Hamiltonian Formalism

Lagrange function:

$$L(\vec{x}, \dot{\vec{x}}, t) = T(\dot{\vec{x}}) - U(\vec{x}, t)$$

kinetic energy potential energy

→ canonical momenta:

$$p_i = \frac{\partial L}{\partial \dot{x}_i}$$

Hamilton function:

$$H(\vec{x}, \vec{p}, t) = \sum_i p_i \cdot \dot{x}_i - L(\vec{x}, \dot{\vec{x}}, t)$$

equation of motion:

$$\rightarrow \quad \frac{dx}{dt} = \frac{\partial H}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

Harmonic Oscillator

Lagrange function:

kinetic energy: $\longrightarrow$ $T(\dot{x}) = \frac{1}{2} \cdot m \cdot \dot{x}^2$

potential energy: $\longrightarrow$ $U(x) = \frac{1}{2} \cdot k \cdot x^2$

canonical momentum: $p = m \cdot \dot{x}$

Hamiltonian: $H(x, p) = \frac{1}{2} \cdot m^{-1} \cdot p^2 + \frac{1}{2} \cdot k \cdot x^2$

integrable systems:

$\longrightarrow$ $$t = \int_{x_0}^{x} \frac{dx}{\frac{\partial H}{\partial p}}$$

E = constant $\longrightarrow$ $p = p(x, E)$

$$x(t) = x_0 \cdot \sin(\sqrt{k/m} \cdot t); \quad p(t) = m \cdot \sqrt{k/m} \cdot x_0 \cdot \cos(\sqrt{k/m} \cdot t)$$

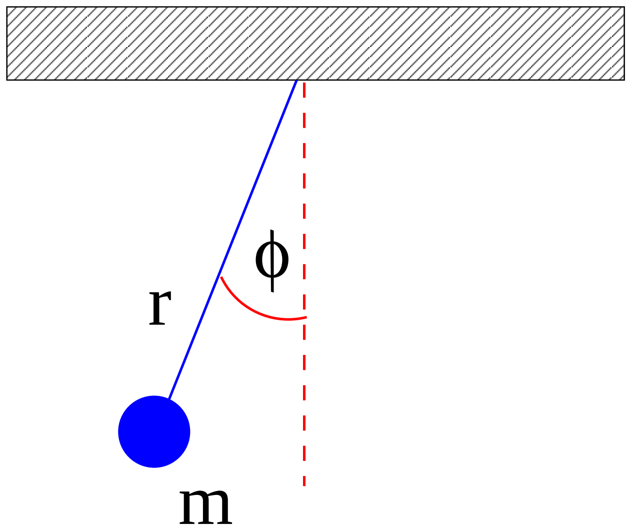
Pendulum

equation of motion:

$$T = \frac{1}{2} \cdot m \cdot (r \cdot \dot{\phi})^2$$

$$U = -m \cdot g \cdot r \cdot \cos(\phi)$$

$$\longrightarrow \quad \underline{p = m \cdot r^2 \cdot \dot{\phi}}$$



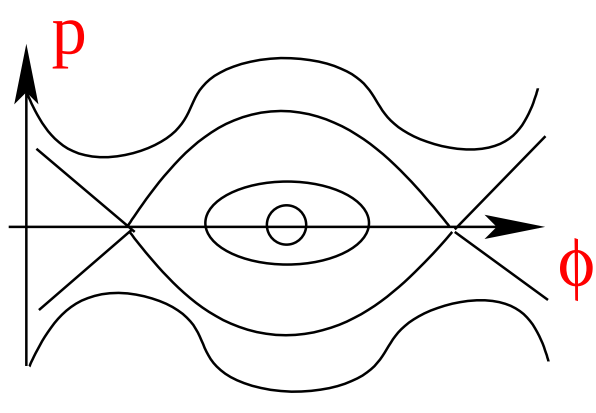
Hamilton function:

$$H(x, p) = \frac{1}{2} \cdot G \cdot p^2 - F \cdot \cos(\phi) \qquad G = 1/(mh^2)$$

$$F = mgh$$

integrable systems:

$$t = \int_{x_0}^{x} \frac{dx}{\frac{\partial H}{\partial p}} \longrightarrow$$



$$\omega_0 = (F \cdot G)^{1/2} \qquad \Delta p_{max} = 2(F / G)^{1/2}$$

Hamiltonian Formalism

accelerator dynamics: $\vec{x} = (x, z)$

(transverse dynamics only)

replace time by s coordinate: $\longrightarrow$ $p_i = \dfrac{d\,x_i}{d\,s}$

$$\longrightarrow \quad H(\vec{x}, \vec{p}) = \frac{1}{2} \cdot \vec{p}^{\,2} + U(\vec{x}, s)$$

equation of motion:

$$\longrightarrow \quad \frac{d\,x_i}{d\,s} = \frac{\partial H}{\partial p_i} \qquad \frac{d\,p_i}{d\,s} = -\frac{\partial H}{\partial x_i}$$

potential energy in EM field:

$$U(\vec{x}, \vec{v}) = \frac{e}{c} \cdot (\vec{v} \cdot \vec{A}) - e \cdot \varphi$$

Hamiltonian Formalism

potential energy in EM field:

$$U(\vec{x},\vec{v}) = \frac{e}{c}\cdot(\vec{v}\cdot\vec{A}) - e\cdot\varphi$$

with:

$$E = -\mathrm{grad}\varphi - \frac{1}{c}\cdot\frac{\partial\vec{A}}{\partial t}$$

$$B = \mathrm{rot}\,\vec{A} \longrightarrow \vec{F} = e\cdot\vec{E} + e\cdot\vec{v}\times\vec{B}$$

quadrupole:

$$A_s = \frac{1}{2}\cdot k_1(s)\cdot(x^2 - y^2);\ A_x = 0;\ A_y = 0$$

sextupole:

$$A_s = \frac{k_2(s)}{6}\cdot(x^3 - 3x\cdot y^2)$$

octupole:

$$A_s = \frac{k_3(s)}{24}\cdot(x^4 - 6x^2\cdot y^2 + y^4)$$

Canonical Transformations

coordinate change:

$$(\vec{x}, \vec{p}) \longrightarrow (\vec{\bar{x}}, \vec{\bar{p}})$$

$\longrightarrow$ arbitrary transformations changes the equation of motion!

generating function F_1:

$$F_1 = f(x, \bar{x}) \rightarrow p_i = \frac{\partial F_1}{\partial x_i} \qquad \bar{p}_i = -\frac{\partial F_1}{\partial \bar{x}_i}$$

$$\longrightarrow \quad p_i = g(x, \bar{x}) \qquad \bar{p}_i = g(x, \bar{x})$$

generating function F_2:

$$F_2 = f(x, \bar{p}) \rightarrow p_i = \frac{\partial F_2}{\partial x_i} \qquad \bar{x}_i = \frac{\partial F_2}{\partial \bar{p}_i}$$

new Hamilton function:

$$\bar{H} = H(\bar{p}, \bar{x}) + \frac{\partial F}{\partial t}$$

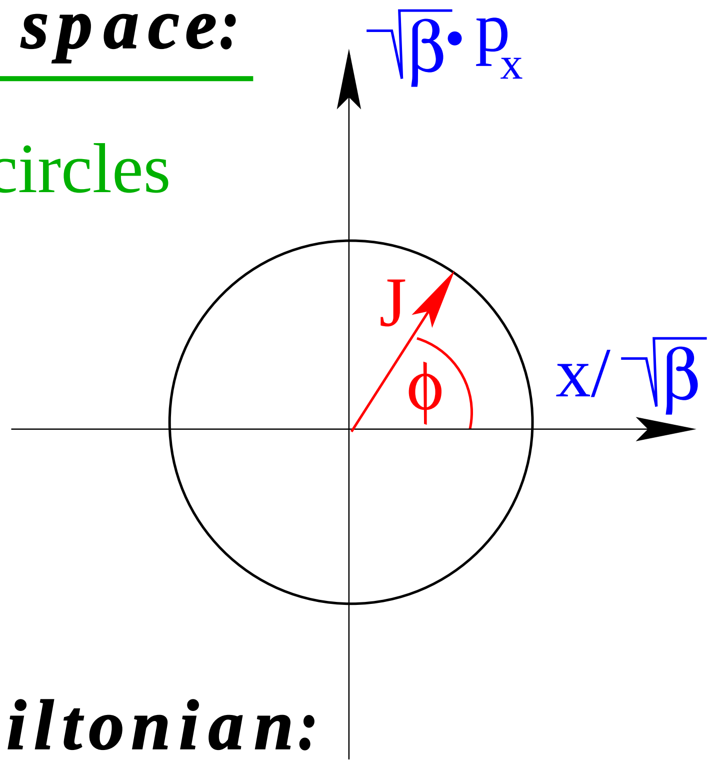
Accelerator Hamiltonian

linear Hamiltonian:

$$H_0(x, p_x, s) = \frac{1}{2} \cdot p_x^2 + \frac{1}{2} \cdot K(s) \cdot x^2$$

normalised phase space:

particle motion lies on circles



Action Angle Hamiltonian:

$$F_1(\hat{x}, \phi, s) = \frac{-1}{2\beta(s)} \cdot x^2 \cdot \left[\tan(\phi) - \frac{\beta(s)}{2}\right]$$

$$\hat{x} = \sqrt{\beta\, 2J} \cdot \cos(\phi) \qquad \hat{p} = -\sqrt{\frac{2J}{\beta}} \cdot \left[\sin(\phi) + \alpha \cos(\phi)\right]$$

$$H(J, \phi, s) = \frac{J}{\beta(s)} \longrightarrow$$ J is a constant of motion

Accelerator Hamiltonian

Action Angle Variable Hamiltonian:

$$H_0(\phi_x, \phi_y, J_x, J_y, s) = \frac{1}{\beta_x(s)} \cdot J_x + \frac{1}{\beta_y(s)} \cdot J_y$$

reduced Hamiltonian:

eliminate 's' dependence

$$F_2(\bar{J}, \phi) = \bar{J} \cdot \left[\frac{2\pi Q}{C} \cdot s - \int_{s_0}^{s} \frac{dt}{\beta(t)} \right] + \bar{J} \cdot \phi$$

$$\longrightarrow \quad \bar{\phi} = \frac{\partial F_2}{\partial \bar{J}} \qquad J = \frac{\partial F_2}{\partial \phi}$$

$$H(\bar{\phi}, \bar{J}, s) = H(\phi, J, s) + \frac{\partial F_2}{\partial s}$$

$$\longrightarrow \quad \boxed{H(\phi_x, \phi_z, J_x, J_y, s) = \frac{2\pi Q_x}{C} \cdot J_x + \frac{2\pi Q_y}{C} \cdot J_y}$$

$\longrightarrow$ action−angle variables

Near Integrable Systems

integrable system:

n-dimensional system

→ we need n invariants of the motion!

linear accelerator model:

3-dimensional system (x, y, s)

→ the linear system is integrable!

(see action angle variables)

perturbation theory:

$$H(\vec{x}, \vec{p}, s) = H_0(\vec{x}, \vec{p}, s) + \varepsilon\, H_1(\vec{x}, \vec{p}, s)$$

with H_0 being an integrable system

→ local invariants?

→ KAM theorem

Perturbation Theory

an additional octupole term:

$$H(\phi\ \ , J\ \ , s) = \frac{2\pi Q}{C} \cdot J + \frac{f_3}{24} \cdot x^4(J,\phi)$$

$$x = (2J\beta)^{1/2} \cdot \cos(\phi);$$

$$x^4 = (2J\beta)^2 \cdot \frac{1}{8} \cdot [3 + 4\cos(2\phi) + \cos(4\phi)]$$

$$H(\phi_x, J_x, s) = \frac{2\pi Q_x}{C} \cdot J_x + \frac{\kappa}{16} \cdot \beta^2 \cdot J_x^2 + ..\cos(\ )$$

the tune changes with the oscillation amplitude!