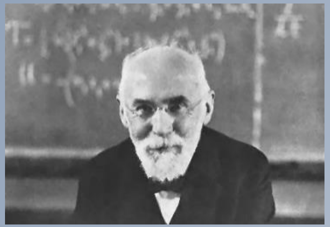
Superconducting RF Systems I

RF basics & principles of RF Superconductivity

Erk JENSEN, CERN



Hendrik A. Lorentz
1853 – 1928

Lorentz force

- A charged particle moving with velocity $\vec{v} = \frac{\vec{p}}{m\,\gamma}$ through an electromagnetic field in vacuum experiences the Lorentz force $\boldsymbol{\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)}$.
- The total energy of this particle is $W = \sqrt{(mc^2)^2 + (pc)^2} = \gamma\, mc^2$, the kinetic energy is $W_{kin} = mc^2(\gamma - 1)$.
- The role of acceleration is to increase W.
- Change of W (by differentiation):

$$W dW = c^2 \vec{p} \cdot d\vec{p} = qc^2 \vec{p} \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) dt = qc^2 \vec{p} \cdot \vec{E}\, dt$$

$$\boldsymbol{dW = q\vec{v} \cdot \vec{E} dt}$$

Note: **Only the electric field can change the particle energy!**

James Clerk Maxwell
1831 – 1879

Maxwell’s equations in vacuum

Source-free:

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 \qquad \nabla \cdot \vec{E} = 0$$

curl (rot, $\nabla \times$) of 3rd equation and $\frac{\partial}{\partial t}$ of 1st equation:

$$\nabla \times \nabla \times \vec{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0.$$

Using the vector identity $\nabla \times \nabla \times \vec{E} = \nabla\nabla \cdot \vec{E} - \nabla^2\vec{E}$ and the 4th Maxwell equation, this yields:

$$\boldsymbol{\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0},$$

i.e. the 4-dimensional Laplace equation.

Homogeneous plane wave

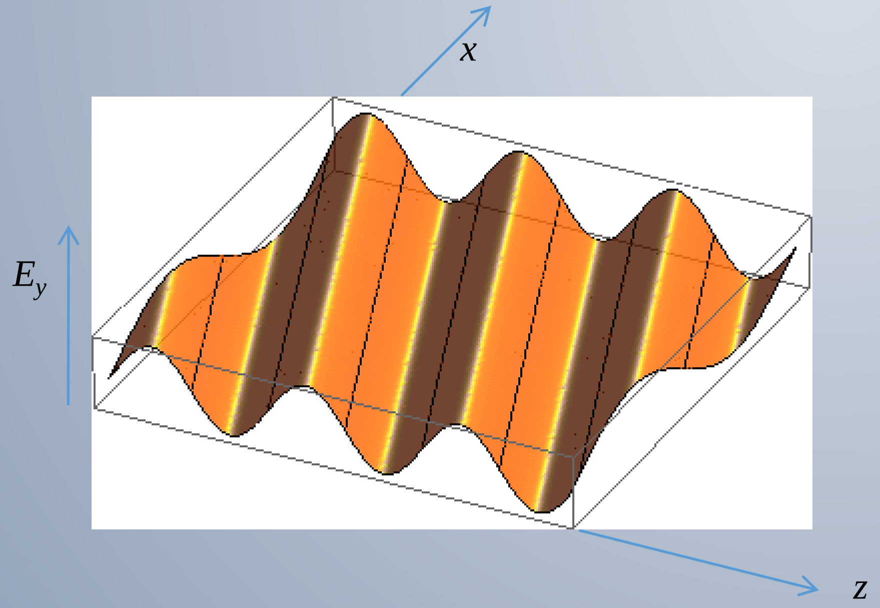
$$\vec{E} \propto \vec{u}_y \cos\left(\omega t - \vec{k} \cdot \vec{r}\right)$$
$$\vec{B} \propto \vec{u}_x \cos\left(\omega t - \vec{k} \cdot \vec{r}\right)$$
$$\vec{k} \cdot \vec{r} = \frac{\omega}{c}(z \cos\varphi + x \sin\varphi)$$

Wave vector $\vec{k}$:

the direction of $\vec{k}$ is the direction of propagation,

the length of $\vec{k}$ is the phase shift per unit length.

$\vec{k}$ behaves like a vector.



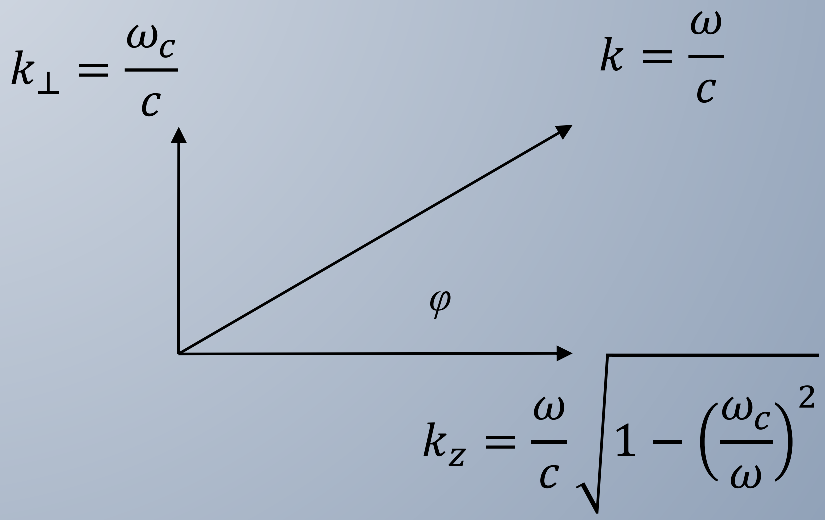
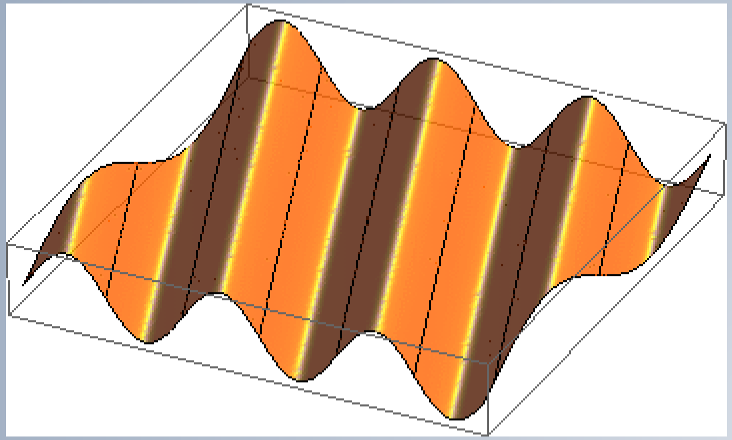



Photo:
Reidar Hahn

Superposition of 2 homogeneous plane waves



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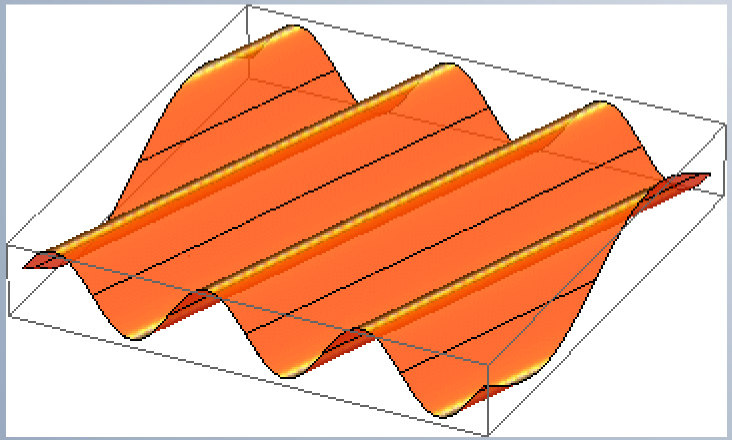

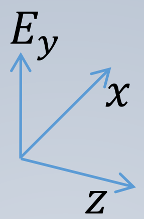

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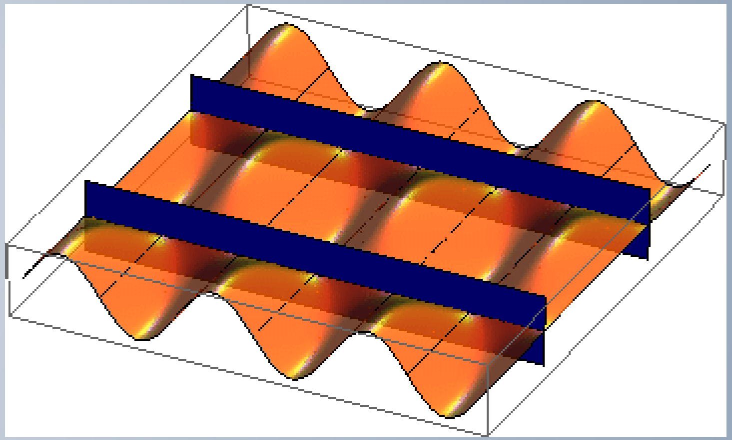


Metallic walls may be inserted where $E_y \equiv 0$

without perturbing the fields.

Note the standing wave in x-direction!

This way one gets a hollow rectangular waveguide.

Photo: Reidar Hahn

Rectangular waveguide

Fundamental (TE_{10} or H_{10}) mode
in a standard rectangular waveguide.

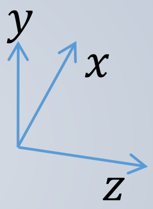
Example 1: "S-band": 2.6 GHz ... 3.95 GHz,

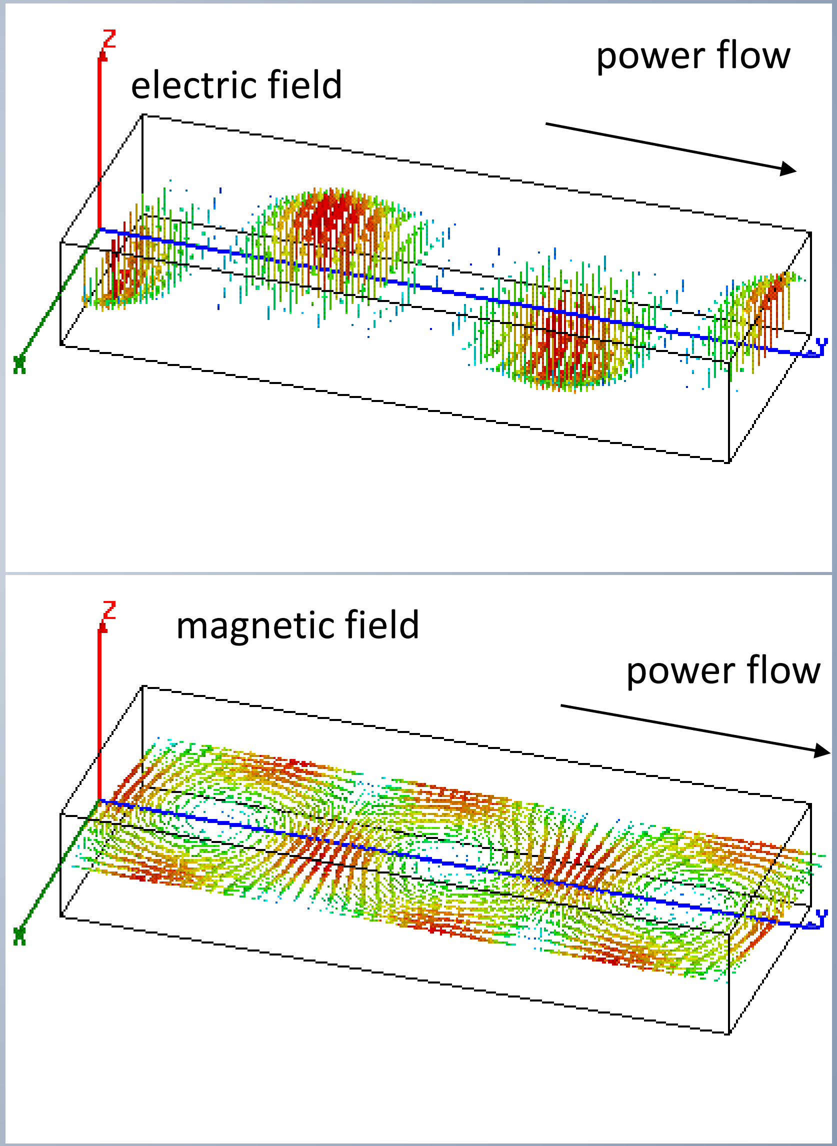
Waveguide type WR284 (2.84" wide), dimensions:
72.14 mm x 34.04 mm.
cut-off: $f_c = 2.078$ GHz.

Example 2: "L-band" : 1.13 GHz ... 1.73 GHz,

Waveguide type WR650 (6.5" wide), dimensions:
165.1 mm x 82.55 mm.
cut-off: $f_c = 0.908$ GHz.

Both these pictures correspond to operation at 1.5 f_c.

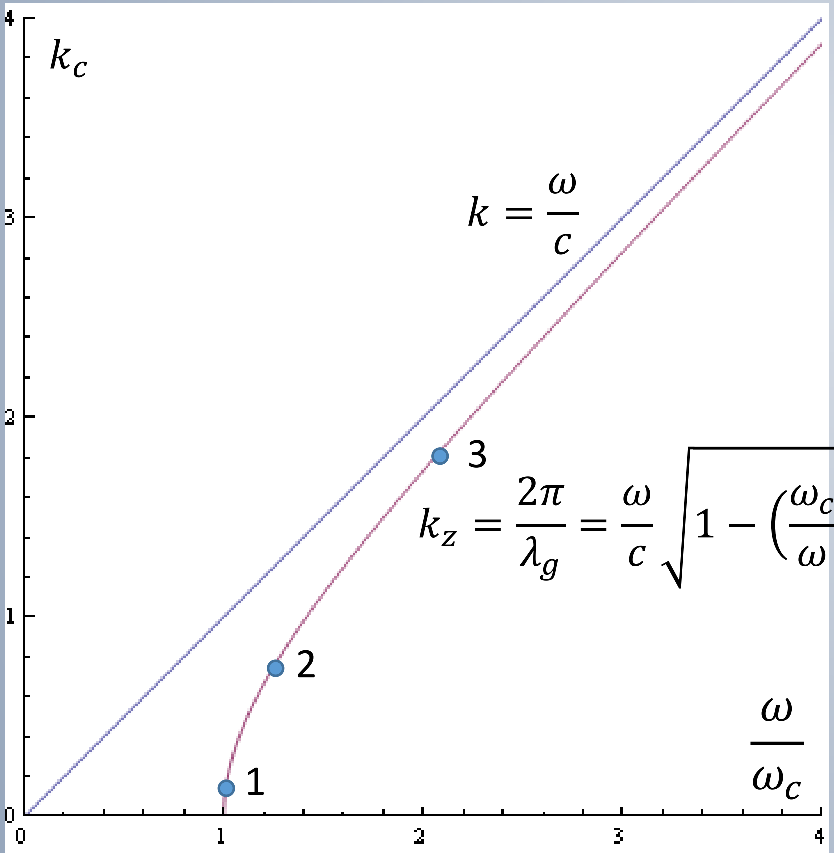




power flow: $\frac{1}{2}\mathrm{Re}\left\{\iint \vec{E} \times \vec{H}^* \cdot \mathrm{d}\vec{A}\right\}$

Waveguide dispersion – phase velocity $v_{\varphi,z}$

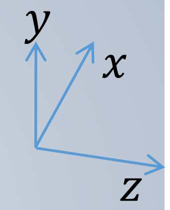
The phase velocity $v_{\varphi,z}$ is the speed at which the crest (or zero-crossing) travels in z-direction.
Note on the 3 animations on the right that, at constant f, $v_{\varphi,z} \propto \lambda_g$. Note also that at $f = f_c$, $v_{\varphi,z} = \infty$!
With $f \to \infty$, $v_{\varphi,z} \to c$!



$$k_z = \frac{2\pi}{\lambda_g} = \frac{\omega}{c}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} = \frac{\omega}{v_{\varphi,z}}$$

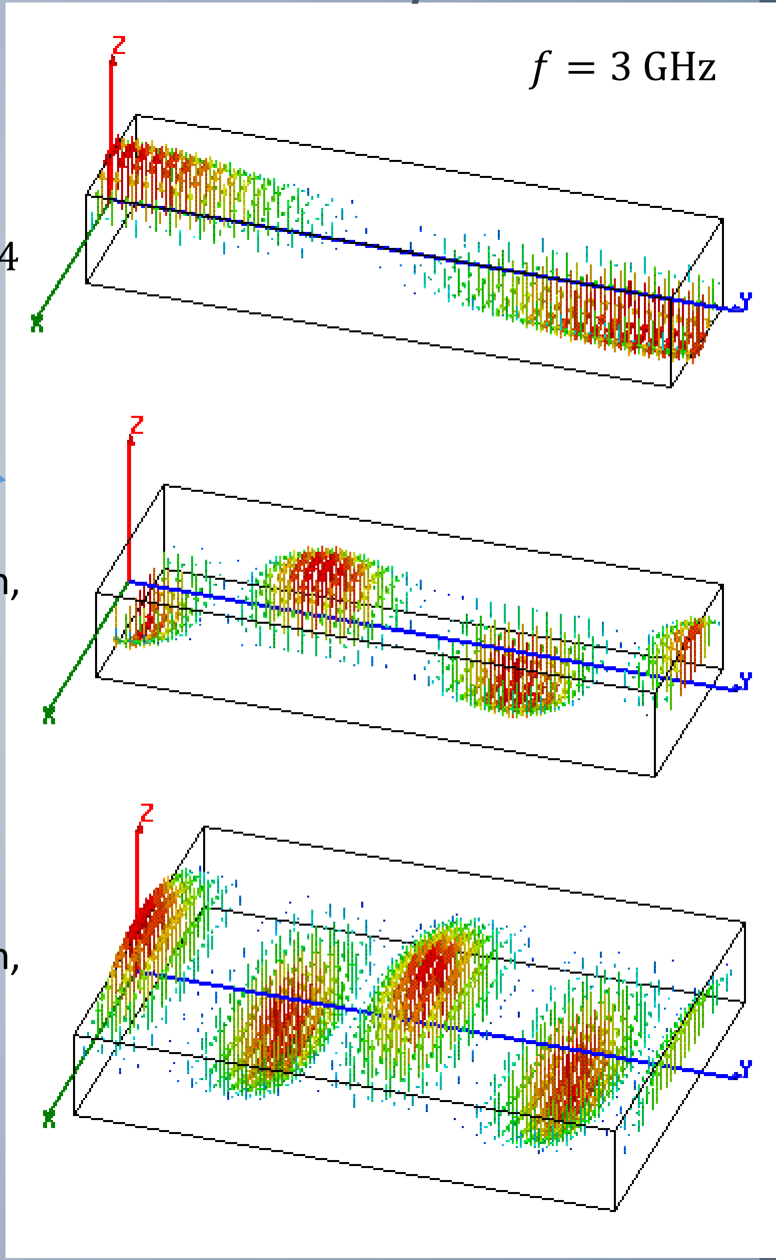
cutoff: $f_c = \frac{c}{2a}, v_{\varphi,z} = \infty$

1:
a = 52 mm,
$f/f_c = 1.04$



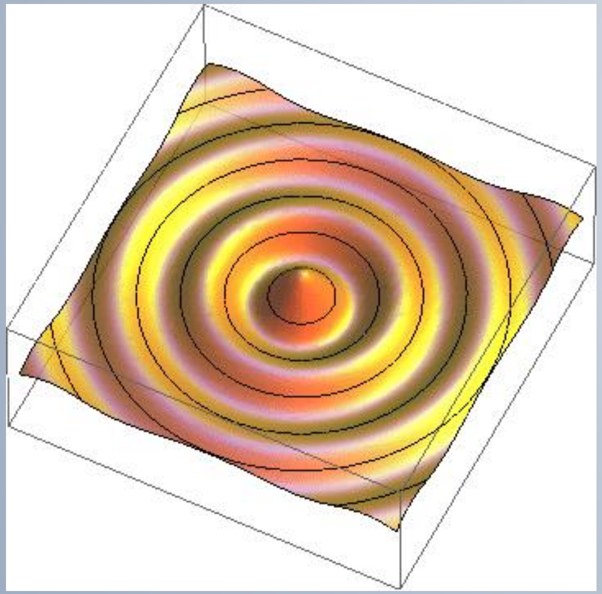
2:
a = 72.14 mm,
$f/f_c = 1.44$

3:
a = 144.3 mm,
$f/f_c = 2.88$

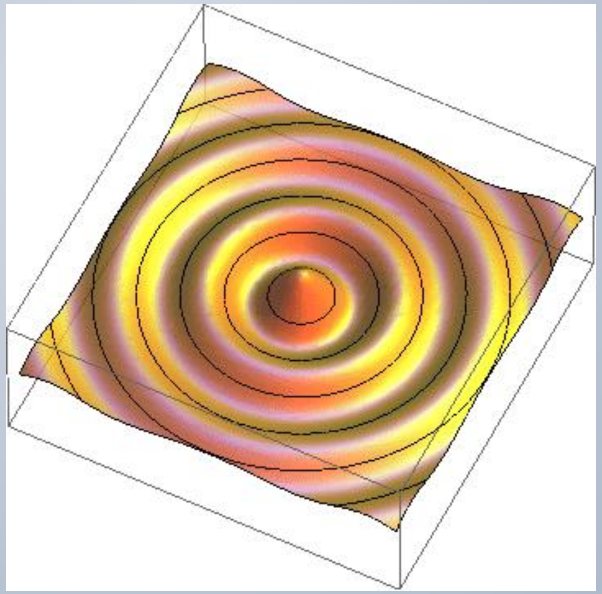


Radial waves

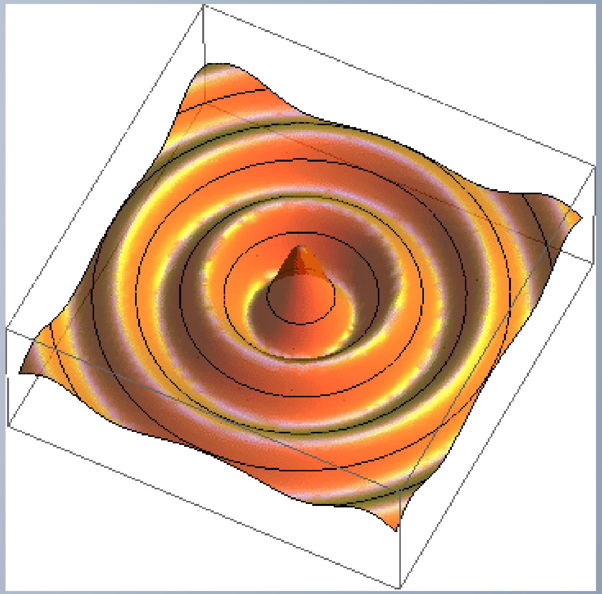
Also radial waves may be interpreted as superposition of plane waves. The superposition of an outward and an inward radial wave can result in the field of a round hollow waveguide.



$$E_z \propto H_n^{(2)}(k_\rho \rho)\cos(n\varphi)$$



$$E_z \propto H_n^{(1)}(k_\rho \rho)\cos(n\varphi)$$

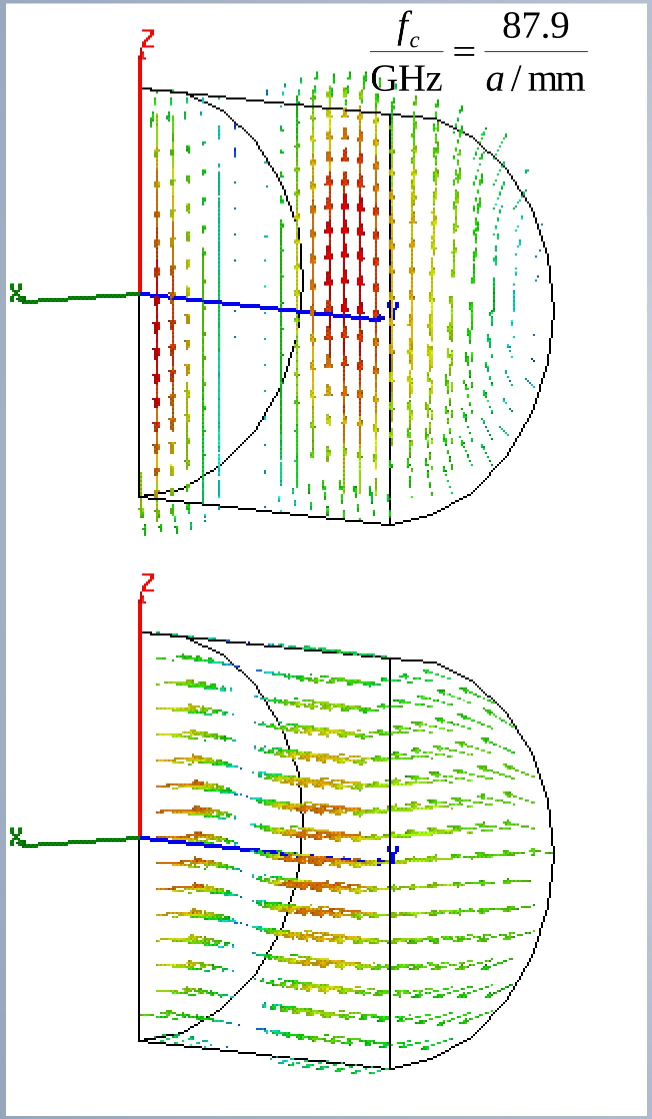


$$E_z \propto J_n(k_\rho \rho)\cos(n\varphi)$$

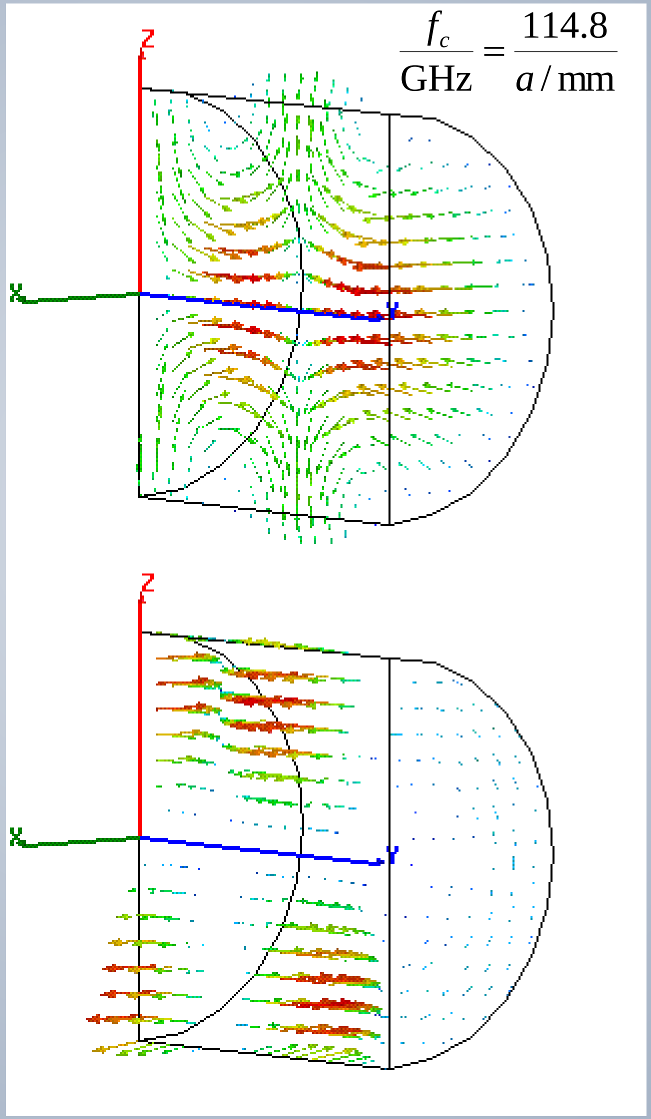
Round waveguide

$f/f_c = 1.44$

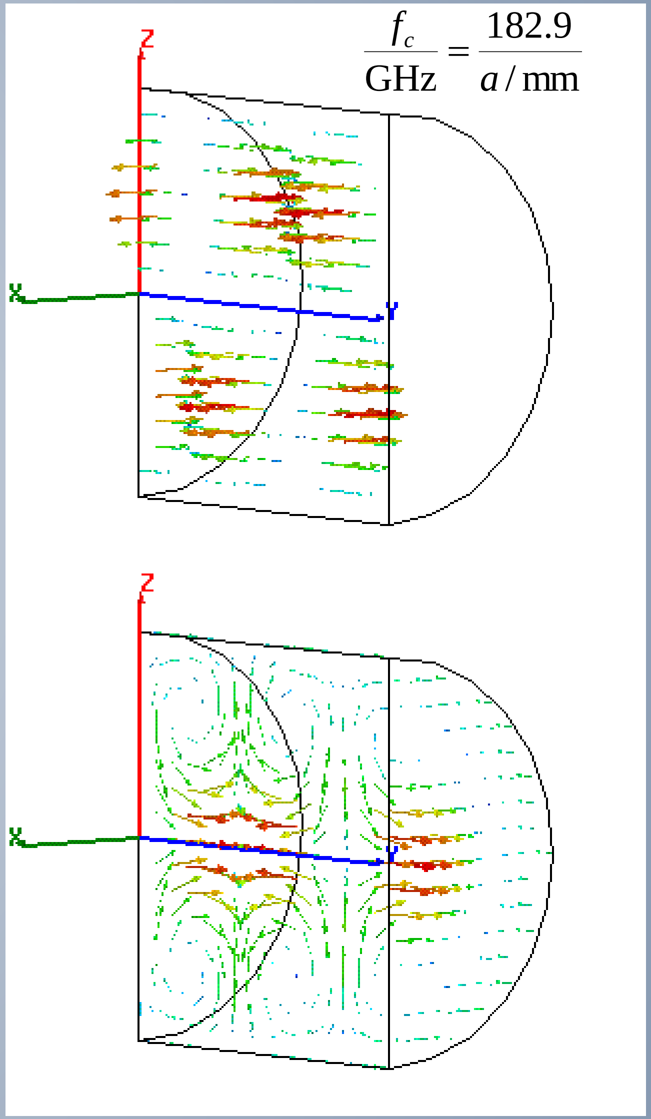
TE$_{11}$ – fundamental



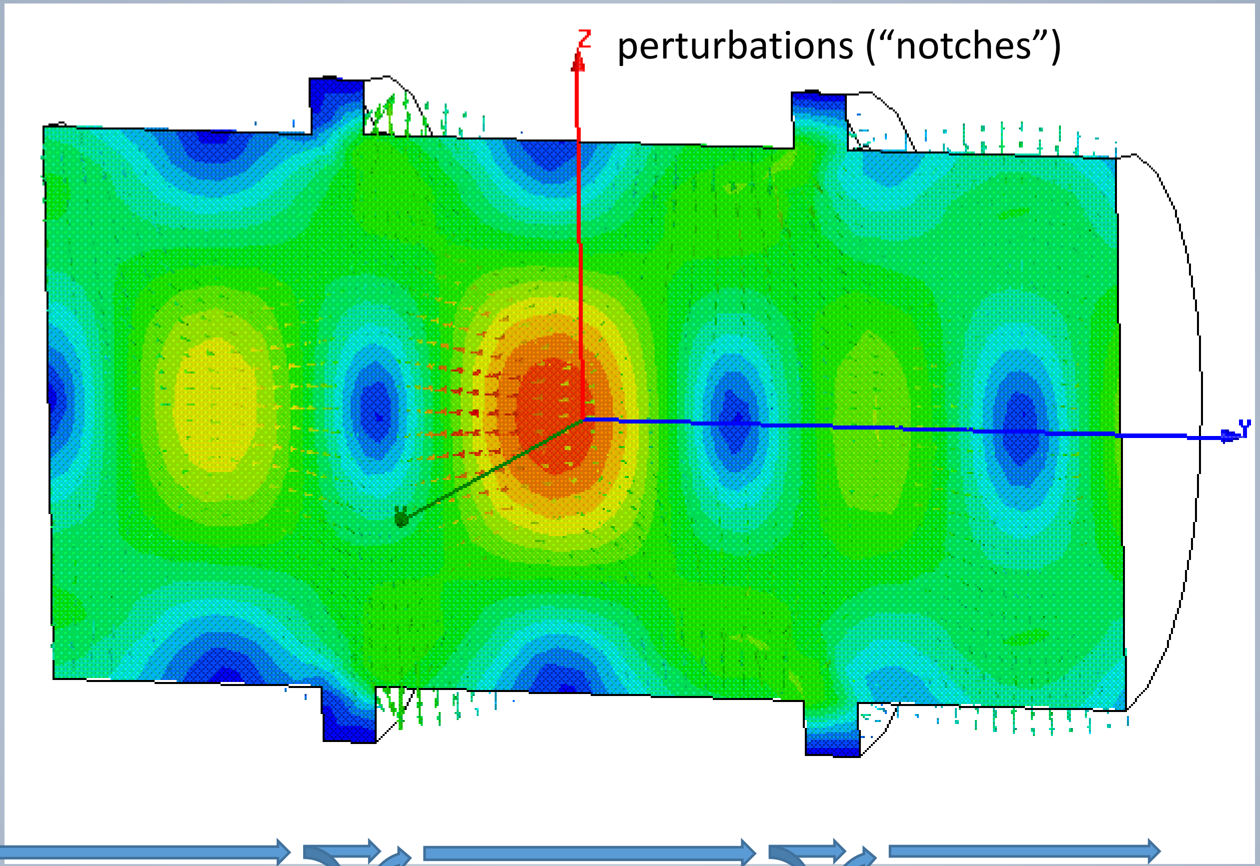
TM$_{01}$ – axial field

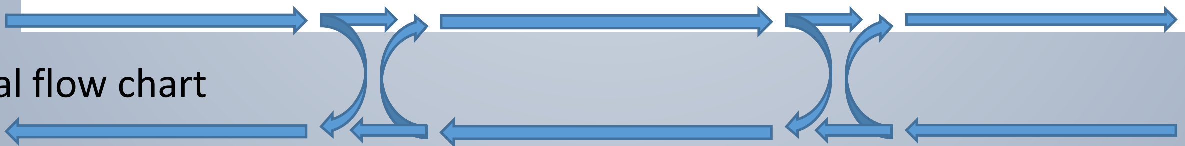


TE$_{01}$ – low loss



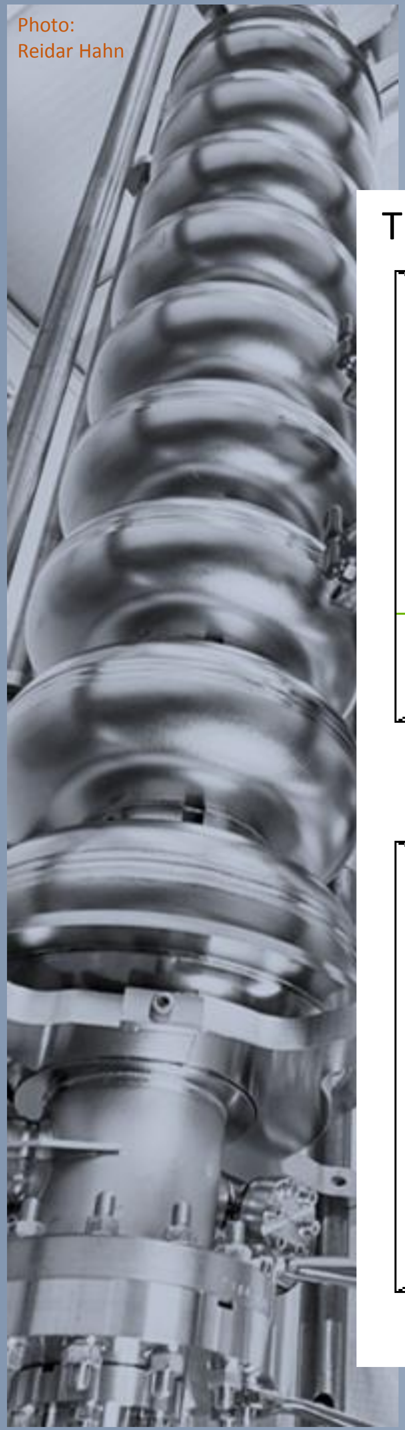
Waveguide perturbed by notches



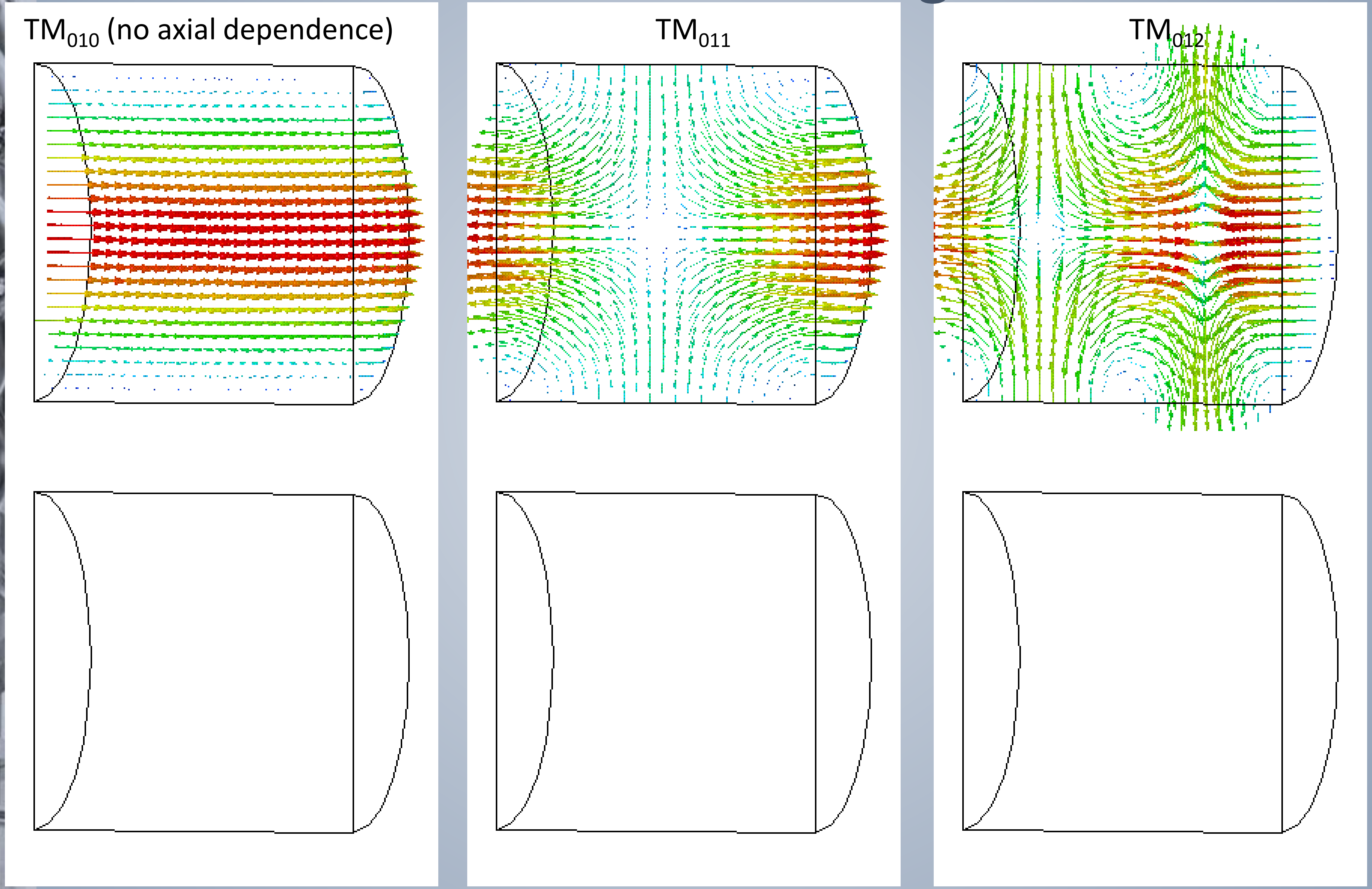


Signal flow chart

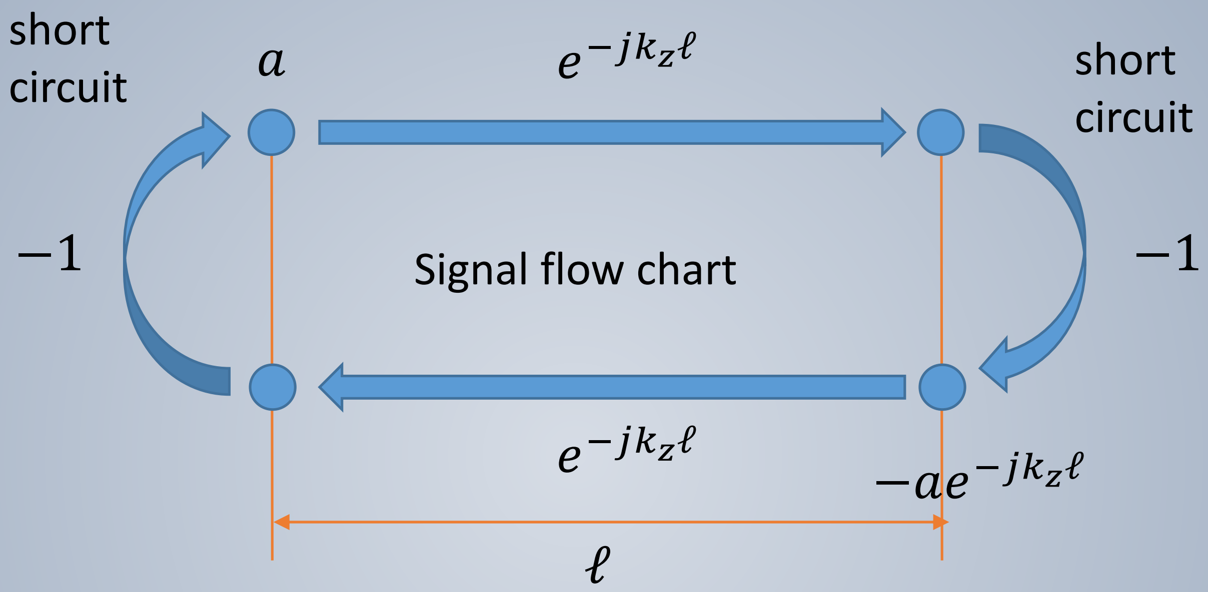
Reflections from notches lead to a superimposed standing wave pattern.
“Trapped mode”



Short-circuited waveguide



Single WG mode between two shorts



Eigenvalue equation for field amplitude a:

$$a = e^{-jk_z 2\ell} a$$

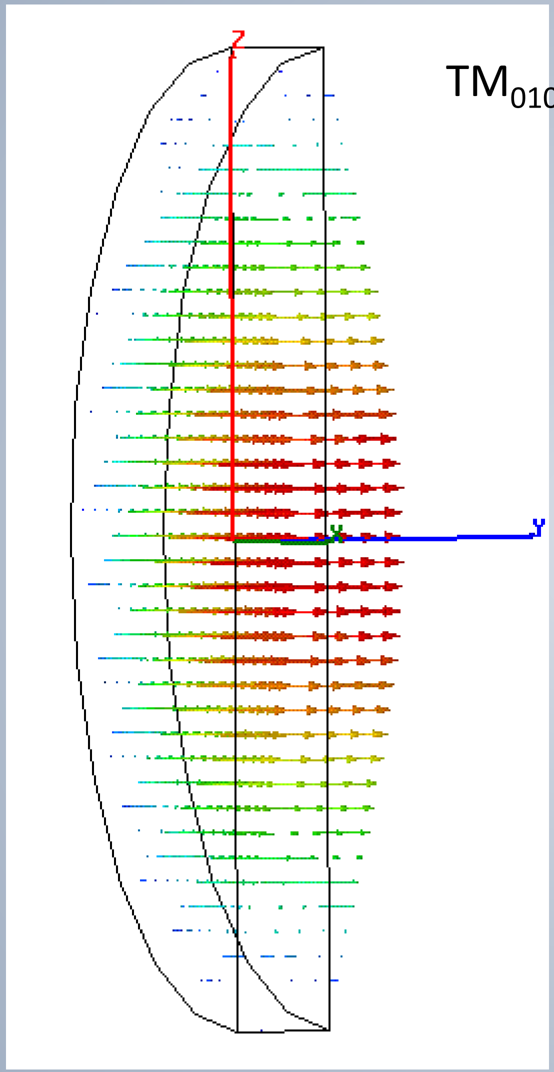
Non-vanishing solutions exist for $2k_z\ell = 2\pi m$:

With $k_z = \frac{\omega}{c}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$, this becomes $f_0^2 = f_c^2 + \left(c\,\frac{m}{2\ell}\right)^2$.

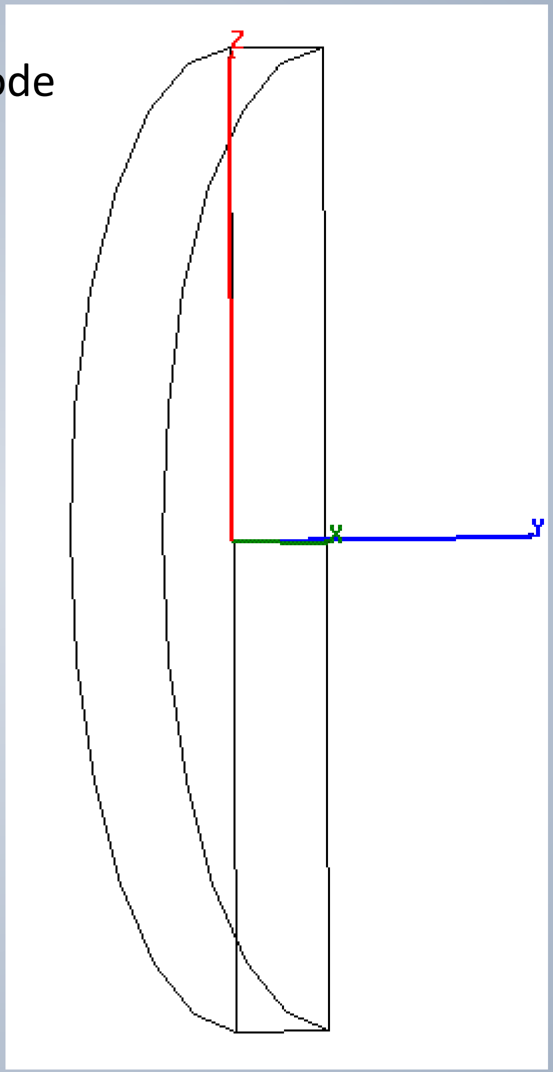

Photo:
Reidar Hahn

Simple pillbox (only 1/2 shown)

TM$_{010}$-mode



electric field (purely axial)

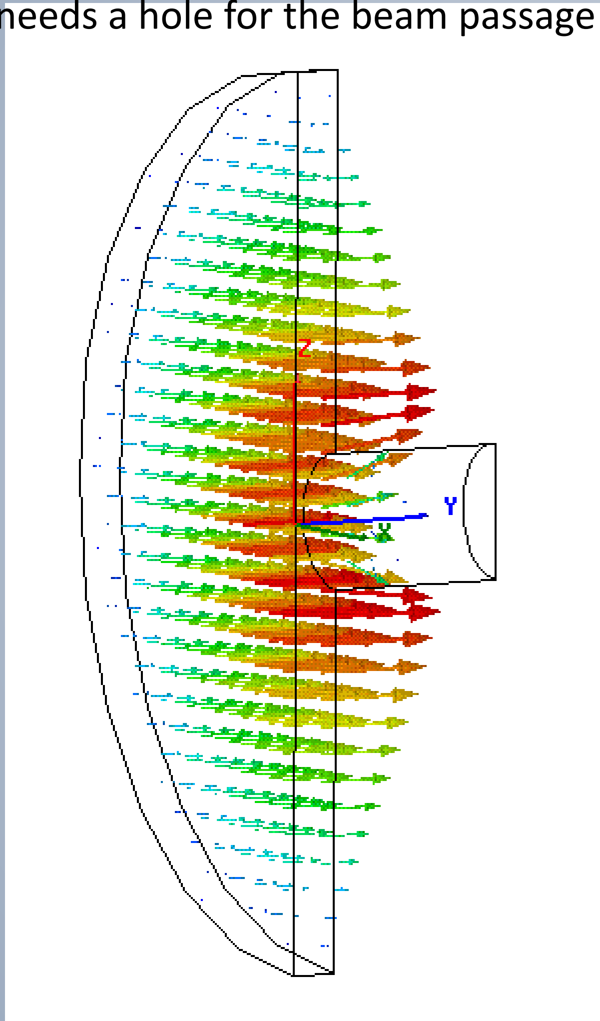


magnetic field (purely azimuthal)

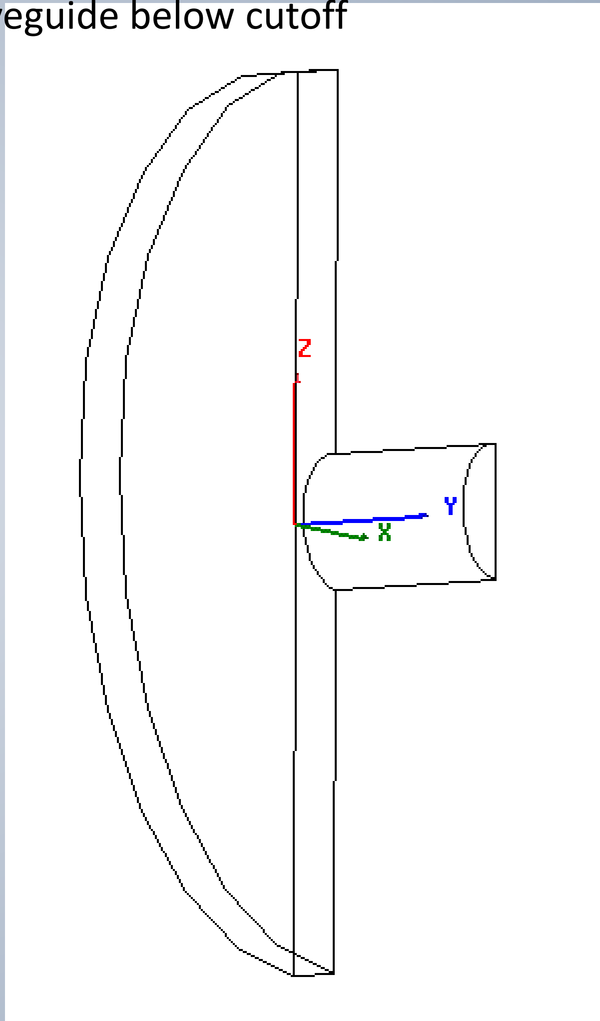
Pillbox with beam pipe

TM_{010}-mode (only 1/4 shown)

One needs a hole for the beam passage – circular waveguide below cutoff



electric field

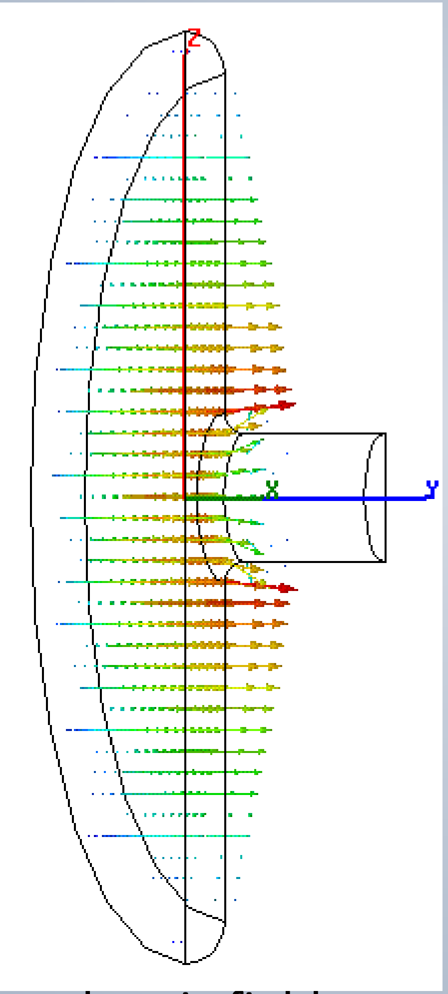


magnetic field

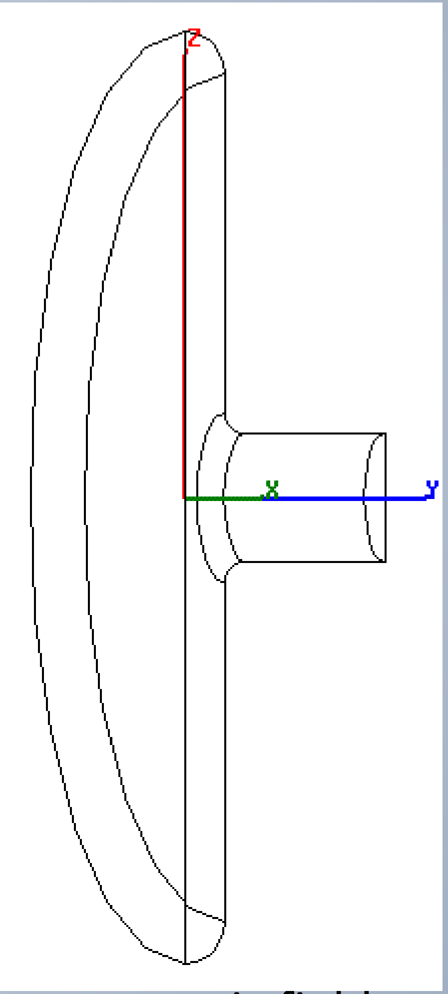
A more practical pillbox cavity

TM_{010}-mode (only 1/4 shown)

Rounding of sharp edges (to reduce field enhancement!)



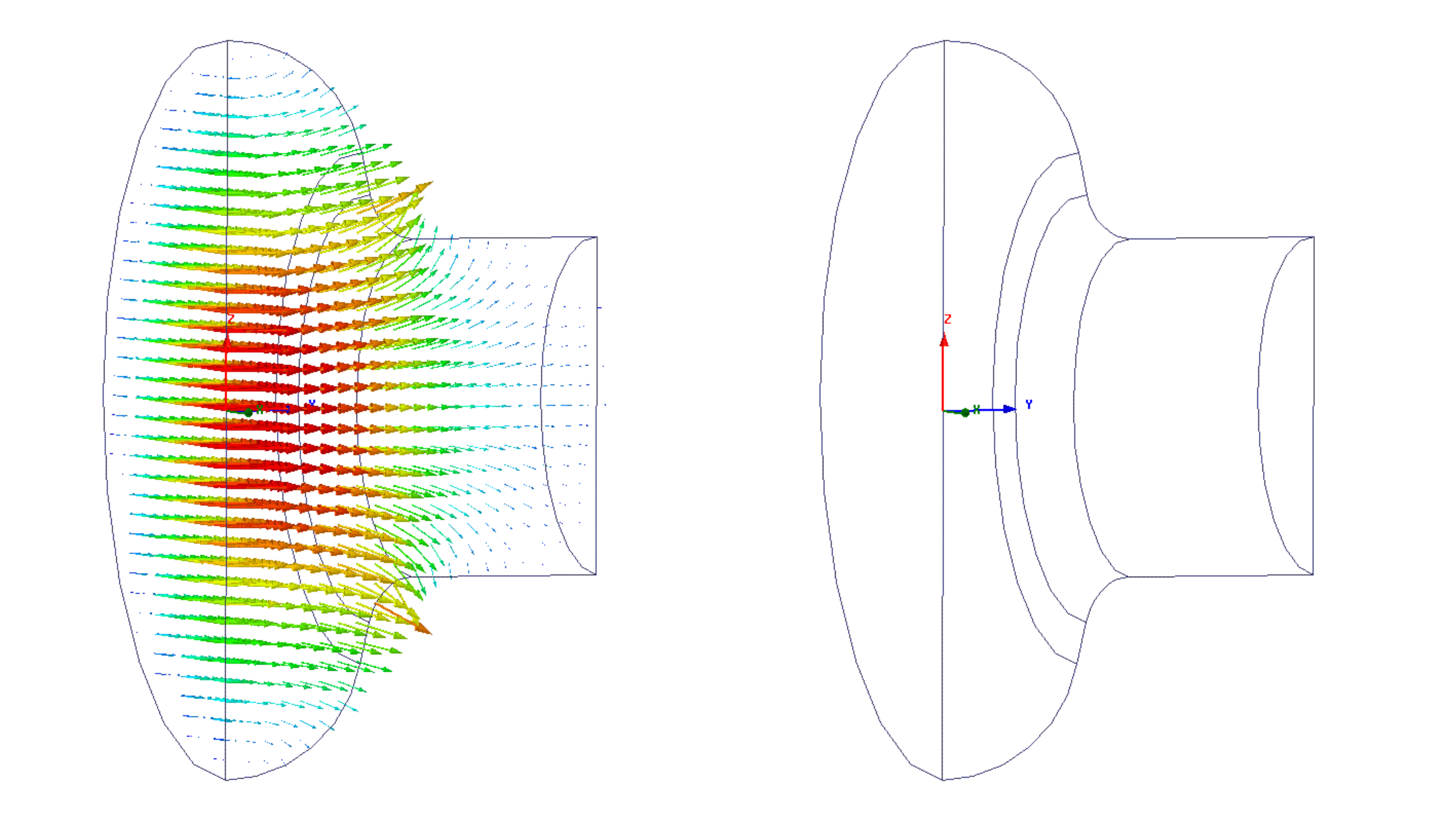
electric field



magnetic field

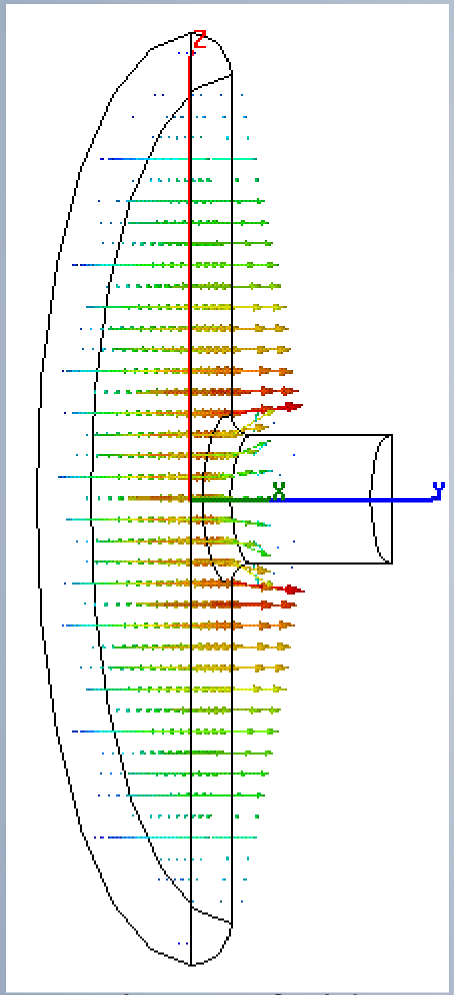
A (real) elliptical cavity

TM_{010}-mode (only 1/4 shown)



electric field magnetic field

Acceleration voltage and R/Q



electric field

- I define

$$V_{acc} = \int_{-\infty}^{\infty} E_z e^{j\frac{\omega}{\beta c}z}\, dz\,.$$

- The exponential factor accounts for the variation of the field while particles with velocity βc are traversing the cavity gap.
- With this definition, V_{acc} is generally complex – this becomes important with more than one gap (cell).
- For the time being we are only interested in $|V_{acc}|$.
- The square of the acceleration voltage $|V_{acc}|^2$ is proportional to the stored energy W; the proportionality constant defines the quantity called "R-upon-Q":

$$\frac{R}{Q} = \frac{|V_{acc}|^2}{2\omega_0 W}\,.$$

› **Attention – different definitions are used in literature!**

Transit time factor

- The transit time factor is the ratio of the acceleration voltage to the (non-physical) voltage a particle with infinite velocity would see:

$$TT = \frac{|V_{acc}|}{\left|\int E_z\, dz\right|} = \frac{\left|\int E_z e^{j\frac{\omega}{\beta c}z}\, dz\right|}{\left|\int E_z\, dz\right|}.$$

- The transit time factor of an ideal pillbox cavity (no axial field dependence) of height (gap length) h is:

$$TT = \frac{\sin\left(\frac{\chi_{01} h}{2a}\right)}{\frac{\chi_{01} h}{2a}}$$

(remember: $\omega_0 = \frac{2\pi c}{\lambda} = \frac{\chi_{01} c}{a}$)

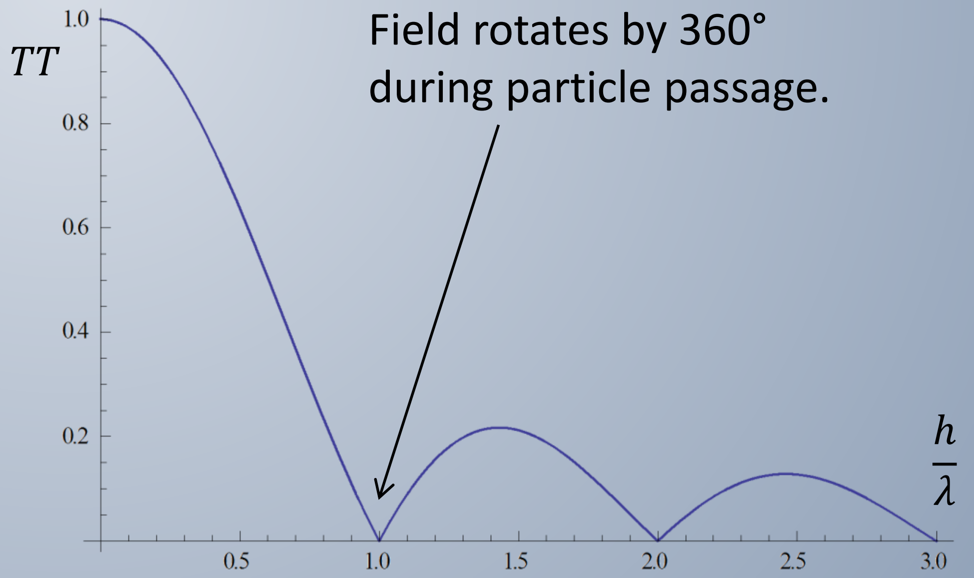



Photo: Reidar Hahn

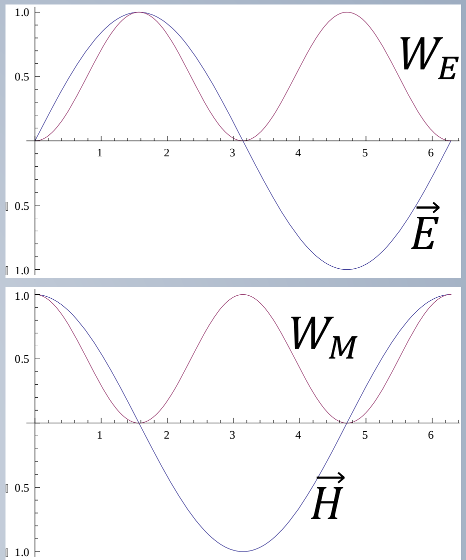
Stored energy

- The energy stored in the electric field is

$$W_E = \iiint\limits_{\text{cavity}} \frac{\varepsilon}{2} \left|\vec{E}\right|^2 dV\,.$$

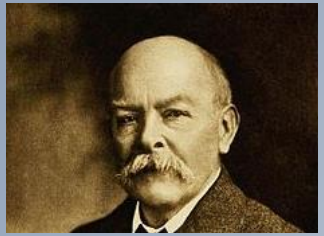
- The energy stored in the magnetic field is

$$W_M = \iiint\limits_{\text{cavity}} \frac{\mu}{2} \left|\vec{H}\right|^2 dV\,.$$



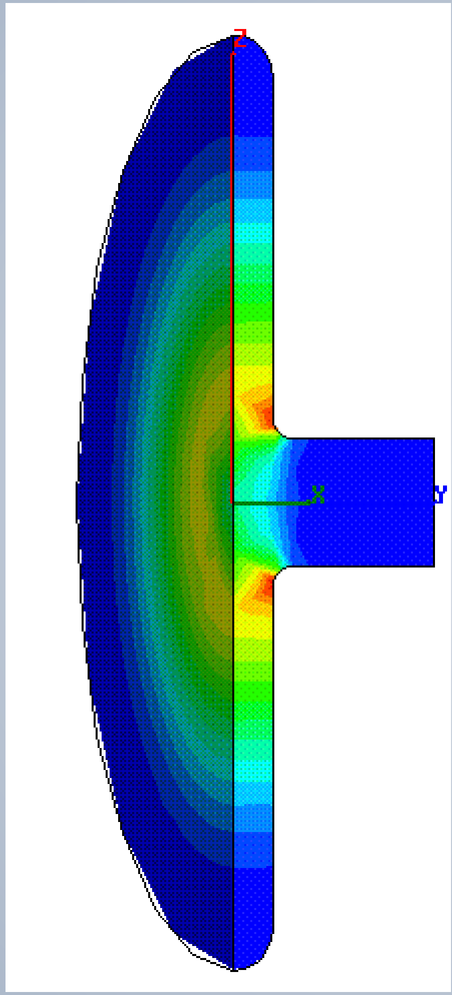
- Since $\vec{E}$ and $\vec{H}$ are $90°$ out of phase, the stored energy continuously swaps from electric energy to magnetic energy.
- On average, electric and magnetic energy must be equal.
- In steady state, the Poynting vector describes this energy flux.
- In steady state, the total energy stored (constant) is

$$W = \iiint\limits_{cavity} \left(\frac{\varepsilon}{2}\left|\vec{E}\right|^2 + \frac{\mu}{2}\left|\vec{H}\right|^2\right) dV.$$

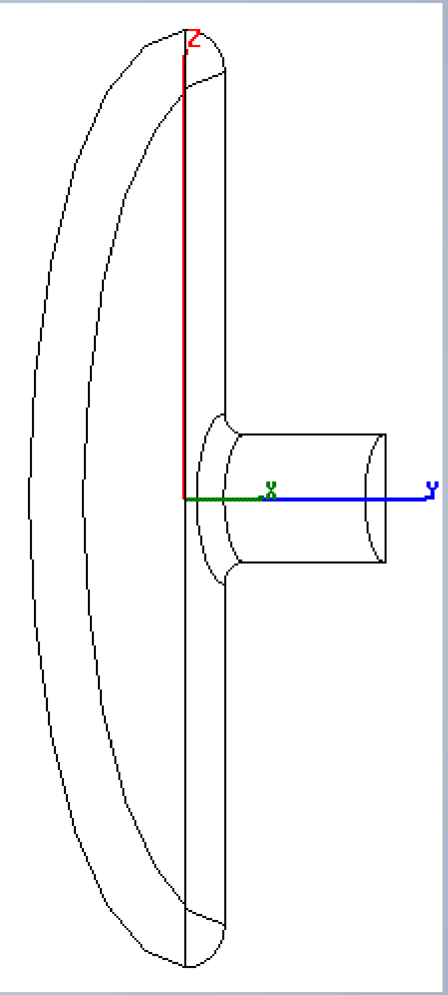


John Henry Poynting
1852 – 1914

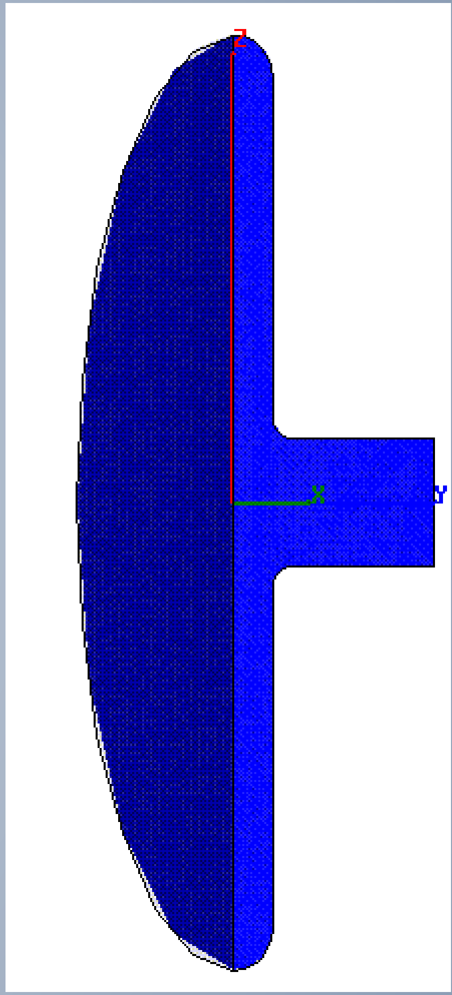
Stored energy and Poynting vector



electric field energy



Poynting vector



magnetic field energy

Photo: Reidar Hahn

Wall losses & Q_0

- The losses P_{loss} are proportional to the stored energy W.
- The tangential $\vec{H}$ on the surface is linked to a surface current $\vec{J}_A = \vec{n} \times \vec{H}$ (flowing in the skin depth $\delta = \sqrt{2 / (\omega\mu\sigma)}$).
- This surface current $\vec{J}_A$ sees a surface resistance R_s, resulting in a local power density $R_s|H_t|^2$ flowing into the wall.
- R_s is related to skin depth δ as $\delta\sigma R_s = 1$.
  - Cu at 300 K has $\sigma \approx 5.8 \cdot 10^7$S/m, leading to $R_s \approx 8$ mΩ at 1 GHz, scaling with $\sqrt{\omega}$.
  - Nb at 2 K has a typical $R_s \approx 10$ nΩ at 1 GHz, scaling with ω^2.
- The total wall losses result from $P_{\text{loss}} = \iint_{wall} R_s|H_t|^2\, dA$.
- The cavity Q_0 (caused by wall losses) is defined as $Q_0 = \frac{\omega_0 W}{P_{\text{loss}}}$.
- Typical Q_0 values:
  - Cu at 300 K (normal-conducting): $\mathcal{O}(10^3 \dots 10^5)$, **should** improve at cryogenic T by a factor $\sqrt{RRR}$.
  - Nb at 2 K (superconducting): $\mathcal{O}(10^9 \dots 10^{11})$

No! Anomalous skin effect!
improves only by a factor ≈ 10!


Photo: Reidar Hahn

Shunt impedance

- Also the power loss P_{loss} is also proportional to the square of the acceleration voltage $|V_{acc}|^2$; the proportionality constant defines the "shunt impedance"

$$R = \frac{|V_{acc}|^2}{2\,P_{\text{loss}}}.$$

› **Attention, also here different definitions are used!**

- Traditionally, the shunt impedance is the quantity to optimize in order to minimize the power required for a given gap voltage.

- Now the previously introduced term "R-upon-Q" makes sense:

$$\left(\frac{R}{Q}\right) = R/Q$$

Geometric factor

- With

$$Q_0 = \frac{\omega_0 W}{\iint\limits_{wall} R_s |H_t|^2 \, dA},$$

and assuming an average surface resistance R_s, one can introduce the "geometric factor" G as

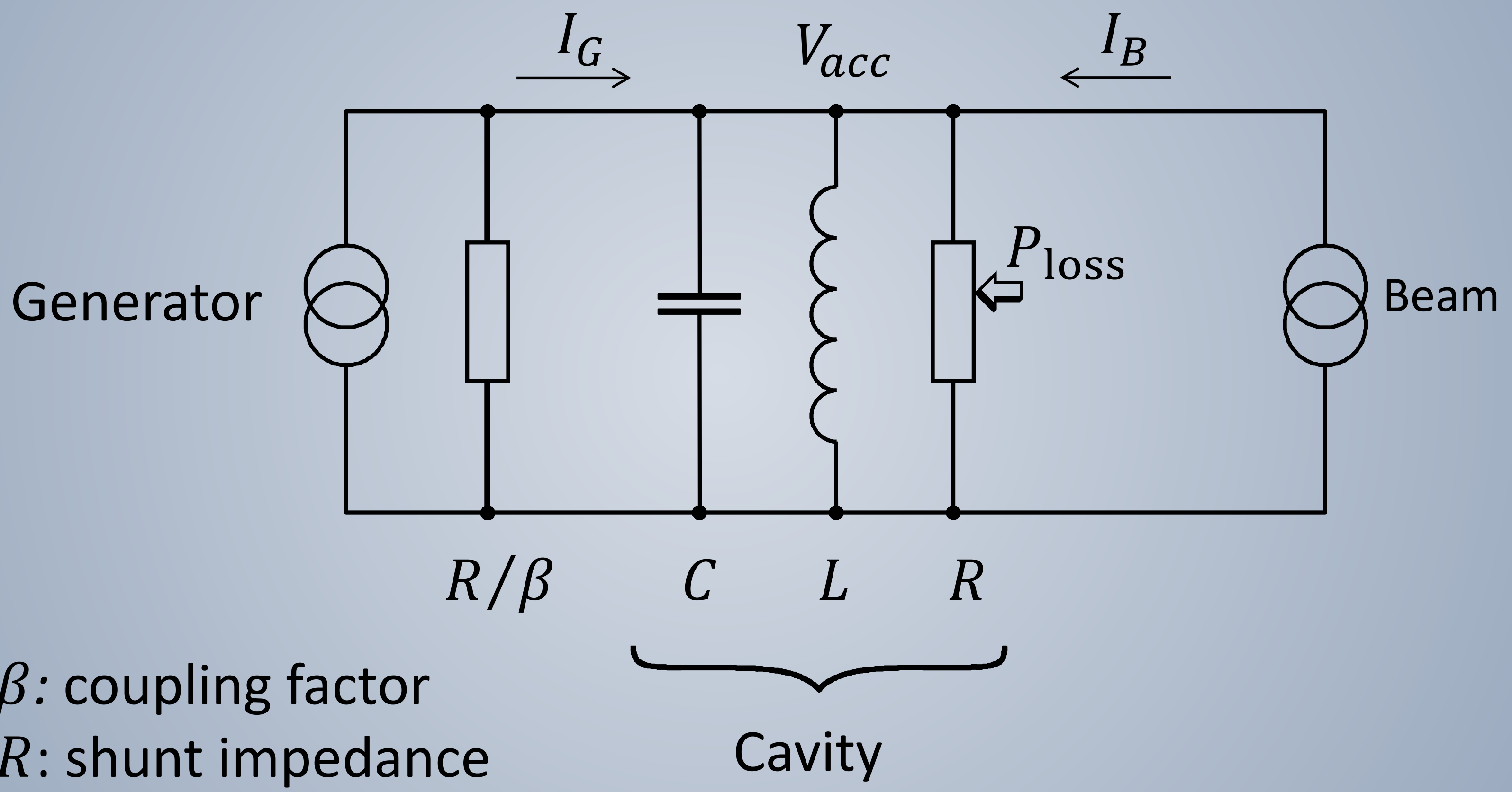
$$G = Q_0 \cdot R_s = \frac{\omega_0 W}{\iint\limits_{wall} |H_t|^2 \, dA}.$$

- G has dimension Ohm, depends only on the cavity geometry (as the name suggests) and typically is $\mathcal{O}(100\ \Omega)$.
- Note that $R_s \cdot R = G \cdot (R/Q)$ (dimension Ω^2, purely geometric)
- G is only used for SC cavities.

Photo:
Reidar Hahn

Cavity resonator – equivalent circuit

Simplification: single mode



β: coupling factor

R: shunt impedance

$\sqrt{L/C} = \frac{R}{Q}$: R-upon-Q

Power coupling - Loaded Q

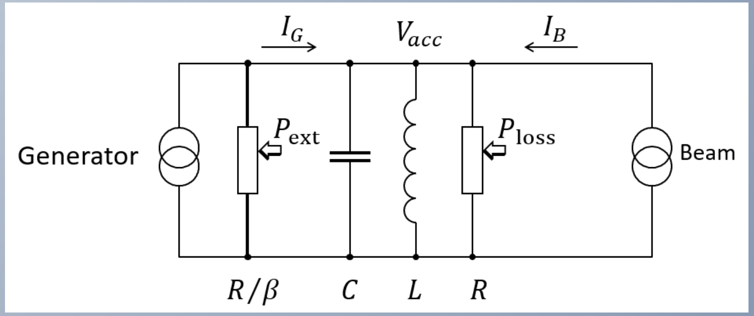
- Note that the generator inner impedance also loads the cavity – for very large Q_0 more than the cavity wall losses.
- To calculate the loaded Q (Q_L), losses have to be added:

$$\frac{1}{Q_L} = \frac{P_{\text{loss}} + P_{\text{ext}} + \cdots}{\omega_0\, W} = \frac{1}{Q_0} + \frac{1}{Q_{ext}} + \frac{1}{\cdots}.$$

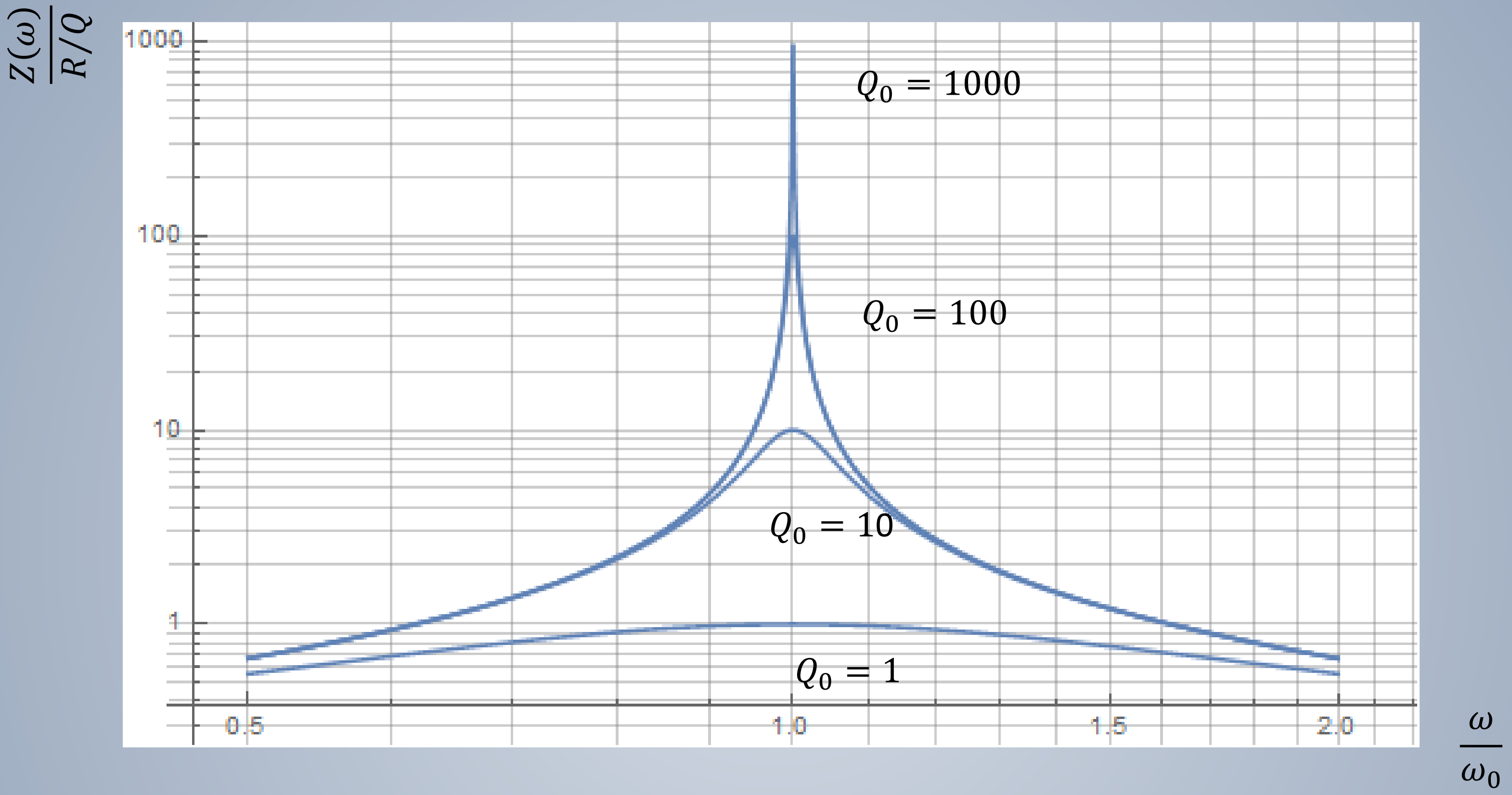
- The coupling factor β is the ratio $P_{\text{ext}}/P_{\text{loss}}$.
- With β, the loaded Q can be written

$$Q_L = \frac{Q_0}{1+\beta}.$$

- For NC cavities, often $\beta = 1$ is chosen (power amplifier matched to empty cavity); for SC cavities, $\beta = \mathcal{O}(10^4 \ldots 10^6)$.



Resonance



- While a high Q_0 results in small wall losses, so less power is needed for the same voltage.
- On the other hand the bandwidth becomes very narrow.
- Note: a 1 GHz cavity with a Q_0 of 10^{10} has a natural bandwidth of 0.1 Hz!
- … to make this manageable, Q_{ext} is chosen much smaller!

Summary: relations V_{acc}, W and P_{loss}

Attention – different definitions are used in literature !

V_{acc}
Accelerating voltage

$$\frac{R}{Q} = \frac{|V_{acc}|^2}{2\omega_0 W}$$

$$k_{\mathrm{loss}} = \frac{\omega_0}{2}\frac{R}{Q} = \frac{|V_{acc}|^2}{4W}$$

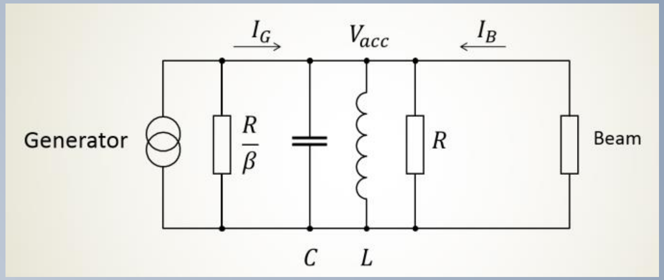
$$R = \frac{|V_{acc}|^2}{2P_{\mathrm{loss}}} = \frac{R}{Q} Q_0$$

W
Energy stored

P_{loss}
wall losses

$$Q_0 = \frac{\omega_0 W}{P_{\mathrm{loss}}}$$

Beam loading

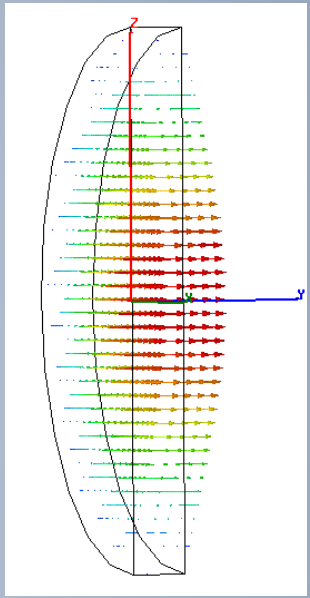


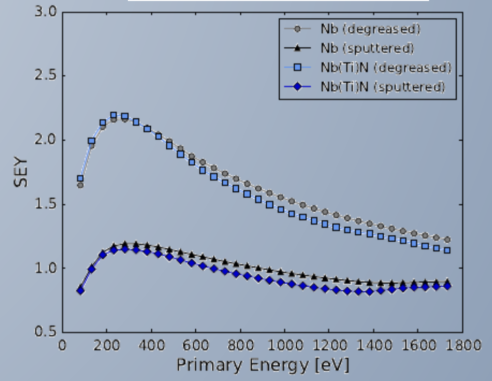
- The beam current “loads” the cavity, in the equivalent circuit this appears as an impedance in parallel to the shunt impedance.
- If the generator is matched to the unloaded cavity ($\beta = 1$), beam loading will (normally) cause the accelerating voltage to decrease.
- The power absorbed by the beam is $-\frac{1}{2}\Re\{V_{acc}I_B^*\}$.
- For high power transfer efficiency RF → beam, beam loading must be high!
- For SC cavities (very large β), the generator is typically matched to the beam impedance!
- Variation in the beam current leads to **transient beam loading**, which requires special care!
- Often the “impedance” the beam presents is strongly reactive – this leads to a detuning of the cavity.

Photo: Reidar Hahn

Multipactor

- The words “multipactor”, “to multipact” and “multipacting” are artificially composed of “multiple” “impact”.
- Multipactor describes a resonant RF phenomenon in vacuum:
  - Consider a free electron in a simple cavity – it gets accelerated by the electric field towards the wall
  - when it impacts the wall, secondary electrons will be emitted, described by the secondary emission yield (SEY)
  - in certain impact energy ranges, more than one electron is emitted for one electron impacting! So the number of electrons can increase
  - When the time for an electron from emission to impact takes exactly ½ of the RF period, resonance occurs – with the SEY>1, this leads to an avalanche increase of electrons, effectively taking all RF power at this field level, depleting the stored energy and limiting the field!
- For this simple “2-point MP”, this resonance condition is reached at $\frac{1}{4\pi}\frac{e}{m}V = (fd)^2$ or $\frac{V}{112\,\mathrm{V}} = \left(\frac{f}{\mathrm{MHz}}\frac{d}{\mathrm{m}}\right)^2$. There exist other resonant bands.

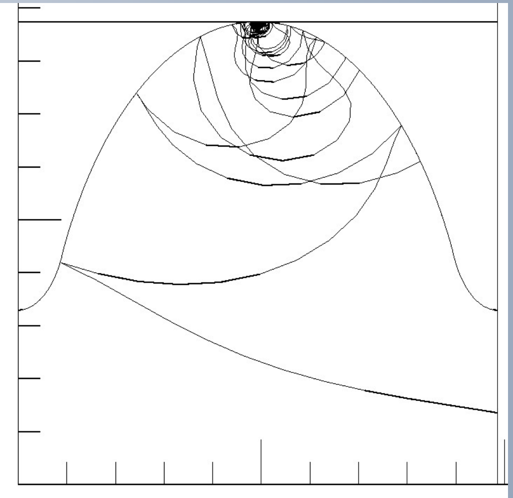




courtesy: Sarah Aull/CERN

Multipactor (contd.)

- Unfortunately, good metallic conductors (Cu, Ag, Nb) all have SEY>1!
- 1-point MP occurs when the electron impact where they were emitted
- Electron trajectories can be complex since both $\vec{E}$ and $\vec{B}$ influence them; computer simulations allow to determine the MP bands (barriers)
- To reduce or suppress MP, a combination of the following may be considered:
  - Use materials with low SEY
  - Optimize the shape of your cavity (→ elliptical cavity)
  - Conditioning (surface altered by exposure to RF fields)
  - Coating (Ti, TiN, NEG, amorphous C …)
  - Clearing electrode (for a superimposed DC electric field)
  - Rough surfaces

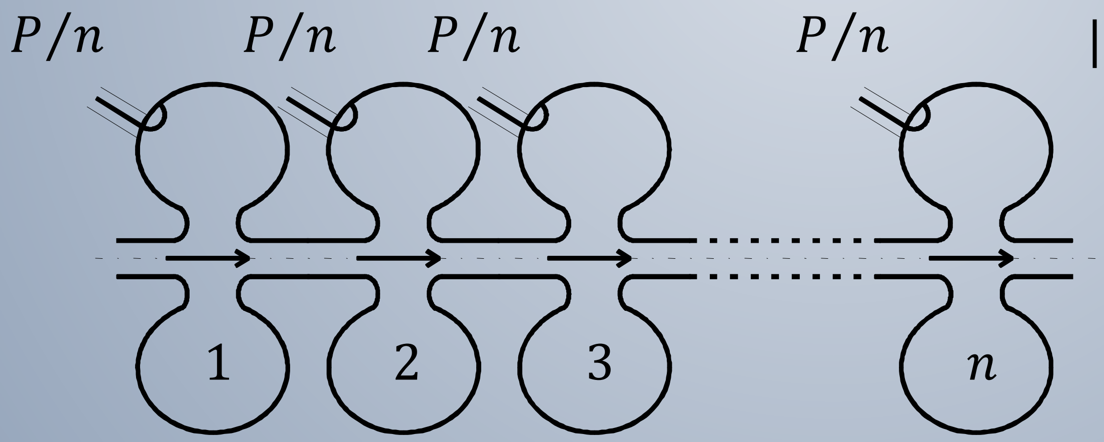


Many gaps

Photo: Reidar Hahn

What do you gain with many gaps?

- The R/Q of a single gap cavity is limited to some $100\ \Omega$.
  Now consider to distribute the available power to n identical cavities: each will receive P/n, thus produce an accelerating voltage of $\sqrt{2RP/n}$. (Attention: phase important!)
  The total accelerating voltage thus increased, equivalent to a total equivalent shunt impedance of nR .

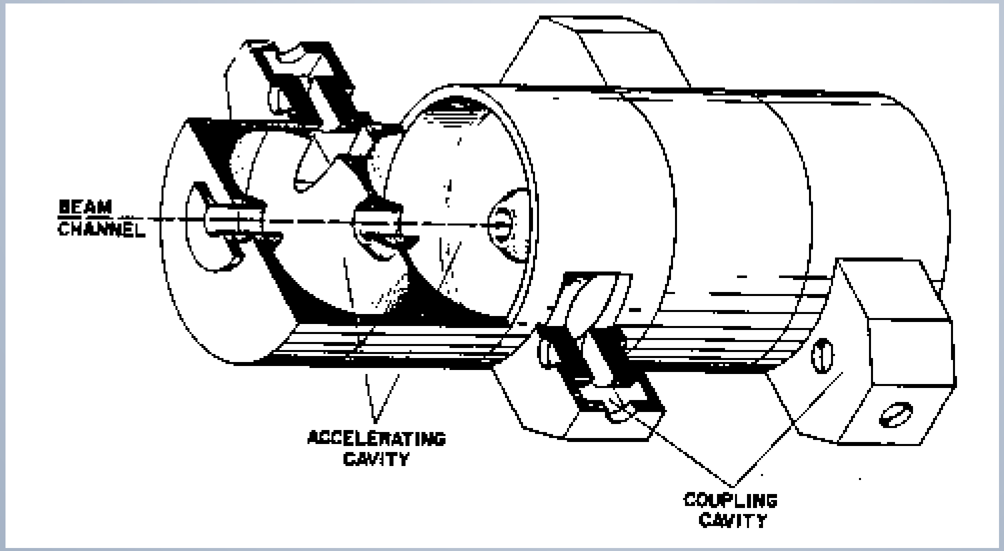


$$|V_{acc}| = n\sqrt{2R\frac{P}{n}} = \sqrt{2(nR)P}$$

Standing wave multi-cell cavity

- Instead of distributing the power from the amplifier, one might as well couple the cavities, such that the power automatically distributes, or have a cavity with many gaps (e.g. drift tube linac).

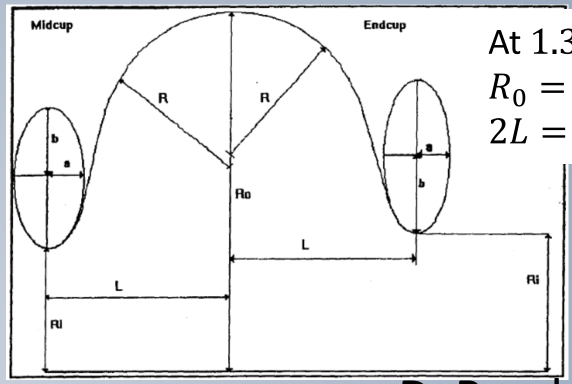
- Coupled cavity accelerating structure (side coupled)



- The phase relation between gaps is important!

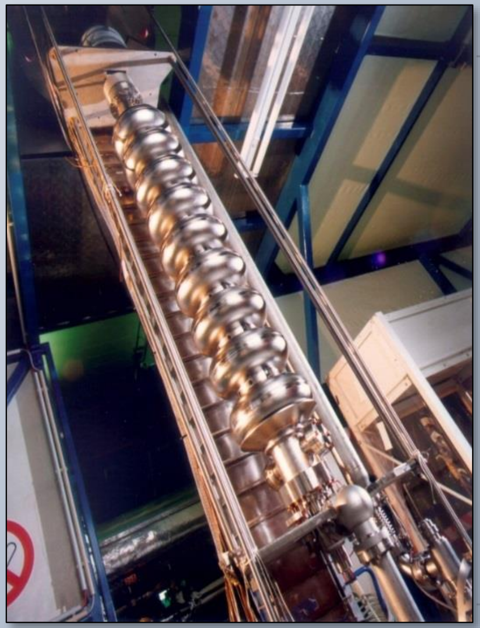
The elliptical cavity

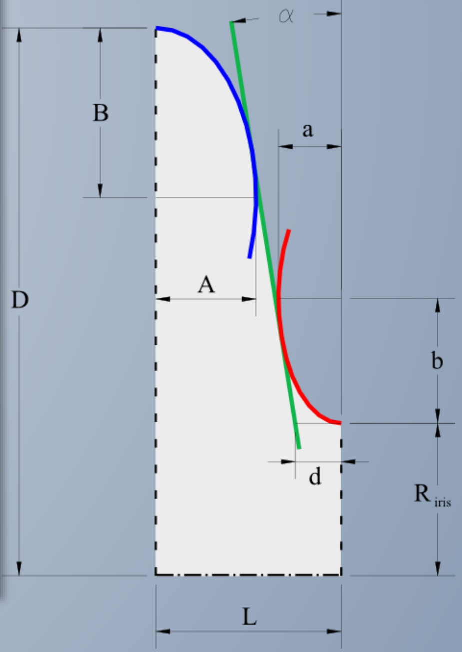
- The elliptical shape was found as optimum compromise between
  - maximum gradient ($E_{acc}/E_{\mathrm{surface}}$)
  - suppression of multipactor
  - mode purity
  - machinability
- A multi-cell elliptical cavity is typically operated in π-mode, i.e. cell length is exactly $\beta\lambda/2$.
- It has become de facto standard, used for ions and leptons! E.g.:
  - ILC/X-FEL: 1.3 GHz, 9-cell cavity
  - SNS (805 MHz)
  - SPL/ESS (704 MHz)
  - LHC (400 MHz)



At 1.3 GHz:
$R_0 = 103.3$ mm,
$2L = 115.3$ mm.

D. Proch, 1993 *)





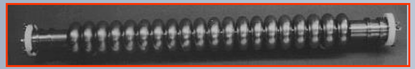
*): http://accelconf.web.cern.ch/AccelConf/SRF93/papers/srf93g01.pdf

Elliptical cavity – the *de facto* standard for SRF

FERMI 3.9 GHz

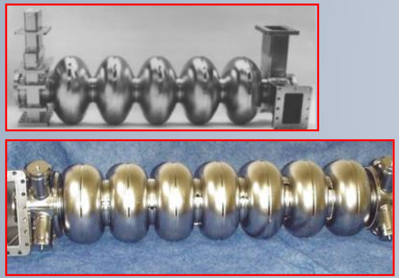
TESLA/ILC 1.3 GHz

S-DALINAC 3 GHz



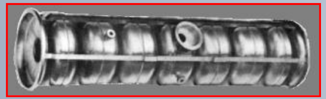
SNS $\beta = 0.61, 0.81,$ 0.805 GHz

CEBAF 1.5 GHz

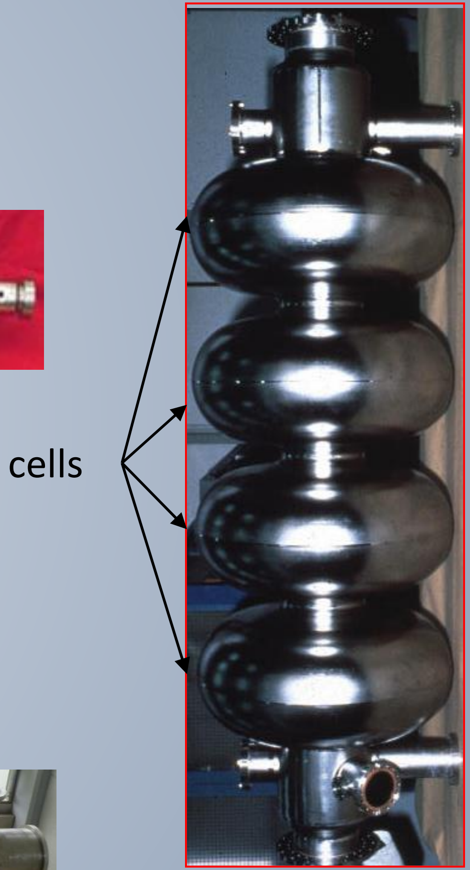


HERA 0.5 GHz

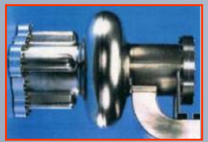
HEPL 1.3 GHz



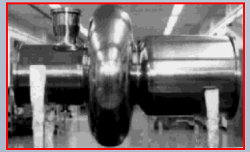
LEP 0.352 GHz



CESR 0.5 GHz



KEK-B 0.5 GHz

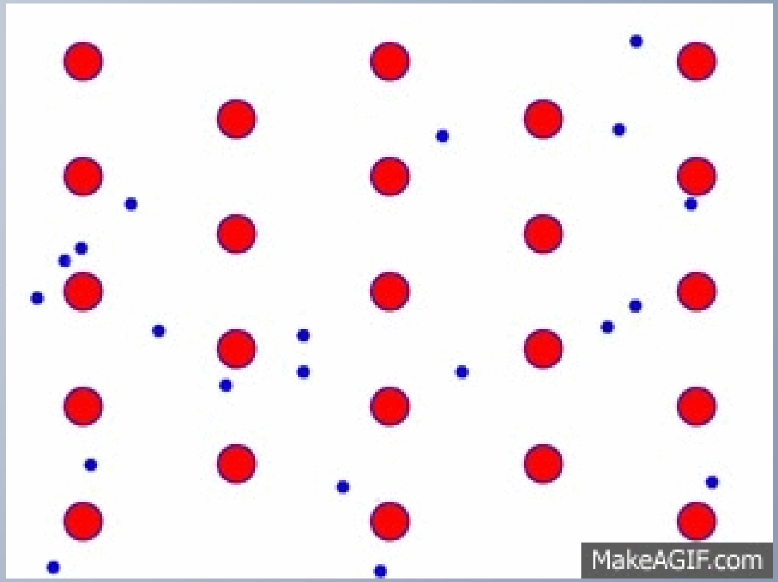


TRISTAN 0.5 GHz

Paul Drude
1863 – 1906

How does electrical conduction work?

- Drude model: In a conductor, the electrons move through an ion lattice, bounce off the ions and get slowed in the process.
- In the presence of E fields only, this is governed by the equation
  $$\frac{d}{dt}p = eE - p/\tau.$$
- In steady state (DC), one gets
  $v = \frac{e}{m}\tau\, E$, which is …
- … Ohm's law: $j_n = nev = \frac{e^2\tau n}{m}E = \sigma E$
  with $\sigma_{DC} = \frac{e^2\tau n}{m}$.
- Typical scattering time $\tau \approx 10^{-14}$s.



The electrons (blue) slowly drift to the right under the influence of a DC electric field


Photo: Reidar Hahn

Drude model extended for RF

- The equation $\frac{d}{dt}p = eE - \frac{p}{\tau}$ can of course be solved for an excitation by a time-varying field $E = \Re\{\hat{E}\ e^{j\omega t}\}$.
- Solution: Assume $p(t)$ to be of the same time-dependence as RHS: $p(t) = \Re\{\hat{p}e^{j\omega\tau}\}e^{j\omega\tau}$ and solve:
- $\left(j\omega + \frac{1}{\tau}\right)\hat{p} = e\hat{E}$ or $\hat{p} = \frac{\tau e\hat{E}}{1+j\omega\tau}$.
- This naturally results in a complex, f-dependent σ:

$$\sigma = \sigma_{DC}\frac{1}{1+j\omega\tau} = \frac{\sigma_{DC}}{1+(\omega\tau)^2}(1-j\omega\tau)$$

- With $\tau \approx 10^{-14}$s, $\omega\tau < 1$ for frequencies up to many THz!

Good (normal-)conductor boundary – skin depth

- A good (normal-)conductor: inside the metal, the surface field $H_{||}$ gives rise to a damped "wave", which "propagates" into the metal with a propagation constant of $k_\perp = \sqrt{-j\omega\mu\sigma}$. The skin-depth is the inverse of the damping constant, the real part of $k_\perp$:
- **Skin depth**: $\delta = \frac{1}{\alpha} = \frac{1}{\Re\{\sqrt{-j\omega\mu\sigma}\}} = \sqrt{\frac{2}{\omega\mu\sigma}}$
- The wave impedance in the metal is $Z = \sqrt{\frac{j\omega\mu}{\sigma}}$, its real part $R_n = \sqrt{\frac{\omega\mu}{2\sigma}}$ is the **surface resistance** and can be used to determine the losses.

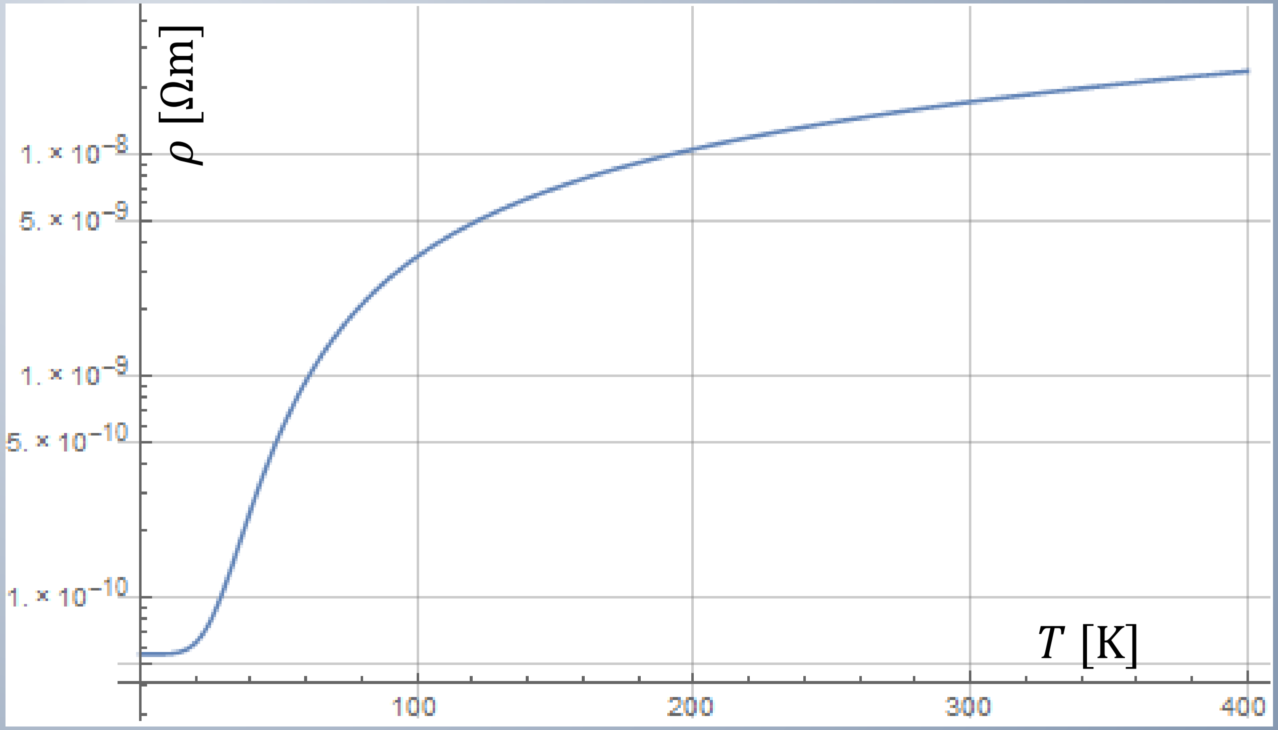
$$\delta R_n \sigma \equiv 1$$

I use the term R_n for the surface resistance for normal conductors, R_s for superconductors.

Temperature dependence of resistivity of metals

$$\rho(T) = \frac{1}{\sigma(T)} = \rho_0 + A\left(\frac{T}{\Theta_D}\right)^5 \iint_0^{\frac{\Theta_D}{T}} \frac{x^5}{(e^x - 1)(1 - e^{-x})} dx$$

Cu resistivity vs. T, assuming $\rho(20\ ^\circ\mathrm{C}) = 16.78\ \mathrm{n\Omega m}$, $RRR = 300$ and $\Theta_D = 343.5\ \mathrm{K}$ (Debye T). [Bloch-Grüneisen formula]

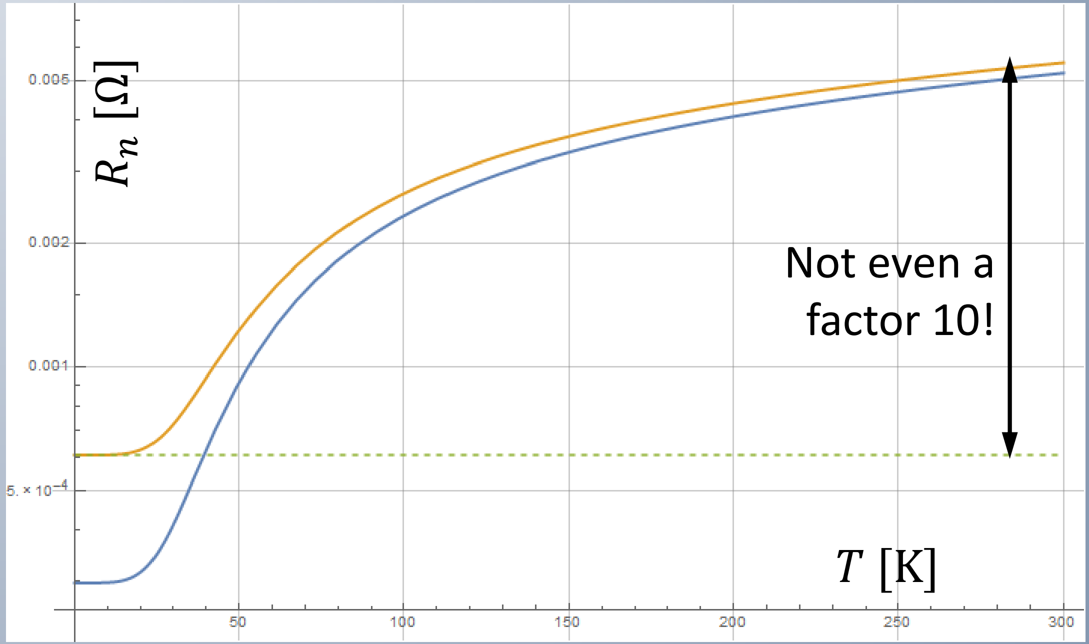


Anomalous skin effect

- The relation between skin depth δ and surface resistance R_n, $\delta R_n \sigma \equiv 1$, is valid only while the mean free path ℓ of the electrons is smaller than the skin depth, $\ell \ll \delta$.
- If the skin depth gets smaller (e.g. at low T, high ω), R_n will be dominated by ℓ and be limited to

$$R_n \approx \left(\sqrt{3}\pi \left(\frac{\ell}{\sigma}\right)\left(\frac{\mu\omega}{4\,\pi}\right)^2\right)^{1/3}$$

Prediction of surface resistance $R_n(T)$ of Cu at 400 MHz with $RRR = 300$ (blue) and correction for anomalous skin effect ($\rho\ell = 6.6 \cdot 10^{-18}\Omega\mathrm{m}^2$) (orange).



Superconductivity

Heike Kammerling Onnes (1853 – 1926)

- Kammerling Onnes investigated the behaviour of metal conductivity at low temperatures and noted in 1911, when measuring the resistivity of Mercury at 4.2 K: “*Kwik nagenoeg nul*” (meaning “mercury almost zero”).

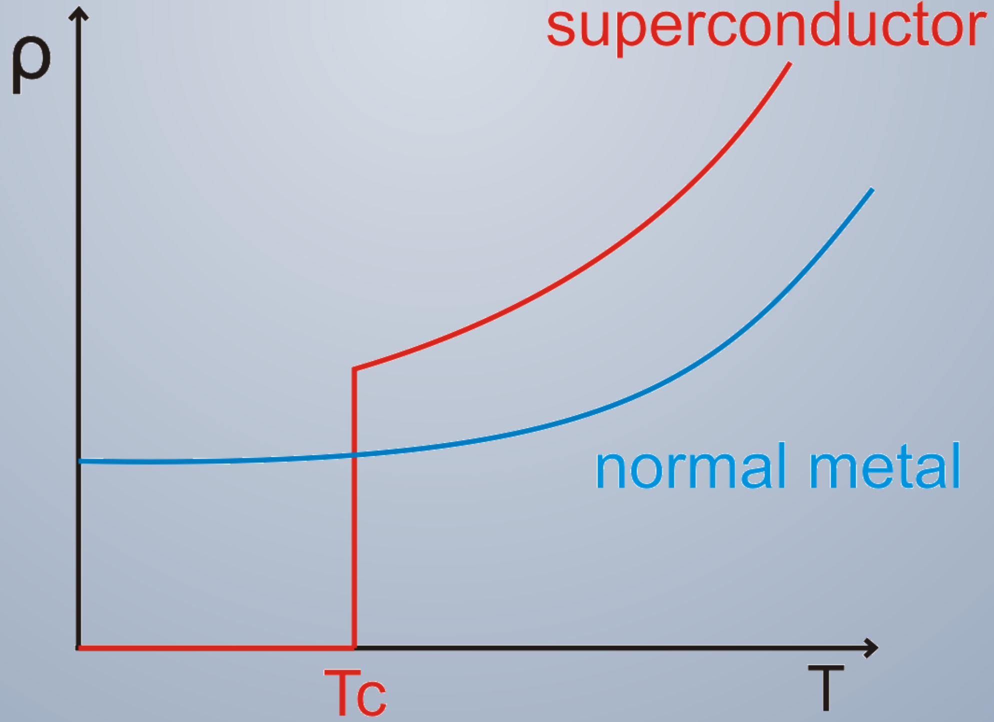


Photo: Reidar Hahn

Critical temperatures of superconductors

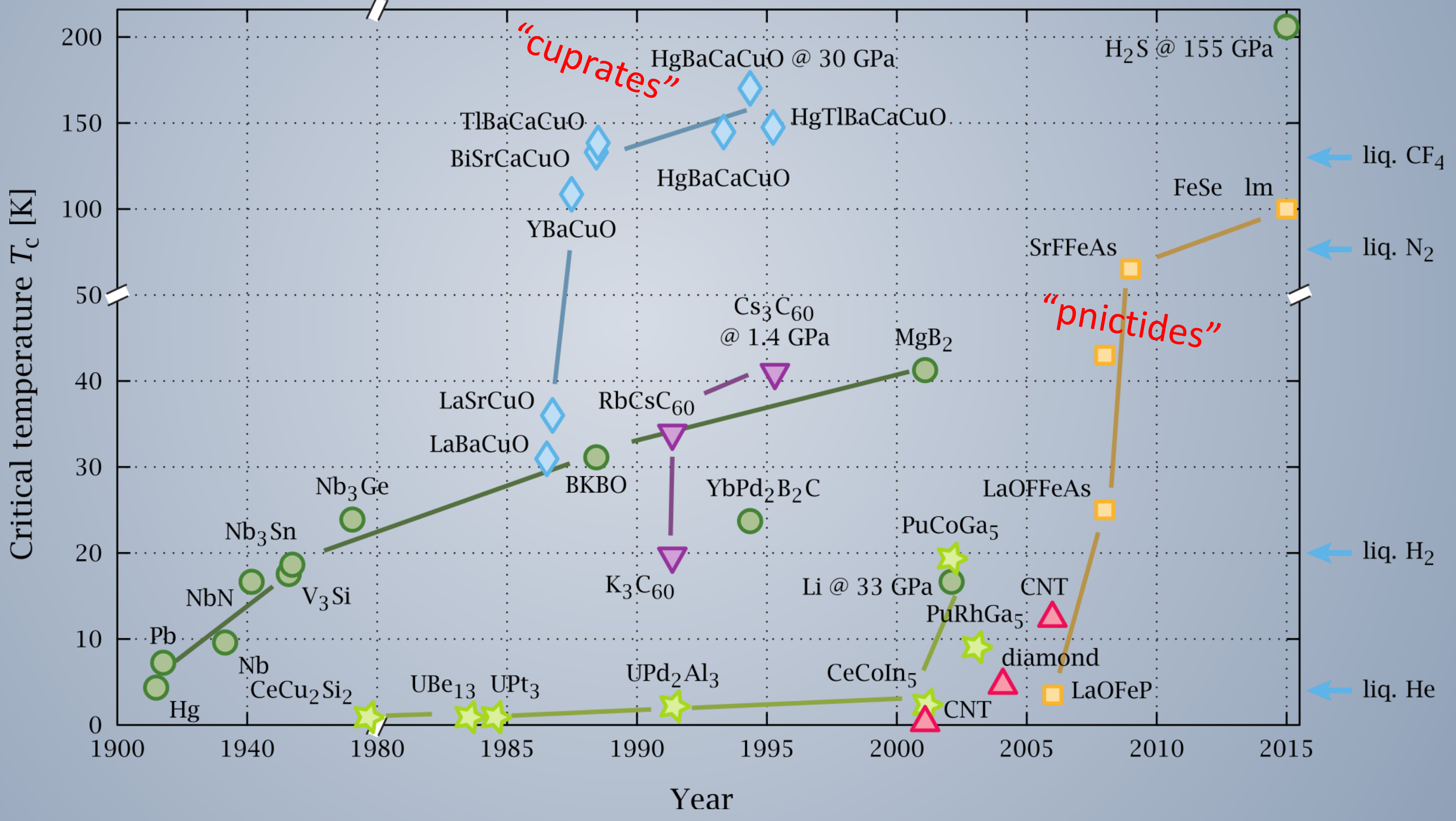
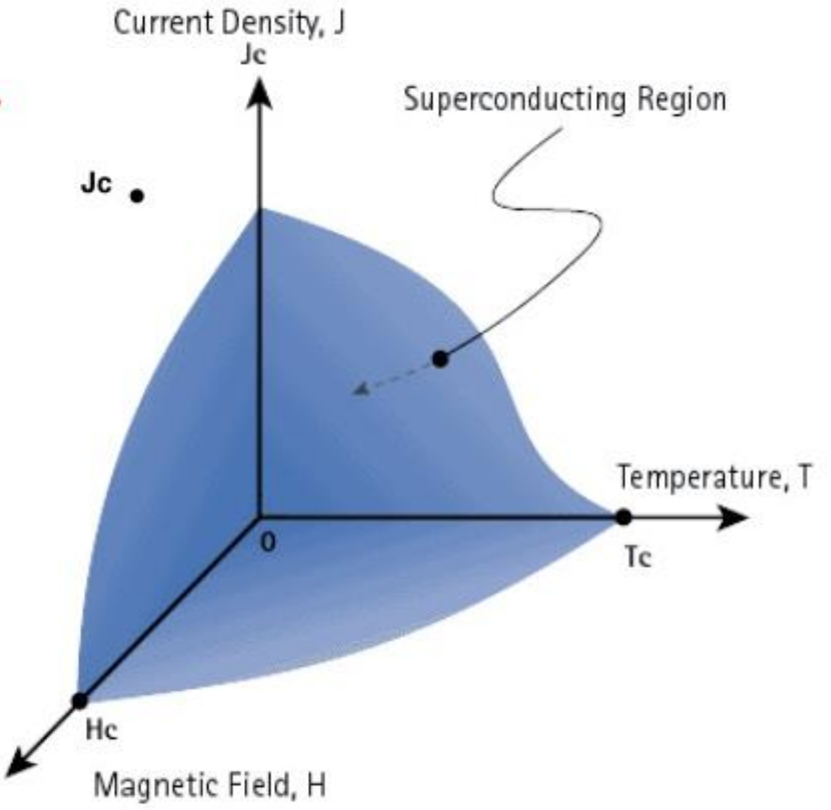



Photo:
Reidar Hahn

Phase diagram of a SC

Critical temp. 'T$_c$'
Critical mag. Field 'H$_c$'
Critical current density 'J$_c$'

are the three parameters to decide SC property



Perfect conductor

- With a magnetic field B_0 at the transition vacuum/superconductor, the following equations hold:
  - 3rd Maxwell equation: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
  - 1st London equation: $\frac{\partial \vec{j_s}}{\partial t} = \frac{n_s e^2}{m} \vec{E}$, where $\vec{j_s} = -n_s e \vec{v}$.
- This results in the equation $\frac{\partial}{\partial t}\left(\nabla \times \vec{j_s} + \frac{n_s e^2}{m} \vec{B}\right) = 0$, or (slightly transformed) in $\frac{\partial}{\partial t}\left(\nabla^2 \vec{B} - \mu \frac{n_s e^2}{m} \vec{B}\right) = 0$
- This could be solved by either a time-independent field, $\frac{\partial}{\partial t} \equiv 0$ or by a field satisfying $\nabla^2 \vec{B} - \mu \frac{n_s e^2}{m} \vec{B} = 0$.

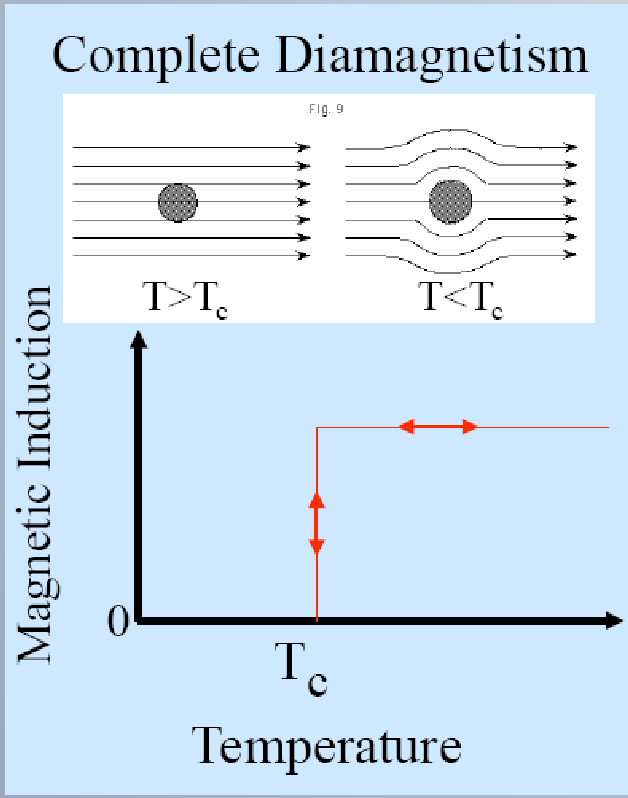
$$\mu \frac{n_s e^2}{m} = \lambda_L^{-2}$$


Photo: Reidar Hahn

Meissner effect (flux expulsion)


Walther Meißner
1882 – 1974

- Observation: During cool-down, when T passes T_c, the magnetic field gets completely expulsed.



- A non-vanishing, time-independent field is not observed and thus can be excluded as non-physical.


Photo:
Reidar Hahn

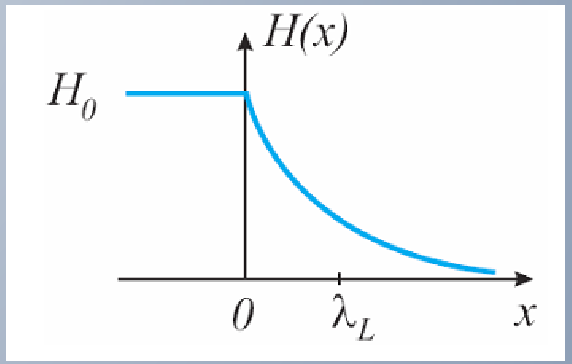
London equations


London brothers
Heinz (1907 – 1970),
Fritz (1900 – 1954)

- Supported by Meissner's observation, the time-independent (non-trivial) solution of the equation $\frac{\partial}{\partial t}\left(\nabla \times \vec{j_s} + \frac{n_s e^2}{m}\vec{B}\right) = 0$ can be excluded as non-physical.
- London equations (London penetration depth: $\lambda_L = \sqrt{\frac{m}{\mu n_s e^2}}$)

1. (zero resistance): $\frac{\partial \vec{j_s}}{\partial t} = \frac{n_s e^2}{m}\vec{E} = \frac{1}{\mu\,\lambda_L^2}\vec{E}$
2. (Meissner effect): $\nabla \times \vec{j_s} + \frac{n_s e^2}{m}\vec{B} = 0$

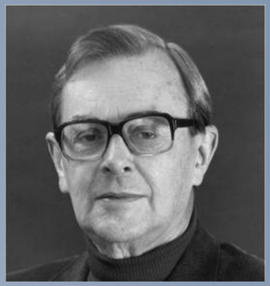
   or $\left(\nabla^2 - \frac{1}{\lambda_L^2}\right)\vec{B} = 0$



vacuum | superconductor

Photo:
Reidar Hahn

The coherence length ξ



Brian Pippard
1920 – 2008

- Pippard found out that λ_L depends on the purity of the material and therefore on the electron mean free path ℓ.
- In 1953, he proposed the coherence length ξ as a new parameter to better describe the characteristic dimension of the electrons wave function in a superconductor.
- In a pure metal, one finds $\xi = \xi_0 \propto \frac{\hbar v_F}{k\,T_c}$,

  with impurities one can approximate $\xi = \left(\frac{1}{\xi_0} + \frac{1}{\ell}\right)^{-1}$ and $\lambda(\ell) = \lambda_L \sqrt{\frac{\xi_0}{\xi}}$.
- "Clean" limit: $\ell \gg \xi$, "dirty" limit: $\ell \ll \xi$.

Characteristic lengths:

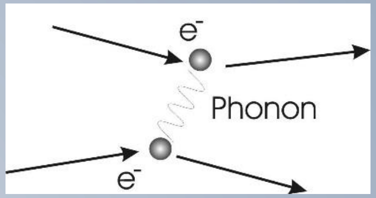
- London penetration depth $\lambda_L = \sqrt{\frac{m}{\mu n_s e^2}}$: the distance up to which magnetic field penetrates into a superconductor if placed in a magnetic field
- Electron mean free path ℓ. …gets smaller with impurities.
- Coherence length $\xi = \left(\frac{1}{\xi_0} + \frac{1}{\ell}\right)^{-1}$: the distance over which the electron wave function extends – the "size" of Cooper pairs (see below).
- Orders of magnitude:
  - λ_L and ξ: tens of nm. (E.g. Nb: $\xi_0 = 39\ \mathrm{nm}$, Pb: $\xi_0 = 83\ \mathrm{nm}$)
  - ℓ: nm (dirty) to $\mu\mathrm{m}$ (clean)
- Sometimes used: $\kappa \equiv \lambda_L / \xi$

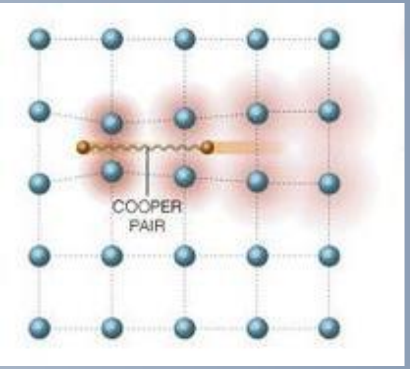
BCS Theory

Bardeen – Cooper – Schrieffer (BCS)

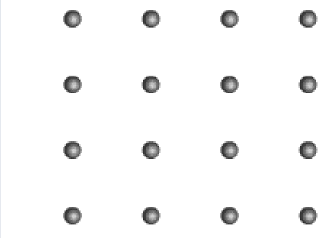
In 1958, Bardeen, Cooper and Schrieffer proposed a theory of superconductivity in which there exists an attractive interaction between electrons, forming “Cooper pairs”.

Cooper pairs

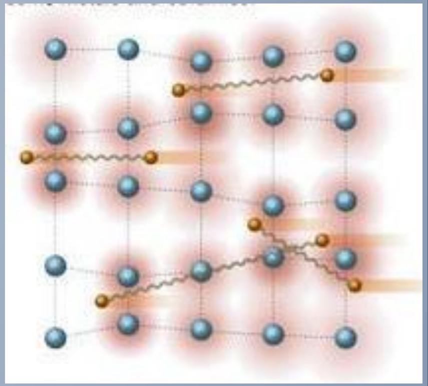




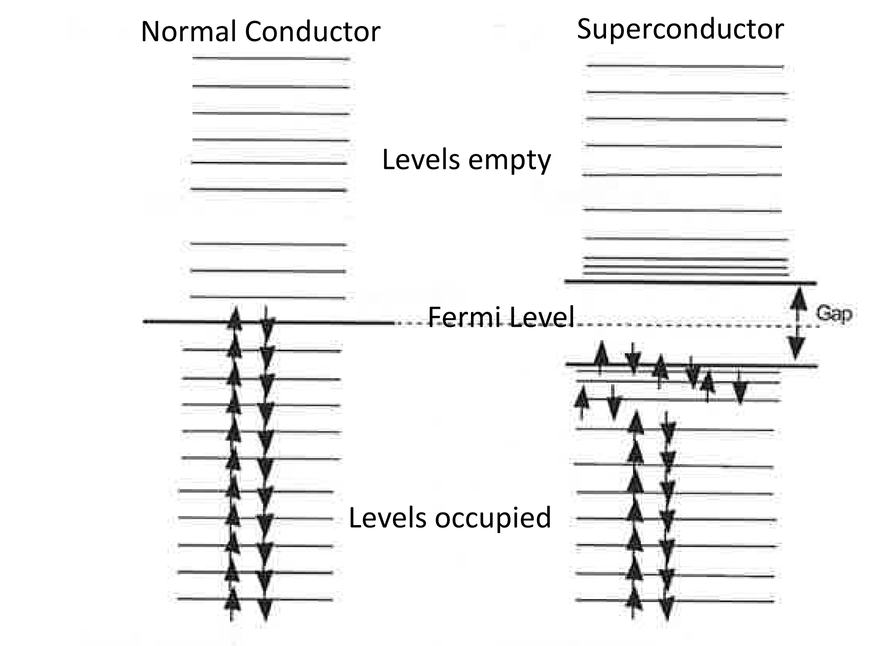
- Positively charged wake due to moving electron attracting nearby atoms (electron-phonon interaction)
- … this wake can attract another nearby electron and …
- → … a Cooper pair is formed.



- Cooper pairs are formed by electrons with opposite momentum and spin.
- Cooper pairs belong all to the same quantum state and have the same energy.
- While electrons are fermions, Cooper pairs are bosons.
- When carrying a current, each Cooper pair acquires a momentum which is the same for all pairs,
- The **total** momentum of the pair remains constant. It can be changed only if the pair is broken, but this requires a minimum energy ΔE.
- While NC electrons are scattered by the ion lattice (cf. Drude model) leading to resistive losses, Cooper pairs are not scattered by ion lattice.



Energy gap

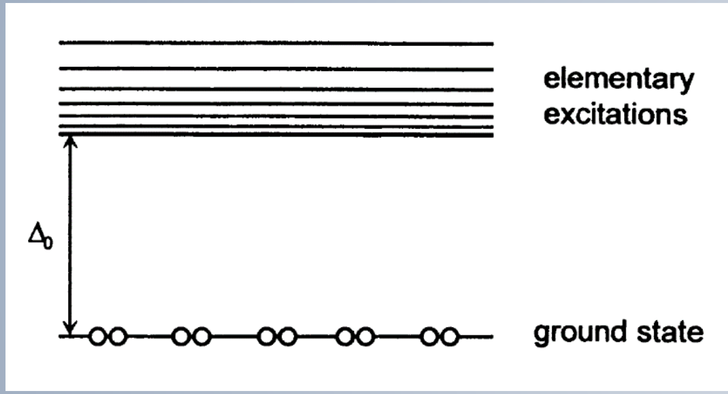


Padamse, Knobloch, Hays: RF Superconductivity for Accelerators, Wiley 2008

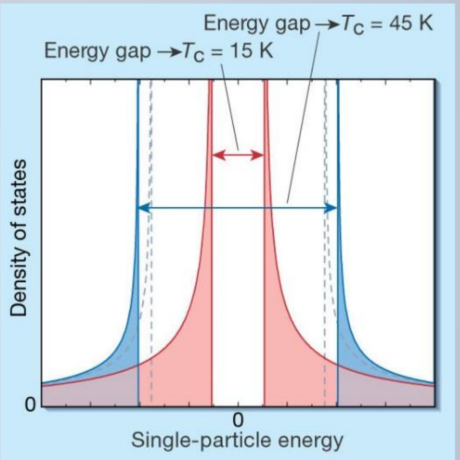
- The electron-phonon interaction changes the density of states.
- Near the Fermi surface an energy gap forms and energy states below get denser – the new ground state (with Cooper-pairs) has a lower energy than the NC ground state.
- For Nb, $\Delta(0)/(k_B T_c) = 1.9$,
  for Pb, $\Delta(0)/(k_B T_c) = 2.4$.

The microscopic BCS theory

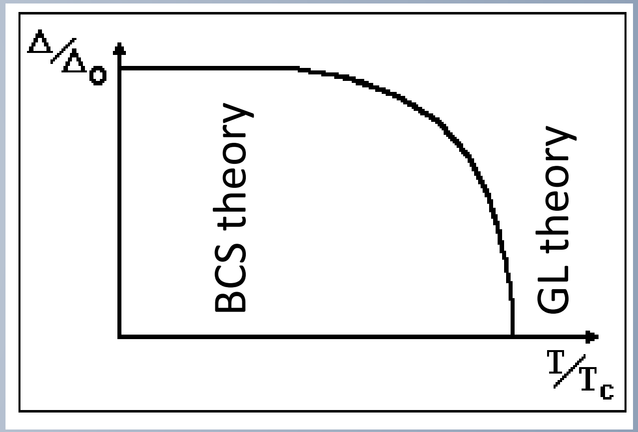
- The superconducting state consists of electron pairs and elementary excitations, quasiparticles, behaving almost as free electrons



The energy gap Δ separates the energy levels of elementary excitations from the ground state level. At 0 K, only the ground state is occupied.



Density of elementary excitations. There are no states within the energy gap Δ.



Temperature dependence of Δ:

$$\Delta(T) \approx \Delta(0\text{ K})\sqrt{\cos\left(\frac{\pi}{2}\left(\frac{T}{T_c}\right)^2\right)}$$

Original BCS theory has been derived using "mean field approximation" – valid for $0 \leq T < T_c$. Ginzburg-Landau theory is valid for $\mathrm{T} \approx T_c$. Gor'kov showed that $\lim_{T \to T_c} (\mathrm{BCS}) = \mathrm{GL}$.


Photo:
Reidar Hahn

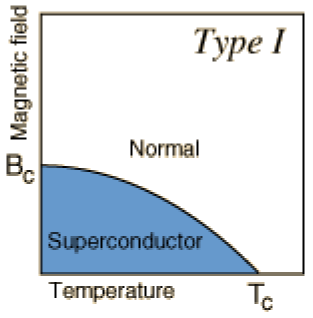
Classification of superconductors

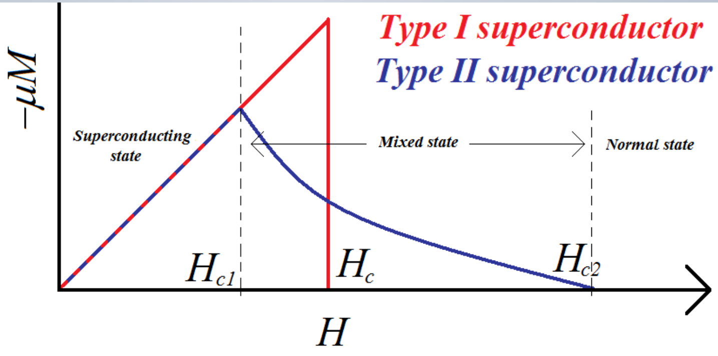
Type-I superconductor:

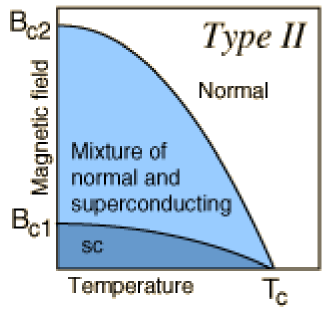
- Meissner effect complete
- Sharp transition NC/SC at field H_c
- $\kappa = \lambda_L/\xi < 1/\sqrt{2}$

Type-II superconductor:

- Meissner effect incomplete
- Two critical fields H_{c1} and H_{c2}
- $\kappa > 1/\sqrt{2}$



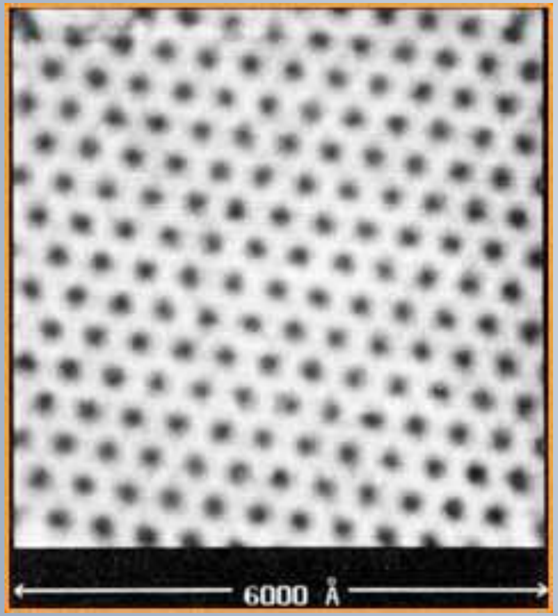




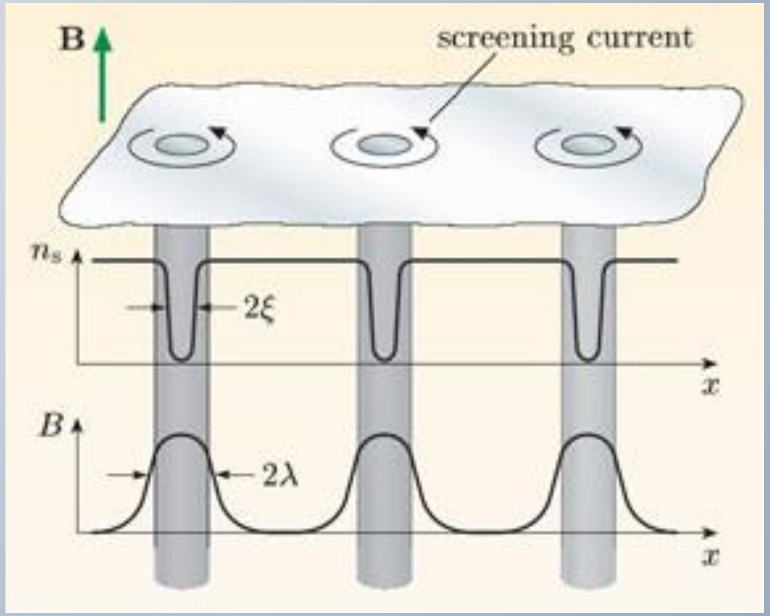
- Examples: Pb, Sn, Hg, Cr, Al
- Examples: Nb, alloys

Flux quantization

- In Type-II superconductors flux tubes are created each carrying one flux quantum (the minimal flux allowed by quantum mechanics)
- Flux tubes are repulsive creating, therefore the vortex lattice



STM image of Vortex lattice, 1989
H. F. Hess et al. Phys. Rev. Lett. 62, 214, 1989

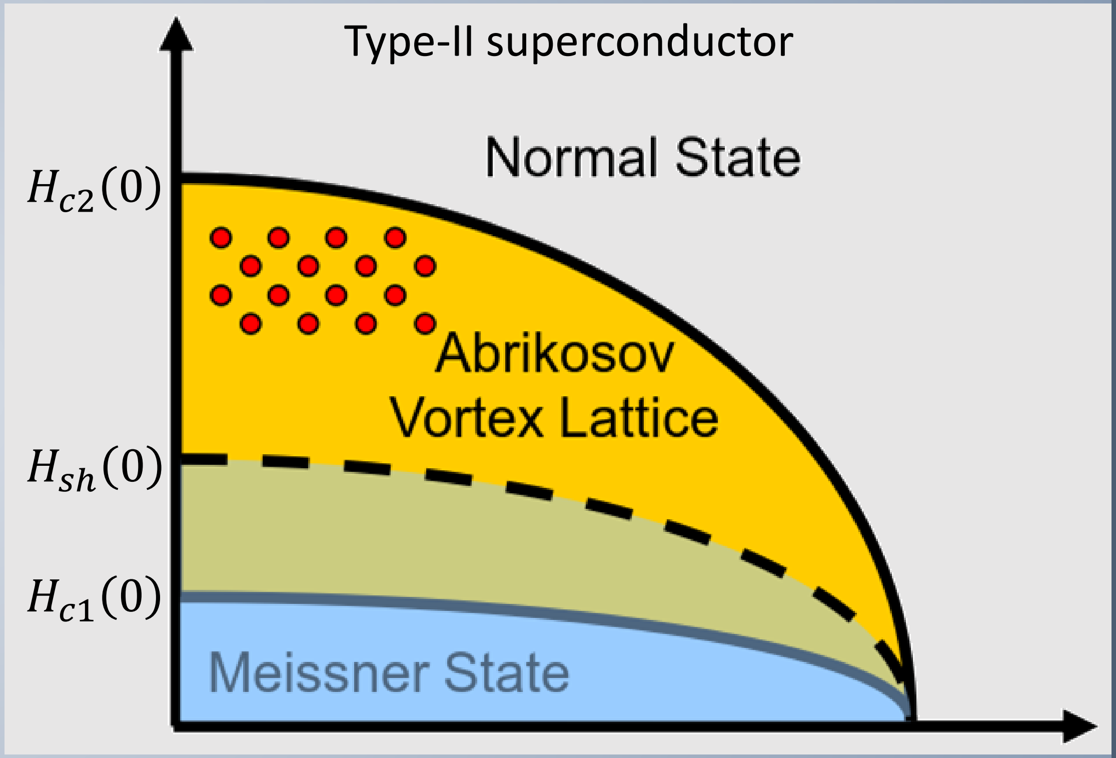


Field limits for type-II superconductors

- $H < H_{c1}$: perfect Meissner state
- $H_{c1} < H < H_{c2}$: penetration and oscillation of vortices give rise to strong dissipation – not useable for RF.

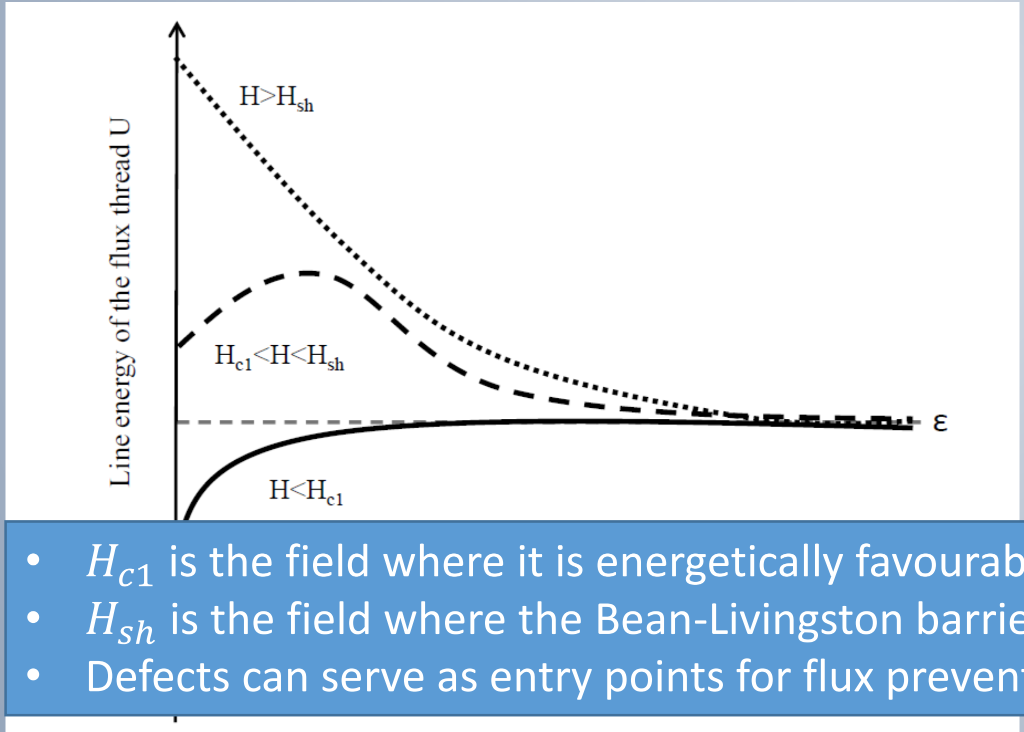
But …

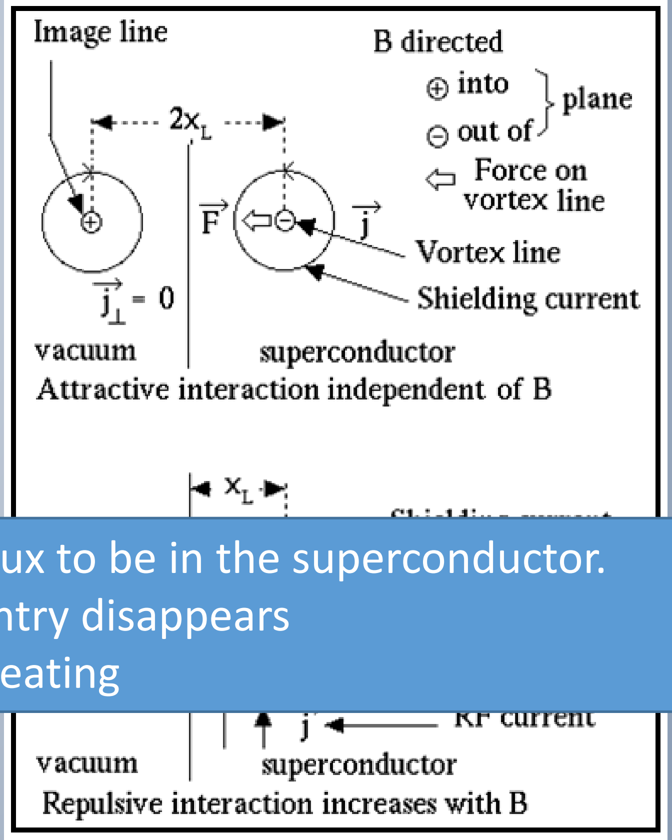
- The Meissner state can remain metastable for $H_{c1} < H < H_{sh}$, if vortices can be prevented from entering (Bean-Livingston barrier).



The superheating field

The superheating field H_{sh} is set by the competition between magnetic pressure (imposed by the external magnetic field), the energy cost to destroy superconductivity, and the attractive force due to the zero-current boundary condition at the interface.





- H_{c1} is the field where it is energetically favourable for the flux to be in the superconductor.
- H_{sh} is the field where the Bean-Livingston barrier for flux entry disappears
- Defects can serve as entry points for flux preventing superheating

Suggested further reading:
B. Liarte et al. - Supercond. Sci. Technol. 30 (2017) 033002

RF case

- The time-dependent magnetic field in the penetration depth λ_L will generate an electric field

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

- At $T > 0\ \mathrm{K}$, there will exist a fraction of unpaired electrons:

$$n_n(T) \propto e^{-\frac{\Delta}{k_B T}}$$

- Since Cooper pairs have inertia, they cannot shield these NC electrons from E, hence

$$R_s > 0.$$


Photo: Reidar Hahn

Two-fluid model

- Proposed by Gorter and Casimir already in 1943: Charge carriers are divided in two subsystems, superconducting carriers of density n_s and normal electrons of density n_n.
- Assume $\frac{n_s}{n} = 1 - \left(\frac{T}{T_c}\right)^4, \frac{n_n}{n} = \left(\frac{T}{T_c}\right)^4, n_s + n_n = n.$
- The total current results from $J = J_s + J_n$

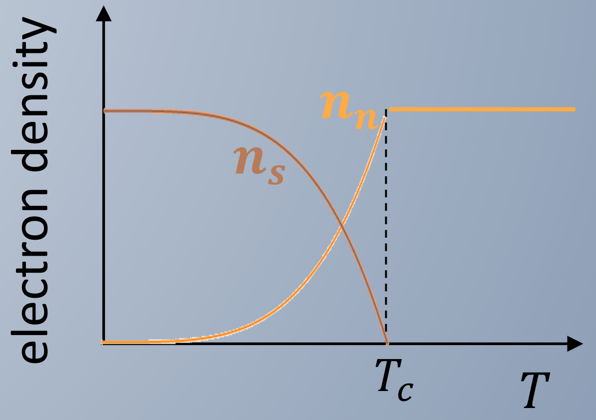


Photo: Reidar Hahn

RF surface impedance of SC

- From 1st London equation: $\frac{\partial}{\partial t}\vec{J}_s = \frac{\vec{E}}{\mu\,\lambda_L^2}$, $\rightarrow J_s = -j\,\frac{1}{\omega\mu\lambda_L^2}\,E$, or $J_s = -j\,\frac{n_s e^2}{m\omega}E.$
- For the unpaired electrons (Ohm's law): $J_n = \frac{n_n\,e^2\tau}{m}E.$
- For the total current:

$$J = J_n + J_s = (\sigma_n - j\sigma_s)E = \left(\frac{n_n\,e^2\tau}{m} - j\,\frac{n_s e^2}{m\,\omega}\right)E$$

- Surface impedance:

$$Z_s = R_s + jX_s = \sqrt{\frac{j\omega\mu}{\sigma_n - j\sigma_s}}.$$

- With the 2-fluid model this results in:

$$R_s = \frac{1}{2}\mu\omega^2\sigma_n\lambda_L^3$$
$$X_s = \omega\,\mu\,\lambda_L$$

RF surface resistance of a SC

We found $R_s = \frac{1}{2}\mu\omega^2\sigma_n\lambda_L^3$ – what does this mean?

- ***Frequency dependence:***
  $R_s \propto \omega^2$: use low frequency cavities to reduce power dissipation!
- ***Temperature dependence:***
  from two-fluid model: $\sigma_n \propto n_n \propto e^{-\frac{\Delta}{k_B T}}$

$$R_s \propto \exp\left(-\frac{\Delta}{k_B T}\right)$$

- There are “better” approximations around, most famous the formulae developed by Halbritter (1970), which approximate R_s as a function of ω, T, ξ_0, λ_L, T_c, Δ, and ℓ, see e.g. here: http://www.lepp.cornell.edu/~liepe/webpage/researchsrimp.html

Halbritter approximation example

SRIMP

This webpage calculates BCS surface resistance under wide range of conditions, and is based on a program by Jurgen Halbritter. [J. Halbritter, Zeitschrift f�r Physik 238 (1970) 466]

Enter material parameters below, and click submit to calculate the BCS surface resistance. Results are given in a new window.
Please be aware that frequencies much lower than 1 MHz may cause substantial processing times (depending on the user's computer).

Submit

Frequency (MHz):	1300
Transition temperature (K):	9.2
DELTA/kTc:	1.86
London penetration depth (A):	330
Coherence length (A):	400
RRR:	300
Accuracy of computation:	.001
Temperature (of operation):	2

Results:

Diffuse Reflection:	Resistance (Ohm):	1.9031321344341478e-8
	Penetration Depth (um):	0.037746828693838295

Input Parameters:

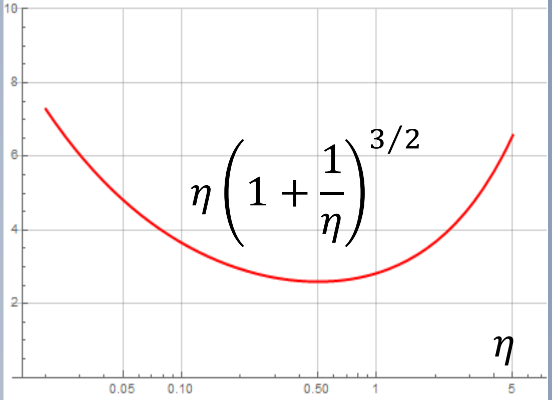
Frequency (MHz):	1300
Transition temperature (K):	9.2
DELTA/kTc:	1.86
London penetration depth (A):	330
Coherence length (A):	400
RRR:	300
Accuracy of computation:	0.001
Temperature (of operation):	2

Be carefureful here. The website suggests 40 nm. The input required is $\pi\xi_0/2$, while $\xi_0 \approx 38$ nm for Nb.

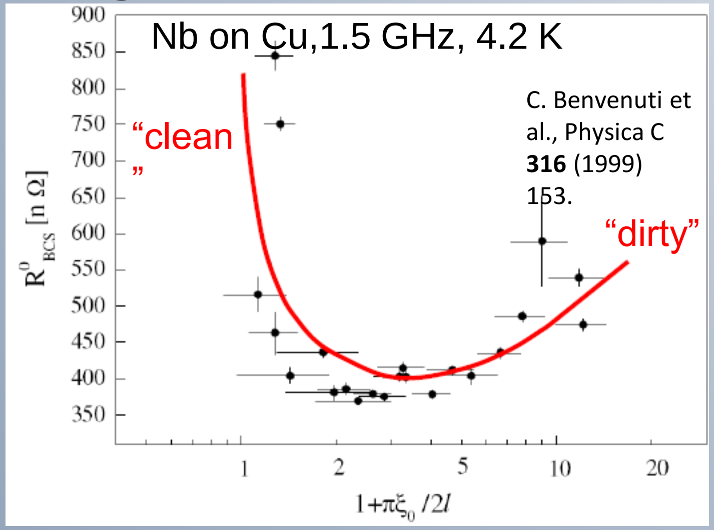
R_s dependence on material purity

- We had introduced above:
  - $\lambda(\ell) = \lambda_L \sqrt{1 + \frac{\xi_0}{\ell}}$
  - $\sigma_1 \propto \ell$

$$R_s \propto \left(1 + \frac{\xi_0}{\ell}\right)^{3/2} \frac{\ell}{\xi_0}$$



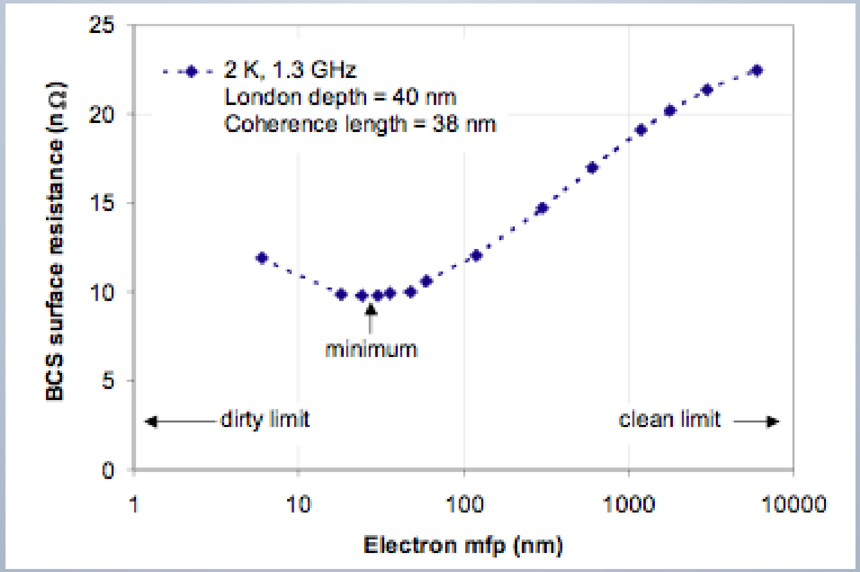
- It follows that R_s has a local minimum at $\ell/\xi_0 = 1/2$.
- This is remarkable: at some point, increasing ℓ (cleaner material) makes things worse!



- This example: Nb films sputtered on Cu
- By changing the sputtering species, the mean free path was varied.
- RRR of niobium on copper cavities can be tuned for lowest R_S.

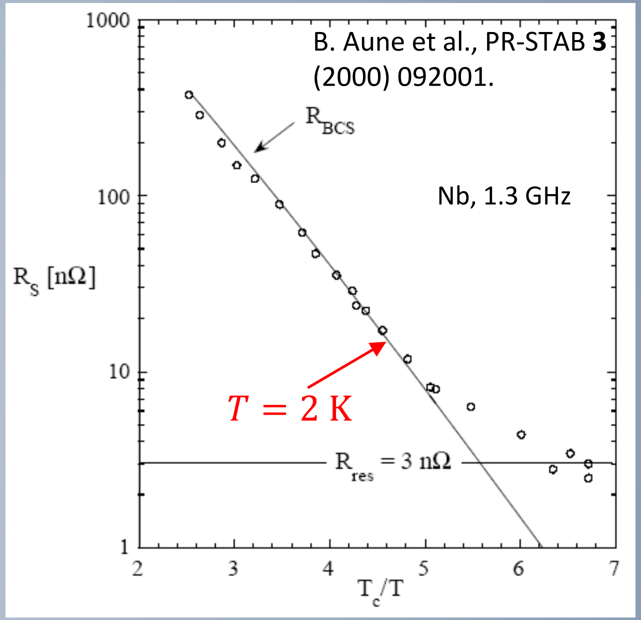
BCS resistance

- $$R_{s,BCS} \approx \frac{A}{2}\omega^2\mu^2\left(\lambda_L\sqrt{1+\frac{\xi}{\ell}}\right)^3 \frac{RRR}{\rho_n(300\ \mathrm{K})}\, e^{-\frac{\Delta}{k_B T_c}\frac{T_c}{T}}$$
$$\approx 1.643\cdot 10^{-5}\,\frac{T_c}{T}\left(\frac{f}{\mathrm{GHz}}\right)^2 e^{-1.92\frac{T_c}{T}}$$



Residual resistance

- In real, technical superconductors, the observed surface resistance deviates from the BCS prediction and can be written as $R_s = R_{BCS} + R_{\mathrm{res}}$.
- Possible contributions to R_{res}:
    - Trapped magnetic flux and thermal currents
    - Lossy oxides, metallic hydrides
    - Normal-conducting precipitates
    - Grain boundaries
    - Interface losses
    - Magnetic impurities
    - Subgap states



For Nb, $R_{res} \approx (1 \dots 10)\ \mathrm{n\Omega}$ often dominates R_s at low f (< 1 GHz) and low T (< 2 K).