LONGITUDINAL beam DYNAMICS RECAP

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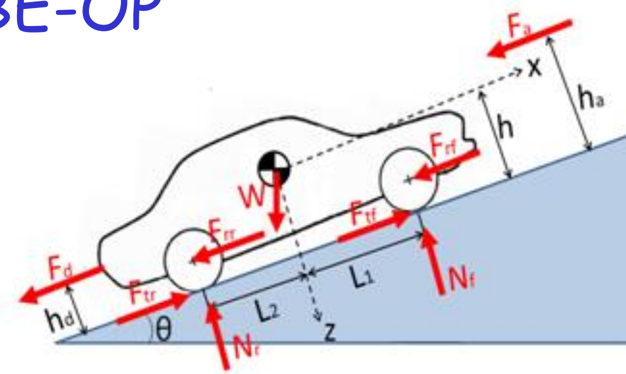
CERN, BE-OP

The CERN Accelerator School



Beam Dynamics and Technologies for Future Colliders
Zürich, 21/2-6/3/2018

Summary of the (1 1/2) lecture:

- Introduction
- Linac: Phase stability and longitudinal oscillations
- Cavities
- Synchrotron:
  - Synchronous Phase
  - Dispersion Effects in Synchrotron
  - Stability and Longitudinal Phase Space Motion
  - Equations of motion
- Injection Matching
- Hamiltonian of Longitudinal Motion

- Appendices: some derivations and details

More related lectures:

- Linear Collider Beam Dynamics – D. Schulte
- Circular Lepton Collider Beam Dynamics/Damping Rings – K.Oide

Particle types and acceleration

The accelerating system will depend upon the evolution of the particle velocity:

- electrons reach a constant velocity (~speed of light) at relatively low energy
- heavy particles reach a constant velocity only at very high energy

  -> we need different types of resonators, optimized for different velocities

  -> the **revolution frequency will vary**, so the RF frequency will be changing

Particle rest mass:

electron	0.511	MeV
proton	938	MeV
^{239}U	~220000	MeV

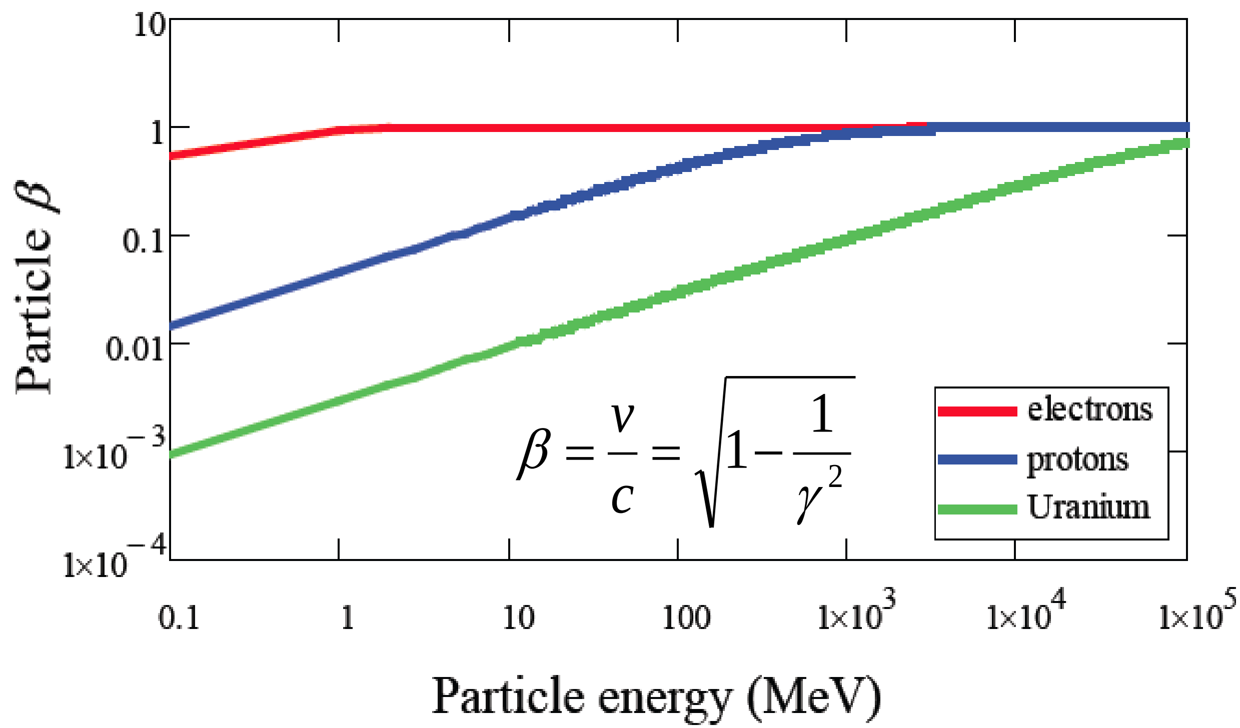
Total Energy: $E = g m_0 c^2$

Relativistic gamma factor:

$$g = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - b^2}}$$

Momentum:

$$p = mv = \frac{E}{c^2} bc = b\frac{E}{c} = bg m_0 c$$



Acceleration + Energy Gain

May the force be with you!

To accelerate, we need a **force in the direction of motion**!

Newton-Lorentz Force on a charged particle:

$$\vec{F} = \frac{d\vec{p}}{dt} = e\left(\vec{E} + \cancel{\vec{v} \times \vec{B}}\right)$$

2[nd] term always perpendicular to motion => no acceleration

Hence, it is necessary to have an electric field E (preferably) along the direction of the initial momentum (z), which changes the momentum p of the particle.

$$\boxed{\frac{dp}{dt} = eE_z}$$

In relativistic dynamics, total energy E and momentum p are linked by

$$E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad dE = v\,dp \qquad \left(2E\,dE = 2c^2 p\,dp \Leftrightarrow dE = c^2 mv/E\,dp = v\,dp\right)$$

The rate of energy gain per unit length of acceleration (along z) is then:

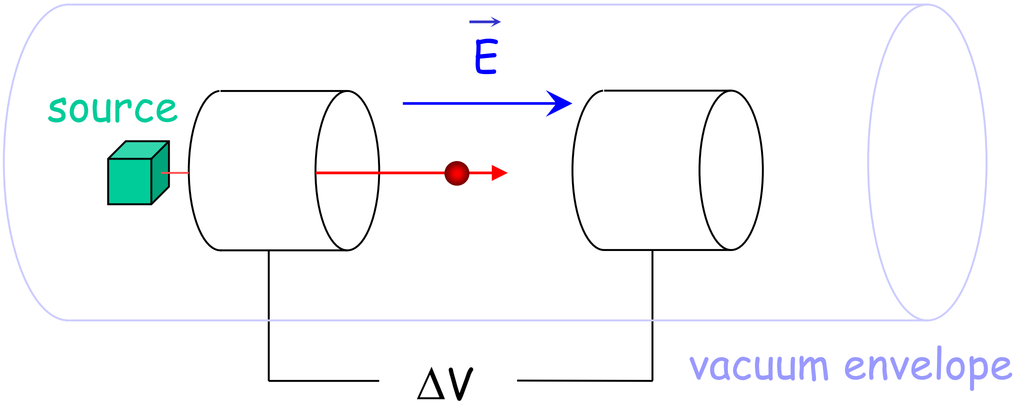
$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

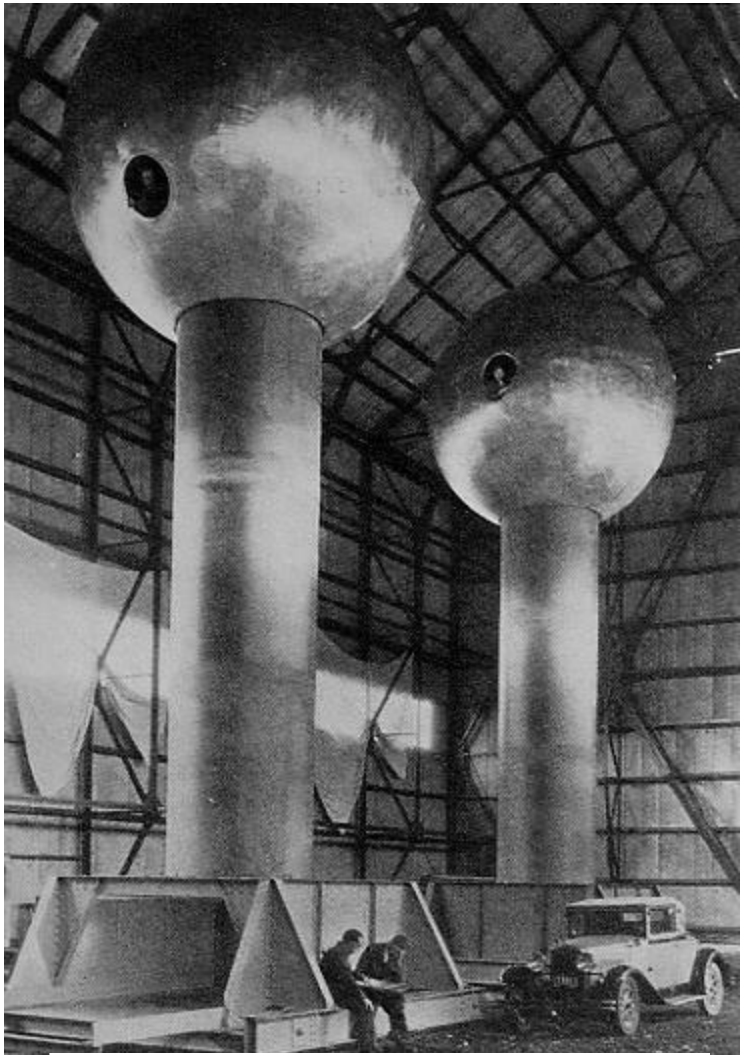
and the kinetic energy gained from the field along the z path is:

$$dW = dE = qE_z\,dz \quad \rightarrow \quad W = q\int E_z\,dz = qV$$

- V is a potential
- q the charge

Electrostatic Acceleration




Van-de-Graaf generator at MIT

Electrostatic Field:

Force: $\vec{F} = \frac{d\vec{p}}{dt} = q\,\vec{E}$

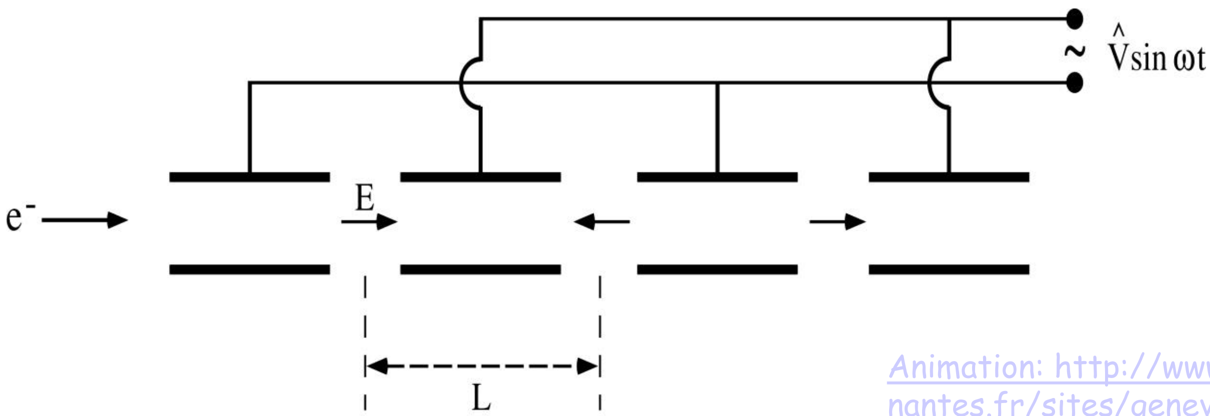
Energy gain: $W = q\,\Delta V$

used for first stage of acceleration:
particle sources, electron guns,
x-ray tubes

Limitation: insulation problems
maximum high voltage (~ 10 MV)

Radio-Frequency (RF) Acceleration

Electrostatic acceleration limited by isolation possibilities => use RF fields



Widerøe-type structure

Animation: http://www.sciences.univ-nantes.fr/sites/genevieve_tulloue/Meca/Charges/linac.html

Cylindrical electrodes (drift tubes) separated by gaps and fed by a RF generator, as shown above, lead to an alternating electric field polarity

Synchronism condition ⟶ $L = v\,T/2$

v = particle velocity
T = RF period

Note: - Drift tubes become longer for higher velocity

- Acceleration only for bunched beam (not continuous)

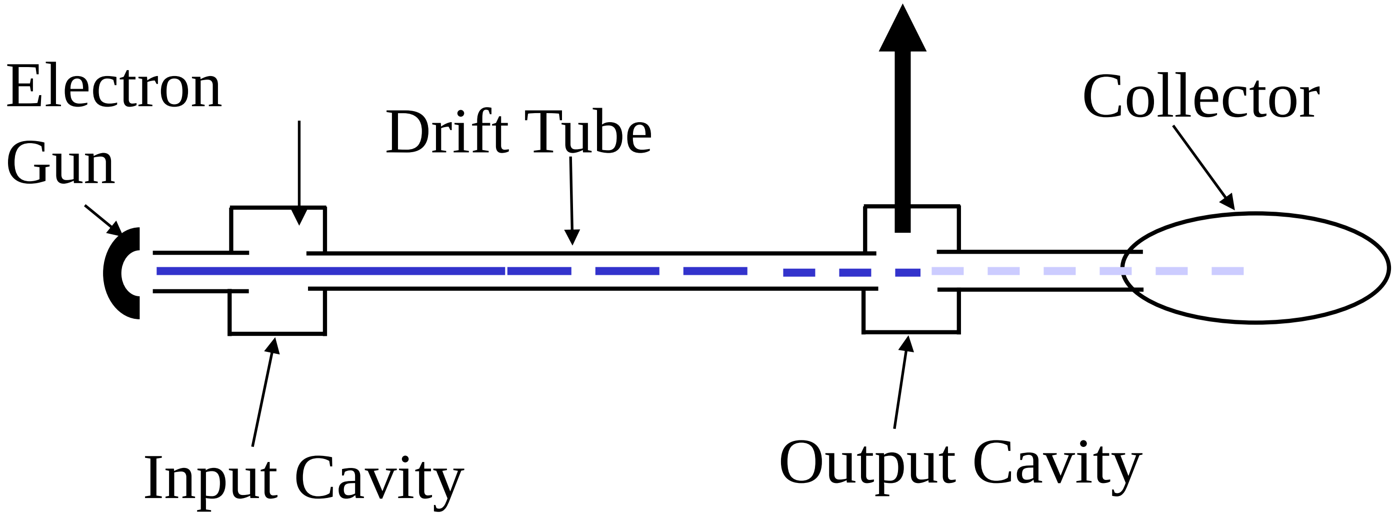
Side remark: Klystrons – Producing the RF

narrow-band vacuum-tube amplifier at microwave frequencies (an electron-beam device).

low-power RF signal at the design frequency excites input cavity

Velocity modulation creates density modulation in the drift tube

Bunched beam excites output cavity and produces high-power RF



U 150 -500 kV
I 100 -500 A
f 0.2 -20 GHz

P_{ave} < 1.5 MW
P_{peak} < 150 MW

efficiency 40-70%

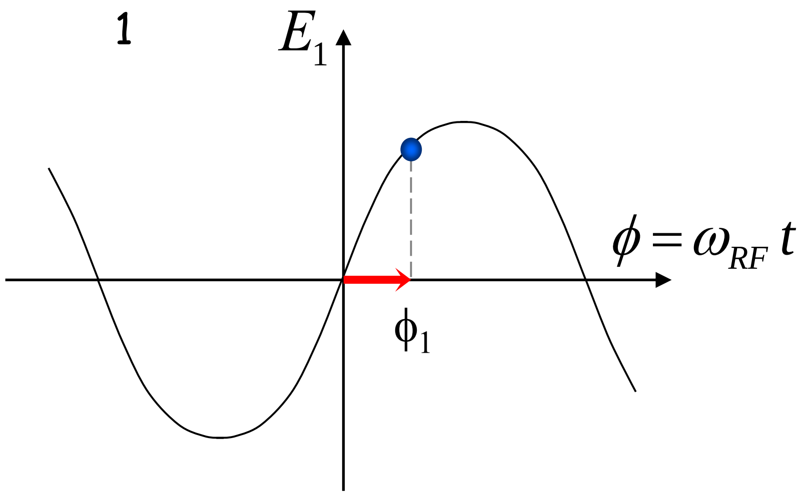
⇒ more in Steffen Döbert's talk

Common Phase Conventions

1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope
2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

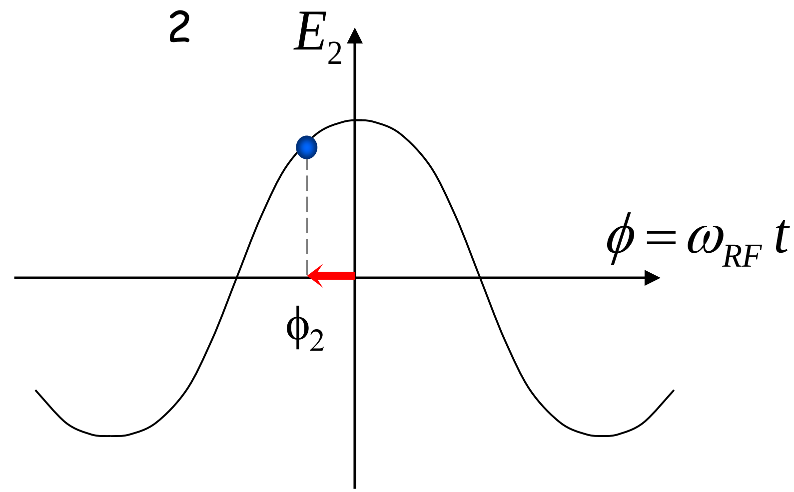
Time t= 0 chosen such that:

1



$$E_1(t) = E_0 \sin\left(W_{RF}\, t\right)$$

2



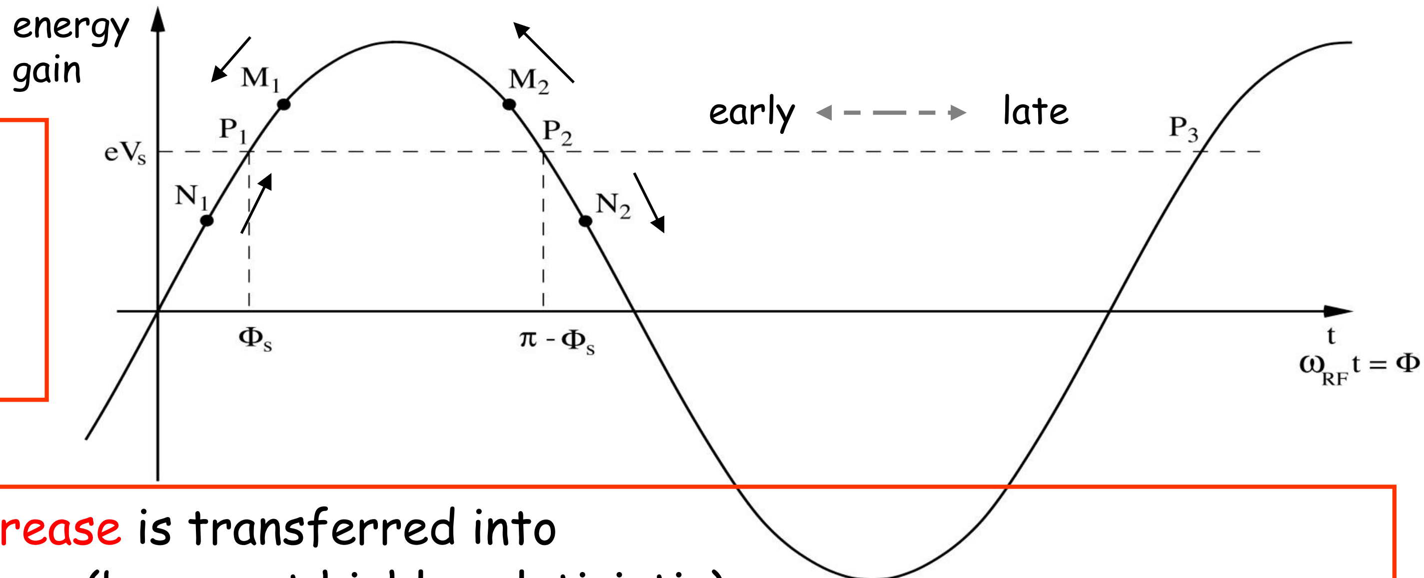
$$E_2(t) = E_0 \cos\left(W_{RF}\, t\right)$$

3. I will stick to **convention 1** in the following to avoid confusion

Principle of Phase Stability and Oscillations

Succession of accelerating gaps, in 2π mode, with synchronous phase Φ_s .

For a 2π mode, the electric field is the same in all gaps at any given time.



If an energy increase is transferred into
a velocity increase (beam not highly relativistic) =>

M_1 & N_1 will move towards P_1 => stable

M_2 & N_2 will go away from P_2 => unstable

Oscillations around stable fix point P_1

Highly relativistic particles have no significant velocity change => oscillation frozen!

Can accelerate at crest for maximum energy gain!

Energy-phase Oscillations (Small Amplitude) (1)

- Rate of energy gain for the synchronous particle:

$$\frac{dE_s}{dz} = \frac{dp_s}{dt} = eE_0 \sin f_s$$

- Use **reduced variables** with respect to synchronous particle

$$w = W - W_s = E - E_s \qquad \varphi = \phi - \phi_s$$

Energy gain: $$\frac{dw}{dz} = eE_0\left[\sin\left(\phi_s + \varphi\right) - \sin\phi_s\right] \approx eE_0 \cos\phi_s . \varphi \quad \left(small\ \varphi\right)$$

- Rate of **phase change** with respect to the synchronous one:

$$\frac{d\varphi}{dz} = \omega_{RF}\left(\frac{dt}{dz} - \left(\frac{dt}{dz}\right)_s\right) = \omega_{RF}\left(\frac{1}{v} - \frac{1}{v_s}\right) \cong -\frac{\omega_{RF}}{v_s^2}\left(v - v_s\right)$$

Leads finally to: $$\frac{d\varphi}{dz} = -\frac{\omega_{RF}}{m_0 v_s^3 \gamma_s^3} w$$

Energy-phase Oscillations (Small Amplitude) (2)

Combining the two 1st order equations into a 2nd order equation gives the equation of a harmonic oscillator:

$$\frac{d^2\varphi}{dz^2} + \Omega_s^2 \varphi = 0$$

with

$$\Omega_s^2 = \frac{eE_0 \omega_{RF} \cos\phi_s}{m_0 v_s^3 \gamma_s^3}$$

Slower for higher energy!

Stable harmonic oscillations imply: $\Omega_s^2 > 0 \quad \text{and real}$

hence: $\cos\phi_s > 0$

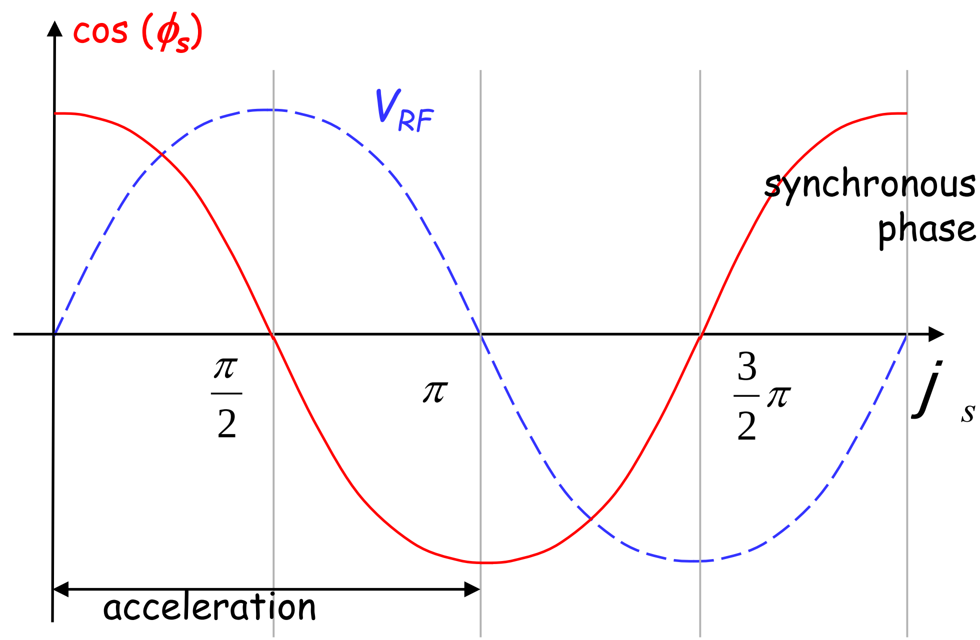
And since acceleration also means:

$$\sin\phi_s > 0$$

You finally get the result for the stable phase range:

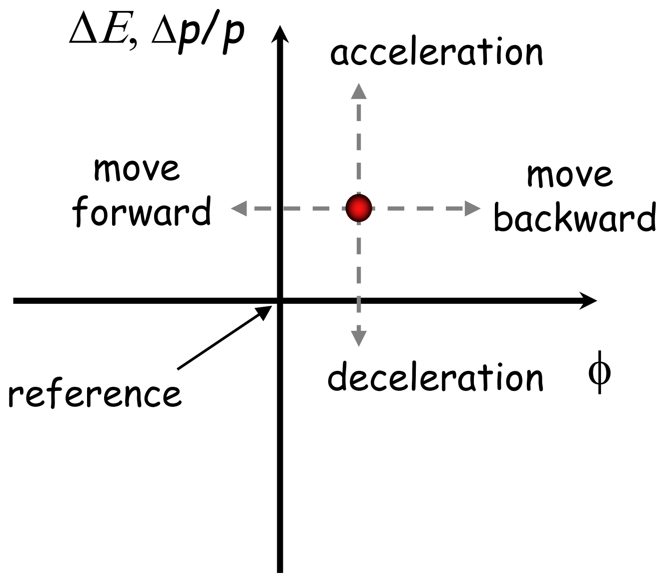
$$0 < \phi_s < \frac{\pi}{2}$$

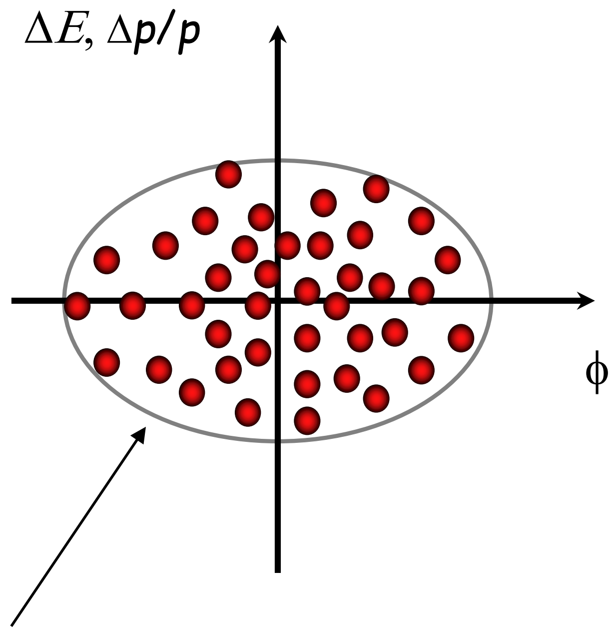
Positive rising RF slope!



Longitudinal phase space

The energy – phase oscillations can be drawn in phase space:





The particle trajectory in the phase space ($\Delta p/p$, ϕ) describes its longitudinal motion.

Emittance: phase space area including all the particles

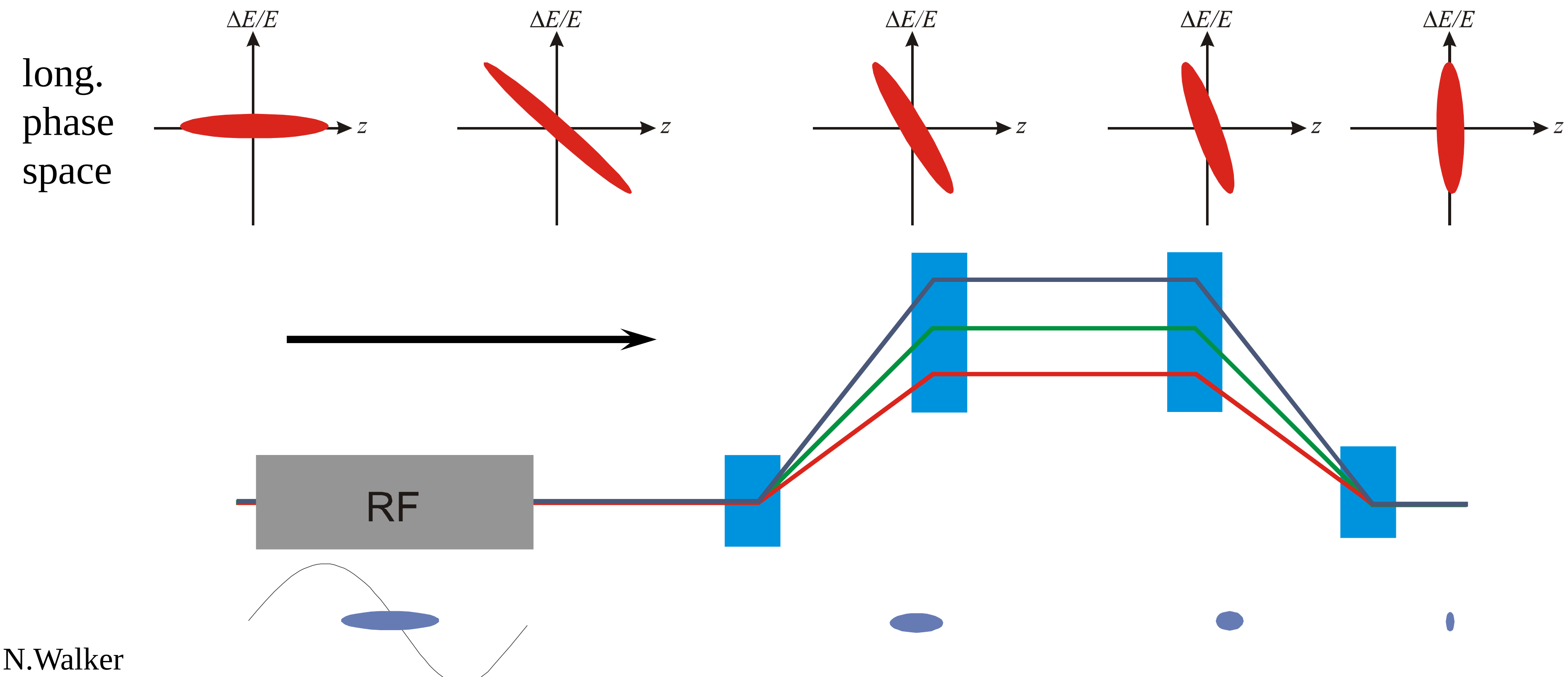
NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

Side remark: Bunch compression

At ultra-relativistic energies (γ>> 1) the longitudinal motion is frozen.
For linear e+/e- colliders, you need very short bunches (few 100-50μm).

Solution: introduce **energy/time correlation** + a magnetic **chicane.**

Increases energy spread in the bunch => chromatic effects
=> compress at low energy before further acceleration to reduce relative $\Delta E/E$

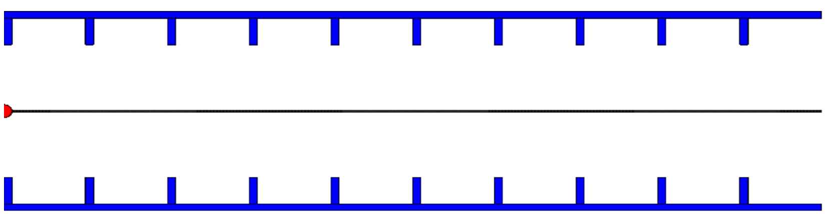


High Energy Linacs - Cavities

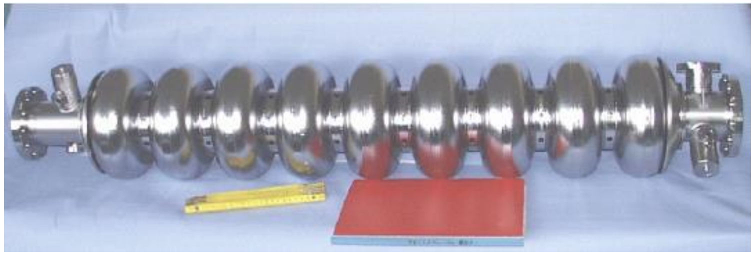
ILC

- Superconducting technology @ 2K
- 1.3 GHz
- 31.5 MV/m gradient
- Standing wave cavity

- bunch sees field:
  $E_z = E_0 \sin(\omega t + \varphi)\sin(kz)$



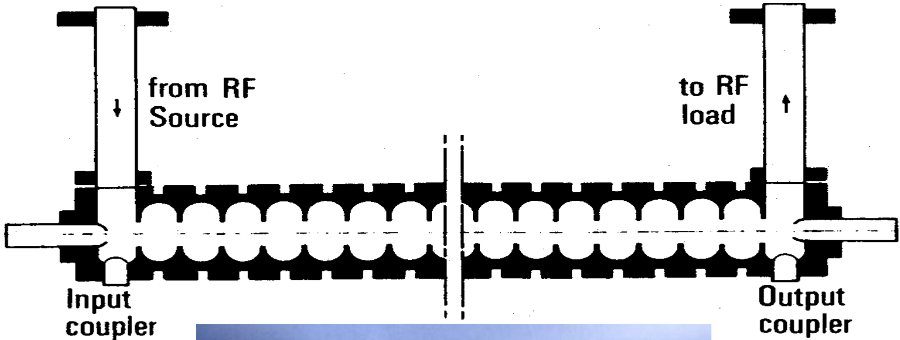
D.Schulte



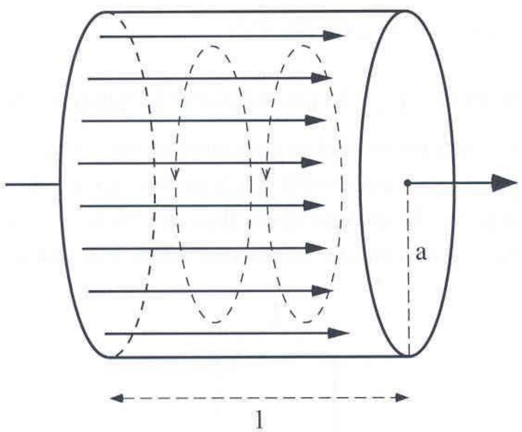
CLIC

- Normalconducting technology
- 12 GHz
- 100 MV/m gradient
- Traveling wave cavity

- bunch sees constant field:
  $E_z = E_0 \cos(\varphi)$



The Pill Box Cavity



$\longrightarrow E_z$ $\quad$ $\dashrightarrow H_\theta$

From Maxwell's equations one can derive the wave equations:

$$\nabla^2 A - \varepsilon_0 \mu_0 \frac{\partial^2 A}{\partial t^2} = 0 \qquad (A = E \text{ or } H)$$

Solutions for E and H in cavities are oscillating modes, at **discrete frequencies**, of types

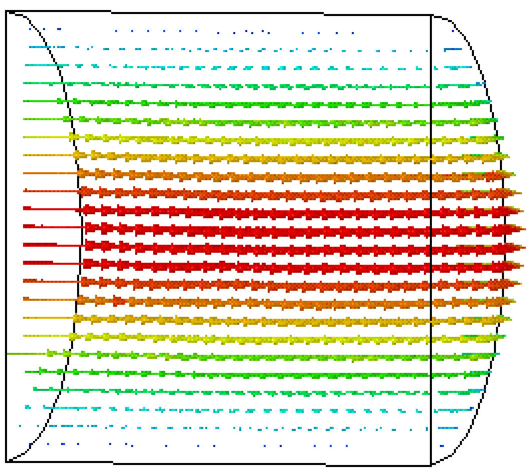
TM_{xyz} (transverse magnetic) or

TE_{xyz} (transverse electric).

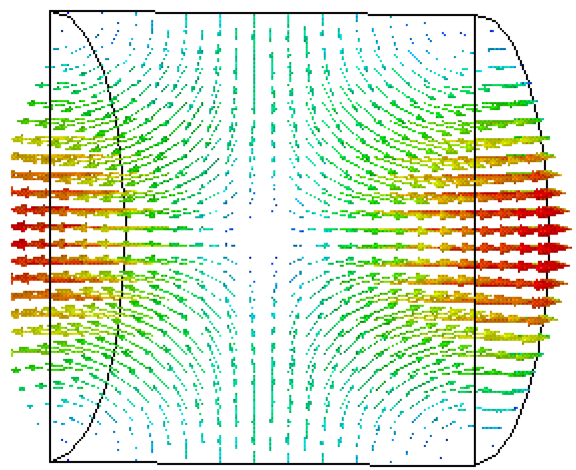
Indices linked to the number of field knots in polar co-ordinates φ, r and z.

TM_{010} (no axial dependence)

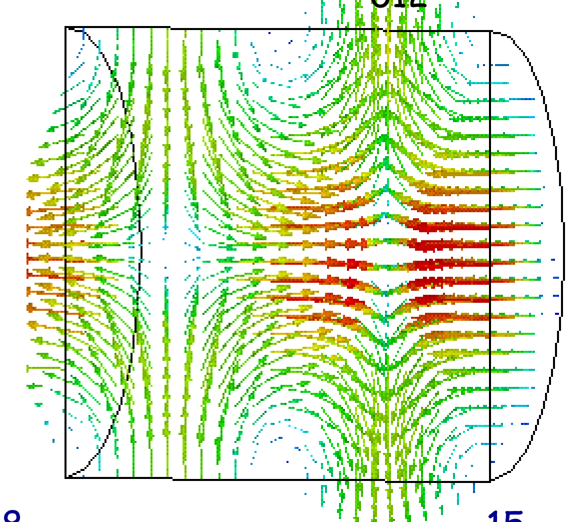
$\vec{E}$



TM_{011}

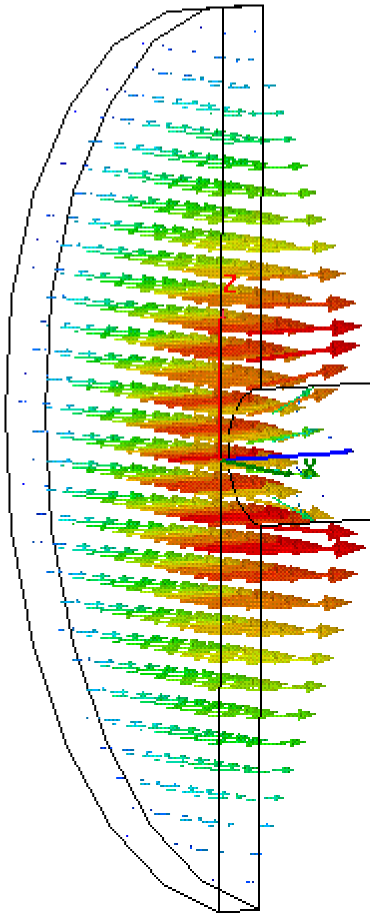


TM_{012}



The Pill Box Cavity (2)

One needs a hole for the beam pipe – circular waveguide below cutoff

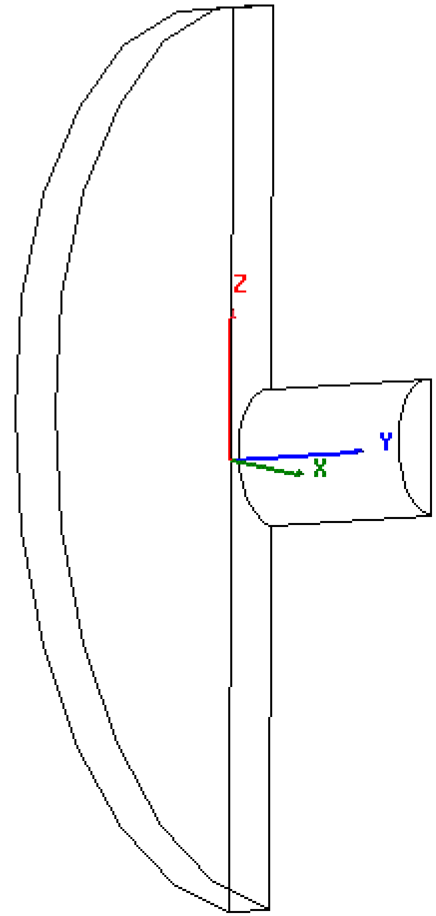


electric field

TM_{010}-mode

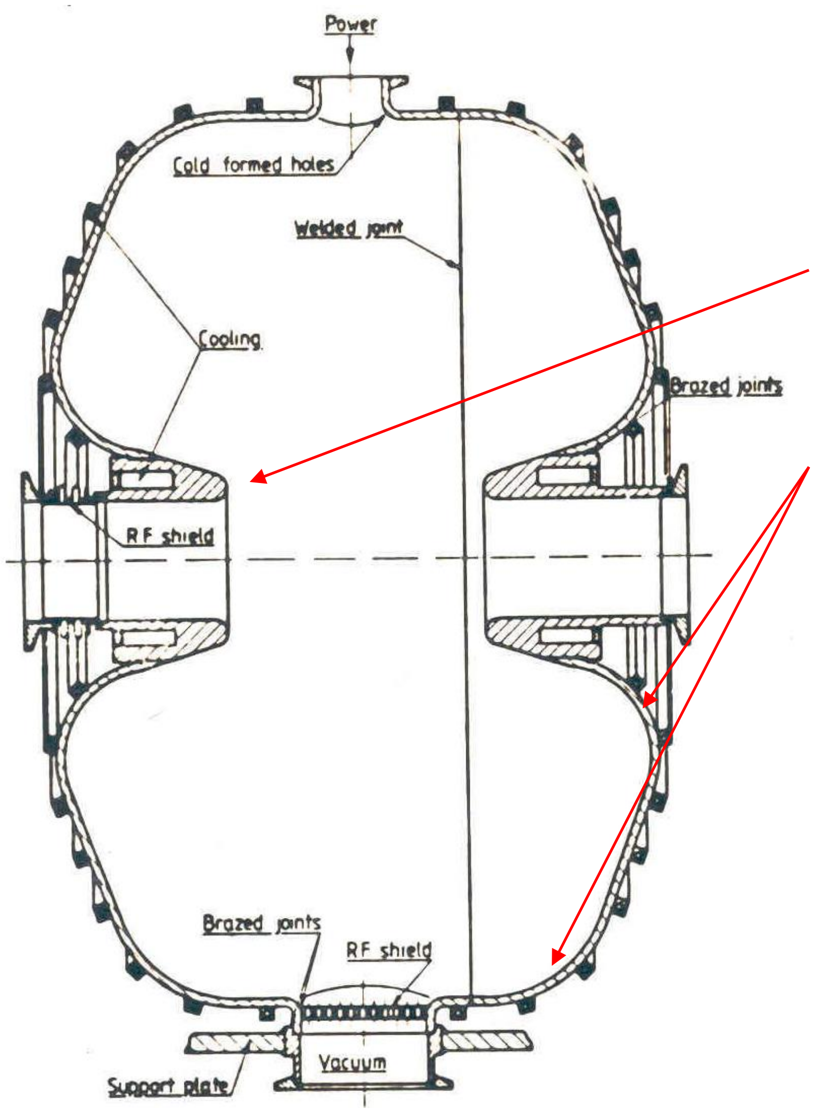
(only 1/4 shown)

Useful for acceleration!



magnetic field

Real Cavity Example



The design of a cavity can be sophisticated in order to improve its performances:

- A nose cone can be introduced in order to concentrate the electric field around the axis

- Round shaping of the corners allows a better distribution of the magnetic field on the surface and a reduction of the Joule losses.
It also prevents from multipactoring effects (e- emission and acceleration).

A good cavity efficiently transforms the RF power into accelerating voltage.

Simulation codes allow precise calculation of the properties.

Transit time factor

The accelerating field varies during the passage of the particle
=> particle does not always see maximum field => effective acceleration smaller

Transit time factor defined as:

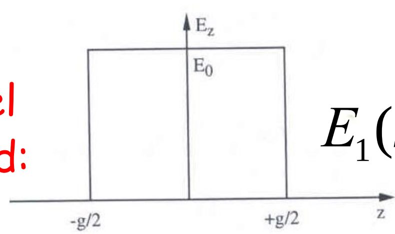
$$T_a = \frac{\text{energy gain of particle with } v = \beta c}{\text{maximum energy gain (particle with } v \to \infty)}$$

In the general case, the transit time factor is:

for $E(s,r,t) = E_1(s,r) \cdot E_2(t)$

$$T_a = \frac{\left| \int_{-\infty}^{+\infty} E_1(s,r)\cos\left(\omega_{RF}\frac{s}{v}\right) \mathrm{d}s \right|}{\int_{-\infty}^{+\infty} E_1(s,r)\,\mathrm{d}s}$$

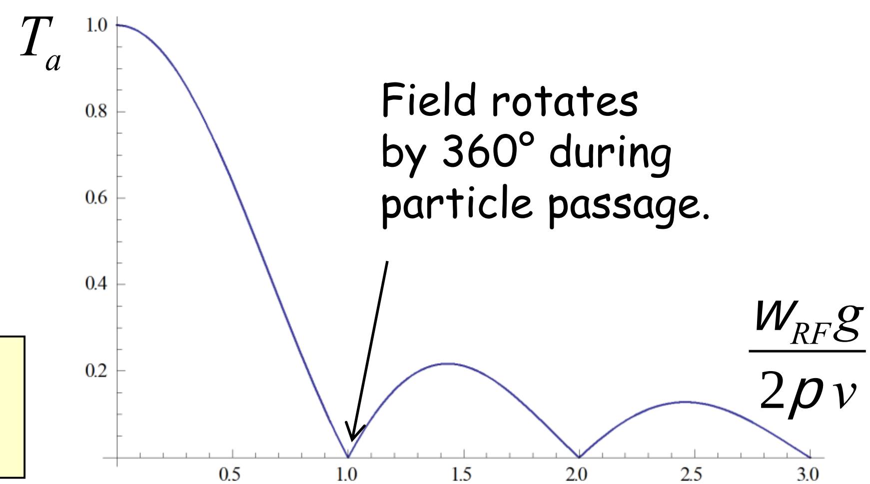
Simple model uniform field:



$$E_1(s,r) = \frac{V_{RF}}{g}$$

follows:

$$T_a = \left| \sin\frac{\omega_{RF}\, g}{2v} \Big/ \frac{\omega_{RF}\, g}{2v} \right|$$



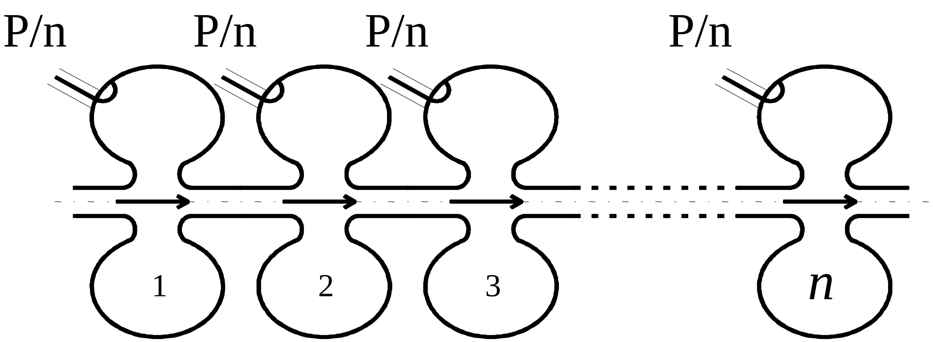
$0 < T_a < 1$, $T_a \to 1$ for $g \to 0$, smaller ω_{RF}
Important for low velocities (ions)

Multi-Cell Cavities

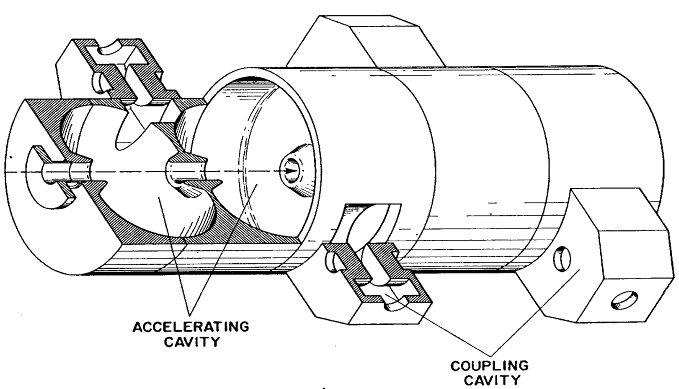
Acceleration of one cavity limited => distribute power over several cells

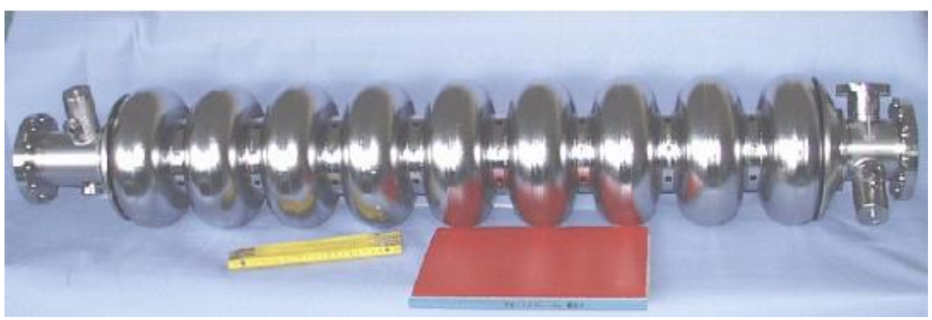
Each cavity receives P/n

Since the field is proportional √P, you get $\sum E_i \propto n\sqrt{P/n} = \sqrt{n}E_0$



Instead of distributing the power from the amplifier, one might as well couple the cavities, such that the power automatically distributes, or have a cavity with many gaps (e.g. drift tube linac).





Multi-Cell Cavities - Modes

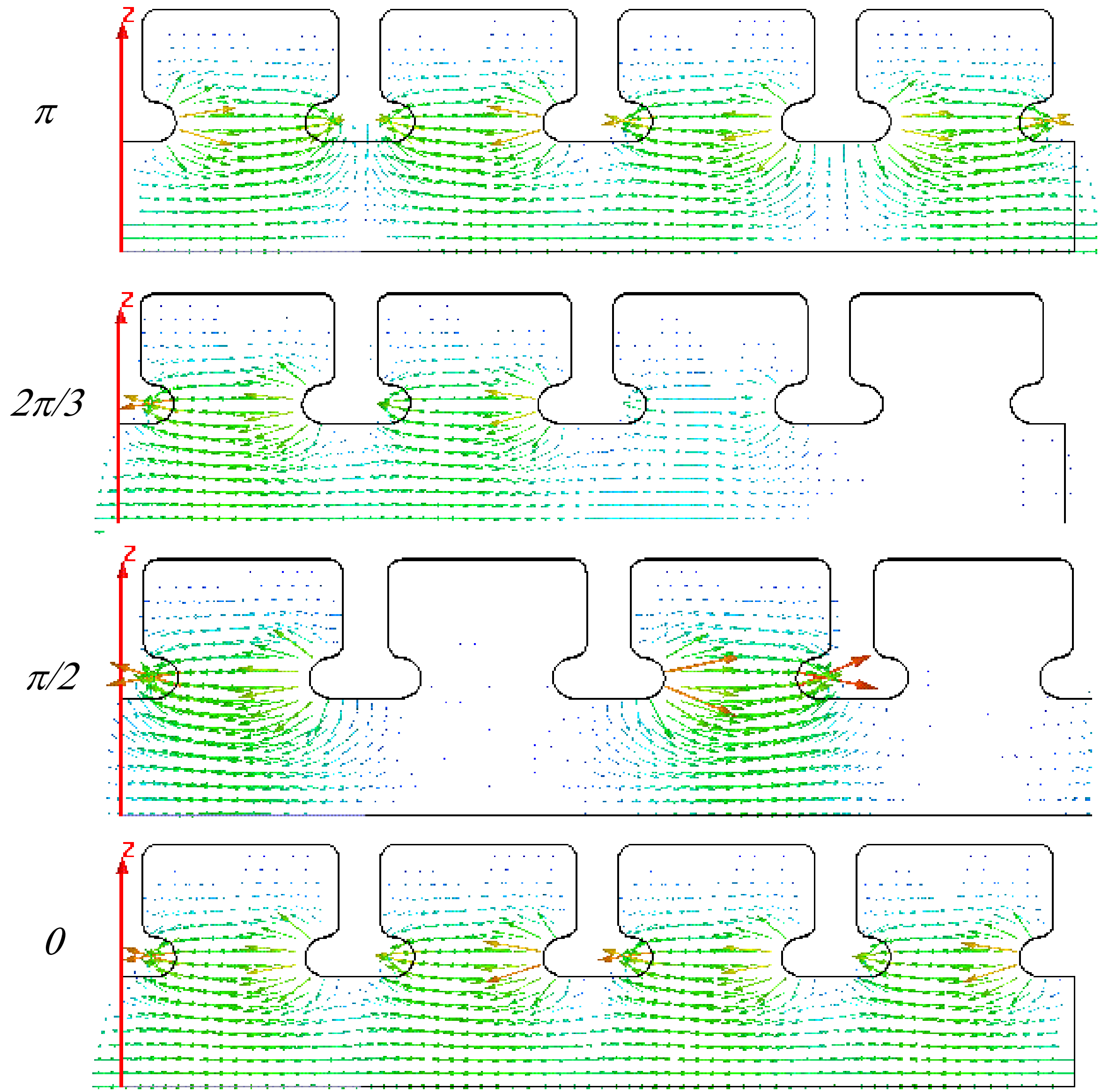
The phase relation between gaps is important!

Coupled harmonic oscillator

=> Modes, named after the phase difference between adjacent cells.

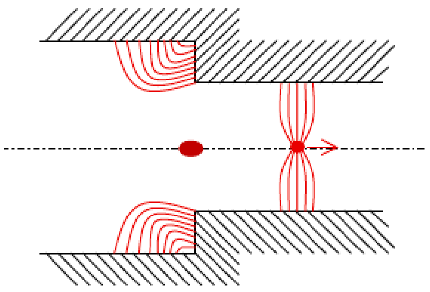
Relates to different synchronism conditions for the cell length L

Mode	L
0 (2π)	βλ
π/2	βλ/4
2π/3	βλ/3
π	βλ/2



Longitudinal Wake Fields - Beamloading

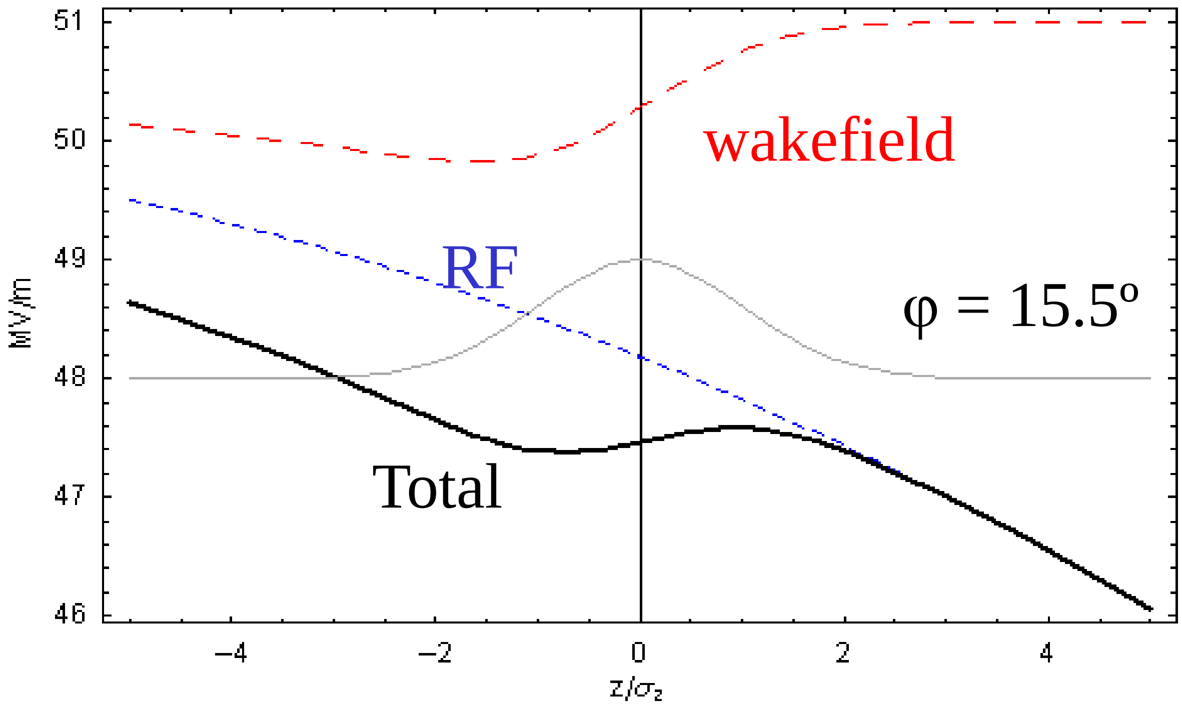
Beam induces wake fields in cavities (in general when chamber profile changing)
⇒ decreasing RF field in cavities
(beam absorbs RF power when accelerated)



Particles within a bunch see a decreasing field
⇒ energy gain different within the single bunch

Locating bunch off-crest at the best RF phase minimises energy spread

Example: Energy gain along the bunch in the NLC linac (TW):

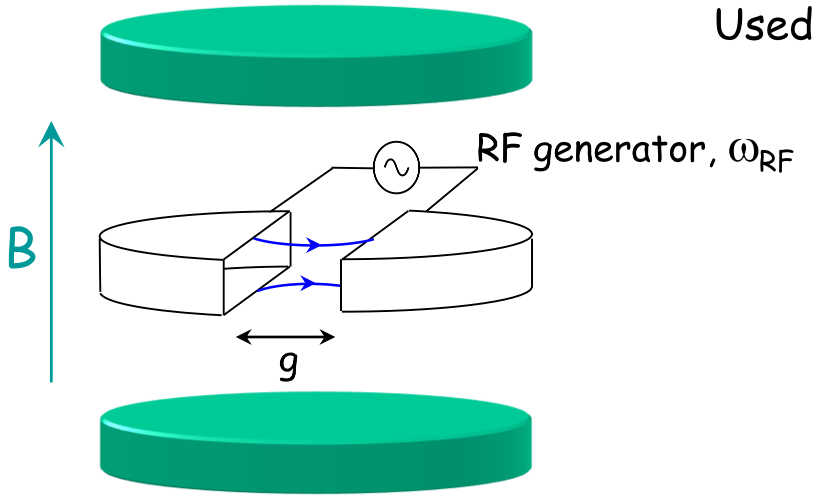


Circular accelerators

Cyclotron

Synchrotron

Circular accelerators: Cyclotron

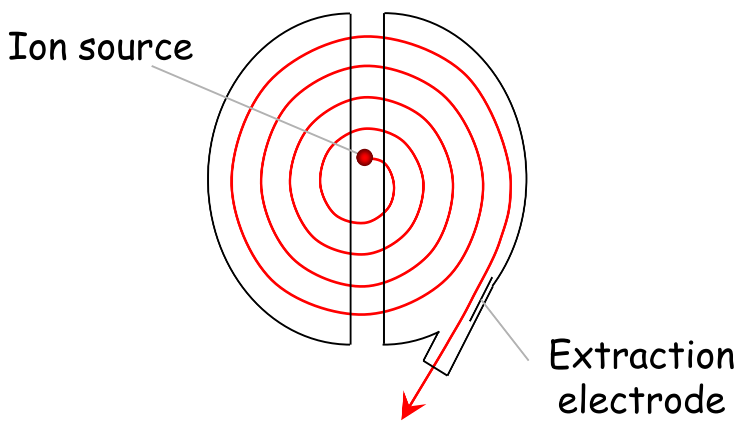


Used for protons, ions

B = constant
ω_{RF} = constant

Synchronism condition

$$\omega_s = \omega_{RF}$$
$$2\pi\ \rho = v_s\ T_{RF}$$



Cyclotron frequency $$\omega = \frac{q\ B}{m_0\ \gamma}$$

1. γ increases with the energy
   $\Rightarrow$ no exact synchronism
2. if $v \ll c \Rightarrow \gamma \cong 1$

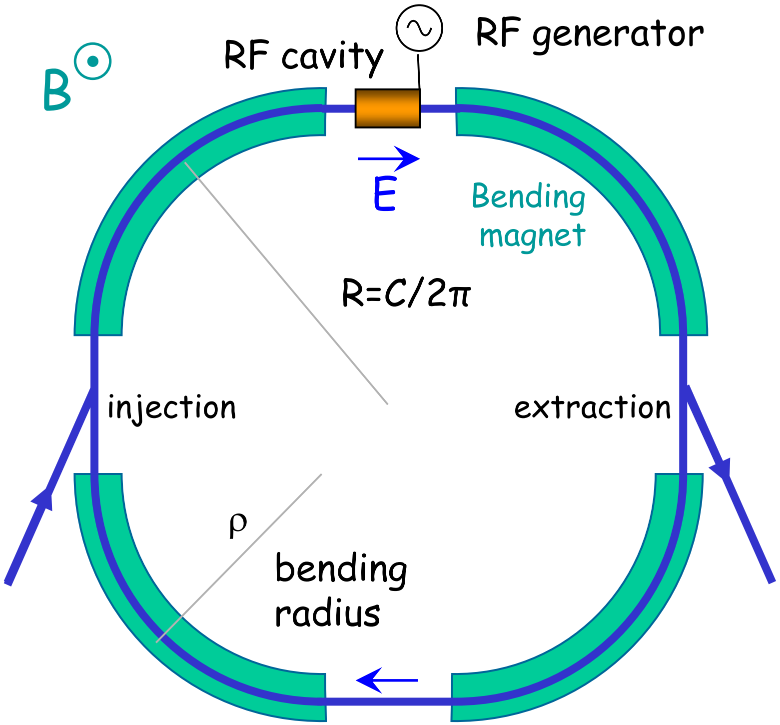
Cyclotron Animation

Animation: http://www.sciences.univ-nantes.fr/sites/genevieve_tulloue/Meca/Charges/cyclotron.html

Circular accelerators: Cyclotron

Courtesy Berkeley Lab,
https://www.youtube.com/watch?v=cutKuFxeXmQ

Circular accelerators: The Synchrotron



1. Constant orbit during acceleration

2. To keep particles on the closed orbit, B should increase with time

3. ω and ω_{RF} increase with energy

RF frequency can be multiple of revolution frequency

$$\omega_{RF} = h\omega$$

Synchronism condition ➡

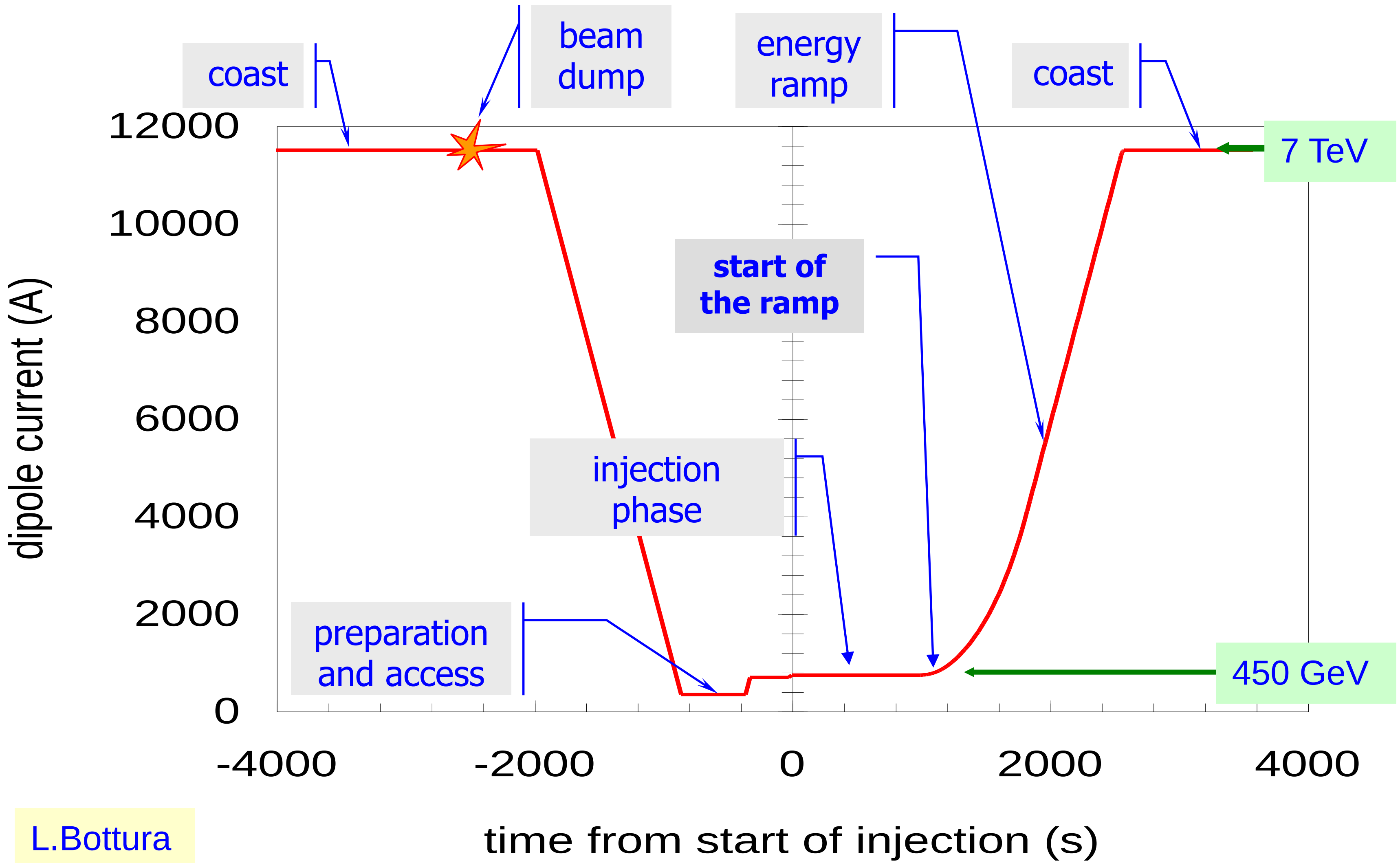
$$T_s = h\,T_{RF}$$

$$\frac{2\pi R}{v_s} = h\,T_{RF}$$

h integer,
harmonic number:
number of RF cycles per revolution

The Synchrotron – LHC Operation Cycle

The magnetic field (dipole current) is increased during the acceleration.



L.Bottura

The Synchrotron – Energy ramping

Energy ramping by increasing the B field (frequency has to follow v):

$$p = eB\rho \quad \Rightarrow \quad \frac{dp}{dt} = e\rho\dot{B} \quad \Rightarrow \quad (\Delta p)_{turn} = e\rho\dot{B}T_r = \frac{2\pi e\rho R\dot{B}}{v}$$

Since: $$E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad \Delta E = v\Delta p$$

$$\boxed{(\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho R\dot{B} = e\hat{V}\sin\phi_s}$$

Stable phase Φ_s changes during energy ramping!

$$\sin\phi_s = 2\pi\rho R\frac{\dot{B}}{\hat{V}_{RF}}$$

$$\boxed{\phi_s = \arcsin\left(2\pi\rho R\frac{\dot{B}}{\hat{V}_{RF}}\right)}$$

- The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference. They have the nominal energy and follow the nominal trajectory.

The Synchrotron – Frequency change

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency :

$$\omega = \frac{\omega_{RF}}{h} = \omega(B, R_s)$$

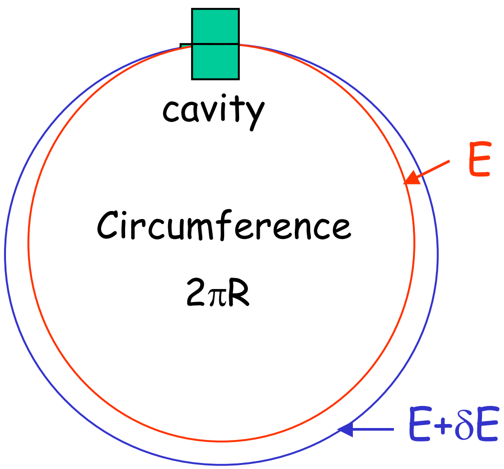
Hence: $$\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi}\frac{ec^2}{E_s(t)}\frac{\rho}{R_s}B(t)$$ (using $p(t) = eB(t)\rho, \quad E = mc^2$)

Since $E^2 = (m_0c^2)^2 + p^2c^2$ the RF frequency must follow the variation of the B field with the law

$$\boxed{\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s}\left\{\frac{B(t)^2}{(m_0c^2/ec\rho)^2 + B(t)^2}\right\}^{1/2}}$$

This asymptotically tends towards $f_r \rightarrow \dfrac{c}{2\pi R_s}$ when B becomes large compared to $m_0c^2/(ec\rho)$ which corresponds to $v \rightarrow c$

Dispersion Effects in a Synchrotron



A particle slightly shifted in momentum will have a

- dispersion orbit and a different orbit length
- a different velocity.

As a result of both effects the revolution frequency changes with a "slip factor η":

$$\eta = \frac{\mathrm{d} f_r / f_r}{\mathrm{d} p / p} \quad Þ$$

p=particle momentum

R=synchrotron physical radius

f_r=revolution frequency

Note: you also find η defined with a minus sign!

Effect from orbit defined by Momentum compaction factor: $\alpha_c = \frac{dL/L}{dp/p}$

Property of the beam optics: (derivation in Appendix)

$$\alpha_c = \frac{1}{L} \int_C \frac{D_x(s)}{\rho(s)} ds_0$$

Dispersion Effects – Revolution Frequency

The two effects of the orbit length and the particle velocity change the revolution frequency as:

$$f_r = \frac{\beta c}{2\pi R} \quad \Rightarrow \quad \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} = \frac{d\beta}{\beta} - \alpha_c \frac{dp}{p}$$

definition of momentum compaction factor

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{dp}{p}$$

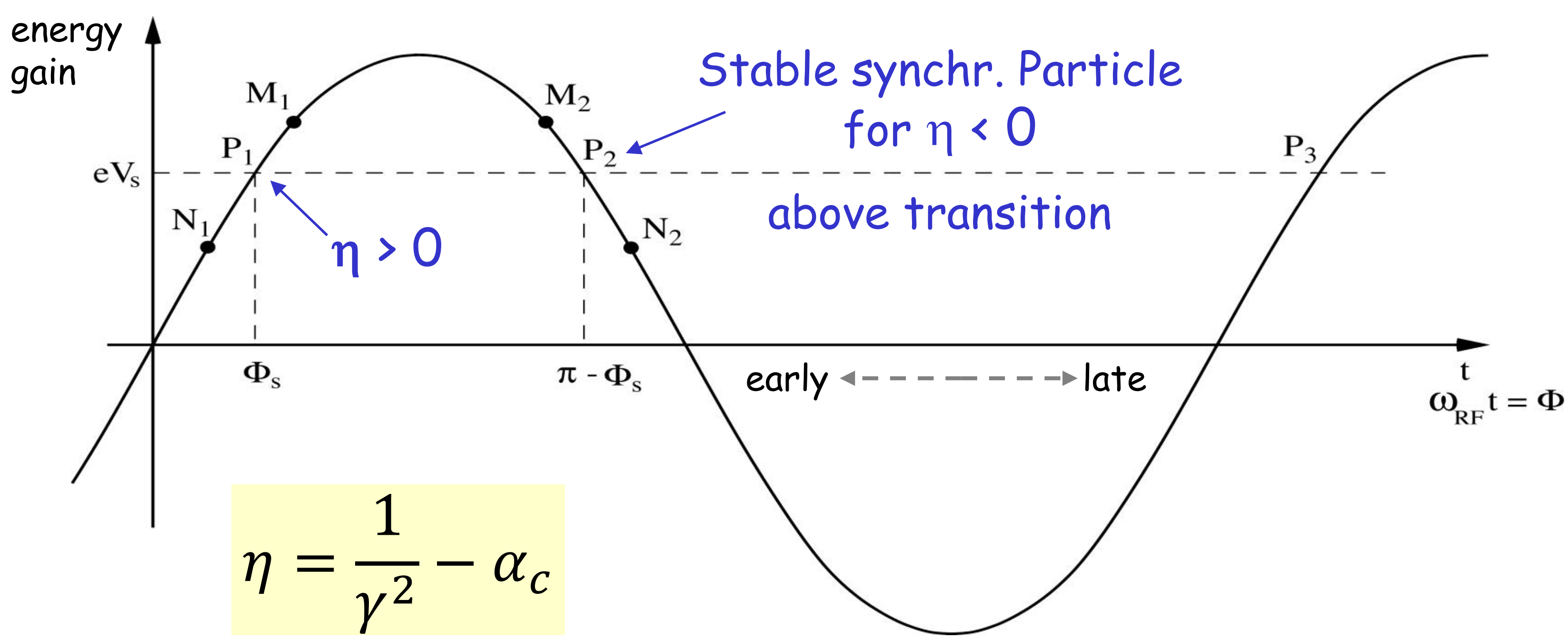
$$p = mv = \beta\gamma\frac{E_0}{c} \quad \Rightarrow \quad \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-1/2}}{\left(1-\beta^2\right)^{-1/2}} = \underbrace{\left(1-\beta^2\right)^{-1}}_{\gamma^2}\frac{d\beta}{\beta}$$

Slip factor: $\eta = \dfrac{1}{\gamma^2} - \alpha_c$ or $\eta = \dfrac{1}{\gamma^2} - \dfrac{1}{\gamma_t^2}$ with $\gamma_t = \dfrac{1}{\sqrt{\alpha_c}}$

At transition energy, $\eta = 0$, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Phase Stability in a Synchrotron

From the definition of η it is clear that an increase in momentum gives
- below transition ($\eta > 0$) a higher revolution frequency (increase in velocity dominates) while
- above transition ($\eta < 0$) a lower revolution frequency ($v \approx c$ and longer path) where the momentum compaction (generally > 0) dominates.

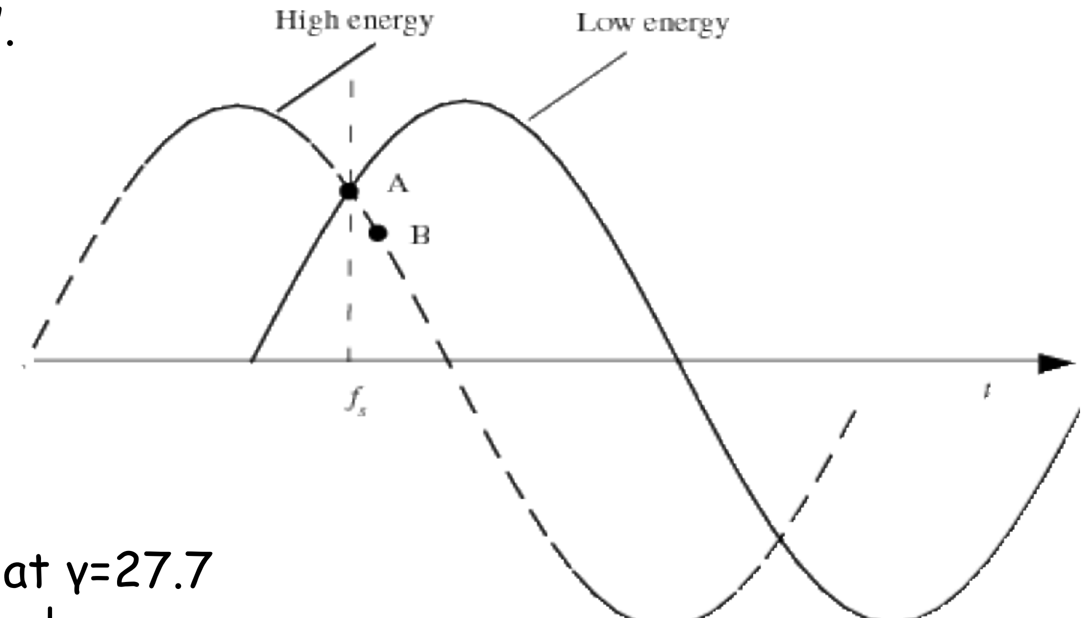


$$\eta = \frac{1}{\gamma^2} - \alpha_c$$

Crossing Transition

At transition, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a 'phase jump'.



$$\alpha_c \sim \frac{1}{Q_x^2} \qquad \gamma_t = \frac{1}{\sqrt{\alpha_c}} \sim Q_x$$

In the PS: γ_t is at ~6 GeV
In the SPS: γ_t= 22.8, injection at γ=27.7
=> no transition crossing!
In the LHC: γ_t is at ~55 GeV, also far below injection energy

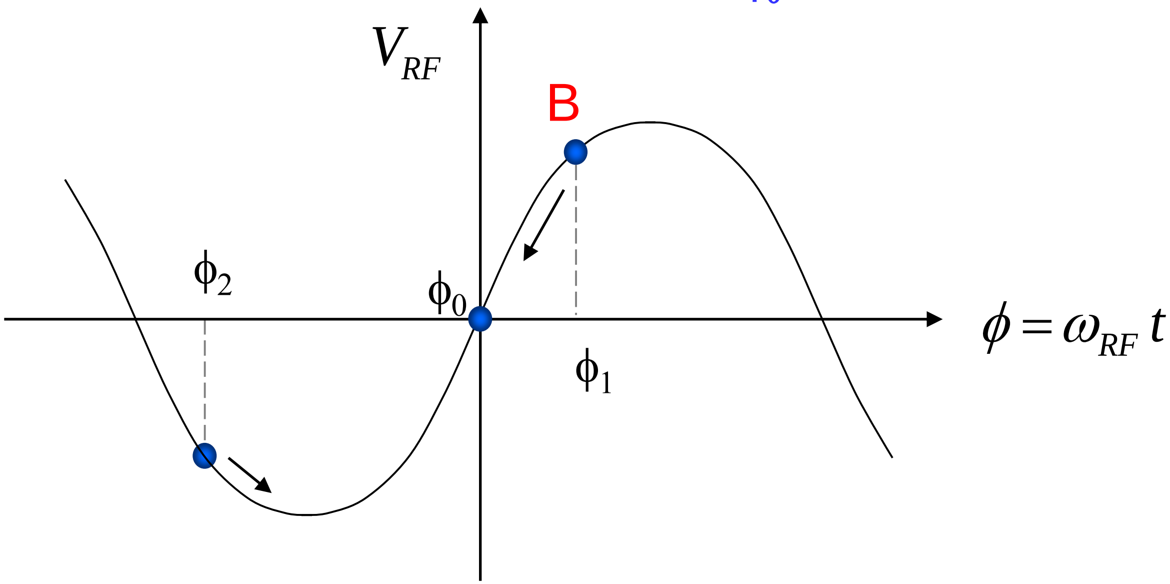
Transition crossing not needed in leptons machines, why?

Dynamics: Synchrotron oscillations

Simple case (no accel.): B = const., below transition $\gamma < \gamma_t$

The phase of the synchronous particle must therefore be $\phi_0 = 0$.

Φ_1 - The particle B is accelerated
- Below transition, an energy increase means an increase in revolution frequency
- The particle arrives earlier – tends toward ϕ_0

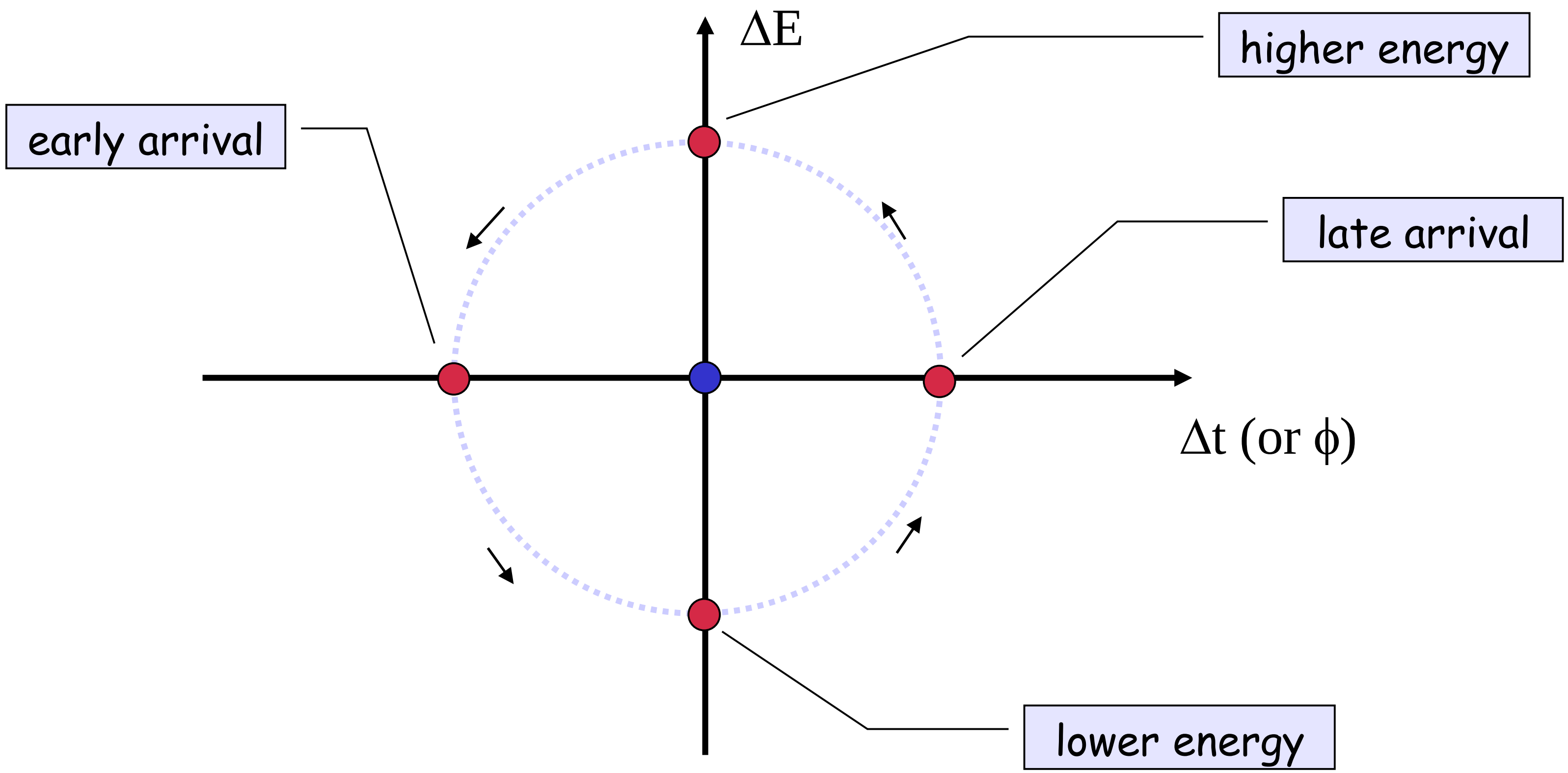


ϕ_2 - The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later – tends toward ϕ_0

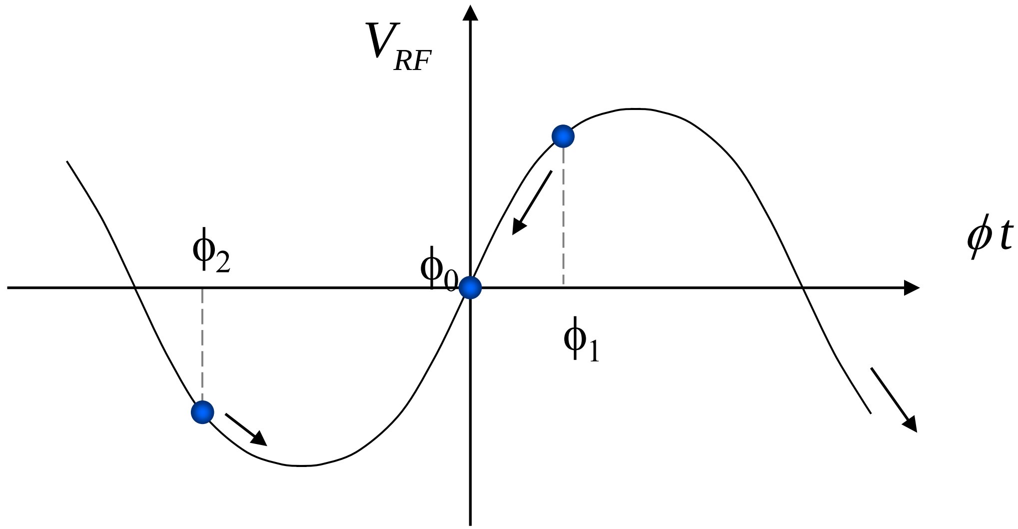
Longitudinal Phase Space Motion

Particle B performs a synchrotron oscillation around synchronous particle A.

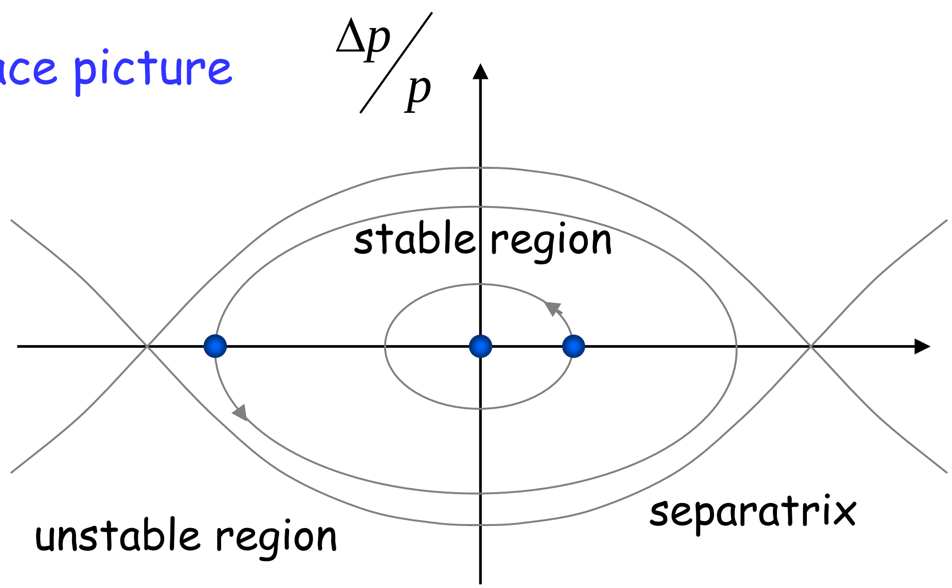
Plotting this motion in longitudinal phase space gives:



Synchrotron oscillations – No acceleration

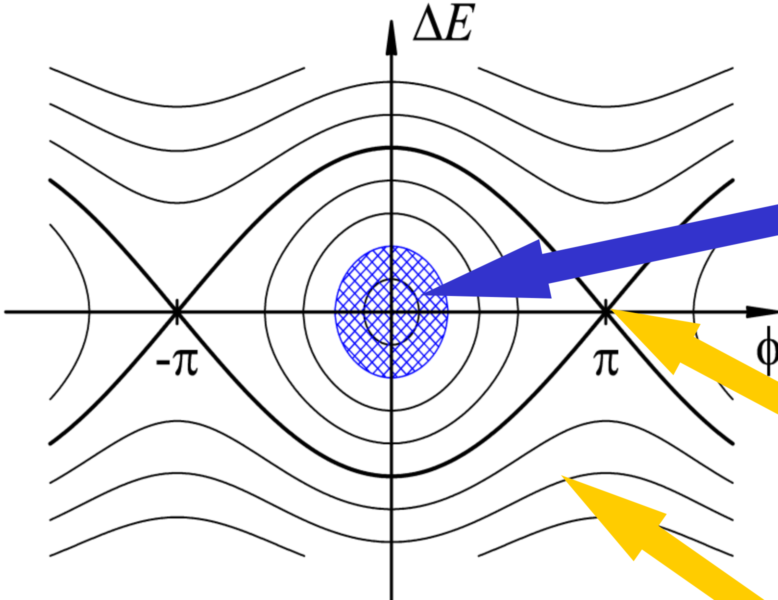


Phase space picture



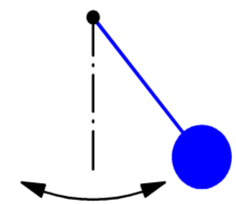
Synchrotron motion in phase space

ΔE-ϕ **phase space** of a stationary bucket
(when there is no acceleration)



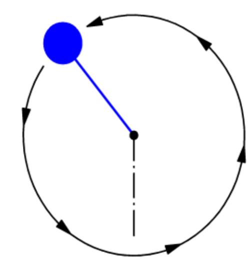
Dynamics of a particle
Non-linear, conservative oscillator $\rightarrow$ e.g. pendulum

Particle **inside the separatrix:**



Particle at the **unstable fix-point**

Particle **outside the separatrix:**

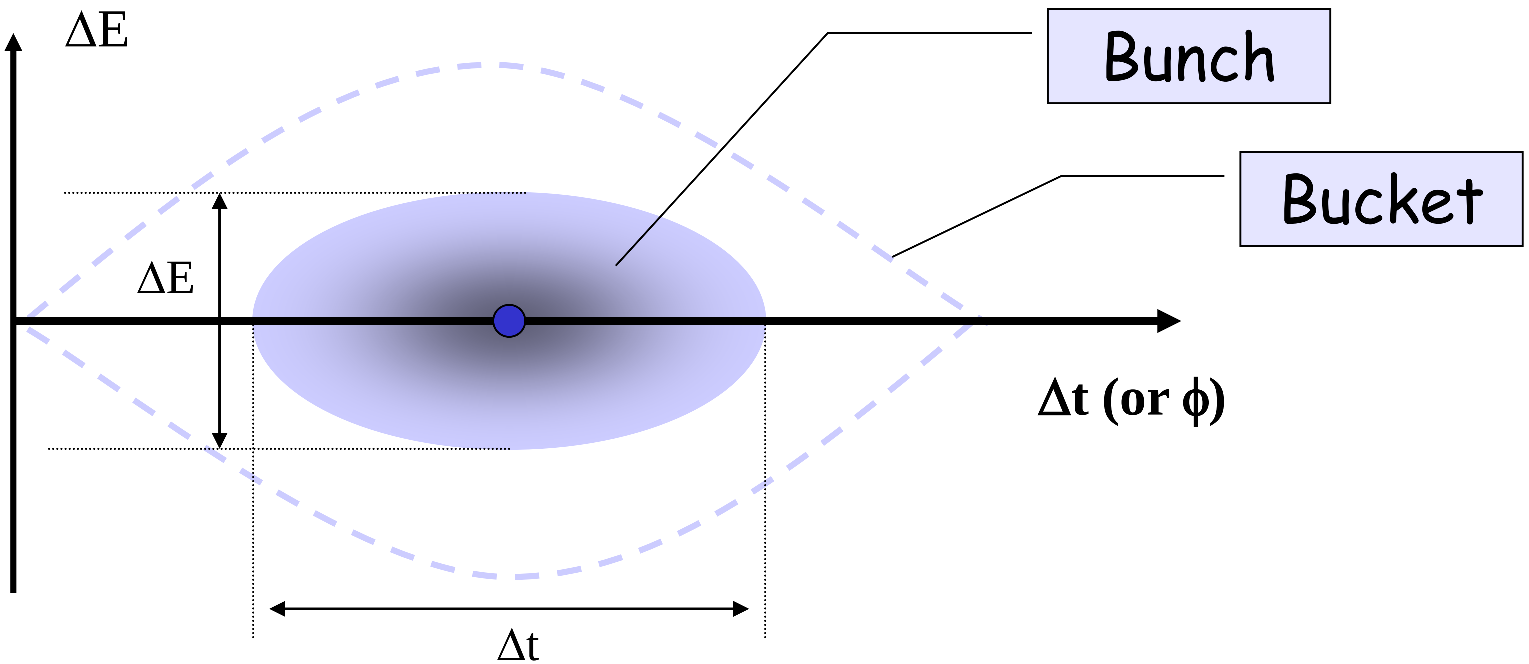


Bucket area: area enclosed by the separatrix

The area covered by particles is the longitudinal emittance.

(Stationary) Bunch & Bucket

The bunches of the beam fill usually a part of the bucket area.



Bucket area = **longitudinal Acceptance** [eVs]

Bunch area = **longitudinal beam emittance** = $4\pi\, \sigma_E\, \sigma_t$ **[eVs]**

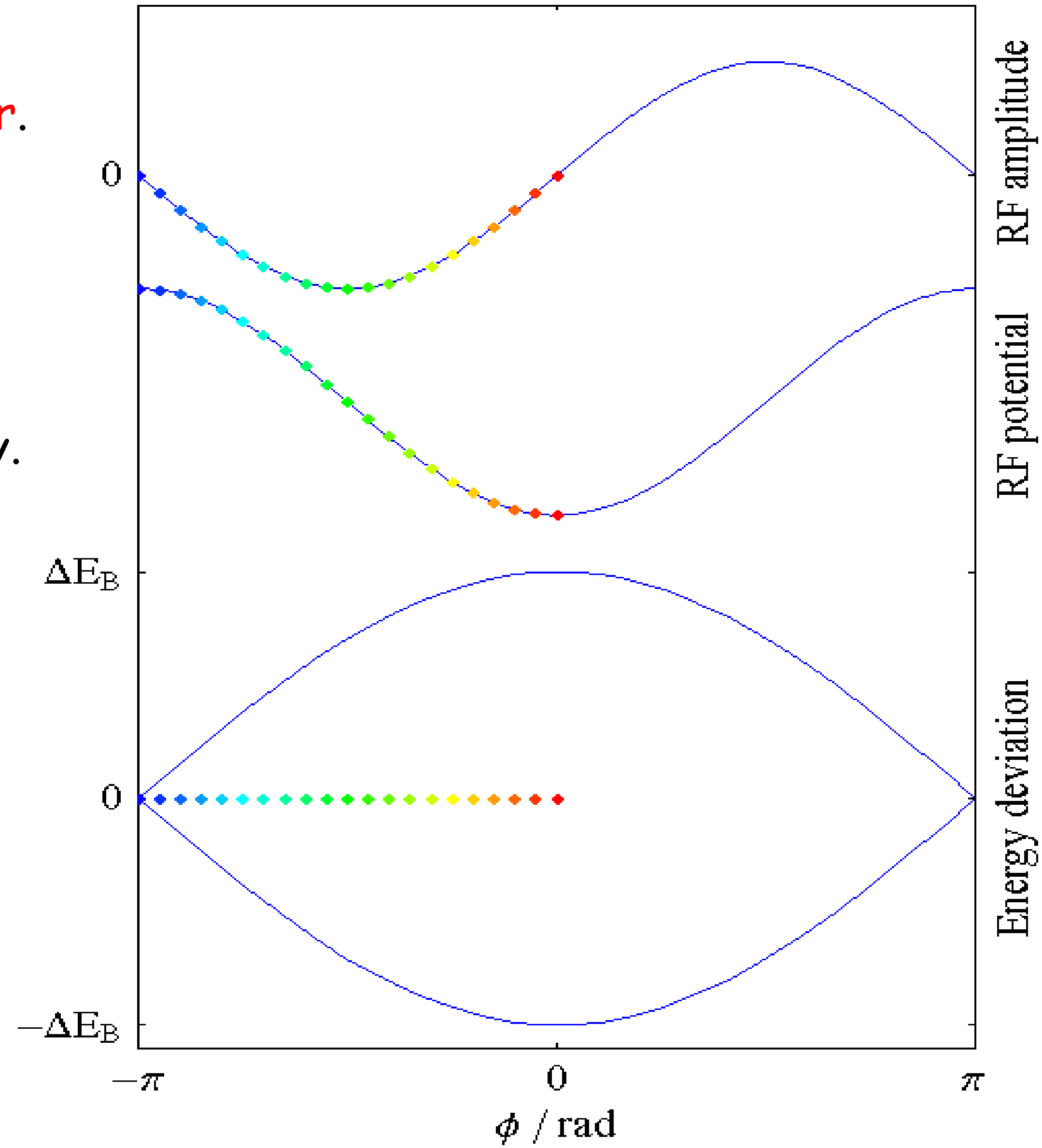
Attention: Different definitions are used!

Synchrotron motion in phase space

The restoring force is non-linear.
⇒ speed of motion depends on position in phase-space

Remark:
Synchrotron frequency much smaller than betatron frequency. It takes a large number of revolutions for one complete oscillation. (Restoring electric force smaller than magnetic force.)

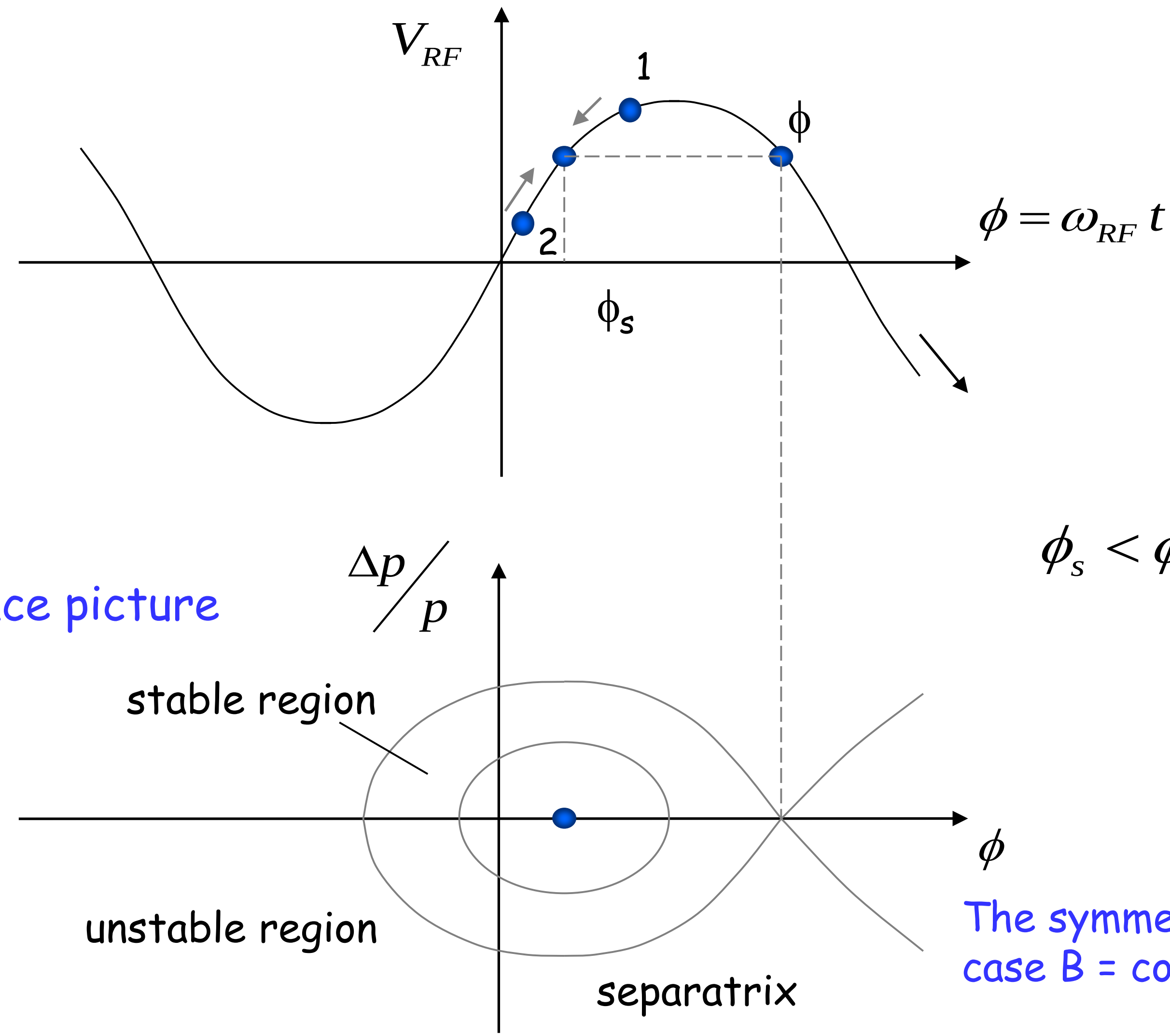
(here shown for a stationary bucket)



Synchrotron oscillations (with acceleration)

Case with acceleration *B increasing*

$\gamma < \gamma_t$

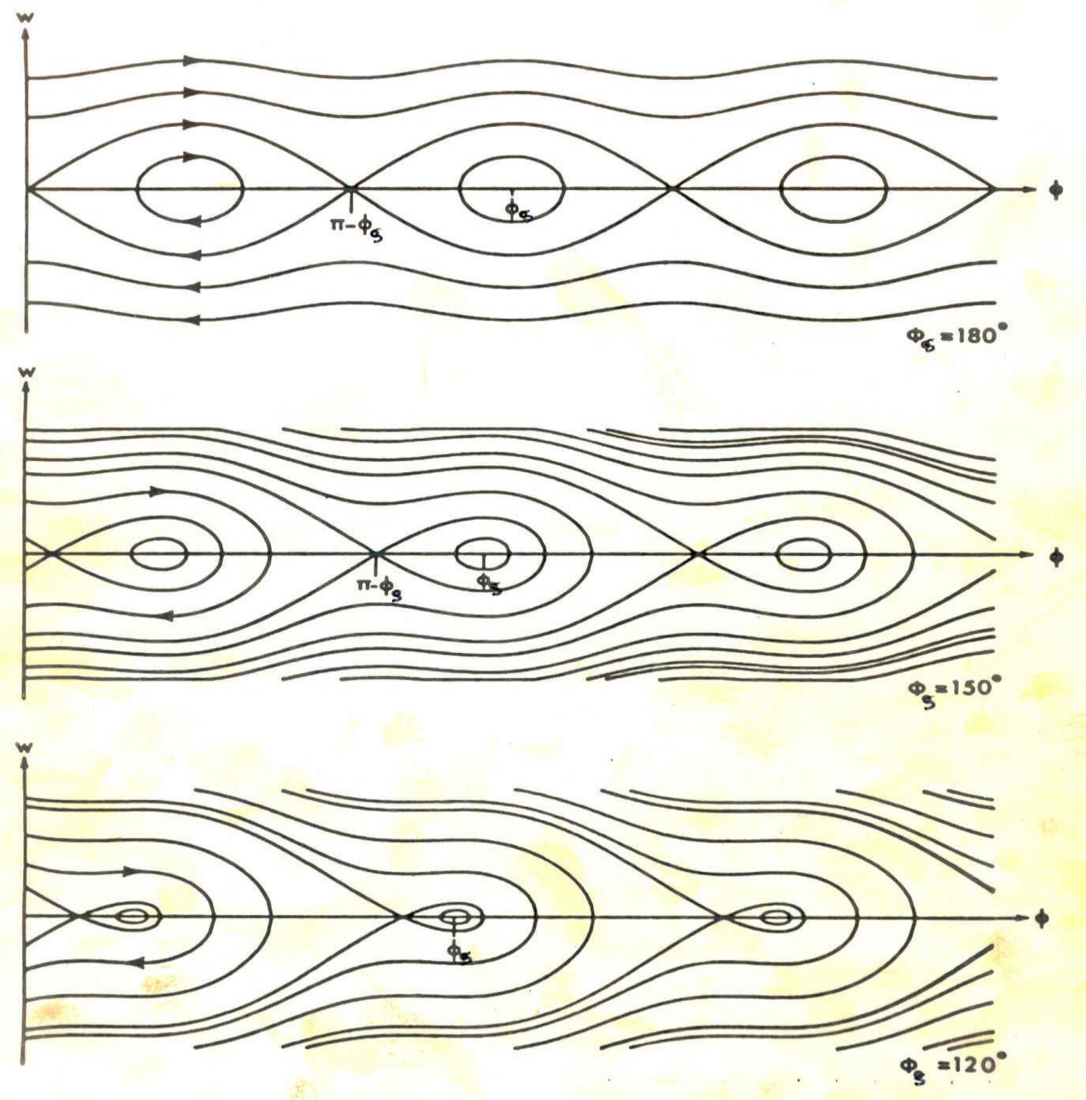


$$\phi_s < \phi < \pi - \phi_s$$

Phase space picture

The symmetry of the case B = const. is lost

RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET". The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which means no acceleration.

During acceleration, the buckets get smaller, both in length and energy acceptance.

=> Injection preferably without acceleration.

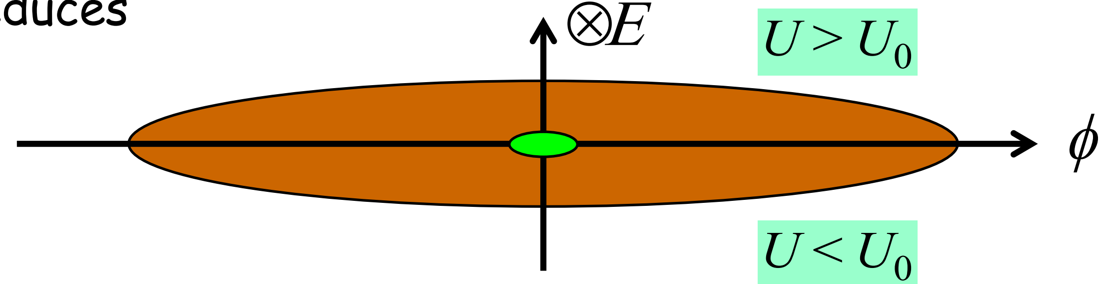
Longitudinal Motion with Synchrotron Radiation

Synchrotron radiation energy-loss energy dependant:

$$U_0 = \frac{4}{3} \square \frac{r_{e,p}}{\left(m_0 c^2\right)^3} \frac{E^4}{\rho}$$

During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces



- when the particle is in the lower half-plane, it loses less energy per turn, but receives U_0 on the average, so its energy deviation gradually reduces

The phase space trajectory spirals towards the origin (limited by quantum excitations)

=> The synchrotron motion is damped toward an equilibrium bunch length and energy spread.

$$\sigma_\tau = \frac{\alpha}{\Omega_s}\left(\frac{\sigma_\varepsilon}{E}\right)$$

More details in the lectures on *Damping Rings*

Longitudinal Dynamics in Synchrotrons

Now we will look more quantitatively at the "synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle.
Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle.
So let's introduce the following reduced variables:

revolution frequency :	$\Delta f_r = f_r - f_{rs}$
particle RF phase :	$\Delta\phi = \phi - \phi_s$
particle momentum :	$\Delta p = p - p_s$
particle energy :	$\Delta E = E - E_s$
azimuth angle :	$\Delta\theta = \theta - \theta_s$

Equations of Longitudinal Motion

In these reduced variables, the equations of motion are (see Appendix):

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}}\frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will simplify in the following…

Small Amplitude Oscillations

Let's assume constant parameters R_s, p_s, ω_s and η:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

with

$$\Omega_s^2 = \frac{h\eta\omega_{rs} e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now small phase deviations from the reference particle:

$$\sin\phi - \sin\phi_s = \sin\left(\phi_s + \Delta\phi\right) - \sin\phi_s \cong \cos\phi_s \Delta\phi$$

(for small $\Delta\phi$)

and the corresponding linearized motion reduces to a harmonic oscillation:

$$\ddot{\Delta\phi} + \Omega_s^2 \Delta\phi = 0$$

where Ω_s is the synchrotron angular frequency.

The synchrotron tune ν_s is the number of synchrotron oscillations per revolution:

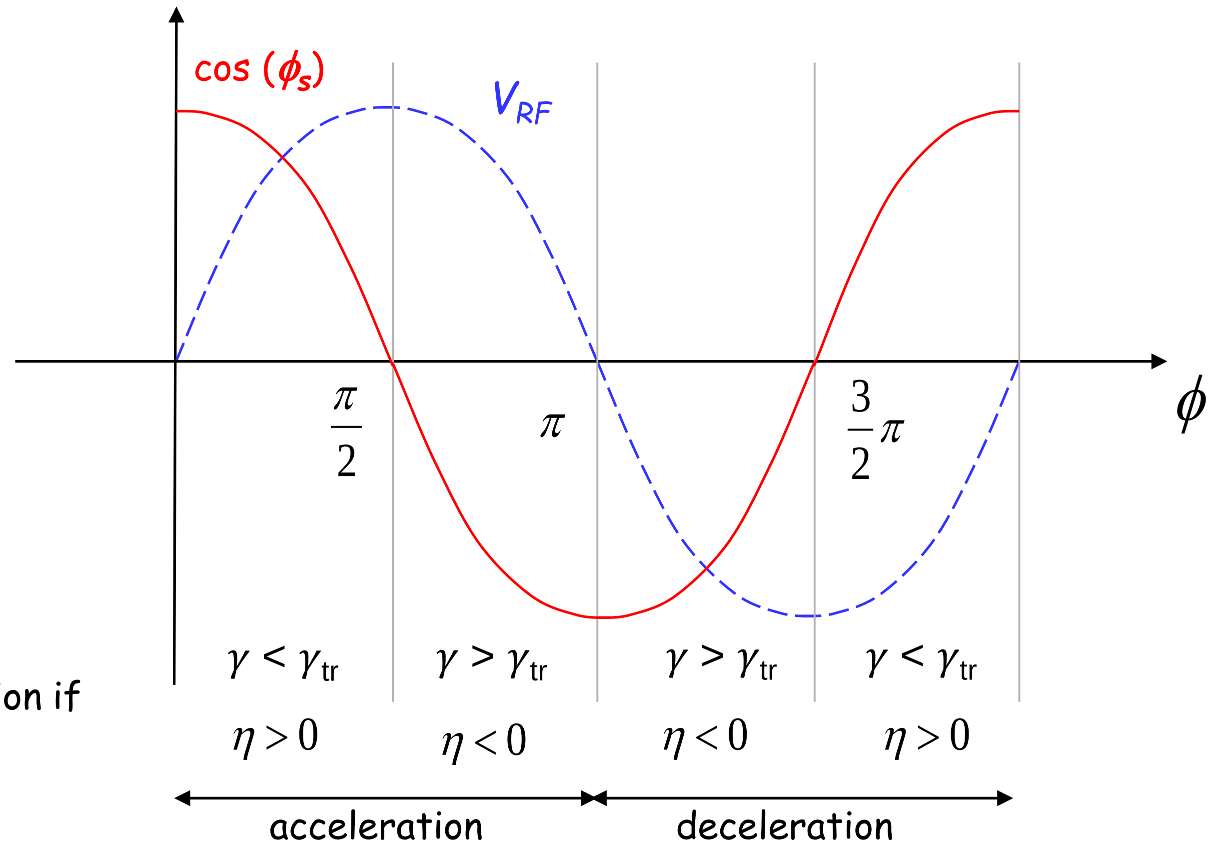
$$\nu_s = \Omega_s / \omega_r$$

See Appendix for large amplitude treatment and further details.

Stability condition for ϕ_s

Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\boxed{\Omega_s^2 = \frac{e\hat{V}_{RF}\,\eta h\,\omega_s}{2\pi R_s p_s}\cos\phi_s} \quad \Rightarrow \quad \Omega_s^2 > 0 \quad \Leftrightarrow \quad \boxed{\eta\cos\phi_s > 0}$$



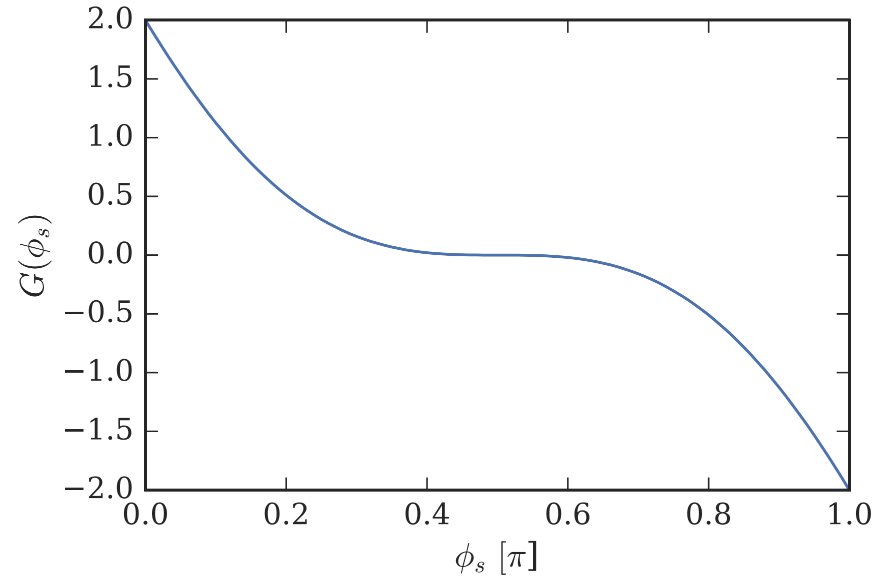
Stable in the region if

Energy Acceptance

From the equation of the separatrix, we can calculate (see appendix) the acceptance in energy:

$$\left(\frac{\Delta E}{E_s}\right)_{\max} = \pm\beta\sqrt{\frac{e\hat{V}}{\pi h \eta E_s} G(\phi_s)}$$

$$G(\phi_s) = \left[2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s\right]$$



This "RF acceptance" depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

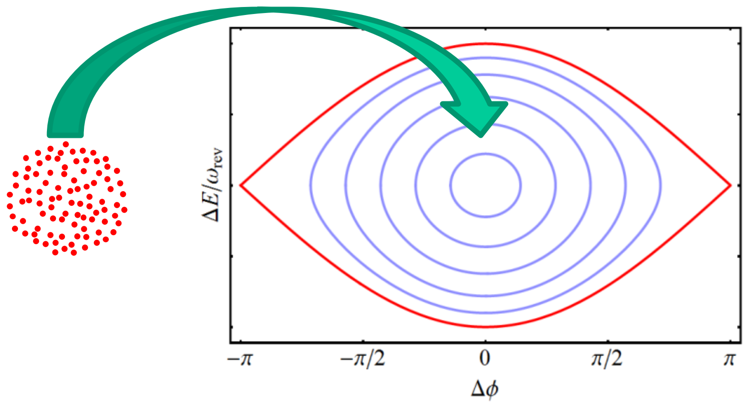
It's largest for ϕ_s=0 and ϕ_s=π (no acceleration, depending on η).

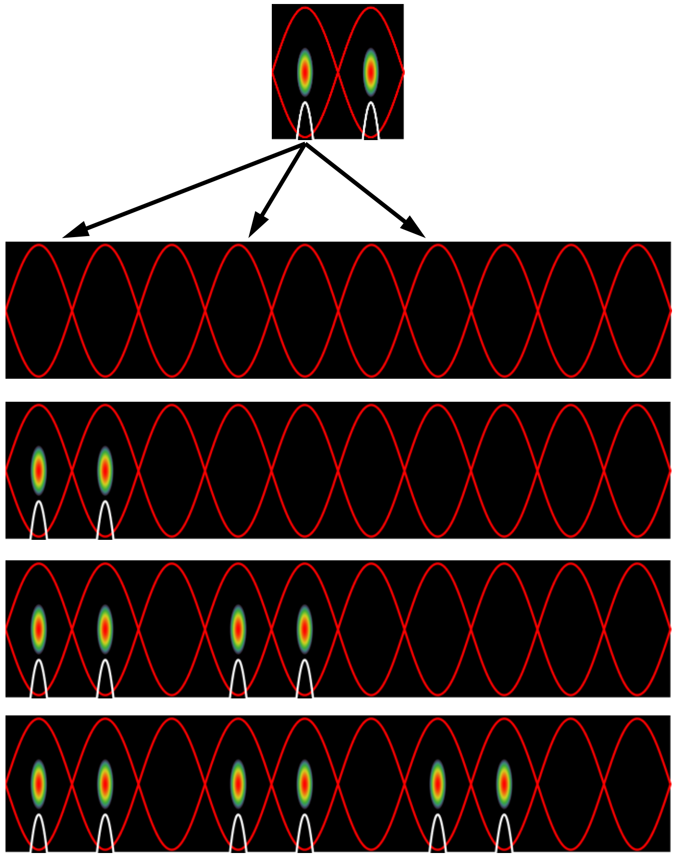
It becomes smaller during acceleration, when ϕ_s is changing

Need a higher RF voltage for higher acceptance.

Injection: Bunch-to-bucket transfer

- Bunch from sending accelerator into the bucket of receiving





Advantages:

→ Particles always subject to longitudinal focusing

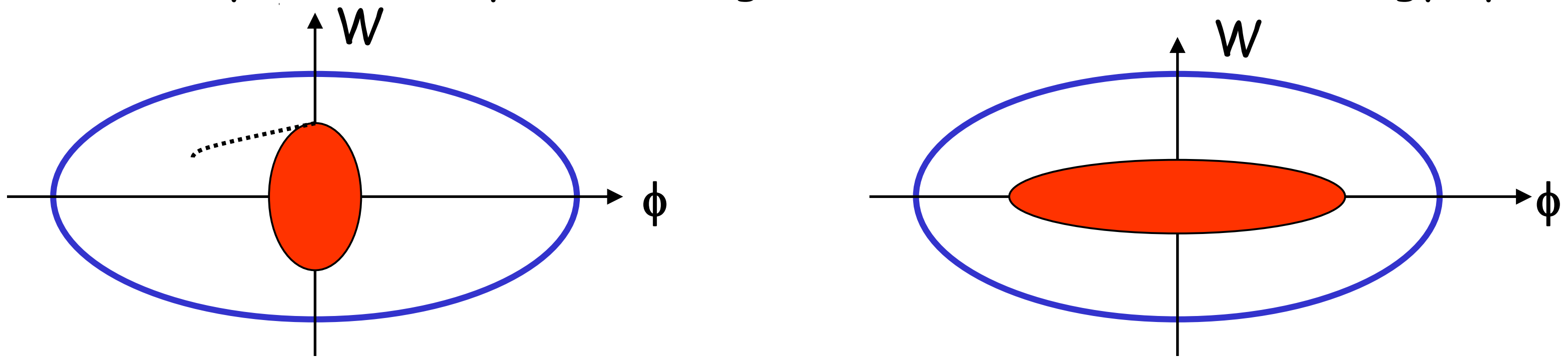
→ No need for RF capture of de-bunched beam in receiving accelerator

→ No particles at unstable fixed point

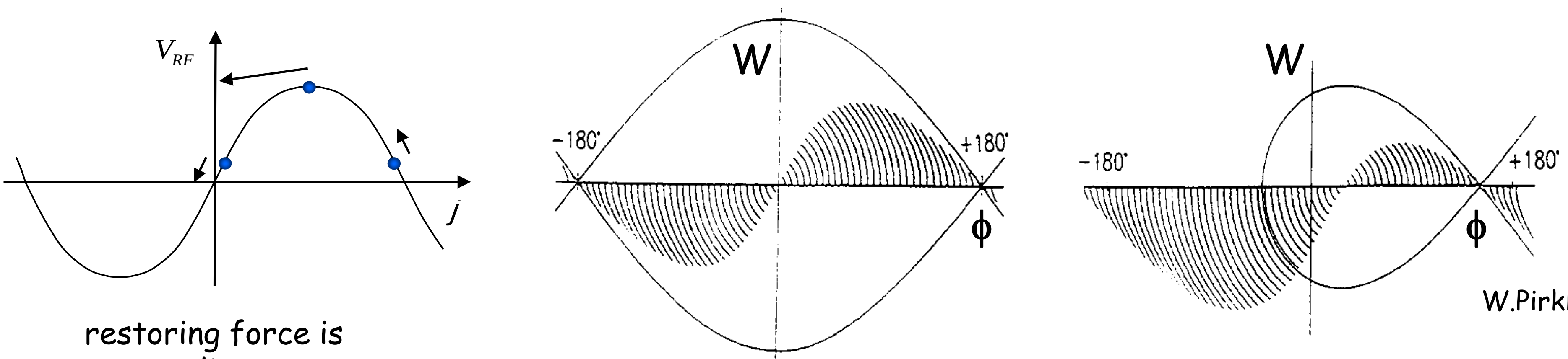
→ Time structure of beam preserved during transfer

Injection: Effect of a Mismatch

Injected bunch: short length and large energy spread
after 1/4 synchrotron period: longer bunch with a smaller energy spread.



For larger amplitudes, the angular phase space motion is slower
(1/8 period shown below)   => can lead to filamentation and emittance growth



restoring force is non-linear

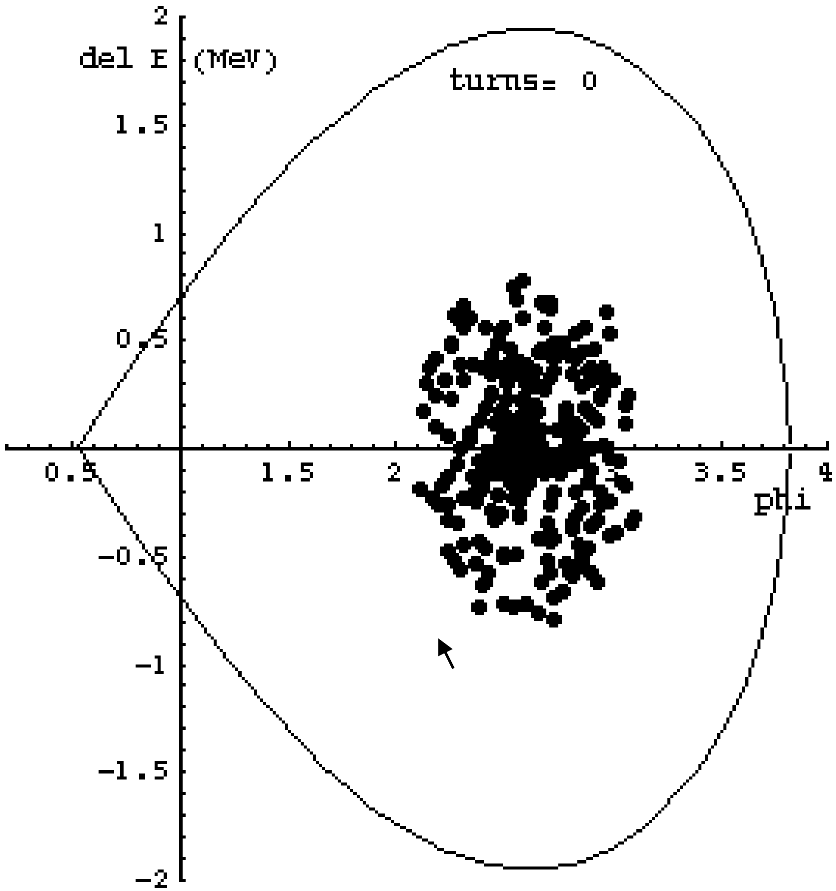
stationary bucket

accelerating bucket

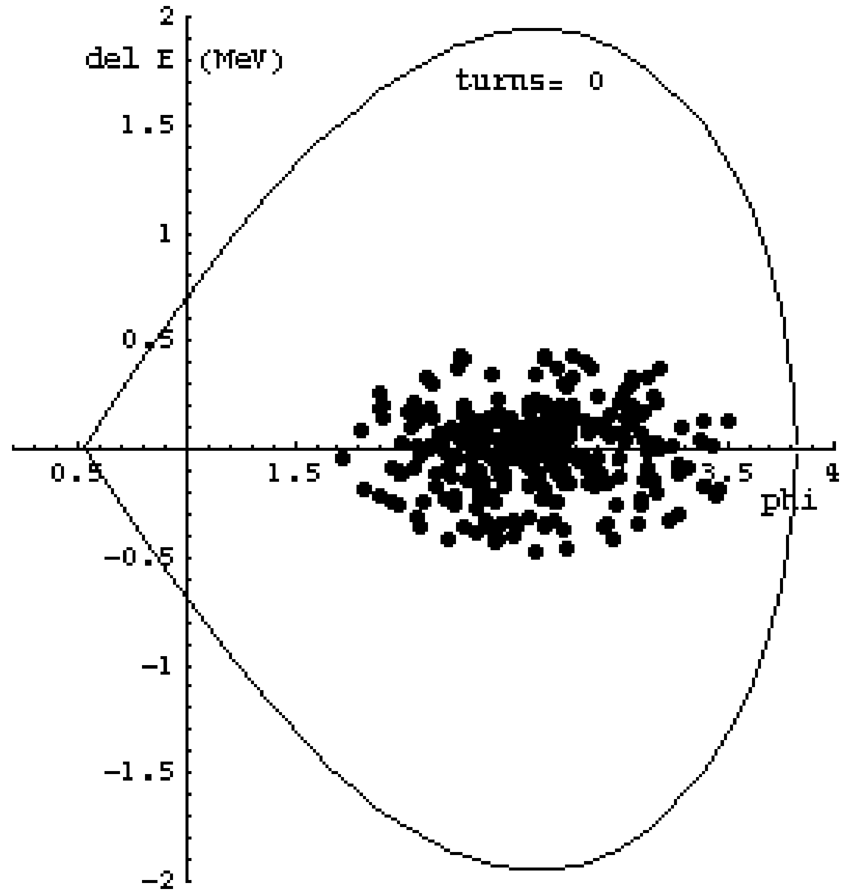
Effect of a Mismatch (2)

Evolution of an injected beam for the first 100 turns.

For a matched transfer, the emittance does not grow (left).



matched beam

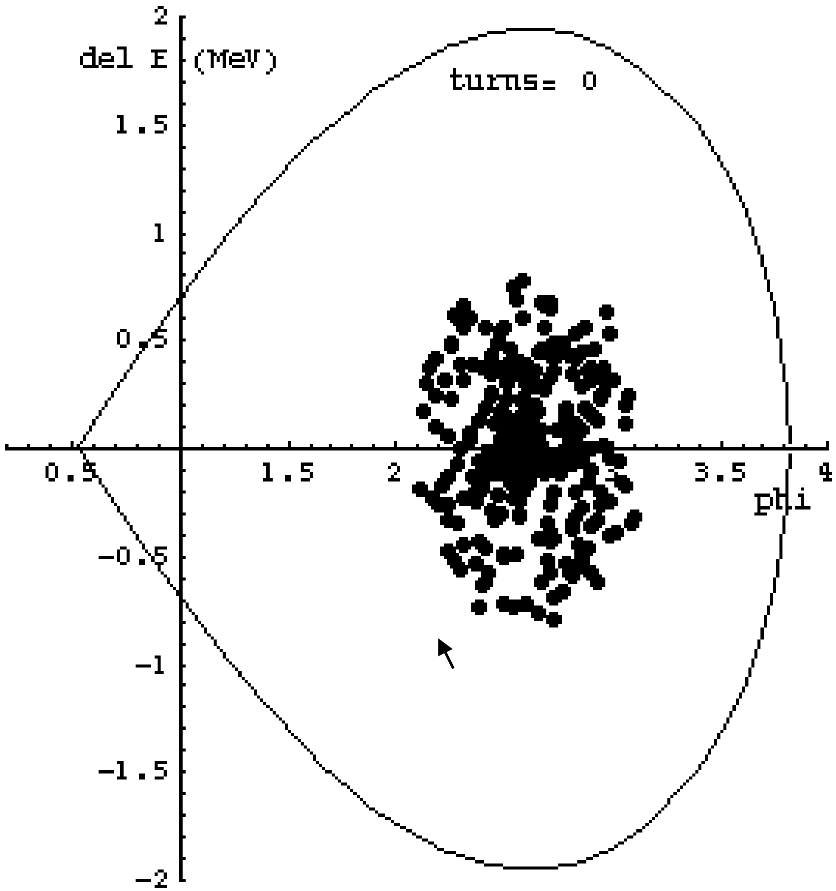


mismatched beam – **bunch length**

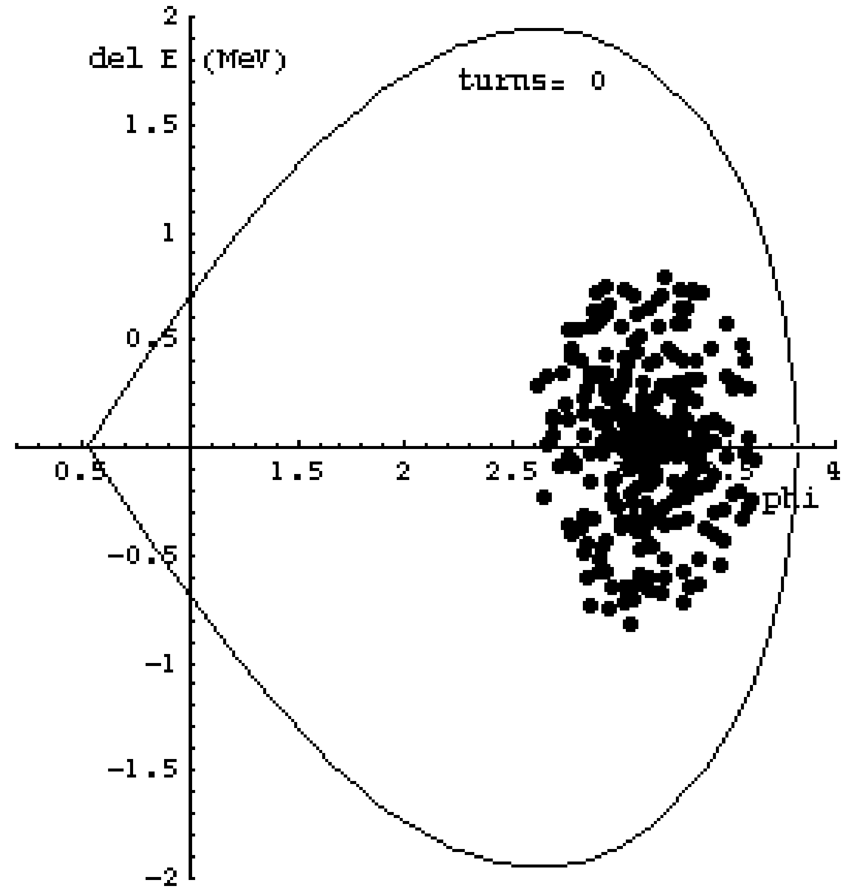
Effect of a Mismatch (3)

Evolution of an injected beam for the first 100 turns.

For a mismatched transfer, the emittance increases (right).



matched beam

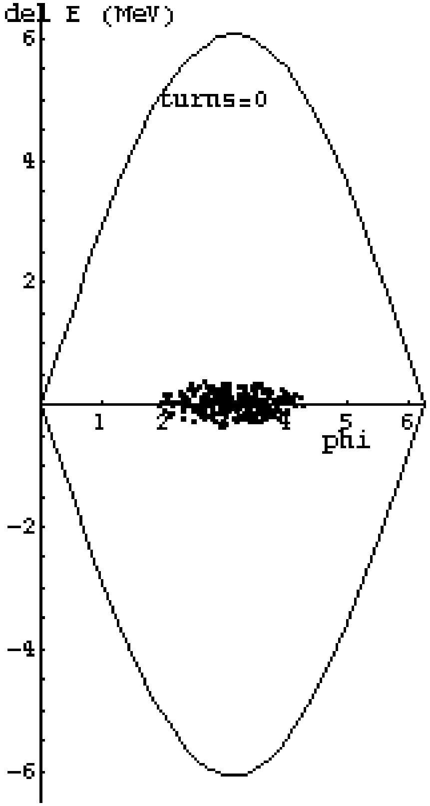


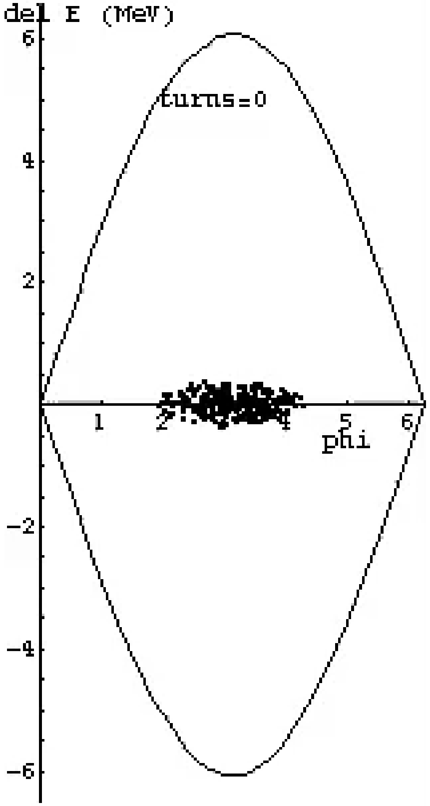
mismatched beam – **phase error**

Bunch Rotation

Phase space motion can be used to make short bunches.

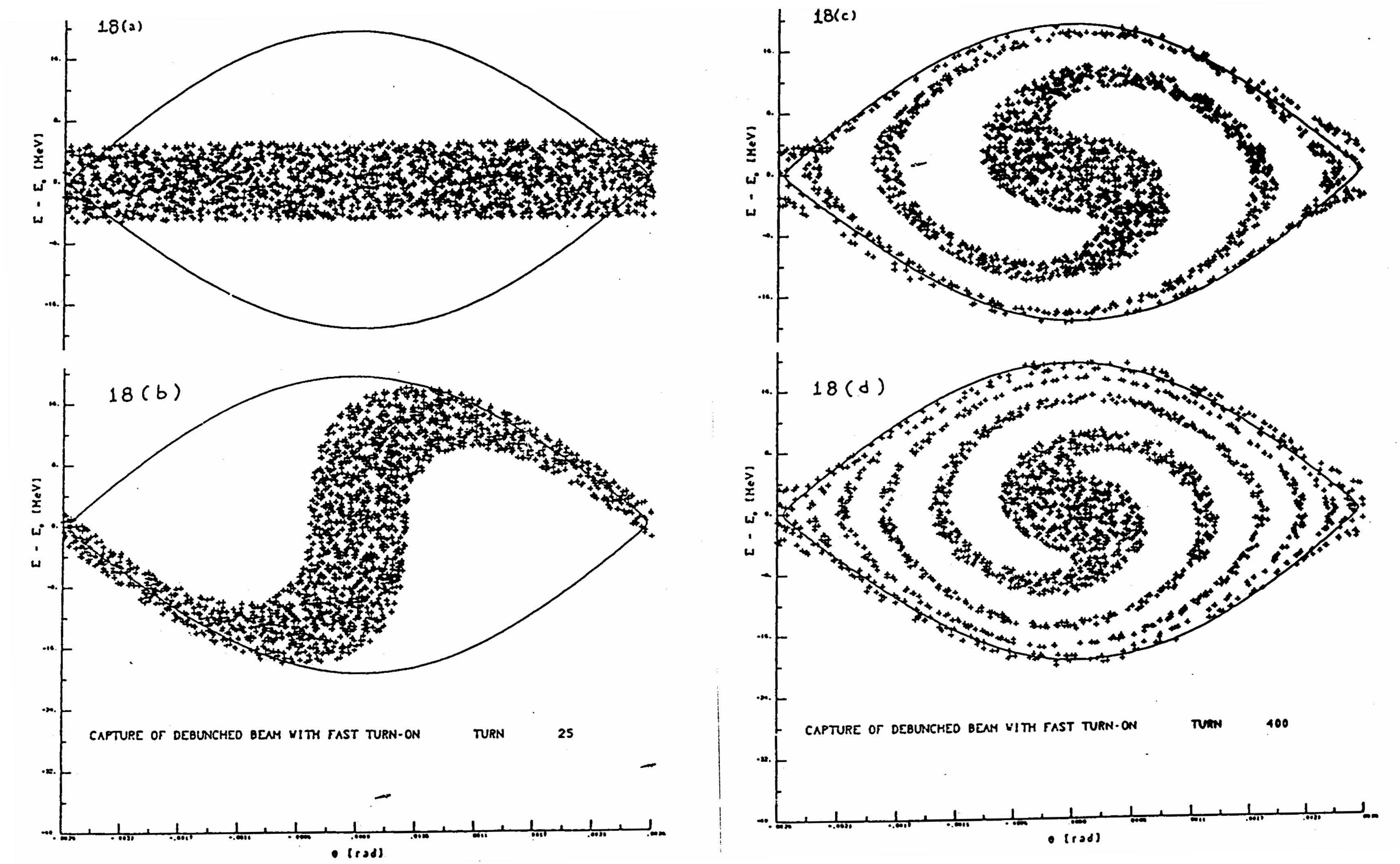
Start with a long bunch and extract or recapture when it's short.



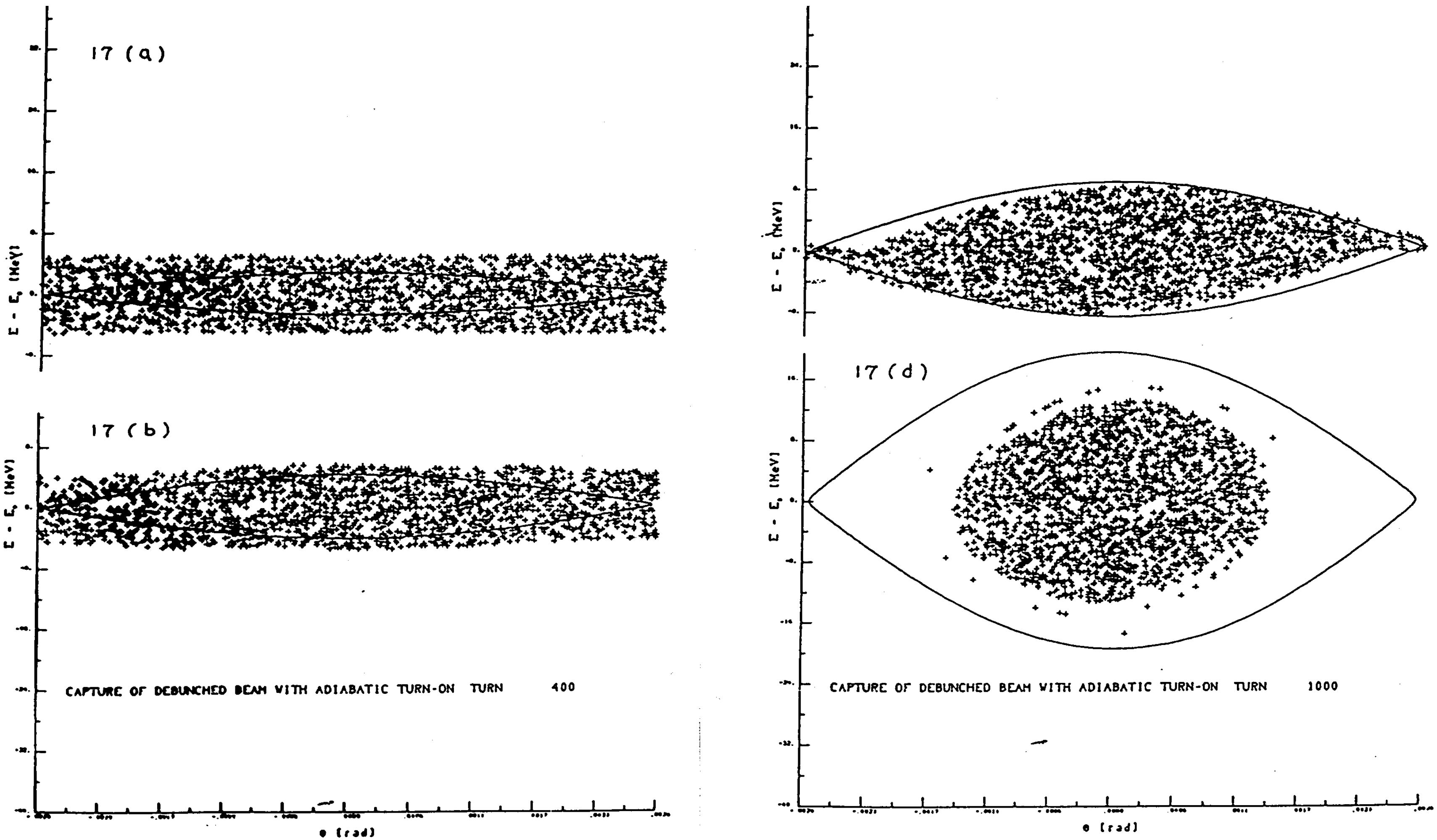


initial beam

Capture of a Debunched Beam with Fast Turn-On



Capture of a Debunched Beam with Adiabatic Turn-On

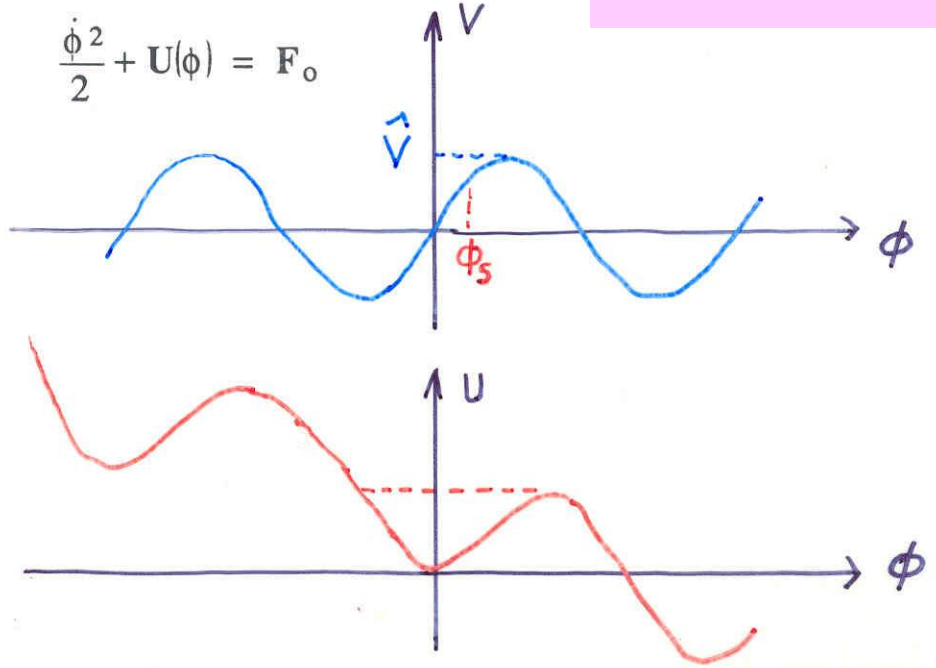


Potential Energy Function

The longitudinal motion is produced by a force that can be derived from a scalar potential:

$$\frac{d^2\phi}{dt^2} = F(\phi) \qquad F(\phi) = -\frac{\partial U}{\partial \phi}$$

$$U = -\int_0^{\phi} F(\phi)\, d\phi = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi \sin\phi_s\right) - F_0$$



The sum of the potential energy and kinetic energy is constant and by analogy represents the total energy of a non-dissipative system.

Hamiltonian of Longitudinal Motion

Introducing a new convenient variable, W, leads to the 1st order equations:

$$W = \frac{\Delta E}{\omega_{rs}} \longrightarrow \quad \frac{d\phi}{dt} = -\frac{h\eta\omega_{rs}}{pR} W$$

$$\frac{dW}{dt} = \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s)$$

The two variables ϕ,W are canonical since these equations of motion can be derived from a Hamiltonian H(ϕ,W,t):

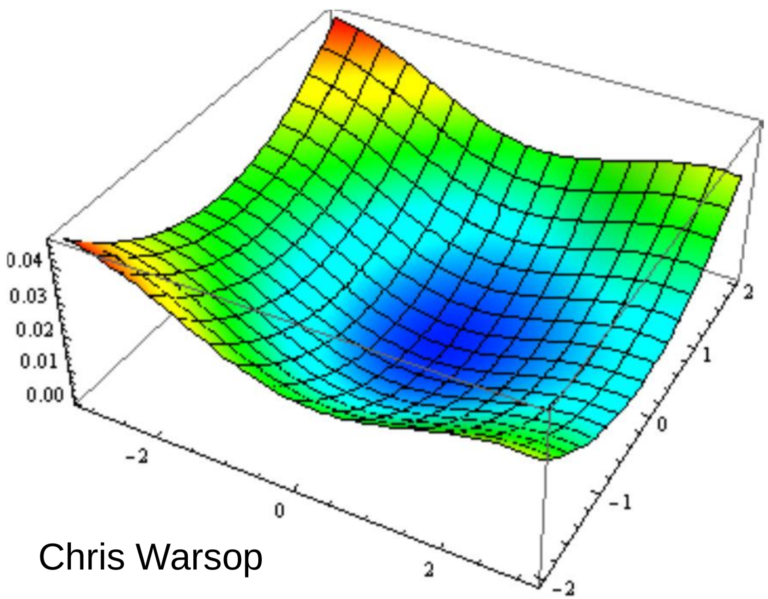
$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} \qquad \frac{dW}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H(\phi, W) = -\frac{1}{2}\frac{h\eta\omega_{rs}}{pR}W^2 + \frac{e\hat{V}}{2\pi}[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s]$$

Hamiltonian of Longitudinal Motion

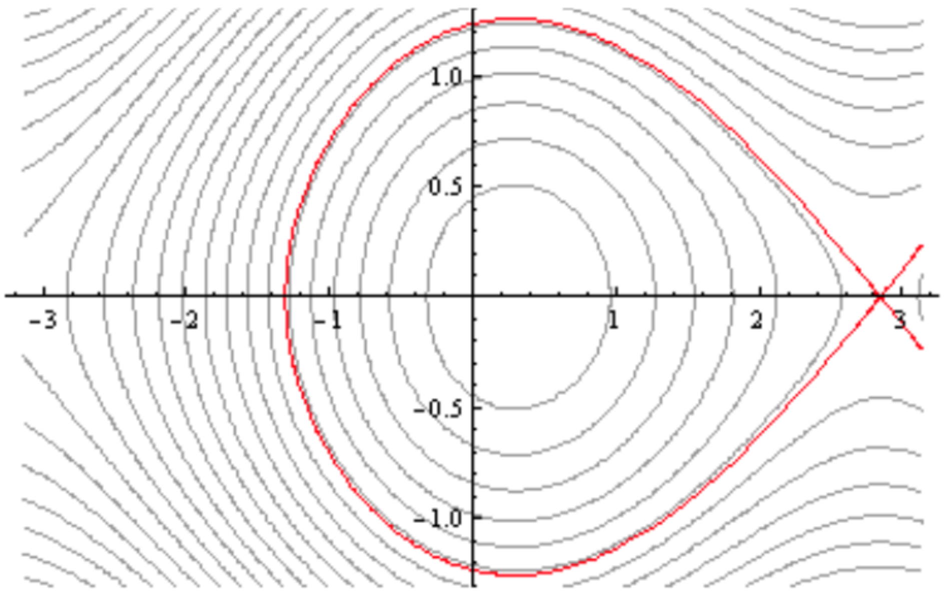
What does it represent? The total energy of the system!

Surface of H (φ,W)



Chris Warsop

Contours of H (φ,W)

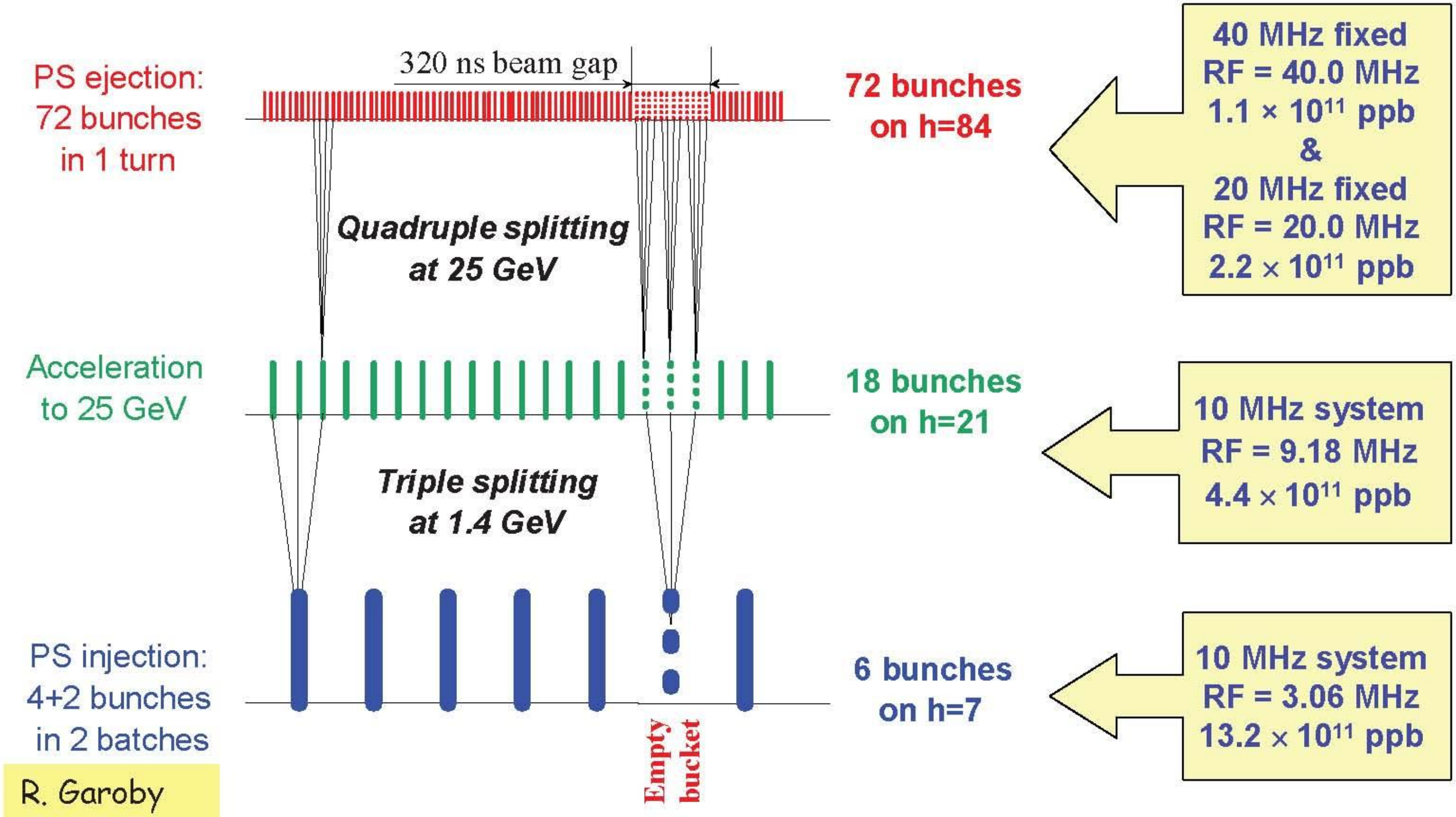


Contours of constant H are particle trajectories in phase space!
(H is conserved)

Hamiltonian Mechanics can help us understand some fairly complicated dynamics (multiple harmonics, bunch splitting, …)

Generating a 25ns LHC Bunch Train in the PS

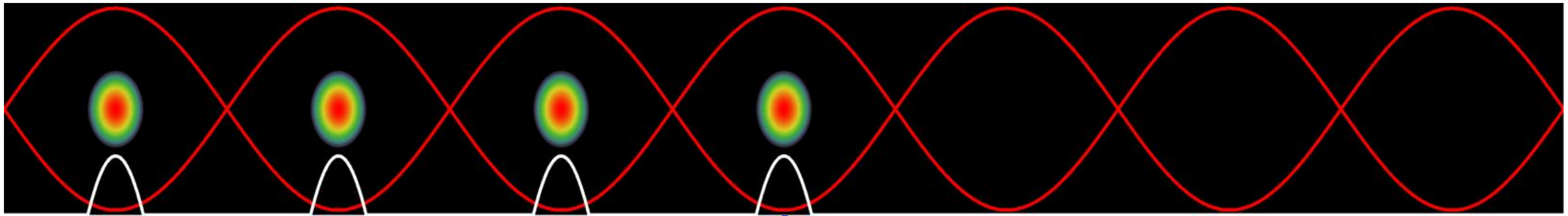
- **Longitudinal bunch splitting (basic principle)**
  - Reduce voltage on principal RF harmonic and simultaneously rise voltage on multiple harmonics (adiabatically with correct phase, etc.)



Use double splitting at 25 GeV to generate 50ns bunch trains instead

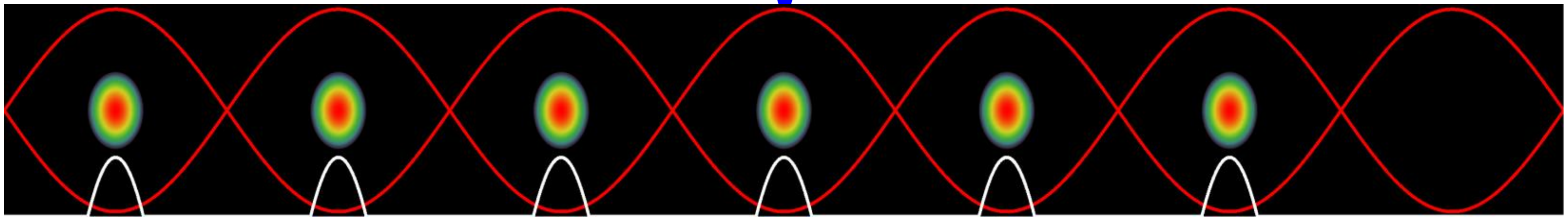
Production of the LHC 25 ns beam

1. Inject four bunches ~ 180 ns, 1.3 eVs

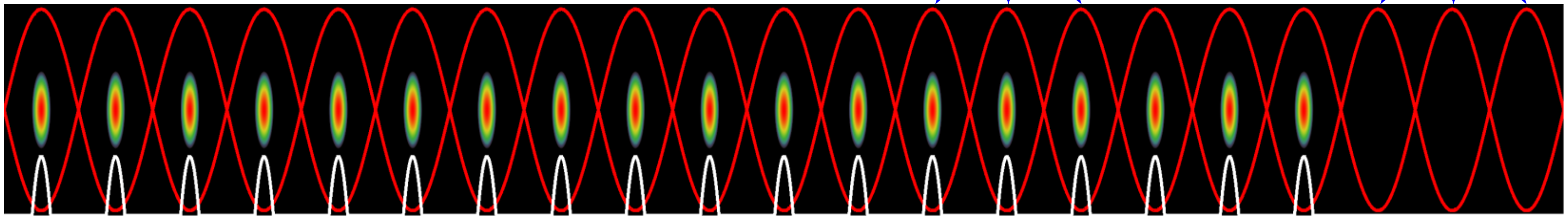


Wait 1.2 s for second injection

2. Inject two bunches



3. Triple split after second injection

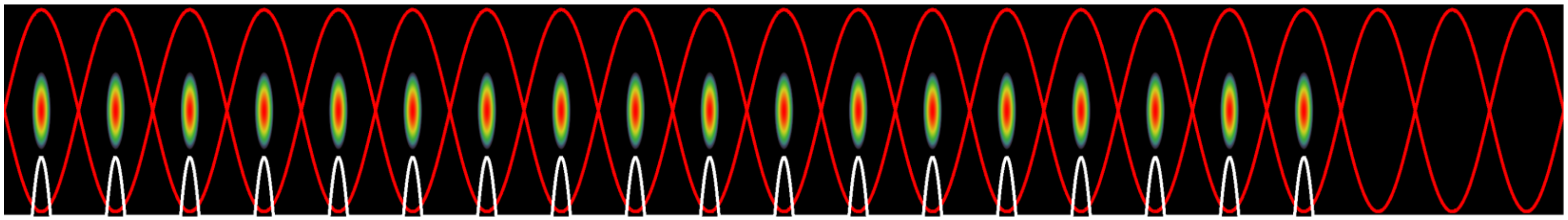


~ 0.7 eVs

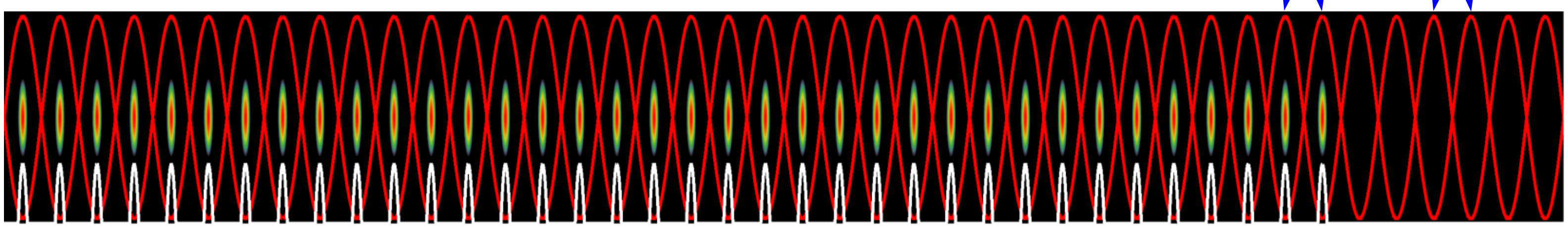
4. Accelerate from 1.4 GeV (E_{kin}) to 26 GeV

Production of the LHC 25 ns beam

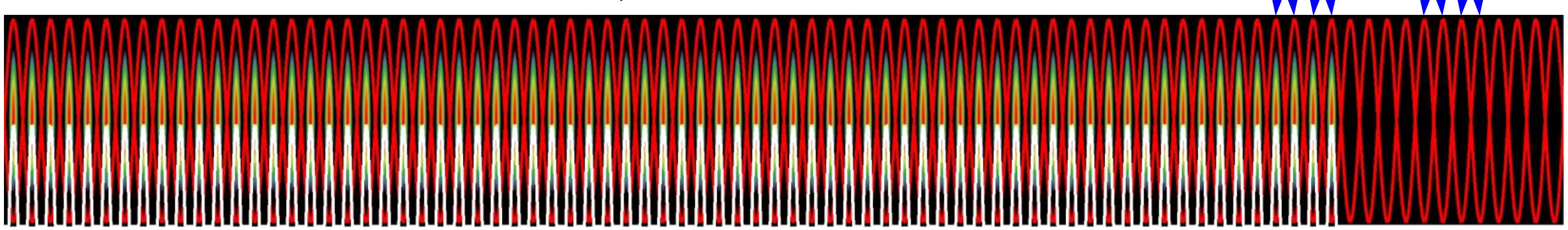
5. During acceleration: longitudinal emittance blow-up: 0.7 – 1.3 eVs



6. Double split ($h21 \rightarrow h42$)

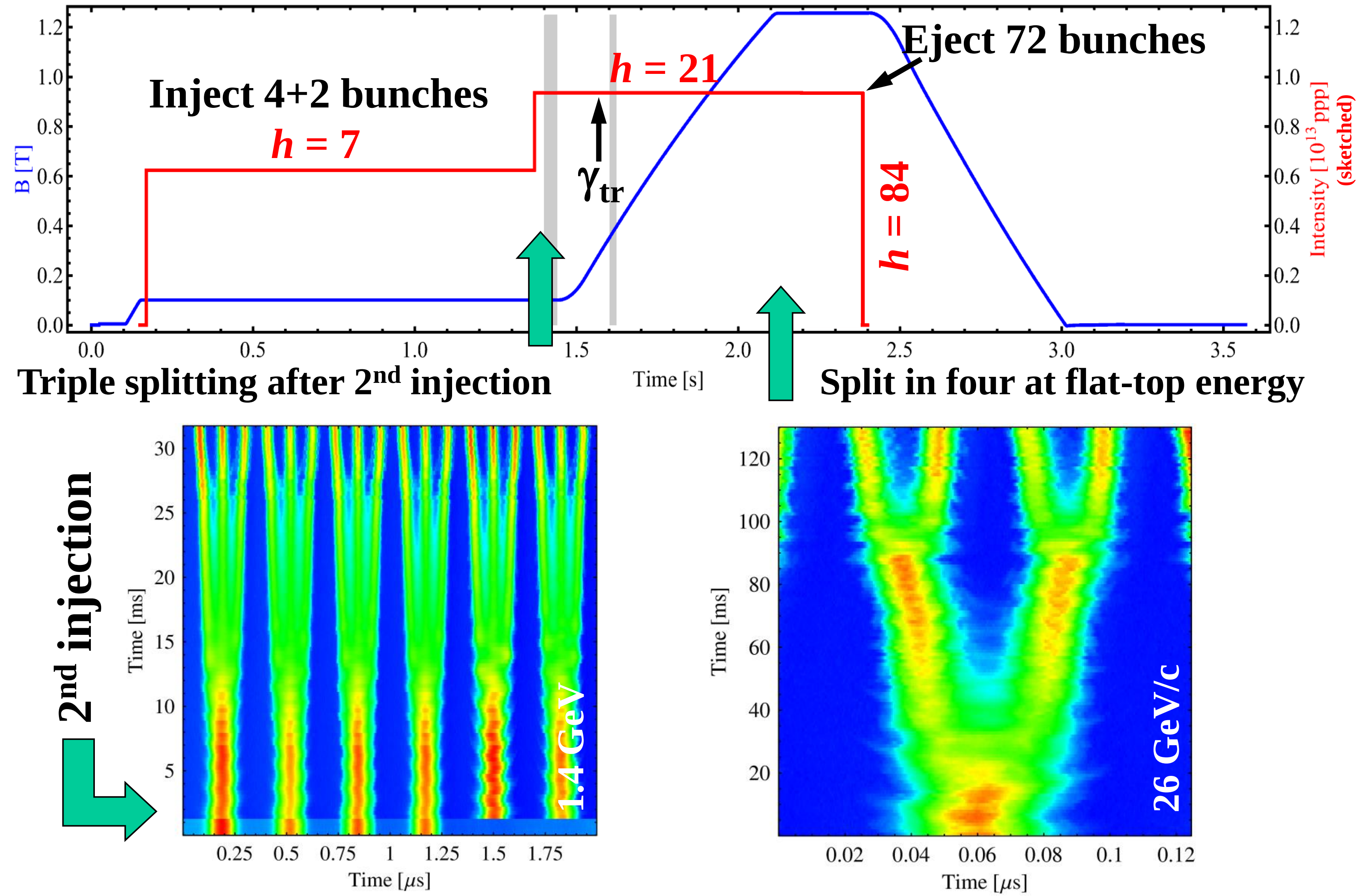


7. Double split ($h42 \rightarrow h84$) **~ 0.35 eVs, 4 ns**



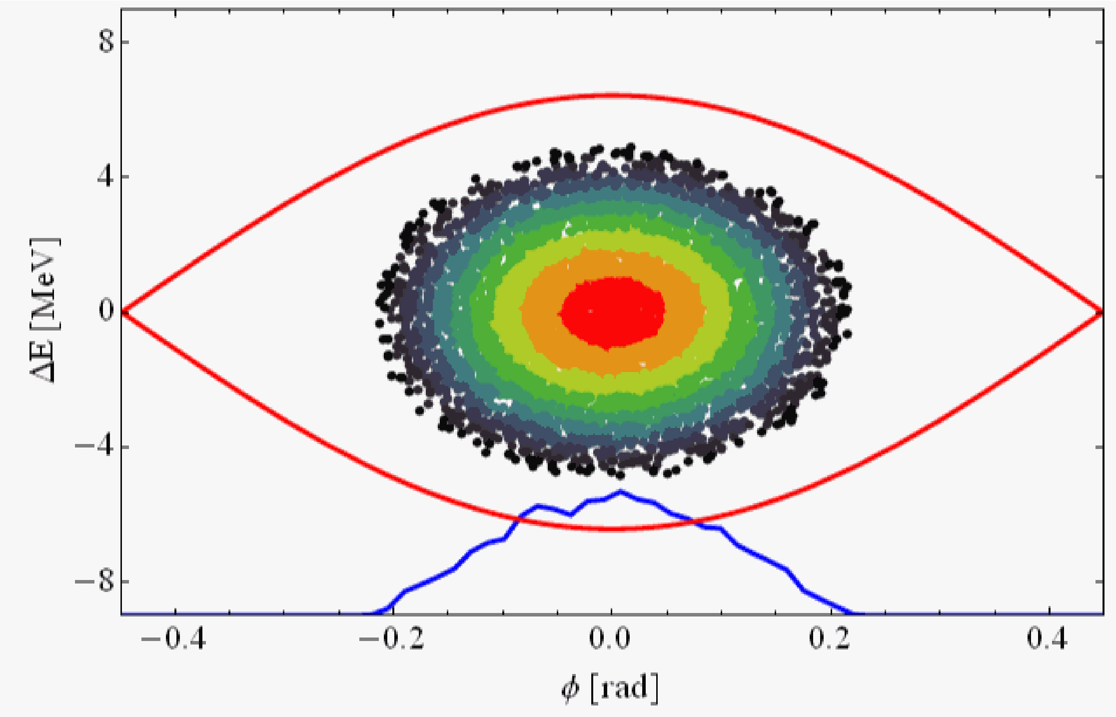
10. Fine synchronization, bunch rotation → Extraction!

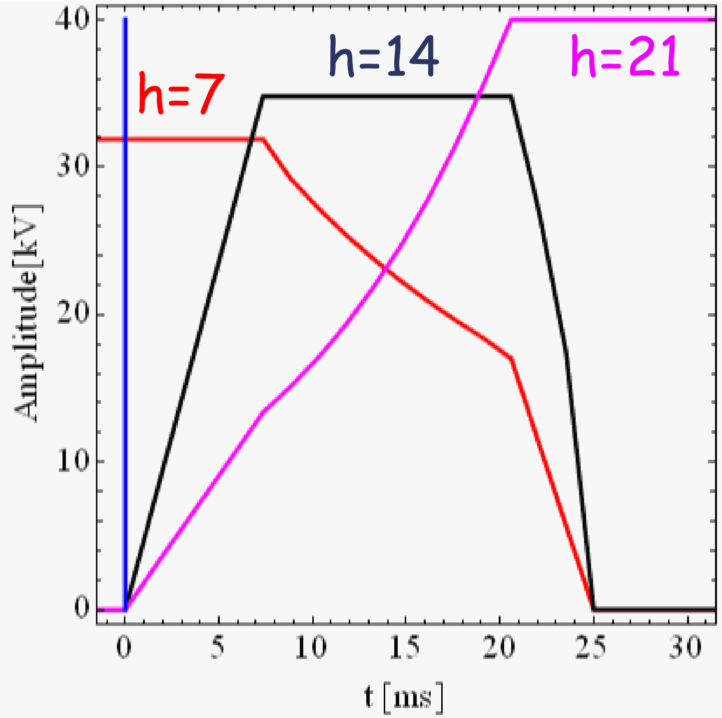
The LHC25 (ns) cycle in the PS



→ Each bunch from the Booster divided by 12 → 6 × 3 × 2 × 2 = 72

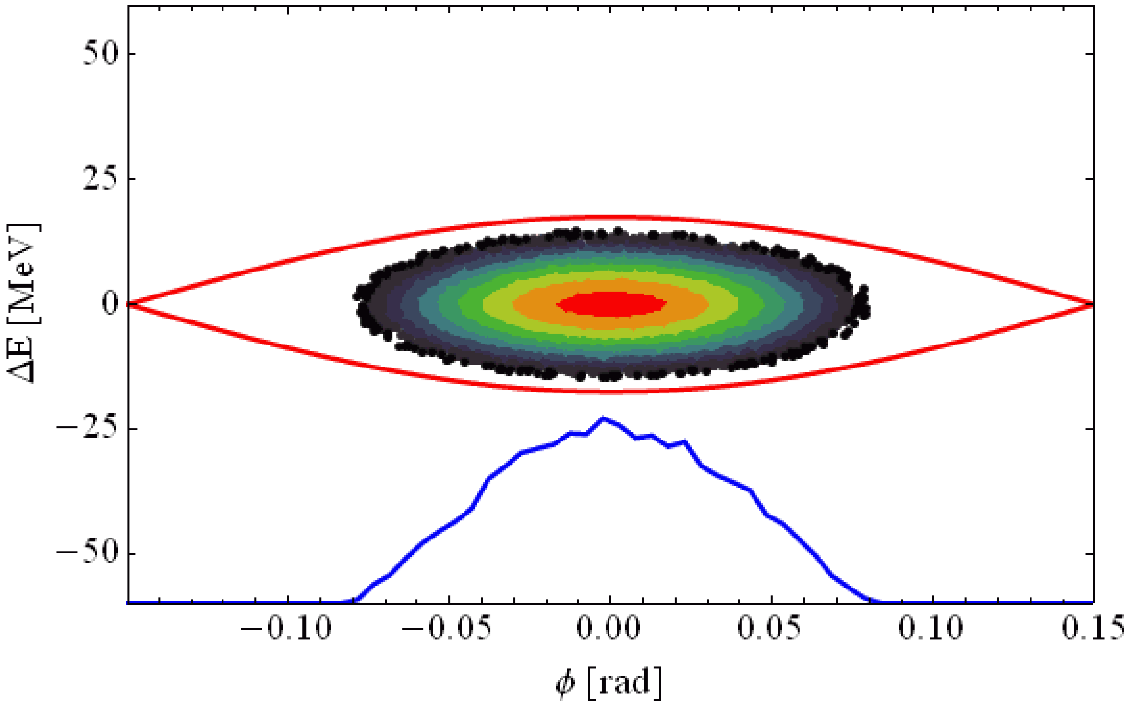
Triple splitting in the PS

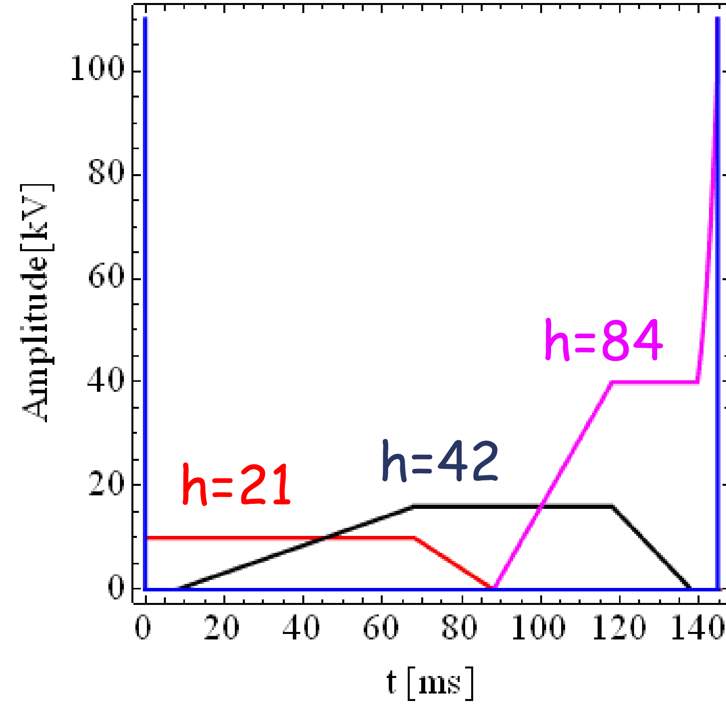




Two times double splitting in the PS

Two times double splitting and bunch rotation:





- Bunch is divided twice using RF systems at h = 21/42 (10/20 MHz) and h = 42/84 (20/40 MHz)
- Bunch rotation: first part h84 only + h168 (80 MHz) for final part

Summary

- Synchrotron oscillations in the longitudinal phase space (E, ϕ) around synchronous phase Φ_s
  - get 'frozen' in a linac at relativistic energies
  - synchronous phase depends on acceleration
  - below or above transition (in synchrotron)
- Bucket is the region in phase space for stable oscillations
  - Bucket size is the largest without acceleration
- to avoid filamentation and emittance increase it is important to
  - match the shape of the bunch to the bucket and
  - inject with the correct phase and energy
- Hamiltonian formalism helpful to understand complex behaviour

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Appendix

- Summary Relativity and Energy Gain
- Cavity parameters
- Momentum compaction factor
- Synchrotron energy-phase oscillations
- Stability condition
- Separatrix stationary bucket
- Large amplitude oscillations
- Bunch matching into stationary bucket

Summary: Relativity + Energy Gain

Newton-Lorentz Force $\vec{F} = \frac{d\vec{p}}{dt} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$

2nd term always perpendicular to motion => no acceleration

Relativistics Dynamics

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \qquad \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$$

$$p = mv = \frac{E}{c^2}\beta c = \beta\frac{E}{c} = \beta\gamma m_0 c$$

$$E^2 = E_0^2 + p^2c^2 \longrightarrow dE = v\,dp$$

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

$$dE = dW = eE_z\,dz \rightarrow W = e\int E_z\,dz$$

RF Acceleration

$$E_z = \hat{E}_z \sin\omega_{RF}t = \hat{E}_z \sin\phi(t)$$

$$\int \hat{E}_z\,dz = \hat{V}$$

$$W = e\hat{V}\sin\phi$$

(neglecting transit time factor)

The field will change during the passage of the particle through the cavity
=> effective energy gain is lower

Cavity Parameters: Quality Factor Q

The total energy stored is $$W = \iiint\limits_{cavity} \left(\frac{\varepsilon}{2}\left|\vec{E}\right|^2 + \frac{\mu}{2}\left|\vec{H}\right|^2\right) dV.$$

- Quality Factor Q (caused by wall losses) defined as

$$Q_0 = \frac{\omega_0 W}{P_{loss}}$$

Ratio of stored energy W and dissipated power P_{loss} on the walls in one RF cycle

The Q factor determines the maximum energy the cavity can fill to with a given input power.
Larger Q => less power needed to sustain stored energy.

The Q factor is 2π times the number of rf cycles it takes to dissipate the energy stored in the cavity (down by 1/e).

- function of the geometry and the surface resistance of the material:
  superconducting (niobium) : $Q= 10^{10}$
  normal conducting (copper) : $Q=10^{4}$

Important Parameters of Accelerating Cavities

- Accelerating voltage V_{acc}

$$V_{acc} = \int_{-\infty}^{\infty} E_z e^{-i\frac{\omega z}{\beta c}} dz$$

Measure of the acceleration

- R upon Q

$$\frac{R}{Q} = \frac{|V_{acc}|^2}{2\omega_0 W}$$

Relationship between acceleration V_{acc} and stored energy W

independent from material!

Attention: Different definitions are used!

- Shunt Impedance R

$$R = \frac{|V_{acc}|^2}{2P_{loss}}$$

Relationship between acceleration V_{acc} and wall losses P_{loss}

depends on
- material
- cavity mode
- geometry

Important Parameters of Accelerating Cavities (cont.)

- Fill Time t_F

  - **standing wave cavities**:

$$P_{loss} = -\frac{dW}{dt} = \frac{\omega}{Q} W$$

Exponential decay of the stored energy W due to losses

$$t_F = \frac{Q}{\omega}$$

time for the field to decrease by 1/e after the cavity has been filled
measure of how fast the stored energy is dissipated on the wall

Several fill times needed to fill the cavity!

  - **travelling wave cavities**:

time needed for the electromagnetic energy to fill the cavity of length L

$$t_F = \int_0^L \frac{dz}{v_g(z)}$$

v_g: velocity at which the energy propagates through the cavity

Cavity is completely filled after 1 fill time!

Cavity parameters

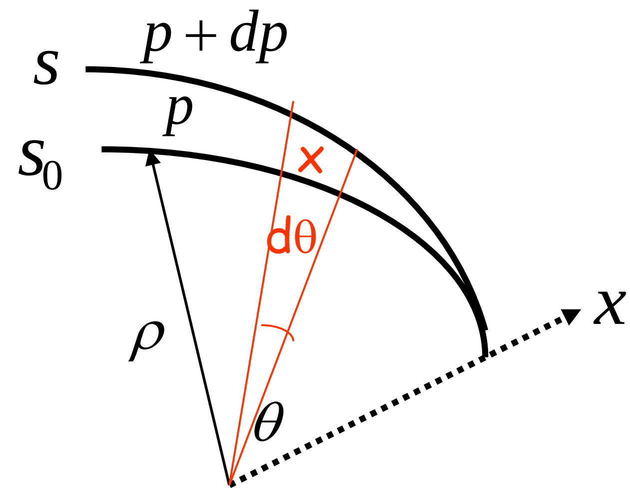
Resonance frequency	$\omega_0 = \frac{1}{\sqrt{L \cdot C}}$	
Transit time factor	$TT = \frac{\left\| \int E_z e^{i\frac{\omega}{\beta c}z}\, dz \right\|}{\left\| \int E_z\, dz \right\|}$	
Q factor	$\omega_0 W = Q\, P_{loss}$	
	Circuit definition	***Linac definition***
Shunt impedance	$\left\|V_{gap}\right\|^2 = 2\, R\, P_{loss}$	$\left\|V_{gap}\right\|^2 = R\, P_{loss}$
R/Q (R-upon-Q)	$\frac{R}{Q} = \frac{\left\|V_{gap}\right\|^2}{2\, \omega_0 W} = \sqrt{L/C}$	$\frac{R}{Q} = \frac{\left\|V_{gap}\right\|^2}{\omega_0 W}$
Loss factor	$k_{loss} = \frac{\omega_0}{2} \frac{R}{Q} = \frac{\left\|V_{gap}\right\|^2}{4W} = \frac{1}{2C}$	$k_{loss} = \frac{\omega_0}{4} \frac{R}{Q} = \frac{\left\|V_{gap}\right\|^2}{4W}$

Appendix: Momentum Compaction Factor

$$\alpha_c = \frac{p}{L}\frac{dL}{dp}$$

$$ds_0 = \rho\, d\theta$$

$$ds = (\rho + x)\, d\theta$$



The elementary path difference from the two orbits is:

definition of dispersion D_x

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho}\frac{dp}{p}$$

leading to the total change in the circumference:

$$dL = \oint_C dl = \oint \frac{x}{\rho} ds_0 = \oint \frac{D_x}{\rho}\frac{dp}{p} ds_0$$

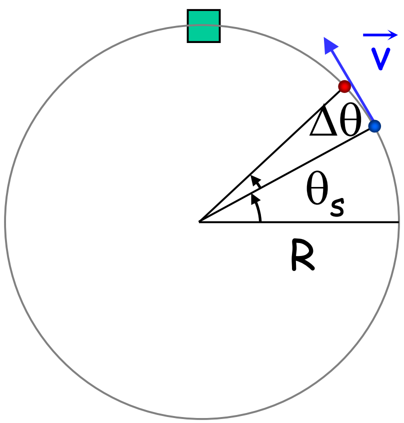
$$\alpha_c = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)} ds_0$$

With ρ=∞ in straight sections we get:

$$\alpha_c = \frac{\langle D_x \rangle_m}{R}$$

$\langle\ \rangle_m$ means that the average is considered over the bending magnet only

Appendix: First Energy-Phase Equation



$$f_{RF} = h f_r \quad \Rightarrow \quad \Delta\phi = -h\,\Delta\theta \quad \textit{with} \quad \theta = \int \omega \, dt$$

particle ahead arrives earlier
=> smaller RF phase

For a given particle with respect to the reference one:

$$\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $$\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega}{dp}\right)_s$$ and $$E^2 = E_0^2 + p^2c^2$$ $$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets:

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

Appendix: Second Energy-Phase Equation

The rate of energy gained by a particle is: $$\frac{dE}{dt} = e\hat{V}\sin\phi\,\frac{\omega_r}{2\pi}$$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{E}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left-hand side to first order:

$$\Delta\left(E T_r\right) \cong E\,\Delta T_r + T_{rs}\,\Delta E = \Delta E\, T_r + T_{rs}\,\Delta E = \frac{d}{dt}\left(T_{rs}\,\Delta E\right)$$

leads to the second energy-phase equation:

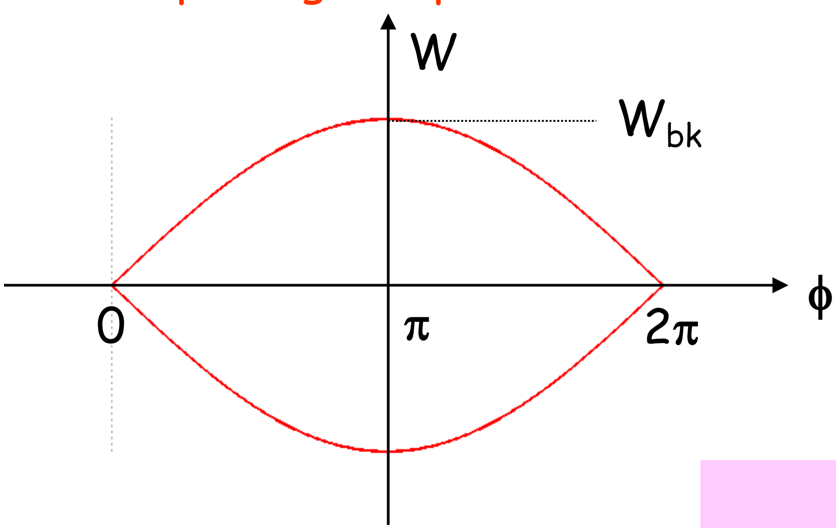
$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\left(\sin\phi - \sin\phi_s\right)$$

Appendix: Stationary Bucket - Separatrix

This is the case $\sin\phi_s = 0$ (no acceleration) which means $\phi_s = 0$ or π . The equation of the separatrix for $\phi_s = \pi$ (above transition) becomes:

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos\phi = \Omega_s^2 \qquad\qquad \frac{\dot{\phi}^2}{2} = 2\Omega_s^2 \sin^2\frac{\phi}{2}$$

Replacing the phase derivative by the (canonical) variable W:



$$W = \frac{\Delta E}{\omega_{rf}} = -\frac{p_s R_s}{h\eta\,\omega_{rf}}\dot{\phi}$$

and introducing the expression for Ω_s leads to the following equation for the separatrix:

with $C = 2\pi R_s$

$$W = \pm \frac{C}{\pi h c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}\sin\frac{\phi}{2} = \pm W_{bk}\sin\frac{\phi}{2}$$

Stationary Bucket (2)

Setting $\phi=\pi$ in the previous equation gives the height of the bucket:

$$W_{bk} = \frac{C}{\pi h c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}$$

This results in the maximum energy acceptance:

$$\Delta E_{\max} = \omega_{rf} W_{bk} = \beta_s\sqrt{2\frac{-e\hat{V}_{RF}E_s}{\pi h\eta}}$$

The area of the bucket is: $A_{bk} = 2\int_0^{2\pi} W\, d\phi$

Since: $\int_0^{2\pi}\sin\frac{\phi}{2}\, d\phi = 4$

one gets: $$A_{bk} = 8W_{bk} = 8\frac{C}{\pi h c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}} \quad\longrightarrow\quad W_{bk} = \frac{A_{bk}}{8}$$

Appendix: Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

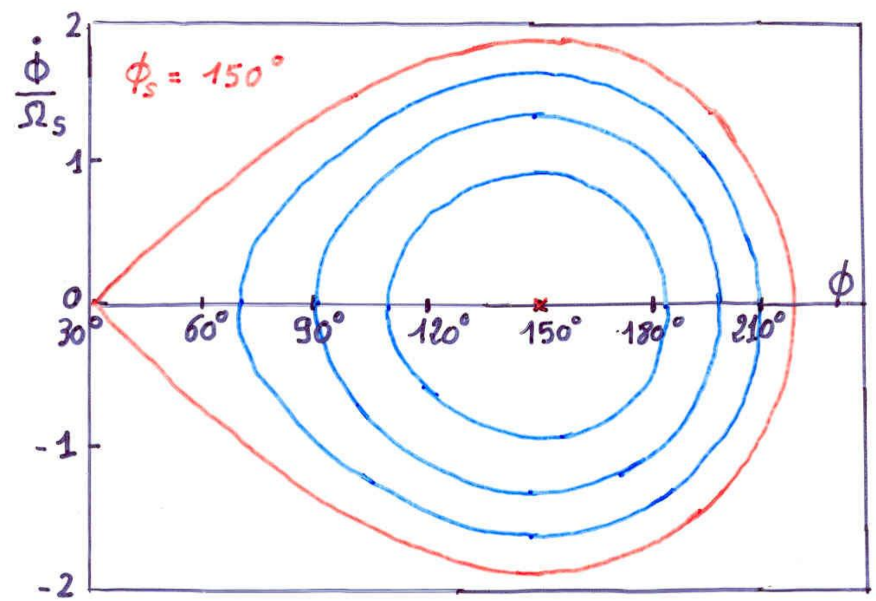
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{\left(\Delta\phi\right)^2}{2} = I'$$

(the variable is $\Delta\phi$, and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches $\pi-\phi_s$ the force goes to zero and beyond it becomes non restoring.
Hence $\pi-\phi_s$ is an extreme amplitude for a stable motion which in the phase space($\frac{\dot\phi}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot\phi^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme when $\ddot{\phi} = 0$, hence corresponding to $\phi = \phi_s$.

Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{\max} = 2\Omega_s^2 \left\{ 2 + \left(2\phi_s - \pi\right)\tan\phi_s \right\}$$

That translates into an acceptance in energy:

$$\boxed{\left(\frac{\Delta E}{E_s}\right)_{\max} = \pm\beta\sqrt{\frac{e\hat{V}}{\pi h \eta E_s} G(\phi_s)}}$$

$$G(\phi_s) = \left[2\cos\phi_s + \left(2\phi_s - \pi\right)\sin\phi_s\right]$$

This “RF acceptance” depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

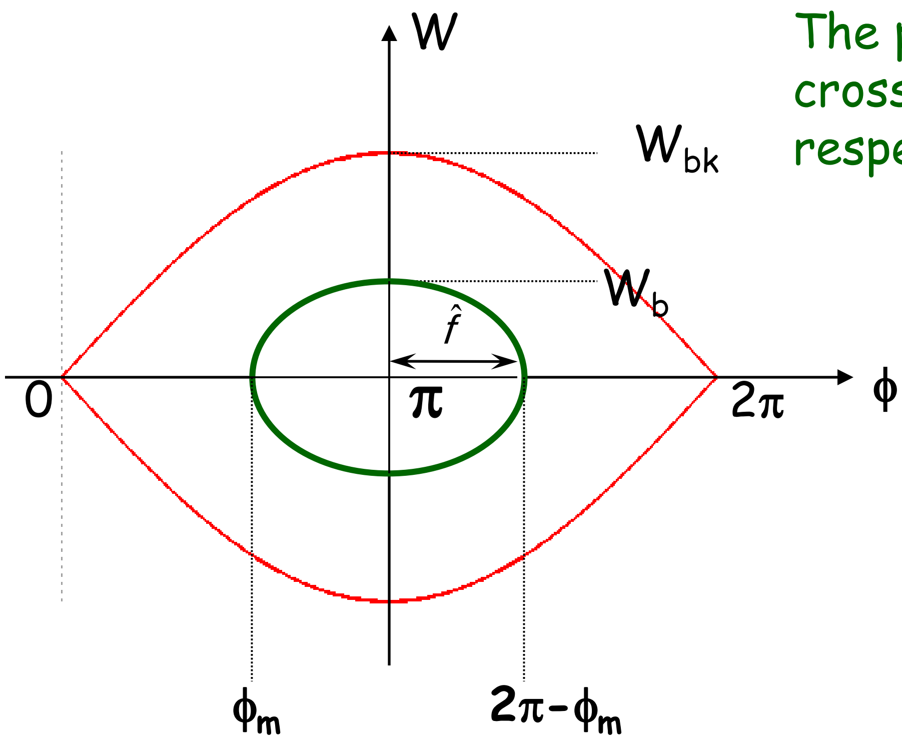
It’s largest for $\phi_s=0$ and $\phi_s=\pi$ (no acceleration, depending on η).

Need a higher RF voltage for higher acceptance.

Bunch Matching into a Stationary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I \xrightarrow{\phi_s = \pi} \boxed{\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = I}$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = \Omega_s^2\cos\phi_m$$

$$\dot{\phi} = \pm\Omega_s\sqrt{2\left(\cos\phi_m - \cos\phi\right)}$$

$$\boxed{W = \pm W_{bk}\sqrt{\cos^2\frac{\phi_m}{2} - \cos^2\frac{\phi}{2}}}$$

$$\cos(\phi) = 2\cos^2\frac{\phi}{2} - 1$$

Bunch Matching into a Stationary Bucket (2)

Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = W_{bk} \cos\frac{\phi_m}{2} = W_{bk} \sin\frac{\hat{\phi}}{2}$$

or:

$$W_b = \frac{A_{bk}}{8} \cos\frac{\phi_m}{2}$$

$$\longrightarrow \quad \left(\frac{\Delta E}{E_s}\right)_b = \left(\frac{\Delta E}{E_s}\right)_{RF} \cos\frac{\phi_m}{2} = \left(\frac{\Delta E}{E_s}\right)_{RF} \sin\frac{\hat{\phi}}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a shorter bunch (ϕ_m close to π, $\hat{\phi}$ small) will require a bigger RF acceptance, hence a higher voltage

For small oscillation amplitudes the equation of the ellipse reduces to:

$$W = \frac{A_{bk}}{16}\sqrt{\hat{\phi}^2 - (\Delta\phi)^2} \longrightarrow \left(\frac{16\,W}{A_{bk}\hat{\phi}}\right)^2 + \left(\frac{\Delta\phi}{\hat{\phi}}\right)^2 = 1$$

Ellipse area is called longitudinal emittance

$$A_b = \frac{\pi}{16} A_{bk} \hat{\phi}^2$$