Lattice Design II: Insertions

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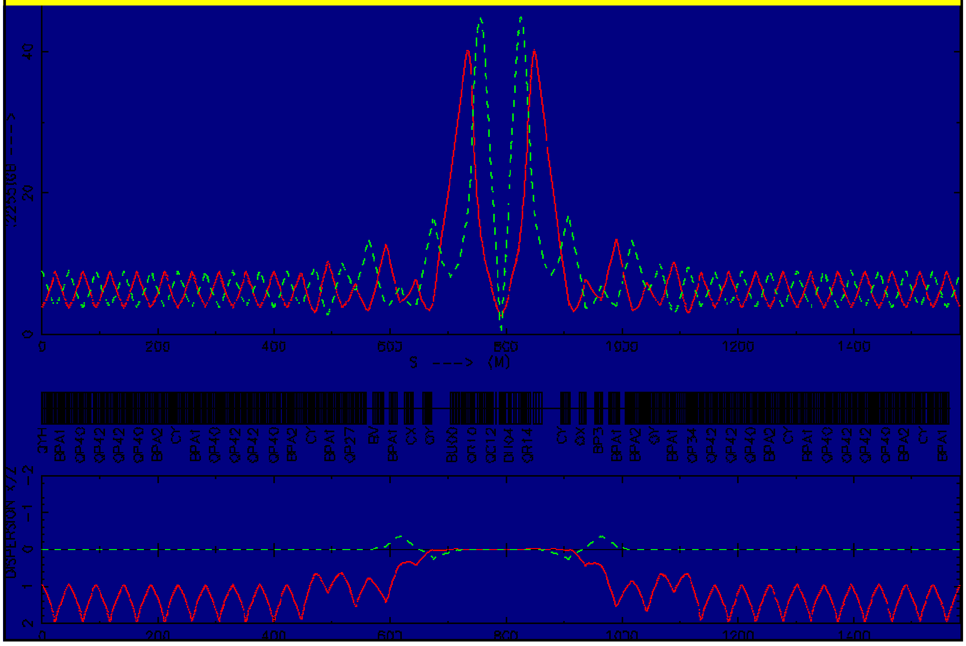
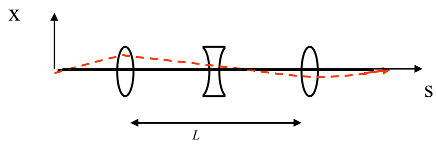


Reminder:

... Particle trajectories in a storage ring are defined by the external fields of the magnets that are placed in the lattice.
The coordinates of this trajectories (with respect to the closed orbit) can be calculated using a matrix formalism that describes the effect of the lattice elements.

equation of motion

$$x'' + K(s) * x = 0$$



particle coordinates

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M * \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

e.g. matrix for a quadrupole lens:

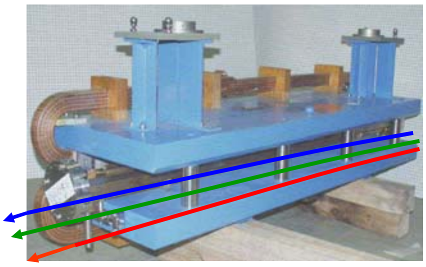
$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\psi & \frac{1}{\sqrt{K}}\sin\psi \\ -\sqrt{K}\sin\psi & \cos\psi \end{pmatrix}$$

$$\psi = s * \sqrt{K} \qquad K = -k + 1/\rho^2$$

Dispersion:

describes the motion of particles with momentum deviation Δp/p

$$x'' + K(s) * x = \frac{1}{\rho}\frac{\Delta p}{p}$$



particles with different momentum are running on a different closed orbit

→special solution of the inhomogeneous differential equation:

$$x_i(s) = D(s) * \Delta p / p$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p}\begin{pmatrix} D \\ D' \end{pmatrix}$$

→ extend the matrices to include D, D´

$$\begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_0$$

Dispersion:

the dispersion function D(s) is (...obviously) defined by the focusing properties of the lattice and from position s_0 to s in the lattice it is given by:

$$D(s) = S(s) * \int_{s_0}^{s} \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int_{s_0}^{s} \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

! weak dipoles → large bending radius → small dispersion

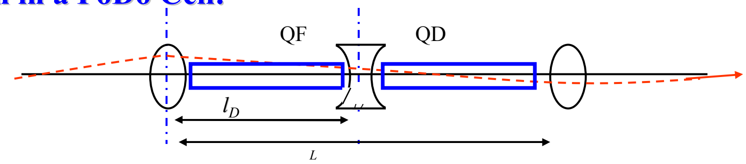
Example: Drift

$$M_D = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \qquad D(s) = S(s) * \int \underbrace{\frac{1}{\rho(\tilde{s})}}_{=0} C(\tilde{s}) d\tilde{s} - C(s) * \int \underbrace{\frac{1}{\rho(\tilde{s})}}_{=0} S(\tilde{s}) d\tilde{s}$$

$$\rightarrow M_D = \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

...in similar way for quadrupole matrices,
!!! in a quite different way for dipole matrix (see appendix)

Dispersion in a FoDo Cell:



!! we have now introduced dipole magnets in the FoDo:

→ we still neglect the weak focusing contribution $1/\rho^2$

→ but take into account $1/\rho$ for the dispersion effect

1.) calculate the matrix of the FoDo half cell in thin lens approximation:

in analogy to the derivations of $\hat{\beta}, \check{\beta}$

* thin lens approximation: $f = \frac{1}{K \ell_Q} >> \ell_Q$

* length of quad negligible $\ell_Q \approx 0, \quad \rightarrow \quad \ell_D = \frac{1}{2} L$

* start at half quadrupole $\frac{1}{\tilde{f}} = \frac{1}{2f}$

matrix of the half cell

$$M_{half\ Cell} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\tilde{f}} & 1 \end{pmatrix} * \begin{pmatrix} 1 & \ell_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ \frac{-1}{\tilde{f}} & 1 \end{pmatrix}$$

... neglecting as usual the weak focusing term of the dipole

$$M_{half\ Cell} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 - \frac{\ell_D}{\tilde{f}} & \ell_D \\ \frac{-\ell_D}{\tilde{f}^2} & 1 + \frac{\ell_D}{\tilde{f}} \end{pmatrix}$$

2.) calculate the dispersion terms D, D´ from the matrix elements

$$D(s) = S(s) * \int_{s0}^{s} \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int_{s0}^{s} \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

$$D(\ell_D) = \ell_D * \frac{1}{\rho} * \int_0^{\ell_D} \left(1 - \frac{s}{\tilde{f}}\right) ds - \left(1 - \frac{\ell_D}{\tilde{f}}\right) * \frac{1}{\rho} * \int_0^{\ell_D} s\, ds$$

$$D(\ell_D) = \frac{\ell_D}{\rho}\left(\ell_D - \frac{\ell_D^2}{2\tilde{f}}\right) - \left(1 - \frac{\ell_D}{\tilde{f}}\right) * \frac{1}{\rho} * \frac{\ell_D^2}{2} = \frac{\ell_D^2}{\rho} - \frac{\ell_D^3}{2\tilde{f}\rho} - \frac{\ell_D^2}{2\rho} + \frac{\ell_D^3}{2\tilde{f}\rho}$$

$$D(\ell_D) = \frac{\ell_D^2}{2\rho}$$

in full analogy on derives for D´:

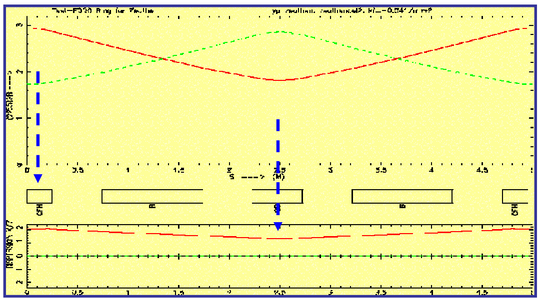
$$D'(\ell_D) = \frac{\ell_D}{\rho}\left(1 + \frac{\ell_D}{2\tilde{f}}\right)$$

and we get the complete matrix including the Dispersion terms D, D‘

$$M_{half\ Cell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\ell_D}{\tilde{f}} & \ell_D & \frac{\ell_D^{\,2}}{2\rho} \\ \frac{-\ell_D}{\tilde{f}^2} & 1 + \frac{\ell_D}{\tilde{f}} & \frac{\ell_D}{\rho}\left(1 + \frac{\ell_D}{2\tilde{f}}\right) \\ 0 & 0 & 1 \end{pmatrix}$$

boundary conditions for the transfer from the center of the foc. to the center of the defoc. quadrupole

$$\begin{pmatrix} \check{D} \\ 0 \\ 1 \end{pmatrix} = M_{1/2} * \begin{pmatrix} \hat{D} \\ 0 \\ 1 \end{pmatrix}$$



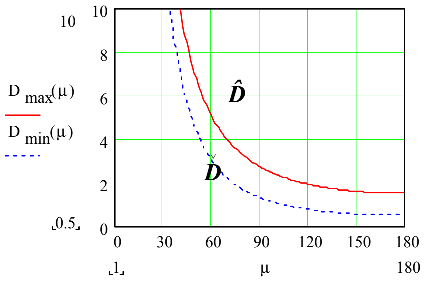
Dispersion in a FoDo Cell

$$\rightarrow \quad \check{D} = \hat{D}\left(1 - \frac{\ell_D}{\tilde{f}}\right) + \frac{\ell_D^{\,2}}{2\rho}$$

$$\rightarrow \quad 0 = -\frac{\ell_D}{\tilde{f}^2} * \hat{D} + \frac{\ell_D}{\rho}\left(1 + \frac{\ell_D}{2\tilde{f}}\right)$$

$$\hat{D} = \frac{\ell_D^{\,2}}{\rho} * \frac{\left(1 + \frac{1}{2}\sin\frac{\mu}{2}\right)}{\sin^2\frac{\mu}{2}} \qquad \check{D} = \frac{\ell_D^{\,2}}{\rho} * \frac{\left(1 - \frac{1}{2}\sin\frac{\mu}{2}\right)}{\sin^2\frac{\mu}{2}}$$

where μ denotes the phase advance of the full cell and $l/f = \sin(\mu/2)$



Nota bene:

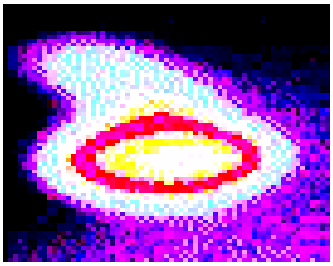
! small dispersion needs strong focusing
→ large phase advance

!! ↔ there is an optimum phase for small β

!!! ...do you remember the stability criterion?
½ trace = cos μ ↔ μ < 180°

!!!! … life is not easy

Reminiszenz: Synchrotron light sources



the emittance of an electron beam in a storage ring:

$$\varepsilon_x = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \gamma^2 \frac{\left\langle \frac{1}{R^3} H(s) \right\rangle}{J_x \left\langle \frac{1}{R^2} \right\rangle}$$

γ = relativistic gamma,
R = bending radius of the dipole magnets
J_x = damping partition number ≈ 1

$$H(s) = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

„low emittance lattices" → low Dispersion lattices
(see lecture of A.Streun in this school)

Chromaticity Correction in lattice cells

Remember Definition $\Delta Q = \xi * \frac{\Delta p}{p}$

in FoDo Cells: $\xi_{Cell} = -\frac{1}{\pi} * \tan\frac{\mu}{2}$

Example HERA: μ ≈ 90° ,	number of cells 104 → ξ ≈ -33
	including mini beta sections …→ ξ ≈ -42…-80
	Δp/p ≈ 0.5*10^{-3} →ΔQ ≈ 0.04
	compare to nominal tune: Q = 0.292

Chromaticity Correction in lattice cells

1.) sort the particles as a function of their momentum $x(s) = D(s) * \frac{\Delta p}{p}$

2.) create magnetic field with linear increasing gradient $\frac{\partial B_y}{\partial x} = g' * x, \quad g' = const$
normalised to the momentum: $m_{6pol} = g'/(p/e)$

3.) calculate resulting quadrupol strength $K = m_{6pol} * x = m_{6pol} * D * \frac{\Delta p}{p}$

4.) resulting overall chromaticity of the storage ring

$$\xi_{total} = \frac{-1}{4\pi} \oint \{K(s)\beta(s) - m(s)D(s)\beta(s)\} ds$$

Hints for the lattice design:

- ! avoid large β values
- ! provide space where $D(s) \neq 0$ for installation of 6pol magnets
- !! put sextupoles at places where β is large → close to the quads
- !!! in general ξ is created at locations in the lattice where it cannot be corrected (… as D(s) = 0)
- !!!! … life is not easy

part of LEP lattice for ξ correction

Lattice Design: Insertions

I.) ... the most complicated one: the drift space

Question to the auditorium: what will happen to the beam parameters α, β, γ if we stop focusing for a while …?

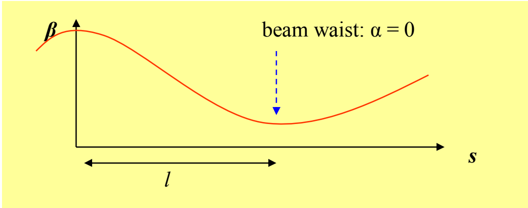
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

transfer matrix for a drift: $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$
$$\alpha(s) = \alpha_0 - \gamma_0 s$$
$$\gamma(s) = \gamma_0$$

„0“ refers to the position of the last lattice element
„s“ refers to the position in the drift

location of the waist:



given the initial conditions $\alpha_0, \beta_0, \gamma_0$: where is the point of smallest beam dimension in the drift … or at which location occurs the beam waist ?

beam waist: $\alpha(\ell) = 0 \quad \rightarrow \quad \alpha_0 = \gamma_0 * \ell$

$$\ell = \frac{\alpha_0}{\gamma_0}$$

beam size at that position:

$$\left.\begin{array}{l}\gamma(\ell) = \gamma_0 \\ \alpha(\ell) = 0\end{array}\right\} \rightarrow \gamma(l) = \frac{1+\alpha^2(\ell)}{\beta(\ell)} = \frac{1}{\beta(\ell)} \qquad \beta(\ell) = 1/\gamma_0$$

β-Function in a Drift:

let's assume we are at a symmetry point in the center of a drift.

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

as $\alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$

and we get for the β function in the neighborhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$ **! ! !**

Nota bene:
1.) this is very bad !!!
2.) this is a direct consequence of the conservation of phase space density (... in our words: ε = const) … and there is no way out.
3.) Thank you, Mr. Liouville !!!

Joseph Liouville, 1809-1882

If we cannot fight against Liouvuille theorem ... at least we can optimise

Optimisation of the beam dimension:

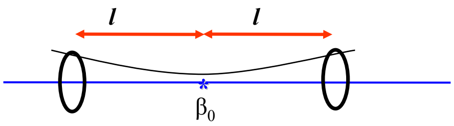
$$\beta(\ell) = \beta_0 + \frac{\ell^2}{\beta_0}$$

Find the β at the center of the drift that leads to the lowest maximum β at the end:

$$\frac{d\hat{\beta}}{d\beta_0} = 1 - \frac{\ell^2}{\beta_0^2} \overset{!}{=} 0$$

$$\rightarrow \quad \beta_0 = \ell$$

$$\rightarrow \quad \hat{\beta} = 2\beta_0$$

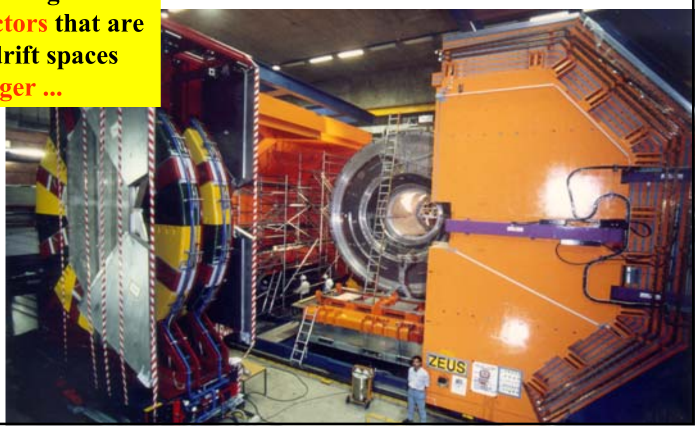


If we choose $\beta_0 = \ell$ we get the smallest β at the end of the drift and the maximum β is just twice the distance ℓ

... clearly there is another problem !!!

Example: Luminosity optics at HERA: $\beta^* = 18$ cm
for smallest β_{max} we have to limit the overall length of the drift to $L = 2*\ell$
$L = 36$ cm

But: ... unfortunately ... in general high energy detectors that are installed in that drift spaces are a little bit bigger ...

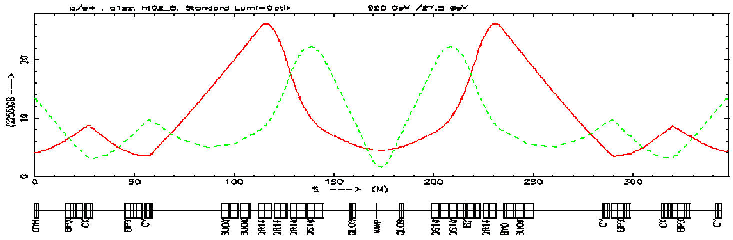


The Mini-β Insertion:

Event rate of a collider ring for a reaction with cross section σ_R : $$R = \sigma_R * L$$

Luminosity: given by the total stored beam currents and the beam size at the collision point (IP)

$$L = \frac{1}{4\pi e^2 f_0 b} * \frac{I_1 * I_2}{\sigma^*_x * \sigma^*_y}$$



How to create a mini β insertion:

- **symmetric drift space (length adequate for the experiment)**
- **quadrupole doublet on each side (as close as possible)**
- **additional quadrupole lenses to match twiss parameters to the periodic cell in the arc**

Mini-β Insertions: Betafunctions

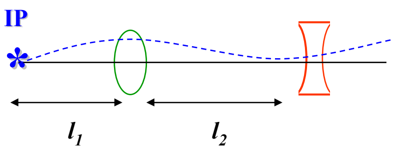
A mini-β insertions is always a kind of special symmetric drift space.
→greetings from Liouville

$$\alpha^* = 0 \quad \rightarrow \quad \beta^* = \frac{\sigma^*}{\sigma'^*}$$

at a symmetry point β is just the ratio of beam dimension and beam divergence.
… which gives us the size of β at the first quadrupole

$$\beta(s) = \beta^* + \frac{s^2}{\beta^*}$$

size of β at the second quadrupole lens (in thin lens approx):
… after some transformations and a couple of beer …



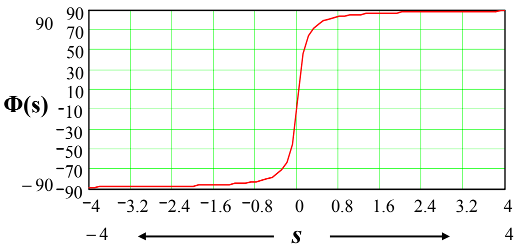
$$\beta(s) = \left(1 + \frac{l_2}{f_1}\right)^2 * \beta^* + \frac{1}{\beta^*}\left(l_1 + l_2 + \frac{l_1 l_2}{f_1}\right)^2$$

Mini-β Insertions: Phase advance

By definition the phase advance is given by: $\Phi(s) = \int \frac{1}{\beta(s)} ds$

Now in a mini β insertion: $\beta(s) = \beta_0 \left(1 + \frac{s^2}{\beta_0^2}\right)$

$$\rightarrow \quad \Phi(s) = \frac{1}{\beta_0} \int_0^L \frac{1}{1 + s^2/\beta_0^2} ds \;=\; \arctan \frac{L}{\beta_0}$$



summing the drift spaces on both sides of the IP the phase advance of a mini β insertion is approximately π, in other words: the tune will increase by half an integer.

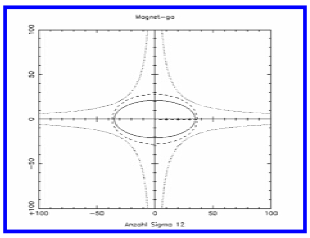
are there any problems ??

sure there are...

* **large β values at the doublet quadrupoles → large contribution to chromaticity ξ … and no local correction**

$$\xi = \frac{-1}{4\pi} \oint \{K(s)\beta(s)\}\, ds$$

* **aperture of mini β quadrupoles can limit the luminosity**



beam envelope at the first mini β quadrupole lens in the HERA proton storage ring

• **field quality and magnet stability most critical at the high β sections**
orbit distortion due to a kick:

$$x(s) = \frac{\sqrt{\beta(s)}}{2 \sin \pi Q} * \oint \sqrt{\beta(\tilde{s})} \frac{1}{\rho(\tilde{s})} \cos(|\phi(\tilde{s}) - \phi(s)| - \pi Q)\, d\tilde{s}$$

→ keep distance „s“ to the first mini β quadrupole as small as possible

Mini-β Insertions: some Guide lines

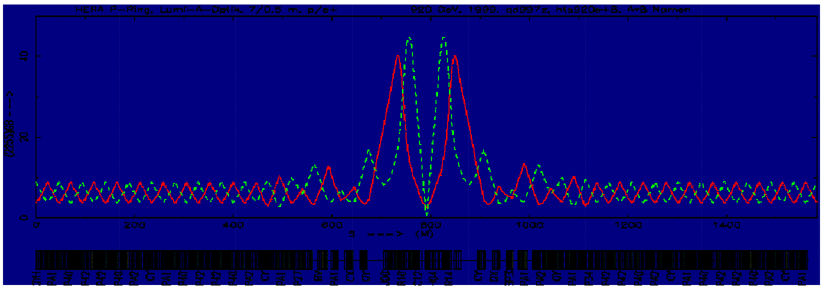
Optics guide line:

- **calculate the periodic solution in the arc**
- **introduce the drift space needed for the insertion device (detector ...)**
- **put a quadrupole doublet (triplet ?) as close as possible**
- **introduce additional quadrupole lenses to match the beam parameters α, β, γ, D, D' o the values at hte beginning of the arc structure**

parameters to be optimised & matched to the periodic solution:

$$\left.\begin{array}{ll} \alpha_x, & \beta_x \\ \alpha_y, & \beta_y \\ D_x, & D_x' \\ Q_x, & Q_y \end{array}\right\} \rightarrow$$

8 individually powered quadrupole magnets are needed to match the insertion (...at least)



Dispersion Suppressors

There are two comments of paramount importance about dispersion:

! it is nasty

!! it is not easy to get rid of it.

remember: oscillation amplitude for a particle with momentum deviation

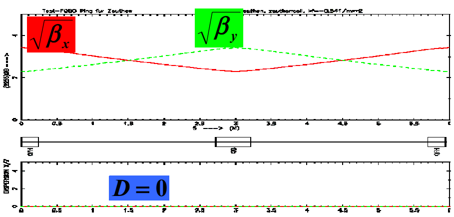
$$x(s) = x_\beta(s) + D(s) * \frac{\Delta p}{p}$$

Example:
beam size at the IP in HERA
average dispersion in the arc:
typical momentum spread:

$$\sigma_x^* = 118\ \mu m, \quad \sigma_y^* = 32\ \mu m$$

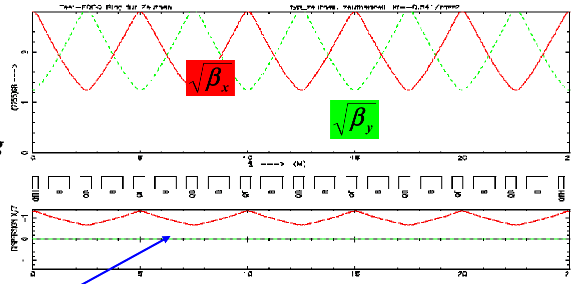
$$\left.\begin{array}{l} \bar{D}(s) = 1.5\ m \\ \frac{\Delta p}{p} \approx 5 * 10^{-4} \end{array}\right\} \rightarrow x_D \approx 0.75\ mm$$

Dispersion spoils the luminosity and leads to additional stop bands (synchro-betatron resonances) in RF sections and at the IP



optical functions of a FoDo Cell without dipoles: D=0

Dispersion Suppressors



Optical functions in a FoDo Cell including the effect of the bending magnets

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{(1 + \frac{1}{2}\sin\frac{\mu}{2})}{\sin^2\frac{\mu}{2}}$$

$$\check{D} = \frac{\ell^2}{\rho} * \frac{(1 - \frac{1}{2}\sin\frac{\mu}{2})}{\sin^2\frac{\mu}{2}}$$

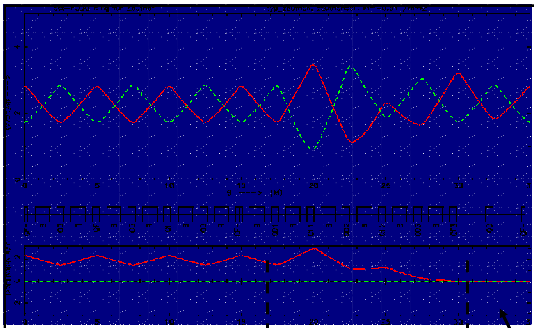
There are several ways to suppress the dispersion function in a lattice:

I.) The straight forward one: use additional quadrupole lenses to match the optical parameters ... including the D(s), D´(s) terms

optical functions to match:

$$\left.\begin{matrix} D(s), \quad D'(s) \\ \beta_x(s), \alpha_x(s) \\ \beta_y(s), \alpha_y(s) \end{matrix}\right\} \rightarrow$$ **6 additional quadrupole lenses required**

Dispersion Suppressors



periodic FoDo structure

matching section including 6 additional quadrupoles

Dispersion free section, regular FoDo without dipoles

Optical functions in a FoDo Cell

•Dispersion suppressed by 2 quadrupole lenses,

•β and α restored to the values of the periodic solution by 4 additional quadrupoles

Advantage:
- easy,
- flexible: it works for any phase advance per cell
- does not change the geometry of the storage ring,
- can be used to match between different lattice structures (i.e. phase advances)

Disadvantage:
- additional power supplies needed (⟶ expensive)
- requires stronger quadrupoles
- due to higher β values: more aperture required

Dispersion Suppressors

II.) The clever way: half bend schemes

Desperate statement of the lecturer:
the mathematical derivation of the dispersion suppressor is a bit awkward.
→put it into the appendix
→state the following conditions:

$$\left.\begin{aligned} \rightarrow\quad & 2\delta_{supr}\, sin^2\left(\frac{n\Phi_C}{2}\right) = \delta_{arc} \\ \rightarrow\quad & sin(n\Phi_C) = 0 \end{aligned}\right\}\quad \delta_{supr} = \frac{1}{2}\delta_{arc}$$

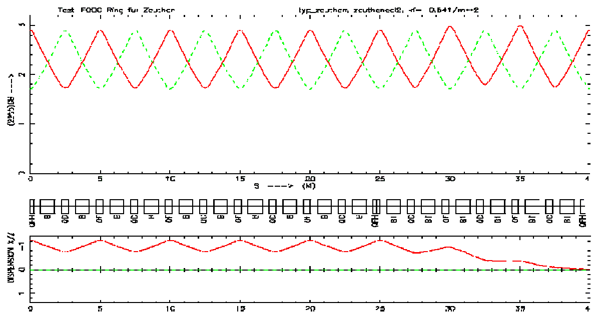
$$n\Phi_C = k * \pi, \quad k = 1,3, \ldots$$

strength of suppressor dipoles is half as strong as that of arc dipoles

in the *n* suppressor cells the phase advance has to accumulate to a odd multiple of π

Example:

phase advance in the arc $\Phi_C = 60°$
number of suppr. cells $n = 3$
$\delta_{suppr} = 1/2\ \delta_{arc}$



Dispersion Suppressors

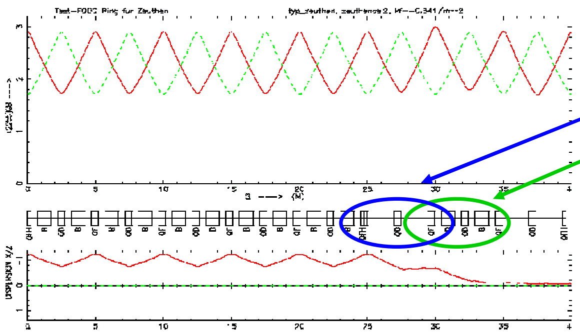
III.) The clever way: missing bend schemes

at the end of the arc: add *m* cells without dipoles followed by *n* regular arc cells.
condition for dispersion suppression:

$$\frac{2m+n}{2}\Phi_C = (2k+1)\frac{\pi}{2}$$

$$sin\frac{n\Phi_C}{2} = \frac{1}{2}, \quad k = 0,2\ldots \quad or$$

$$sin\frac{n\Phi_C}{2} = \frac{-1}{2}, \quad k = 1,3\ldots$$



Example:

phase advance in the arc $\Phi_C = 60°$
number of suppr. cells $m = 1$
number of regular cells $n = 1$

Resume‘

1.) Dispersion in a FoDo cell:

small dispersion ↔ large bending radius
short cells
strong focusing

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{(1+\frac{1}{2}\sin\frac{\mu}{2})}{\sin^2\frac{\mu}{2}}$$

2.) Chromaticity of a cell:

small ξ ↔ weak focusing
small β

$$\xi_{total} = \frac{-1}{4\pi}\oint\{K(s)\beta(s) - m(s)D(s)\beta(s)\}\,ds$$

3.) Position of a waist at the cell end:

α_0 β_0 = values at the end of the cell

$$\ell = \frac{\alpha_0}{\gamma_0} \qquad \beta(\ell) = \frac{1}{\gamma_0}$$

4.) β function in a drift

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

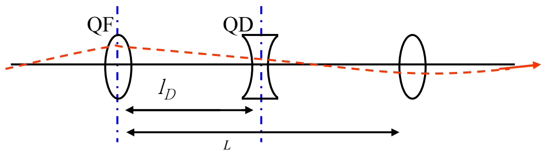
5.) Mini β insertion

small β↔ short drift space required
phase advance ≈ 180 °

$$\beta(\ell) = \beta_0 + \frac{\ell^2}{\beta_0}$$

Appendix: some usefull formulae in more detail

Dispersion in a FoDo Cell:



in analogy to the derivations of $\hat{\beta}, \check{\beta}$

* thin lens approximation: $f = \frac{1}{Kl_Q} >> l_Q$

* length of quad negligible $l_Q \approx 0, \quad \rightarrow \quad l_D = \frac{1}{2}L$

* start at half quadrupole $\frac{1}{\tilde{f}} = \frac{1}{2f}$

$$M_{Half\,Cell} = M_{\frac{QD}{2}} * M_B * M_{\frac{QF}{2}}$$

matrix of the half cell

$$M = \begin{pmatrix} 1 & 0 \\ \frac{1}{\tilde{f}} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ \frac{-1}{\tilde{f}} & 1 \end{pmatrix}$$

... neglecting as usual the weak focusing term of the dipole

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 - \frac{l}{\tilde{f}} & l \\ \frac{-l}{\tilde{f}^2} & 1 + \frac{l}{\tilde{f}} \end{pmatrix}$$

$$D(l) = S(l) * \int \frac{1}{\rho(s)} C(s) ds - C(l) * \int \frac{1}{\rho(s)} S(s) ds$$

$$D(l) = l * \frac{1}{\rho} * \int_0^l \left(1 - \frac{s}{\tilde{f}} \right) ds - \left(1 - \frac{l}{\tilde{f}} \right) * \frac{1}{\rho} * \int_0^l s\, ds$$

$$= \frac{l}{\rho} \left(l - \frac{l^2}{2\tilde{f}} \right) - \left(1 - \frac{l}{\tilde{f}} \right) * \frac{1}{\rho} * \frac{l^2}{2} = \frac{l^2}{\rho} - \frac{l^3}{2\tilde{f}\rho} - \frac{l^2}{2\rho} + \frac{l^3}{2\tilde{f}\rho}$$

$$D(l) = \frac{l^2}{2\rho}$$

Expression for D´

$$D'(l) = S'(l) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C'(l) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

$$D'(l) = (1 + \frac{l}{\tilde{f}}) * \frac{1}{\rho} * \int_0^l \left(1 - \frac{s}{\tilde{f}} \right) ds + \frac{l}{\tilde{f}^2} * \frac{1}{\rho} * \int_0^l s\, ds$$

$$= \left(1 + \frac{l}{\tilde{f}} \right) * \frac{1}{\rho} * (l - \frac{l^2}{2\tilde{f}}) + \frac{l}{\tilde{f}^2} * \frac{1}{\rho} * \frac{l^2}{2}$$

$$= \left(\frac{1}{\rho} + \frac{l}{\tilde{f}\rho} \right) * (l - \frac{l^2}{2\tilde{f}}) + \frac{l^3}{2\rho\tilde{f}^2}$$

$$= \frac{l}{\rho} + \frac{l^2}{2\tilde{f}\rho}$$

$$D'(l) = \frac{l}{\rho} (1 + \frac{l}{2\tilde{f}})$$

Complete Matrix including the terms for D, D':

$$M = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{l}{\tilde{f}} & l & \frac{l^2}{2\rho} \\ \frac{-l}{\tilde{f}^2} & 1+\frac{l}{\tilde{f}} & \frac{l}{\rho}(1+\frac{l}{2\tilde{f}}) \\ 0 & 0 & 1 \end{pmatrix}$$

require boudary conditions for half cell solution:

$$\begin{pmatrix} \check{D} \\ 0 \\ 1 \end{pmatrix} = M_{1/2} * \begin{pmatrix} \hat{D} \\ 0 \\ 1 \end{pmatrix} \qquad \begin{aligned} &\rightarrow \check{D} = \hat{D}(1-\frac{l}{\tilde{f}}) + \frac{l^2}{2\rho} && (1) \\ &\rightarrow 0 = -\frac{l}{\tilde{f}^2} * \hat{D} + \frac{l}{\rho}(1+\frac{l}{2\tilde{f}}) && (2) \end{aligned}$$

$$(1)-(2)*\tilde{f} \qquad \rightarrow \check{D} = \hat{D} + \frac{l^2}{2\rho} - \frac{l\tilde{f}}{\rho}(1+\frac{l}{2\tilde{f}})$$

$$\check{D} = \hat{D} + \frac{l^2}{2\rho} - \frac{l\tilde{f}}{\rho} + \frac{l^2}{2\rho} = \underline{\underline{\hat{D} - \frac{l\tilde{f}}{\rho}}}$$

put result in (1)

$$\rightarrow \hat{D} - \frac{l\tilde{f}}{\rho} = \hat{D} - \hat{D}\frac{l}{\tilde{f}} + \frac{l^2}{2\rho}$$

$$\frac{l\tilde{f}}{\rho} + \frac{l^2}{2\rho} = \hat{D}\frac{l}{\tilde{f}}$$

$$\hat{D} = \frac{\tilde{f}^2}{\rho} + \frac{\tilde{f}l}{2\rho} = \frac{\tilde{f}^2}{\rho}(1+\frac{l}{2\tilde{f}})$$

remember: $\frac{l}{\tilde{f}} = \sin\frac{\mu}{2}$

$$\rightarrow \hat{D} = \frac{\tilde{f}^2}{\rho}(1+\frac{1}{2}\sin\frac{\mu}{2}) \qquad \rightarrow \check{D} = \frac{\tilde{f}^2}{\rho}(1-\frac{1}{2}\sin\frac{\mu}{2})$$

$$\hat{D} = \frac{\tilde{f}^2}{\rho}\frac{l^2}{l^2}(1+\frac{1}{2}\sin\frac{\mu}{2})$$

$$\boxed{\hat{D} = \frac{l^2}{\rho}\left(\frac{1+\frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right) \qquad \check{D} = \frac{l^2}{\rho}\left(\frac{1-\frac{1}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right)}$$

Dispersion: **Example: Dipole sector magnet** $\psi = \ell * \sqrt{K}$ $k = 0, \quad K = 1/\rho^2$

K describes in this case only the weak focusing term in the dipole

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\psi & \frac{1}{\sqrt{K}}\sin\psi \\ -\sqrt{K}\sin\psi & \cos\psi \end{pmatrix} = \begin{pmatrix} \cos\frac{\ell}{\rho} & \rho\sin\frac{\ell}{\rho} \\ \frac{-1}{\rho}\sin\frac{\ell}{\rho} & \cos\frac{\ell}{\rho} \end{pmatrix}$$

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

$$= \rho \sin\left(\frac{\ell}{\rho}\right) * \frac{1}{\rho} * \rho * \sin\left(\frac{\ell}{\rho}\right) - \cos\left(\frac{\ell}{\rho}\right) * \frac{1}{\rho} * \rho * \left(1 - \cos\left(\frac{\ell}{\rho}\right)\right) * \rho$$

$$= \rho \sin^2\left(\frac{\ell}{\rho}\right) + \rho * \cos\left(\frac{\ell}{\rho}\right) * (\cos\left(\frac{\ell}{\rho}\right) - 1)$$

$$= \rho\left\{\sin^2\left(\frac{\ell}{\rho}\right) + \cos^2\left(\frac{\ell}{\rho}\right) - \cos\left(\frac{\ell}{\rho}\right)\right\}$$

Dispersion: Example: Dipole sector magnet

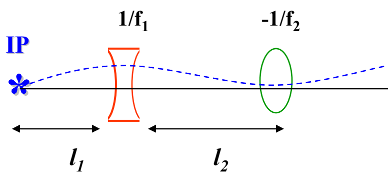
$$D(l) = \rho * (1 - \cos\frac{\ell}{\rho}) \qquad D'(l) = \sin(\frac{\ell}{\rho})$$

Using these expressions for D and D‘ the extended matrix of a dipole sector magnet is given by:

$$M = \begin{pmatrix} \cos\psi & \rho\sin\psi & \rho(1-\cos\psi) \\ \frac{-1}{\rho}\sin\psi & \cos\psi & \sin\psi \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad \psi = \ell/\rho$$

The Mini-β Insertion: value of the β function at the position of the second lens

thin lens approximation for the mini β doublet:



marices for the elements in the mini-β region

$$M_{D1} = \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix} \quad , \quad M_{D2} = \begin{pmatrix} 1 & l_2 \\ 0 & 1 \end{pmatrix}$$

$$M_{F1} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_1} & 1 \end{pmatrix} \quad , \quad M_{F2} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{pmatrix}$$

$-1/f_2$=hor. fok. lens

$$M_{total} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_2 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ \frac{1}{f_1} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix}$$

$$M_{total} = \begin{pmatrix} 1 + \frac{l_2}{f_1} & l_1 + l_2 + \frac{l_1 l_2}{f_1} \\ \frac{1}{f_1} - \frac{1}{f_2} - \frac{l_2}{f_1 f_2} & -\frac{l_1}{f_2} - \frac{l_1 l_2}{f_1 f_2} - \frac{l_2}{f_2} + 1 + \frac{l_1}{f_1} \end{pmatrix}$$

transformation of twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

$$\beta(s) = C^2 * \beta_0 - 2SC * \alpha_0 + S^2 * \gamma_0$$

now we add the boundary condition for a symmetric problem: $\alpha_0 = 0$ $\quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$

$$\beta(s) = C^2 * \beta_0 + S^2 / \beta_0$$

using the matrix elements calculcated above for the doublet:

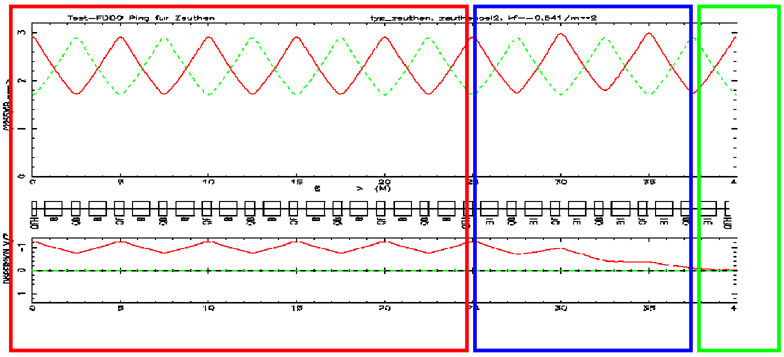
$$\beta(s) = (1 + \frac{l_2}{f_1})^2 * \beta_0 + \frac{1}{\beta_0} \left(l_1 + l_2 + \frac{l_1 l_2}{f_1} \right)^2$$

Dispersion Suppressors

... the calculation in full detail (for purists only)

1.) the lattice is split into 3 parts: ***(Gallia divisa est in partes tres)***

* periodic solution of the arc	periodic β, periodic dispersion D
* section of the dispersion suppressor	periodic β, dispersion vanishes
* FoDo cells without dispersion	periodic β, D = D′ = 0



2.) calculate the dispersion D in the periodic part of the lattice

transfer matrix of a periodic cell:

$$M_{0\to s}=\begin{pmatrix}\sqrt{\frac{\beta_s}{\beta_0}}(\cos\phi+\alpha_0\sin\phi) & \sqrt{\beta_s\beta_0}\sin\phi\\ \frac{(\alpha_0-\alpha_s)\cos\phi-(1+\alpha_0\alpha_s)\sin\phi}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos\phi-\alpha_s\sin\phi)\end{pmatrix}$$

for the transformation from one symmetriy point to the next (i.e. one cell) we have: Φ_C = phase advance of the cell, α = 0 at a symmetry point. The index "*c*" refers to the periodic solution of one cell.

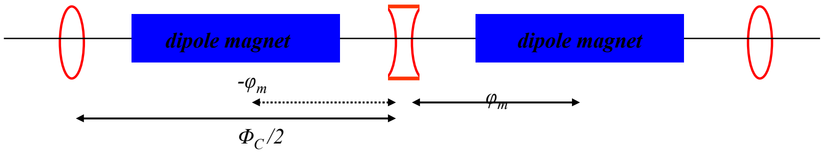
$$M_{Cell}=\begin{pmatrix}C & S & D\\ C' & S' & D'\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}\cos\Phi_C & \beta_C\sin\Phi_C & D(l)\\ \frac{-1}{\beta_C}\sin\Phi_C & \cos\Phi_C & D'(l)\\ 0 & 0 & 1\end{pmatrix}$$

The matrix elements D and D‘ are given by the C and S elements in the usual way:

$$D(l)=S(l)*\int_0^l\frac{1}{\rho(\tilde s)}C(\tilde s)d\tilde s-C(l)*\int_0^l\frac{1}{\rho(\tilde s)}S(\tilde s)d\tilde s$$

$$D'(l)=S'(l)*\int_0^l\frac{1}{\rho(\tilde s)}C(\tilde s)d\tilde s-C'(l)*\int_0^l\frac{1}{\rho(\tilde s)}S(\tilde s)d\tilde s$$

here the values C(l) and S(l) refer to the symmetry point of the cell (middle of the quadrupole) and the integral is to be taken over the dipole magnet where $\rho \neq 0$. For ρ = const the integral over C(s) and S(s) is approximated by the values in the middle of the dipole magnet.



Transformation of C(s) from the symmetry point in the foc. quad to the center of the dipole:

$$C_m = \sqrt{\frac{\beta_m}{\beta_C}} \cos \Delta\Phi = \sqrt{\frac{\beta_m}{\beta_C}} \cos\left(\frac{\Phi_C}{2} \pm \varphi_m\right) \qquad S_m = \beta_m \beta_C \sin\left(\frac{\Phi_C}{2} \pm \varphi_m\right)$$

where β_C is the periodic β function at the beginning and end of the cell, β_m its value at the middle of the dipole and φ_m the phase advance from the quadrupole lens to the dipole center.

Now we can solve the intergal for D and D':

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

$$D(l) = \beta_C \sin \Phi_C * \frac{L}{\rho} * \sqrt{\frac{\beta_m}{\beta_C}} * \cos\left(\frac{\Phi_C}{2} \pm \varphi_m\right) - \cos \Phi_C * \frac{L}{\rho} \sqrt{\beta_m \beta_C} * \sin\left(\frac{\Phi_C}{2} \pm \varphi_m\right)$$

$$D(l) = \delta \sqrt{\beta_m \beta_C} \left\{ \sin \Phi_C \left[\cos\left(\frac{\Phi_C}{2} + \varphi_m\right) + \cos\left(\frac{\Phi_C}{2} - \varphi_m\right) \right] - \right.$$
$$\left. - \cos \Phi_C \left[\sin\left(\frac{\Phi_C}{2} + \varphi_m\right) + \sin\left(\frac{\Phi_C}{2} - \varphi_m\right) \right] \right\}$$

I have put $\delta = L/\rho$ for the strength of the dipole

remember the relations $\cos x + \cos y = 2 \cos \frac{x+y}{2} * \cos \frac{x-y}{2}$

$\sin x + \sin y = 2 \sin \frac{x+y}{2} * \cos \frac{x-y}{2}$

$$D(l) = \delta \sqrt{\beta_m \beta_C} \left\{ \sin \Phi_C * 2 \cos \frac{\Phi_C}{2} * \cos \varphi_m - \cos \Phi_C * 2 \sin \frac{\Phi_C}{2} * \cos \varphi_m \right\}$$

$$D(l) = 2\delta \sqrt{\beta_m \beta_C} * \cos \varphi_m \left\{ \sin \Phi_C * \cos \frac{\Phi_C}{2} * - \cos \Phi_C * \sin \frac{\Phi_C}{2} \right\}$$

remember: $\sin 2x = 2 \sin x * \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$$D(l) = 2\delta \sqrt{\beta_m \beta_C} * \cos \varphi_m \left\{ 2 \sin \frac{\Phi_C}{2} * \cos^2 \frac{\Phi_C}{2} - (\cos^2 \frac{\Phi_C}{2} - \sin^2 \frac{\Phi_C}{2}) * \sin \frac{\Phi_C}{2} \right\}$$

$$D(l) = 2\delta\sqrt{\beta_m \beta_C} * \cos\varphi_m * \sin\frac{\Phi_C}{2}\left\{2\cos^2\frac{\Phi_C}{2} - \cos^2\frac{\Phi_C}{2} + \sin^2\frac{\Phi_C}{2}\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m \beta_C} * \cos\varphi_m * \sin\frac{\Phi_C}{2}$$

in full analogy one derives the expression for D‘:

$$D(l) = 2\delta\sqrt{\beta_m / \beta_C} * \cos\varphi_m * \cos\frac{\Phi_C}{2}$$

As we refer the expression for D and D‘ to a periodic struture, namly a FoDo cell we require periodicity conditons:

$$\begin{pmatrix} D_C \\ D'_C \\ 1 \end{pmatrix} = M_C * \begin{pmatrix} D_C \\ D'_C \\ 1 \end{pmatrix}$$

and by symmetry: $D'_C = 0$

With these boundary conditions the Dispersion in the FoDo is determined:

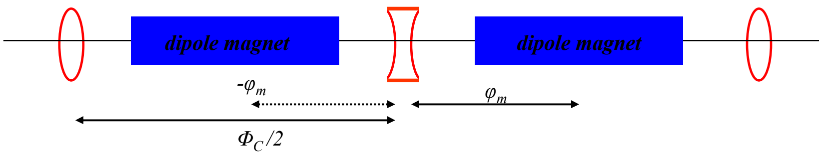
$$D_C * \cos\Phi_C + \delta\sqrt{\beta_m \beta_C} * \cos\varphi_m * 2\sin\frac{\Phi_C}{2} = D_C$$

(A1) $$D_C = \delta\sqrt{\beta_m \beta_C} * \cos\varphi_m / \sin\frac{\Phi_C}{2}$$

This is the value of the periodic dispersion in the cell evaluated at the position of the dipole magnets.

3.) Calculate the dispersion in the suppressor part:

We will now move to the second part of the dispersion suppressor: The section where ... starting from D=D‘=0 the dispesion is generated ... or turning it around where the Dispersion of the arc is reduced to zero.
The goal will be to generate the dispersion in this section in a way that the values of the periodic cell that have been calculated above are obtained.



The relation for D, generated in a cell still holds in the same way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})d\tilde{s}$$

as the dispersion is generated in a number of *n* cells the matrix for these *n* cells is

$$M_n = M_C^n = \begin{pmatrix} \cos n\Phi_C & \beta_C \sin n\Phi_C & D_n \\ \frac{-1}{\beta_C}\sin n\Phi_C & \cos n\Phi_C & D'_n \\ 0 & 0 & 1 \end{pmatrix}$$

replacing the integral over the n cells of the suppressor by the sum over the n cells we obtain for D:

$$D_n = \beta_C \sin n\Phi_C * \delta_{sup\,r} * \sum_{i=1}^{n} \cos(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m) * \sqrt{\frac{\beta_m}{\beta_C}} -$$

$$- \cos n\Phi_C * \delta_{sup\,r} * \sum_{i=1}^{n} \sqrt{\beta_m \beta_C} * \sin(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m)$$

$$D_n = \sqrt{\beta_m\beta_C} * \sin n\Phi_C * \delta_{supr} * \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2} \pm \varphi_m) - \sqrt{\beta_m\beta_C} * \delta_{supr} * \cos n\Phi_C \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2} \pm \varphi_m)$$

remember: $\sin x + \sin y = 2\sin\frac{x+y}{2} * \cos\frac{x-y}{2}$ $\qquad \cos x + \cos y = 2\cos\frac{x+y}{2} * \cos\frac{x-y}{2}$

$$D_n = \delta_{sup\,r} * \sqrt{\beta_m\beta_C} * \sin n\Phi_C * \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2}) * 2\cos\varphi_m -$$

$$- \delta_{sup\,r} * \sqrt{\beta_m\beta_C} * \cos n\Phi_C \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2}) * 2\cos\varphi_m$$

$$D_n = 2\delta_{sup\,r} * \sqrt{\beta_m\beta_C} * \cos\varphi_m \left\{ \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2}) * \sin n\Phi_C - \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2}) * \cos n\Phi_C \right\}$$

$$D_n = 2\delta_{sup\,r} * \sqrt{\beta_m\beta_C} * \cos\varphi_m \left\{ \sin n\Phi_C \left\{ \frac{\sin\frac{n\Phi_C}{2} * \cos\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}} \right\} - \cos n\Phi_C * \left\{ \frac{\sin\frac{n\Phi_C}{2} * \sin\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}} \right\} \right\}$$

$$D_n = \frac{2\delta_{sup\,r} * \sqrt{\beta_m\beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ \sin n\Phi_C * \sin\frac{n\Phi_C}{2} * \cos\frac{n\Phi_C}{2} - \cos n\Phi_C * \sin^2\frac{n\Phi_C}{2} \right\}$$

set for more convenience $x = n\Phi_C/2$

$$D_n = \frac{2\delta_{sup\,r} * \sqrt{\beta_m\beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ \sin 2x * \sin x * \cos x - \cos 2x * \sin^2 x \right\}$$

$$D_n = \frac{2\delta_{sup\,r} * \sqrt{\beta_m\beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ 2\sin x \cos x * \cos x \sin x - (\cos^2 x - \sin^2 x)\sin^2 x \right\}$$

(A2)

$$D_n = \frac{2\delta_{sup\,r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin^2\frac{n\Phi_C}{2}$$

and in similar calculations:

$$D'_n = \frac{\delta_{sup\,r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin n\Phi_C$$

This expression gives the dispersion generated in a certain number of *n* cells as a function of the dipole kick δ in these cells.
At the end of the dispersion generating section, the value obtained for D(s) and D'(s) has to be equal to the value of the periodic solution:

→equating *(A1)* and *(A2)* gives the conditions for the matching of the periodic dispersion in the arc to the values D = D'= 0 afte the suppressor.

$$D_n = \frac{2\delta_{sup\,r} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin^2\frac{n\Phi_C}{2} = \delta_{arc}\sqrt{\beta_m \beta_C} * \frac{\cos\varphi_m}{\sin\frac{\Phi_C}{2}}$$

$$\left.\begin{aligned} \rightarrow \quad & 2\delta_{sup\,r}\sin^2\left(\frac{n\Phi_C}{2}\right) = \delta_{arc} \\ \rightarrow \quad & \sin(n\Phi_C) = 0 \end{aligned}\right\} \quad \delta_{sup\,r} = \frac{1}{2}\delta_{arc}$$

and at the same time the phase advance in the arc cell has to obey the relation:

$$n\Phi_C = k * \pi, \quad k = 1,3, ...$$