Introduction to Transverse Beam Optics

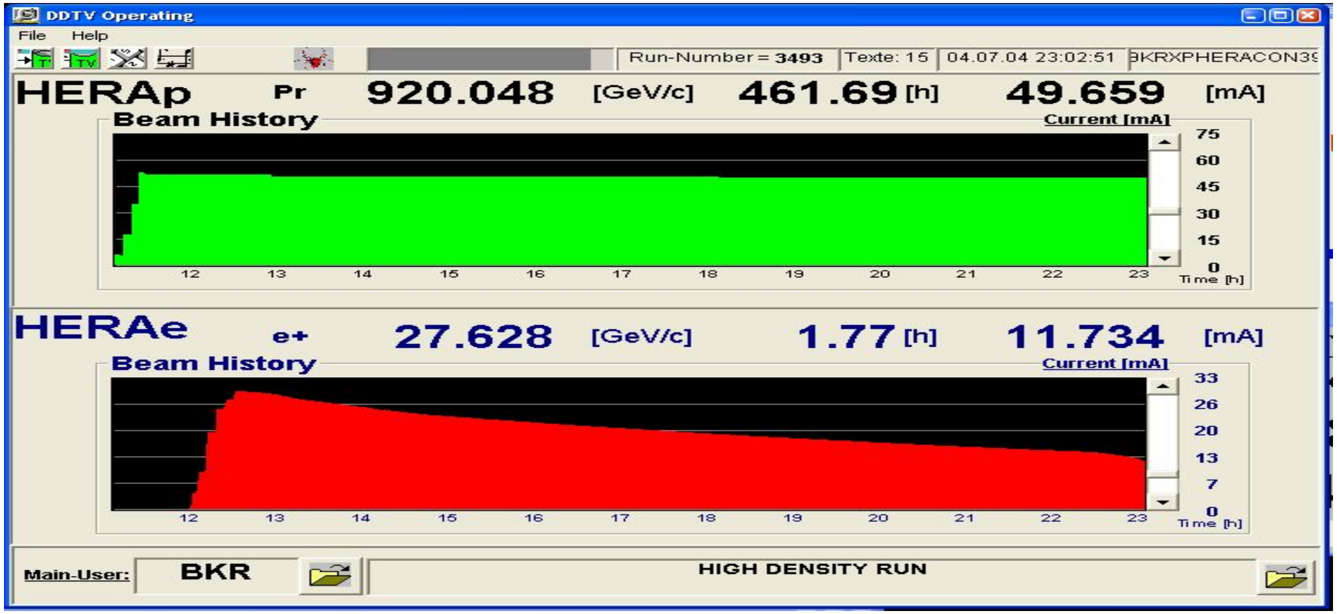
Bernhard Holzer, DESY-HERA

The Ideal World

I.) Magnetic Fields and Particle Trajectories

Luminosity Run of a typical storage ring:

HERA Storage Ring: Protons accelerated and stored for 12 hours
distance of particles travelling at about $v \approx c$
$L = 10^{10}$-10^{11} km
... several times Sun - Pluto and back



→ *guide the particles on a well defined orbit („design orbit")*

→ *focus the particles to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.*

Transverse Beam Dynamics:

0.) Introduction and Basic Ideas

„ ... in the end and after all it should be a kind of circular machine“

→ need transverse deflecting force

Lorentz force $\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$

typical velocity in high energy machines: $v \approx c \approx 3 * 10^8 \, m/s$

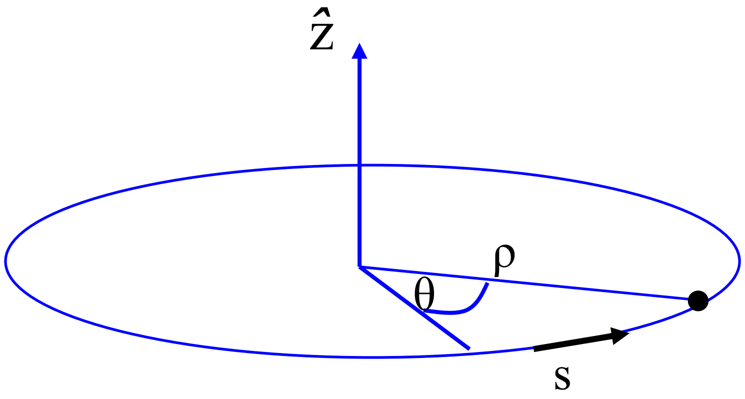
old greek dictum of wisdom:

if you are clever, you use magnetic fields in an accelerator wherever it is possible.

But remember: magn. fields act allways perpendicular to the velocity of the particle

→ only bending forces, → no „beam acceleration“

The ideal circular orbit



circular coordinate system

condition for circular orbit:

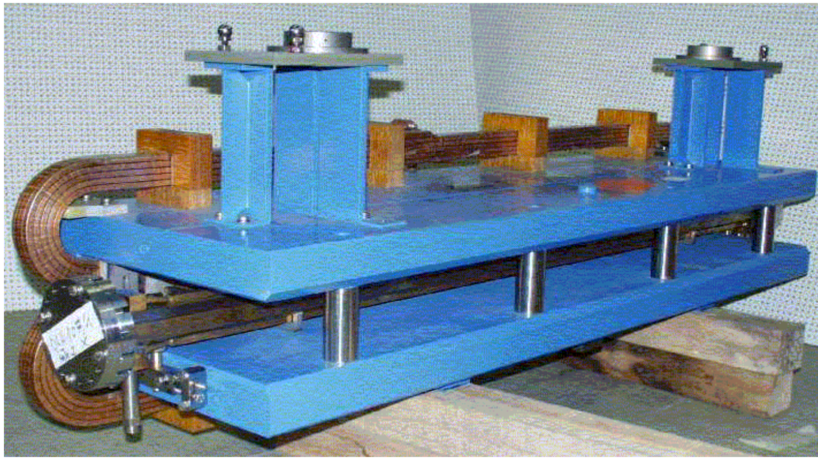
Lorentz force $F_L = e * v * B$

centrifugal force $F_{Zentr} = \frac{\gamma\, m_0\, v^2}{\rho}$

$$\frac{\gamma m_0 \cancel{v}^2}{\rho} = e * \cancel{v} * B$$

$$\boxed{\frac{p}{e} = B * \rho}$$

1.) The Magnetic Guide Field



Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes

$$B = \frac{\mu_0 \, n \, I}{h}$$

Normalise magnetic field to momentum:

$$\frac{p}{e} = B * \rho \quad \longrightarrow \quad \frac{1}{\rho} = e * \frac{B}{p}$$

convenient units:

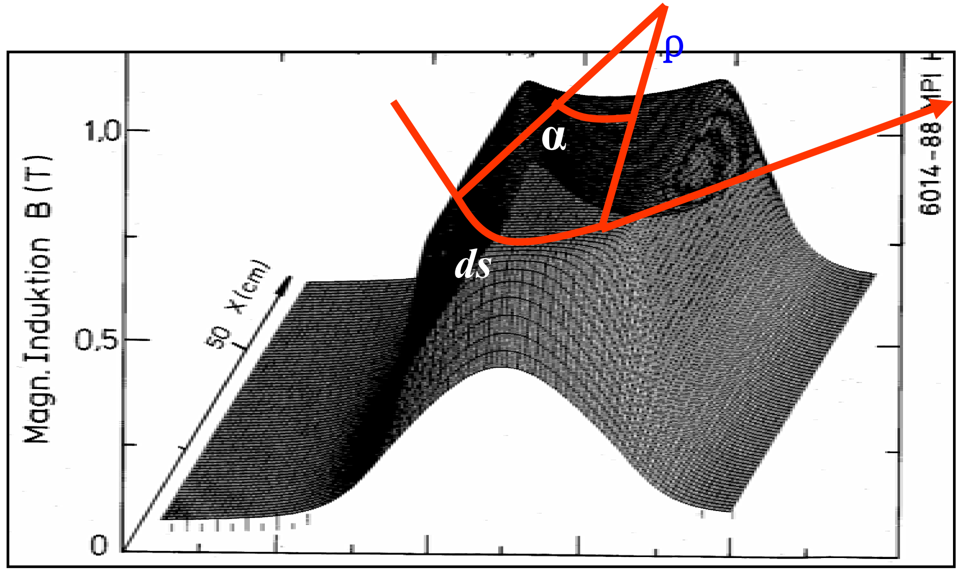
$$\boldsymbol{B} = [\boldsymbol{T}] = \left[\frac{\boldsymbol{Vs}}{\boldsymbol{m}^2}\right] \qquad \boldsymbol{p} = \left[\frac{\boldsymbol{GeV}}{\boldsymbol{c}}\right]$$

Example LHC:

$$\left.\begin{aligned} \boldsymbol{B} &= 8.3\,\boldsymbol{T} \\ \boldsymbol{p} &= 7000\,\frac{\boldsymbol{GeV}}{\boldsymbol{c}} \end{aligned}\right\}$$

$$\frac{1}{\rho} = \boldsymbol{e} * \frac{8.3\, \boldsymbol{Vs}/\boldsymbol{m}^2}{7000 * 10^9\, \boldsymbol{eV}/\boldsymbol{c}} = \frac{8.3\, \boldsymbol{s} * 3 * 10^8\, \boldsymbol{m}/\boldsymbol{s}}{7000 * 10^9\, \boldsymbol{m}^2}$$

Magnetic field of a dipole magnet:



field map of a storage ring dipole magnet

$$\frac{1}{\rho} = 0.333 * \frac{8.3}{7000} \, 1/\boldsymbol{m}$$

„radius of curvature,
bending strength"

rule of thumb:

$$\boxed{\frac{1}{\rho} \approx 0.3 * \frac{\boldsymbol{B}[\boldsymbol{T}]}{\boldsymbol{p}[\boldsymbol{GeV}/\boldsymbol{c}]}}$$

ρ = Bending radius of a dipole magnet

1/ ρ= „bending strength"

Quadrupole Magnets:

required: ***focusing forces*** ***to keep trajectories in vicinity of the ideal orbit***

linear increasing Lorentz force

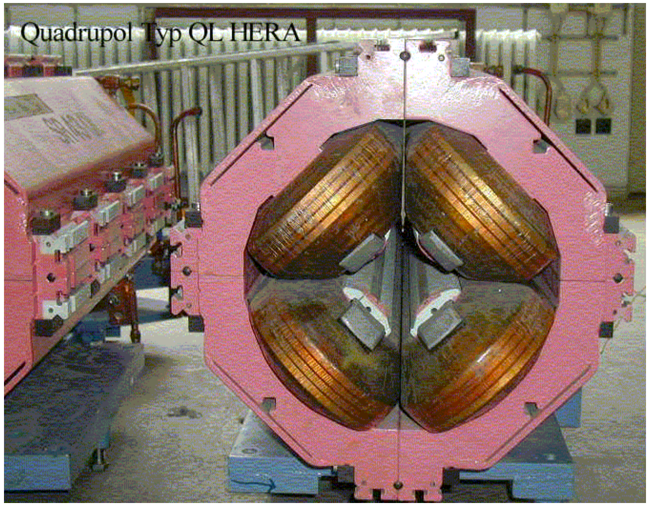
linear increasing magnetic field $\boldsymbol{B_z = g * x} \qquad \boldsymbol{B_x = g * z}$

normalised quadrupole field:

gradient of a quadrupole magnet:

$$g = \frac{2\mu_0 n I}{r^2}$$

$$\rightarrow \quad \boxed{k = \frac{g}{p/e}}$$



… what about the vertical plane:

$$\vec{\nabla} \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t} = 0$$

Maxwell:

$$\Rightarrow \quad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x}$$

2.) The equation of motion:

Linear approximation:

**** ideal particle** → **design orbit***

**** any other particle** → **coordinates x, z small quantities***
$x,z \ll \rho$

→ magnetic guide field: only linear terms in x & z of B have to be taken into account

Taylor Expansion of the B field:

$$B_z(x) = B_{z0} + \frac{dB_z}{dx} x + \frac{1}{2!}\frac{d^2 B_z}{dx^2} x^2 + \frac{1}{3!}\frac{eg''}{dx^3} + \ldots$$

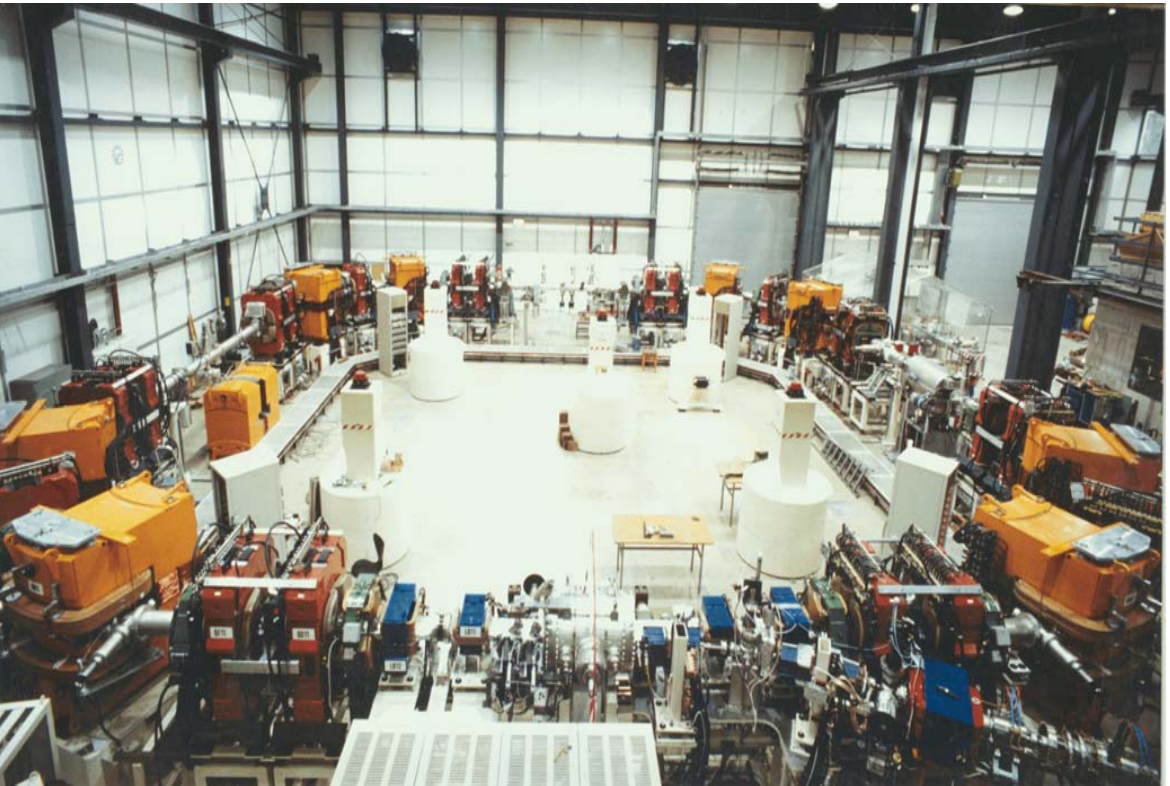
normalise to momentum
$p/e = B\rho$

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0 \rho} + \frac{g * x}{p/e} + \frac{1}{2!}\frac{eg'}{p/e} + \frac{1}{3!}\frac{eg''}{p/e} + \ldots$$

The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k * x + \cancel{\frac{1}{2!} m x^2} + \cancel{\frac{1}{3!} n x^3} + \ldots$$

only terms linear in x, z taken into account ***dipole fields***
quadrupole fields



Separate Function Machines:

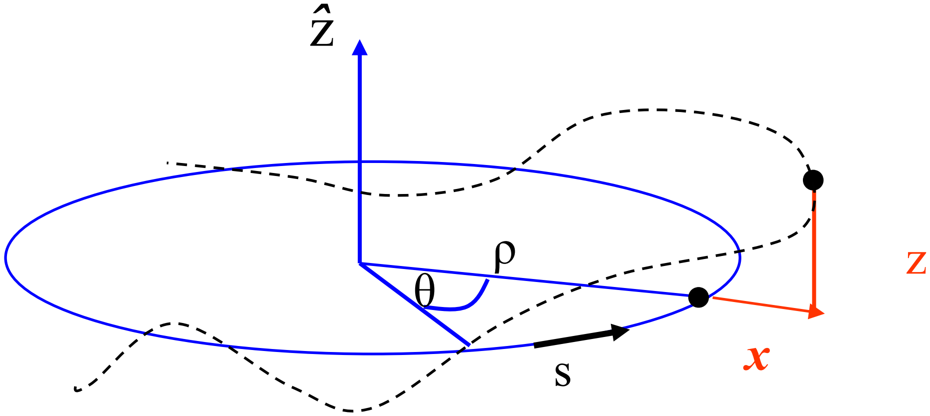
Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

** **man sieht nur dipole und quads → linear***

Equation of Motion:



Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

Force: $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

$$F = mv^2 / \rho$$

general trajectory: $\rho \rightarrow \rho + x$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_z\,v$$

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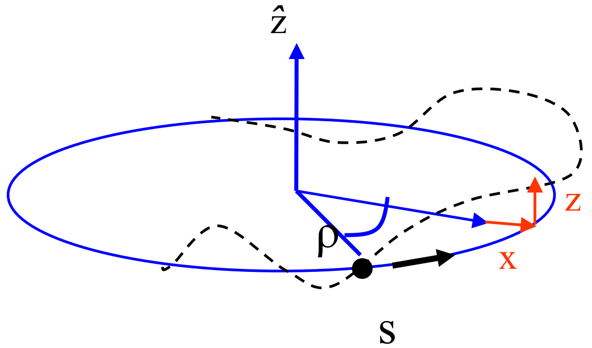
1 2



1 $$\frac{d^2}{dt^2}(x+\rho) = \frac{d^2}{dt^2}x$$ ***… as ρ = const***

2 ***remember: x ≈ mm , ρ ≈ m … → develop for small x***

$$\frac{1}{x+\rho} \approx \frac{1}{\rho}\left(1-\frac{x}{\rho}\right)$$

Taylor Expansion

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} * f'(x_0) + \frac{(x-x_0)^2}{2!} * f''(x_0) + \quad \dots$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_z v$$

guide field in linear approx.

$$B_z = B_0 + x\frac{\partial B_z}{\partial x}$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = ev\left\{B_0 + x\frac{\partial B_z}{\partial x}\right\} \qquad \Bigg| \; : m$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m}$$

independent variable: t → s

$$\frac{dx}{dt} = \frac{dx}{ds} * \frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds} * \frac{ds}{dt}\right) = \frac{d}{ds}\left(\underbrace{\frac{dx}{ds}}_{x'} * \underbrace{\frac{ds}{dt}}_{v}\right)\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = x'' * v^2 + \cancel{\frac{dx}{ds} * \frac{dv}{ds}} * v$$

$$x''v^2 - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m} \qquad \Bigg| \; : v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{e\ B_0}{mv} + \frac{e\ x\ g}{mv}$$

$$m\,v = p$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{x\ g}{p/e}$$

normalize to momentum of particle

$$\frac{B_0}{p/e} = -\frac{1}{\rho}$$

$$\frac{g}{p/e} = k$$

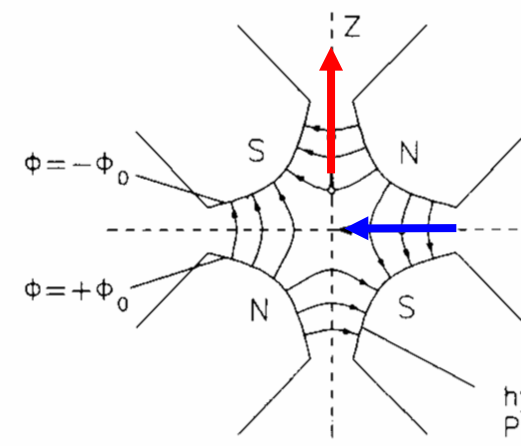
$$x'' - \cancel{\frac{1}{\rho}} + \frac{x}{\rho^2} = -\cancel{\frac{1}{\rho}} + k\ x$$

$$\boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = 0}$$

* ***Equation for the vertical motion:***

$$\frac{1}{\rho^2} = 0$$ ***no dipoles … in general …***

$$k \leftrightarrow -k$$ ***quadrupole field changes sign***



$$\boxed{z'' + k \cdot z = 0}$$

Remarks:

* $$x'' + \left(\frac{1}{\rho^2} - k\right) \cdot x = 0$$

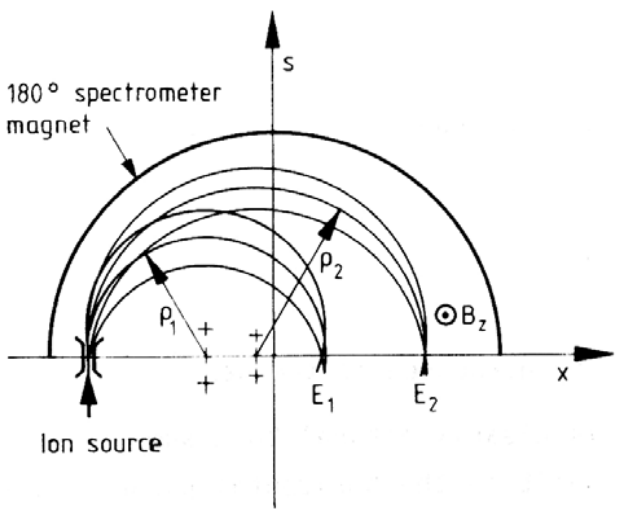
… there seems to be a focusing even without a quadrupole gradient

„weak focusing of dipole magnets"

$$k = 0 \quad \Rightarrow \quad x'' = -\frac{1}{\rho^2} * x$$

even without quadrupoles there is a retriving force (i.e. focusing) in the bending plane of the dipole magnets

… in large machines it is weak. (!)



Mass spectrometer: particles are separated according to their energy and focused due to the 1/ρ effect of the dipole

* Hard Edge Model:

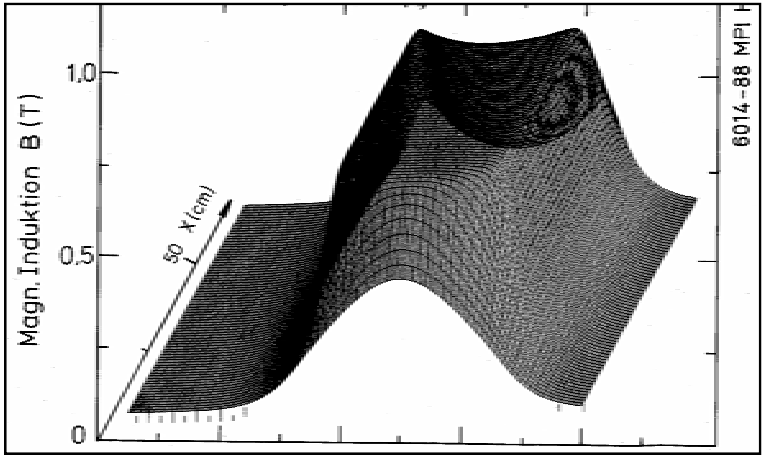
$$x'' + (\frac{1}{\rho^2} - k) \cdot x = 0$$

… this equation is not correct !!!

$$x''(s) + \left\{ \frac{1}{\rho^2(s)} - k(s) \right\} * x(s) = 0$$

bending and focusing fields … are functions of the independent variable „s“

!



magnetic field of a storage ring dipole

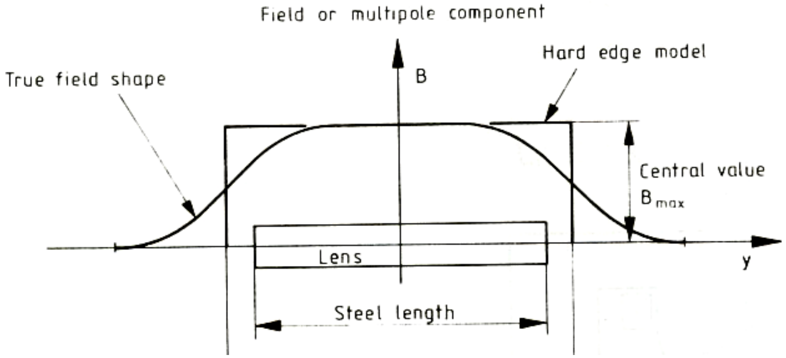
k = const within a magnet

Inside a magnet the focusing properties are constant !

$$\frac{1}{\rho} = const$$

$$k = const$$

$$B * l_{eff} = \int_0^{l_{mag}} B ds$$



3.) Solution of Trajectory Equations

Define … hor. plane: $K = 1/\rho^2 - k$

… vert. Plane: $K = k$

$$y'' + K * y = 0$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: $$x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)$$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1\omega \sin(\omega s) + a_2\omega \cos(\omega s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) - a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \longrightarrow \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)$$

determine a_1 , a_2 by boundary conditions:

$$s = 0 \longrightarrow \begin{cases} x(0) = x_0 \quad , \quad a_1 = x_0 \\ x'(0) = x'_0 \quad , \quad a_2 = \dfrac{x'_0}{\sqrt{K}} \end{cases}$$

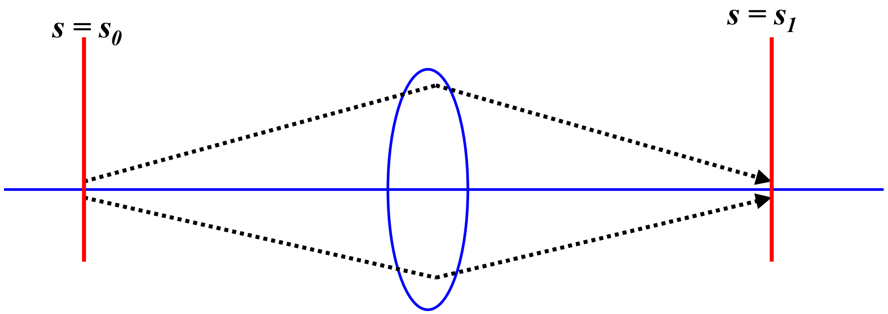
Hor. Focusing Quadrupole $K > 0$:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

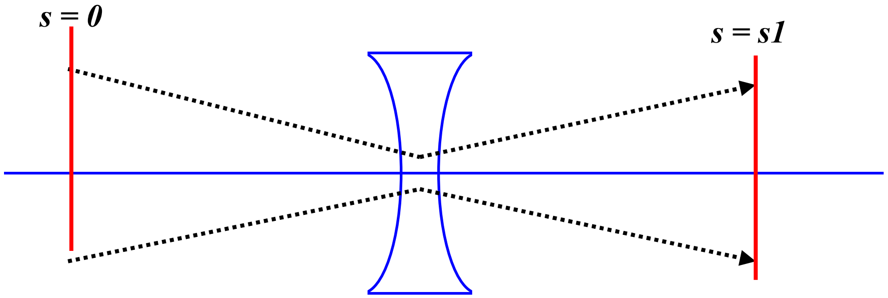
For convenience expressed in matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \dfrac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole:



$$x'' - K * x = 0$$

Remember from school:

$$f(s) = \cosh(s) \quad , \qquad f'(s) = \sinh(s)$$

Ansatz: $\quad x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\,l \\ \sqrt{|K|}\sinh\sqrt{|K|}\,l & \cosh\sqrt{|K|}\,l \end{pmatrix}$$

drift space:

K = 0

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & z is uncoupled“

Thin Lens Approximation:

matrix of a quadrupole lens

$$M = \begin{pmatrix} \cos\sqrt{|k|}\,l & \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}\,l \\ -\sqrt{|k|}\sin\sqrt{|k|}\,l & \cos\sqrt{|k|}\,l \end{pmatrix}$$

in many practical cases we have the situation:

$$f = \frac{1}{k l_q} >> l_q$$

... focal length of the lens is much bigger than the length of the magnet

limes: $l \to 0$ ***while keeping k*l = const***

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \qquad M_z = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}$$

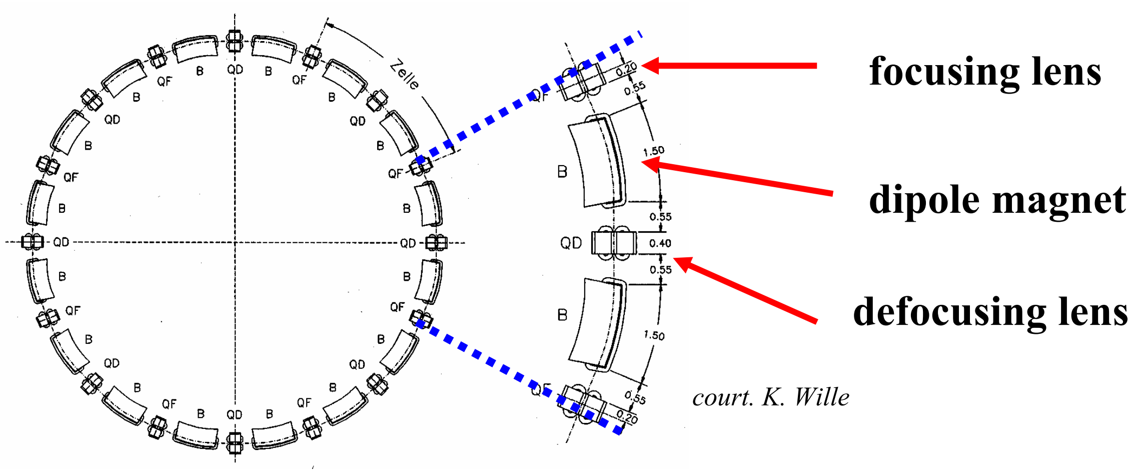
... useful for fast (and in large machines still quite accurate) „back on the envelope calculations“ ... and for the guided studies !

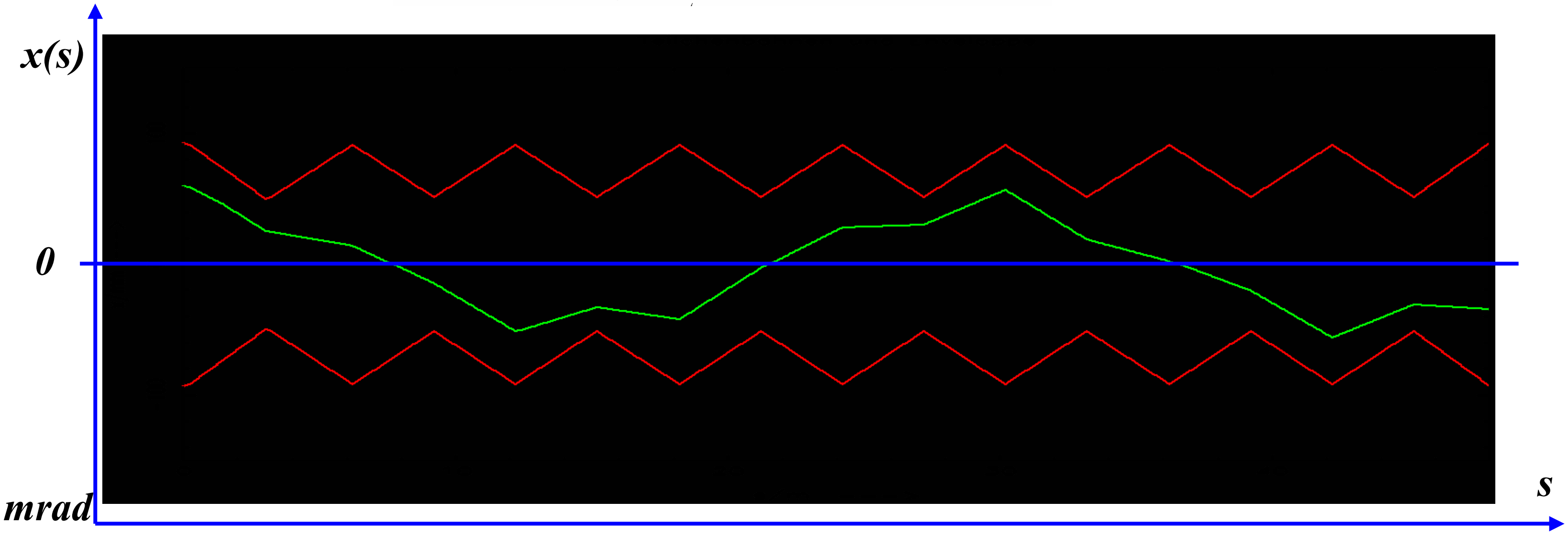
Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_{D} * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s2,s1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$





typical values in a strong foc. machine:

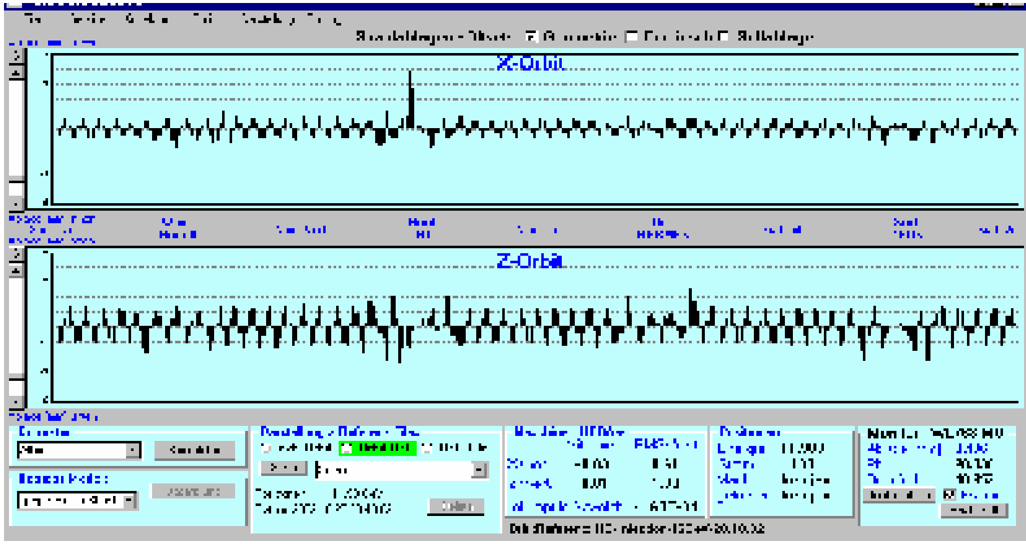
$x \approx mm, x' \leq mrad$

4.) Orbit & Tune:

Tune: number of oscillations per turn

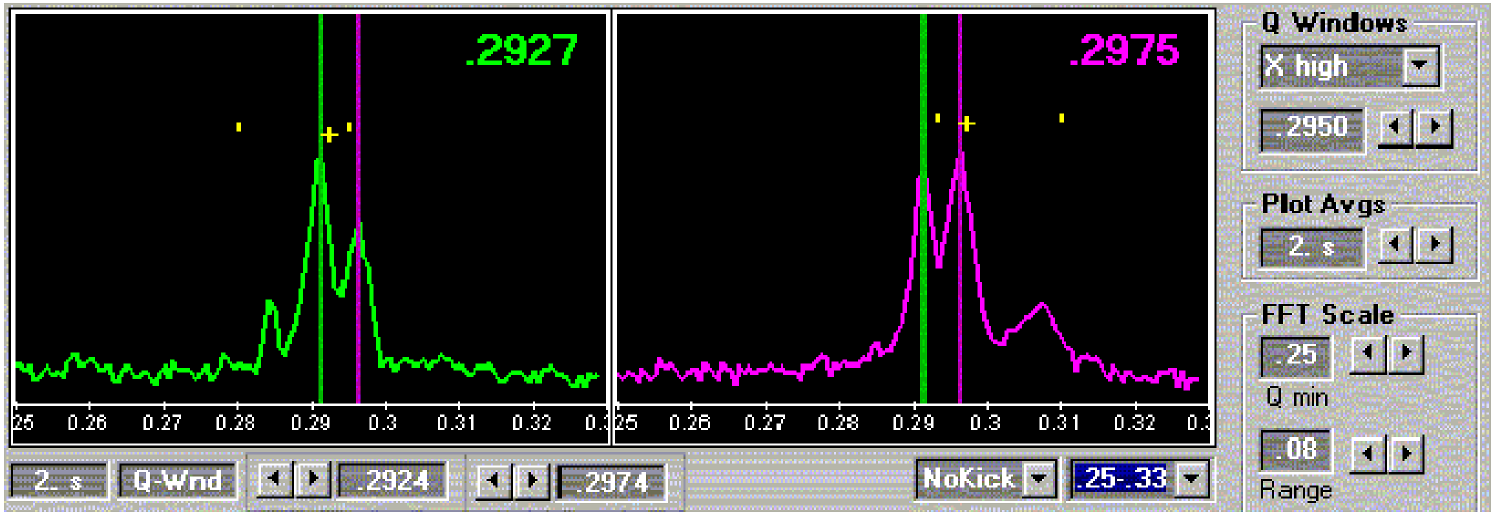
31.292
32.297

Relevant for beam stability:
non integer part



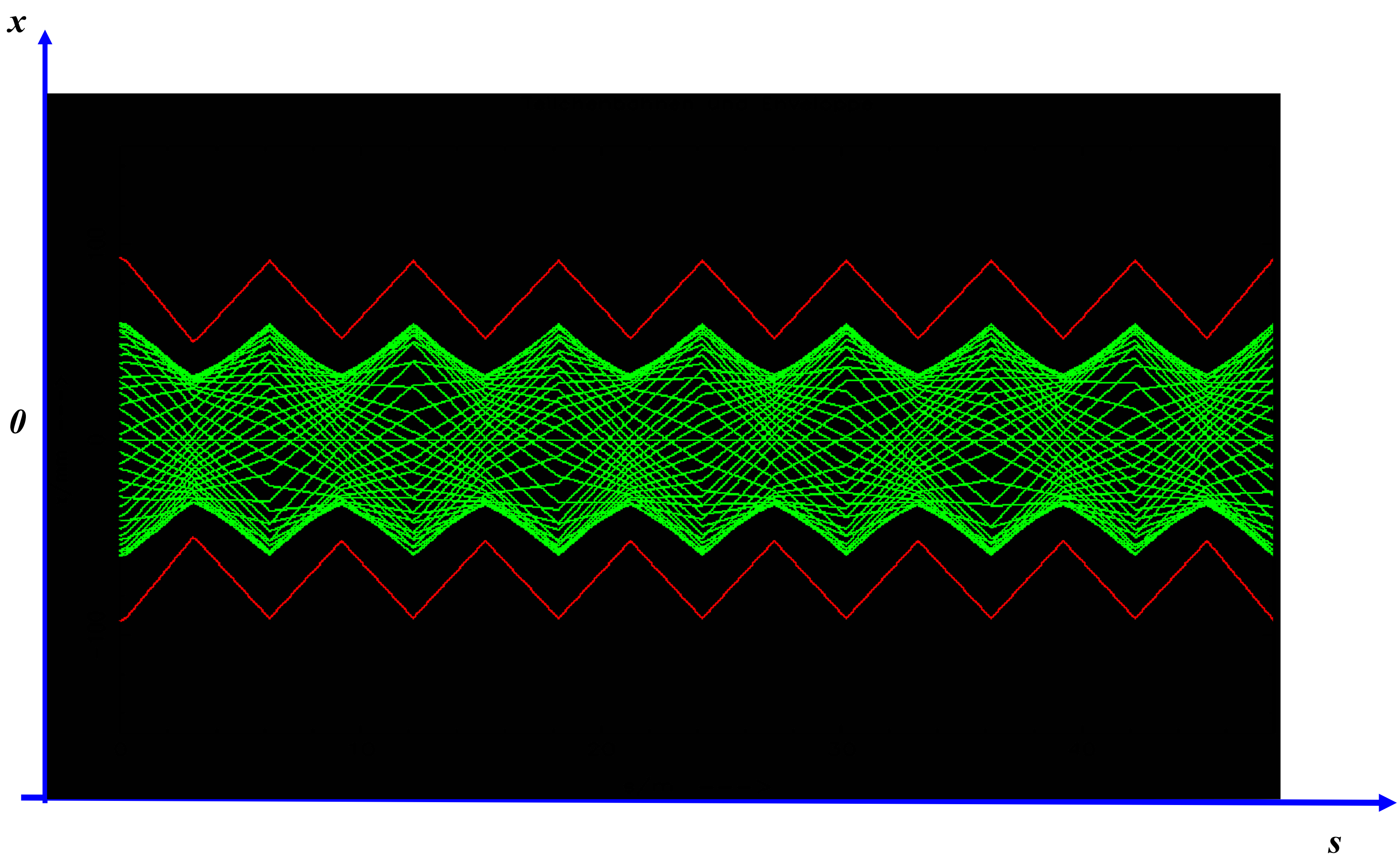
HERA revolution frequency: 47.3 kHz

$$0.292 * 47.3 \ kHz = 13.81 \ kHz$$



Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $$x''(s) - k(s)x(s) = 0$$

restoring force $\neq$ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

5.) The Beta Function

General solution of Hill´s equation:

(i) $$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.
For one complete revolution: number of oscillations per turn „Tune“

$$Q_y = \frac{1}{2\pi} \cdot \oint \frac{ds}{\beta(s)}$$

6.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\begin{cases} (1) \quad x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s)+\phi) \\ (2) \quad x'(s) = -\dfrac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} * \{\alpha(s) * \cos(\psi(s)+\phi) + \sin(\psi(s)+\phi)\} \end{cases}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon} * \sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

Insert into (2) and solve for ε

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

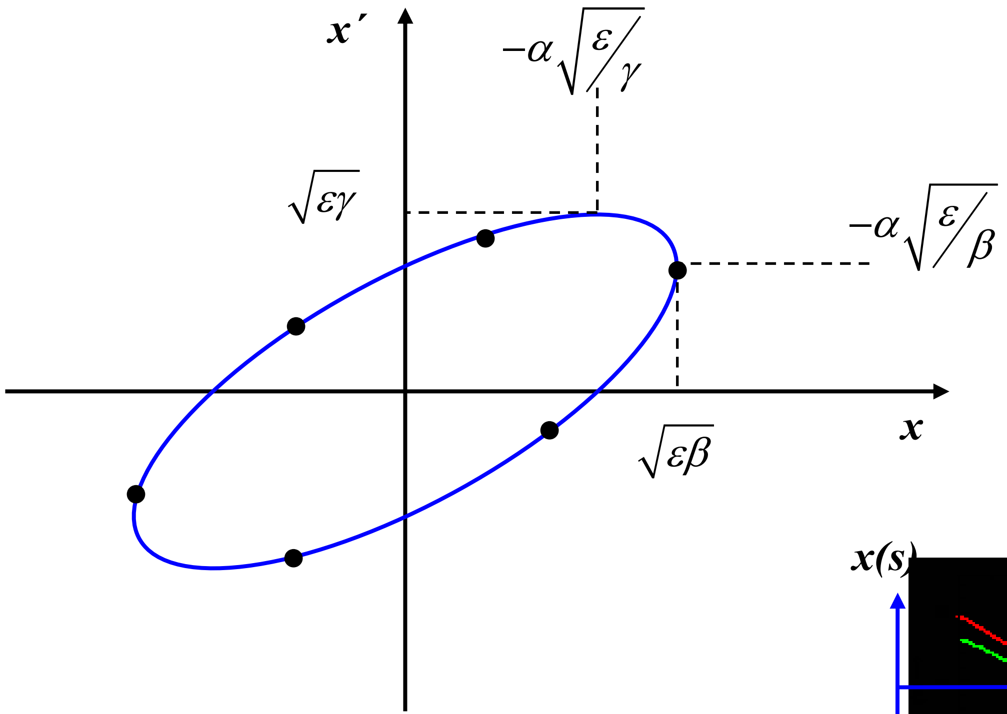
***** ε is a constant of the motion … it is independent of „s“**

***** parametric representation of an ellipse in the x x‘ space**

***** shape and orientation of ellipse are given by α, β, γ**

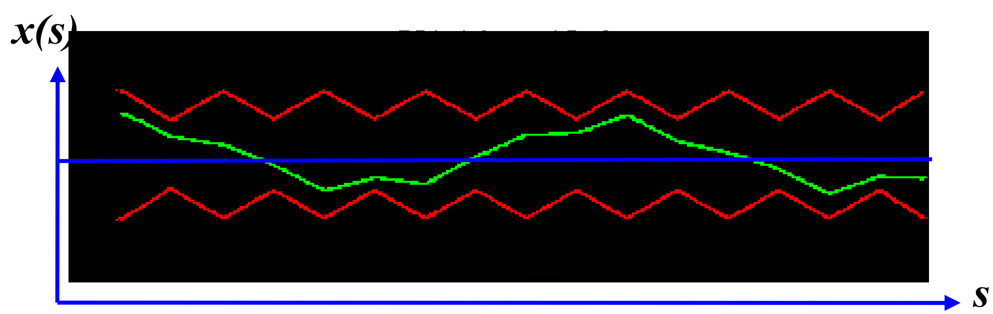
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$



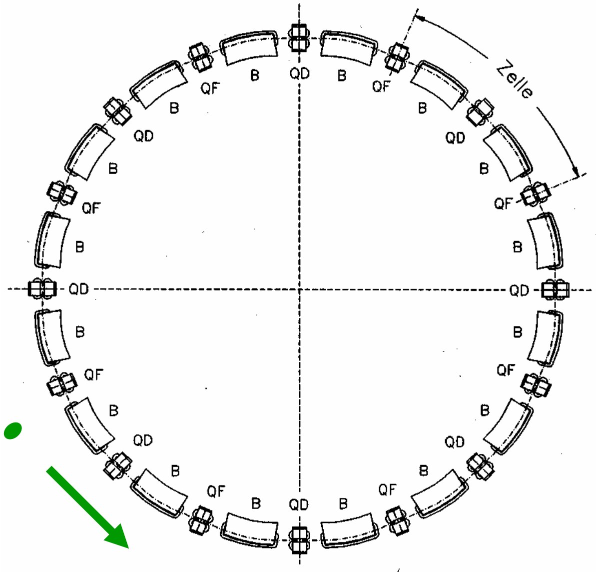
***Liouville:* in reasonable storage rings**
area in phase space is constant.

A = π*ε=const



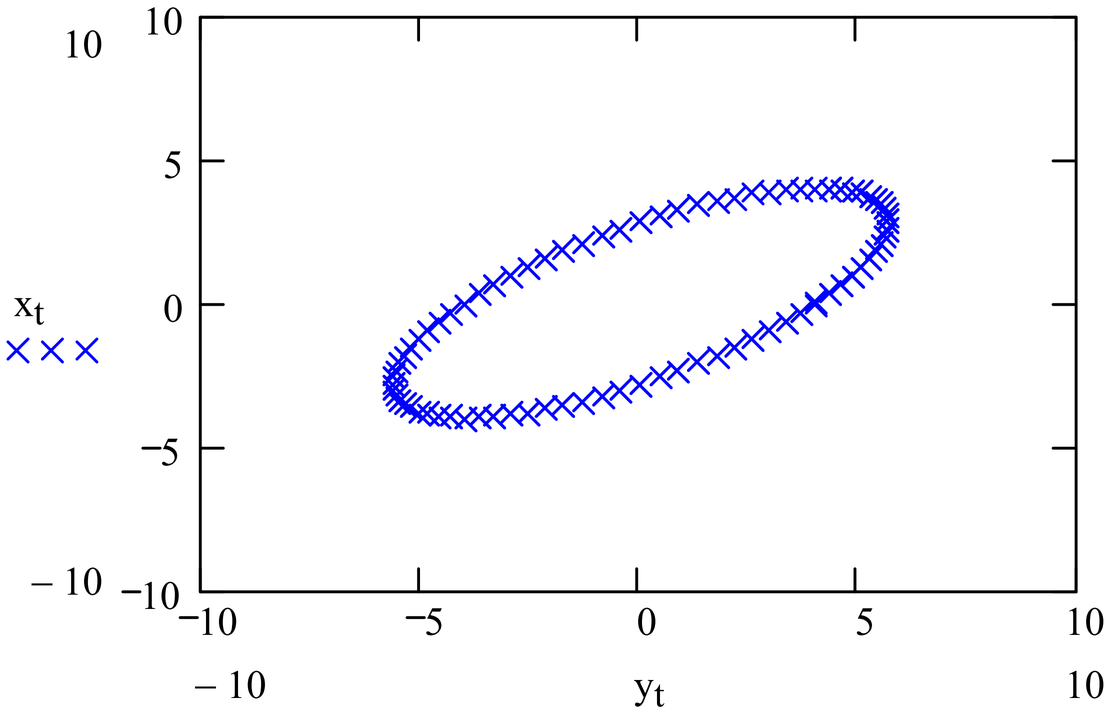
***ε beam emittance* = *woozilycity* of the particle ensemble, *intrinsic beam parameter*,**
cannot be changed by the foc. properties.
***Scientifiquely spoken:* area covered *in transverse x, x´ phase space* … and it is constant !!!**

Particle Tracking in a Storage Ring



note for each turn x, x´ at a given position „s_1" and plot in the phase space diagram

… and you will get an ellipse:



7.) Résumé:

beam rigidity: $B \cdot \rho = p/q$

bending strength of a dipole: $\frac{1}{\rho}\left[m^{-1}\right] = \frac{0.2998 \cdot B_0(T)}{p(GeV/c)}$

focusing strength of a quadrupole: $k\left[m^{-2}\right] = \frac{0.2998 \cdot g}{p(GeV/c)}$

focal length of a quadrupole: $f = \frac{1}{k \cdot l_q}$

equation of motion: $x'' + Kx = \frac{1}{\rho}\frac{\Delta p}{p}$

matrix of a foc. quadrupole: $x_{s2} = M \cdot x_{s1}$

$$M = \begin{pmatrix} \cos\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}l \\ -\sqrt{|K|}\sin\sqrt{|K|}l & \cos\sqrt{|K|}l \end{pmatrix} , \qquad M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

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II.) Emittance & Betafunction

1.) The Beam Emittance

General solution of Hill´s equation:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right]$$

β(s) = periodic function given by focusing properties of the lattice

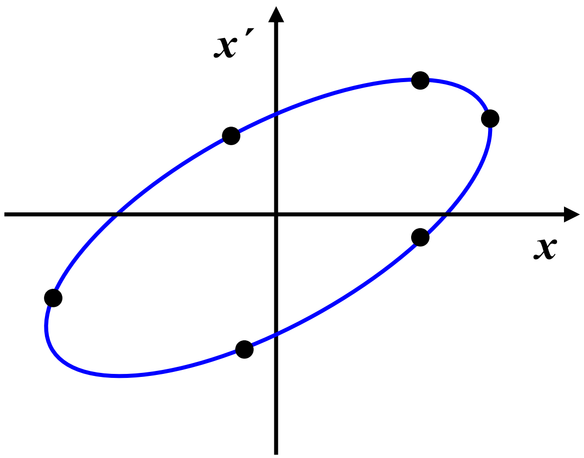
$$\beta(s+L) = \beta(s)$$

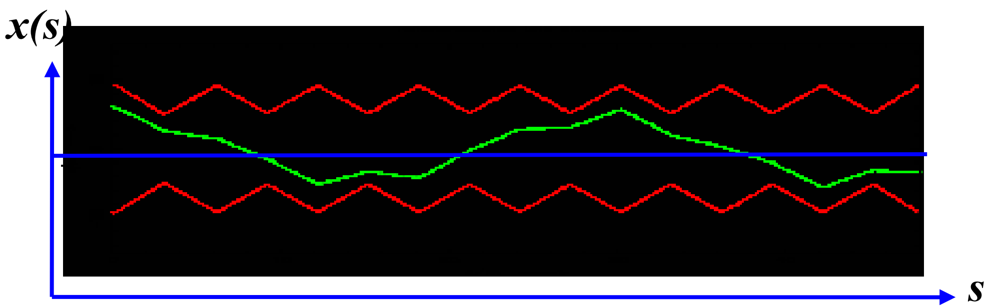
ε = constant, determined by initial conditions of the particle ensemble.

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

Liouville: in reasonable storage rings area in phase space is constant.

A = π*ε=const





ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.

Scientifiquely spoken: area covered in transverse x, x´ phase space … and it is constant !!!

Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ $\longrightarrow$ ***x´ at that position …?***

… put $\hat{x}(s)$ ***into*** $\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$ ***and solve for x´***

$$\varepsilon = \gamma \cdot \varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

$$\longrightarrow \quad x' = -\alpha \cdot \sqrt{\varepsilon / \beta}$$

* ***A high β-function means a large beam size and a small beam divergence.*** ***!***
… et vice versa !!!

* ***In the middle of a quadrupole β is maximum,***
α = zero

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $x' = 0$

… and the ellipse is flat

Phase Space Ellipse

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

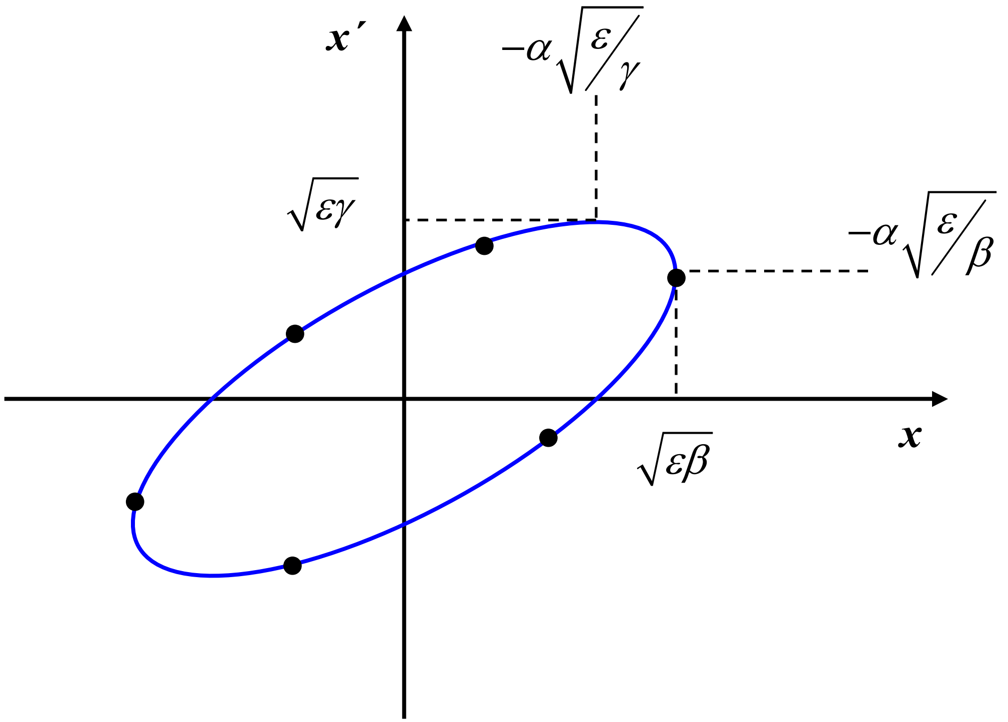
$$\longrightarrow \quad \varepsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot xx' + \beta \cdot x'^2$$

… solve for x´ $\quad x'_{1,2} = \dfrac{-\alpha \cdot x \pm \sqrt{\varepsilon\beta - x^2}}{\beta}$

… and determine $\hat{x}'$ ***via:*** $\quad \dfrac{dx'}{dx} = 0$

$$\longrightarrow \quad \hat{x}' = \sqrt{\varepsilon\gamma}$$

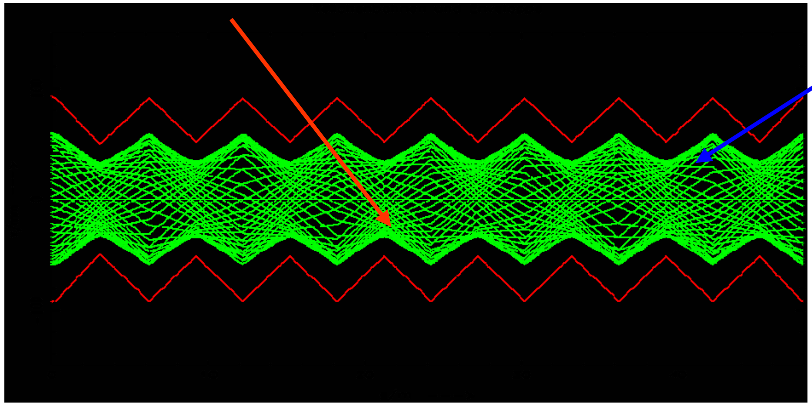
$$\longrightarrow \quad \hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}$$



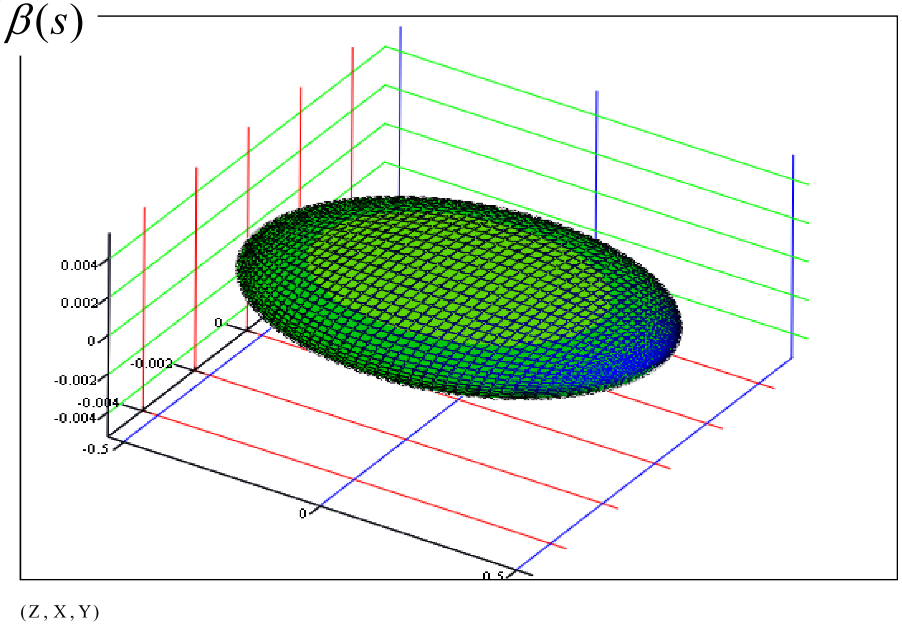
Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi)$$

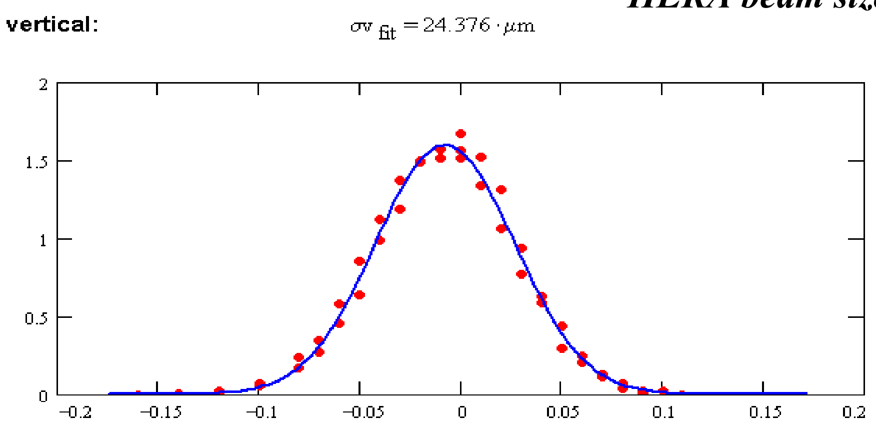
$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, N ≈ 10 [11] per bunch



HERA beam size



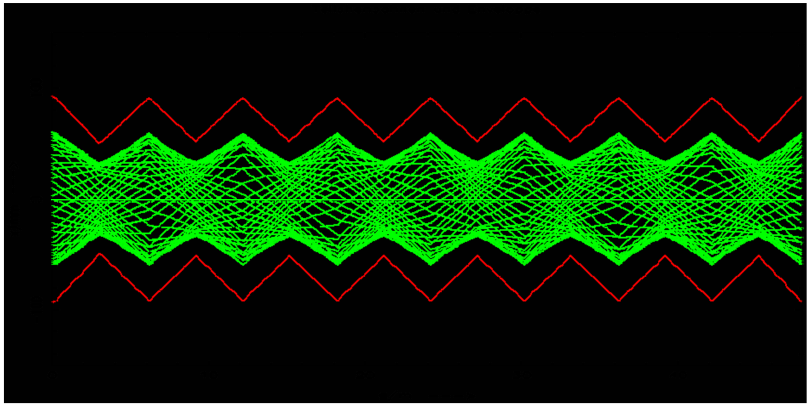
Gauß Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

particle at distance ***1 σ from centre:***

surrounding ***68.3 % of all beam particles***

Emittance of the Particle Ensemble:



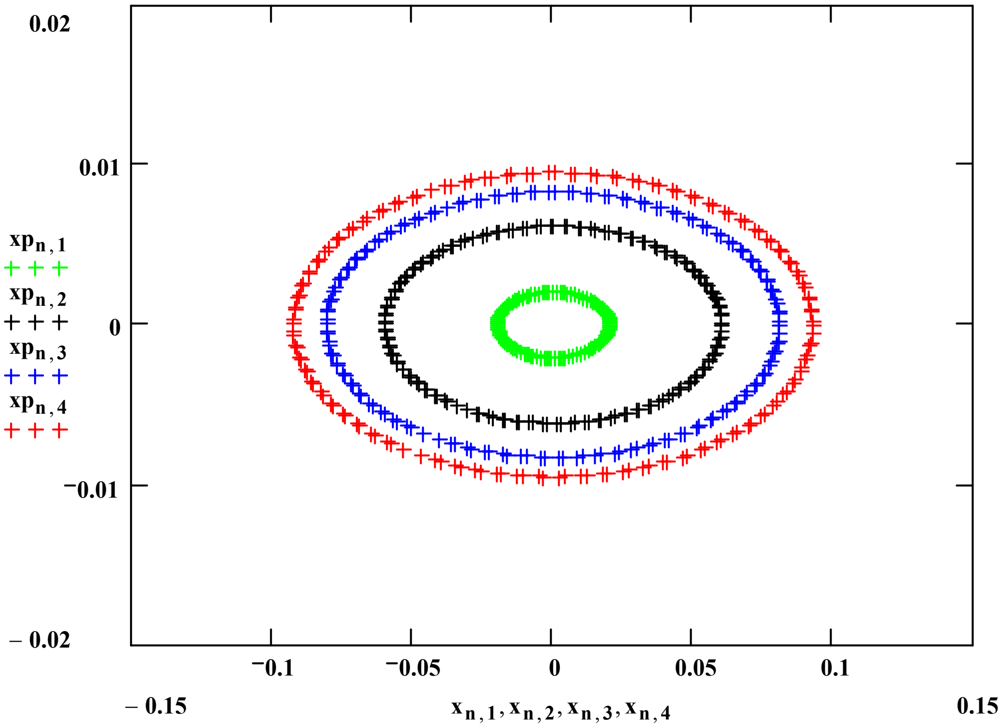
Example: HERA

beam parameters in the arc

$$\beta(x) \approx 80\ m$$

$$\varepsilon \approx 7 * 10^{-9}\ rad \cdot m \qquad (\leftrightarrow 1\,\sigma)$$

$$\sigma = \sqrt{\varepsilon\beta} \approx 0.75\ mm$$



2.) Transfer Matrix M … *yes we had the topic already*

general solution of Hill´s equation

$$\begin{cases} x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \\ x'(s) = \dfrac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\} + \sin\{\psi(s)+\phi\}\right] \end{cases}$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point s(0) = s_0, where we put $\Psi(0) = 0$

$$\left.\begin{aligned} \cos\phi &= \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \\ \sin\phi &= -\frac{1}{\sqrt{\varepsilon}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{aligned}\right\}$$

inserting above …

$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}}\,\{\cos\psi_s + \alpha_0 \sin\psi_s\}\, x_0 + \left\{\sqrt{\beta_s \beta_0}\,\sin\psi_s\right\} x_0'$$

$$x'(s) = \frac{1}{\sqrt{\beta_s \beta_0}}\left\{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s\right\} x_0 + \sqrt{\frac{\beta_0}{\beta_s}}\,\{\cos\psi_s - \alpha_s \sin\psi_s\}\, x_0'$$

which can be expressed ... for convenience ... ***in matrix form***

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***

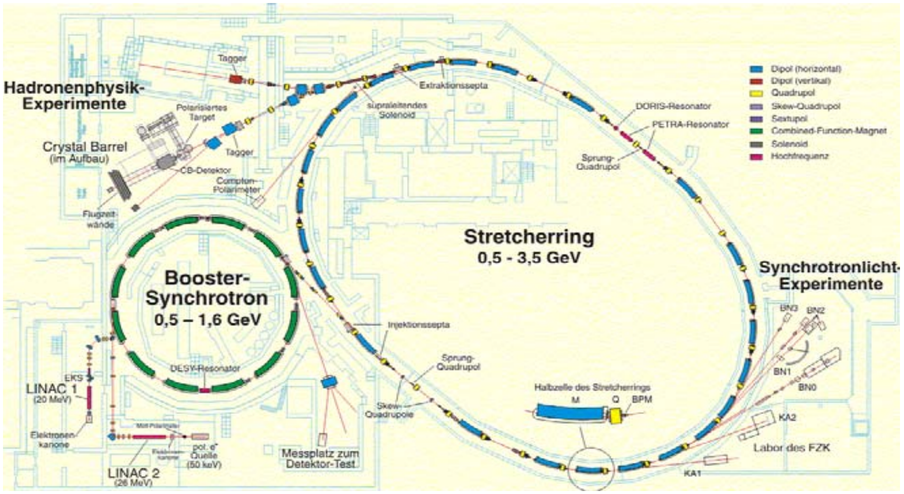
* ***and nothing but the α β γ at these positions.***

* ***... !***

**** Äquivalenz der Matrizen***

3.) Periodic Lattices

$$M=\begin{pmatrix} \sqrt{\dfrac{\beta_s}{\beta_0}}\left(\cos\psi_s+\alpha_0\sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \dfrac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\dfrac{\beta_0}{\beta s}}\left(\cos\psi_s-\alpha_s\sin\psi_s\right) \end{pmatrix}$$



Delta Electron Storage Ring

„This rather formidable looking matrix simplifies considerably if we consider one complete turn …“

$$M(s)=\begin{pmatrix} \cos\psi_{turn}+\alpha_s\sin\psi_{turn} & \beta_s\sin\psi_{turn} \\ -\gamma_s\sin\psi_{turn} & \cos\psi_{turn}-\alpha_s\sin\psi_{turn} \end{pmatrix}$$

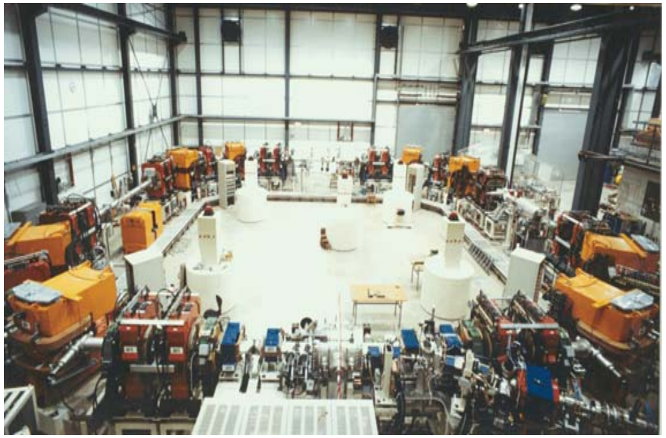
$$\psi_{turn}=\int_s^{s+L}\frac{ds}{\beta(s)}$$

ψ_{turn} = phase advance per period

$$Q=\frac{1}{2\pi}*\oint\frac{ds}{\beta(s)}$$

***Tune:* Phase advance per turn in units of 2π**

Stability Criterion:



Question: ***what will happen, if we do not make too many mistakes and your particle performs one complete turn ?***

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{1}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Matrix for N turns:

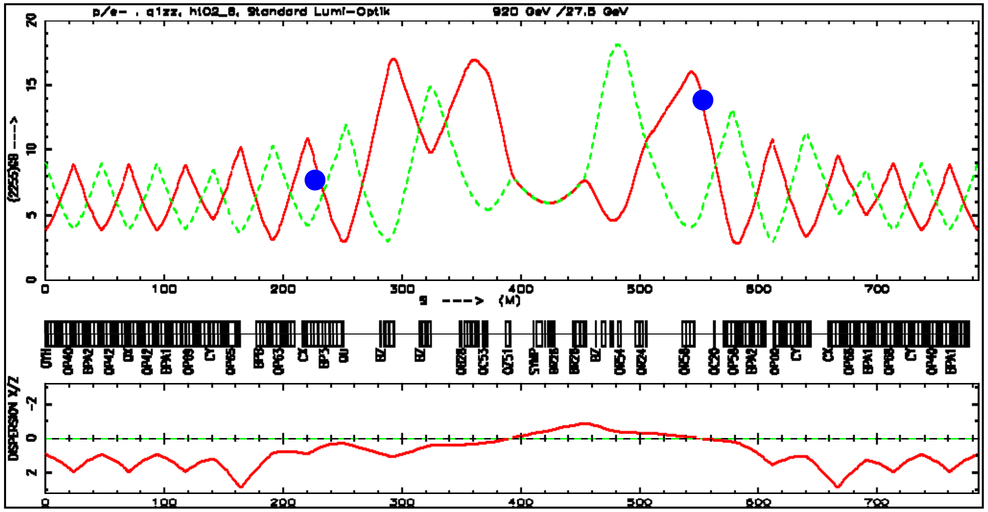
$$M^N = (1 \cdot \cos\psi + J \cdot \sin\psi)^N = 1 \cdot \cos N\psi + J \cdot \sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

$$\psi = real \quad \leftrightarrow \quad |\cos\psi| \le 1 \quad \leftrightarrow \quad Tr(M) \le 2$$

4.) Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s



Betafunction in a storage ring

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{S} \\ \boldsymbol{C'} & \boldsymbol{S'} \end{pmatrix}$$

since ε = const (Liouville):

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

... remember W = CS´-SC´ = 1

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

$$\rightarrow \quad x_0 = S'x - Sx'$$

$$x_0' = -C'x + Cx'$$

... inserting into ε

$$\varepsilon = \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

sort via x, x´and compare the coefficients to get ....

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

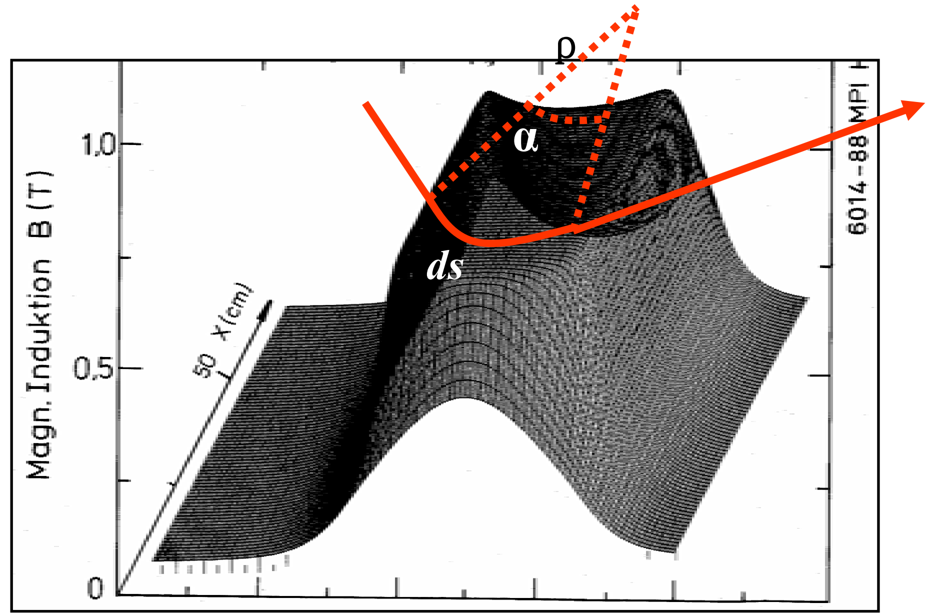
3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)

5.) Lattice Design:

„… how to build a storage ring“

$$B * \rho = p / q$$



field map of a storage ring dipole magnet

Circular Orbit: dipole magnets to define the geometry

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$

The angle run out in one revolution must be 2π, so

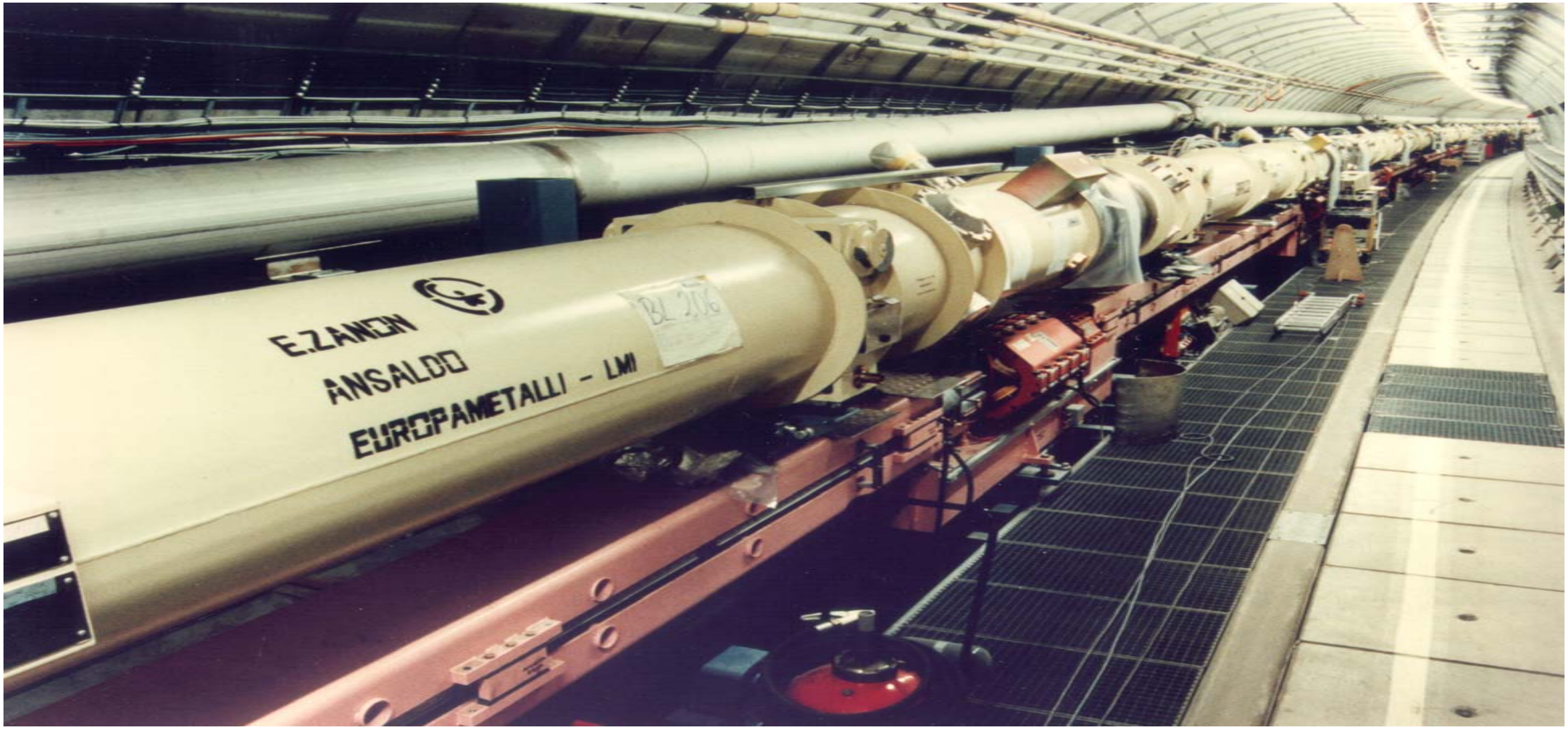
… for a full circle

$$\alpha = \frac{\int Bdl}{B * \rho} = 2\pi \quad \rightarrow \quad \int Bdl = 2\pi * \frac{p}{q}$$

… defines the integrated dipole field around the machine.

Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ *is usually required !!*

Example HERA:



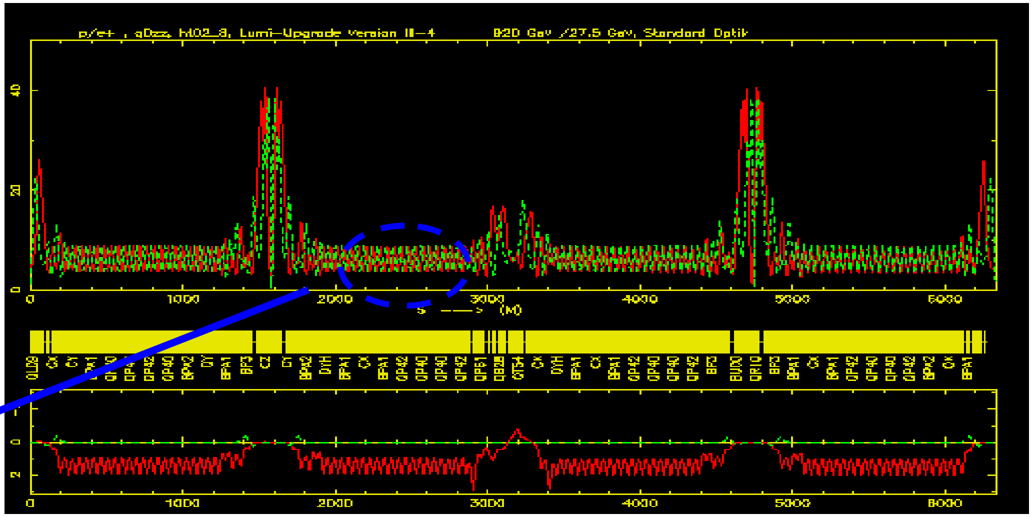
920 GeV Proton storage ring
dipole magnets N = 416
l = 8.8m
q = +1 e

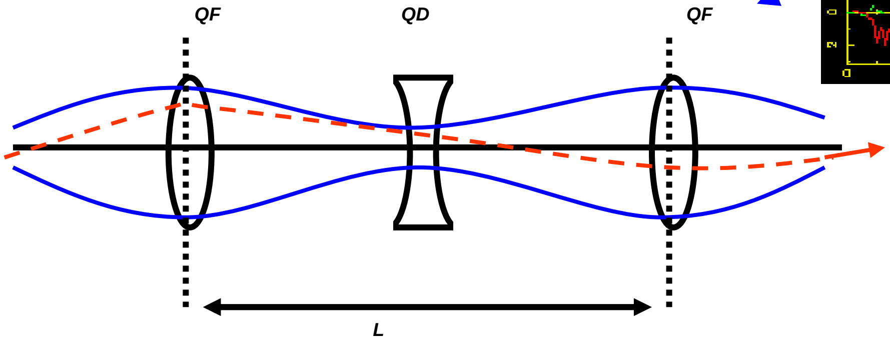
$$\int B dl \approx N * l * B = 2\pi\ p / q$$

$$B \approx \frac{2\pi * 920 * 10^9 eV}{416 * 3 * 10^8 \frac{m}{s} * 8.8m * e} \approx 5.15\ Tesla$$

The FoDo-Lattice

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)





Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell $\mu = 45°$,
→ calculate the twiss parameters for a periodic solution

Can we understand, what the optics code is doing?

matrices

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l_q) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l_q) \\ -\sqrt{K} \sin(\sqrt{K} * l_q) & \cos(\sqrt{K} * l_q) \end{pmatrix}, \qquad M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix}$$

strength and length of the FoDo elements

K = +/- 0.54102 m^{-2}
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qfh}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for one period gives us all the information that we need !

1.) is the motion stable?

$$trace(M_{FoDo}) = 1.415 \rightarrow \quad \underline{\underline{< 2}}$$

2.) Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\gamma\sin\psi & \cos\psi - \alpha\sin\psi \end{pmatrix} \rightarrow$$

$$\cos(\psi) = \frac{1}{2}\,Trace(M) = 0.707$$

$$\psi = arc\cos(\frac{1}{2}\,Trace(M)) = \underline{\underline{45^\circ}}$$

3.) hor β-function

$$\beta = \frac{M_{1,2}}{\sin\psi} = \underline{\underline{11.611\,m}}$$

4.) hor α-function

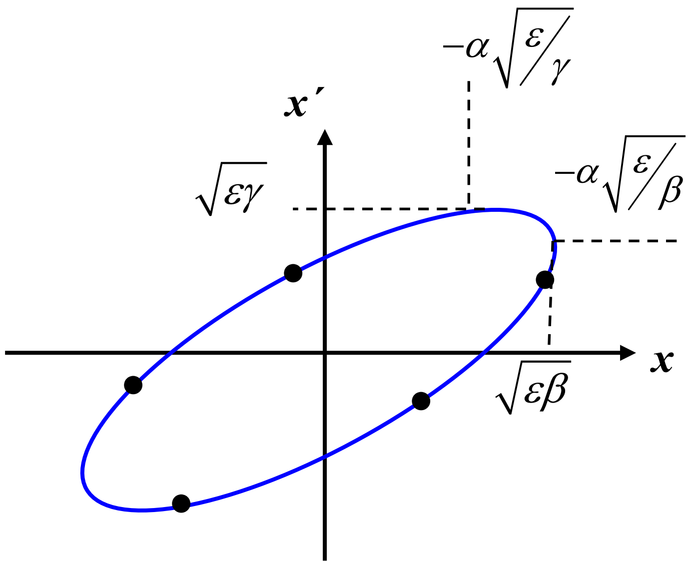
$$\alpha = \frac{M_{1,1} - \cos\psi}{\sin\psi} = \underline{\underline{0}}$$

The „ not so ideal world "

III.) Acceleration and Momentum Spread

1.) Liouville during Acceleration

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$



Beam Emittance ***corresponds to the area covered in the x, x´ Phase Space Ellipse***

Liouville: ***Area in phase space is constant.***

But: ε ≠ const !

Classical Mechanics:

phase space* = *diagram of the two canonical variables

position & ***momentum***

x p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

$$\boldsymbol{q = position = x}$$
$$\boldsymbol{p = momentum = mc\gamma\beta_x}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $\int p\, dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\boldsymbol{\beta_x = v_x / c}$

$$\int pdq = mc \int \gamma\beta_x dx$$

$$\int pdq = mc\gamma\beta \underbrace{\int x'dx}_{\varepsilon}$$

$$\boxed{\Rightarrow \quad \varepsilon = \int x'dx \propto \frac{1}{\beta\gamma}}$$

the beam emittance shrinks during acceleration $\boldsymbol{\varepsilon \sim 1/\gamma}$

Nota bene:

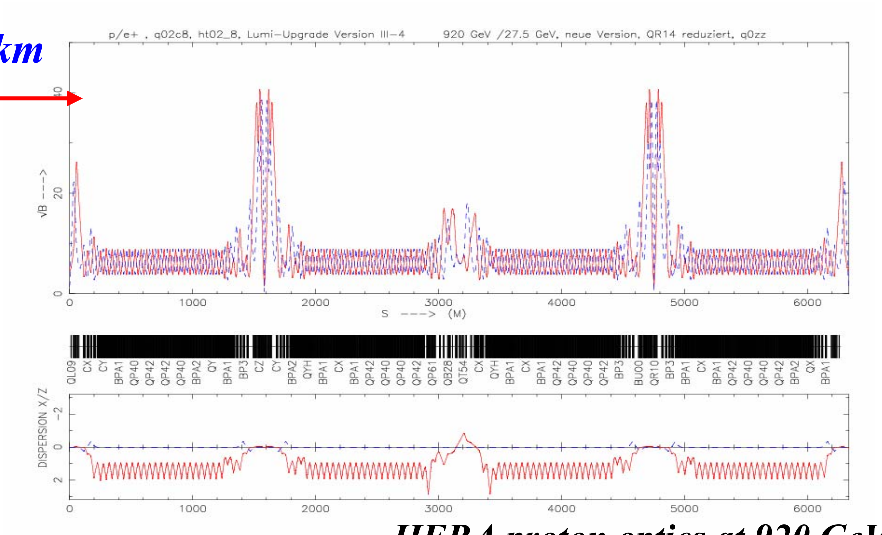
1.) *A proton machine … or an electron linac … needs the highest aperture at injection energy !!!*
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

$$\sigma = \sqrt{\varepsilon\beta}$$

2.) *At lowest energy the machine will have the major aperture problems,*
→ here we have to minimise $\hat{\beta}$

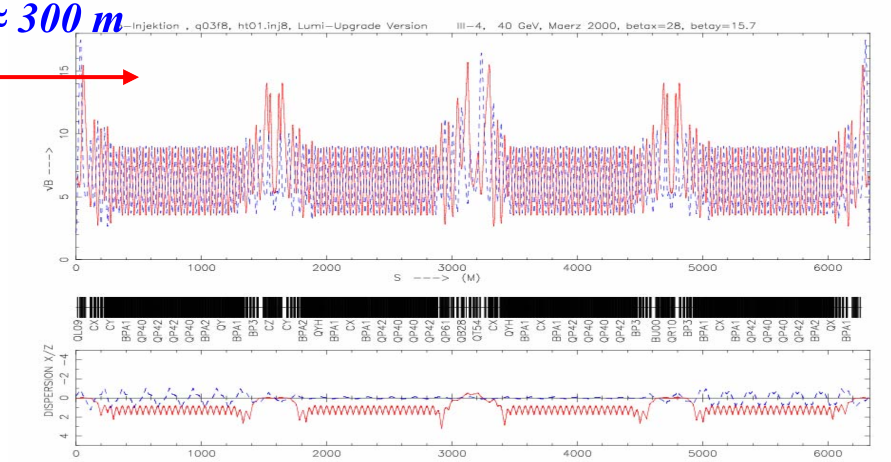
3.) *we need different beam optics adopted to the energy:*
A Mini Beta concept will only be adequate at flat top.

$\beta \approx 1{,}8km$



HERA proton optics at 920 GeV

$\beta \approx 300\ m$

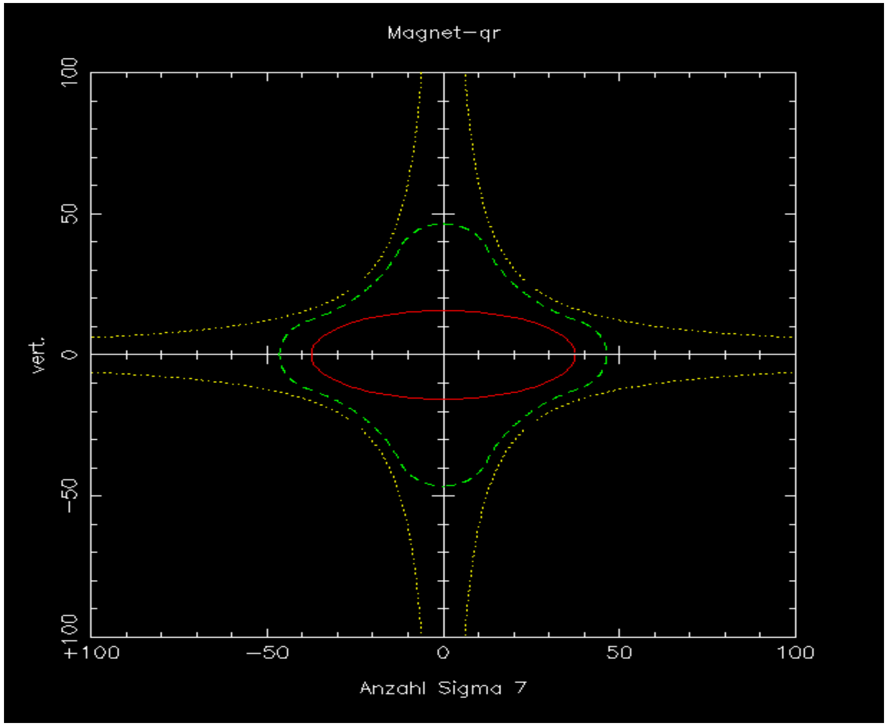


HERA proton optics at 40 GeV

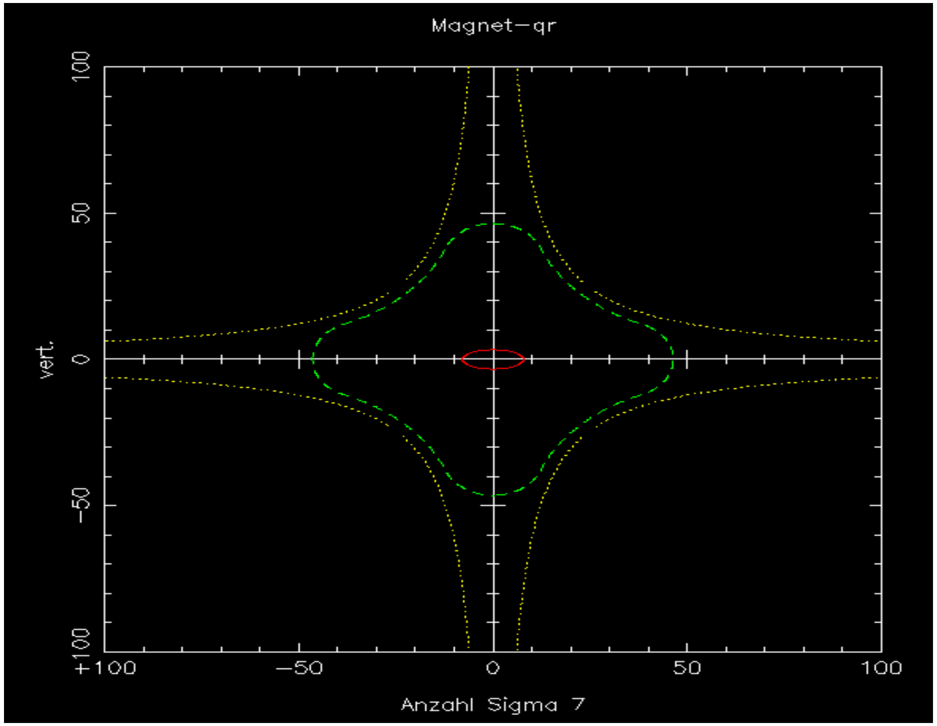
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance ε *(40GeV)* $= 1.2 * 10^{-7}$
ε *(920GeV)* $= 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

The „ not so ideal world “

2.) The „ $\Delta p / p \neq 0$" Problem

ideal accelerator: *all particles will see the same accelerating voltage.*

$\rightarrow \Delta p / p = 0$

„nearly ideal“ accelerator: *Cockroft Walton or van de Graaf*

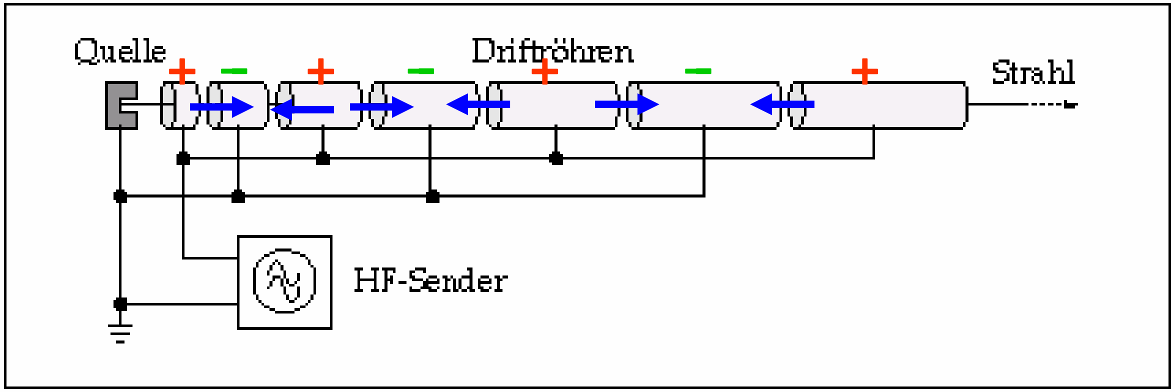
$\Delta p / p \approx 10^{-5}$

Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

Linear Accelerator

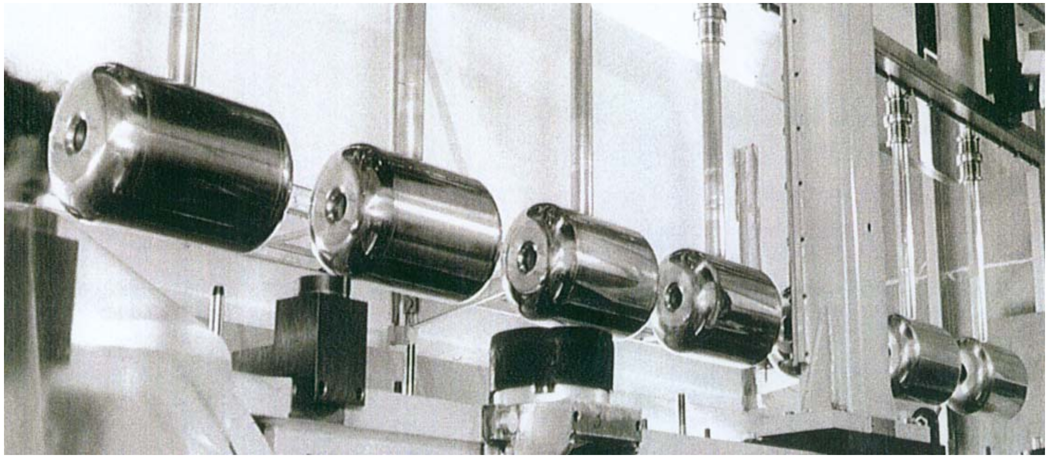
1928, Wideroe *schematic Layout:*



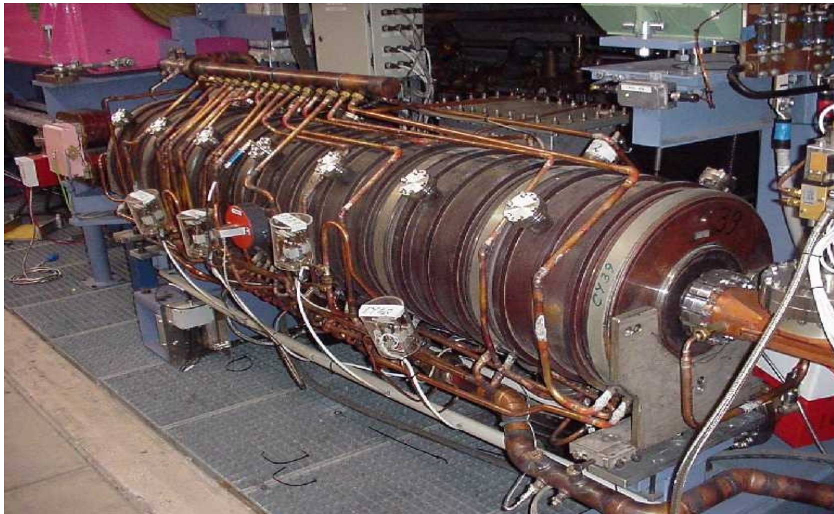
Energy Gain per „Gap":

$$W = q * U_0 * \sin \omega_{RF} t$$

drift tube structure at a proton linac



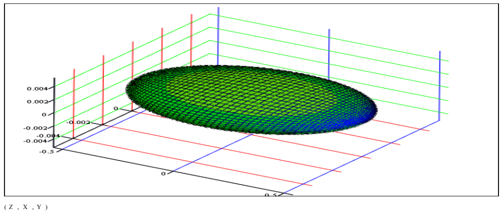
500 MHz cavities in an electron storage ring



* ***RF Acceleration:*** ***multiple application of the same acceleration voltage; brillant idea to gain higher energies***

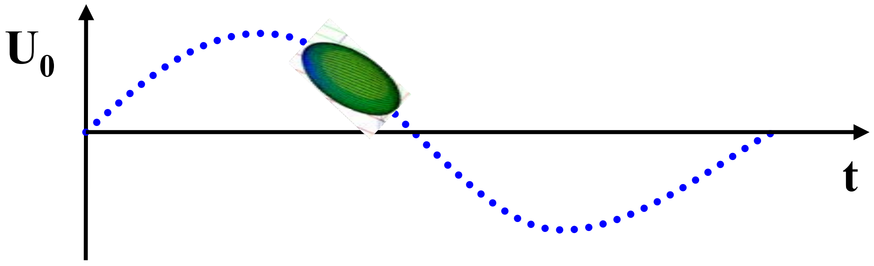
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Example: HERA RF:

Bunch length of Electrons ≈ 1cm



$$\lambda = 60\ cm$$

$$\nu = 500\ MHz$$
$$c = \lambda * \nu$$

$$\lambda = 60\ cm$$

$$\sin(90^o) = 1$$
$$\sin(84^o) = 0.994$$

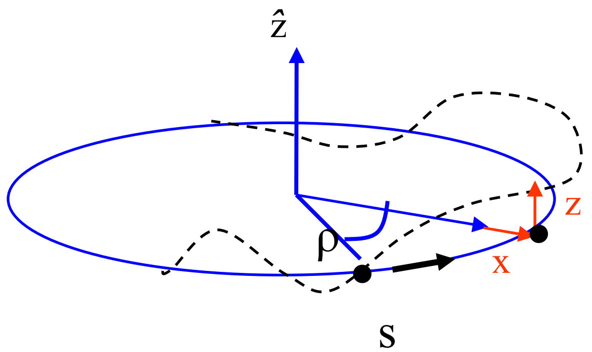
$$\frac{\Delta U}{U} = 6 * 10^{-3}$$

typical momentum spread of an electron bunch: $\frac{\Delta p}{p} \approx 1 * 10^{-3}$

3.) Dispersion: trajectories for $\Delta p / p \neq 0$

Question: ***do you remember last session, page 11 ? … sure you do***

Force acting on the particle



$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_z\,v$$

remember: $x \approx mm$, $\rho \approx m$ … → develop for small x

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_z v$$

consider only linear fields, and change independent variable: $t \rightarrow s$ $\quad B_z = B_0 + x\frac{\partial B_z}{\partial x}$

$$x'' - \frac{1}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{e\;B_0}{mv} + \frac{e\;x\;g}{mv}$$

$p = p_0 + \Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error

$$\Delta p \ll p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{eB_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{xeg}{p_0}}_{k*x} - \underbrace{xeg\frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k*x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k*x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0} * \frac{1}{\rho} \longrightarrow \boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0} * \frac{1}{\rho}}$$

***Momentum spread* of the beam adds a term on the r.h.s. of the equation of motion.**
→ *inhomogeneous differential equation.*

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

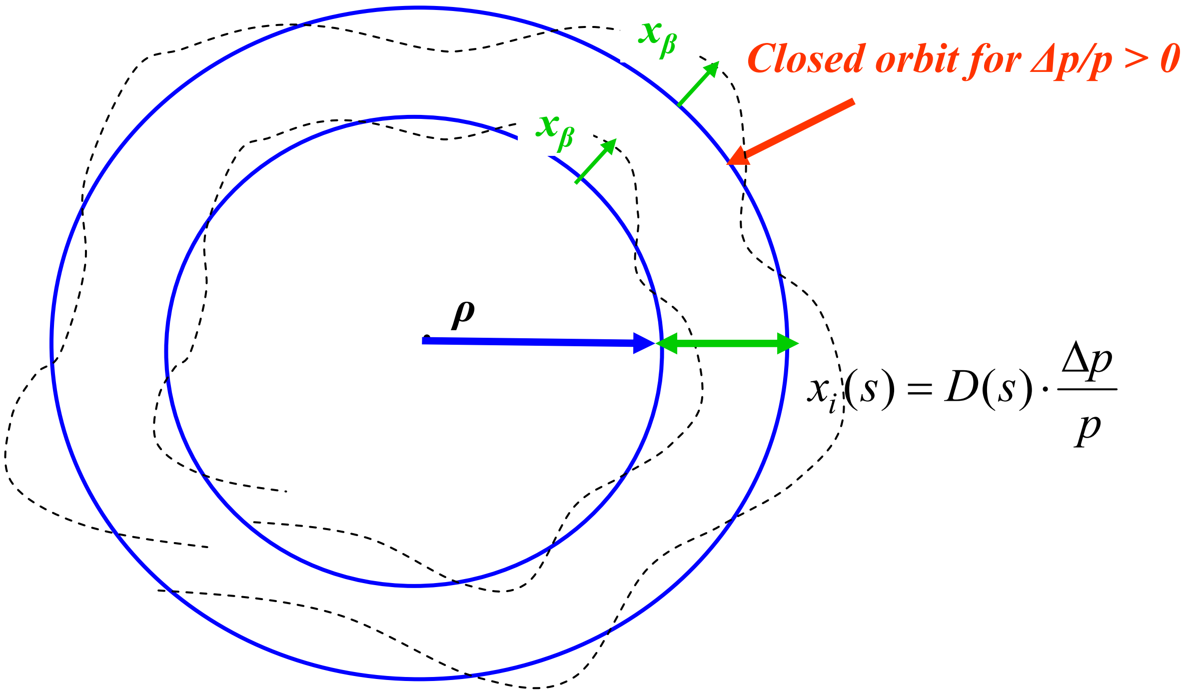
**** is that special orbit, an ideal particle would have for Δp/p = 1***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogeneous dipole field



Matrix formalism:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

Resume´:

beam emittance $$\varepsilon \propto \frac{1}{\beta\gamma}$$

beta function in a drift $$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

… and for α = 0 $$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

particle trajectory for Δp/p ≠ 0
inhomogenious equation $$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0} * \frac{1}{\rho}$$

… and its solution $$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

Introduction to Transverse Beam Optics

The „ not so ideal world "

Bernhard Holzer, DESY-HERA

IV.) Errors in Fields and Gradient

1.) Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

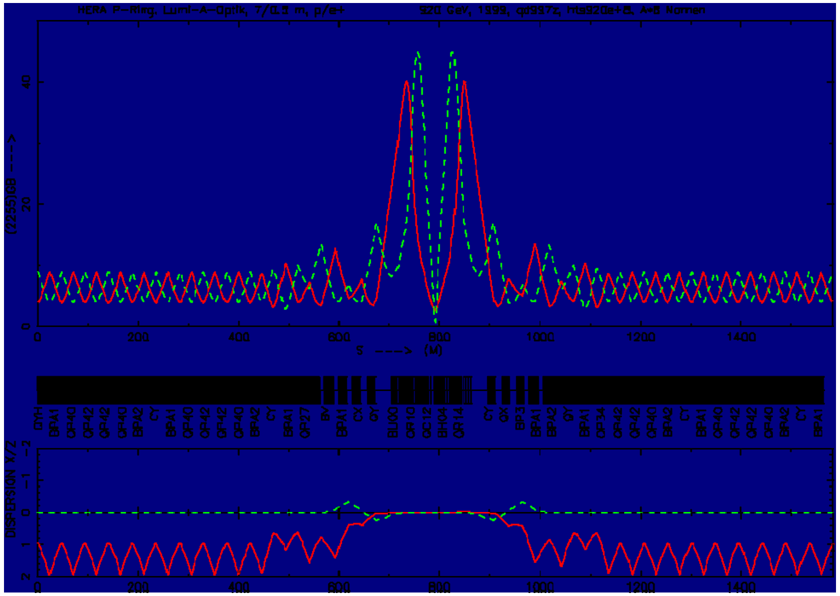
**** is that special orbit, an ideal particle would have for Δp/p = 1***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example HERA

$$\left.\begin{aligned} x_\beta &= 1 \ldots 2\, mm \\ D(s) &\approx 1 \ldots 2\, m \\ \Delta p/p &\approx 1 \cdot 10^{-3} \end{aligned}\right\}$$

Amplitude of Orbit oscillation

contribution due to Dispersion ≈ beam size

Calculate D, D´

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\, d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\, d\tilde{s}$$

* ***see appendix: solution of inh. dgl***

Example: Drift

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$D(s) = S(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})d\tilde{s}}_{=0} - C(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})d\tilde{s}}_{=0}$$

$$M_{Drift} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example: Dipole

$$M_{Dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix} \quad \rightarrow$$

$$D(s) = \rho \cdot (1 - \cos\frac{l}{\rho})$$

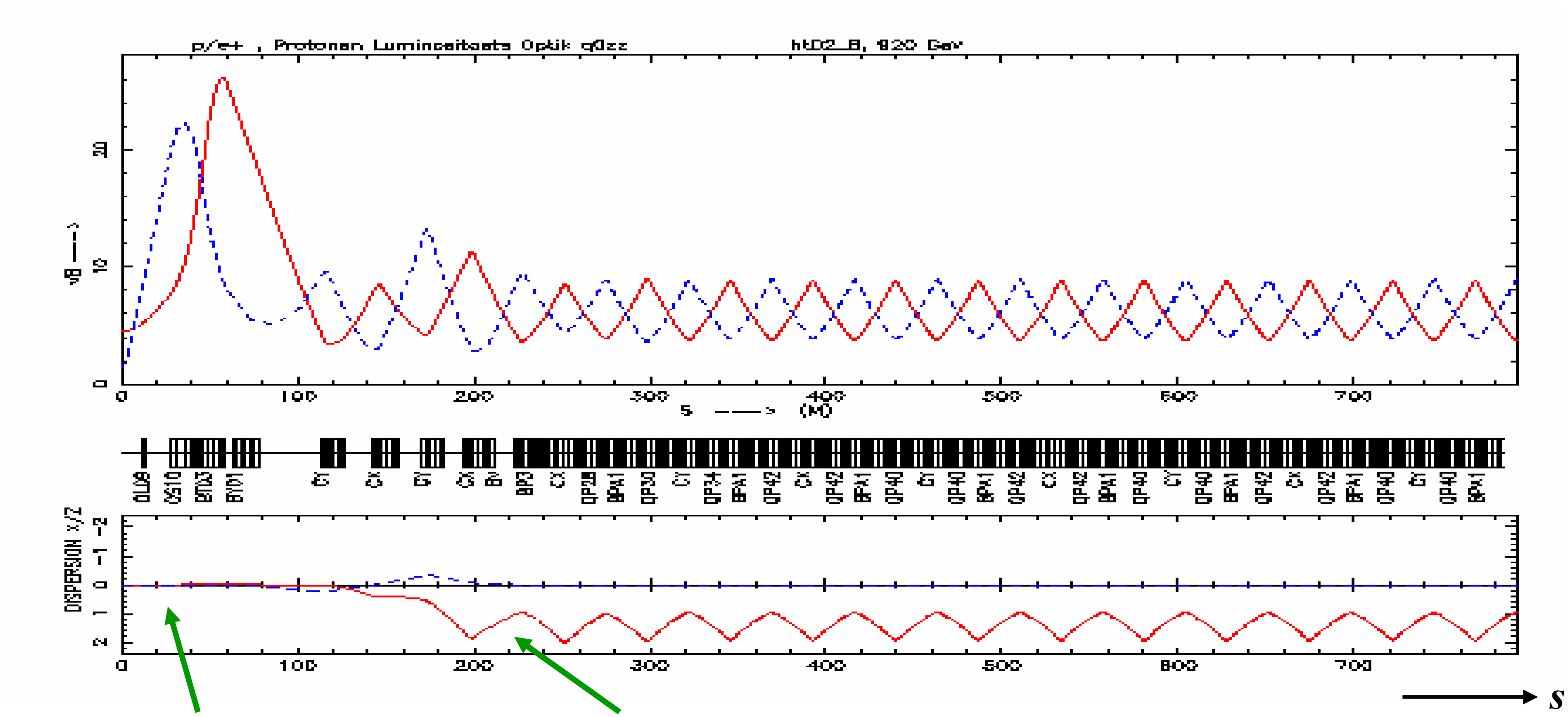
$$D'(s) = \sin\frac{l}{\rho}$$

Example: Dispersion, calculated by an optics code for a real machine

$$x_D = D(s) * \frac{\Delta p}{p}$$

* *D(s) is created by the dipole magnets*

… and afterwards focused by the quadrupole fields



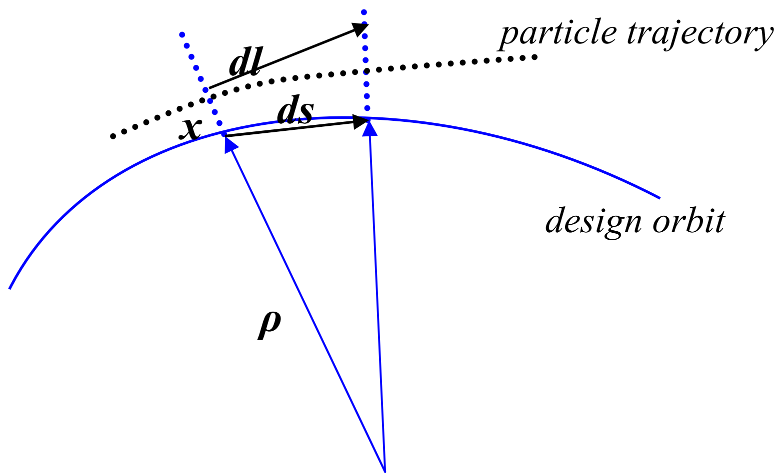
$\longrightarrow$ s

Mini Beta Section,
→ no dipoles !!!

D(s) ≈ 1 … 2 m

2.) Momentum Compaction Factor: α_{cp}

particle with a displacement x to the design orbit
→ path length dl ...



$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \quad dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$

circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s)\frac{\Delta p}{p}$$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

**** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.***

Definition: $$\frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p}$$

$$\rightarrow \quad \alpha_{cp} = \frac{1}{L} \oint \left(\frac{D(s)}{\rho(s)} \right) ds$$

For first estimates assume: $$\frac{1}{\rho} = const$$

$$\int_{dipoles} D(s) ds = l_{dipoles} \cdot \langle D \rangle_{dipole}$$

$$\alpha_{cp} = \frac{1}{L} l_{dipoles} \langle D \rangle \frac{1}{\rho} = \frac{1}{L} 2\pi\rho \langle D \rangle \frac{1}{\rho} \quad \rightarrow$$

$$\alpha_{cp} \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Assume: $v \approx c$

$$\rightarrow \frac{\delta T}{T} = \frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p}$$

α_{cp} combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

3.) Quadrupole Errors

go back to Lecture I, page 1

single particle trajectory

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{QF} * \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

Solution of equation of motion

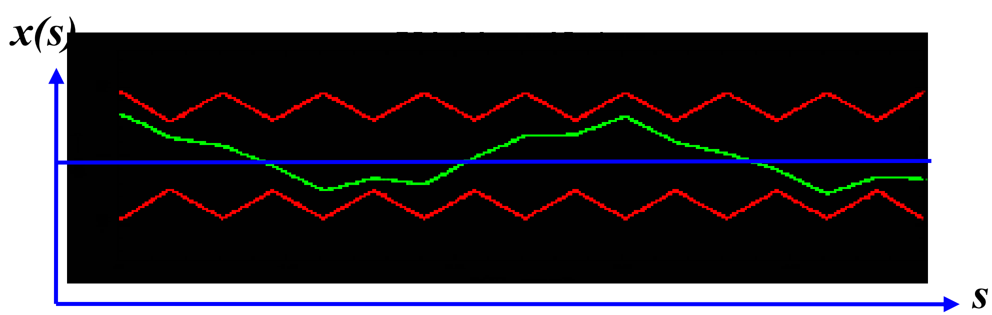
$$x = x_0 * \cos(\sqrt{k} * l) + x_0' * \frac{1}{\sqrt{k}} \sin(\sqrt{k} * l)$$

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{k} * l) & \frac{1}{\sqrt{k}} \sin(\sqrt{k} * l) \\ -\sqrt{k} \sin(\sqrt{k} * l) & \cos(\sqrt{k} * l) \end{pmatrix} \quad , \quad M_{thinlens} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

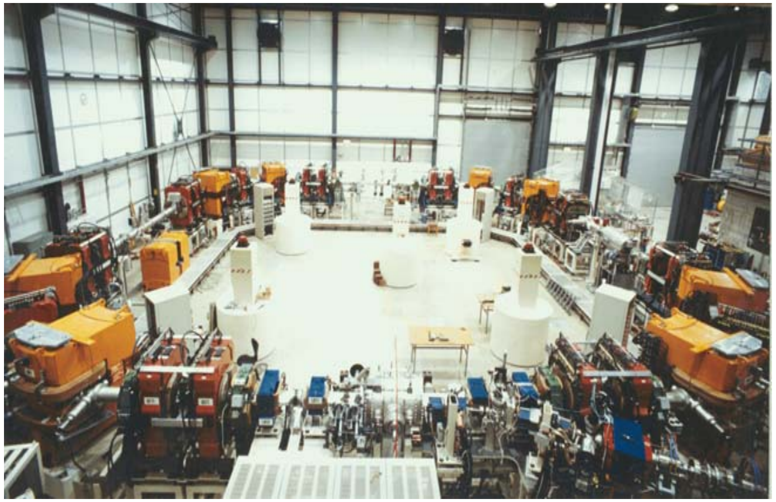
$$M_{turn} = M_{QF} * M_{D1} * M_{QD} * M_{D2} * M_{QF} \; \ldots$$

Definition: ***phase advance of the particle oscillation per revolution in units of 2π is called tune***

$$Q = \frac{\psi_{turn}}{2\pi}$$



Matrix in Twiss Form



Transfer Matrix from point „0" in the lattice to point „s":

$$M(s)=\begin{pmatrix} \sqrt{\dfrac{\beta_s}{\beta_0}}(\cos\psi_s+\alpha_0\sin\psi_s) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \dfrac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\dfrac{\beta_0}{\beta_s}}(\cos\psi_s-\alpha_0\sin\psi_s) \end{pmatrix}$$

For one complete turn the Twiss parameters have to obey periodic bundary conditions:

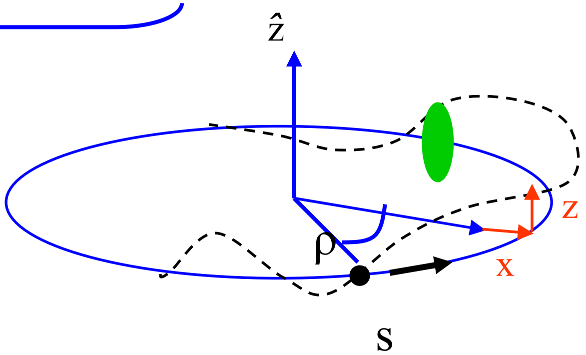
$$\beta(s+L)=\beta(s)$$
$$\alpha(s+L)=\alpha(s)$$
$$\gamma(s+L)=\gamma(s)$$

$$M(s)=\begin{pmatrix} \cos\psi_{turn}+\alpha_s\sin\psi_{turn} & \beta_s\sin\psi_{turn} \\ -\gamma_s\sin\psi_s & \cos\psi_{turn}-\alpha_s\sin\psi_{turn} \end{pmatrix}$$

Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

$$\boldsymbol{M}_{dist} = \boldsymbol{M}_{\Delta k} \cdot \boldsymbol{M}_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta \boldsymbol{k ds} & 1 \end{pmatrix}}_{\textit{quad error}} \cdot \underbrace{\begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}}_{\textit{ideal storage ring}}$$



$$\boldsymbol{M}_{dist} = \begin{pmatrix} \cos\psi_0 + \alpha \sin\psi_0 & \beta \sin\psi_0 \\ \Delta \boldsymbol{k ds}(\cos\psi_0 + \alpha \sin\psi_0) - \gamma \sin\psi_0 & \Delta \boldsymbol{k ds} \beta \sin\psi_0 + \cos\psi_0 - \alpha \sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$\boldsymbol{Trace}(\boldsymbol{M}) = 2\cos\psi = 2\cos\psi_0 + \Delta \boldsymbol{k ds} \beta \sin\psi_0$$

Quadrupole error → Tune Shift

$$\boxed{\psi = \psi_0 + \Delta\psi} \longrightarrow \cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta k ds\, \beta \sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos\psi_0 * \underbrace{\cos\Delta\psi}_{\approx 1} - \sin\psi_0 * \underbrace{\sin\Delta\psi}_{\approx \Delta\psi} = \cos\psi_0 + \frac{k ds\, \beta \sin\psi_0}{2}$$

$$\Delta\psi = \frac{k ds\, \beta}{2}$$

and referring to Q instead of ψ:

$$\psi = 2\pi Q$$

$$\boxed{\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k(s) \beta(s) ds}{4\pi}}$$

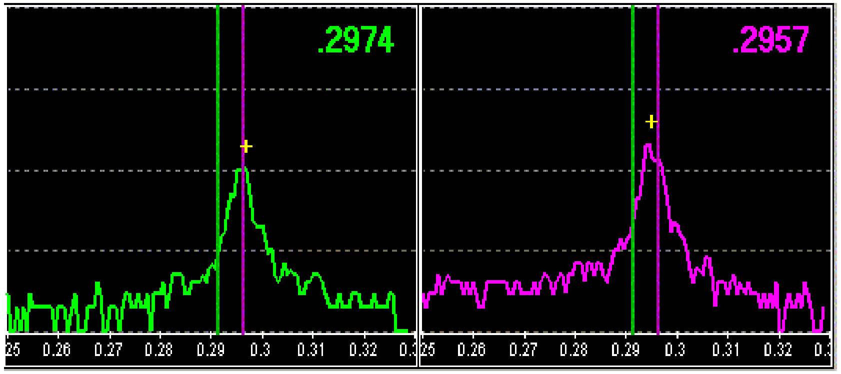
! the tune shift is proportional to the β-function at the quadrupole

!! field quality, power supply tolerances etc are much tighter at places where β is large

!!! mini beta quads: β ≈ 1900 m
arc quads: β ≈ 80 m

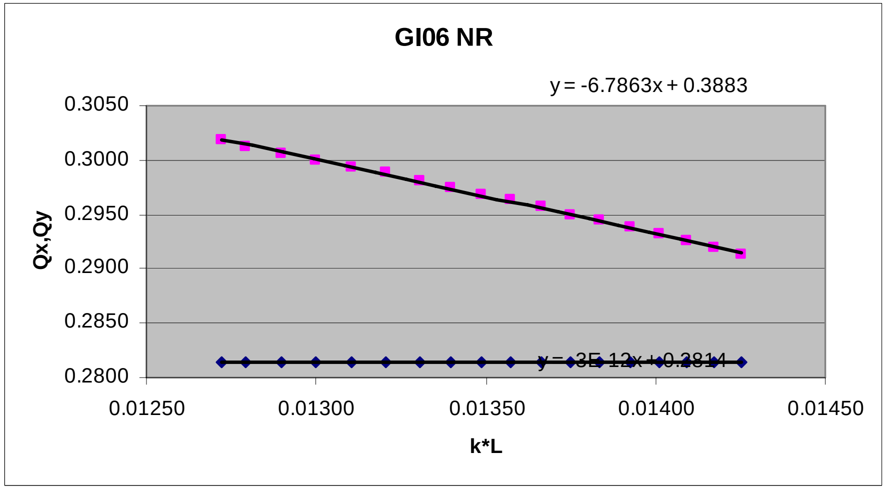
!!!! β is a measure for the sensitivity of the beam

a quadrupol error leads to a shift of the tune:



$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k \, \beta(s)}{4\pi} ds \approx \frac{\Delta k l_{quad} \bar{\beta}}{4\pi}$$

Example: measurement of β in a storage ring: tune spectrum



Error in the β function:

$$\Delta\beta(s_0) = \frac{-\beta_0}{2\sin 2\pi Q} \int_{s1}^{s1+l} \beta(s_1) \Delta k \cos\left(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\right) ds$$

Nota bene:

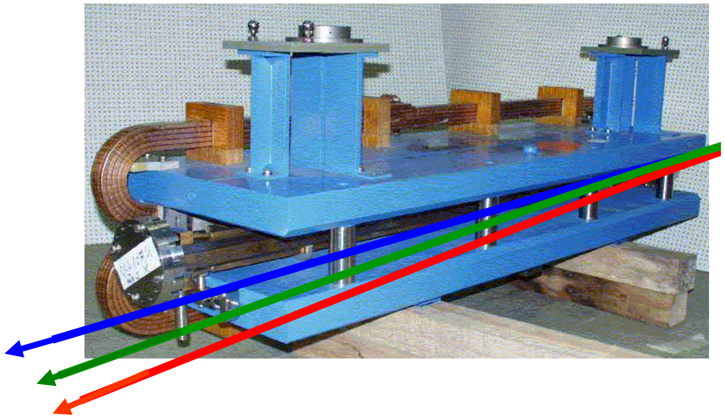
$$\Delta\beta \propto \begin{cases} \Delta k \\ \beta(s_1) \\ \beta(s_0) \end{cases}$$

4.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

dipole magnet

$$\alpha = \frac{\int B * dl}{p/e}$$



$$x_D = D(s) * \frac{\Delta p}{p}$$

focusing lens

$$k = \frac{g}{p/e}$$

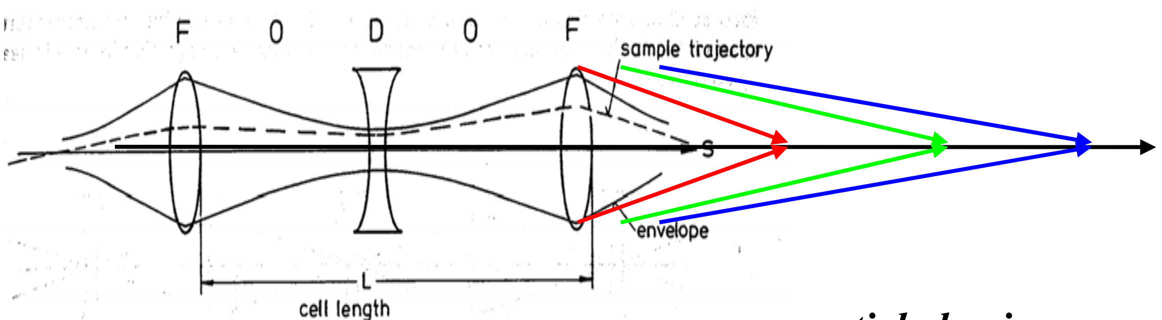


Figure 29: FODO cell

particle having ...

to high energy

to low energy

ideal energy

Chromaticity: ξ

$$k = \frac{g}{p/e} \qquad\qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}\left(1 - \frac{\Delta p}{p_0}\right) * g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

… which acts like a quadrupole error in the machine and leads to a tune spread:

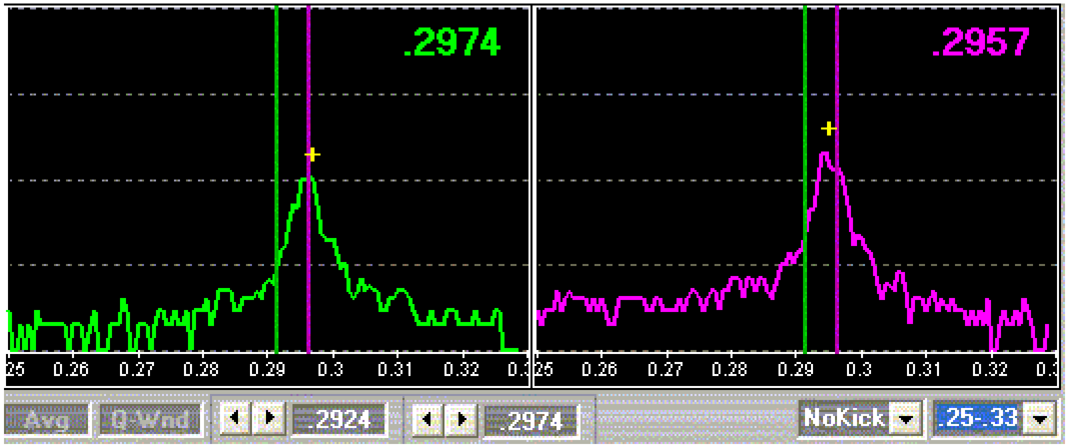
$$dQ = -\frac{1}{4\pi}\frac{\Delta p}{p_0} k_0 \beta(s) ds$$

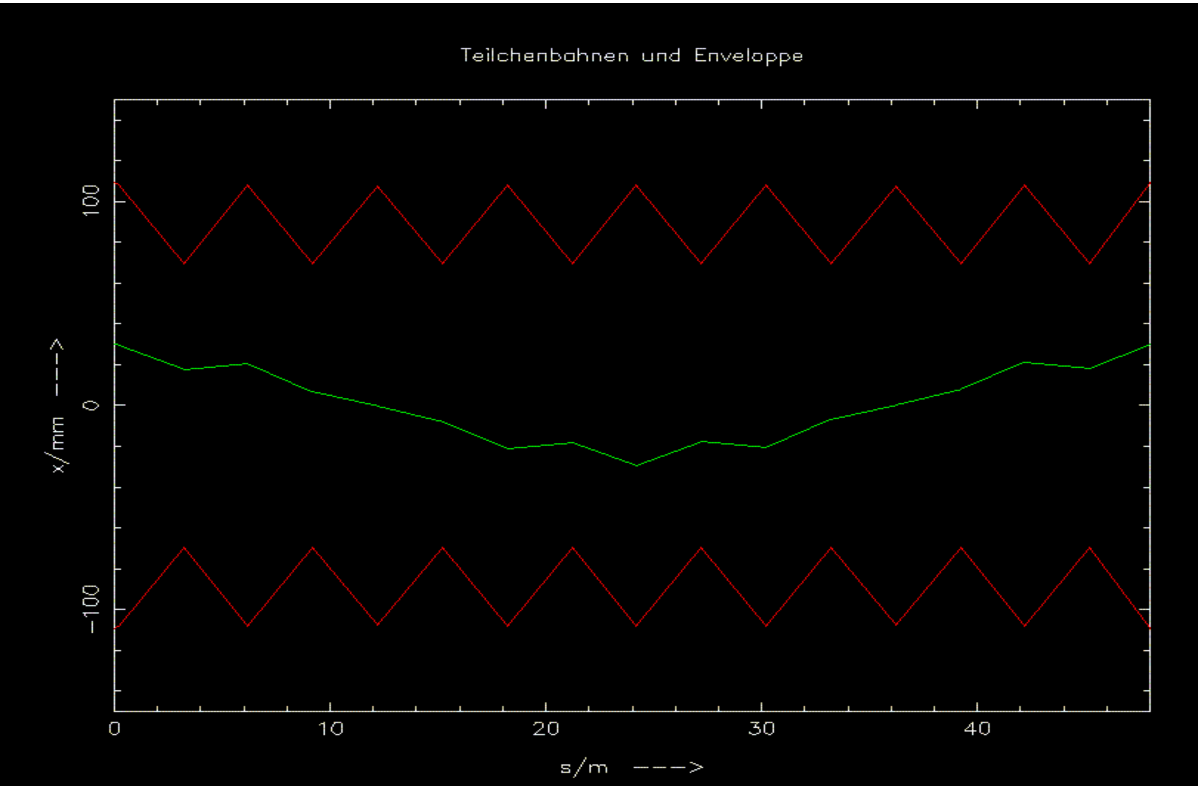
definition of chromaticity:

$$\boxed{\Delta Q = \xi \cdot \frac{\Delta p}{p_0}}$$

Where is the Problem ?

Tunes and Resonances





avoid resonance conditions:

$$m*Q_x+n*Q_y+l*Q_s = integer$$

… for example: $1*Q_x=1$

… and now again about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

ξ is a number indicating the size of the tune spot in the working diagram,
ξ is always created if the beam is focussed
→ it is determined by the focusing strength k of all quadrupoles

$$\xi = -\frac{1}{4\pi} * \oint k(s)\beta(s)ds$$

k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: HERA

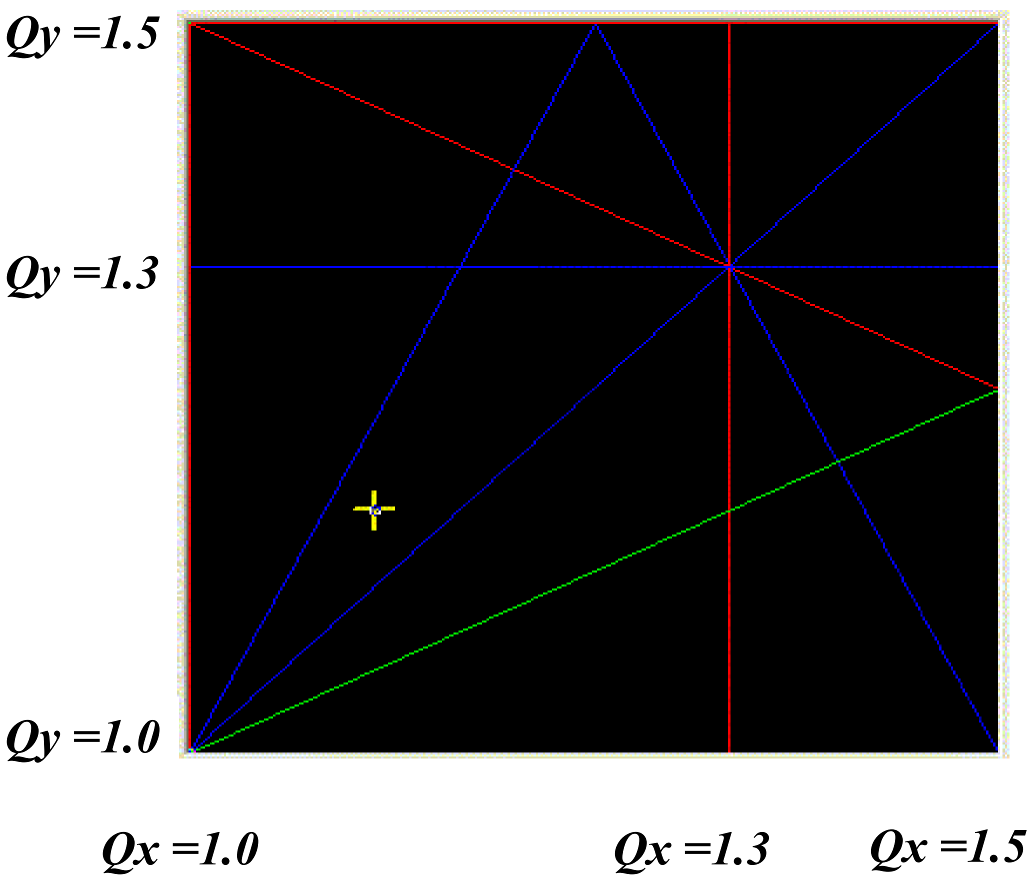
HERA-p: $\xi = -70 \dots -80$
$\Delta p/p = 0.5 * 10^{-3}$
$\Delta Q = 0.257 \dots 0.337$

→ Some particles get very close to resonances and are lost

in other words: the tune is not a point
it is a cow pat.

Tune and Resonances

$$m*Q_x+n*Q_y+l*Q_s = integer$$



HERA e Tune diagram up to 3rd order

… and up to 7th order

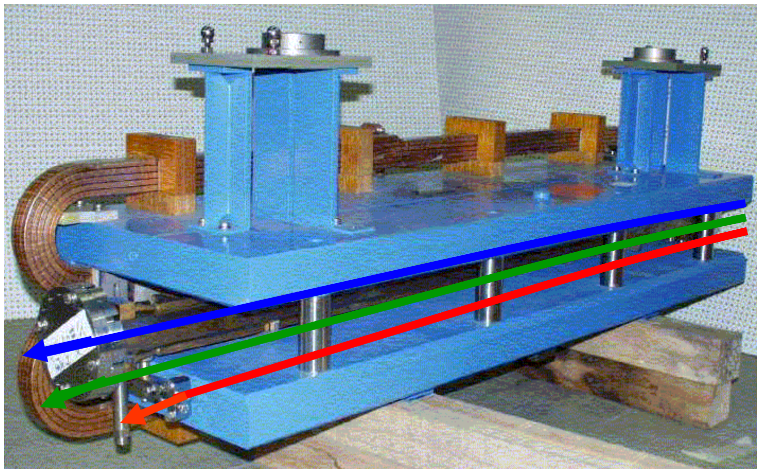
Homework for the operateurs:
find a nice place for the tune
where against all probability
the beam will survive

Correction of ξ:

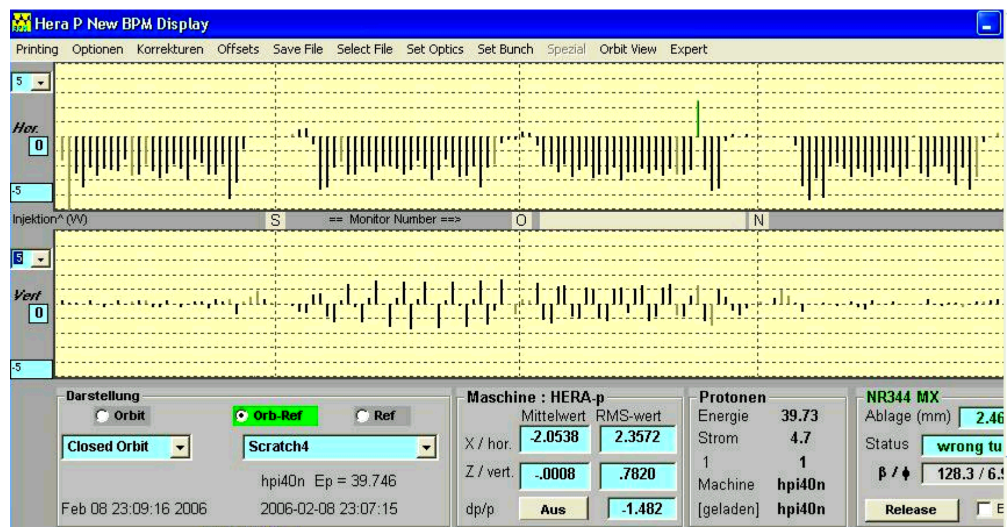
Need: additional quadrupole strength for each momentum deviation Δp/p

1.) sort the particles acording to their momentum

$$x_D(s) = D(s)\,\frac{\Delta p}{p}$$



… using the dispersion function



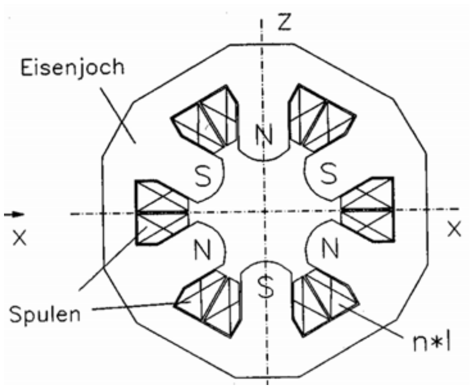
Correction of ξ:

2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left.\begin{aligned} B_x &= \tilde{g}xz \\ B_z &= \frac{1}{2}\tilde{g}(x^2 - z^2) \end{aligned}\right\} \qquad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

linear rising „gradient“:

Sextupole Magnet:



normalised quadrupole strength:

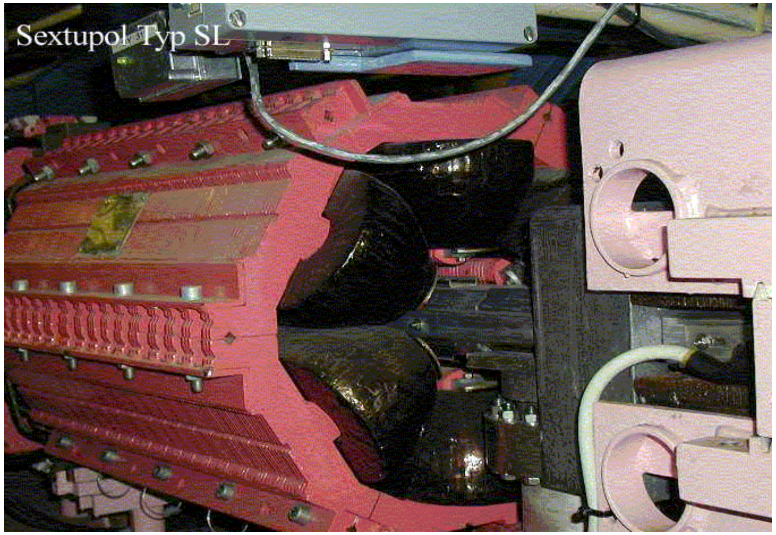
$$k_{sext} = \frac{\tilde{g}x}{p/e} = m_{sext}.x$$

$$k_{sext} = m_{sext}.D\frac{\Delta p}{p}$$

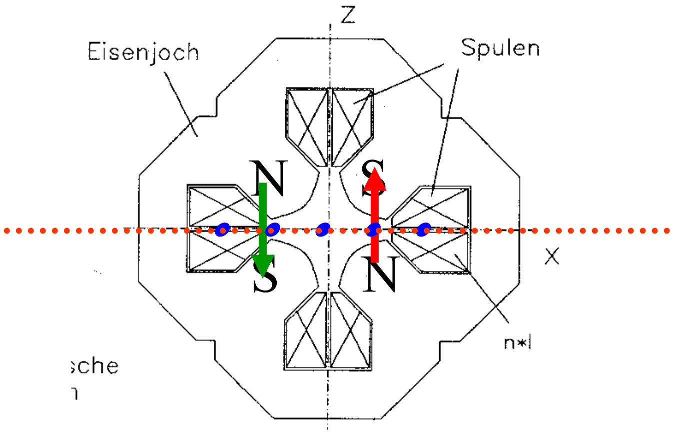
corrected chromaticity:

$$\xi = \frac{-1}{4\pi}\oint\{k(s) - mD(s)\}\beta(s)ds$$

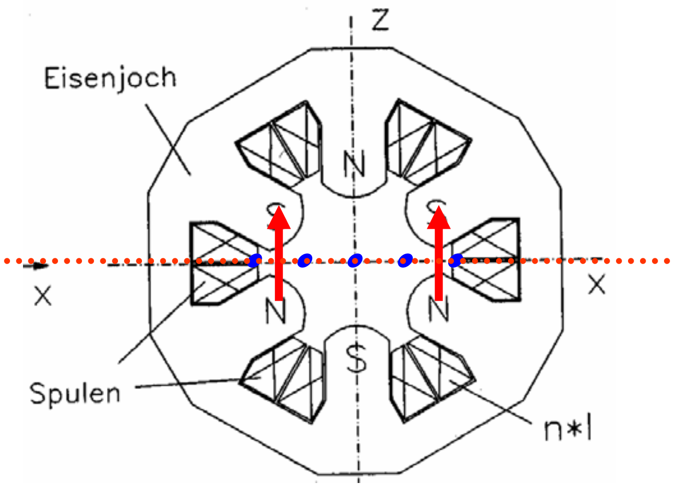
sextupole magnet in a storage ring
… placed close to the quadrupole lens



quadrupole magnet



sextupole magnet



The „ not so ideal world "

5.) Insertions

... the most complicated one: the drift space

Question to the audience: what will happen to the beam parameters α, β, γ if we stop focusing for a while ...?

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_S = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

transfer matrix for a drift:

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \longrightarrow$$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$
$$\alpha(s) = \alpha_0 - \gamma_0 s$$
$$\gamma(s) = \gamma_0$$

β-Function in a Drift:

let's assume we are at a symmetry point in the center of a drift.

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

as $\alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$

and we get for the β function in the neighborhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

! ! !

Nota bene:

1.) this is very bad !!!

2.) this is a direct consequence of the conservation of phase space density (... in our words: ε = const) ... and there is no way out.

3.) Thank you, Mr. Liouville !!!

Joseph Liouville, 1809-1882

β-Function in a Drift:

If we cannot fight against Liouville theorem ... at least we can optimise

Optimisation of the beam dimension:

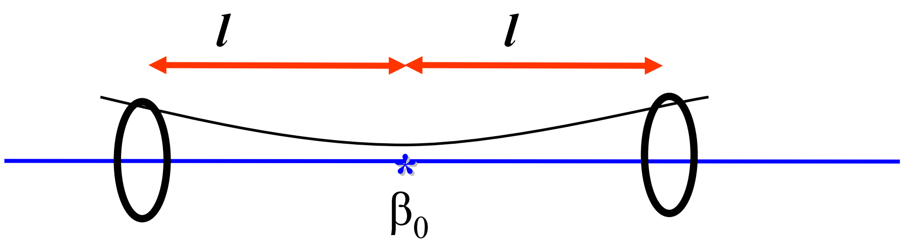
$$\beta(\ell) = \beta_0 + \frac{\ell^2}{\beta_0}$$

Find the β at the center of the drift that leads to the lowest maximum β at the end:

$$\frac{d\hat{\beta}}{d\beta_0} = 1 - \frac{\ell^2}{\beta_0^2} \overset{!}{=} 0$$

$$\rightarrow \beta_0 = \ell$$

$$\rightarrow \hat{\beta} = 2\beta_0$$



If we choose $\beta_0 = \ell$ we get the smallest β at the end of the drift and the maximum β is just twice the distance ℓ

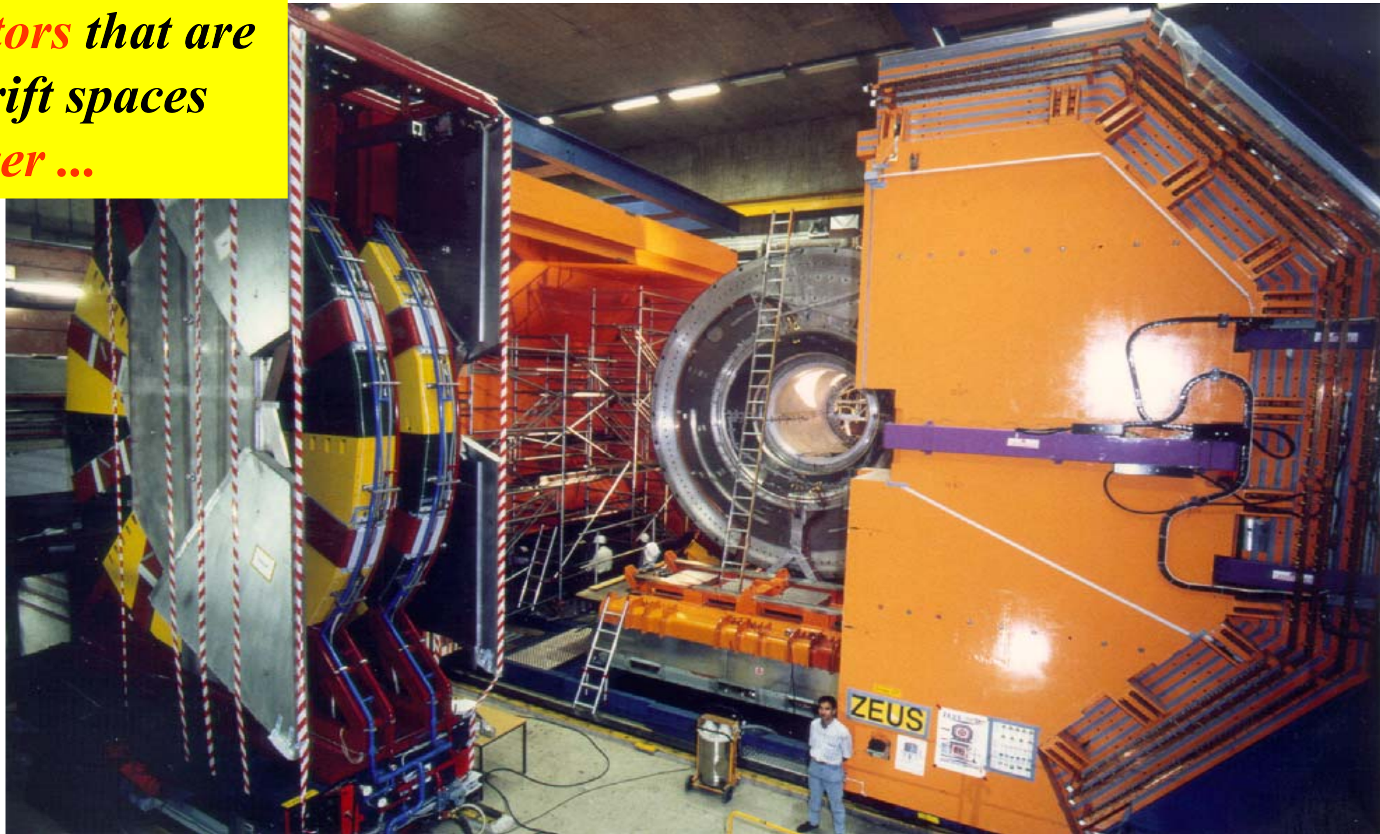
... clearly there is another problem !!!

Example: Luminosity optics at HERA: $\beta^* = 18\ cm$

for smallest β_{max} we have to limit the overall length of the drift to $L = 2*\ell$

$L = 36\ cm$

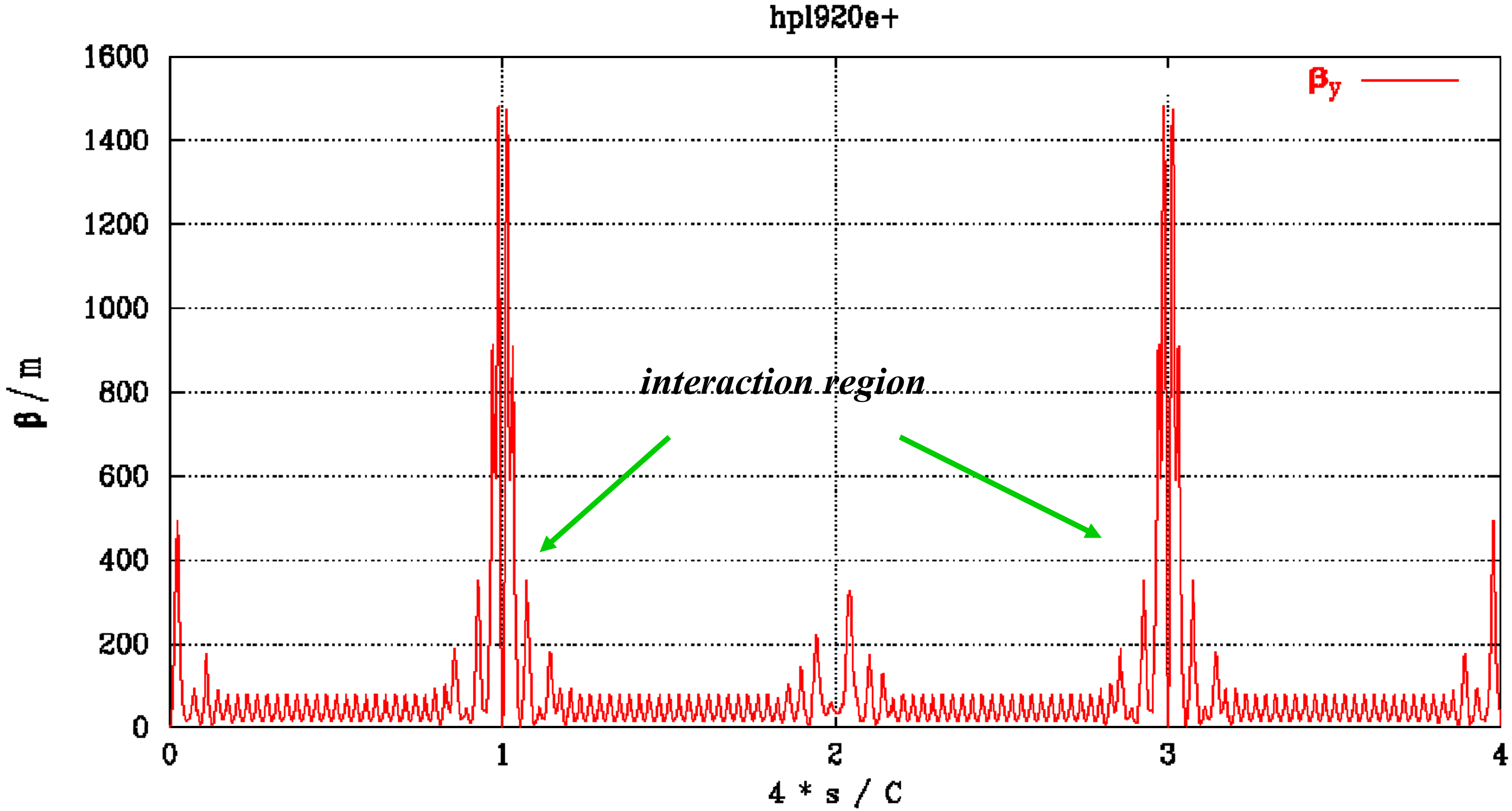
But:** ... unfortunately ... in general **high energy detectors** that are installed in that drift spaces **are a little bit bigger ...



… and now back to the Chromaticity

$$\xi = \frac{-1}{4\pi} \oint k(s)\beta(s)ds$$

question: main contribution to ξ in a lattice … ?



Resume´:

quadrupole error: ***tune shift***

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s) * l_{quad} * \overline{\beta}}{4\pi}$$

beta beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l} \beta(s_1)\Delta k \cos\left(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\right) ds$$

chromaticity

$$\Delta Q = \xi \cdot \frac{\Delta p}{p_0}$$

$$\xi := \frac{1}{4\pi} * \oint k(s)\beta(s)\,ds$$

momentum compaction

$$\frac{\delta l_\varepsilon}{L} = \alpha_{cp}\frac{\Delta p}{p}$$

$$\alpha_{cp} \approx \frac{2\pi}{L}\left\langle D\right\rangle \approx \frac{\left\langle D\right\rangle}{R}$$

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Appendix:

Dispersion:
Solution of the inhomogenious equation of motion

Ansatz:

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})d\tilde{s}$$

$$D'(s) = S' * \int \frac{1}{\rho} C\, dt + \cancel{S\frac{1}{\rho}C} - C' * \int \frac{1}{\rho} S\, dt - \cancel{C\frac{1}{\rho}S}$$

$$D'(s) = S' * \int \frac{C}{\rho} dt \; - \; C' * \int \frac{S}{\rho} dt$$

$$D''(s) = S'' * \int \frac{C}{\rho} d\tilde{s} + S'\frac{C}{\rho} \; - \; C'' * \int \frac{S}{\rho} d\tilde{s} \; - \; C'\frac{S}{\rho}$$

$$= S'' * \int \frac{C}{\rho} d\tilde{s} - \; C'' * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}\underbrace{\left(CS' \quad - S\,C'\right)}_{= \det M = 1}$$

remember: for Cs) and S(s) to be independent solutions the Wronski determinant has to meet the condition

$$W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \neq 0$$

and as it is independent of the variable „s"

$$\frac{dW}{ds}=\frac{d}{ds}(CS'-SC')=CS''-SC''=-K(CS-SC)=0$$

we get for the initital conditions that we had chosen ...

$$\left.\begin{array}{ll} C_0=1, & C_0'=0 \\ S_0=0, & S_0'=1 \end{array}\right\} \quad W=\begin{vmatrix} C & S \\ C' & S' \end{vmatrix}=1$$

$$D''=S''*\int\frac{C}{\rho}d\tilde{s}-\ C''*\int\frac{S}{\rho}d\tilde{s}+\frac{1}{\rho}$$

remember: S & C are solutions of the homog. equation of motion:

$$S''+K*S=0$$

$$C''+K*C=0$$

$$D''=-K*S*\int\frac{C}{\rho}d\tilde{s}+\ K*C*\int\frac{S}{\rho}d\tilde{s}+\frac{1}{\rho}$$

$$D''=-K*\underbrace{\left\{S\int\frac{C}{\rho}d\tilde{s}+C\int\frac{S}{\rho}d\tilde{s}\right\}}_{=D(s)}+\frac{1}{\rho}$$

$$D''=-K*D+\frac{1}{\rho}$$

... or

$$D''+K*D=\frac{1}{\rho}$$

qed