Beam dynamics basics in RF linacs

Nicolas PICHOFF

France
CEA-Direction d'Île de France
Service de Physique des Accélérateurs et Applications (SP2A)

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- RF cavity
  - Resonant cavity (modes, properties)
  - Energy gain
  - Shunt impedance
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  - Longitudinal particle motion
  - Particle motion in linear forces (phase advance, Twiss parameters, rms dimensions, matching)
  - Matched-mismatched beam

Why RF linacs

Goal of an accelerator : Accelerate a wanted beam within the lower cost

wanted : particle, energy, emittance, intensity, time structure
cost : construction, operation

Main competitors : RF linacs, Synchrotrons, Cyclotrons...

RF linacs : Particles accelerated on a linear path with RF cavities.

Advantages : High current, high duty-cycle, low synchrotron radiation losses.
Drawbacks : High room & cavities consumption, no synchrotron radiation damping

Main use of linacs : Low energy injectors, high intensity protons beam, high energy lepton colliders.

RF resonant cavity

Goal : Give kinetic energy to the beam

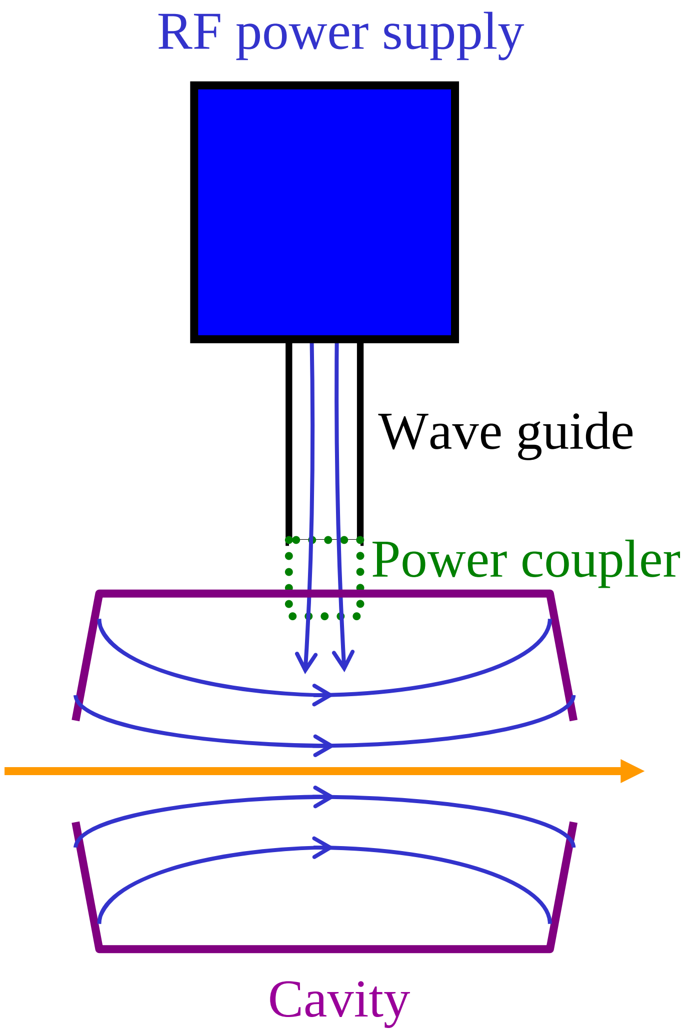
Basic principle

- Conductor enclosing a close volume,

- Maxwell equations + *Boundary conditions* allow possible electromagnetic field E_n/B_n configurations each oscillating with a given frequency f_n : a ***resonant mode***. The field is a weighted superposition of these modes.

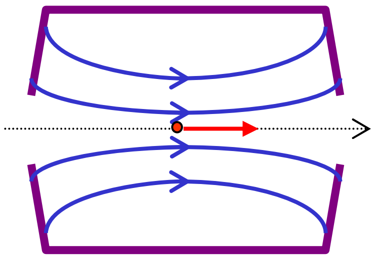
- The wanted (accelerating) mode is excited at the good frequency and position from a RF power supply through a power coupler,

- The phase of the electric field is adjusted to accelerate the beam.



Energy gain

Energy gained by a particle in a cavity :



$$\Delta W = \int qEz(s) \cdot \cos(\phi(s)) \cdot ds$$

with : $\phi(s) = \phi_0 + \omega \cdot t = \phi_0 + \frac{\omega}{c} \int_{s_0}^{s} \frac{ds}{\beta_z(s)}$

$$\boxed{\Delta W = qV_0 \cdot T \cdot \cos\phi_p}$$

with : $V_0 = \int |Ez(s)| \cdot ds$ The cavity voltage

ϕ_p The average phase

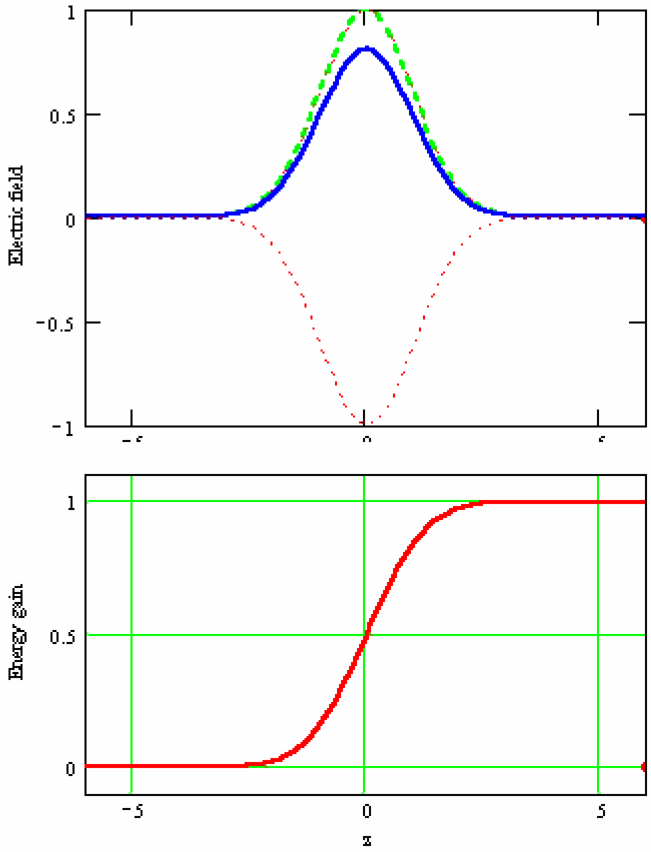
$\boxed{0 < T < 1}$ Transit-time factor

Example 1 : The transit time factor in a one-cell cavity

Fast particle : $T \cong 1$ $\phi_p = 0$

Electric Field

Energy gain



Field amplitude

Field in the cavity with time

Field seen by a non synchronous particle

Energy gain

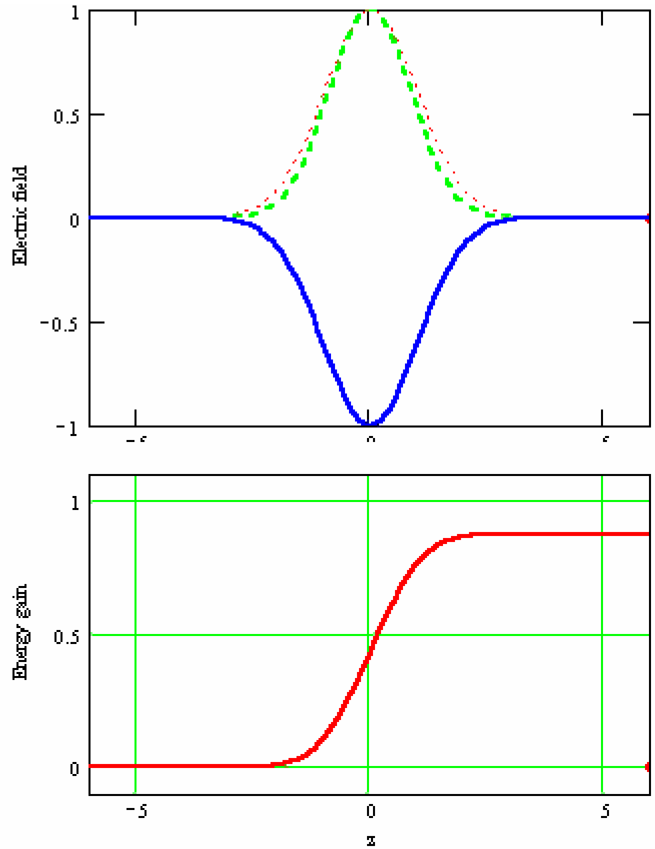
$$\Delta W = qV_0 \cdot T(\overline{\beta})$$

Example 1 : The transit time factor in a one-cell cavity

Medium fast particle : $T \cong 0.85$

Electric Field

Energy gain



Field amplitude

Field in the cavity with time

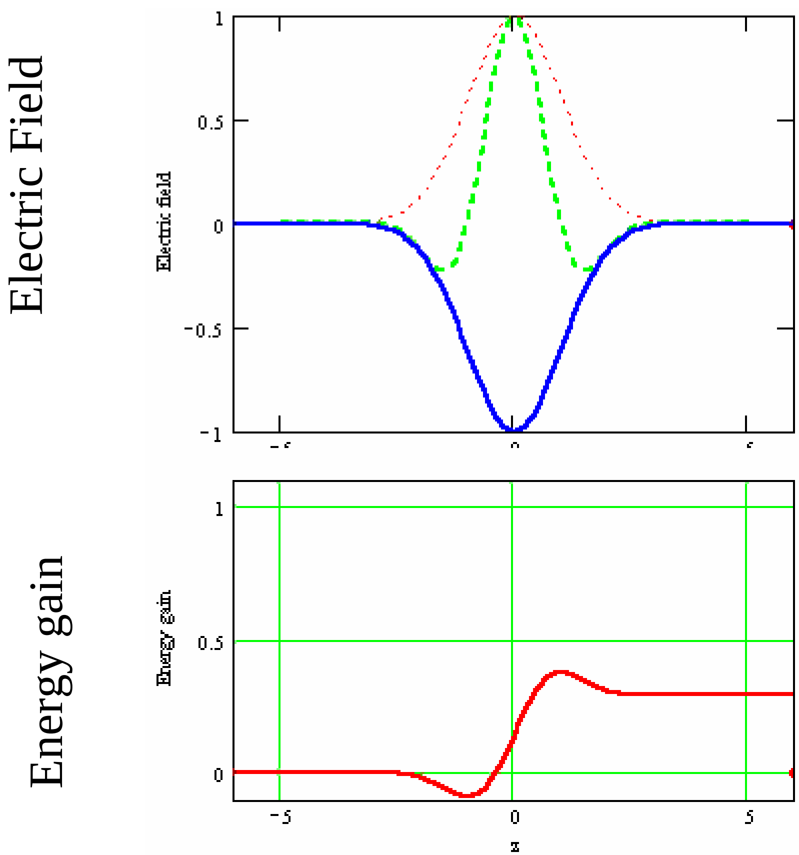
Field seen by a non synchronous particle

Energy gain

$$\Delta W = qV_0 \cdot T\left(\overline{\beta}\right)$$

Example 1 : The transit time factor in a one-cell cavity

Slow particle : $T \cong 0.3$



Field amplitude

Field in the cavity with time

Field seen by a non synchronous particle

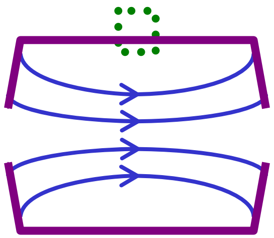
Energy gain

$$\Delta W = qV_0 \cdot T\left(\overline{\beta}\right)$$

RF cavity shunt impedance

Cavity voltage $\boldsymbol{V_0}$:

$$V_0 = \int \hat{E}_z(z) \cdot dz$$



Dissipated power $\boldsymbol{P_d}$: Mean power dissipated in conductor over one RF period

Shunt impedance $\boldsymbol{R}$:

$$R = \frac{V_0^2}{2 \cdot P_d}$$

$$\boxed{P_d = \frac{1}{2} \cdot V_0^2 / Z}$$

ΔW_{max} : Maximum energy that can be gained by a particle in the cavity

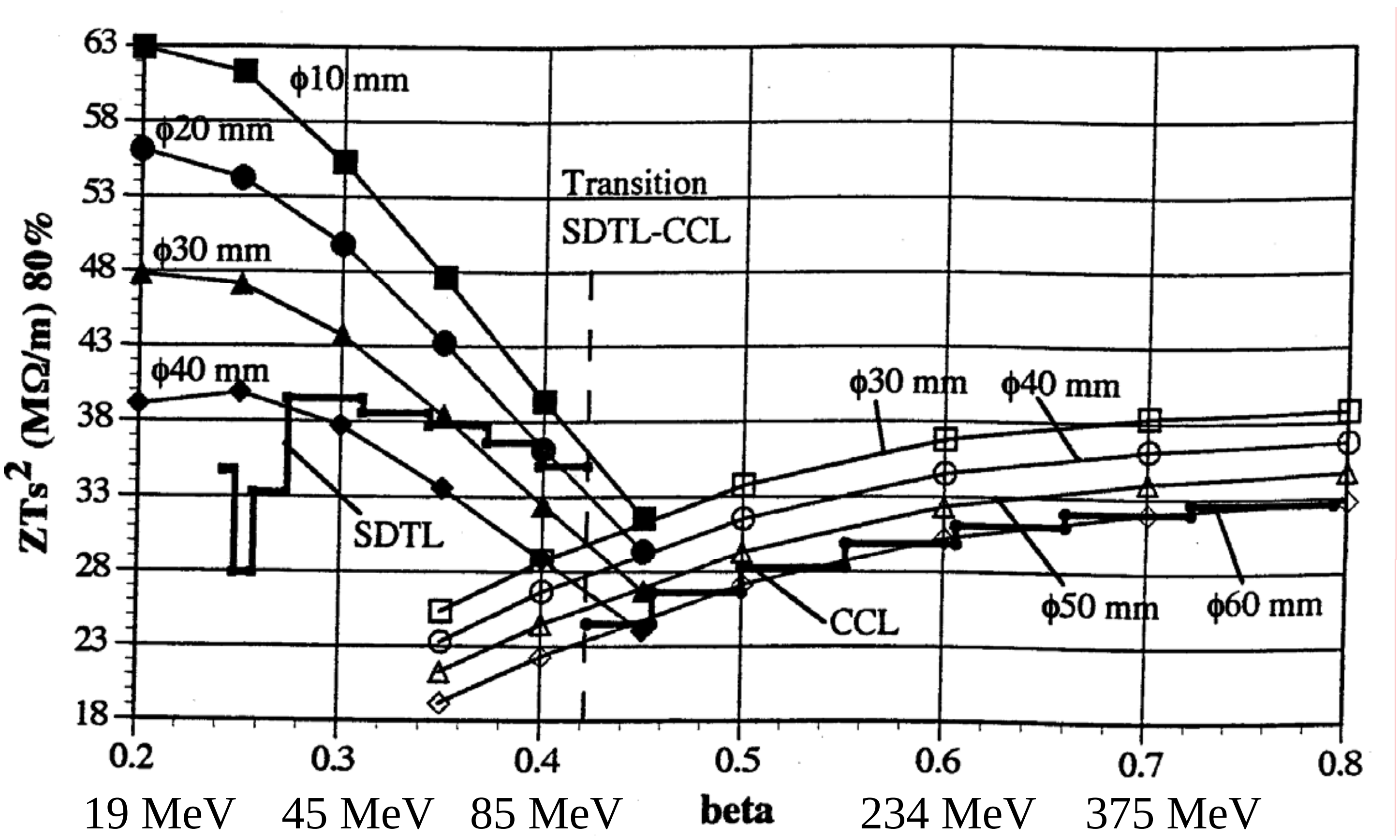
$$\Delta W_{max} = qV_0 \cdot T$$

$\boldsymbol{T}$: Transit time factor

$\Rightarrow$ Effective shunt impedance :

$$\boxed{ZT^2 = \frac{\Delta W_{max}^2}{2P_d}}$$

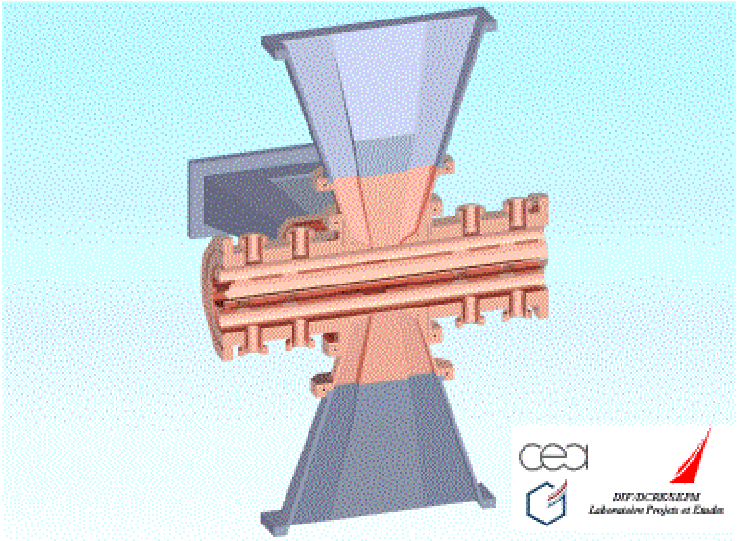
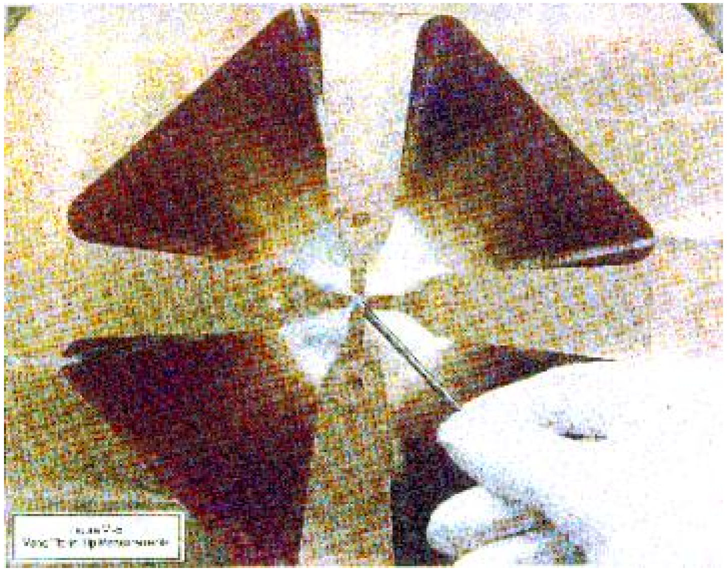
Example of use of effective shunt impedance ZT^2



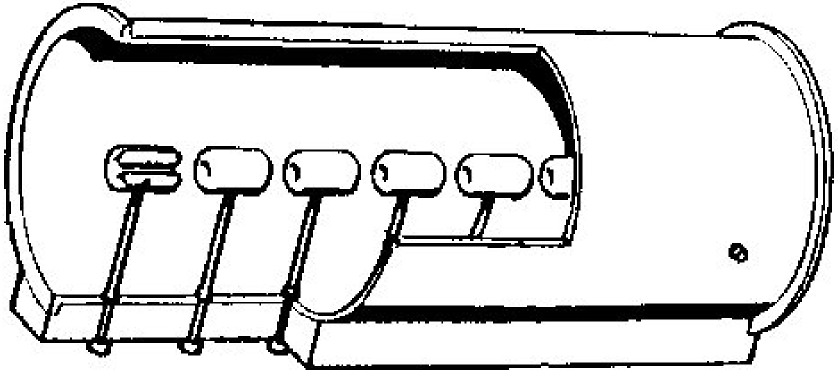
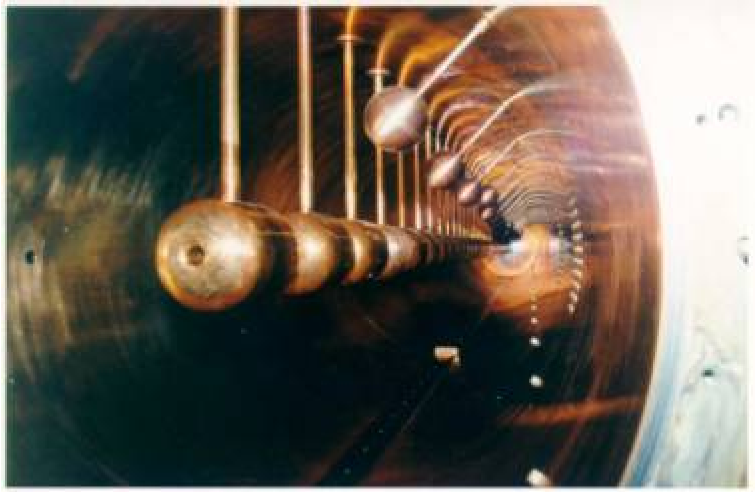
The effective shunt impedance per unit length of the structures can been used to set the transition energy between sections.

Various types of cavity : Tanks

RFQ (low energy ~50 keV-7 MeV)

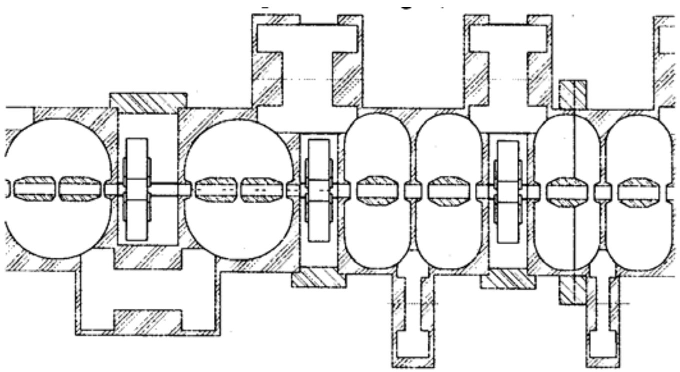



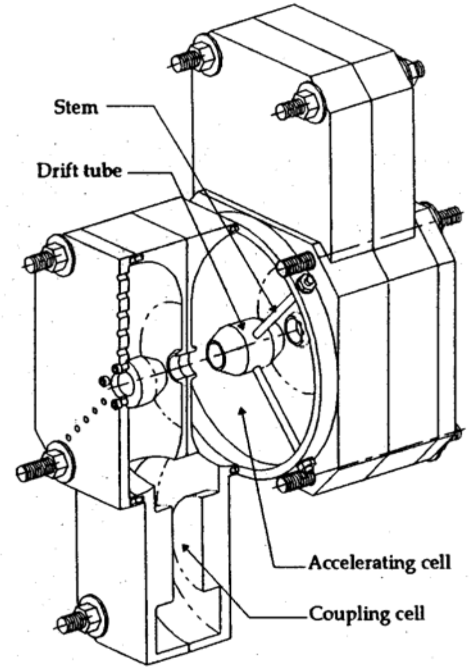
DTL (medium energy ~5-100 MeV)

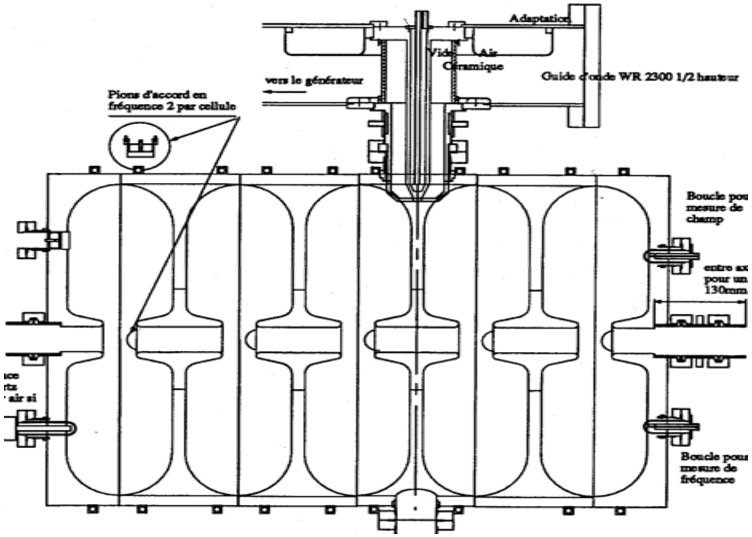
Various types of cavity : Coupled cavity

CCDTL (medium energy ~5-100 MeV)





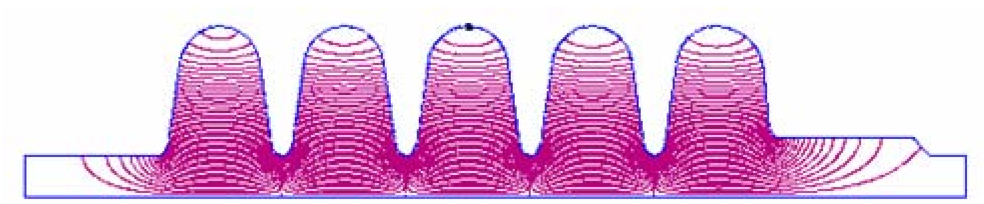
CCL (high energy ~80 MeV-2 GeV)

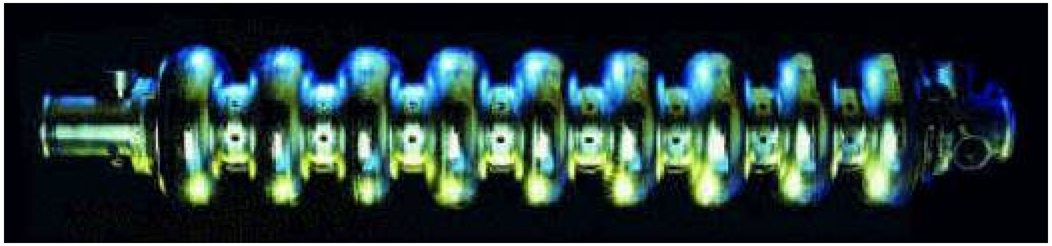


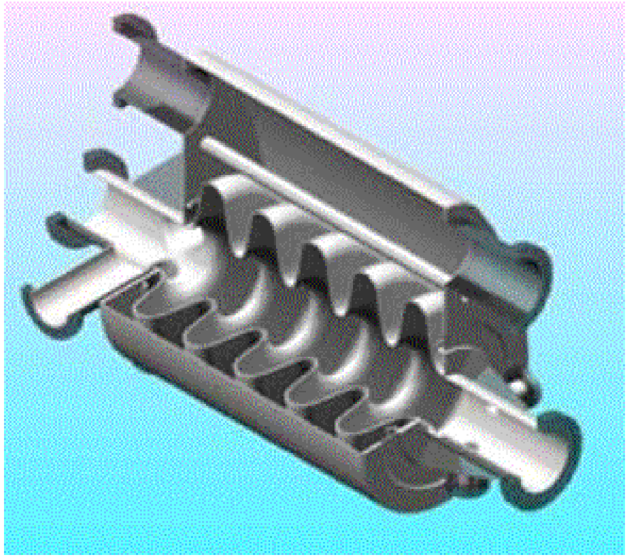
Various types of cavity : Superconducting

Spoke (medium energy ~20-100 MeV)

Elliptical (high energy ~100MeV- 2 GeV)







The synchronous particle - Linac design

A linac is designed with a hypothetical *synchronous particle*. Its phase in a cavity is called the *synchronous phase* :

- The synchronous particle absolute phase φ_i and the velocity β_{si-1} being known at the entrance of cavity *i*, its RF phase ϕ_i is calculated to get the wanted synchronous phase ϕ_{si}.
- The new velocity β_{si} of the particle can be calculated from : $\boxed{\Delta W_i = qV_0 T \cdot \cos\phi_s}$

    ① if the phase difference between cavities *i* and *i+1* is given, the distance D_i between them is adjusted to get the wanted synchronous phase ϕ_s in cavity *i+1*.

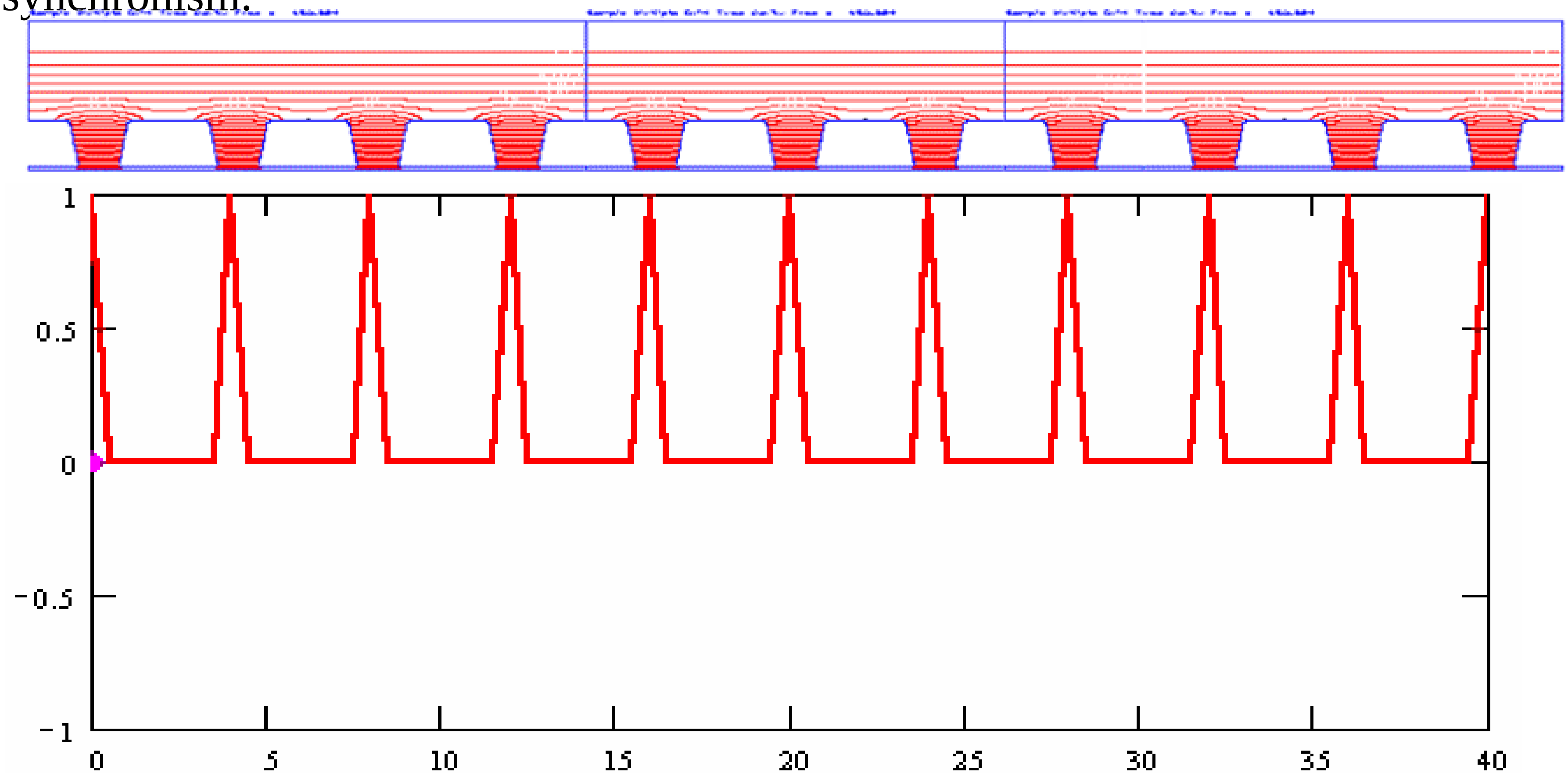
    ② if the distance D_i between cavities *i* and *i+1* is set, the RF phase ϕ_{i+1} of cavity *i+1* is calculated to get the wanted synchronous phase ϕ_s in it.

RF phase	ϕ_{i-1}		ϕ_i		ϕ_{i+1}
Particle velocity		β_{si-1}		β_{si}	
Distances		D_{i-1}		D_i	
Cavity number	*i-1*		*i*		*i+1*

Synchronism condition : $\boxed{\phi_{i+1} - \phi_i + 2\pi n = \omega \cdot \dfrac{D_i}{\beta_{s\,i}\, c}}$

① Linac with coupled cavities

Coupled cavities have the same phase. Distances between them are adjusted for synchronism.



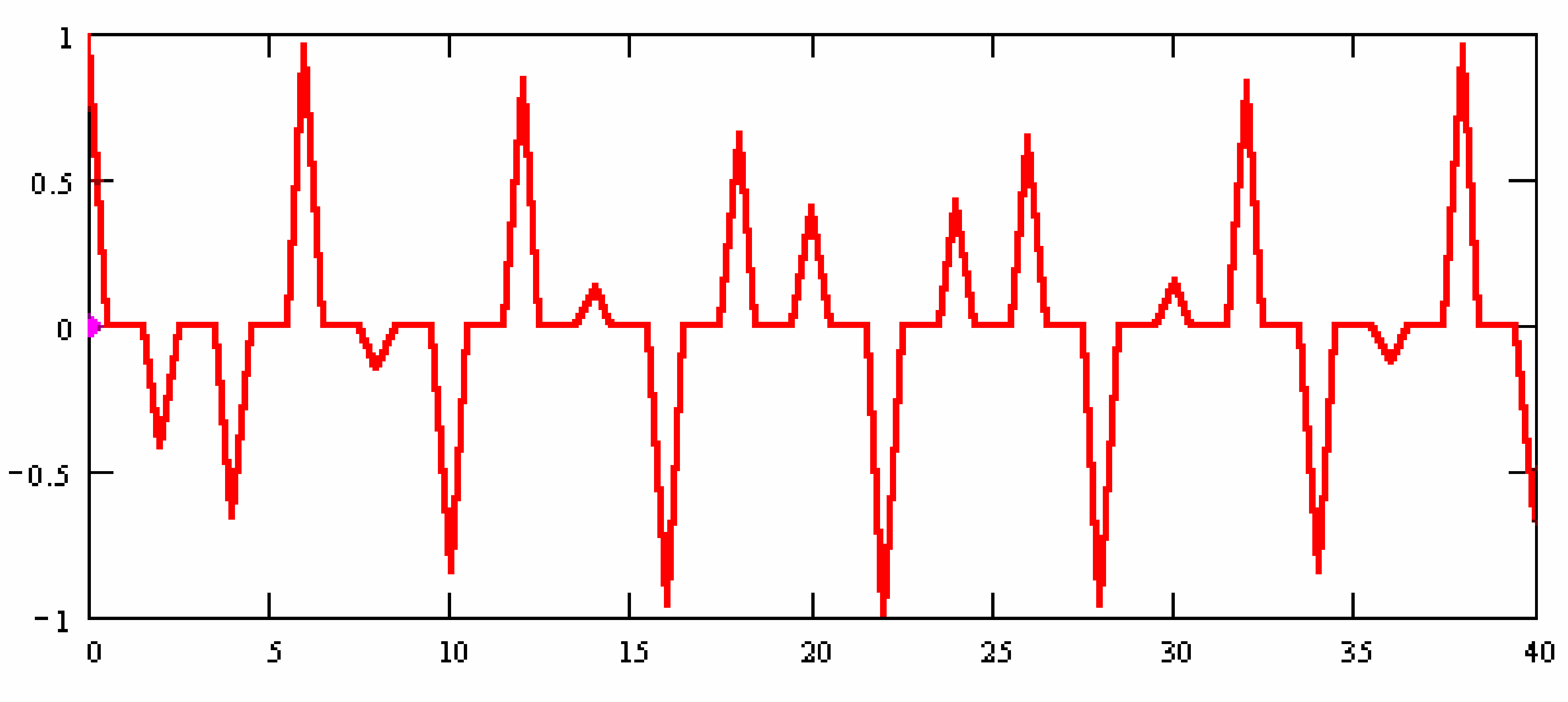
Field in cavities

Particle synchronous with the field — Its energy gain

Particle not synchronous with the field — Its energy gain

② Linac with independently phased cavities

The distance between the cavities is given. Cavities are phased to accelerate a given particle.



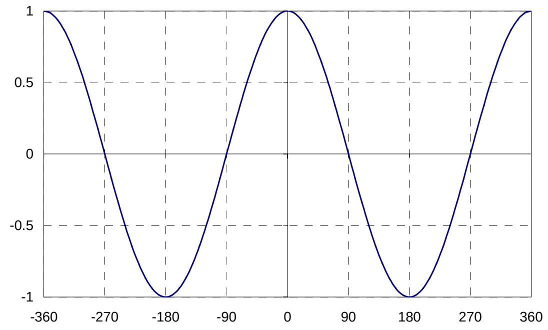
Field in cavities

Particle synchronous with the field — Its energy gain

Particle not synchronous with the field — Its energy gain

Choice of the synchronous phase

Acceleration condition : The field should accelerate the particle



$$\Delta W > 0 \Rightarrow q V_0 T \cdot \cos\phi_p > 0$$

$$qV_0 > 0 : \ \phi_p \in [-90°, 90°]$$

$$qV_0 < 0 : \ \phi_p \in [90°, 270°]$$

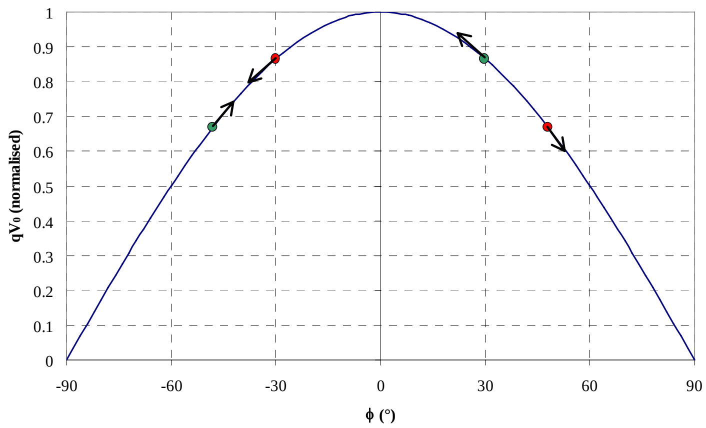
Stability condition : Late particles should gain more energy that early ones

$$\frac{d\Delta W}{d\phi_p} > 0 \Rightarrow qV_0 T \cdot \sin\phi_p < 0$$

$$qV_0 > 0 : \ \phi_p \in [-180°, 0°]$$

$$qV_0 < 0 : \ \phi_p \in [0°, 180°]$$

$$\boxed{\begin{array}{l} qV_0 > 0 : \ \phi_p \in [-90°, 0°] \\ qV_0 < 0 : \ \phi_p \in [90°, 180°] \end{array}}$$



General equations of motion

$$\frac{d\vec{p}}{dt} = q \cdot \left(\vec{v} \times \vec{B} + \vec{E}\right) = \vec{F}$$

$$+ \qquad p_w = \beta_w \gamma \cdot mc$$

$$+ \qquad dt = ds / \beta_z c$$

$$\Rightarrow \begin{cases} \dfrac{d\gamma\beta_x}{ds} = \dfrac{F_x}{mc^2\beta_z} = \dfrac{d\gamma\beta_z x'}{ds} \\ \dfrac{d\gamma\beta_y}{ds} = \dfrac{F_y}{mc^2\beta_z} = \dfrac{d\gamma\beta_z y'}{ds} \\ \dfrac{d\gamma\beta_z}{ds} = \dfrac{F_z}{mc^2\beta_z} \end{cases}$$

$\beta_w c$ is the particle velocity along w direction
γ is the particle reduced energy,
q and m its charge and rest mass.
x and y are transverse directions,
s is the abscissa along longitudinal direction z,
x' and y' are called the particle slopes.

$$\boxed{x'' + \frac{d\gamma\beta_z / ds}{\gamma\beta_z} x' = \frac{F_x}{mc^2\gamma\beta_z^2}}$$

These equations are <u>non linear</u>, <u>coupled</u> and <u>damped</u>.
Each element (cavity, quadrupole …) contributes to the force.

A few words on longitudinal coordinates

• If the beam is described at a given time *t* (useful for space-charge calculation and time-varying fields), particles are generally represented by their position *z* in the bunch and their velocity *z'* relative to the synchronous particle :

$$\begin{cases} z(t) = s(t) - s_s(t) \\ z'(t) = \dfrac{\beta_z(t) - \beta_s(t)}{\beta_s(t)} = \dfrac{v_z - v_s}{v_s} = \dfrac{p_z - p_s}{p_s} \end{cases}$$

• If the beam is described at a given abscissa *s* (easier for transport into elements), particles are generally represented by their phase φ in the bunch and their kinetic energy *w* relative to the synchronous particle :

$$\begin{cases} \varphi(s) = \phi(s) - \phi_s(s) \\ w(s) = W(s) - W_s(s) \end{cases}$$

One remarks that kinetic energy is not only a longitudinal particle property but it is often assumed as being totally longitudinal (paraxial approximation).

Longitudinal motion (1)

- At a given abscissa, a particle arrives at a given time : $t = \dfrac{\phi}{\omega_{rf}}$
- It has a given kinetic energy : $W = (\gamma - 1) \cdot mc^2$
- The evolution of these variables is given by :

$$\begin{cases} \dfrac{dW(s)}{ds} = qE_z(s) \cdot \cos\left(\phi(s) - \phi_{rf}\right) \\ \dfrac{d\phi(s)}{ds} = \dfrac{2\pi}{\beta(s)\lambda_{rf}} \end{cases}$$

n.b. Joel Leduff is using sinus

- The linac is designed with a *synchronous particle*, for which the phase law in the cavities has been fixed by the designer : $\phi_s(s)$.

- Beam particles are referred to this synchronous particle :

$$\begin{cases} \varphi(s) = \phi(s) - \phi_s(s) \\ w(s) = W(s) - W_s(s) \end{cases}$$

Longitudinal motion (2)

- The evolution of these variables is given by :

$$\begin{cases} \dfrac{d\varphi}{ds} = \dfrac{2\pi}{\lambda_{rf}}\left(\dfrac{1}{\beta} - \dfrac{1}{\beta_s}\right) \\ \dfrac{dw}{ds} = qE_z(s)\cdot\left(\cos\left(\phi(s) - \phi_{rf}\right) - \cos\left(\phi_s(s) - \phi_{rf}\right)\right) \end{cases}$$

- Giving finally :

$$\begin{cases} \dfrac{d\varphi}{ds} = -2\pi\cdot\dfrac{w}{(\beta_s\gamma_s)^3\cdot mc^2\cdot\lambda} = \dfrac{\partial H_{\phi w}}{\partial w} \\ \dfrac{dw}{ds} = -q\cdot E_z(s)\cdot\left(\cos\left(\phi_s - \phi_{rf}\right)\cdot(1-\cos\varphi) + \sin\left(\phi_s - \phi_{rf}\right)\cdot\sin\varphi\right) = -\dfrac{\partial H_{\phi w}}{\partial\varphi} \end{cases}$$

- $H_{\varphi\omega}$ is the motion hamiltonian given by :

$$\begin{aligned} H_{\varphi w} = &-\frac{2\pi}{(\beta_s\gamma_s)^3\cdot mc^2\cdot\lambda}\cdot\frac{w^2}{2} \\ &-q\cdot E_z(s)\cdot\left(\sin\left(\phi_s - \phi_{rf}\right)\cdot(\cos\varphi - 1) + \cos\left(\phi_s - \phi_{rf}\right)\cdot(\sin\varphi - \varphi)\right) \end{aligned}$$

- Particles follow curves for which $H_{\varphi w}$ =Cst.

Longitudinal motion linearisation

- The longitudinal motion can be linearised assuming : $\varphi << 1$

- In these conditions :

$$\begin{cases} \dfrac{d\varphi}{ds} = -2\pi \cdot \dfrac{w}{(\beta_s \gamma_s)^3 \cdot mc^2 \cdot \lambda} = \dfrac{\partial H_{\phi w,l}}{\partial w} \\ \dfrac{dw}{ds} = -q \cdot E_z(s) \cdot \sin(\phi_s - \phi_{rf}) \cdot \varphi = -\dfrac{\partial H_{\phi w,l}}{\partial \varphi} \end{cases}$$

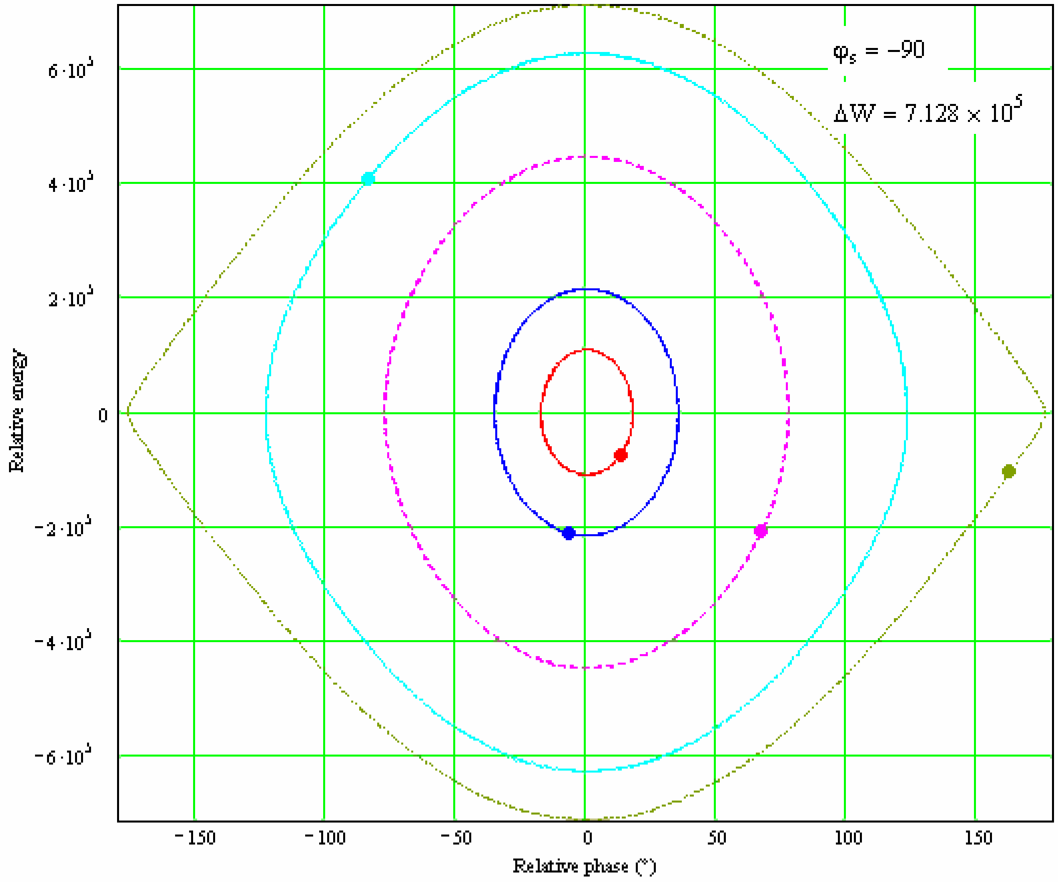
- $H_{\varphi\omega,l}$ is the motion hamiltonian given by :

$$H_{\varphi w,l} = -\frac{2\pi}{(\beta_s \gamma_s)^3 \cdot mc^2 \cdot \lambda} \cdot \frac{w^2}{2} + q \cdot E_z(s) \cdot \sin(\phi_s - \phi_{rf}) \cdot \frac{\varphi^2}{2}$$

- Particles follow curves for which $H_{\varphi w}$ =Cste, which are ellipses.

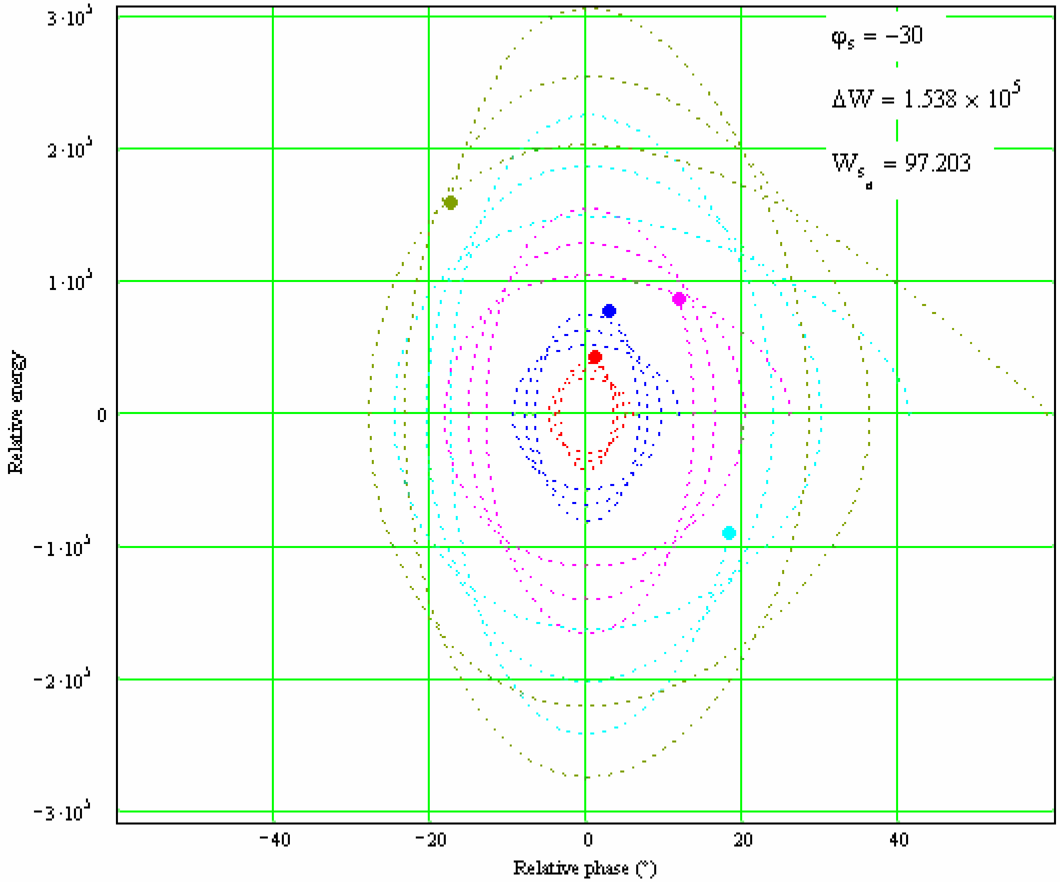
Longitudinal motion - No acceleration (φ_s=-90°)

- Let's assume that $H_{\varphi w}$ is not a function of s (E_z = Cst; $\beta_s\gamma_s$ = Cst).



Longitudinal motion - φ_s=-30° - With $\beta_s\gamma_s$ change

- Let's assume that $H_{\varphi w}$ is not a function of s (E_z = Cst; $\beta_s\gamma_s \neq$ Cst).



Particle motion in linear periodic force

In the highest simplification level, the particle motion along direction w (x, y or φ) can be considered : undamped, uncoupled, linear, and periodic.

$$w'' + \cancel{\frac{d\gamma\beta_z/ds}{\gamma\beta_z}} w' - \frac{F_w(w,\cancel{u,...})}{mc^2\gamma\beta_z^2} = 0$$

=Cste

Hill equation : $\dfrac{d^2w}{ds^2} + k_w(s)\cdot w = 0$ $\qquad k_w(s+S) = k_w(s)$

Giving : $w(s) = \sqrt{\varepsilon_0 \beta_{wm}(s)} \cdot \cos(\mu(s) + \mu_0)$

with : μ_0 and ε_0 constant, $\mu(s) = \mu_0 + \displaystyle\int_{s_0}^{s} \frac{ds}{\beta_{wm}(s)}$, and : $\beta_{wm}(s+S) = \beta_{wm}(s)$

In the (w, w') phase-space, the particle is moving on an ellipse of equation :

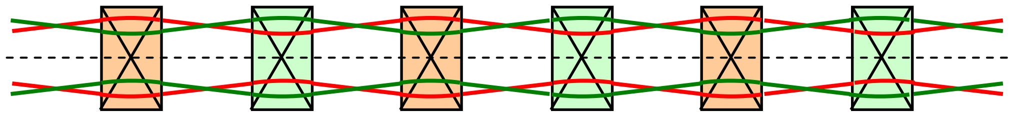
$$\gamma_{wm}(s)\cdot w^2 + 2\alpha_{wm}(s)\cdot ww' + \beta_{wm}(s)\cdot w'^2 = A^2$$

with : $\alpha_{wm}(s) = -\dfrac{1}{2}\dfrac{d\beta_{wm}}{ds}$ and $\gamma_{wm}(s) = \dfrac{1+\alpha_{wm}(s)^2}{\beta_{wm}(s)}$ Channel Courant-Snyder parameters.

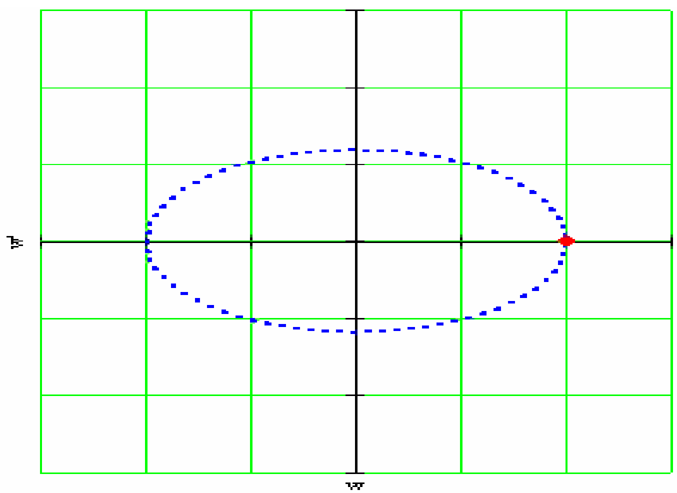
Example : Particle motion in a FODO Lattice

Particle

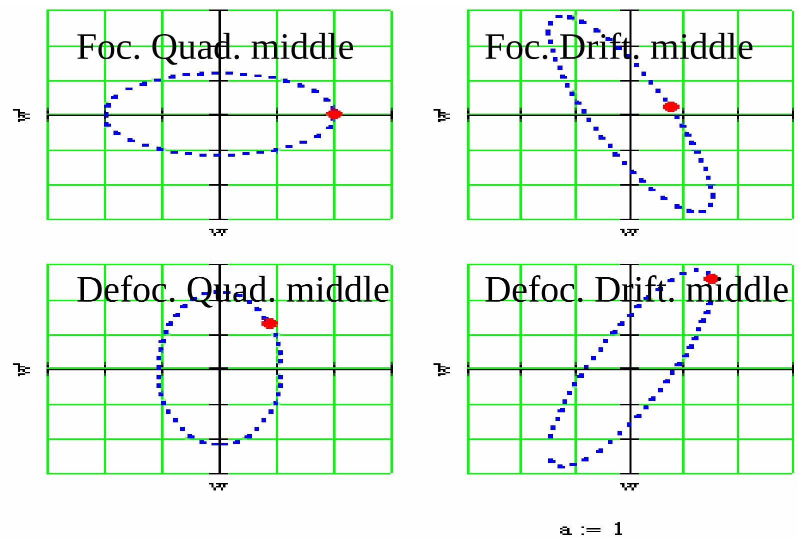
Particle ellipse

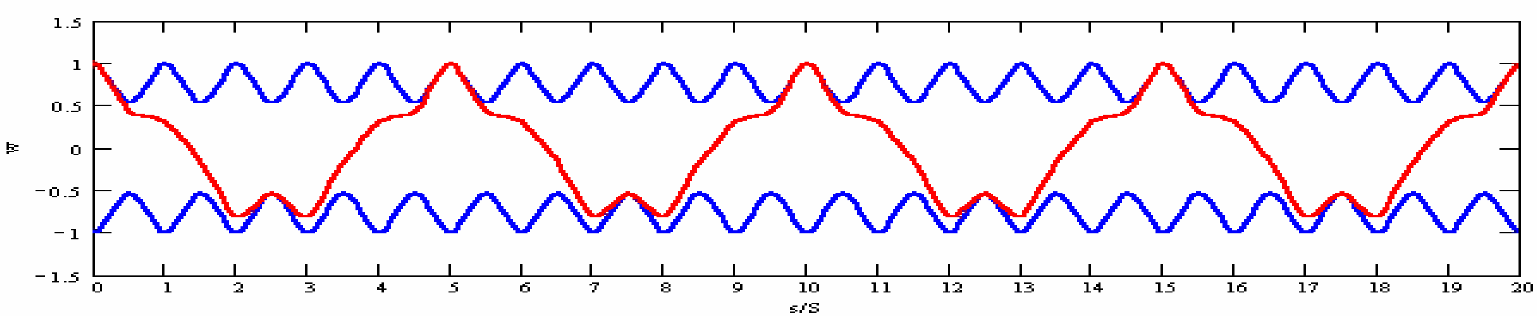


Phase-space trajectory



Phase-space periodic looks





Particle trajectory

Particle ellipse maximum size

RMS dimensions and Beam Twiss parameters

The rms dimensions of the beam are defined statistically as followed :

rms size : $\tilde{w} = \sqrt{\left\langle \left(w - \langle w \rangle \right)^2 \right\rangle}$

rms slope : $\tilde{w}' = \sqrt{\left\langle \left(w' - \langle w' \rangle \right)^2 \right\rangle}$

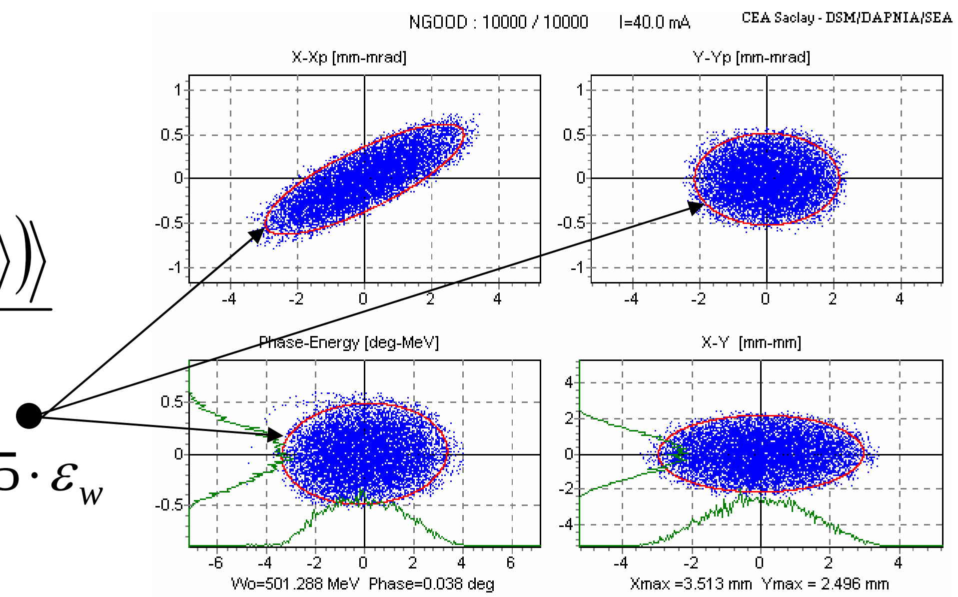
rms emittance : $\tilde{\varepsilon}_w = \sqrt{\tilde{w}^2 \cdot \tilde{w}'^2 - \left\langle \left(w - \langle w \rangle \right) \cdot \left(w' - \langle w' \rangle \right) \right\rangle^2}$

The beam Twiss parameters are then :

$$\beta_w = \frac{\tilde{w}^2}{\tilde{\varepsilon}_w} \qquad \gamma_w = \frac{\tilde{w}'^2}{\tilde{\varepsilon}_w}$$

$$\alpha_w = -\frac{\left\langle \left(w - \langle w \rangle \right) \cdot \left(w' - \langle w' \rangle \right) \right\rangle}{\tilde{\varepsilon}_w}$$

$$\gamma_w \cdot w^2 + 2\alpha_w \cdot ww' + \beta_w \cdot w'^2 = 5 \cdot \varepsilon_w$$



Beam matching

The beam is matched when :

$$\beta_w = \beta_{wm}$$

$$\alpha_w = \alpha_{wm}$$

$$\gamma_w = \gamma_{wm}$$

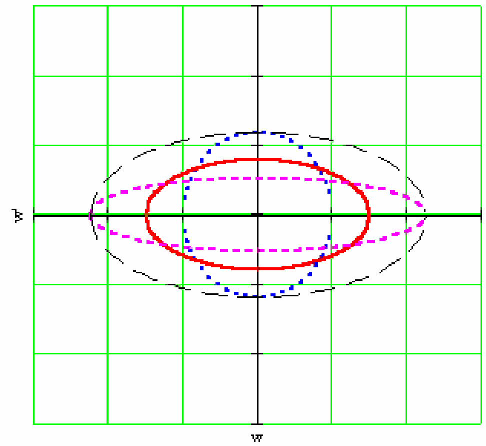
Matched beam

Bigger input beam

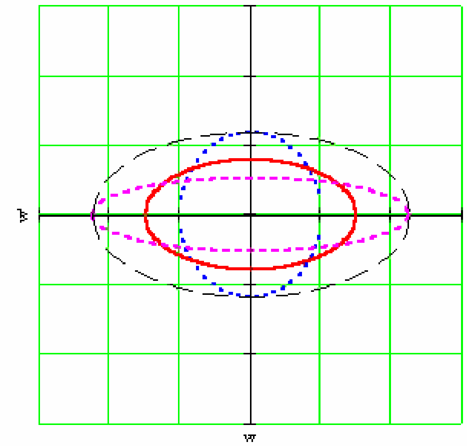
Smaller input beam

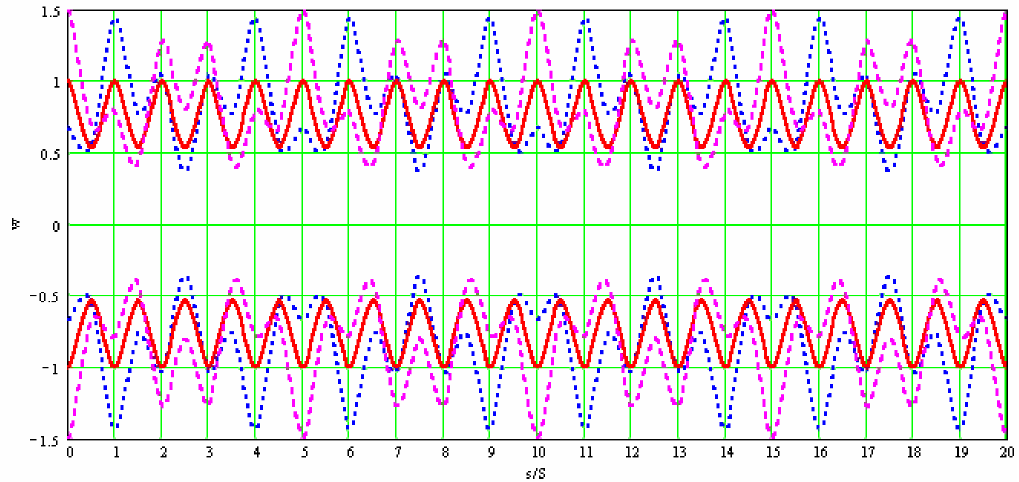
Phase-space scanned by the mismatched beams

Phase-space trajectory



Phase-space periodic looks



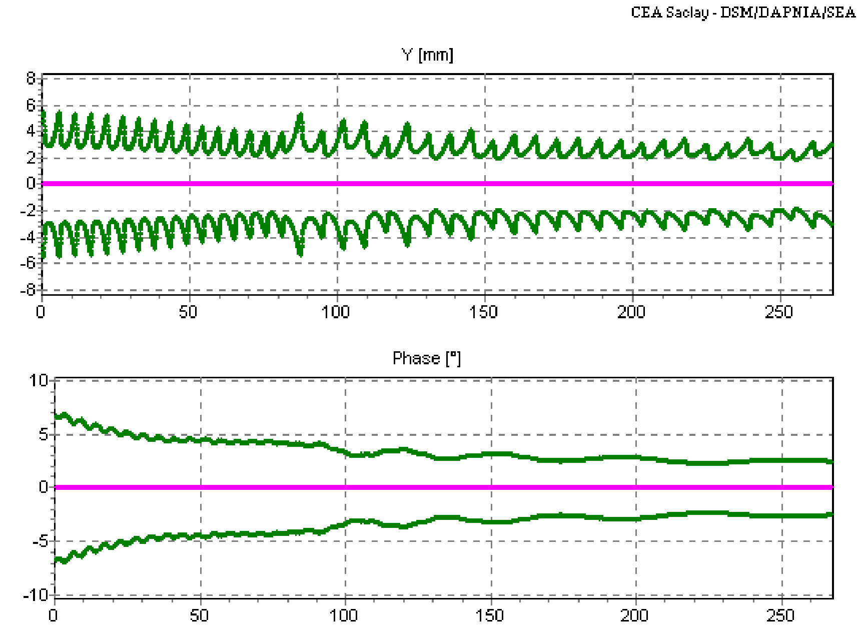


50% mismatched beam

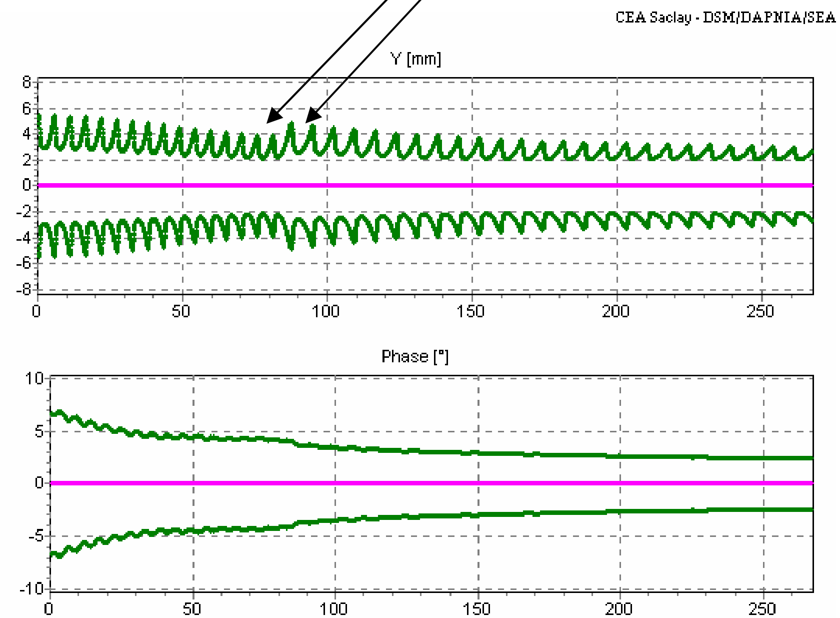
Beam matching (2)

The matching is done between sections by changing focusing force with quadrupoles (transverse) and cavities (longitudinal).

Calculations are made with « envelope codes » where the beam is modelled by its rms dimensions. This type of code calculates automatically the focusing strength that matches the beam.



Non matched beam



Matched beam

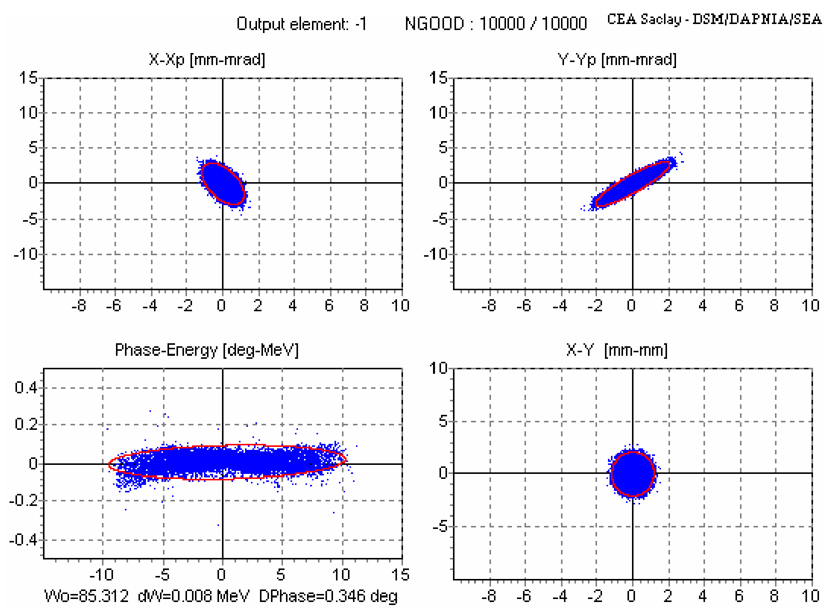
Mismatched Beam

Mismatching :

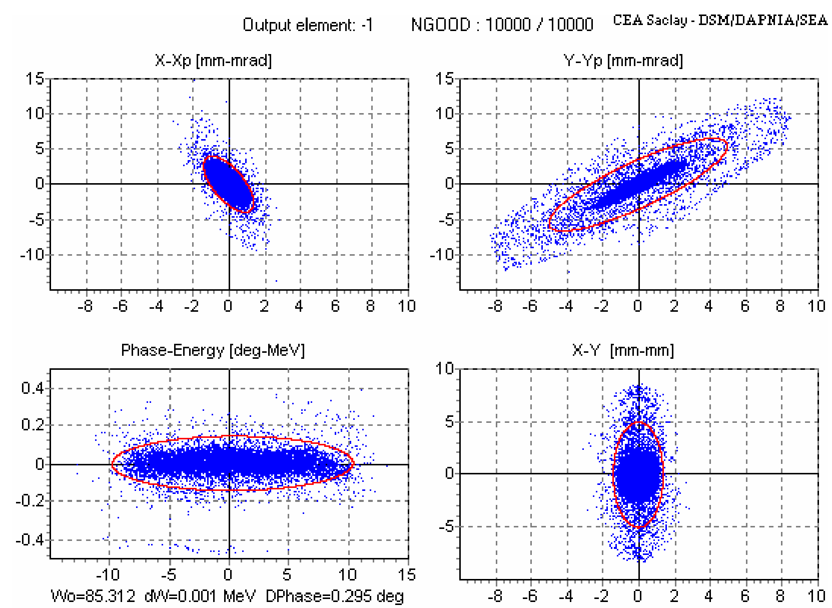
Emittance growth & Halo formation through :

- non linear forces (external or space-charge),
- resonance of some particle motion with core oscillation (space-charge).

<u>Ex</u>: 100 mA proton beam through a 5-85 MeV DTL



Matched beam



Mismatched beam

Non-linear forces

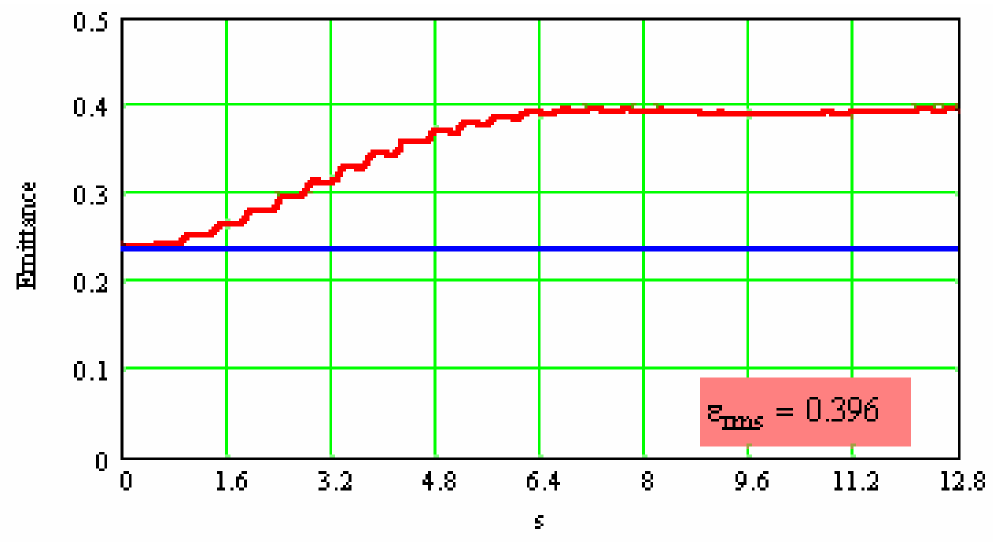
When the confinement force is non linear (multipole, longitudinal, space-charge) , the particle phase advance depends on the oscillation amplitude A :

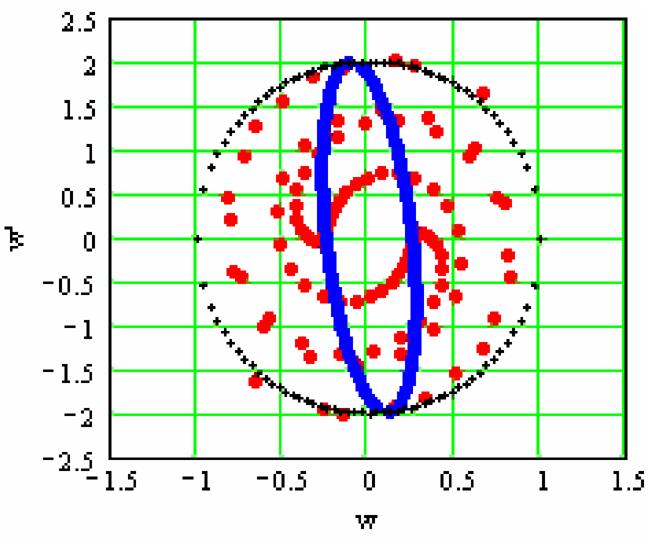
$$\sigma = \sigma(A)$$

$$\frac{d^2 w}{ds^2} + k_w(s,w) \cdot w = 0$$

This phenomenon is known as the tune spread.

Particle do not rotate at the same speed in the phase-space : possible filamentation





Linear force   Non linear force

Summary

• Linacs are competitive for low energy, high current, high duty cycle beams or very very high energy light-particles (e^+-e^-) colliders.

• Acceleration and longitudinal confinement is done with RF resonant cavities, transverse confinement with quadrupoles (except at very low energy).

• Cavities are pieces of metal (Cu or Nb) whose shape is optimised to accelerate the particles at the RF frequency with the higher efficiency (ZT^2 as high as possible) and the lower cost.

• RF phases in cavities are adjusted with respect to a synchronous particle to accelerate the beam and keep it bunched (synchronous phase choice).

• There exists a matched beam at input of each periodic structure

• Forces are linearised to calculate the beam matching to the structure.

• non linear forces have to be taken into account to quantified the effect of the errors on the emittance growth (loss of luminosity or halo formation).