# Lattice Design in Particle Accelerators

**Bernhard Holzer, DESY**

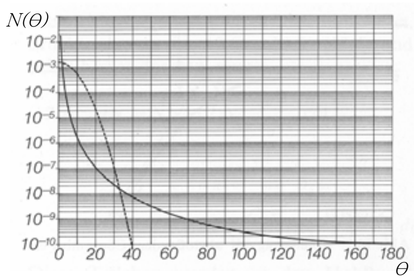
## Historical note:

... Particle acceleration where lattice design is not needed

$$N(\theta) = \frac{N_i n t Z^2 e^4}{(8\pi\varepsilon_0)^2 r^2 K^2} * \frac{1}{\sin^4(\theta/2)}$$

Rutherford Scattering, 1906

Using radioactive particle sources:
α-particles of some MeV energy




## Historical note:

II.) The „infancy of particle accelerators“

1923 Betatron,
1928 Cockroft-Walton Generator,
1929 Cyclotron
1932 Van de Graaf Generator, etc

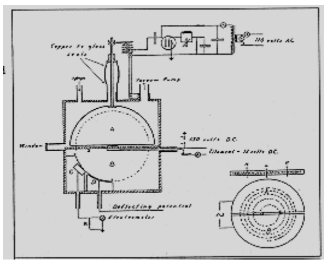
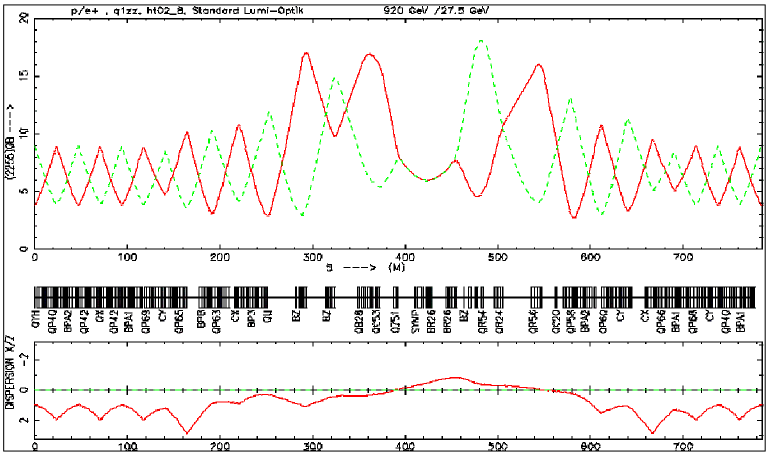


***Diagram of the first cyclotron by Lawrence and Livingstone***


***12 MV-Tandem van de Graaf Accelerator***

1952: Courant, Livingston, Snyder: Theory of strong focusing in particle beams → Ted Wilson in this school




**Lattice design: design and optimisation of the principle elements of an accelerator ... *the lattice cells***

## Lattice Design: Prerequisites

... I will start very easy ... and slowly the topic will become more and more ~~difficult~~ ... interesting

Lorentz Force: $\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$

neglecting electrical fields: $\vec{F} = q * (\vec{v} \times \vec{B})$

High energy accelerators → circular machines

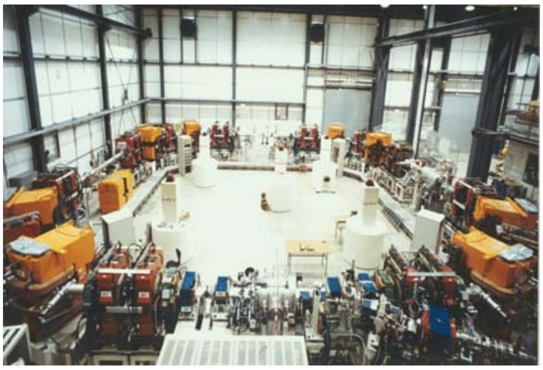
somewhere in the lattice we need a number of dipole magnets, that are bending the design orbit to a closed ring

In a constant external magnetic field the particle trajectory will be a part of a circle and ...

... the centrifugal force will be equal to the Lorentz force

$$e * v * B = \frac{mv^2}{\rho} \qquad \rightarrow e * B = \frac{mv}{\rho} = p / \rho$$

$$\rightarrow \quad B * \rho = p / e$$



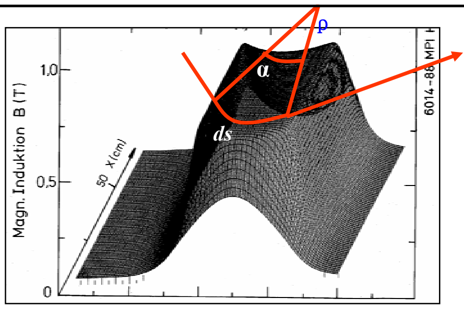
p = momentum of the particle,
ρ = curvature radius

B*ρ is called the “beam rigidity”

***Example: Heavy Ion Storage Ring: TSR***
***8 dipole magnets of equal bending strength***

## Circular Orbit:

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} \qquad \alpha = \frac{B * dl}{B * \rho}$$




*field map of a storage ring dipole magnet*

The angle swept out in one revolution must be 2π, so

$$\alpha = \frac{\int Bdl}{B * \rho} = 2\pi \quad \text{... for a full circle} \qquad \rightarrow \int Bdl = 2\pi * \frac{p}{q}$$

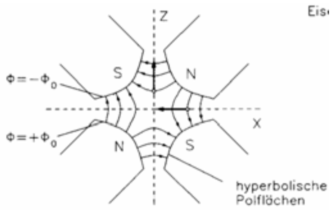
***Example HERA:***
920 GeV Proton storage ring
number of dipole magnets N = 416
$l$ = 8.8m
q = +1 e

$$\int Bdl \approx N * l * B = 2\pi \, p / q$$

$$B \approx \frac{2\pi * 920 * 10^9 eV}{416 * 3 * 10^8 \frac{m}{s} * 8.8m * e} \approx 5.15 \, Tesla$$

## Focusing forces and particle trajectories:

Magnetic field in a quadrupole magnet




$$B_x = -g * z \,, \qquad B_z = -g * x$$

leads to a linear retrieving force on the particle.

Relating the fields to their optical effect: normalise to the particles momentum:

$$\textbf{dipole magnet}: \frac{1}{\rho} = \frac{B}{p/e}$$

$$\textbf{quadrupole lens}: k = \frac{g}{p/e} \qquad \textbf{focal length}: f := \frac{1}{k*l}$$

Under the influence of the focusing and defocusing forces the differential equation of the particles trajectory can be developed:

$$x'' + K * x = 0 \qquad K = -k + 1/\rho^2 \quad \textbf{hor. plane}$$

$$K = k \quad \textbf{vert. plane}$$

Example:

HERA Ring: Circumference:

Circumference: $C_0$ =6335 m
Bending radius: ρ = 580 m
Quadrupol Gradient: G= 110T/m

→ k = 33.64∗$10^{-3}$ /m²
→ 1/ρ² = 2.97 ∗$10^{-6}$ /m²


*the two storage rings of the HERA collider*

! **For estimates in large accelerators the weak focusing term 1/ρ² can in general be neglected ...**

## Single particle trajectories:

**Equation of motion**

$$y''+K(s)*y=0$$

$$y(s)=y_0*\cos(\sqrt{|K|}*s)+\frac{y'_0}{\sqrt{|K|}}*\sin(\sqrt{|K|}*s)$$

$$y'(s)=-y_0*\sqrt{|K|}*\sin(\sqrt{|K|}*s)+y'_0*\cos(\sqrt{|K|}*s)$$

Or written more convenient in matrix form: $\begin{pmatrix} y \\ y' \end{pmatrix}_s = M*\begin{pmatrix} y \\ y' \end{pmatrix}_0$

**Matrices of lattice elements**

**Hor. focusing Quadrupole Magnet**

$$M_{QF}=\begin{pmatrix} \cos(\sqrt{|K|}*l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}*l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}*l) & \cos(\sqrt{|K|}*l) \end{pmatrix}$$

**Hor. defocusing Quadrupole Magnet**

$$M_{QD}=\begin{pmatrix} \cosh(\sqrt{|K|}*l) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}*l) \\ \sqrt{|K|}\sinh(\sqrt{|K|}*l) & \cosh(\sqrt{|K|}*l) \end{pmatrix}$$

**Drift space** $M_{Drift}=\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$

## Periodic Lattices:

In the case of periodic lattices the transfer matrix can be expressed as a function of a set of periodic parameters α, β, γ

$$M(s)=\begin{pmatrix} \cos\mu+\alpha_S\sin\mu & \beta_S\sin\mu \\ -\gamma_S\sin\mu & \cos(\mu)-\alpha_S\sin\mu \end{pmatrix}$$

$$\mu=\int_s^{s+L}\frac{dt}{\beta(t)}$$

*μ = phase advance per period:*

For stability of the motion in periodic lattice structures it is required that

$$|trace(M)|<2$$

In terms of these new periodic parameters the solution of the equation of motion is

$$y(s)=\sqrt{\varepsilon}*\sqrt{\beta(s)}*\cos(\Phi(s)-\delta)$$

$$y'(s)=\frac{-\sqrt{\varepsilon}}{\sqrt{\beta}}*\{\sin(\Phi(s)-\delta)+\alpha\cos(\Phi(s)-\delta)\}$$

The new parameters α, β, γ can be transformed through the lattice via the matrix elements defined above.

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_0$$

Question: What does that mean ????

... and here starts the lattice design !!!

Question: What does that mean ????

Most simple example: drift space

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

particle coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$\rightarrow x(l) = x_0 + l * x_0'$$

$$\rightarrow x'(l) = x_0'$$

transformation of twiss parameters:

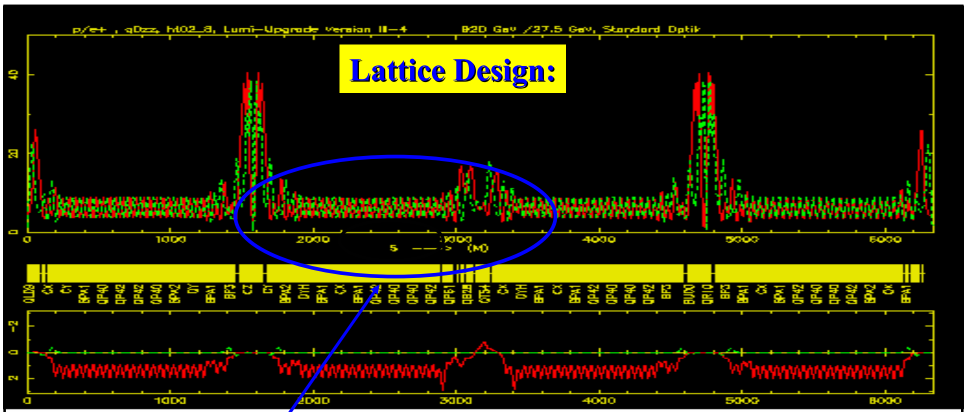
$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_0$$

$$\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0$$

Stability ...?

$$|trace(M)| = 1 + 1 = 2$$

→A periodic solution doesn't exist in a magnetic lattice built exclusively of drift spaces.



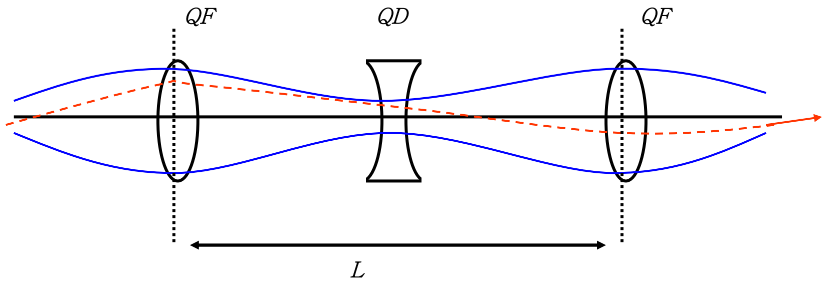

**Arc:** regular **(periodic) magnet structure:**

**bending magnets → define the energy of the ring**
**main focusing & tune control, chromaticity correction,**
**multipoles for higher order corrections**

**Straight sections: drift spaces for injection, dispersion suppressors,**
**low beta insertions, RF cavities, etc....**
**... and the high energy experiments if they cannot be avoided**

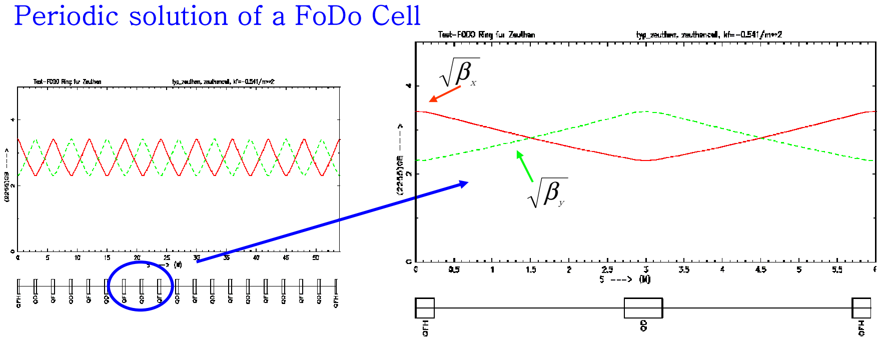
## The FoDo-Lattice

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nota bene: nothing = elements that can be neglected on first sight: drift, bending magnet, RF structures ... and especially experiments...)




Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell $\mu = 45°$,
→ calculate the twiss parameters for a periodic solution

## Periodic solution of a FoDo Cell




*Output of the optics program:*

| NR | TYP | LENGTH | STRENGTH | BETAX | ALFAX | PHIX | BETAZ | ALFAZ | PHIZ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | IP | 0 | 0.00E+00 | 11.611 | 0 | 0 | 5.295 | 0 | 0 |
| 1 | QFH | 0.25 | -5.41E-01 | 11.228 | 1.514 | 0.0035 | 5.488 | -0.78 | 0.007 |
| 2 | QD | 3.251 | 5.41E-01 | 5.4883 | -0.781 | 0.0699 | 11.23 | 1.514 | 0.066 |
| 3 | QFH | 6.002 | -5.41E-01 | 11.611 | 0 | 0.125 | 5.295 | 0 | 0.125 |
| 4 | IP | 6.002 | 0.00E+00 | 11.611 | 0 | 0.125 | 5.295 | 0 | 0.125 |
| QX= | 0.125 | QZ= | 0.125 | | | | | | |

$$0.125 * 2\pi = 45°$$

## ... Can we understand, what the optics code is doing?

The action of a magnet on the beam is given by the transfer matrix:

$$M_{QF} = \begin{pmatrix} cos(\sqrt{|K|} * l) & \frac{1}{\sqrt{|K|}} sin(\sqrt{|K|} * l) \\ -\sqrt{|K|}\, sin(\sqrt{|K|} * l) & cos(\sqrt{|K|} * l) \end{pmatrix}$$

Input: strength and length of the FoDo elements

K =+/- 0.54102 $m^{-2}$
lq = 0.5 m
ld = 2.5 m

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{QFH} * M_{LD} * M_{QD} * M_{LD} * M_{QFH}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

**The transfer matrix for 1 period gives us all the information that we need !**

**1.) is the motion stable?** $|trace(M_{FoDo})| = 1.415 \quad \rightarrow \quad \underline{\underline{< 2}}$

**2.) Phase advance per cell**

$$M(s) = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos(\mu) - \alpha\sin\mu \end{pmatrix} \quad \rightarrow \quad \cos(\mu) = \frac{1}{2} * trace(M) = 0.707$$

$$\mu = a\cos\left(\frac{1}{2} * trace(M)\right) = \underline{\underline{45°}}$$

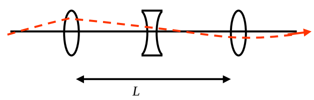
**3.) hor β-function**

$$\beta = \frac{M(1,2)}{\sin(\mu)} = \underline{\underline{11.611\ m}}$$

**4.) hor α-function**

$$\alpha = \frac{M(1,1) - \cos(\mu)}{\sin(\mu)} = \underline{\underline{0}}$$

**Some perls of wisdom about lattice cells:**




1.) think first ...

* a first estimate of the FoDo parameters can and should be done before we run our optics codes.

* we can learn a lot without doing to many too sophisticated calculations

* the optic codes will never tell us whether a lattice cell is technically feasible

* at the very beginning we have to define parameters that lead at least to a stable periodic solution

2.) some rules of „thumb“... to start with: the tune Q

phase advance per cell (i.e. )period: $\mu = \int_s^{s+L} \frac{dt}{\beta(t)}$

Tune := phase advance of the machine in units of $2\pi$ $\qquad Q := N * \frac{\mu}{2\pi} = \frac{1}{2\pi} * \oint \frac{ds}{\beta(s)}$

$$\rightarrow Q \approx \frac{1}{2\pi} * \frac{2\pi \bar{R}}{\bar{\beta}} = \bar{R}/\bar{\beta}$$

The tune is roughly given by the mean bending radius of the circular accelerator divided by the mean β-function

3.) can we do it a little bit easier ?
We can: the thin lens approximation

**Matrix of a focusing quadrupole magnet:** $$M_{QF} = \begin{pmatrix} \cos(\sqrt{|K|}*l) & \frac{1}{\sqrt{K}}\sin(\sqrt{|K|}*l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}*l) & \cos(\sqrt{|K|}*l) \end{pmatrix}$$

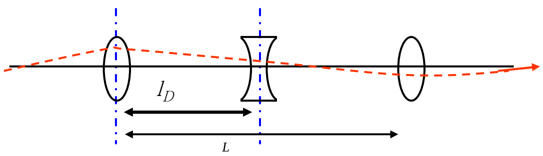
**If the focal length $f$ is much larger than the length of the quadrupole magnet,**

$$f = 1/kl_Q \gg l_Q$$

**the transfer matrix can be aproximated using** $kl_Q = const,\ l_Q \to 0$

$$M = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

FoDo in thin lens approximation




Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

$$M_{halfCell} = M_{QD/2} M_{lD} M_{QF/2}$$

$$M_{halfCell} = \begin{pmatrix} 1 & 0 \\ 1/\tilde{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix}$$

$l_D$ indicates the length of the drift which is now just half the cell length

$$l_D = L/2$$

$$M_{halfCell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

and starting at the middle of a quadrupole the focal length of the half quad is

$$\tilde{f} = 2f$$

For the second half cell set $f \rightarrow -f$

## FoDo in thin lens approximation

**Matrix for the complete FoDo cell:**

$$M = \begin{pmatrix} 1+{l_D}/{\tilde f} & l_D \\ -{l_D}/{\tilde f^2} & 1-{l_D}/{\tilde f} \end{pmatrix} * \begin{pmatrix} 1-{l_D}/{\tilde f} & l_D \\ -{l_D}/{\tilde f^2} & 1+{l_D}/{\tilde f} \end{pmatrix}$$

**Multiplying out we get ...**

$$M = \begin{pmatrix} 1-\frac{2l_D^2}{\tilde f^2} & 2l_D(1+\frac{l_D}{\tilde f}) \\ 2(\frac{l_D^2}{\tilde f^3}-\frac{l_D}{\tilde f^2}) & 1-2\frac{l_D^2}{\tilde f^2} \end{pmatrix}$$

**Now we know, that the phase advance is related to the transfer matrix by**

$$\cos\mu = \frac{1}{2}trace\,(M) = {1}/{2} * (2-\frac{4l_D^2}{\tilde f^2}) = 1-\frac{2l_D^2}{\tilde f^2}$$

**After some beer and with a little bit of trigonometric gymnastics**

$$\cos(x) = \cos^2({x}/{2}) - \sin^2(x/2) = 1-2\sin^2({x}/{2})$$

We can calculate the phase advance as a function of the FoDo parameter

$$\cos(\mu) = 1-2\sin^2(\mu/2) = 1-\frac{2l_D^2}{\tilde f^2}$$

$$|\sin(\mu/2)| = l_D/\tilde f = \frac{L_{Cell}}{2\tilde f}$$

$$|\sin(\mu/2)| = \frac{L_{Cell}}{4f}$$

*Example:* 45-degree Cell

$L_{Cell} = l_{QF} + l_D + l_{QD} + l_D = 0.5m+2.5m+0.5m+2.5m = 6m$

$1/f = k*l_Q = 0.5m*0.541\ m^{-2} = 0.27\ m^{-1}$

$$\sin(\mu/2) \approx \frac{L_{Cell}}{4f} = 0.405$$

$$\rightarrow \mu \approx 47.8°$$

$$\rightarrow \beta \approx 11.4m$$

*Remember:*
*Exact calculation yields:*

$$\mu = 45°$$

$$\beta = 11.6m$$

## Stability in a FoDo structure

transfer matrix for the complete cell

$$M = \begin{pmatrix} 1-\frac{2l_D^2}{\tilde{f}^2} & 2l_D(1+\frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3}-\frac{l_D}{\tilde{f}^2}) & 1-2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

**Stability requires:** $|trace(M)| < 2$ $\quad |trace(M)| = \left|2-\frac{4l_D^2}{\tilde{f}^2}\right| < 2$

$$\rightarrow \quad f > \frac{L_{cell}}{4}$$

For stability the focal length has to be larger than a quarter of the cell length !!

*SPS Lattice*

## Transformation matrix in terms of the Twiss parameters

en detail

Transformation of the coordinate vector (x,x´) in a magnet

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \qquad M_{QF} = \begin{pmatrix} cos(\sqrt{|K|}*l) & \frac{1}{\sqrt{|K|}} sin(\sqrt{|K|}*l) \\ -\sqrt{|K|}\, sin(\sqrt{|K|}*l) & cos(\sqrt{|K|}*l) \end{pmatrix}$$

General solution of the equation of motion

$$x(s) = \sqrt{\varepsilon * \beta(s)} * cos(\phi(s)+\varphi)$$

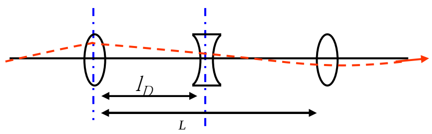
$$x'(s) = -\sqrt{\varepsilon / \beta(s)} * \{\alpha(s) cos(\phi(s)+\varphi) + sin(\phi(s)+\varphi)\}$$

Transformation of the coordinate vector (x,x´) expressed as a function of the twiss parameters

$$M_{0\rightarrow S} = \begin{pmatrix} \sqrt{\frac{\beta_S}{\beta_0}}(cos\phi + \alpha_S \sin\phi) & \sqrt{\beta_S\beta_0}\, sin\phi \\ \frac{(\alpha_0-\alpha_S)cos\phi - (1+\alpha_0\alpha_S) sin\phi}{\sqrt{\beta_S\beta_0}} & \sqrt{\frac{\beta_0}{\beta_S}}(cos\phi - \alpha_S \sin\phi) \end{pmatrix}$$

Transfer matrix for half a FoDo cell:

$$M_{halfCell} = \begin{pmatrix} 1 - {l_D}/{\tilde{f}} & l_D \\ -{l_D}/{\tilde{f}^2} & 1 + {l_D}/{\tilde{f}} \end{pmatrix}$$




Compare to the twiss parameter form of M where Φ denotes the phase advance through the half cell

$$M = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\phi + \alpha \sin\phi) & \sqrt{\beta\beta_0}\sin\phi \\ \frac{(\alpha_0 - \alpha)\cos\phi - (1 + \alpha_0\alpha)\sin\phi}{\sqrt{\beta\beta_0}} & \sqrt{\frac{\beta_0}{\beta}}(\cos\phi - \alpha\sin\phi) \end{pmatrix}$$

In the middle of a foc (defoc) quadrupole of the FoDo we allways have α=0, and the half cell will lead us from $\beta_{max}$ to $\beta_{min}$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}}\cos\phi & \sqrt{\hat{\beta}\check{\beta}}\sin\phi \\ \frac{-1}{\sqrt{\hat{\beta}\check{\beta}}}\sin\phi & \sqrt{\frac{\hat{\beta}}{\check{\beta}}}\cos\phi \end{pmatrix}$$

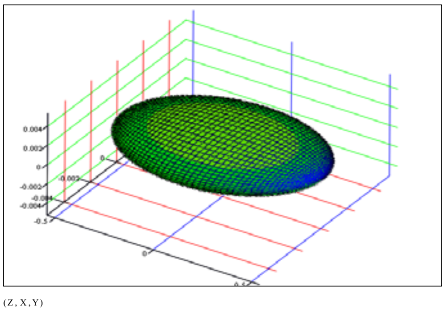
Solving for $\beta_{max}$ and $\beta_{min}$ and remembering that ….

$$\left|\sin\frac{\mu}{2}\right| = \frac{l_D}{\tilde{f}} = \frac{L}{4f}$$

$$\frac{S'}{C} = \frac{\hat{\beta}}{\check{\beta}} = \frac{1 + l_D/\tilde{f}}{1 - l_D/\tilde{f}} = \frac{1 + \sin\frac{\mu}{2}}{1 - \sin\frac{\mu}{2}}$$

$$\frac{S}{C'} = \hat{\beta}\check{\beta} = \tilde{f}^2 = \frac{l_D^2}{\sin^2\frac{\mu}{2}}$$

$$\rightarrow \quad \boxed{\hat{\beta} = \frac{(1 + \sin\frac{\mu}{2})L}{\sin\mu}} \; ! \qquad \boxed{\check{\beta} = \frac{(1 - \sin\frac{\mu}{2})L}{\sin\mu}} \; !$$



(Z, X, Y)

**The maximum and minimum values if the β-function are solely determined by the phase advance and the length of the cell.**

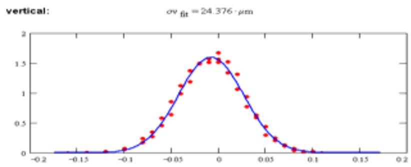
**Longer cells lead to larger β**

*typical shape of a proton bunch in the HERA FoDo Cell*

## Optimisation of the FoDo Phase advance: Beam dimension

In both planes a gaussian particle distribution is assumed given by the beam emittance ε and the β-function

$$\sigma = \sqrt{\varepsilon\beta}$$



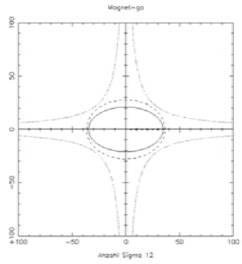

*measured beam size at the HERA IP*

In general proton beams are ***„round“*** in the sense that

$$\varepsilon_x \approx \varepsilon_y$$

So for highest aperture we have to minimise the β-function in both planes:

$$r^2 = \varepsilon_x\beta_x + \varepsilon_y\beta_y$$



*typical beam envelope, vacuum chamber and pole shape in a foc. Quadrupole lens in HERA*

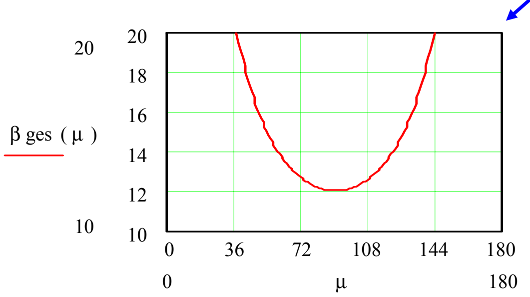
## Optimising the FoDo phase advance

$$r^2 = \varepsilon_x\beta_x + \varepsilon_y\beta_y$$

search for the phase advance μ that results in a minimum of the sum of the beta's

$$\hat{\beta} + \check{\beta} = \frac{(1+\sin\frac{\mu}{2})*L}{\sin\mu} + \frac{(1-\sin\frac{\mu}{2})*L}{\sin\mu}$$

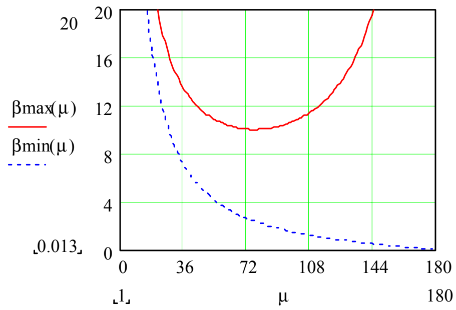
$$\hat{\beta} + \check{\beta} = \frac{2L}{\sin\mu} \qquad \frac{d}{d\mu}\left(2L/\sin\mu\right) = 0$$




$$\frac{L}{\sin^2\mu} * \cos\mu = 0 \quad \rightarrow \quad \mu = 90°$$

Nota bene: electron beams are typicalle flat, $\varepsilon_y \approx 2 \ldots 10\ \%\ \varepsilon_y$
→ optimise only $\beta_{hor}$

$$\frac{d}{d\mu}(\hat{\beta}) = \frac{d}{d\mu}\frac{L(1+\sin\frac{\mu}{2})}{\sin\mu} = 0 \quad \rightarrow \quad \mu \approx 76°$$

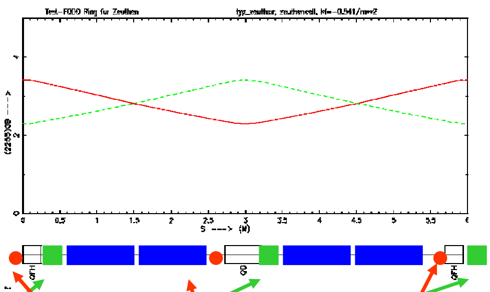



*red curve:* $\beta_{max}$
*blue curve:* $\beta_{min}$
*as a function of the phase advance μ*

## Orbit distortions in a periodic lattice

field error of a dipole/distorted quadrupole

$$\rightarrow \quad \delta(mrad) = \frac{ds}{\rho} = \frac{\int Bds}{p/e}$$



the particle will follow a new closed trajectory, the distorted orbit:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q}\oint\sqrt{\beta(\tilde{s})}\frac{1}{\rho(\tilde{s})}\cos(|\phi(\tilde{s}) - \phi(s)| - \pi Q)\,d\tilde{s}$$

* the orbit amplitude will be large if the β function at the location of the kick is large
$\beta(\tilde{s})$ indicates the sensitivity of the beam → here orbit correctors should be placed in the lattice
* the orbit amplitude will be large at places where in the lattice β(s) is large → here beam position monitors should be installed

## Chromaticity in the FoDo Lattice

**Definition** $\Delta Q = \xi * \frac{\Delta p}{p}$

The chromaticity describes an optical error of quadrupole lenses: For a given magnetic field, i.e. gradient particles with smaller momentum will feel a stronger focusing force and vice versa.

**For small momentum errors Δp/p the focusing parameter k can be written as**

$$k(p) = \frac{g}{p/e} = g * \frac{e}{p_0 + \Delta p}$$

$$k(p) \approx \frac{e}{p_0}\left(1 - \frac{\Delta p}{p}\right) * g = k_0 + \Delta k \quad \rightarrow \quad \Delta k = -k_0 \frac{\Delta p}{p}$$

**This describes a quadrupole error that leads to a tune shift of ...**

$$\Delta Q = \frac{1}{4\pi} \int \Delta k \beta(s) ds = \frac{-1}{4\pi} \frac{\Delta p}{p} \int k_0 \beta(s) ds$$

**ξ contribution in the lattice** $\xi = -\frac{1}{4\pi} \int \beta(s) * k(s) ds$

## Chromaticity in the FoDo Lattice

$$\xi = -\frac{1}{4\pi} \int \beta(s) * k(s) ds$$

$$\xi \approx -\frac{1}{4\pi} N * \frac{\hat{\beta} - \check{\beta}}{f_Q} = -\frac{1}{4\pi} N * \frac{1}{f_Q} * \left\{ \frac{L\left(1 + \sin\frac{\mu}{2}\right) - L\left(1 - \sin\frac{\mu}{2}\right)}{\sin\mu} \right\}$$

**using some trigonometric transformations ... ξ can be expressed in a very simple form:**

$$\xi = -\frac{1}{4\pi} N \frac{1}{f_Q} \frac{2L\sin\frac{\mu}{2}}{\sin\mu} = -\frac{1}{4\pi} N \frac{1}{f_Q} \frac{L\sin\frac{\mu}{2}}{f_Q \sin\frac{\mu}{2}\cos\frac{\mu}{2}}$$

**remember ...** $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$\xi_{Cell} = -\frac{1}{4\pi} \frac{1}{f_Q} \frac{L\tan\frac{\mu}{2}}{f_Q \sin\frac{\mu}{2}}$$

**putting ...** $\sin\frac{\mu}{2} = \frac{L}{4 f_Q}$

Contribution of one FoDo Cell to the chromaticity of the ring:

$$\boxed{\xi_{Cell} = -\frac{1}{\pi} * \tan\frac{\mu}{2}}$$

## Resumé

1.) Dipole strength: $\int B ds = N * B_0 * l_{eff} = 2\pi \frac{p}{q}$

$l_{eff}$ **effective magnet length, N number of magnets**

2.) Stability condition: $|Trace(M)| < 2$

**for periodic structures within the lattice / at least for the transfer matrix of the complete circular machine**

3.) Transfer matrix for periodic cell $M(s) = \begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos(\mu) - \alpha(s)\sin\mu \end{pmatrix}$

**α,β,γ depend on the position s in the ring, μ (phase advance) is independent of s**

4.) Thin lens approximation: $M_{QF} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_Q} & 1 \end{pmatrix}$ $\quad f_Q = \frac{1}{k_Q l_Q}$

**focal length of the quadrupole magnet** $f_Q = 1/(k_Q l_Q) >> l_Q$

5.) Tune (rough estimate): $Q \approx \frac{\overline{R}}{\overline{\beta}}$

$\overline{R}, \overline{\beta}$ **average values of radius and β-function**

6.) Phase advance per FoDo cell $\left|\sin\frac{\mu}{2}\right| = \frac{L_{Cell}}{4 f_Q}$
(thin lens approx)

$L_{Cell}$ **length of the complete FoDo cell,** $f_Q$ **focal length of the quadrupole,** $\mu$ **phase advance per cell**

7.) Stability in a FoDo cell $f_Q > \frac{L_{Cell}}{4}$
(thin lens approx)

8.) Beta functions in a FoDo cell $\hat{\beta} = \frac{(1 + \sin\frac{\mu}{2}) L_{Cell}}{\sin\mu}$ $\quad \check{\beta} = \frac{(1 - \sin\frac{\mu}{2}) L_{Cell}}{\sin\mu}$
(thin lens approx)

$L_{Cell}$ **length of the complete FoDo cell,** $\mu$ **phase advance per cell**

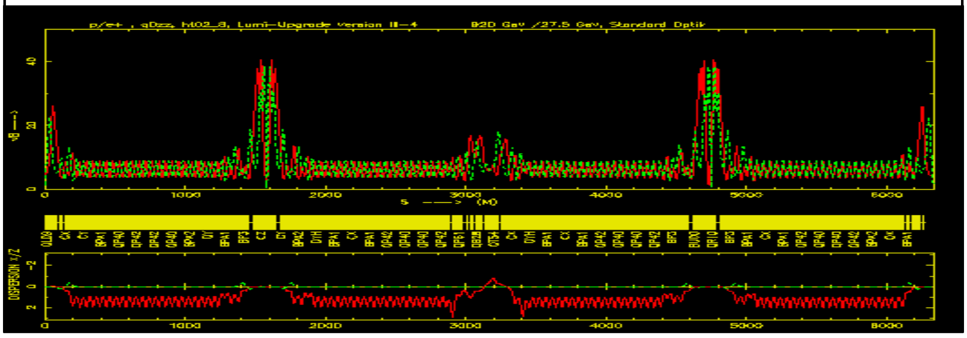
**Resumé** **periodic section („the arc“ in a large accelerator):**

→ parameters of the lattice cell defined by 2 quadrupole strengths (sometimes even only by one if in series)

→ for a given tune (phase advance) the twiss parameters are fixed et vice versa.

→ tune adjustments, variations of cell lengths, matching of twiss parameters creation of symmetry points in the lattice ...

**.... need more degrees of freedom, i.e. additional free quadrupole lenses, varying drift lengths etc.**




APPENDIX

Single particle trajectories:

$$y'' + K * y = 0$$

**The differential equation for the particle movement can be solved by the Ansatz ...**

$$y = a_1 * cos(\omega * s) + a_2 * sin(\omega * s)$$

$$y' = -a_1\omega * sin(\omega * s) + a_2\omega * cos(\omega * s)$$
$$y'' = -a_1\omega^2 * cos(\omega * s) - a_2\omega^2 * sin(\omega * s)$$
$$= -\omega^2 * y$$

$$\rightarrow K = \omega^2, \quad \omega = \sqrt{K}$$

So we get for the equation of motion in a storage ring

$$y(s) = a_1 * cos(\sqrt{K} * s) + a_2 * sin(\sqrt{K} * s)$$

Equation of motion

$$y(s) = a_1 * cos(\sqrt{K} * s) + a_2 * sin(\sqrt{K} * s)$$

The parameters $a_1$ and $a_2$ refer to the individual particle and are determined by boundary conditions.

$$y(0) = y_0 \quad \rightarrow a_1 = y_0$$
$$y'(0) = y_0' \quad \rightarrow a_2 = \frac{y_0'}{\sqrt{K}}$$

resulting in

$$y(s) = y_0 * cos(\sqrt{K} * s) + \frac{y'_0}{\sqrt{K}} * sin(\sqrt{K} * s)$$
$$y'(s) = -y_0 * \sqrt{K} * sin(\sqrt{K} * s) + y'_0 * cos(\sqrt{K} * s)$$

Or written more convenient in matrix form:

$$\begin{pmatrix} y \\ y' \end{pmatrix}_s = M * \begin{pmatrix} y \\ y' \end{pmatrix}_0 , \qquad M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

## Matrices of lattice elements

$$M_{QF} = \begin{pmatrix} cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} sin(\sqrt{K} * l) \\ -\sqrt{K}\, sin(\sqrt{K} * l) & cos(\sqrt{K} * l) \end{pmatrix}$$

Hor. focusing
Quadrupole Magnet

$$M_{QD} = \begin{pmatrix} cosh(\sqrt{K} * l) & \frac{1}{\sqrt{K}} sinh(\sqrt{K} * l) \\ \sqrt{K}\, sinh(\sqrt{K} * l) & cosh(\sqrt{K} * l) \end{pmatrix}$$

Hor. defocusing
Quadrupole Magnet

$$M_{Drift} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

Drift space

$$K = -k + \frac{1}{\rho^2}$$ in horizontal plane

$$K = k$$ in vertical plane

## Transformation of the principal trajectories in terms of the Twiss parameters

General solution of the equation of motion

(1)

$$x(s) = \sqrt{\varepsilon * \beta(s)} * cos(\phi(s) + \varphi)$$

$$x'(s) = -\sqrt{\varepsilon / \beta(s)} * \{\alpha(s) cos(\phi(s) + \varphi) + sin(\phi(s) + \varphi)\}$$

Using theorems of trigonometric functions

*sin(a+b) = sin(a) cos(b)+ cos(a) sin(b) ...*

(2)

$$x(s) = \sqrt{\varepsilon * \beta(s)} * \{cos\,\phi(s) cos(\varphi) - sin\phi(s) sin(\varphi)\}$$

$$x'(s) = -\sqrt{\varepsilon / \beta(s)} * \{\alpha(s) cos\,\phi(s) cos(\varphi) - \alpha(s) sin\phi(s) sin(\varphi) + $$
$$+ sin\phi(s) cos(\varphi) + cos\,\phi(s) sin(\varphi)\}$$

Set initial conditions: $x(0)=x_0$, $x'(0)=x'_0$,
$\beta(0)=\beta_0$, $\alpha(0)=\alpha_0$, $\Phi(0)=0$

(3)

$$cos\,\phi = \frac{x_0}{\sqrt{\varepsilon\beta_0}} \qquad sin\phi = \frac{-1}{\sqrt{\varepsilon}}(x'_0\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}})$$

Inserting in (1) $x(s)=\sqrt{\frac{\beta(s)}{\beta_0}}\{\cos\phi(s)+\alpha_0\sin\phi(s)\}x_0+\sqrt{\beta(s)\beta_0}\sin\phi(s)x_0'$

$$x'(s)=\sqrt{\frac{1}{\beta(s)\beta_0}}\{(\alpha_0-\alpha(s))\cos\phi(s)-(1+\alpha_0\alpha(s)\sin\phi(s)\}x_0$$

$$+\sqrt{\frac{\beta_0}{\beta(s)}}\{\cos\phi(s)-\alpha(s)\sin\phi(s)\}x_0'$$

So again we have got a matrix that transforms the orbit vector ($x_0, x'_0$) into ($x(s)$, $x'(s)$)

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$M=\begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\phi+\alpha_0\sin\phi) & \sqrt{\beta_s\beta_0}\sin\phi \\ \frac{(\alpha_0-\alpha_s)\cos\phi-(1+\alpha_0\alpha_s)\sin\phi}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos\phi-\alpha_s\sin\phi) \end{pmatrix}$$