Relativity for Accelerator Physicists

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Overview

- The principle of special relativity
- Lorentz transformation and its consequences
- 4-vectors: position, velocity, momentum, invariants. Derivation of $E=mc^2$
- Examples of the use of 4-vectors
- Inter-relation between β and γ, momentum and energy
- An accelerator problem in relativity

Reading

- W. Rindler: Introduction to Special Relativity (OUP 1991)
- D. Lawden: An Introduction to Tensor Calculus and Relativity
- N.M.J. Woodhouse: Special Relativity (Springer 2002)
- A.P. French: Special Relativity, MIT Introductory Physics Series (Nelson Thomes)

Historical background

- Groundwork by Lorentz in studies of electrodynamics, with crucial concepts contributed by Einstein to place the theory on a consistent basis.
- Maxwell's equations (1863) attempted to explain electromagnetism and optics through wave theory
  - light propagates with speed $c = 3\times10^8$ m/s in "ether" but with different speeds in other frames
  - the ether exists solely for the transport of e/m waves
  - Maxwell's equations not invariant under Galilean transformations
  - To avoid setting e/m apart from classical mechanics, assume light has speed *c* only in frames where source is at rest
  - And the ether has a small interaction with matter and is carried along with astronomical objects

Nonsense! Contradicted by:

- Aberration of star light (small shift in apparent positions of distant stars)
- Fizeau's 1859 experiments on velocity of light in liquids
- Michelson-Morley 1907 experiment to detect motion of the earth through ether
- Suggestion: perhaps material objects contract in the direction of their motion

$$L(v) = L_0\left(1-\frac{v^2}{c^2}\right)^{1/2}$$

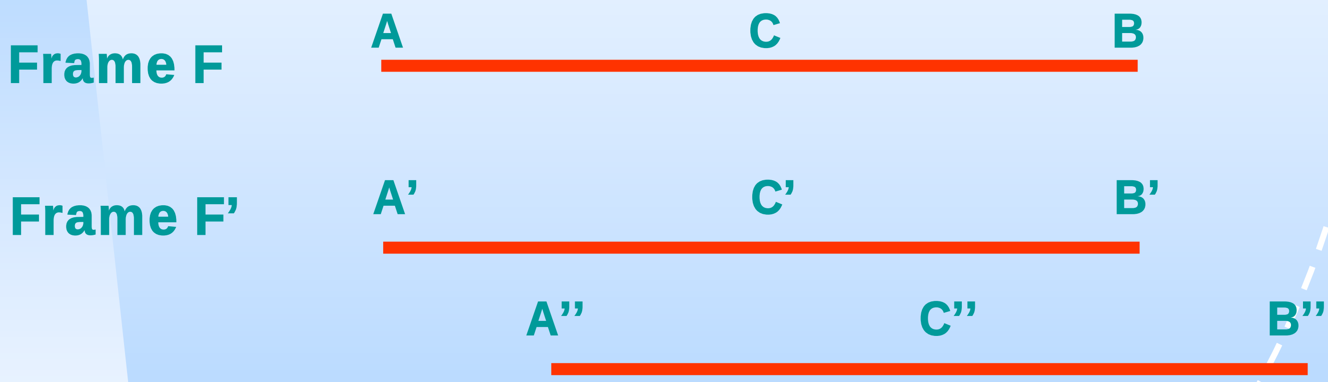
This was the last gasp of ether advocates and the germ of Special Relativity led by Lorentz, Minkowski and Einstein.

The Principle of Special Relativity

- A frame in which particles under no forces move with constant velocity is “inertial”
- Consider relations between inertial frames where measuring apparatus (rulers, clocks) can be transferred from one to another.
- Behaviour of apparatus transferred from F to F' is independent of mode of transfer
- Apparatus transferred from F to F', then from F' to F'', agrees with apparatus transferred directly from F to F''.
- *The Principle of Special Relativity states that all physical laws take equivalent forms in related inertial frames, so that we cannot distinguish between the frames.*

Simultaneity

- Two clocks A and B are synchronised if light rays emitted at the same time from A and B meet at the mid-point of AB



- Frame F' moving with respect to F. Events simultaneous in F cannot be simultaneous in F'.
- Simultaneity is <u>not</u> absolute but frame dependent.

The Lorentz Transformation

- Must be linear to agree with standard Galilean transformation in low velocity limit
- Preserves wave fronts of pulses of light,

i.e. $P \equiv x^2 + y^2 + z^2 - c^2t^2 = 0$

whenever $Q \equiv x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$

- Solution is the **Lorentz transformation** from frame F (t,x,y,z) to frame F'(t',x',y',z') moving with velocity v along the x-axis:

$$\left.\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\right\} \quad \text{where} \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Outline of Derivation

Set $t' = \alpha t + \beta x$

$x' = \gamma x + \delta t$

$y' = \varepsilon y$

$z' = \varsigma z$

Then $P = kQ$

$$\Leftrightarrow c^2 t'^2 - x'^2 - y'^2 - z'^2 = k\left(c^2 t^2 - x^2 - y^2 - z^2\right)$$

$$\Rightarrow c^2(\alpha t + \beta x)^2 - (\gamma x + \delta t)^2 - \varepsilon^2 y^2 - \varsigma^2 z^2 = k\left(c^2 t^2 - x^2 - y^2 - z^2\right)$$

Equate coefficients of x,y,z,t.

Isotropy of space $\Rightarrow k = k(\vec{v}) = k(|\vec{v}|) = \pm 1$

Apply some common sense (e.g. $\varepsilon, \varsigma, k = +1$ and not -1)

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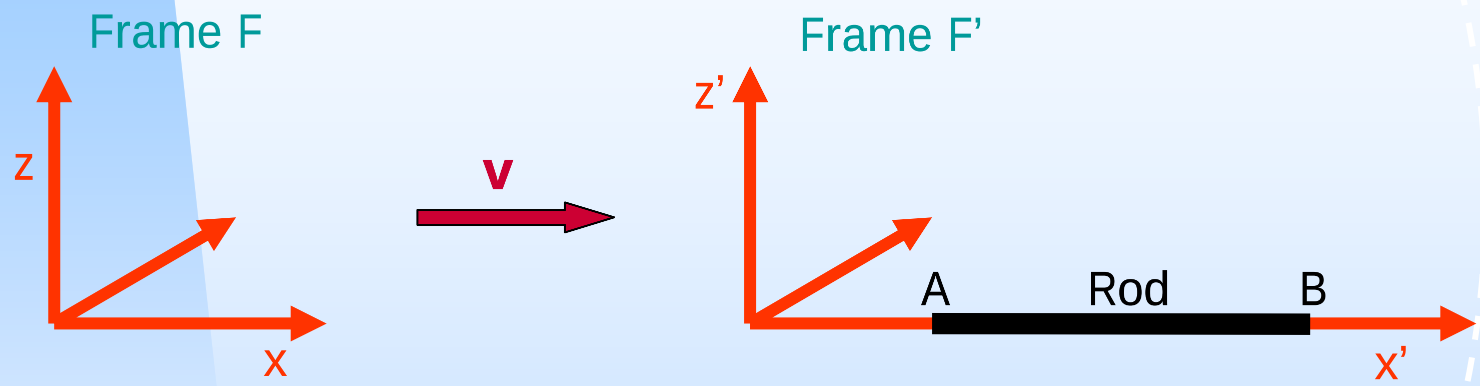
Consequences: length contraction



Rod AB of length L' fixed in F' at x'_A, x'_B. What is its length measured in F?

Must measure positions of ends in F at the same time, so events in F are (t,x_A) and (t,x_B). From Lorentz:

$$x'_A = \gamma\left(x_A - vt\right) \qquad x'_B = \gamma\left(x_B - vt\right)$$

$$L' = x'_B - x'_A = \gamma\left(x_B - x_A\right) = \gamma L > L$$

Moving objects appear contracted in the direction of the motion

Consequences: time dilatation

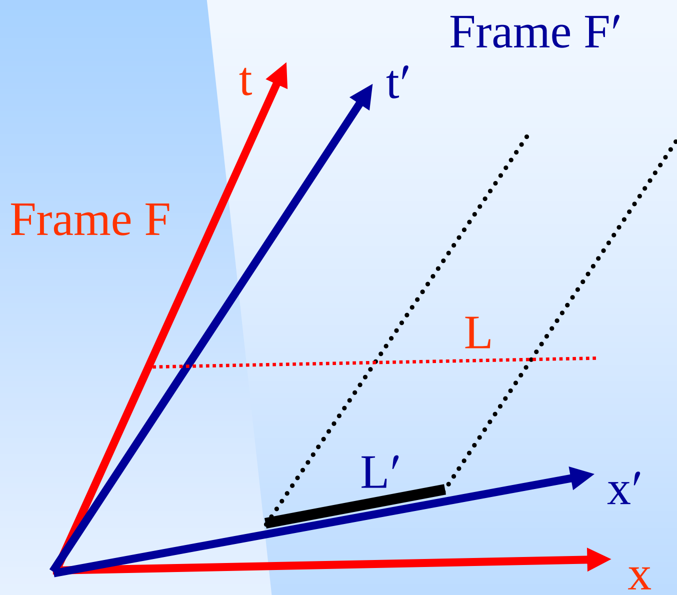
- Clock in frame F at point with coordinates (x,y,z) at different times t_A and t_B
- In frame F' moving with speed v, Lorentz transformation gives

$$t'_A = \gamma\left(t_A - \frac{vx}{c^2}\right) \qquad t'_B = \gamma\left(t_B - \frac{vx}{c^2}\right)$$

- So: $\Delta t' = t'_B - t'_A = \gamma(t_B - t_A) = \gamma \Delta t > \Delta t$

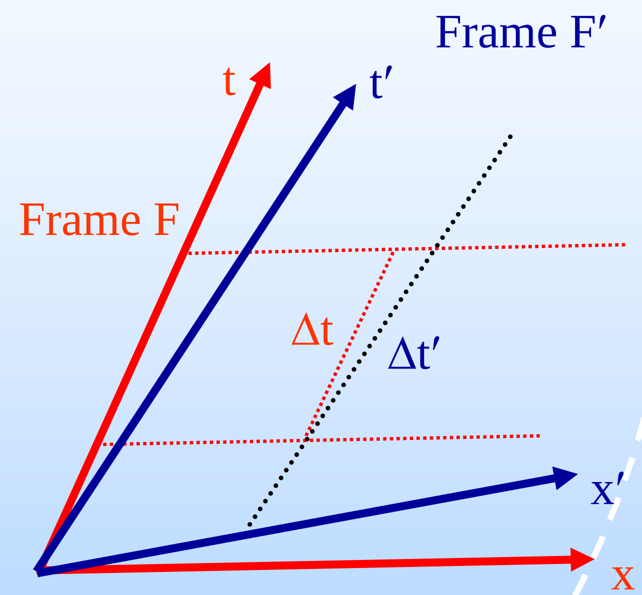
Moving clocks appear to run slow

Schematic Representation of the Lorentz Transformation



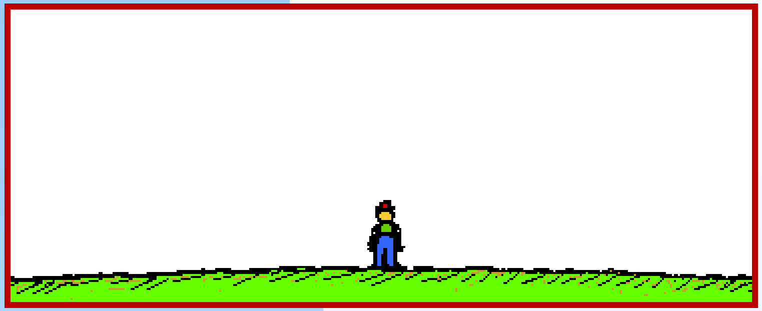
Length contraction $L<L'$

Rod at rest in F′. Measurement in F at fixed time t, along a line parallel to x-axis

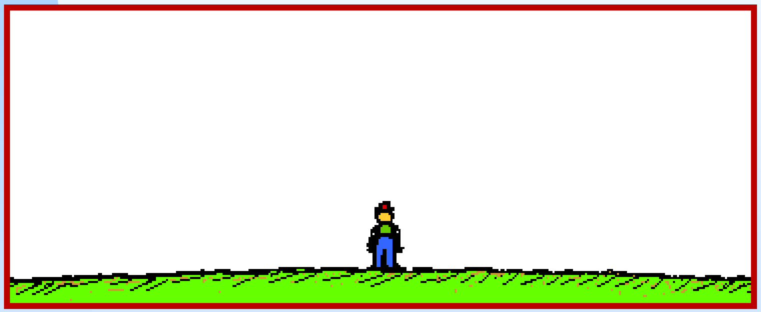


Time dilatation: $\Delta t<\Delta t'$

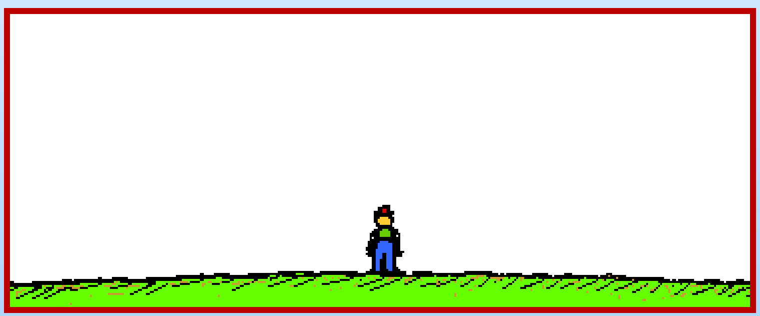
Clock at rest in F. Time difference in F′ from line parallel to x′-axis



v = 0.8c



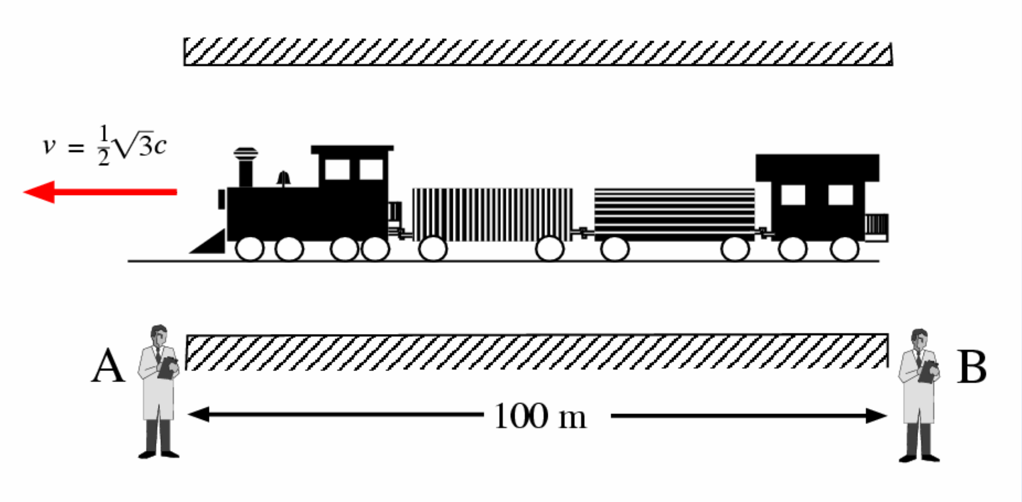
v = 0.9c



v = 0.99c

v = 0.9999c

Example: High Speed Train



- All clocks synchronised.
- Observers A and B at exit and entrance of tunnel say the train is moving, has contracted and has length

$$\frac{100}{\gamma} = 100 \times \left(1 - \frac{v^2}{c^2}\right)^{1/2} = 100 \times \left(1 - \frac{3}{4}\right)^{1/2} = 50\text{m}$$

- But the tunnel is moving relative to the driver and guard on the train and they say the train is 100 m in length but the tunnel has contracted to 50 m

Questions

- If A’s clock reads zero as the driver exits tunnel, what does B’s clock read when the guard goes in?

Moving train length 50m, so driver has still 50m to travel before his clock reads 0. Hence clock reading is

$$-\frac{50}{v} = -\frac{100}{\sqrt{3}c} \approx -200\,\mathrm{ns}$$

- What does the guard’s clock read as he goes in?

To the guard, tunnel is only 50m long, so driver is 50m past the exit as guard goes in. Hence clock reading is

$$+\frac{50}{v} = +\frac{100}{\sqrt{3}c} \approx +200\,\mathrm{ns}$$

- Where is the guard when his clock reads 0?

Guard’s clock reads 0 when driver’s clock reads 0, which is as driver exits the tunnel. To guard and driver, tunnel is 50m, so guard is 50m from the entrance in the train’s frame, or 100m in tunnel frame.

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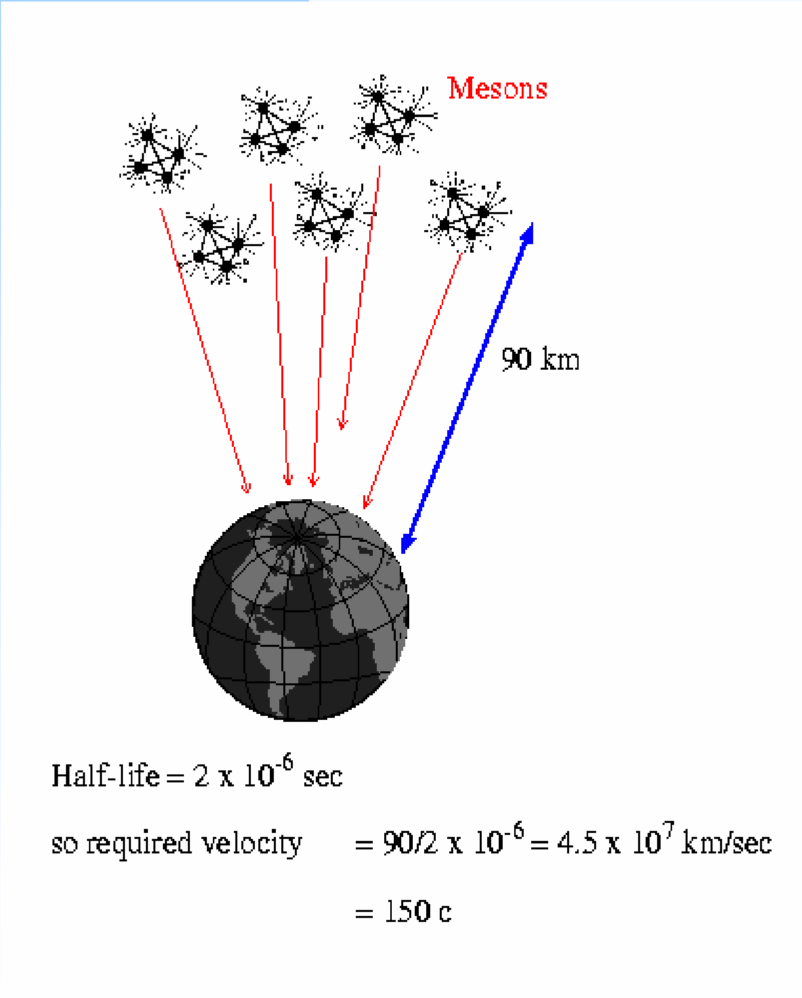
Example: π-mesons



- Mesons are created in the upper atmosphere, 90km from earth. Their half life is τ=2 μs, so they can travel at most 2 ×10^{-6}c=600m before decaying. So how do more than 50% reach the earth's surface undecayed?
- Mesons see distance contracted by γ, so

$$v\tau \approx \left(\frac{90}{\gamma}\right)\text{km}$$

- Earthlings say mesons' clocks run slow so their half-life is γτ and

$$v(\gamma\tau) \approx 90\,\text{km}$$

- Both give

$$\frac{\gamma v}{c} = \frac{90\text{ km}}{c\tau} = 150, \quad v \approx c, \quad \gamma \approx 150$$

Invariants

- An invariant is a quantity that has the same value in all inertial frames.
- Lorentz transformation is based on invariance of

$$c^2t^2-(x^2+y^2+z^2)=(ct)^2-\vec{x}^2$$

- Write in terms of the 4-position vector $\mathrm{X}=\left(ct,\vec{x}\right)$ as $\mathrm{X}\bullet\mathrm{X}$

If $\quad \mathrm{X}=(x_0,\vec{x}),\ \mathrm{Y}=(y_0,\vec{y}),$ define the invariant product

$$\mathrm{X}\bullet\mathrm{Y}=x_0y_0-\vec{x}\cdot\vec{y}$$

- Fundamental invariant (preservation of speed of light):

$$c^2\Delta t^2-\Delta x^2-\Delta y^2-\Delta z^2=c^2\Delta t^2\left(1-\frac{\Delta x^2+\Delta y^2+\Delta z^2}{c^2\Delta t^2}\right)$$

$$=c^2\Delta t^2\left(1-\frac{v^2}{c^2}\right)\quad=c^2\left(\frac{\Delta t}{\gamma}\right)^2$$

- τ is the invariant proper time where $\Delta\tau=\Delta t/\gamma$
- τ is time in the rest frame

4-Vectors

- The Lorentz transformation can be written in matrix form

$$\boxed{\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}} \quad \Rightarrow \quad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

An object made up of 4 elements which transforms like X is called a 4-vector

(analogous to the 3-vector of classical mechanics)

Position 4-vector $\mathrm{X} = (ct, \vec{x})$

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4-Vectors in S.R. Mechanics

- Velocity: $$V = \frac{dX}{d\tau} = \gamma \frac{dX}{dt} = \gamma \frac{d}{dt}(ct, \vec{x}) = \gamma\,(c, \vec{v})$$
- Note invariant $$V \bullet V = \gamma^2 (c^2 - \vec{v}^2) = c^2$$
- Momentum $$P = m_0 V = m_0 \gamma (c, \vec{v}) = (mc, \vec{p})$$

$m = m_0 \gamma$ is relativistic mass

$\vec{p} = m_0 \gamma \vec{v} = m\vec{v}$ is the 3-momentum

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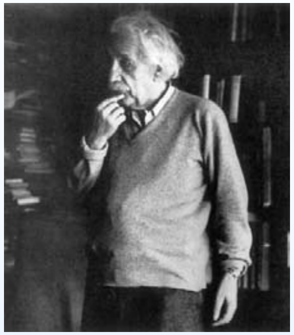
Example of Transformation: Addition of Velocities

- A particle moves with velocity $\vec{u} = (u_x, u_y, u_z)$ in frame F, so has 4-velocity $V = \gamma_u (c, \vec{u})$
- Add velocity $\vec{v} = (v, 0, 0)$ by transforming to frame F′ to get new velocity $\vec{w}$.
- Lorentz transformation gives $(t \leftrightarrow \gamma, \quad \vec{x} \leftrightarrow \gamma\vec{u})$

$$\begin{aligned}
\gamma_w &= \gamma_v \left(\gamma_u + \frac{v \gamma_u u_x}{c^2} \right) \\
\gamma_w w_x &= \gamma_v \left(\gamma_u u_x + v \gamma_u \right) \\
\gamma_w w_y &= \gamma_u u_y \\
\gamma_w w_z &= \gamma_u u_z
\end{aligned}
\quad \Rightarrow \quad
w_x = \frac{u_x + v}{1 + \dfrac{v u_x}{c^2}}$$

$$w_y = \frac{u_y}{\gamma_v \left(1 + \dfrac{v u_x}{c^2} \right)} \qquad w_z = \frac{u_z}{\gamma_v \left(1 + \dfrac{v u_x}{c^2} \right)}$$

Einstein's relation



- Momentum invariant $P \bullet P = m_0^2\,(V \bullet V) = m_0^2 c^2$
- Differentiate $P \bullet \dfrac{dP}{d\tau} = 0 \quad \Rightarrow \quad V \bullet \dfrac{dP}{d\tau} = 0$
- From Newton's 2nd Law expect 4-Force given by

$$F = \frac{dP}{d\tau} = \gamma \frac{dP}{dt} = \gamma \frac{d}{dt}(mc, \vec{p}) = \gamma \left(c\frac{dm}{dt}, \frac{d\vec{p}}{dt} \right) = \gamma \left(c\frac{dm}{dt}, \vec{f} \right)$$

- But $V \bullet \dfrac{dP}{d\tau} = 0 \quad \Rightarrow \quad V \bullet F = 0$
- So $\dfrac{d}{dt}\left(mc^2\right) - \vec{v} \cdot \vec{f} = 0$

Rate of doing work, $\vec{v} \cdot \vec{f}$ = rate of change of kinetic energy

Therefore kinetic energy

$$T = mc^2 + \text{constant} = m_0 c^2 (\gamma - 1)$$

$E=mc^2$ is total energy

Basic quantities used in Accelerator calculations

Relative velocity $\beta = {}^{\mathrm{v}}\!/\!_{\mathrm{c}}$

Velocity $v = \beta \mathrm{c}$

Momentum $p = mv = m_0 \gamma \beta c$

Kinetic energy $T = (m\text{-}m_0)c^2 = m_0 c^2 (\gamma - 1)$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \left(1 - \beta^2\right)^{-\frac{1}{2}}$$

$$\Rightarrow \quad (\beta\gamma)^2 = \frac{\gamma^2 v^2}{c^2} = \gamma^2 - 1 \quad \Rightarrow \quad \beta^2 = \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

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Velocity as a function of energy

$$T = m_0(\gamma - 1)c^2$$

$$\gamma = 1 + T/_{m_0 c^2}$$

$$\beta = \sqrt{1 - 1/_{\gamma^2}}$$

$$p = m_o c\beta\gamma$$

For small $\mathrm{v}/_{\mathrm{c}}$, $\quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$

so $\quad T = m_0 c^2(\gamma - 1) \approx \frac{1}{2} m_0 v^2$

Relationships between small variations in parameters ΔE, ΔT, Δp, Δβ, Δγ

$$(\beta\gamma)^2 = \gamma^2 - 1$$

$$\Rightarrow \quad \beta\gamma\,\Delta(\beta\gamma) = \gamma\,\Delta\gamma$$

$$\Rightarrow \quad \beta\,\Delta(\beta\gamma) = \Delta\gamma \qquad (1)$$

$$\frac{1}{\gamma^2} = 1 - \beta^2$$

$$\Rightarrow \quad \frac{1}{\gamma^3}\Delta\gamma = \beta\,\Delta\beta \qquad (2)$$

$$\frac{\Delta p}{p} = \frac{\Delta(m_0\gamma\beta c)}{m_0\gamma\beta c} = \frac{\Delta(\beta\gamma)}{\beta\gamma}$$

$$= \frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma} = \frac{1}{\beta^2}\frac{\Delta E}{E}$$

$$= \gamma^2\frac{\Delta\beta}{\beta}$$

$$= \frac{\gamma}{\gamma+1}\frac{\Delta T}{T} \qquad \text{(exercise)}$$

	$\dfrac{\Delta\beta}{\beta}$	$\dfrac{\Delta p}{p}$	$\dfrac{\Delta T}{T}$	$\dfrac{\Delta E}{E} = \dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta\beta}{\beta} =$	$\dfrac{\Delta\beta}{\beta}$	$\dfrac{1}{\gamma^2}\dfrac{\Delta p}{p}$ $\dfrac{\Delta p}{p} - \dfrac{\Delta\gamma}{\gamma}$	$\dfrac{1}{\gamma(\gamma+1)}\dfrac{\Delta T}{T}$	$\dfrac{1}{\beta^2\gamma^2}\dfrac{\Delta\gamma}{\gamma}$ $\dfrac{1}{\gamma^2-1}\dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta p}{p} =$	$\gamma^2\dfrac{\Delta\beta}{\beta}$	$\dfrac{\Delta p}{p}$	$\dfrac{\gamma}{\gamma+1}\dfrac{\Delta T}{T}$	$\dfrac{1}{\beta^2}\dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta T}{T} =$	$\gamma(\gamma+1)\dfrac{\Delta\beta}{\beta}$	$\left(1+\dfrac{1}{\gamma}\right)\dfrac{\Delta p}{p}$	$\dfrac{\Delta T}{T}$	$\dfrac{\gamma}{\gamma-1}\dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta E}{E} =$	$(\beta\gamma)^2\dfrac{\Delta\beta}{\beta}$	$\beta^2\dfrac{\Delta p}{p}$	$\left(1-\dfrac{1}{\gamma}\right)\dfrac{\Delta T}{T}$	$\dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta\gamma}{\gamma} =$	$(\gamma^2-1)\dfrac{\Delta\beta}{\beta}$	$\dfrac{\Delta p}{p} - \dfrac{\Delta\beta}{\beta}$		

4-Momentum Conservation

- Equivalent expression for 4-momentum $P = m_0\gamma(c,\vec{v}) = (mc,\vec{p}) = \left(E/c,\vec{p}\right)$
- Invariant $m_0^2c^2 = P \bullet P = \dfrac{E^2}{c^2} - \vec{p}^{\,2}$

$$\boxed{E^2/c^2 = m_0^2c^2 + \vec{p}^{\,2}}$$

- Classical momentum conservation laws → conservation of 4-momentum. Total 3-momentum and total energy are conserved.

$$\sum_{\text{particles, i}} P_i = \text{constant}$$

$$\Rightarrow \sum_{\text{particles, i}} E_i \text{ and } \sum_{\text{particles,i}} \vec{p}_i \text{ constant}$$

Example of use of invariants

- Two particles have equal rest mass m_0.
  - Frame 1: one particle at rest, total energy is E_1.
  - Frame 2: centre of mass frame where velocities are equal and opposite, total energy is E_2.

Problem: Relate E_1 to E_2

Total energy E_1

(Fixed target experiment)

$$P_1 = \left(\frac{E_1 - m_0 c^2}{c}, \vec{p} \right) \qquad P_2 = \left(m_0 c, 0 \right)$$

Total energy E_2

(Colliding beams expt)

$$P_1 = \left(\frac{E_2}{2c}, \vec{p}\,' \right) \qquad P_2 = \left(\frac{E_2}{2c}, -\vec{p}\,' \right)$$

$$\text{Invariant:} \quad P_2 \bullet \left(P_1 + P_2 \right)$$

$$m_0 c \times \frac{E_1}{c} - 0 \times p = \frac{E_2}{2c} \times \frac{E_2}{c} + p' \times 0$$

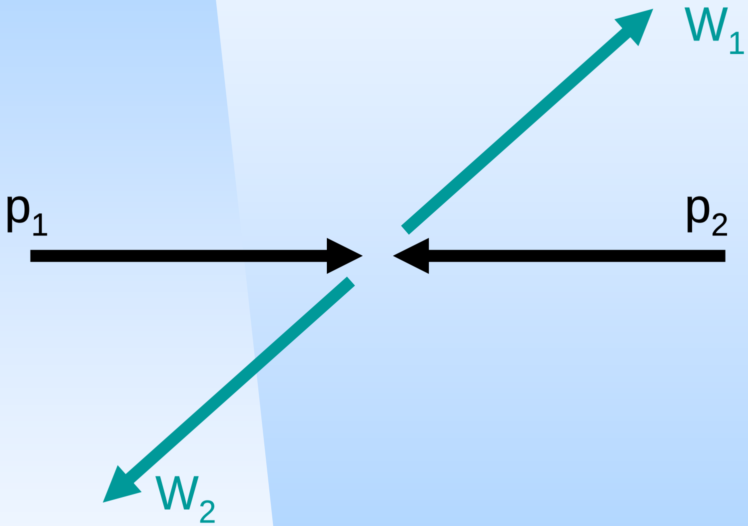
$$\Rightarrow \qquad 2 m_0 c^2 E_1 = E_2^2$$

Accelerator Problem

- In an accelerator, a proton p_1 with rest mass m_0 collides with an anti-proton p_2 (with the same rest mass), producing two particles W_1 and W_2 with equal mass $M_0=100m_0$
  - Expt 1: p_1 and p_2 have equal and opposite velocities in the lab frame. Find the minimum energy of p_2 in order for W_1 and W_2 to be produced.
  - Expt 2: in the rest frame of p_1, find the minimum energy E' of p_2 in order for W_1 and W_2 to be produced

Note: $E^2/c^2 = \vec{p}^{\,2} + m_0^2 c^2$ $\Rightarrow$ same m_0, same p mean same E.

Experiment 1



Total 3-momentum is zero before collision and so is zero after impact

4-momenta before collision:

$$P_1 = \left(E/c, \vec{p} \right) \quad P_2 = \left(E/c, -\vec{p} \right)$$

4-momenta after collision:

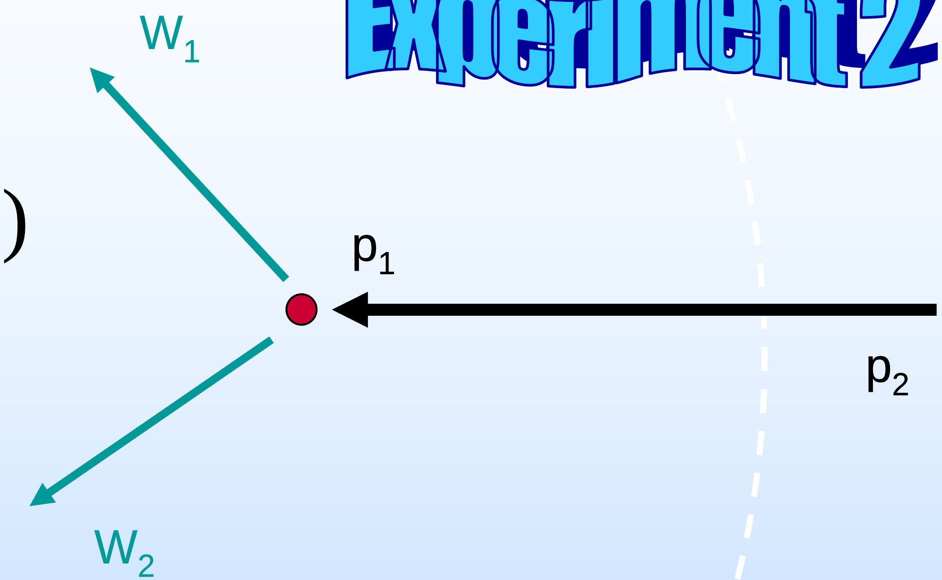
$$P_1 = \left(E'/c, \vec{q} \right) \quad P_2 = \left(E'/c, -\vec{q} \right)$$

Energy conservation $\Rightarrow$ *E=E′* > rest energy = M_0c^2 = 100 m_0c^2

Experiment 2

Before collision:

$$P_1 = \left(m_0 c, \vec{0}\right) \quad P_2 = \left(E'/c, \vec{p}\right)$$



Total energy is

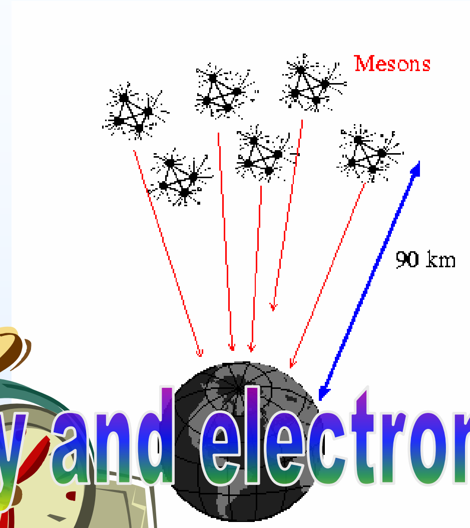
$$E_1 = E' + m_0 c^2$$

Use previous result $2m_0c^2 E_1 = E_2^2$ to relate E_1 to total energy E_2 in C.O.M frame

$$2m_0 c^2 E_1 = E_2^2$$

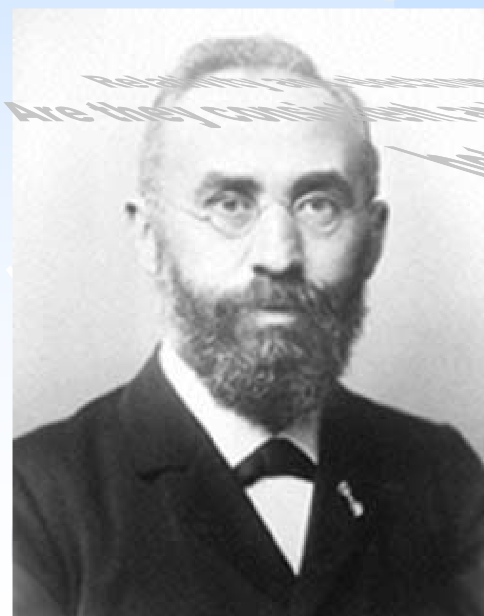
$$\Rightarrow \quad 2m_0 c^2 \left(E' + m_0 c^2\right) = (2E)^2 > \left(200\, m_0 c^2\right)^2$$

$$\Rightarrow \quad E' > \left(2\times10^4 - 1\right) m_0 c^2 \approx 20{,}000\, m_0 c^2$$



Relativity and electromagnetism.

Are they consistent and how do fields transform between moving frames?



Invariants

4-vectors