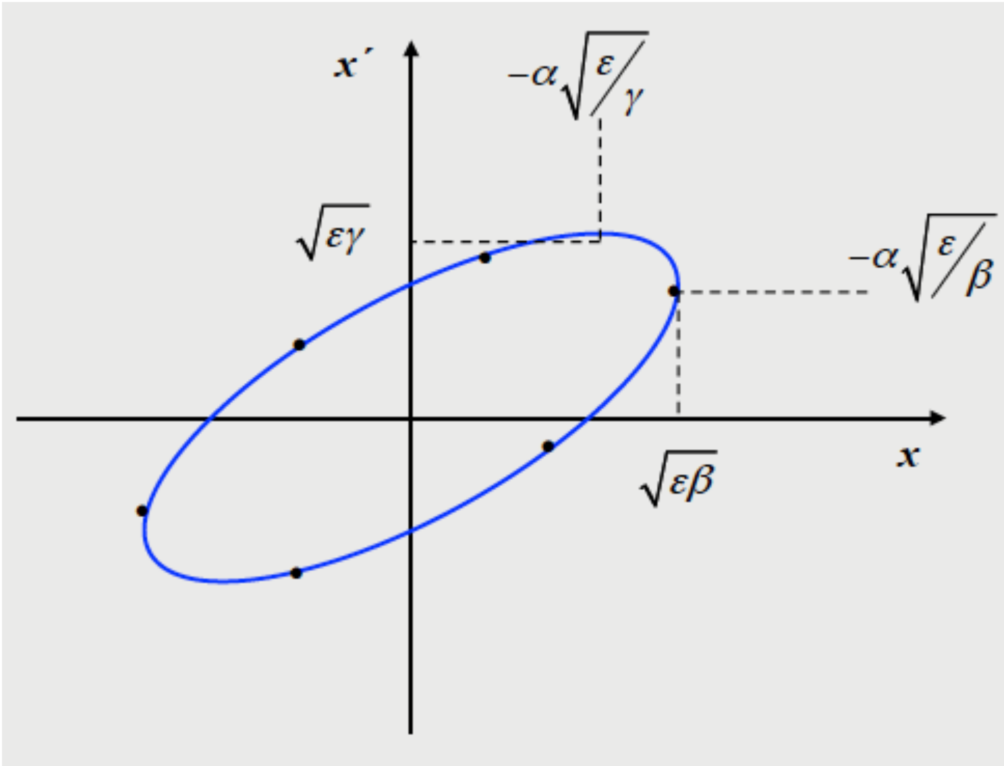
$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i},$$

Geometric emittance: ε_g

Ellipse equation: $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_g$

Twiss parameters: $\beta\gamma - \alpha^2 = 1$ $\beta' = -2\alpha$

Ellipse area: $A = \pi\varepsilon_g$



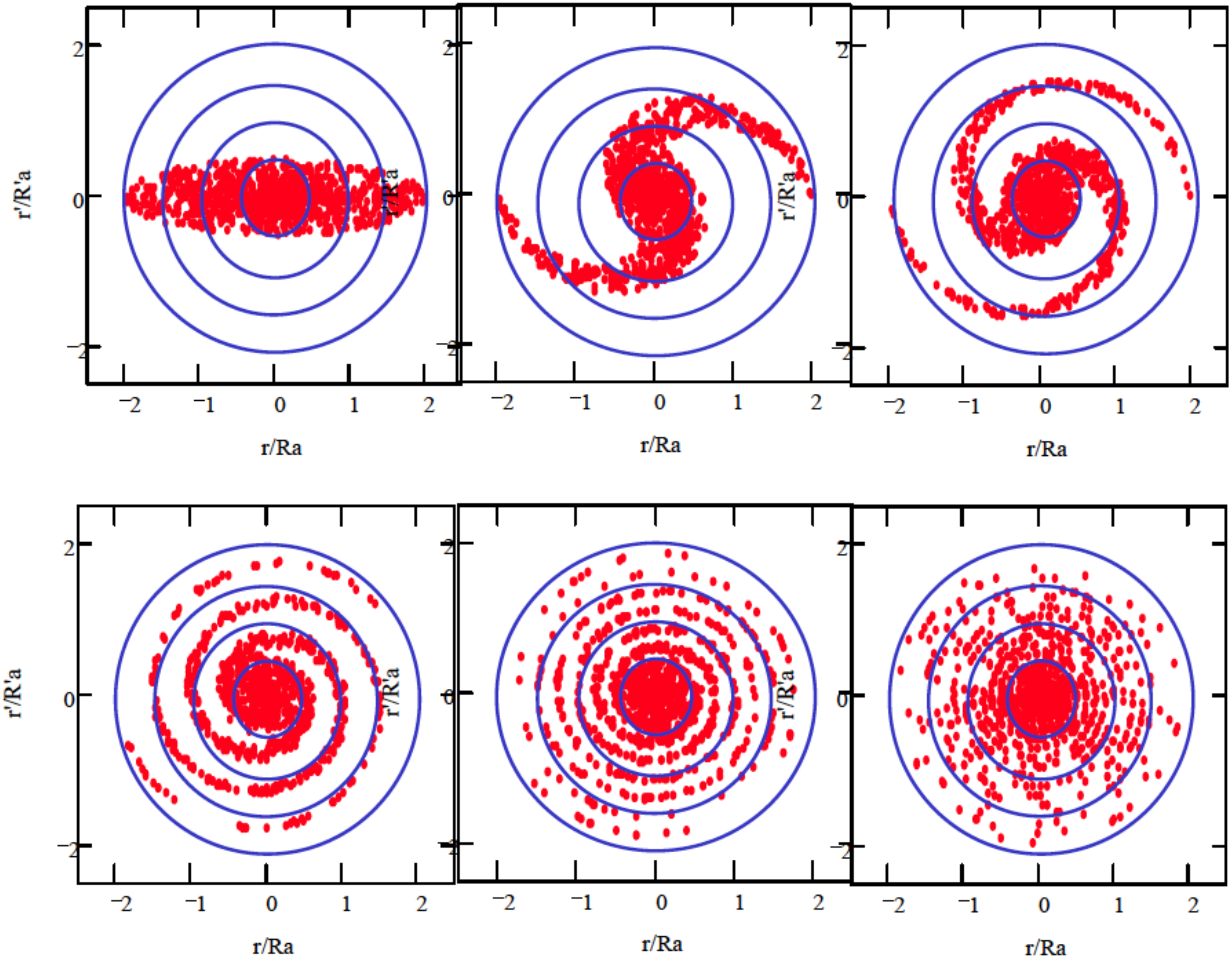
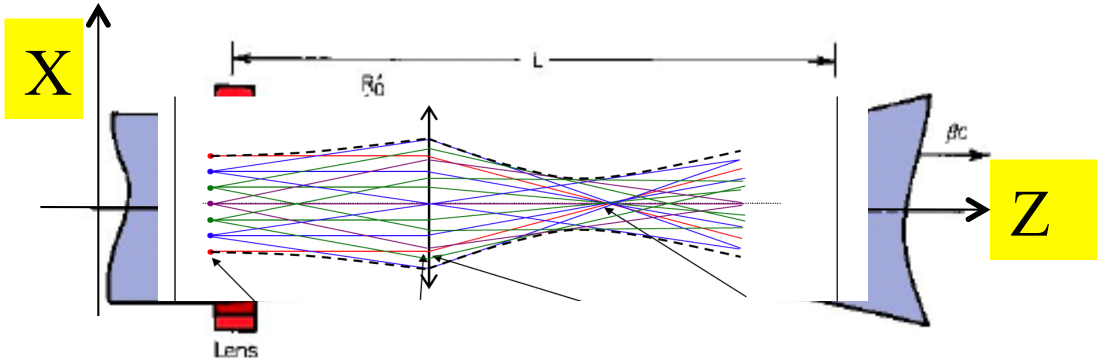
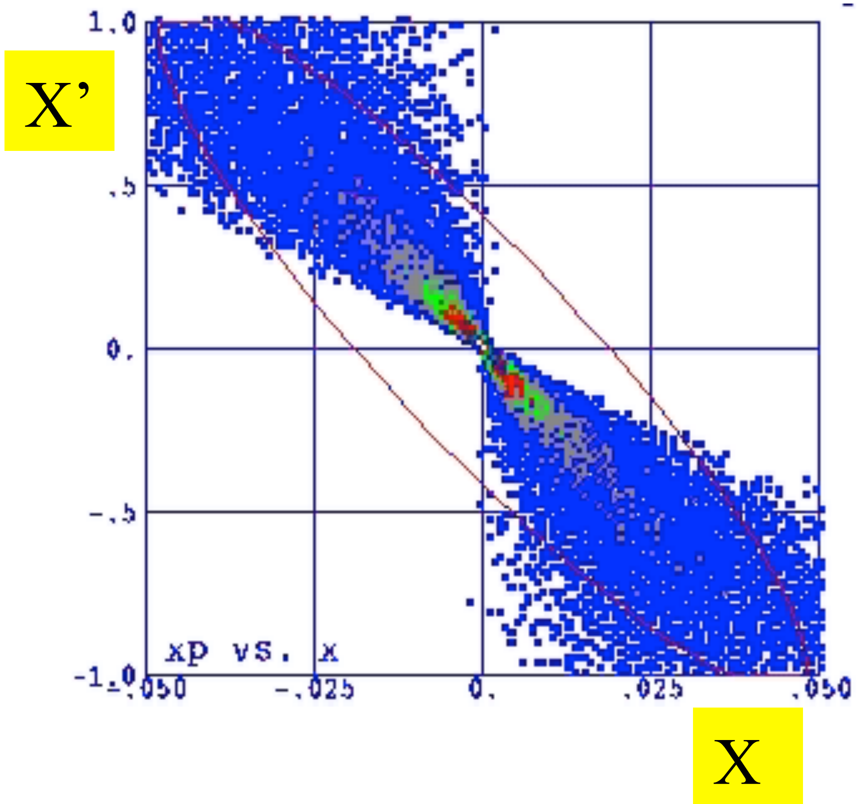


Fig. 17: Filamentation of mismatched beam in non-linear force

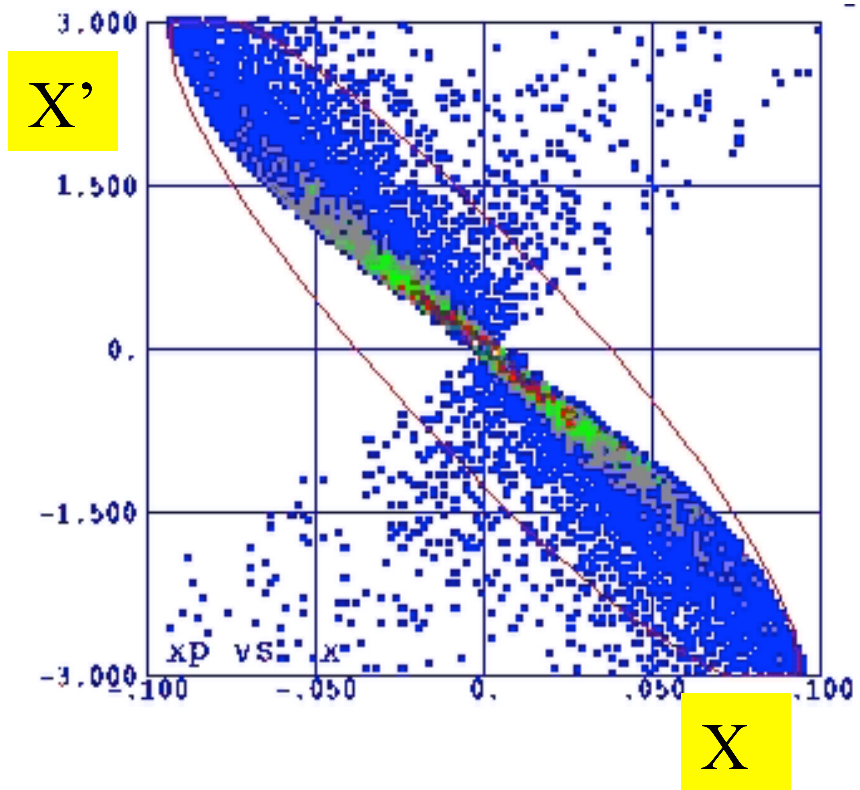
Phase space evolution



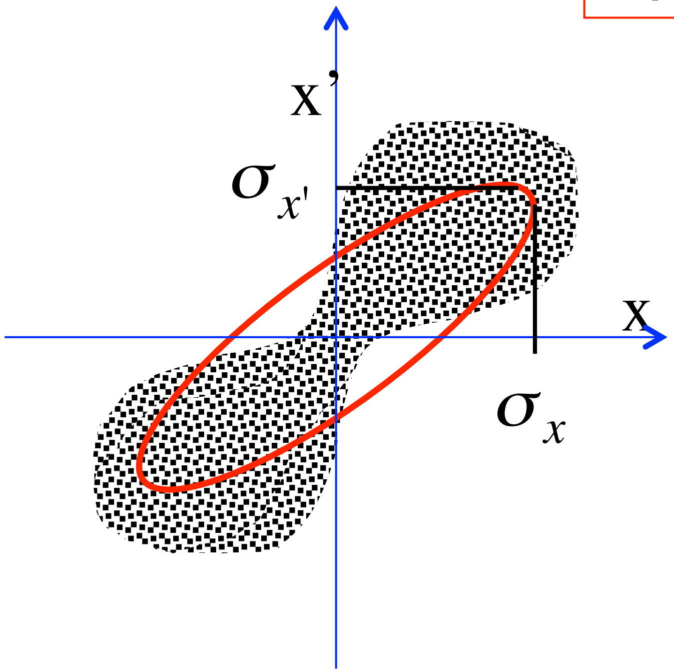
No space charge => cross over



With space charge => no cross over



rms emittance ε_{rms}



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,x')\,dx\,dx' = 1 \qquad f'(x,x') = 0$$

rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,x')\,dx\,dx'$$

Define rms emittance:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}$$

such that:

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

Since: $\beta' = -2\alpha$

it follows:
$$\alpha = -\frac{1}{2\varepsilon_{rms}}\frac{d}{dz}\langle x^2 \rangle = -\frac{\langle x x' \rangle}{\varepsilon_{rms}} = -\frac{\sigma_{xx'}}{\varepsilon_{rms}}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma'_x = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

$$\sigma_{xx'} = \langle xx' \rangle = -\alpha \varepsilon_{rms}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

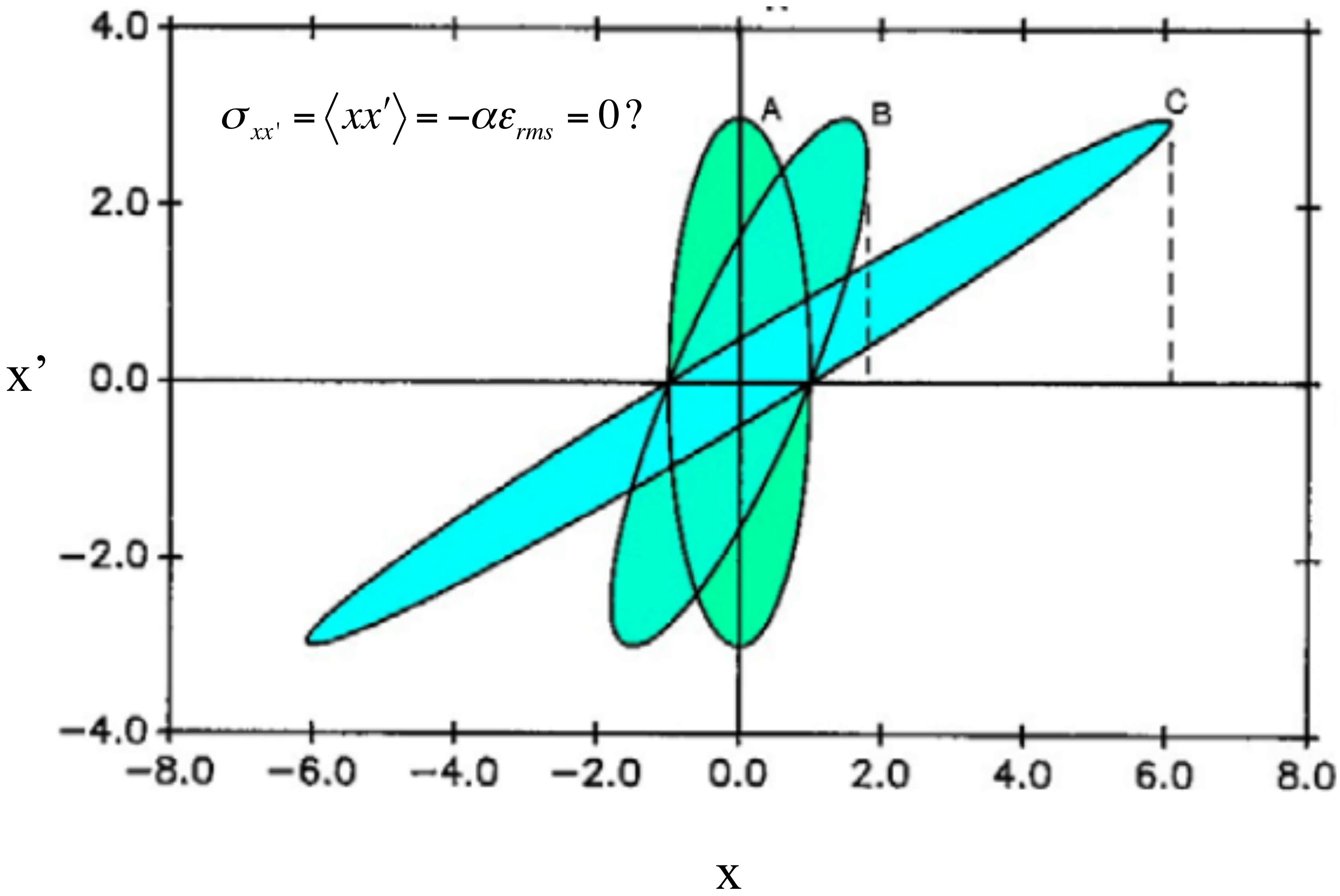
Substituting α, β, γ we get $\frac{\sigma_{x'}^2}{\varepsilon_{rms}} \frac{\sigma_x^2}{\varepsilon_{rms}} - \left(\frac{\sigma_{xx'}}{\varepsilon_{rms}} \right)^2 = 1$

We end up with the definition of rms emittance in terms of the second moments of the distribution:

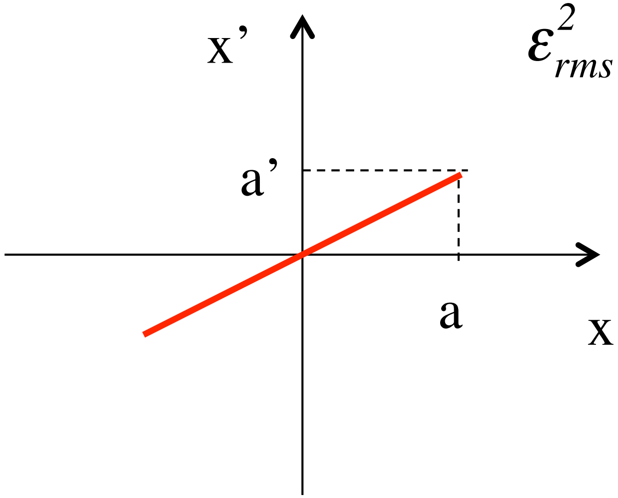
$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \right)}$$

$$x' = \frac{p_x}{p_z}$$

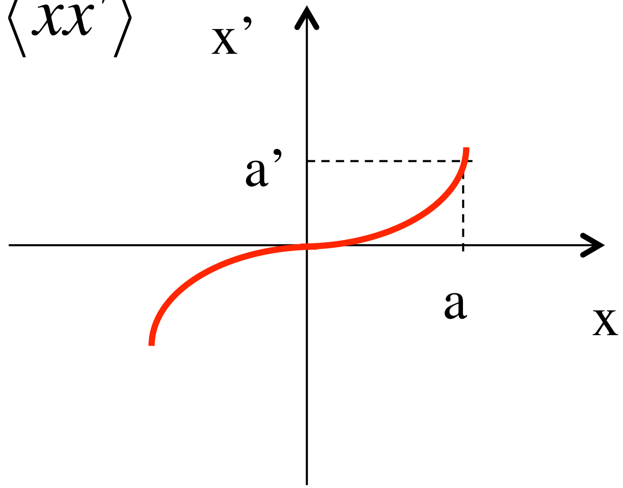
Which distribution has no correlations?



What does rms emittance tell us about beam phase space distributions under the effect of linear or non-linear forces?



$$\varepsilon_{rms}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2$$



Assuming a generic x, x' correlation of the type: $x' = Cx^n$

$$\varepsilon_{rms}^2 = C^2 \left(\langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

When $n = 1$ ==> $\varepsilon_{rms} = 0$

When $n \neq 1$ ==> $\varepsilon_{rms} \neq 0$

Constant under linear transformation only

$$\frac{\mathrm{d}}{\mathrm{d}z}\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = 2\langle xx'\rangle\langle x'^2\rangle + 2\langle x^2\rangle\langle x'\rangle\langle x''\rangle - 2\langle xx''\rangle\langle xx'\rangle = 0$$

For linear transformations, $x'' = -k_x^2 x$, and the right-hand side of the equation is

$$2k_x^2\langle x^2\rangle\langle xx'\rangle - 2\langle x^2\rangle\langle xx'\rangle k_x^2 = 0,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}z}\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = 0$$

And without acceleration:

$$x' = \frac{p_x}{p_z}$$

Normalized rms emittance: $\varepsilon_{n,rms}$

Canonical transverse momentum: $p_x = p_z x' = m_o c\beta\gamma x'$ $\qquad p_z \approx p$

$$\varepsilon_{n,rms} = \sqrt{\sigma_x^2\sigma_{p_x}^2 - \sigma_{xp_x}^2} = \frac{1}{m_o c}\sqrt{\left(\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2\right)}$$

Liouville theorem: the density of particles *n*, or the volume V occupied by a given number of particles in phase space (x,p_x,y,p_y,z,p_z) remains invariant under conservative forces.

$$\frac{dn}{dt} = 0$$

Rms emittance instead is invariant only under linear forces => It is not a Liouvillian invariant

Limit of single particle emittance

Limits are set by Quantum Mechanics on the knowledge of the two conjugate variables (x,p_x). According to Heisenberg:

$$\sigma_x \sigma_{p_x} \geq \frac{\hbar}{2}$$

This limitation can be expressed by saying that the state of a particle is not exactly represented by a point, but by a small uncertainty volume of the order of $\hbar^3$ in the 6D phase space.

In particular for a single electron in 2D phase space it holds:

$$\varepsilon_{n,rms} = \frac{1}{m_o c}\sqrt{\sigma_x^2 \sigma_{p_x}^2 - \sigma_{xp_x}^2} \Rightarrow \begin{cases} = 0 & \text{classical limit} \\ \geq \dfrac{1}{2}\dfrac{\hbar}{m_o c} = \dfrac{\lambda\!\!\!^{-}_c}{2} = 1.9 \times 10^{-13}\, m & \text{quantum limit} \end{cases}$$

Where $\lambda\!\!\!^{-}_c$ is the reduced Compton wavelength.

- The rms emittance concept
- WARNiNG: Energy spread contribution

Normalized and un-normalized emittances

$$p_x = p_z x' = m_o c \beta\gamma x'$$

$$\varepsilon_{n,rms} = \frac{1}{m_o c}\sqrt{\left(\langle x^2\rangle\langle p_x^2\rangle - \langle x p_x\rangle^2\right)} = \sqrt{\left(\langle x^2\rangle\langle (\beta\gamma x')^2\rangle - \langle x\beta\gamma x'\rangle^2\right)} = \langle\beta\gamma\rangle\varepsilon_{rms}$$

Assuming **small energy** spread within the beam, the normalized and un-normalized emittances can be related by the above approximated relation.

This approximation that is often used in conventional accelerators may be strongly misleading when adopted to describe beams with significant energy spread, as the one at present produced by plasma accelerators.

When the correlations between the energy and transverse positions are negligible (as in a drift without collective effects) we can write:

$$\varepsilon_{n,rms}^2 = \langle \beta^2\gamma^2 \rangle \langle x^2 \rangle \langle x'^2 \rangle - \langle \beta\gamma \rangle^2 \langle xx' \rangle^2$$

Considering now the definition of relative energy spread:

$$\sigma_\gamma^2 = \frac{\langle \beta^2\gamma^2 \rangle - \langle \beta\gamma \rangle^2}{\langle \beta\gamma \rangle^2}$$

which can be inserted in the emittance definition to give:

$$\varepsilon_{n,rms}^2 = \langle \beta^2\gamma^2 \rangle \sigma_\gamma^2 \langle x^2 \rangle \langle x'^2 \rangle + \langle \beta\gamma \rangle^2 \left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \right)$$

Assuming relativistic electrons (β=1) we get:

$$\varepsilon_{n,rms}^2 = \langle \gamma^2 \rangle \left(\sigma_\gamma^2 \sigma_x^2 \sigma_{x'}^2 + \varepsilon_{rms}^2 \right)$$

$$\varepsilon_{n,rms}^2 = \langle \gamma^2 \rangle \left(\sigma_\gamma^2 \sigma_x^2 \sigma_{x'}^2 + \varepsilon_{rms}^2 \right)$$

Geometric emittance

At the plasma-vacuum interface is of the same order of magnitude as for conventional accelerators at low energies; however, due to the rapid increase of the bunch size, it becomes predominant compared to the second term.

Migliorati et al., Phys. Rev. STAB 16, 011302 (2013)

- The rms emittance concept
- Energy spread contribution
- rms envelope equation

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma'_x = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

$$\sigma_{xx'} = \langle xx' \rangle = -\alpha \varepsilon_{rms}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

Substituting α, β, γ we get $\frac{\sigma_{x'}^2}{\varepsilon_{rms}} \frac{\sigma_x^2}{\varepsilon_{rms}} - \left(\frac{\sigma_{xx'}}{\varepsilon_{rms}} \right)^2 = 1$

We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \right)}$$

$$x' = \frac{p_x}{p_z}$$

Envelope Equation without Acceleration

Now take the derivatives:

$$\frac{d\sigma_x}{dz} = \frac{d}{dz}\sqrt{\langle x^2 \rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\langle x^2 \rangle = \frac{1}{2\sigma_x} 2\langle xx' \rangle = \frac{\sigma_{xx'}}{\sigma_x}$$

$$\frac{d^2\sigma_x}{dz^2} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\langle x'^2 \rangle + \langle xx' \rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{\sigma_{x'}^2 + \langle xx'' \rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

And simplify:

$$\sigma_x'' = \frac{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x}$$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term.

$$\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

$$\frac{\varepsilon_{rms}^2}{\sigma_x^3} \approx \frac{T}{V} \approx P$$

Beam Thermodynamics

Kinetic theory of gases defines temperatures in each directions and global as:

$$k_B T_x = m\left\langle v_x^2 \right\rangle \qquad T = \frac{1}{3}\left(T_x + T_y + T_z\right) \qquad E_k = \frac{1}{2}m\left\langle v^2 \right\rangle = \frac{3}{2}k_B T$$

Definition of beam temperature in analogy:

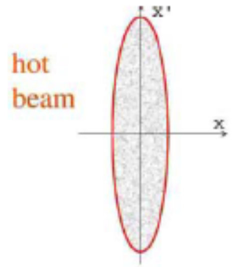
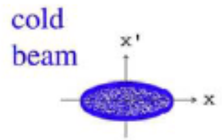
$$k_B T_{beam,x} = \gamma m_o \left\langle v_x^2 \right\rangle \qquad \left\langle v_x^2 \right\rangle = \beta^2 c^2 \left\langle x'^2 \right\rangle = \beta^2 c^2 \sigma_{x'}^2 = \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}$$

We get:

$$k_B T_{beam,x} = \gamma m_o \left\langle v_x^2 \right\rangle = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}$$

$$P_{beam,x} = n k_B T_{beam,x} = n\gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = N_T \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_L \sigma_x^2}$$

$$k_B T_{beam,x} = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}$$

Property	Hot beam	Cold beam
ion mass (m_o)	heavy ion	light ion
ion energy ($\beta\gamma$)	high energy	low energy
beam emittance (ε)	large emittance	small emittance
lattice properties ($\gamma_{x,y} \approx 1/\beta_{x,y}$)	strong focus (low β)	high β
phase space portrait		

Electron Cooling: Temperature relaxation by mixing a hot ion beam with co-moving cold (light) electron beam.

Particle Accelerators
1973, Vol. 5, pp. 61–65

EMITTANCE, ENTROPY AND INFORMATION

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$$S = kN \log(\pi\varepsilon)$$

Beam drifting in the free space

$$\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

Lets now consider for example the simple case with $\langle xx'' \rangle = 0$ describing a beam drifting in the free space.

The envelope equation reduces to: $$\sigma_x^3 \sigma_x'' = \varepsilon_{rms}^2$$

With initial conditions σ_o, σ_o' at z_o, depending on the upstream transport channel, the equation has a hyperbolic solution:

$$\sigma(z) = \sqrt{\left(\sigma_o + \sigma_o'(z - z_o)\right)^2 + \frac{\varepsilon_{rms}^2}{\sigma_o^2}(z - z_o)^2}$$

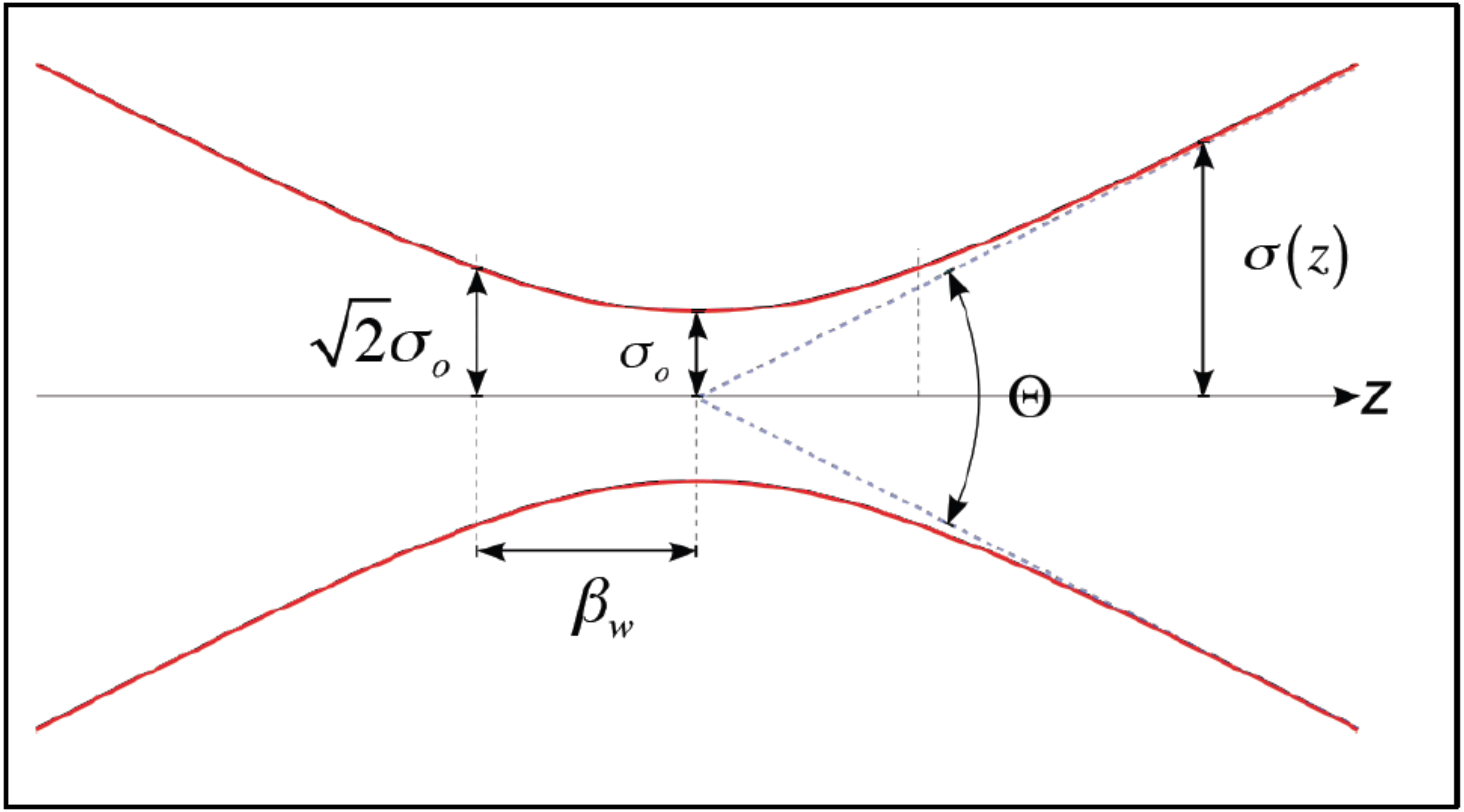
Considering the case $\sigma'_o = 0$ (beam at waist)

and using the definition $\sigma_x = \sqrt{\beta \varepsilon_{rms}}$

the solution is often written in terms of the β function as:

$$\sigma(z) = \sigma_o \sqrt{1 + \left(\frac{z - z_o}{\beta_w}\right)^2}$$

This relation indicates that without any external focusing element the beam envelope increases from the beam waist by a factor $\sqrt{2}$ with a characteristic length $\beta_w = \dfrac{\sigma_o^2}{\varepsilon_{rms}}$



For an effective transport of a beam with finite emittance is mandatory to make use of some external force providing beam confinement in the transport or accelerating line.

$$\sigma(z)=\sqrt{\left(\sigma_o+\sigma_o'(z-z_o)\right)^2+\frac{\varepsilon_{rms}^2}{\sigma_o^2}(z-z_o)^2}$$

At waist holds also the relation: $\varepsilon_{rms}^2=\sigma_{o,x}^2\sigma_{o,x'}^2$ $\sigma_o'=0$

that leads to: $\sigma_x^2(z)\approx\sigma_{o,x'}^2(z-z_o)^2$

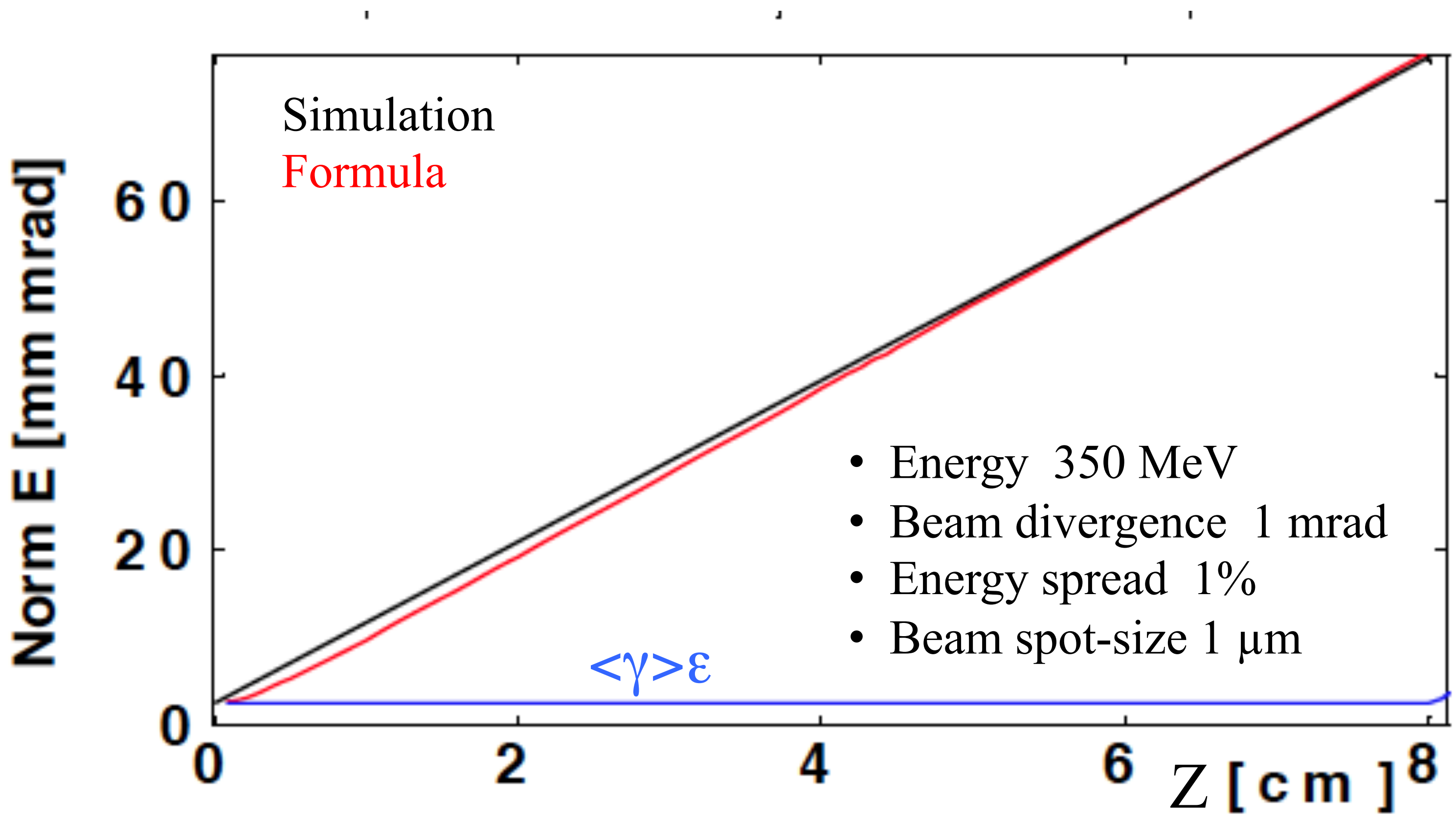
$$\varepsilon_{n,rms}^2=\langle\gamma^2\rangle\left(\sigma_\gamma^2\sigma_x^2\sigma_{x'}^2+\varepsilon_{rms}^2\right)=\langle\gamma^2\rangle\left(\sigma_\gamma^2\sigma_{o,x'}^4(z-z_o)^2+\varepsilon_{rms}^2\right)$$

showing that beams with large energy spread an divergence undergo a significant normalized emittance growth even in a drift

Migliorati et al., Phys. Rev. STAB 16, 011302 (2013)

$$\varepsilon_{n,rms}^2 = \langle \gamma^2 \rangle \left(\sigma_\gamma^2 \sigma_x^2 \sigma_{x'}^2 + \varepsilon_{rms}^2 \right) = \langle \gamma^2 \rangle \left(\sigma_\gamma^2 \sigma_{o,x'}^4 (z - z_o)^2 + \varepsilon_{rms}^2 \right)$$

showing that beams with large energy spread an divergence undergo a significant normalized emittance growth even in a drift



Envelope Equation with Linear Focusing

$$\sigma_x'' - \frac{\langle x x'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

Assuming that each particle is subject only to a linear focusing force, without acceleration: $x'' + k_x^2 x = 0$

take the average over the entire particle ensemble $\langle x x'' \rangle = -k_x^2 \langle x^2 \rangle$

$$\sigma_x'' + k_x^2 \sigma_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

We obtain the rms envelope equation with a linear focusing force in which, unlike in the single particle equation of motion, the rms emittance enters as defocusing pressure like term.

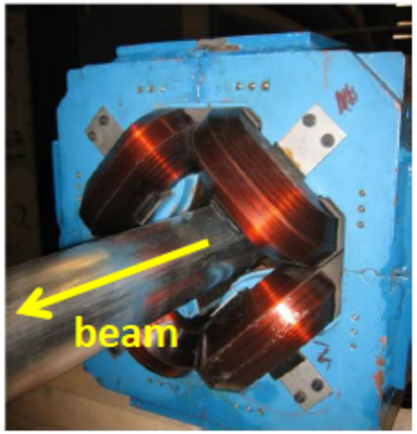
MAGNETIC QUADRUPOLE

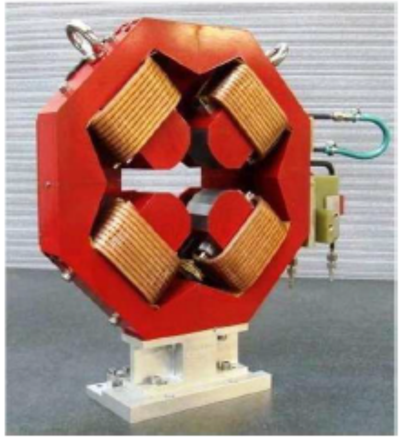
Quadrupoles are used to focalize the beam in the transverse plane. It is a 4 poles magnet:

⇒B=0 in the center of the quadrupole

⇒The B intensity increases linearly with the off-axis displacement.

⇒If the quadrupole is focusing in one plane is defocusing in the other plane

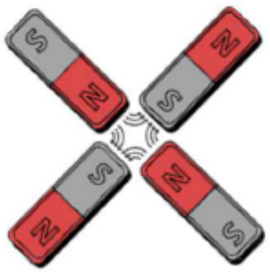


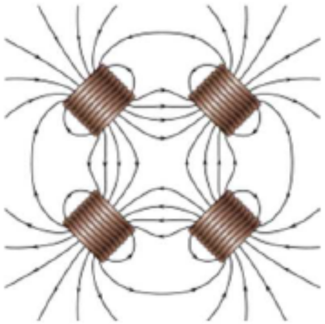


Electromagnetic quadrupoles G <100 T/m

$$\begin{cases} B_x = G \cdot y \\ B_y = G \cdot x \end{cases} \Rightarrow \begin{cases} F_y = qvG \cdot y \\ F_x = -qvG \cdot x \end{cases}$$

$$G = \text{quadrupole gradient} \left[\frac{T}{m} \right]$$





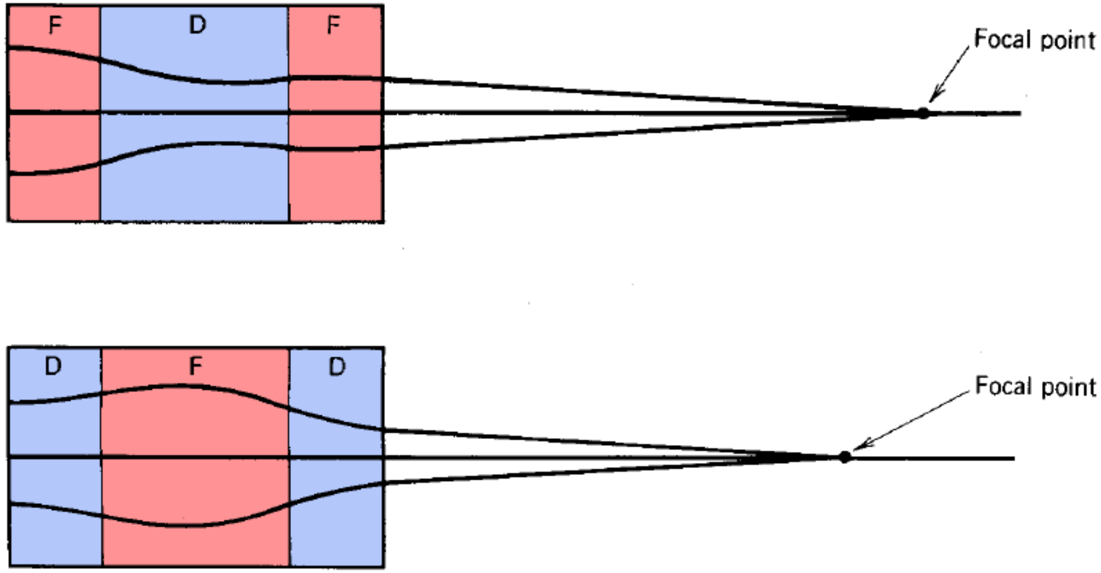
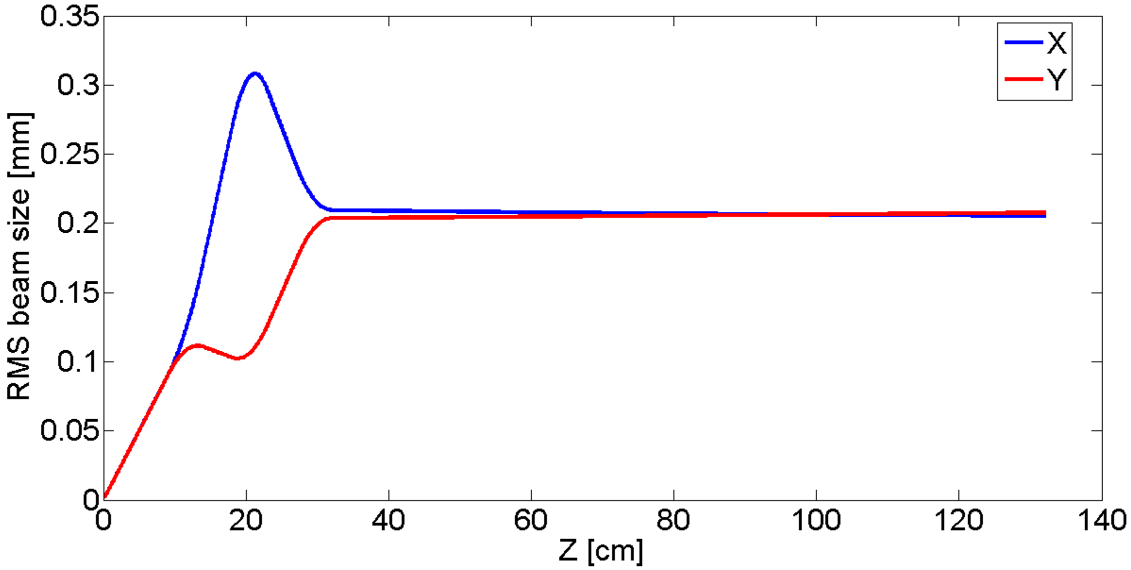


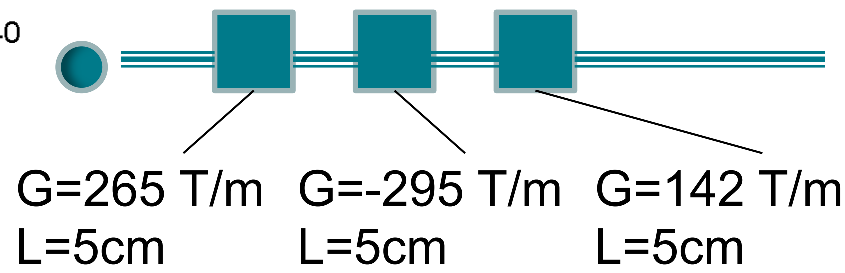
Figure 8.8 Improved stigmatic properties of a quadrupole triplet lens. Orbits of particles initially parallel to the axis projected in the x and y planes.

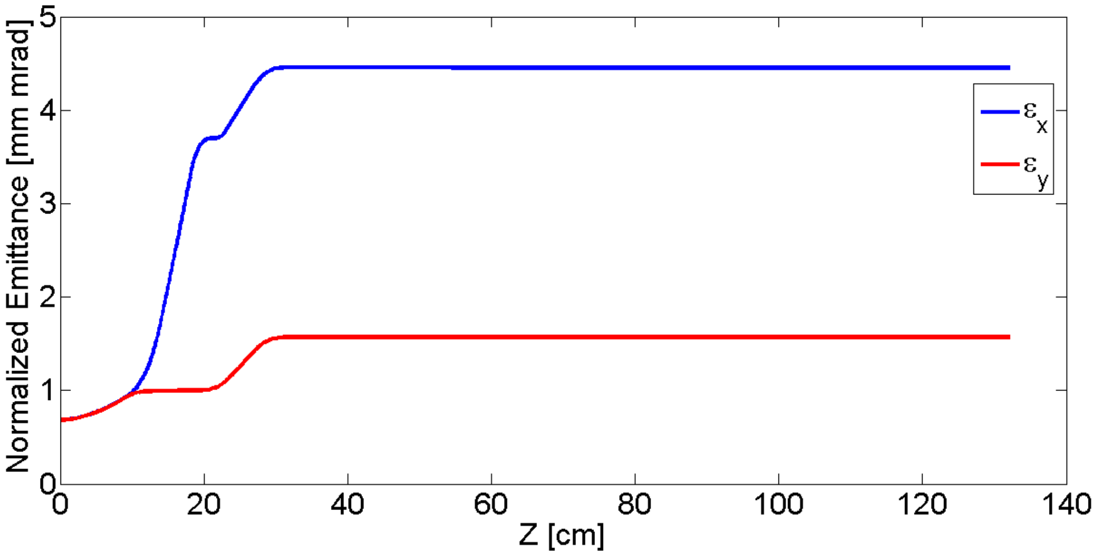
Beam transport line simulated with TSTEP



Beam transport line based on a triplet-lattice.
Beam parameters are:

- Energy 350 MeV
- Beam divergence 1 mrad
- Energy spread 1%
- Beam spot-size 1 μm

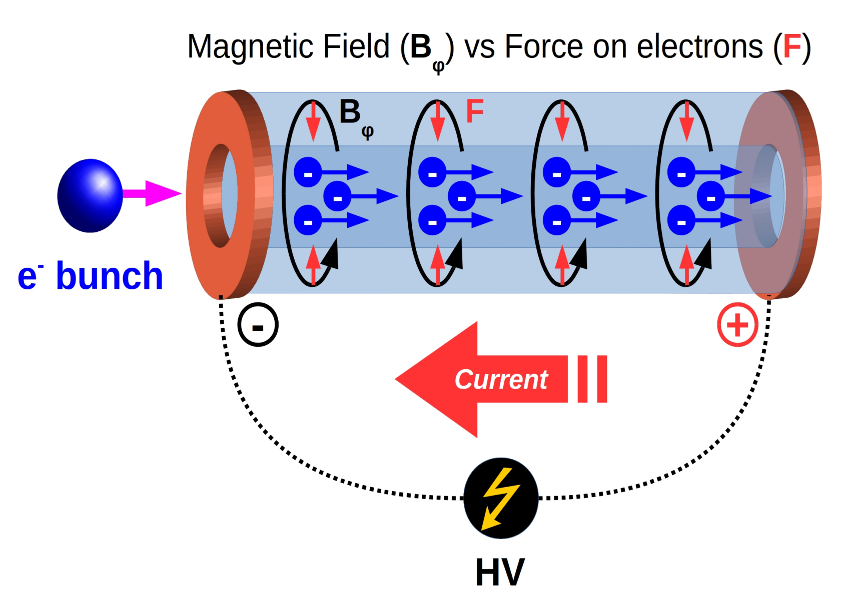




Keeping the **beam size under control** is possible, **but normalized emittance grows** throughout the beamline.

$$\Delta\varepsilon_{n,rms} = \langle\gamma\rangle\left|\left(\sigma_\gamma k_q l_q + \sigma'_o\right)\sigma_o^2 + \sigma_o\sigma'_o\right|$$

Active Plasma Lens



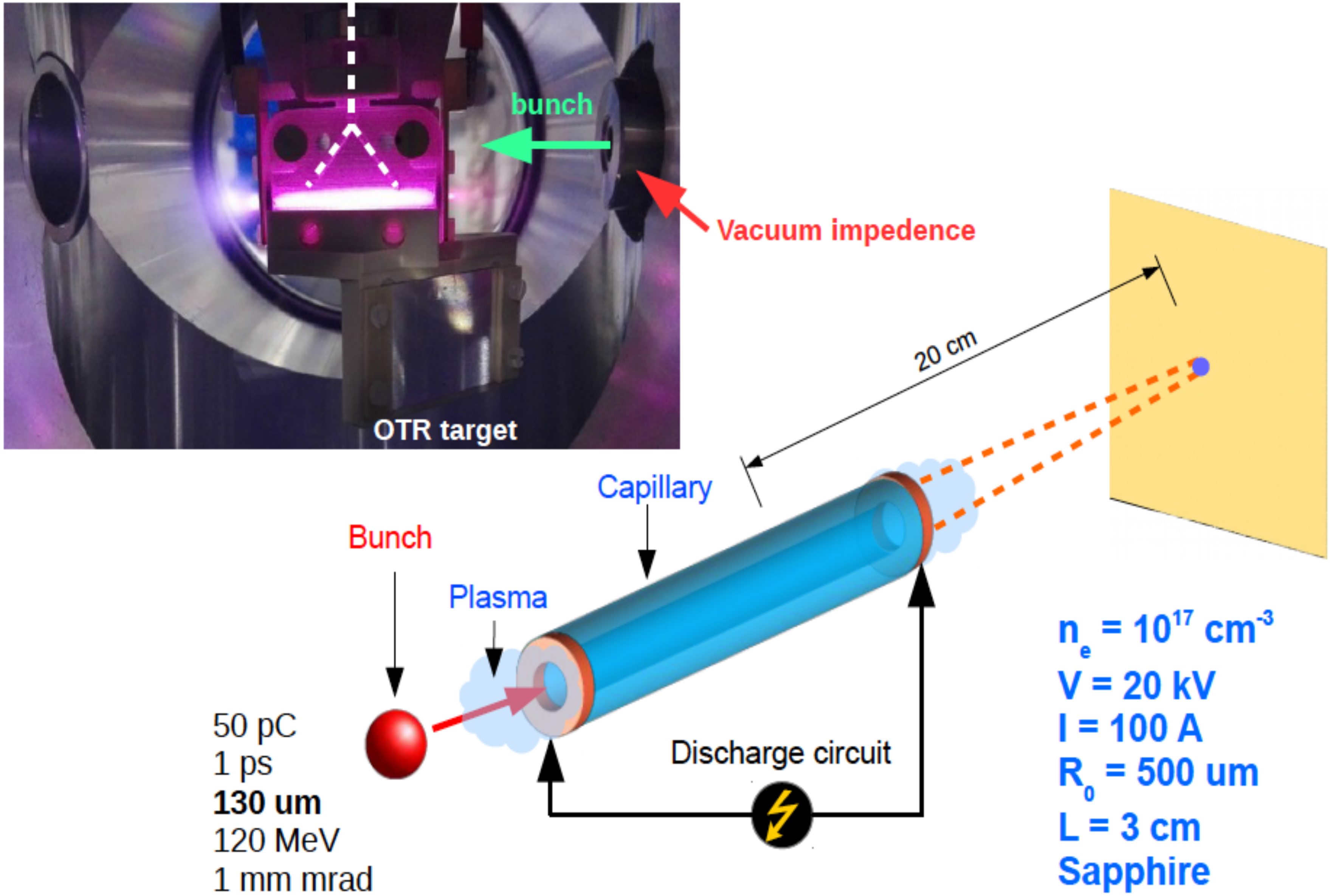
$$F_r = ec\left(\frac{\mu_o I_c}{2\pi R_c^2}\right) r = ecB'_\vartheta r$$

$$\frac{K_{cap}}{\gamma} = \frac{eB'_\vartheta}{\gamma mc} = \frac{2I_c}{\gamma I_A R_c^2}$$

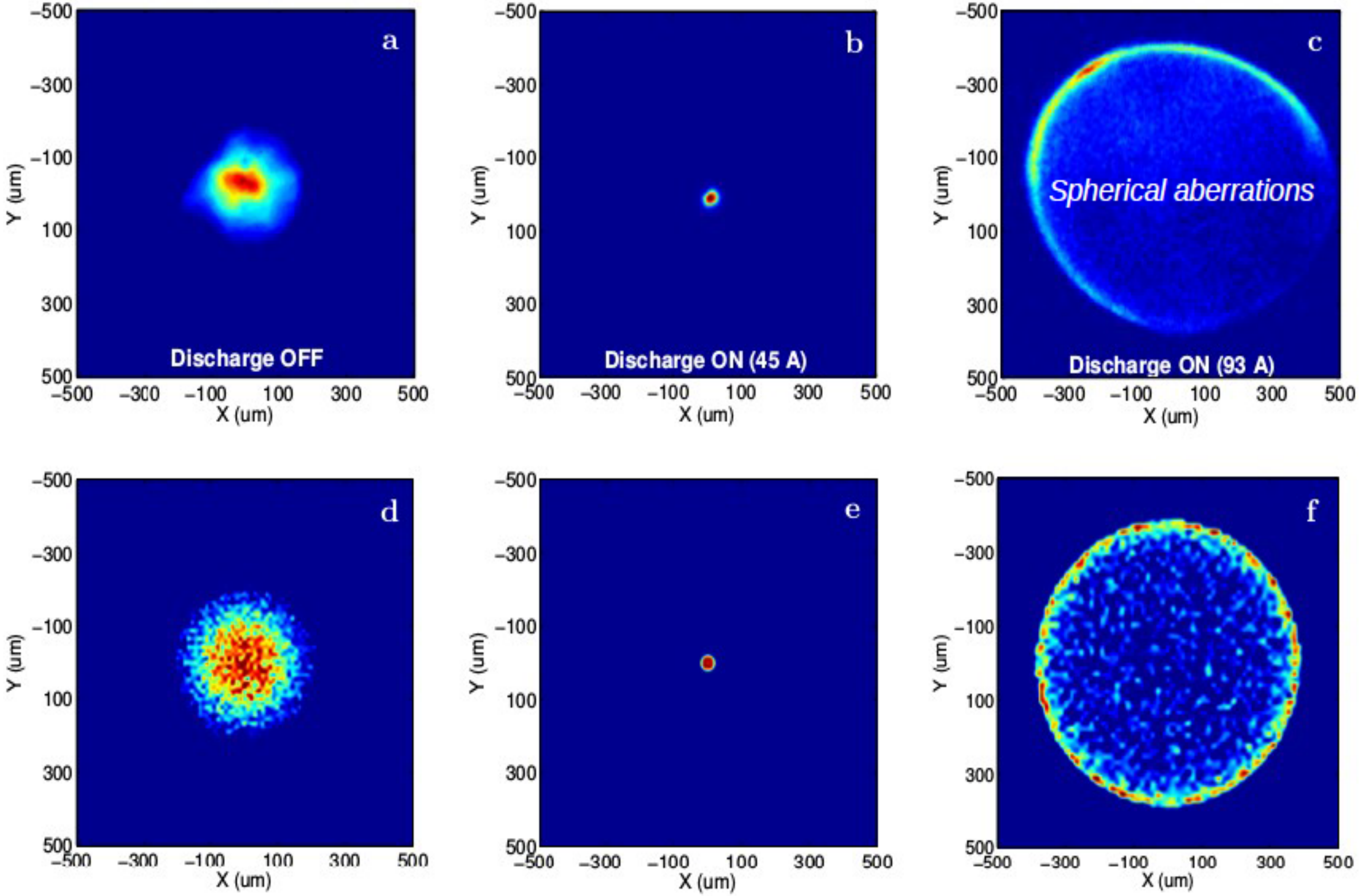
$$I_A = \frac{4\pi\varepsilon_o m_o c^3}{e} = 17kA$$

$$\sigma''_x + \frac{K_{cap}}{\gamma}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2\sigma_x^3}$$

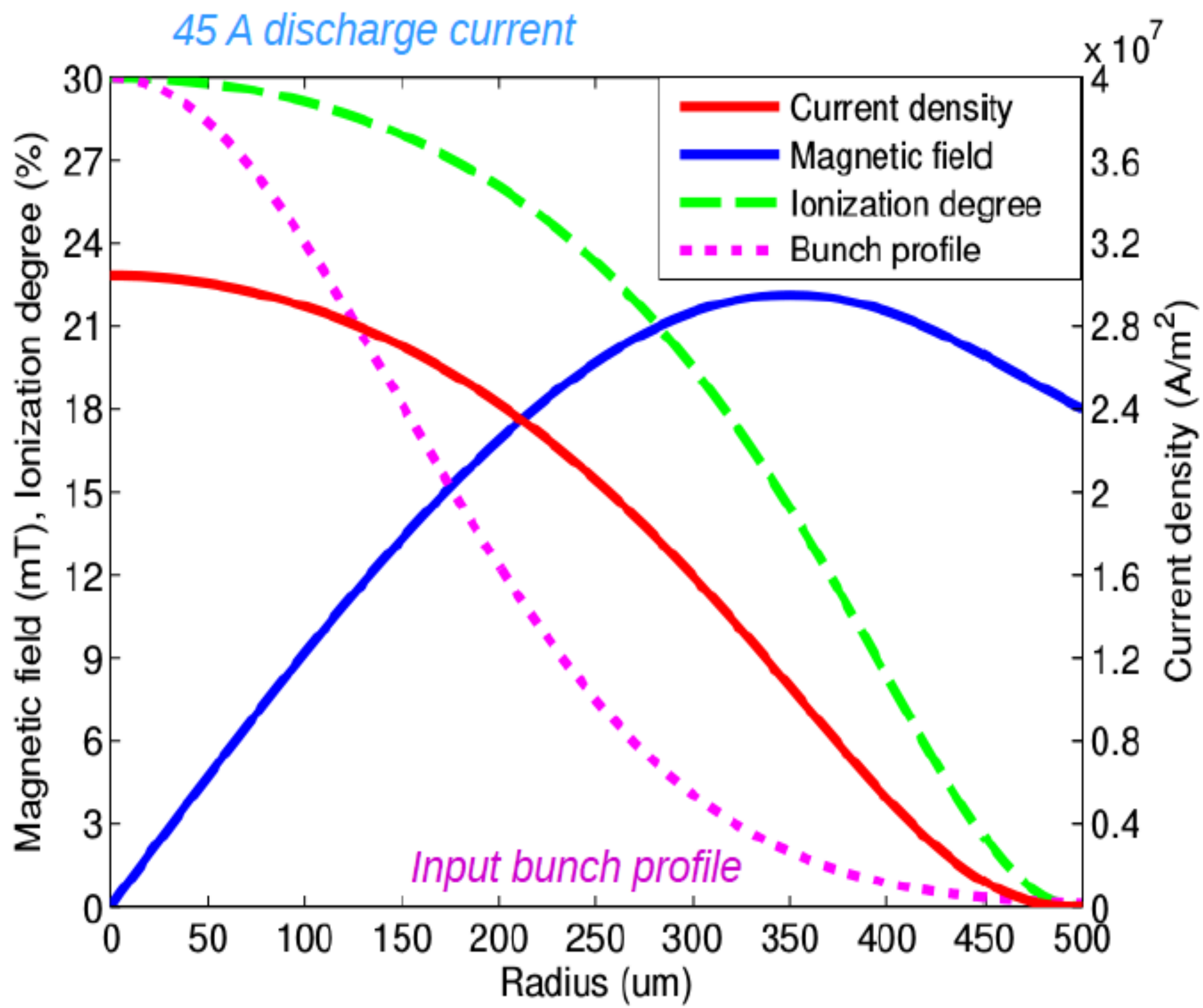
Experimental layout



Results vs simulations



Nonlinear focusing field

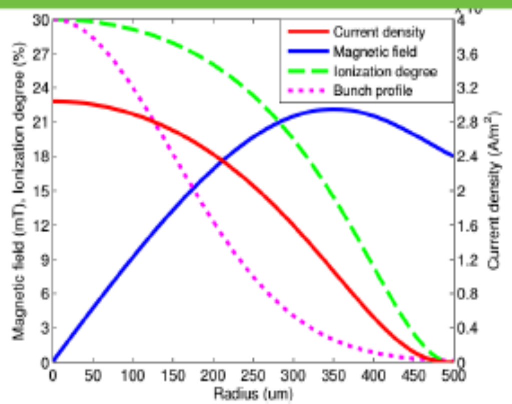


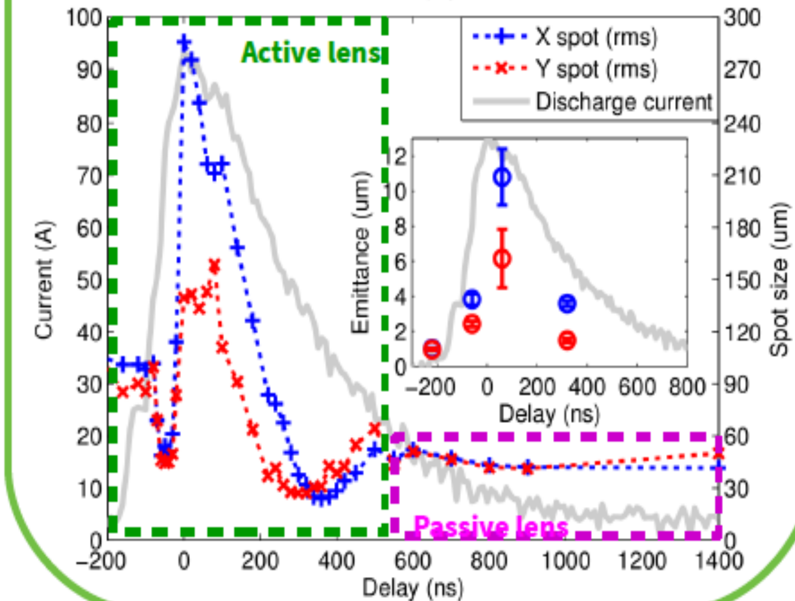
SPARC LAB

Experimental results

INFN Istituto Nazionale di Fisica Nucleare Laboratori Nazionali di Frascati

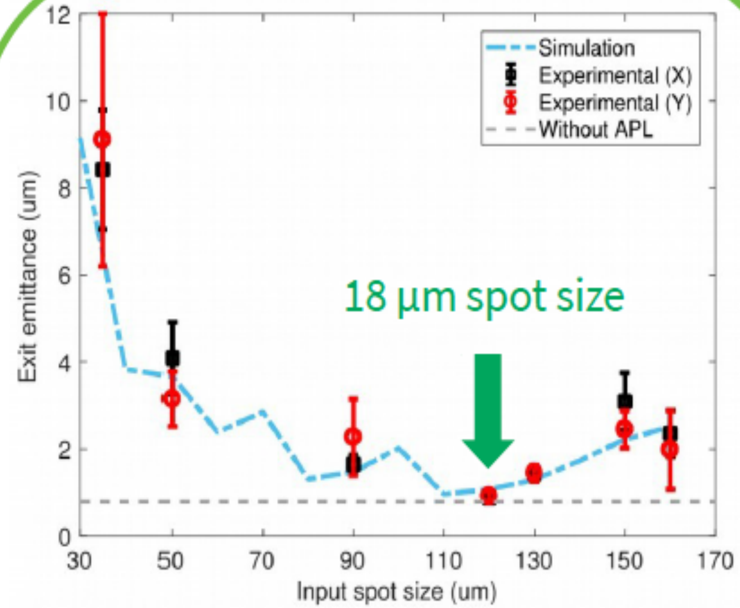
Demonstration of emittance growth

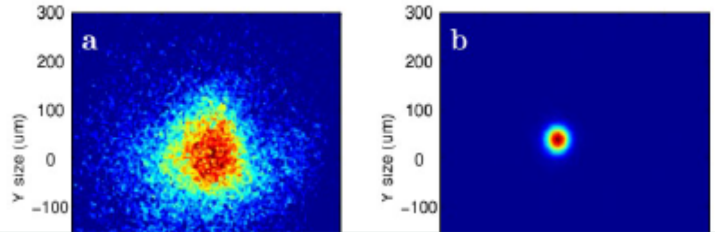




Pompili, R., et al. Applied Physics Letters 110.10 (2017): 104101.
Marocchino, A., et al. Applied Physics Letters 111.18 (2017): 184101.

Demonstration of emittance preservation





PHYSICAL REVIEW LETTERS

Accepted Paper

Focusing of high-brightness electron beams with active-plasma lenses

Phys. Rev. Lett.

R. Pompili et al.

Accepted 11 October 2018

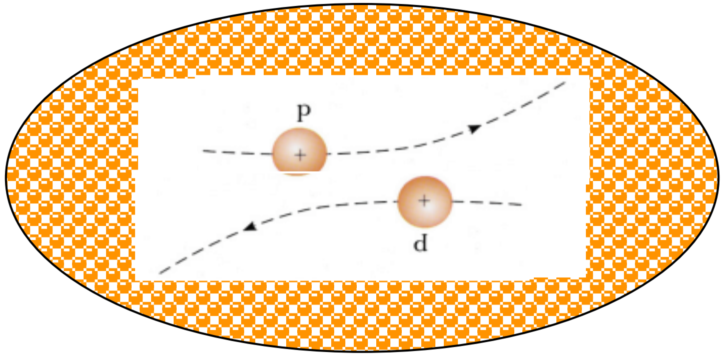
OUTLINE

- The rms emittance concept
- Energy spread contribution
- rms envelope equation
- Space charge foces

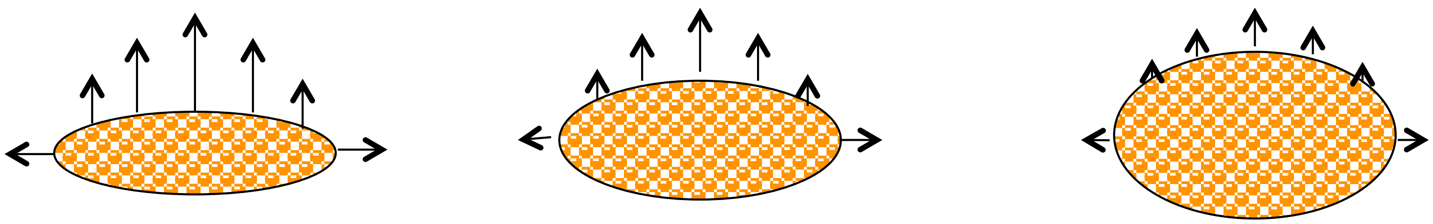
Space Charge: What does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**



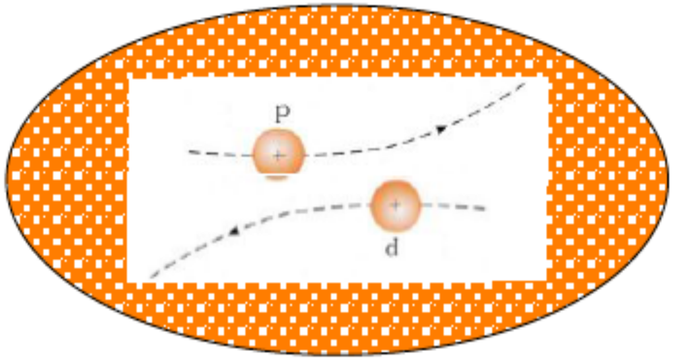
2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects**



The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**

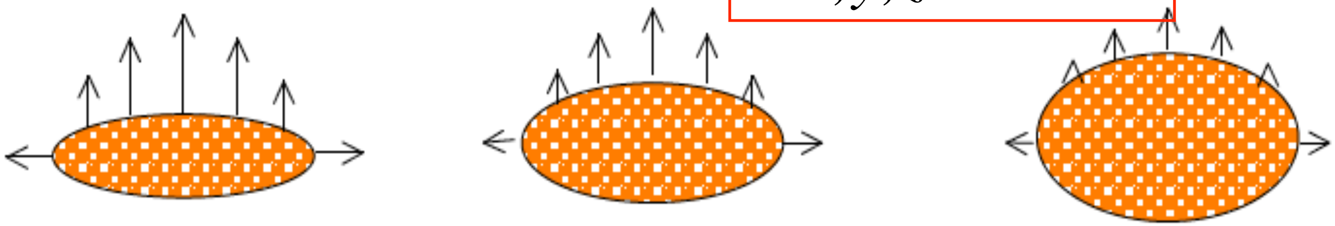
$$\lambda_D = \sqrt{\frac{\varepsilon_o k_B T}{e^2 n}}$$



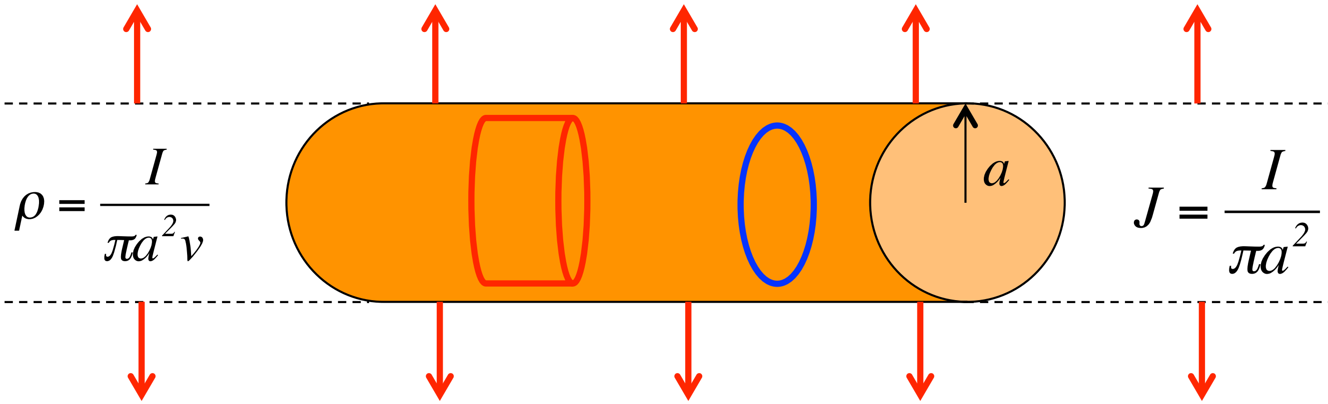
$$\sigma_{x,y,z} << \lambda_D$$

2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects, Single Component Cold Plasma**

$$\sigma_{x,y,z} >> \lambda_D$$



Continuous Uniform Cylindrical Beam Model



$$\rho = \frac{I}{\pi a^2 v}$$

$$J = \frac{I}{\pi a^2}$$

Gauss's law

$$\int \varepsilon_o E \cdot dS = \int \rho dV$$

$$E_r = \frac{I}{2\pi\varepsilon_o a^2 v} r \quad \textbf{\textit{for}} \quad r \le a$$

$$E_r = \frac{I}{2\pi\varepsilon_o v} \frac{1}{r} \quad \textbf{\textit{for}} \quad r > a$$

$$B_\vartheta = \frac{\beta}{c} E_r$$

Ampere's law

$$\int B \cdot dl = \mu_o \int J \cdot dS$$

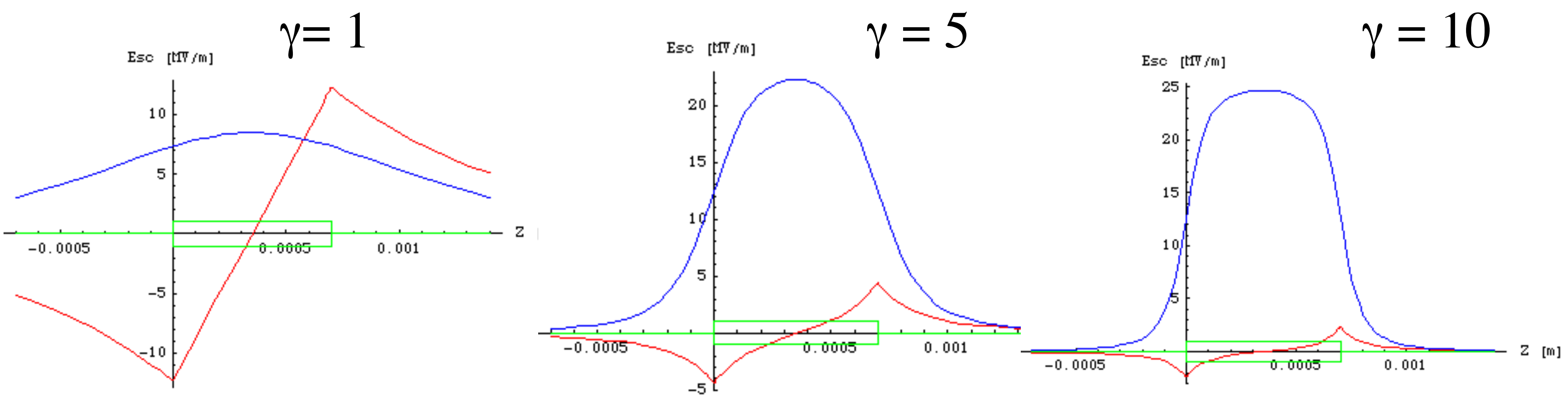
$$B_\vartheta = \mu_o \frac{Ir}{2\pi a^2} \quad \textbf{\textit{for}} \quad r \le a$$

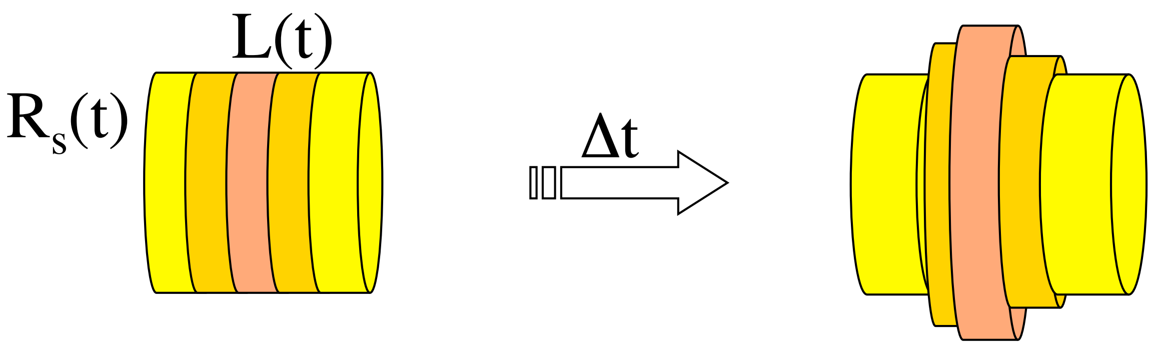
$$B_\vartheta = \mu_o \frac{I}{2\pi r} \quad \textbf{\textit{for}} \quad r > a$$

Bunched Uniform Cylindrical Beam Model

$$E_z(0,s,\gamma) = \frac{I}{2\pi\gamma\varepsilon_0 R^2 \beta c} h(s,\gamma)$$

$$E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$





$$E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$

Lorentz Force

$$F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1-\beta^2\right)E_r = \frac{eE_r}{\gamma^2}$$

$$B_\vartheta = \frac{\beta}{c} E_r$$

is a **linear** function of the transverse coordinate

$$\frac{dp_r}{dt} = F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g(s,\gamma)$$

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore space charge defocusing is primarily a non-relativistic effect. Using $R=2\sigma_x$ for a uniform distribution:

$$F_x = \frac{eIx}{8\pi\gamma^2\varepsilon_0\sigma_x^2\beta c} g(s,\gamma)$$

Envelope Equation with Space Charge

Single particle transverse motion:

$$\frac{dp_x}{dt} = F_x \qquad p_x = p\,x' = \beta\gamma m_o c x'$$

$$\frac{d}{dt}(px') = \beta c \frac{d}{dz}(p\,x') = F_x$$

$$x'' = \frac{F_x}{\beta c p}$$

$$F_x = \frac{eIx}{8\pi\gamma^2\varepsilon_0\sigma_x^2\beta c} g(s,\gamma)$$

$$x'' = \frac{k_{sc}(s,\gamma)}{\sigma_x^2} x$$

$$k_{sc} = \frac{2I}{I_A} g(s,\gamma)$$

$$I_A = \frac{4\pi\varepsilon_o m_o c^3}{e}$$

Now we can calculate the term $\langle x x'' \rangle$ that enters in the envelope equation

$$\sigma''_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\langle x x'' \rangle}{\sigma_x} \qquad \langle x x'' \rangle = \frac{k_{sc}}{\sigma_x^2} \langle x^2 \rangle = k_{sc}$$

Including all the other terms the envelope equation reads:

Space Charge De-focusing Force

$$\sigma''_x + k^2 \sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

Emittance Pressure

External Focusing Forces

Laminarity Parameter: $\rho = \frac{(\beta\gamma)^2 k_{sc} \sigma_x^2}{\varepsilon_n^2}$

The beam undergoes two regimes along the accelerator

$$\sigma''_x + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

ρ>>1 Laminar Beam

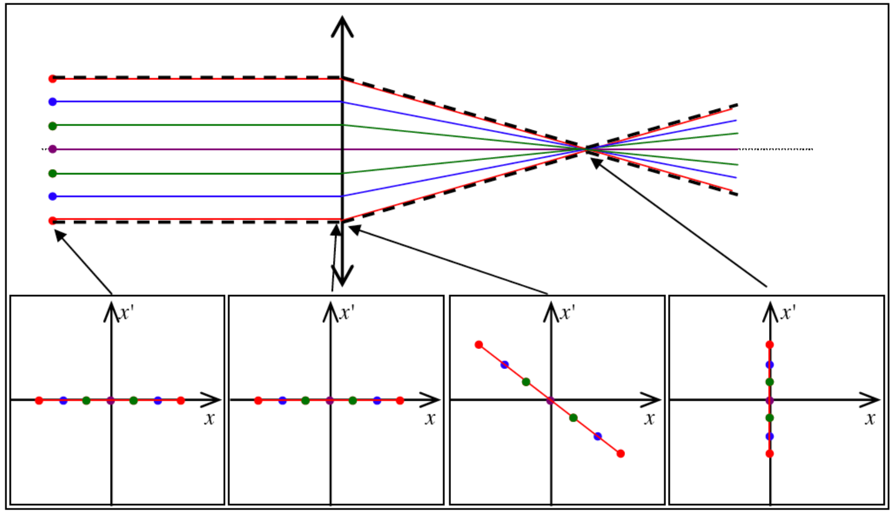


Fig. 10: Particle trajectories in laminar beam

$$\sigma''_x + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

ρ<<1 Thermal Beam

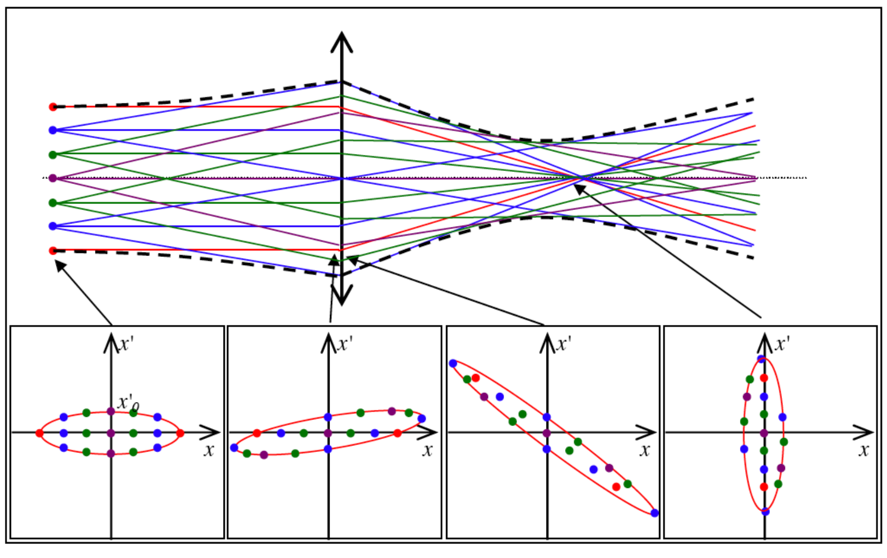


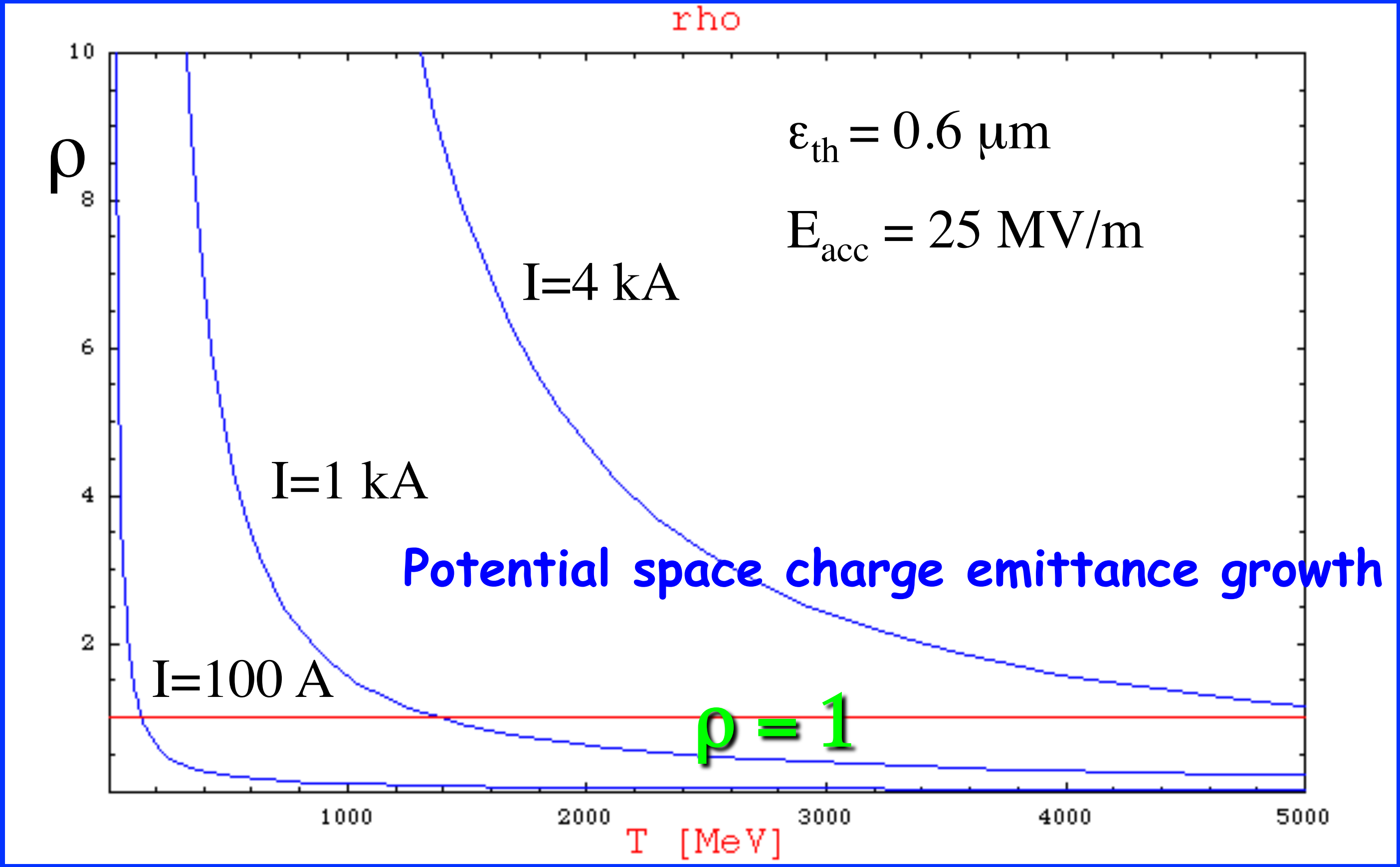
Fig. 11: Particle trajectories in non-zero emittance beam

Laminarity parameter

$$\rho = \frac{2I\sigma^2}{\gamma I_A \varepsilon_n^2} \equiv \frac{2I\sigma_q^2}{\gamma I_A \varepsilon_n^2} = \frac{4I^2}{\gamma'^2 I_A^2 \varepsilon_n^2 \gamma^2}$$

Transition Energy (ρ=1)

$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$

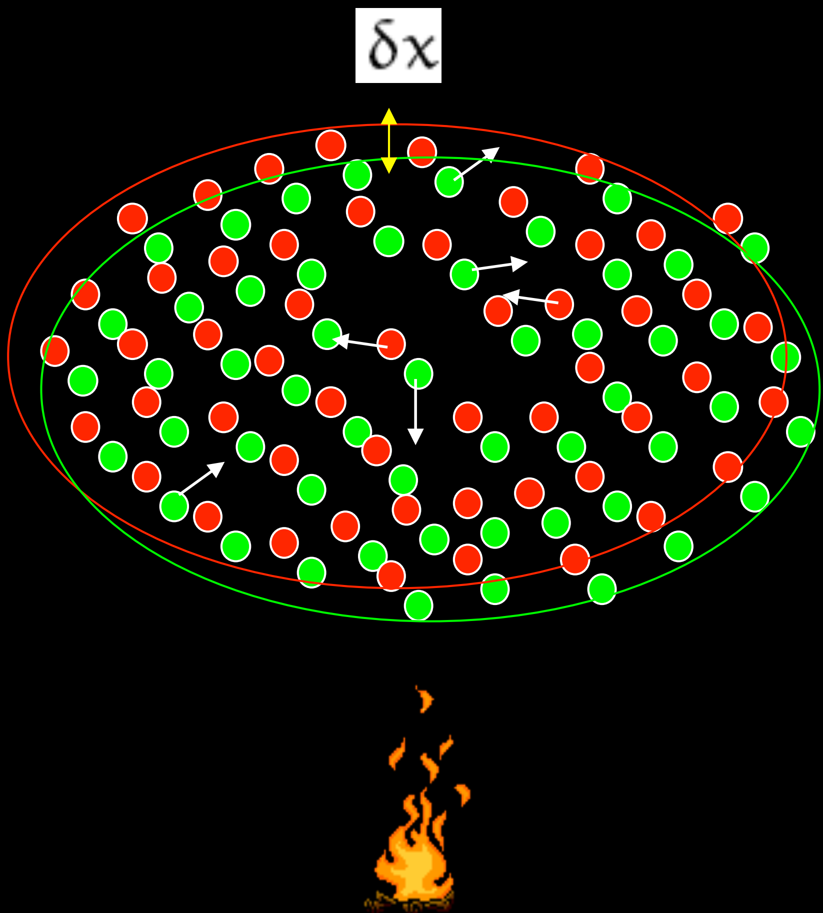


OUTLINE

- The rms emittance concept
- Energy spread contribution
- rms envelope equation
- Space charge forces
- Beam emittance oscillations

Surface charge density

$$\sigma = e\,n\,\delta x$$



Surface electric field

$$E_x = -\sigma/\epsilon_0 = -e\,n\,\delta x/\epsilon_0$$

Restoring force

$$m\,\frac{d^2\delta x}{dt^2} = e\,E_x = -m\,\omega_p{}^2\,\delta x$$

Plasma frequency

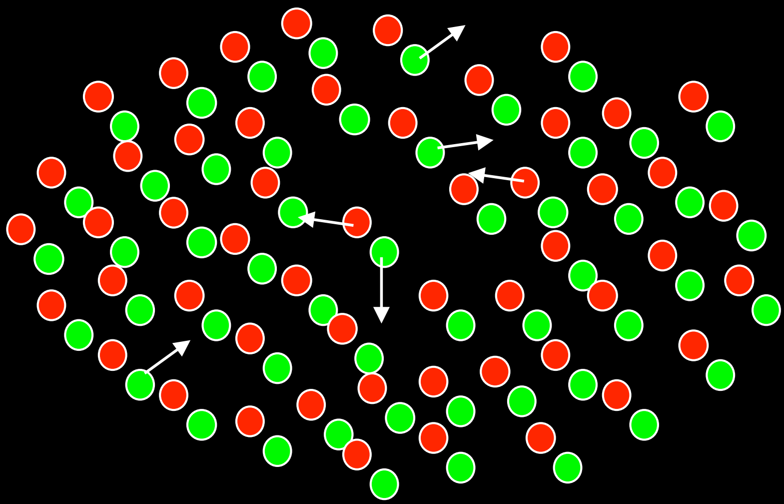
$$\omega_p{}^2 = \frac{n\,e^2}{\epsilon_0\,m}$$

Plasma oscillations

$$\delta x = (\delta x)_0\,\cos\,(\omega_p\,t)$$

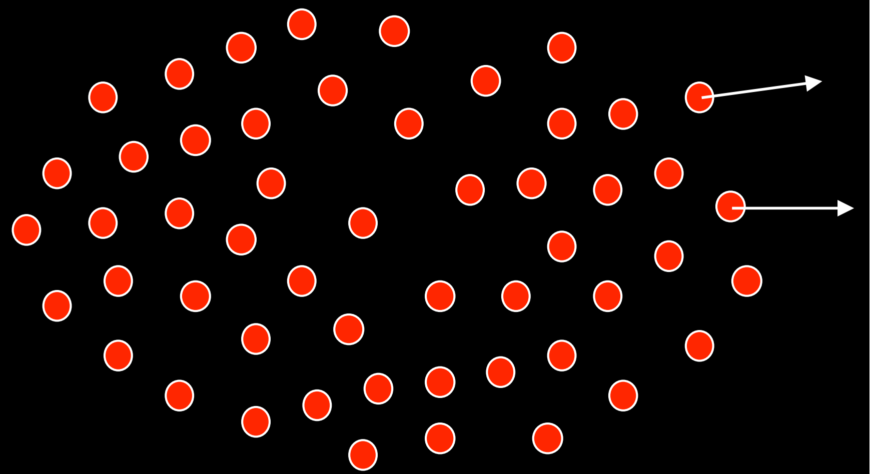
Neutral Plasma

- Oscillations
- Instabilities
- EM Wave propagation



Single Component Cold Relativistic Plasma

Magnetic focusing



Magnetic focusing

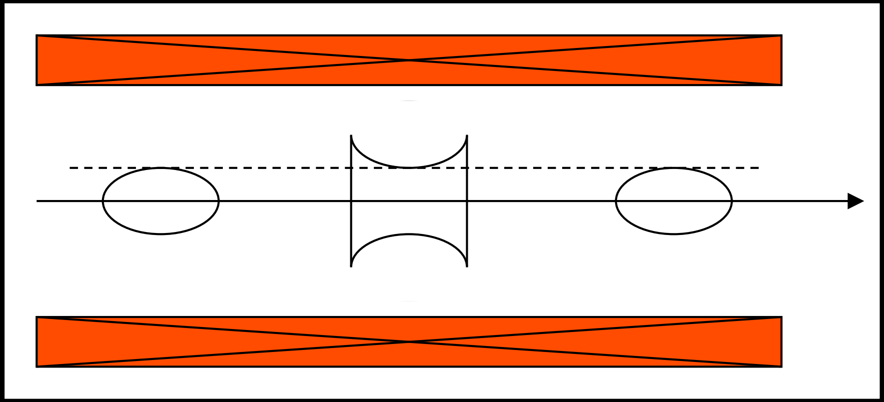
$$\sigma'' + k_s^2 \sigma = \frac{k_{sc}(s,\gamma)}{\sigma}$$

Single Component Relativistic Plasma

$$k_s = \frac{qB}{2mc\beta\gamma}$$

Equilibrium solution:

$$\sigma_{eq}(s,\gamma) = \frac{\sqrt{k_{sc}(s,\gamma)}}{k_s}$$



Small perturbation:

$$\sigma(\zeta) = \sigma_{eq}(s) + \delta\sigma(s)$$

$$\delta\sigma''(s) + 2k_s^2 \delta\sigma(s) = 0$$

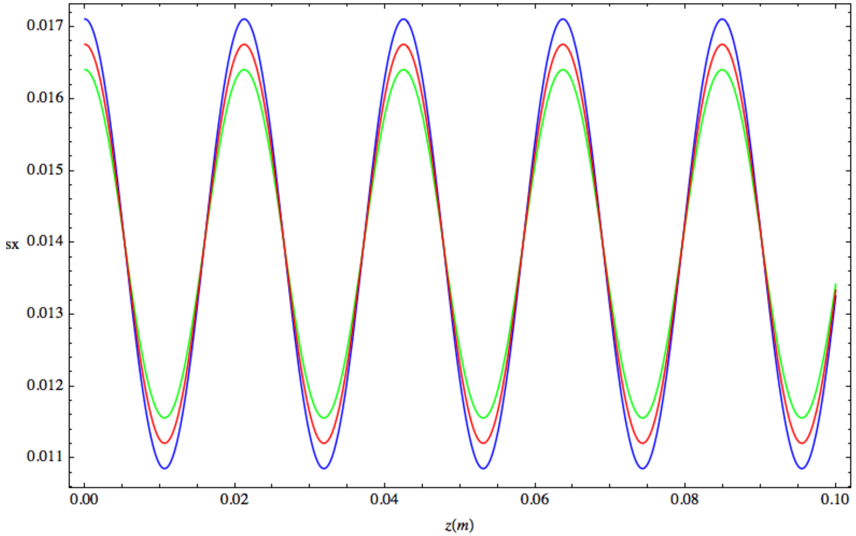
$$\delta\sigma(s) = \delta\sigma_o(s)\, cos\left(\sqrt{2}k_s z\right)$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

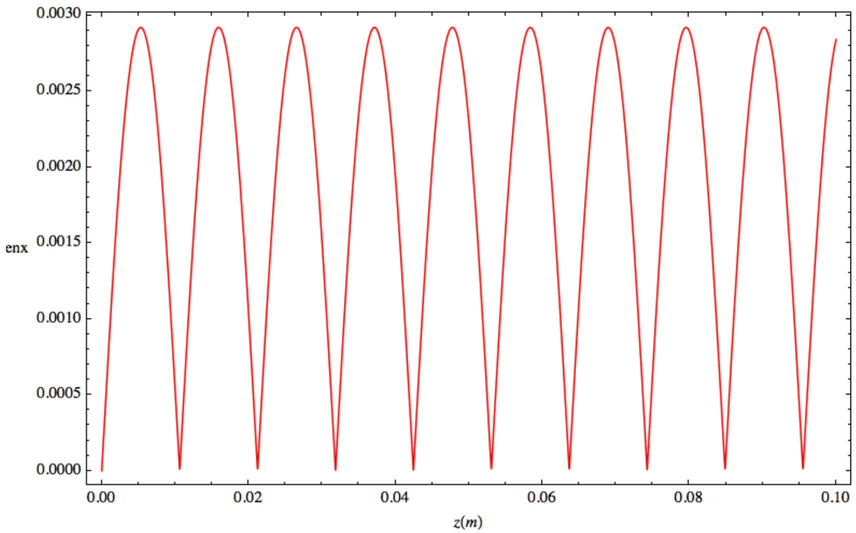
$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma_o(s)\, cos\left(\sqrt{2}k_s z\right)$$

Envelope oscillations drive Emittance oscillations

$\sigma(z)$



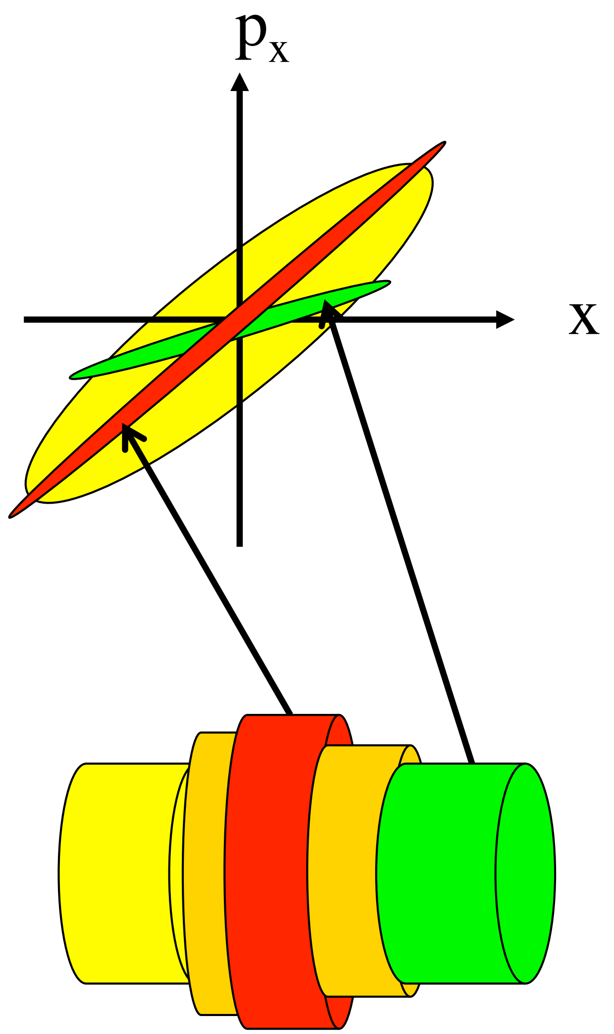
$\varepsilon(z)$



$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}{}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \right)} \approx \left| sin\left(\sqrt{2} k_s z \right) \right|$$

Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam

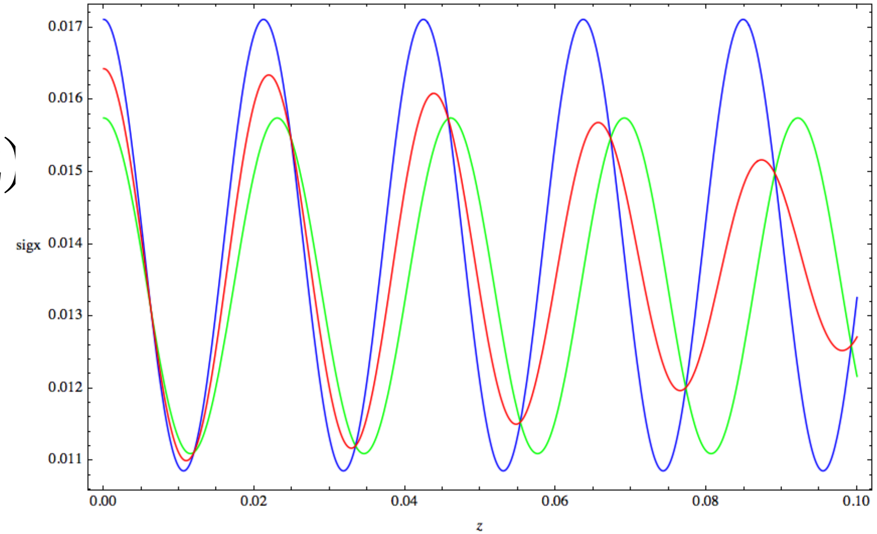
Projected Phase Space



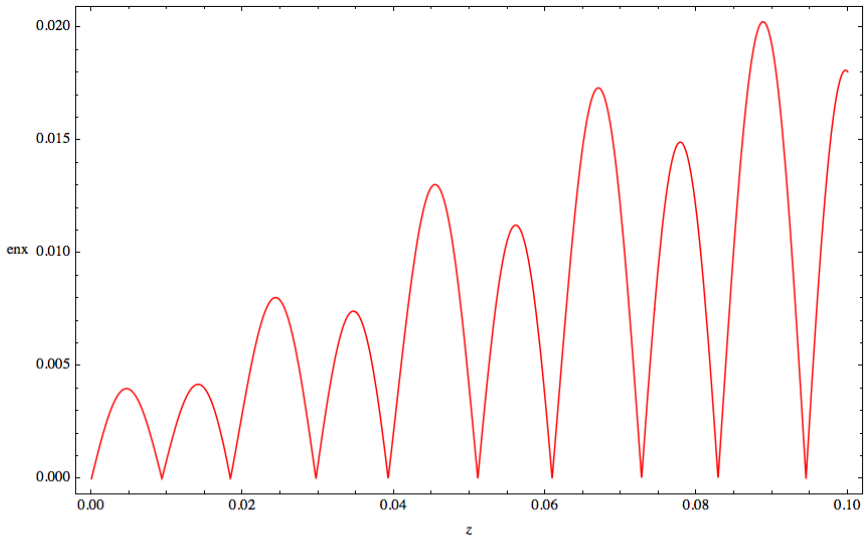
Slice Phase Spaces

energy spread induces decoherence

$\sigma(z)$



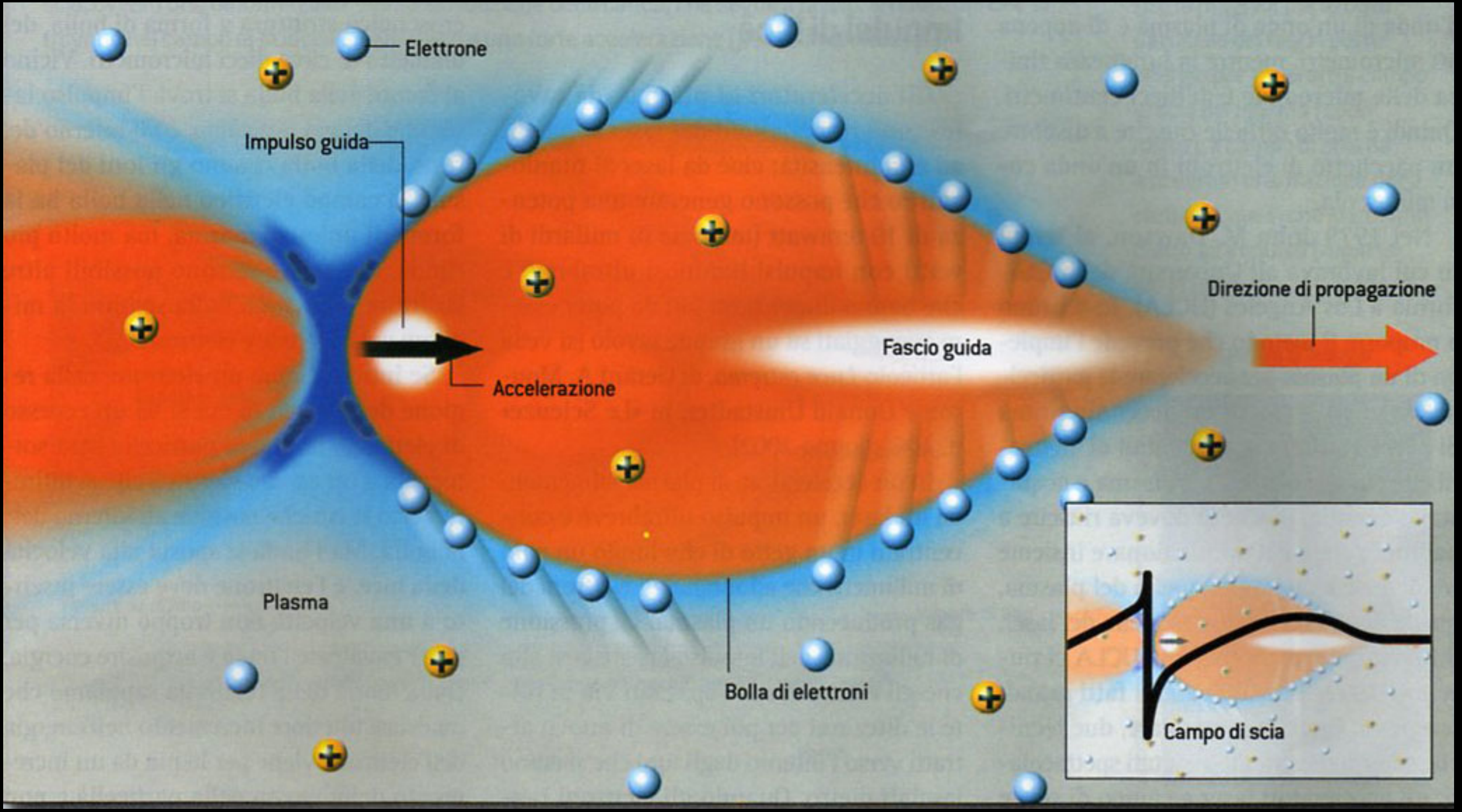
$\varepsilon(z)$



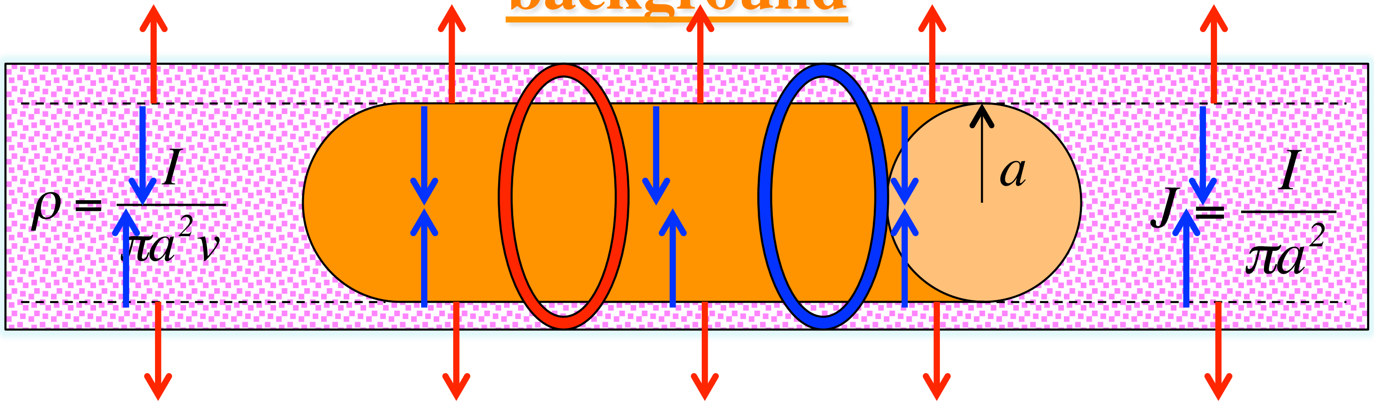
OUTLINE

- The rms emittance concept
- Energy spread contribution
- rms envelope equation
- Space charge forces
- Beam emittance oscillations
- Adiabatic plasma matching

Plasma Accelerator



Continuous Uniform Cylindrical Beam Model with ionized gas background



f_e : charge neutralisation factor

$$E_r = \frac{I(1-f_e)}{2\pi\varepsilon_o a^2 v} r \quad \text{for} \quad r \leq a$$

$$E_r = \frac{I(1-f_e)}{2\pi\varepsilon_o v} \frac{1}{r} \quad \text{for} \quad r > a$$

f_m : current neutralisation factor

$$B_\vartheta = \mu_o \frac{I(1-f_m)}{2\pi a^2} r \quad \text{for} \quad r \leq a$$

$$B_\vartheta = \mu_o \frac{I(1-f_m)}{2\pi a^2} \frac{a^2}{r} \quad \text{for} \quad r > a$$

Lorentz Force

$$F_r = e\left(E_r - \beta c B_\vartheta\right) = \frac{eE_r}{\gamma^2}\left(1 - \gamma^2 f_e + \beta^2\gamma^2 f_m\right)$$

Generalized Envelope Equation

$$\sigma'' + \frac{k^2}{\gamma}\sigma = \frac{2I}{I_A\gamma^3\sigma}\left(1 - \gamma^2 f_e + \gamma^2 f_m\right) + \frac{\varepsilon_n^2}{\gamma^2\sigma^3}$$

$$\beta = 1$$

Equilibrium solution

$$\sigma'' = \frac{2I}{I_A \gamma^3 \sigma}\left(1 - \gamma^2 f_e + \gamma^2 f_m\right) + \frac{\varepsilon_n^2}{\gamma^2 \sigma^3}$$

$$\gamma' = K = 0$$

$$\left(1 - \gamma^2 f_e + \gamma^2 f_m\right) \frac{<}{>} 0 \qquad \frac{\Rightarrow \text{focusing}}{\Rightarrow \text{defocusing}}$$

$$\sigma = \sqrt{\frac{I_A \gamma \varepsilon_n^2}{2I\left(\gamma^2 f_e - \gamma^2 f_m - 1\right)}}$$

Adiabatic Plasma Matching

$$\sigma'' = \frac{2I\left(1-\gamma^2 f_e + \gamma^2 f_m\right)}{I_A \gamma^3 \sigma} + \frac{\varepsilon_n^2}{\gamma^2 \sigma^3}$$

$$\begin{cases} f_e(z) = \dfrac{n_p(z)}{n_e} \\ f_m = 0 \end{cases}$$

$$L_s \equiv n_0(z)/n_0'(z) >> \beta_{qeq}(z)$$

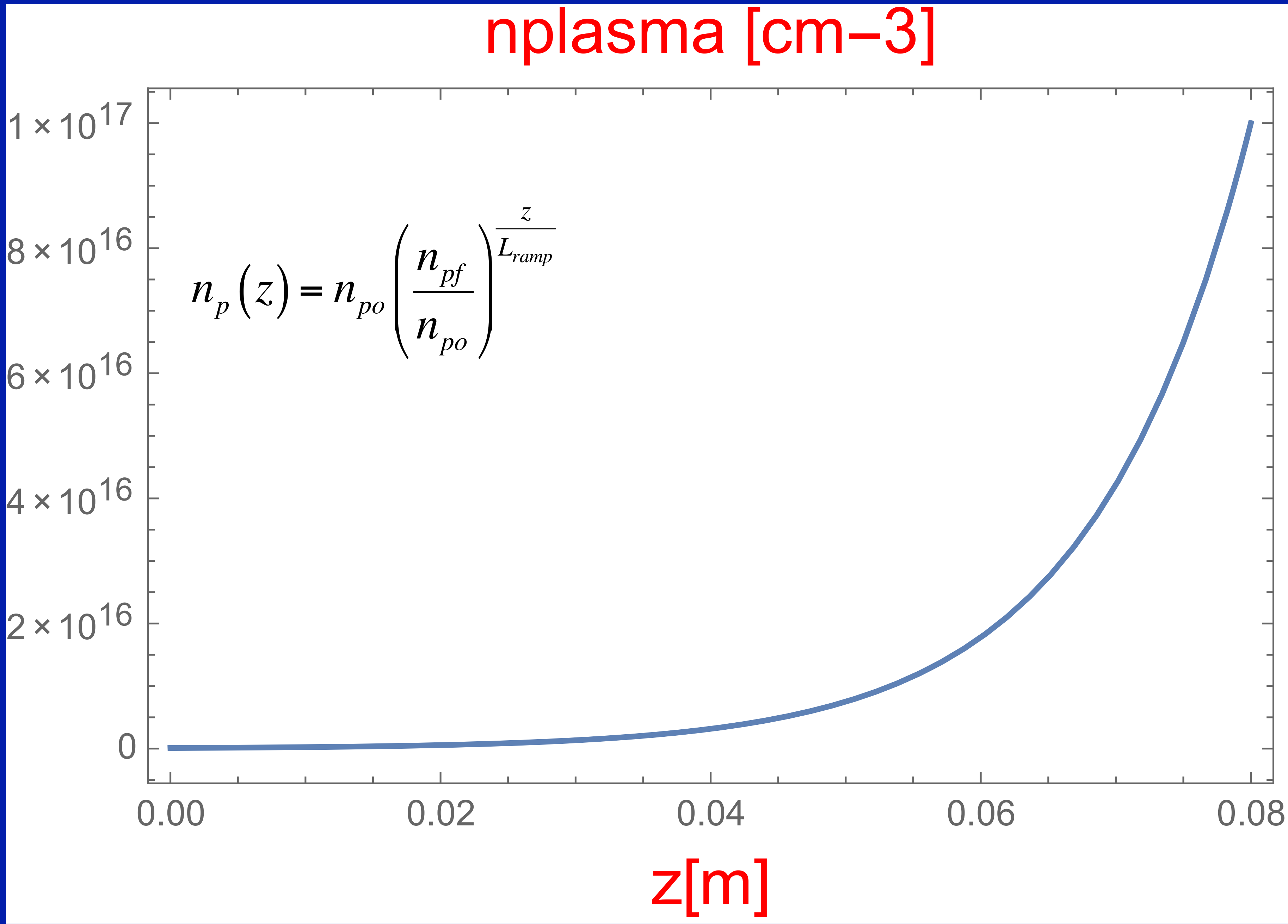
$$\sigma'' + \frac{2I}{I_A \gamma\ \sigma} \frac{n_p(z)}{n_e} = \frac{\varepsilon_n^2}{\gamma^2 \sigma^3}$$

$$\sigma'' + \frac{k_p^2(z)}{2\gamma} \sigma = \frac{\varepsilon_n^2}{\gamma^2 \sigma^3}$$

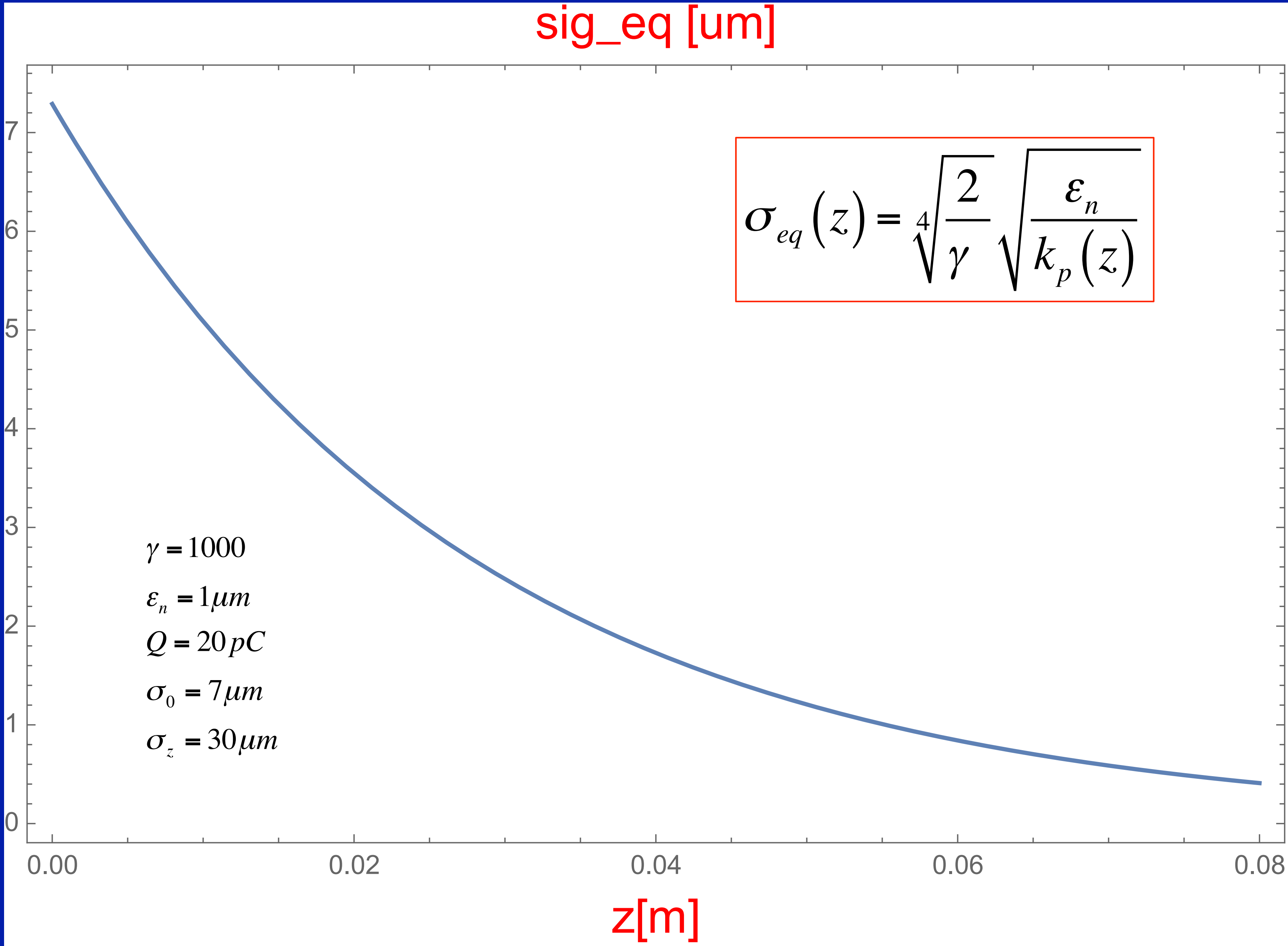
$$I = ecn_e \pi \sigma^2$$

$$k_p^2(z) = \frac{e^2 n_p(z)}{\varepsilon_o mc^2}$$

$$\sigma_{eq}(z) = \sqrt[4]{\frac{2}{\gamma}} \sqrt{\frac{\varepsilon_n}{k_p(z)}}$$

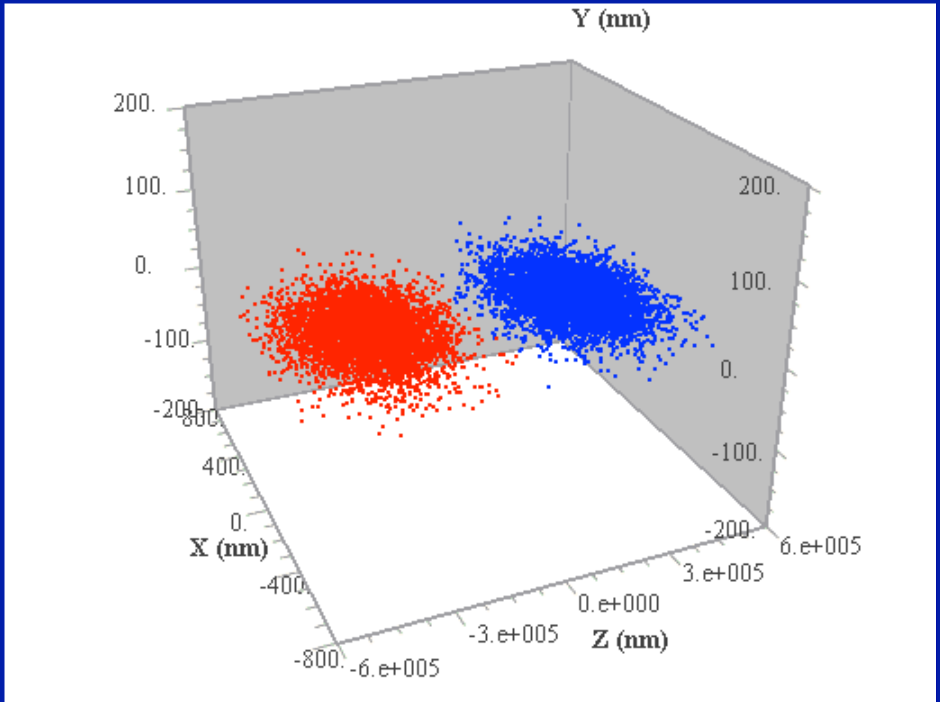


$$n_p(z) = n_{po}\left(\frac{n_{pf}}{n_{po}}\right)^{\frac{z}{L_{ramp}}}$$


sig_eq [um]
$\sigma_{eq}(z) = \sqrt[4]{\frac{2}{\gamma}}\sqrt{\frac{\varepsilon_n}{k_p(z)}}$
$\gamma = 1000$
$\varepsilon_n = 1\mu m$
$Q = 20\,pC$
$\sigma_0 = 7\mu m$
$\sigma_z = 30\mu m$
0
1
2
3
4
5
6
7
0.00
0.02
0.04
0.06
0.08
z[m]

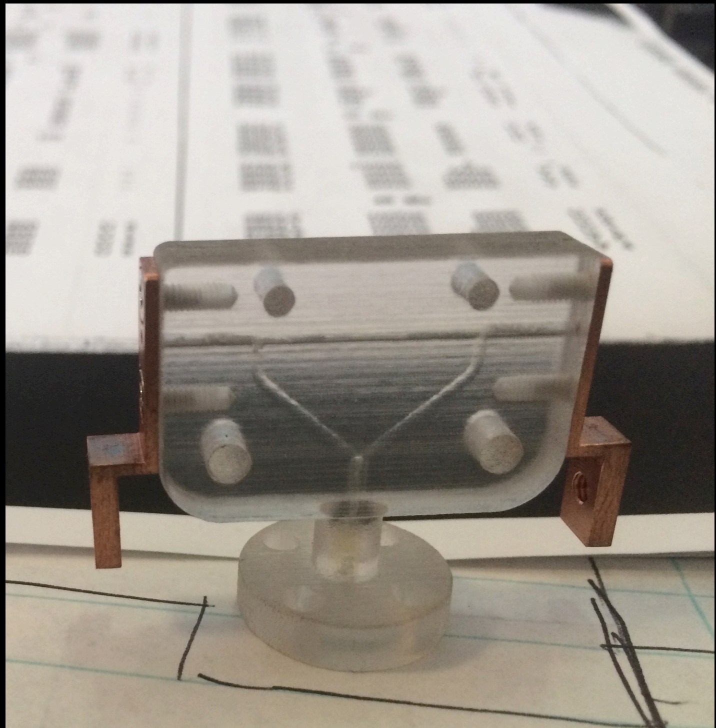
Self - Pinch in the Final Focus of a e^+e^- Collider

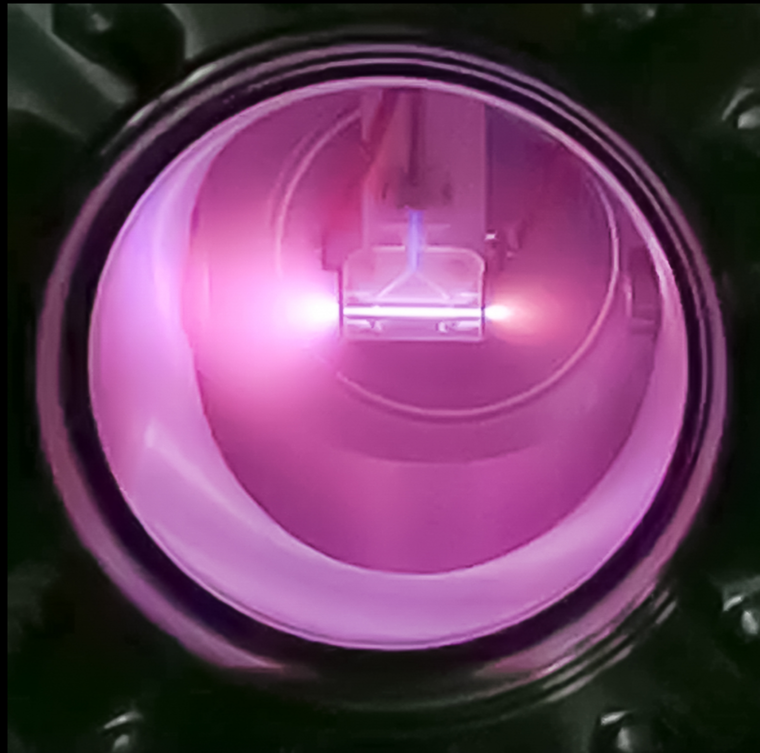
$$\left.\begin{array}{l} f_e = 1 \\ f_m = -1 \end{array}\right\} \Rightarrow \left(1-\gamma^2-\beta^2\gamma^2\right) \equiv -2\beta^2\gamma^2 << 0$$



$$\sigma_{eq} = \sqrt{\frac{I_A \varepsilon_n^2}{4 I \gamma}}$$

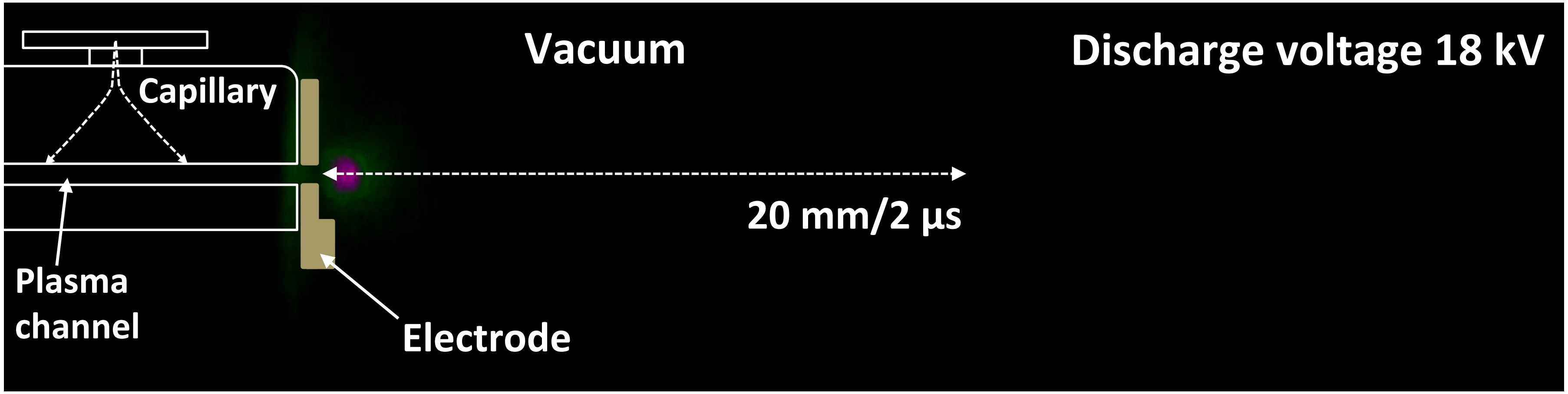
Capillary discharge





Plasma plumes

- **20 images separated by 100 ns, so 2 μs of total observation time of the plasma plumes**
- **The ICCD camera area is 1024 x 256 pixel**



- Both plama plumes can reach a total expansion length around 40 mm (20 mm each one) that is comparable with the channel length of 30 mm, so they can strongly affect the beam properties that passes through the capillary

- Temperature, pressure and plasma density, inside and outside the gas-filled capillary plasma source, represent essential parameters that have to be investigated to understand the plasma evolution and how it can affect the electron beam.

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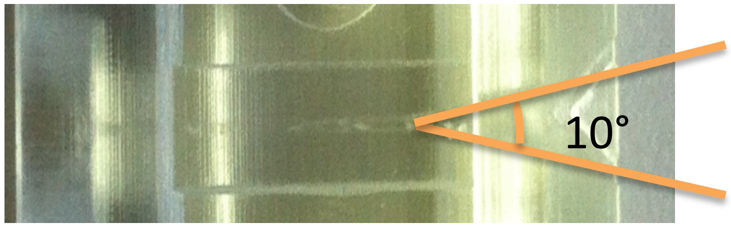
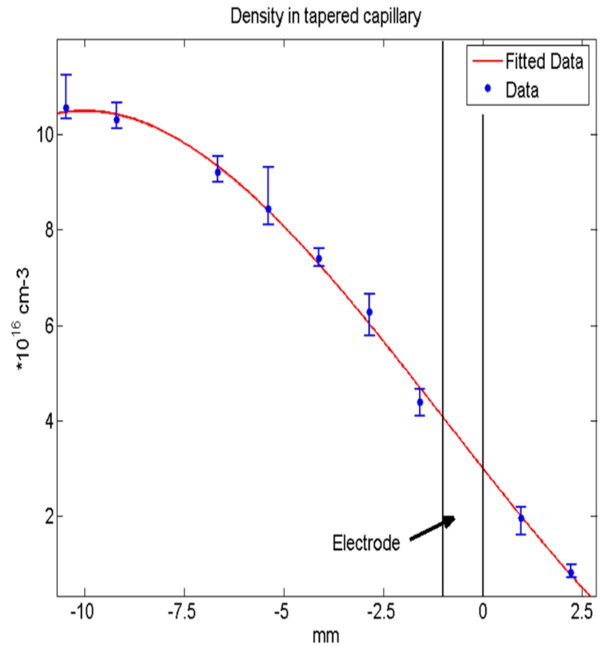
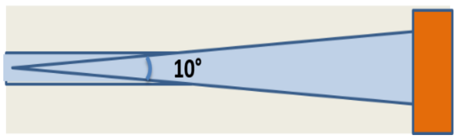
Tapered capillaries

Local control of the plasma density is required to match the laser/electron beam into the plasma.
Tapering the capillary diameter is the easiest way to change locally the density.

By monotonically varying the radius of the capillary it is possible to change the density.







Kaganovich et al., Appl. Phys. Lett. 75, 772–774 (1999).

Studies on plasma tapering are currently in progress in the SPARC_LAB Plasma lab.

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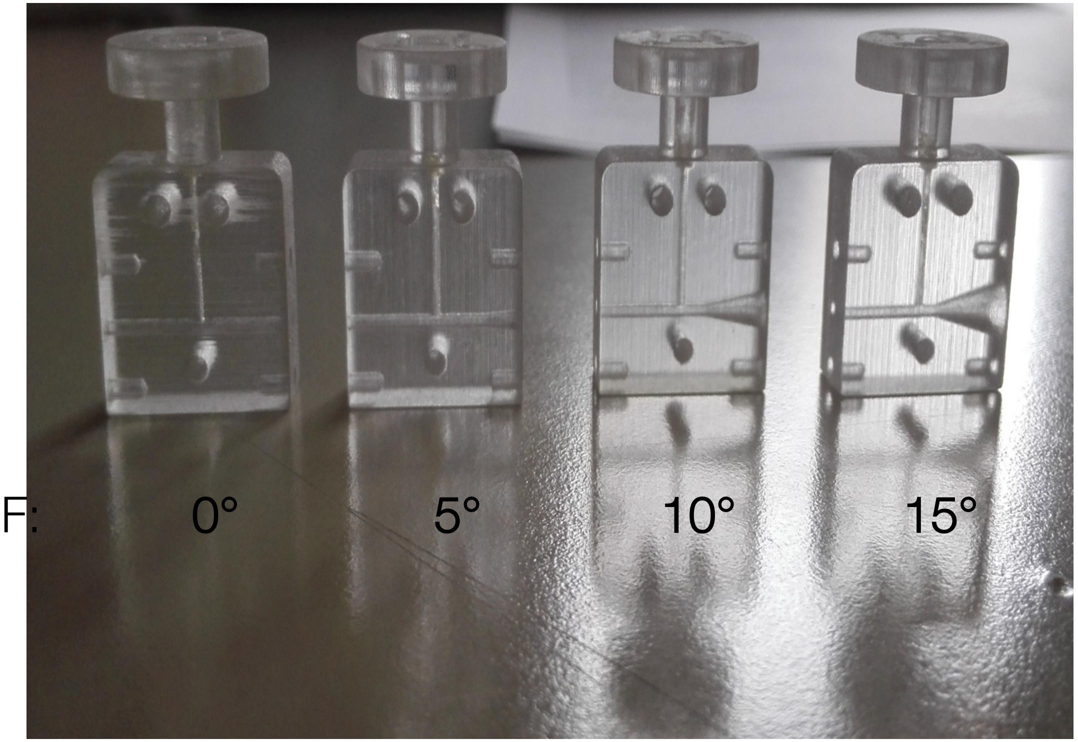
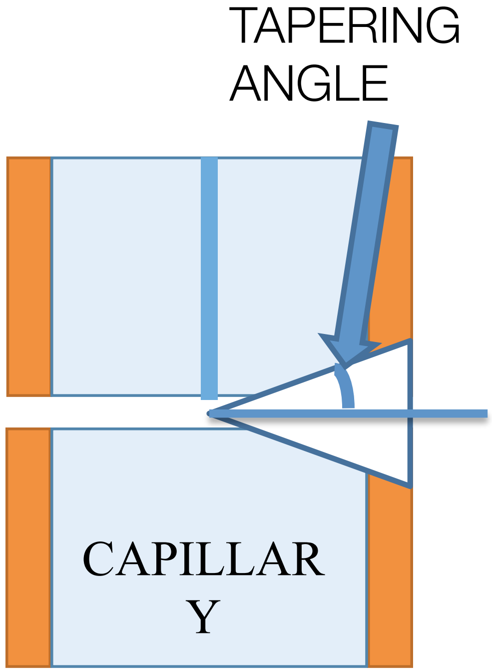


TAPERING OF: 0° 5° 10° 15°



Envelope Equation with Acceleration

$$\frac{dp_x}{dt} = \frac{d}{dt}\left(px'\right) = \beta c \frac{d}{dz}\left(px'\right) = 0$$

$$p = \beta\gamma m_o c$$

$$x'' + \frac{p'}{p}x' = 0$$

$$x'' = -\frac{(\beta\gamma)'}{\beta\gamma}x'$$

$$\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx''\rangle}{\sigma_x}$$

$$\langle xx''\rangle = -\frac{(\beta\gamma)'}{\beta\gamma}\langle xx'\rangle = -\frac{(\beta\gamma)'}{\beta\gamma}\sigma_{xx'} = -\frac{(\beta\gamma)'}{\beta\gamma}\sigma_x\sigma_x'$$

Space Charge De-focusing Force

$$\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma}\sigma_x' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

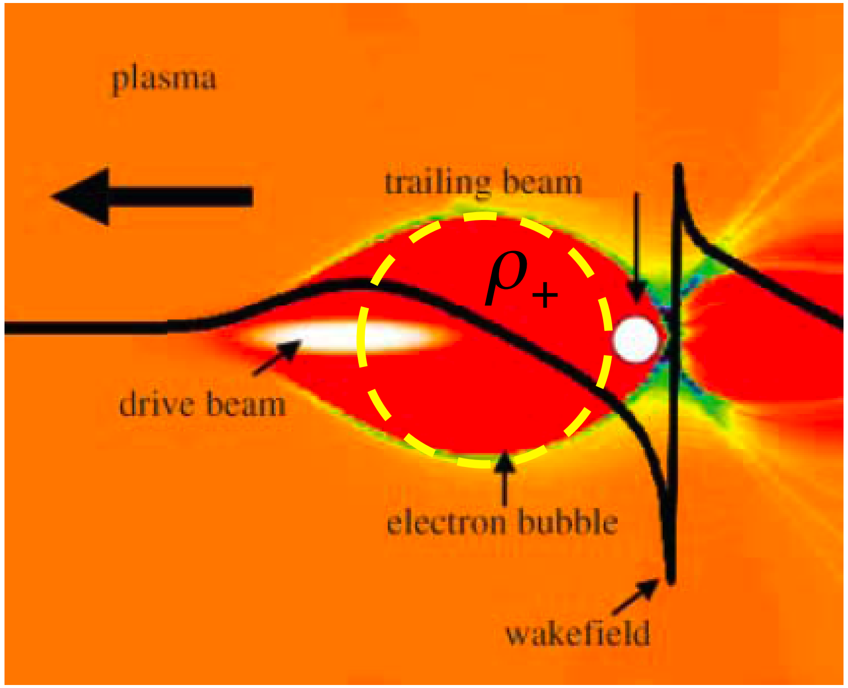
Adiabatic Damping

Emittance Pressure

Other External Focusing Forces

$$\varepsilon_n = \beta\gamma\varepsilon_{rms}$$

Envelope equation in a plasma accelerator



$$R_{sphere} \approx \frac{\lambda_p}{2}$$ Bubble radius

$$n_1 \approx n_{drive}$$ Bubble density

$$E_r = \frac{e n_1}{3\varepsilon_o} r$$ Radial field

$$F_r = e\left(E_r - \cancel{\beta c B_\vartheta}\right) = eE_r = \frac{e^2 n_1}{3\varepsilon_0} r$$

$$x'' = \frac{F_x}{\beta c p} = \frac{e^2 n_1 x}{3\varepsilon_o \gamma m c^2} = \frac{k_p^2}{3\gamma} x \qquad k_p^2 = \frac{e^2 n_1}{\varepsilon_o m c^2} \qquad \left\langle x x'' \right\rangle = \frac{k_p^2}{\gamma}\left\langle x^2 \right\rangle = \frac{k_p^2}{\gamma}\sigma_x^2$$

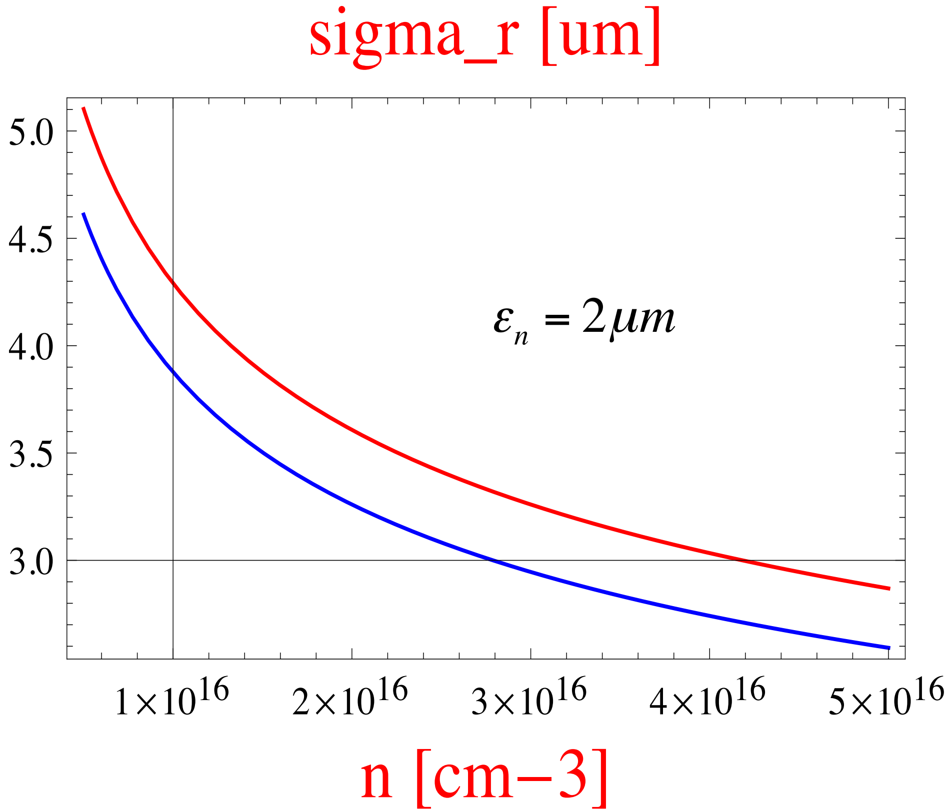
$$\sigma_x'' + \cancel{\frac{\gamma'}{\gamma}}\sigma_x' + \frac{k_p^2}{3\gamma}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2 \sigma_x^3} + \cancel{\frac{k_{sc}^o}{\gamma^3 \sigma_x}}$$

$$\sigma_x'' + \frac{k_p^2}{3\gamma}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2 \sigma_x^3}$$

Looking for an equilibrium solution of the form: $\sigma = \sigma_\varepsilon$

We get the matching condition with the plasma:

$$\sigma_\varepsilon = \sqrt[4]{\frac{3}{\gamma}}\sqrt{\frac{\varepsilon_n}{k_p}}$$



Perturbation around the equilibrium solution:

$$\sigma_\varepsilon = \sqrt[4]{\frac{3}{\gamma}}\sqrt{\frac{\varepsilon_n}{k_p}}$$

$$\sigma = \sigma_\varepsilon + \delta\sigma$$

$$\sigma_x'' + \frac{k_p^2}{3\gamma}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2\sigma_x^3}$$

==>

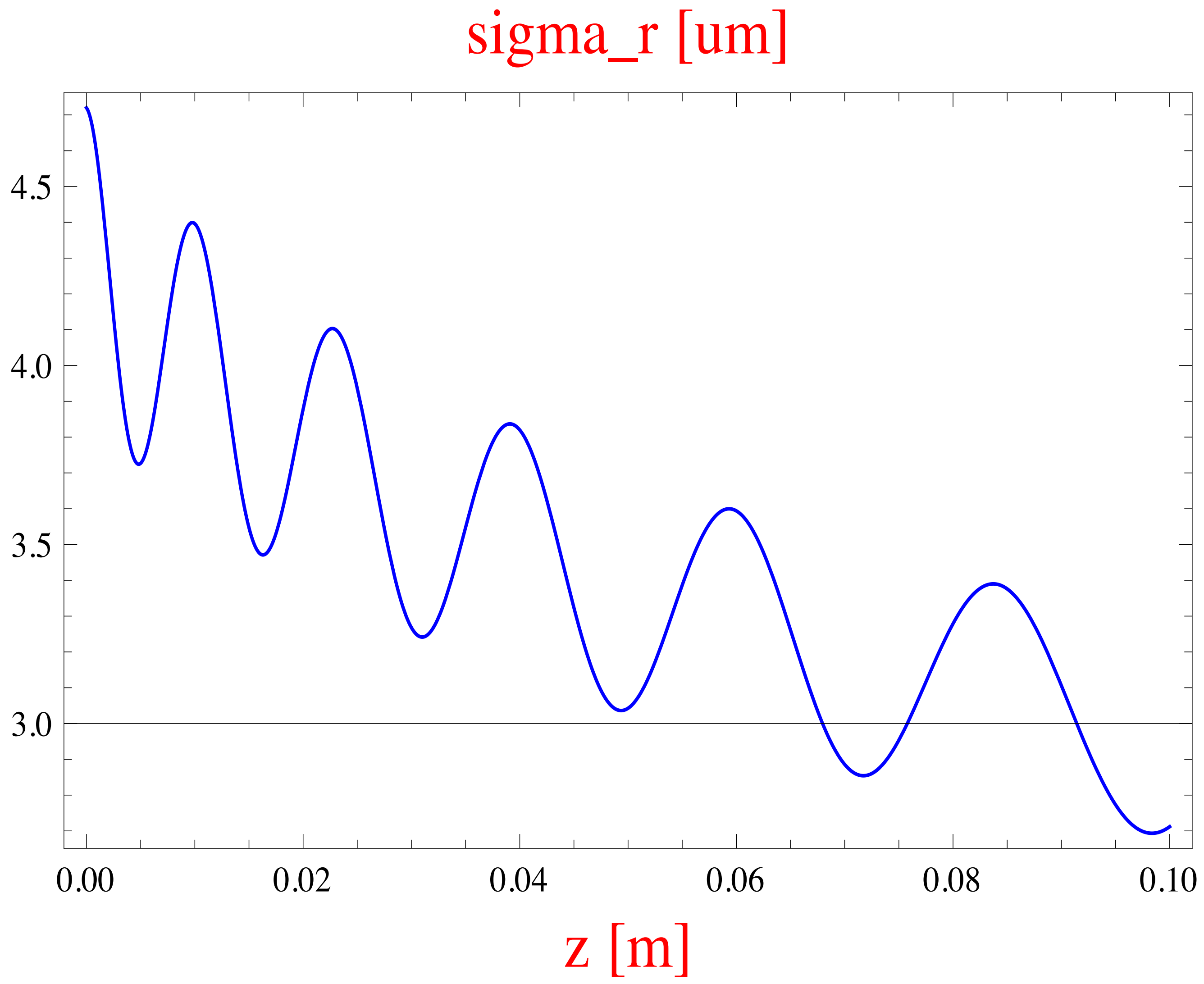
$$\delta\sigma_x'' + \frac{4}{3}\frac{k_p^2}{\gamma}\delta\sigma_x = 0$$

$$\delta\sigma(z) = \delta\sigma_o \cos\left(\sqrt{\frac{4}{3\gamma}}k_p z\right)$$

$$\sigma = \sqrt[4]{\frac{3}{\gamma}}\sqrt{\frac{\varepsilon_n}{k_p}} + \delta\sigma_o \cos\left(\sqrt{\frac{4}{3\gamma}}k_p z\right)$$

$$\sigma = \sqrt[4]{\frac{3}{\gamma}}\sqrt{\frac{\varepsilon_n}{k_p}} + \delta\sigma_o \cos\left(\sqrt{\frac{4}{3\gamma}}k_p z\right)$$

$$\frac{\delta\sigma}{\sigma_\varepsilon} = 10\%$$



Transverse emittance growth in staged laser-wakefield acceleration

T. Mehrling,[1] J. Grebenyuk,[2] F. S. Tsung,[3] K. Floettmann,[2] and J. Osterhoff[1,*]

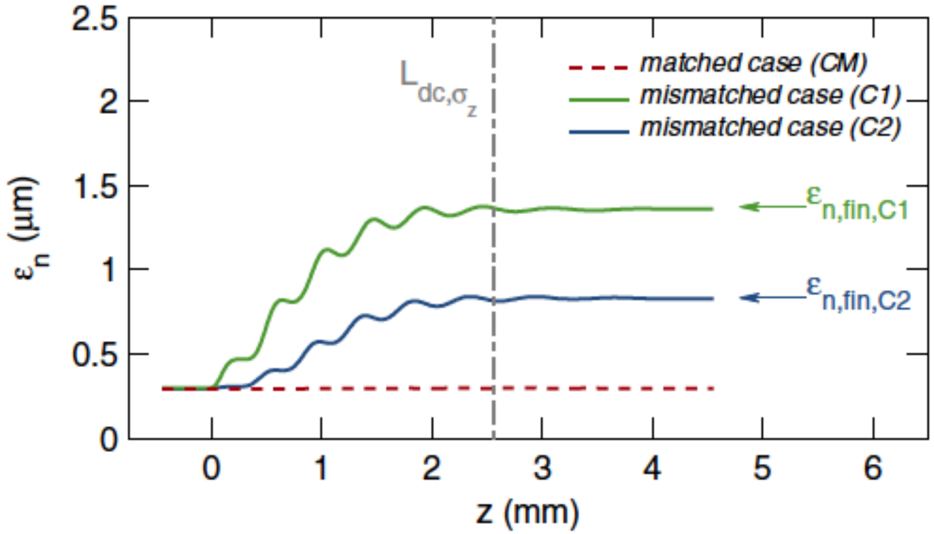


FIG. 3. Evolution of the normalized emittance ϵ_n in PIC simulations for the three considered cases. Arrows show the analytic predictions of the emittance growth. The betatron-decoherence length for the injection phase in the simulations $k_p\xi_0 = 1.00$ relative to position z_0 is indicated by the dash-dotted line.

$$\epsilon_{n,\mathrm{fin}} = \frac{\epsilon_{n,\mathrm{init}}}{2}\left(\frac{1+\alpha^2}{\beta^*} + \beta^*\right).$$