MULTI-PARTICLE EFFECTS

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Albert Hofmann

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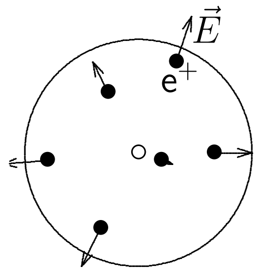
1) INTRODUCTION

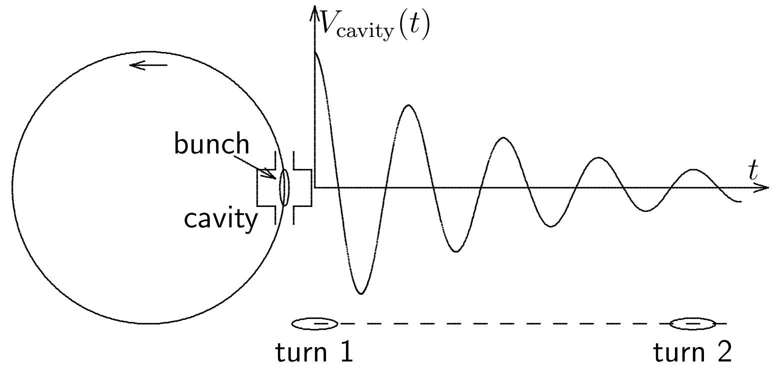
The single particle motion is given by external guide fields (dipoles, quadrupoles, RF-system), initial conditions and synchrotron radiation.

A beam of many particles represents a charge and current and creates electromagnetic **self fields**. They act back on the particles directly as **space-charge** effect, repel them and counteract the focusing. In most cases they induce charges and currents in the surroundings (vacuum chamber) which represent an **impedance**. They are a source of fields which act back on the beam. This can cause: a **frequency shift** (change of the betatron or synchrotron frequency), an increase of an initial disturbance, **instability** or a **change of the particle distribution**, (bunch lengthening) all due to a **collective** action by many particles.

Examples: Space-charge field repels particles, reduces focusing



A circulating bunch induces fields in a passive cavity which oscillate and act back on the next turn, Depending on the phase the initial perturbation is increased or decreased.



2) SPACE CHARGE EFFECT

Introduction

The many charged particles in an intensive beam present a space-charge and produce electromagnetic self fields which affects dynamics being otherwise determined by the guide fields of the magnetic lattice and RF-system. Assuming weak self fields we treat their effects as a perturbation. In the transverse case this shifts the betatron frequencies (tunes).

For the **direct space charge effect** the conducting vacuum chamber is neglected, E and B-fields are obtained directly. The first is repelling and defocuses while the Lorentz force of the second focuses. The balance between them becomes more perfect as the particle velocity v approaches c.

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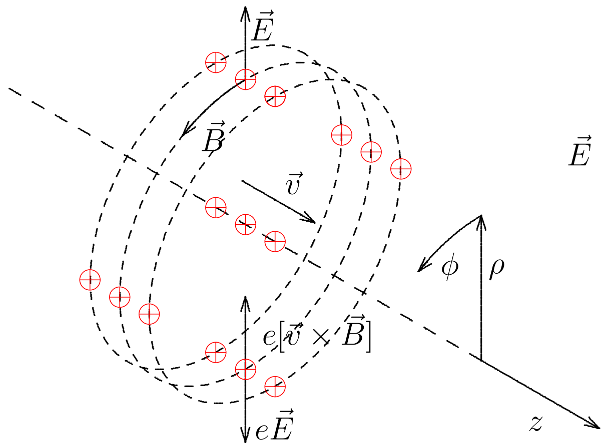
Conducting boundaries modify the field giving an **indirect space charge field** calculated with help of image charges. Here the balance between E and B-effects is perturbed and the effect is important also for $v \to c$

For a rigid **(coherent)** oscillation of the beam as a whole the direct space charge represents an internal force which does not influence this motion, however the indirect wall effect does.



Direct space charge effect

Fields and forces

A uniform (unbunched) beam of circular cross section, radius a, uniform charge and current densities η and $\vec{J}$, charge per unit length $\lambda = \pi a^2 \eta$, moves with $v_z = \beta c$, $I = \beta c \lambda$. It creates cylindrically symmetric fields $\vec{E} = [E_\rho,\ 0,\ 0]$ and $\vec{B} = [0,\ E_\phi,\ 0]$ determined by:

$$\boxed{\begin{array}{c} \text{div}\, \vec{E} = \eta/\epsilon_0 \\ \iiint \text{div}\, \vec{E}\, dV = \iint \vec{E} d\vec{S} \end{array}} \qquad \boxed{\begin{array}{c} \text{curl}\, \vec{B} = \mu_0 \vec{J} \\ \oint \vec{B} \cdot d\vec{s} = \iint \text{curl}\, \vec{B} \cdot d\vec{S} \end{array}}$$

On cylinder, radius $\rho \le a$, int. elements, volume $dV = 2\pi\rho dz d\rho$, area $dS_\rho = 2\pi\rho dz$, $dS_z = 2\pi\rho d\rho$,$d\vec{s} = [0,\ 1,\ 0]\rho d\phi$, have fields inside and outside beam

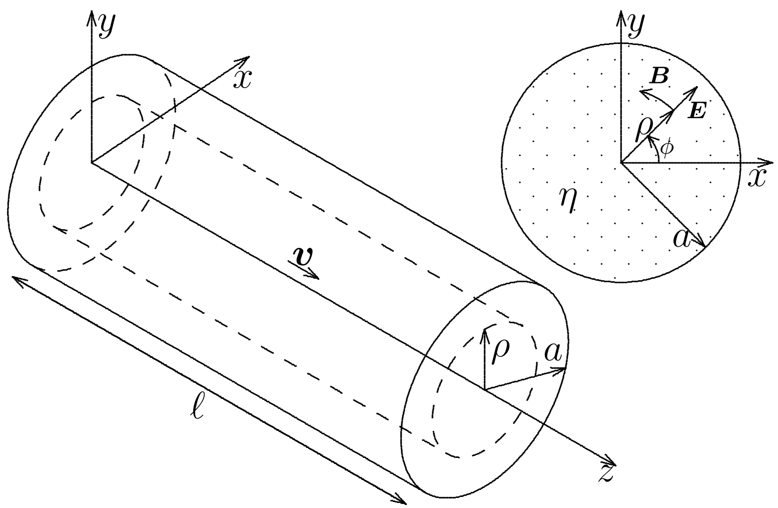
$$\boxed{\begin{array}{rl} & 2\pi\rho\ell E_\rho = \pi\rho^2\ell\eta/\epsilon_0 \\ E_\rho = & \dfrac{\eta\rho}{2\epsilon_0} = \dfrac{\lambda}{2\pi\epsilon_0}\dfrac{\rho}{a^2} \\ E_\rho = & \dfrac{\eta a^2}{2\epsilon_0\rho} = \dfrac{\lambda}{2\pi\epsilon_0}\dfrac{1}{\rho} \end{array}} \begin{array}{c} \\ \text{in} \\ \text{out} \end{array} \boxed{\begin{array}{rl} & 2\pi\rho B_\phi = \pi\rho^2\mu_0 J_z \\ B_\phi = & \dfrac{\beta\eta\rho}{2\epsilon_0 c} = \dfrac{\mu_0 I}{2\pi}\dfrac{\rho}{a^2} \\ B_\phi = & \dfrac{\beta\eta a^2}{2\epsilon_0 c\rho} = \dfrac{\mu_0 I}{2\pi}\dfrac{1}{\rho} \end{array}}$$

.

The force on a particle in the beam

$$\vec{F} = F_E + F_B = e\left(\vec{E} + [\vec{v} \times \vec{B}]\right)$$
$$= \frac{e\eta}{2\epsilon_0}(1-\beta^2)\vec{\rho} = \frac{eI}{2\pi\epsilon_0 c\beta\gamma^2}\frac{\vec{\rho}}{a^2}.$$

We have $\vec{F} \propto \vec{\rho}$ resulting in a linear defocusing effect. It is proportional to $1/\gamma^2$ and vanishes as $\beta \to 1$.



Uniform focusing

Simple case: uniform density η, circular cross section radius a, machine radius R, revolution frequency $\omega_0 = \beta c/R$, uniform focusing with tunes Q_x and Q_y and constant beta functions

$$\beta_x \approx \langle \beta_x \rangle \approx R/Q_x \ , \ \beta_y \approx \langle \beta_y \rangle \approx R/Q_y.$$

Without space charge equation of motion is

$$d^2y/dt^2 + \omega_\beta^2 y = \ddot{y} + Q_{y0}^2\omega_0^2 y = 0,$$

Space charge gives additional acceleration

$$\ddot{x}_{sc} = \frac{F_x}{m_0\gamma} = \frac{e\eta x}{2\epsilon_0\gamma^2 m_0\gamma} \ , \ \ddot{y}_{sc} = \frac{e\eta y}{2\epsilon_0\gamma^2 m_0\gamma}$$

$$\ddot{y} + \left(Q_{y0}^2\omega_0^2 - \frac{e\eta}{2\epsilon_0 m_0\gamma^3}\right) y = \ddot{y} + Q_y^2\omega_0^2 y = 0,$$

same for x. We express new tune Q_y as

$$\frac{Q_y^2}{Q_{y0}^2} = \left(1 + \frac{\Delta Q}{Q_{y0}}\right)^2 = \left(1 - \frac{e\eta}{2\epsilon_0 Q_{y0}^2\omega_0^2 m_0\gamma^3}\right)$$

We assume small effect, $\Delta Q_y/Q_{y0} \ll 1$

$$\Delta Q_y \approx -\frac{e\eta}{4\epsilon_0 Q_{y0}\omega_0^2 m_0\gamma^3}$$

We introduce the classical electron radius

$$r_0 = \frac{e^2}{4\pi\epsilon_0 m_0 c^2} = \begin{array}{l} 1.54 \cdot 10^{-18}\,\mathrm{m\ for\ protons} \\ 2.82 \cdot 10^{-15}\,\mathrm{m\ for\ electrons} \end{array}$$

$$\Delta Q_y = -\frac{\pi c^2 r_0 \eta}{\omega_0^2 Q_{y0} e\gamma^3} = -\frac{r_0 c I}{e\omega_0^2 Q_{y0} a^2\beta\gamma^3}.$$

Using beta function β_y, emittance $\mathcal{E}_y$ and ω_0

$$\beta_y \approx R/Q_{y0} \ , \ \mathcal{E}_y \approx a^2/\beta_y \ , \ \omega_0 = \beta c/R.$$

gives tune shift in practical parameters

$$\Delta Q_y = -\frac{r_0 R I}{ec\mathcal{E}_y\beta^3\gamma^3} \ , \ \Delta Q_x = -\frac{r_0 R I}{ec\mathcal{E}_x\beta^3\gamma^3}.$$

The space charge force for uniform η is linear giving the same tune shift to all particles. The betatron frequency of the internal (incoherent) particle motion is changed.

Non-uniform focusing

A realistic focusing lattice has F- and D-quads and beta-functions which depend on path s. We still approximate for a circular beam but with radius $a(s)$. We use the local focusing parameter K_y, derivative of curvature $1/\rho_0$, to describe space-charge strength

$$K_y = -\frac{d(1/\rho_0)}{dy} \approx -\frac{1}{y}\frac{d^2y}{ds^2} = -\frac{1}{y\beta^2c^2}\frac{d^2y}{dt^2}$$
$$= -\frac{1}{\beta^2c^2}\frac{e\eta}{2\epsilon_0 m_0\gamma^3} = -\frac{2r_0 I}{eca^2\beta^3\gamma^3}$$

The tune shift by a local weak lens of length ds

$$d(\Delta Q_y) = \frac{K_y\beta_y}{4\pi}ds = -\frac{r_0 I}{2\pi ce\mathcal{E}_y\beta^3\gamma^3}ds.$$

Interesting: partial tune shift by local space-charge depends on emittance $\mathcal{E}_y = a^2/\beta_y$. but not on β_y and a separately. Small β_y gives small a and strong force but reduced focusing.

With emittance $\mathcal{E}_y = a^2/\beta_y$ being invariant with s around the ring the total tune shift is

$$\Delta Q_y = \oint d(\Delta Q_y) = \oint \frac{K_y\beta_y}{4\pi}ds = -\frac{r_0 I R}{ce\mathcal{E}_y\beta^3\gamma^3}.$$

It decreases with the third power of the Lorentz factor γ. The cancelation between the magnetic and electric forces contributes a power of two and the stiffness of the beam a power of one to this dependence. For given current I the tune shift increases with machine radius R, but expressing I with particles number N_b gives

$$I = \frac{eN_b\omega_0}{2\pi} = \frac{eN_b\beta c}{2\pi R}$$
$$\Delta Q_x = -\frac{r_0 N_b}{2\pi\mathcal{E}_x\beta^2\gamma^3},$$

with the charge eN_b of the whole beam.

Elliptic beam cross section

Uniform η and elliptic cross section with half-axes $a,\ b$ gives fields and forces inside (L. Teng)

$$\vec{E} = [E_x,\ E_y,\ E_z]] \ = \ \frac{I}{\pi\epsilon_0(a+b)\beta c}\left[\frac{x}{a}, \frac{y}{b}, 0\right]$$
$$\vec{B} = [B_x,\ B_y,\ B_z] \ = \ \frac{\mu_0 I}{\pi(a+b)}\left[-\frac{y}{b}, \frac{x}{a}, 0\right]$$

which satisfies div $\vec{E} = \eta/\epsilon_0$, curl $\vec{B} = \mu_0 \vec{J}$.

$$\vec{F} = e\Big[\vec{E} + [\vec{v}\times\vec{B}]\Big] = \frac{I\,[(x/a),\ (y/b),\ 0]\,.}{\pi\epsilon_0\beta c\gamma^2(a+b)}$$

This linear space-charge force gives tune shifts

$$\Delta Q_x \ = \ -\frac{r_0 I}{\pi e c\beta^3\gamma^3\mathcal{E}_x}\oint\frac{a}{a+b}ds$$
$$\Delta Q_y \ = \ -\frac{r_0 I}{\pi e c\beta^3\gamma^3\mathcal{E}_y}\oint\frac{b}{a+b}ds$$

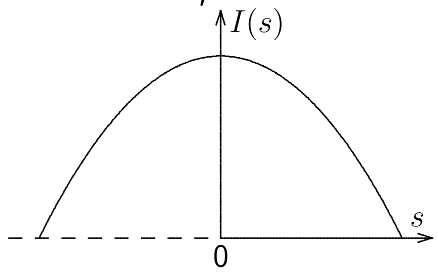
Since a/b depends on s the local tune shift contribution depends also weakly on s.

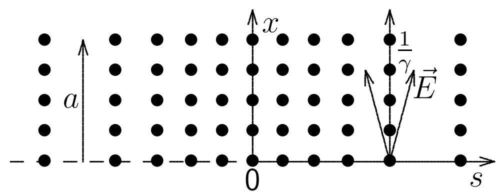
Bunched beams

In bunched beams the current $I(s)$ depends on longitudinal distance s from the bunch center. The space-charge force is more complicated, however, since electric field of a relativistic particle is mainly transverse with opening angle $\approx 1/\gamma$ and $\propto I(s)$ as long it changes little over a distance a/γ

$$\Delta Q_y = -\frac{r_0 R I(s)}{ec\mathcal{E}_y\beta^3\gamma^3}\ ,\quad \Delta Q_x = -\frac{r_0 R I(s)}{ec\mathcal{E}_x\beta^3\gamma^3},$$

Tune shift depends now on longitudinal particle position in bunch. This leads to a tune spread and, since particles execute synchrotron oscillations, to a tune modulation.





Non-uniform distribution

Back to a continuous (unbunched) beam. An uniform charge density gives a linear defocusing force $F_x \propto x$ and same tune shift for each particle. A general transverse charge distribution can give a non-linear force, making the tune shift dependent on the betatron oscillation amplitude and resulting in a tune spread. Example: circular, Gaussian beam with

$$\eta(\rho) = \frac{I}{2\pi\beta c\sigma^2} e^{-\frac{\rho^2}{2\sigma^2}}$$

Gauss Theorem $\iiint \mathrm{div}\, \vec{E}\, dV = \iint \vec{E} d\vec{S}$ gives field components and a radial force

$$2\pi\rho E_\rho = \frac{1}{\epsilon_0}\int \rho\eta d\rho = \frac{I}{2\pi\epsilon_0\beta c\sigma^2}\int \rho e^{-\frac{\rho^2}{2\sigma^2}} d\rho$$

$$E_\rho = \frac{I}{2\pi\epsilon_0\beta c}\frac{1}{\rho}\left(1 - e^{-\frac{\rho^2}{2\sigma^2}}\right), \; B_\phi = \frac{\beta E_\rho}{c}$$

$$F_\rho = \frac{eI}{2\pi\epsilon_0\beta c\gamma^2}\frac{1}{\rho}\left(1 - e^{-\frac{\rho^2}{2\sigma^2}}\right).$$

In this non-linear force the oscillation is not harmonic but more complicated. We develop F_ρ in powers of ρ

$$F_\rho \approx \frac{eI}{2\pi\epsilon_0\beta c\gamma^2}\frac{\rho}{2\sigma^2}\left(1 - \frac{\rho^2}{4\sigma^2} + \; ...\right).$$

The lowest order gives the small amplitude tune shift

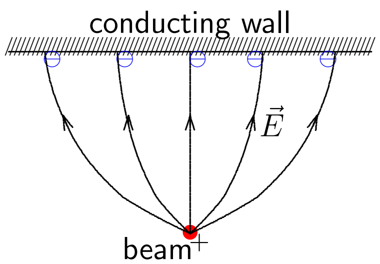
$$\Delta Q_x \approx -\frac{r_0 I R}{ec\beta^3\gamma^3}\frac{\beta_x}{2\sigma^2}.$$

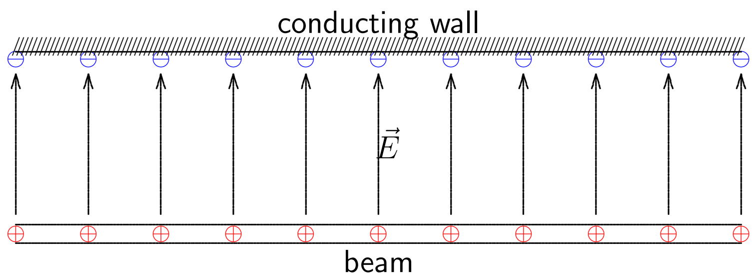
in a Gaussian beam which is the same as the one of an uniform beam with radius $a = \sqrt{2}\sigma$.

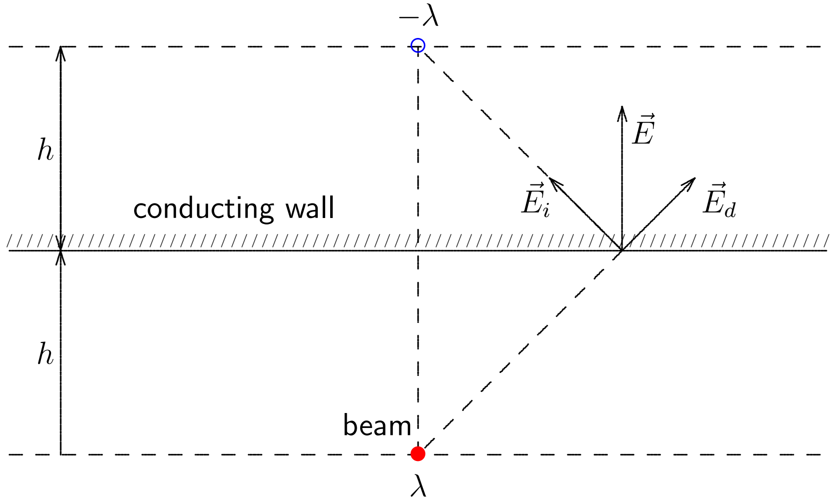
Indirect space-charge effect — influence of the chamber wall

Conducting boundary

The conducting vacuum chamber imposes a perpendicular electric field as boundary condition, $E_\parallel = 0$, on surface. This $E_\perp$-field induces surface charges. For a continuous beam they are static and don't represent a current. As chamber wall we use conducting horizontal plate at distance h from the beam line charge λ. To calculate the field we introduce an image charge $-\lambda$ at distance h behind wall which cancels $E_\parallel$ on the wall. Inside the vacuum chamber the field has the beam as its source and satisfies the wall boundary condition.







Field: direct $\vec{E}_d$, image $\vec{E}_i$, total $\vec{E}$ inside vacuum chamber:

$$\vec{E} = \vec{E}_d + \vec{E}_i$$

$$\text{div}\vec{E}_d = \frac{\eta}{\epsilon_0}, \quad \text{div}\vec{E}_i = 0, \quad \text{div}\vec{E} = \frac{\eta}{\epsilon_0}$$

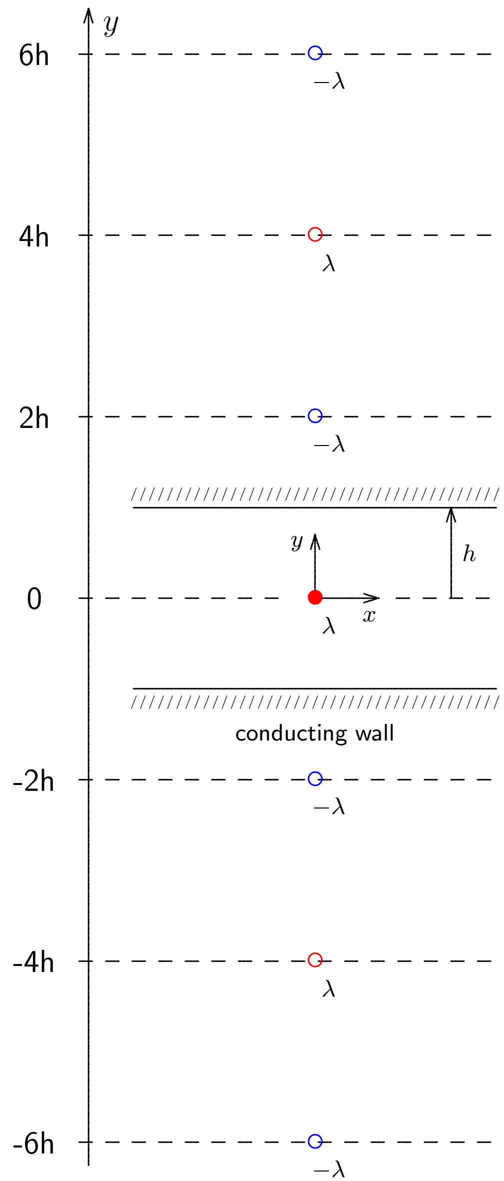
on the chamber wall:

$$E_{d\parallel} = -E_{i\parallel}\ , \ E_\parallel = 0.$$

Conducting vacuum chamber

The vacuum chamber represents two conducting boundaries at distances $\pm h$ from the beam. To satisfy $E_\parallel = 0$ on them we need not only an image charge of the beam behind the upper and lower boundaries but also secondary image charges of the primary images with alternating polarities. The fields due to each image line charge λ_{in} are calculated only close to the beam to first order of x and y (quadrupole field). Starting with the vertical field of the nth image pair at distance $\pm 2nh$ and summing over n

$$
\begin{aligned}
E_{iny} &= (-1)^n \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{2nh+y} - \frac{1}{2nh-y}\right) \\
&= (-1)^n \frac{\lambda}{2\pi\epsilon_0}\frac{-2y}{(2nh)^2} = -\frac{\lambda y}{4\pi\epsilon_0 h^2}\frac{(-1)^n}{n^2} \\
E_{iy} &= -\frac{\lambda y}{4\pi\epsilon_0 h^2}\sum_1^\infty \frac{(-1)^n}{n^2} = \frac{\lambda y}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12},
\end{aligned}
$$



We got the field of all charges

$$E_{iy} = -\frac{\lambda y}{4\pi\epsilon_0 h^2}\sum_1^\infty \frac{(-1)^n}{n^2} = \frac{\lambda y}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12},$$

The horizontal field is obtained from

$$\text{div}\vec{E}_i = \frac{\partial E_{ix}}{\partial x} + \frac{\partial E_{iy}}{\partial y} = 0$$

$$\frac{\partial E_{ix}}{\partial x} = -\frac{\partial E_{iy}}{\partial y} = -\frac{\lambda}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12},$$

or

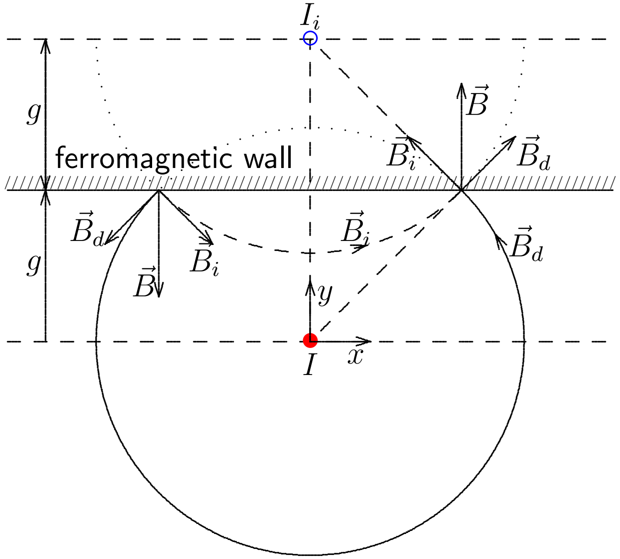
$$E_{i_x} = -\frac{\lambda y}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12}.$$

The boundary condition of the conducting plates does not affect the magnetic field and there is no relativistic compensation for the forces due to image charges. The total forces and tune shifts are, using $I = \lambda\beta c$.

$$\begin{aligned}
F_x &= \frac{2e\lambda y}{2\pi\epsilon_0}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2}\right) \\
F_y &= \frac{2e\lambda y}{2\pi\epsilon_0}\left(\frac{1}{2a^2\gamma^2} + \frac{\pi^2}{48h^2}\right) \\
\Delta Q_x &= -\frac{2r_0 IR\langle\beta_x\rangle}{ec\beta^3\gamma}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2}\right) \\
\Delta Q_y &= -\frac{2r_0 IR\langle\beta_y\rangle}{ec\beta^3\gamma}\left(\frac{1}{2a^2\gamma^2} + \frac{\pi^2}{48h^2}\right),
\end{aligned}$$

The first term, direct space-charge effect, decreases as $1/\gamma^2$ due to the cancelation between electric and magnetic forces and depends on the beam size. The second term, indirect effect of the wall, has no cancelation and does in our approximation not depend on beam size but on the distance h of the plate. Both decrease in addition as $1/\gamma$ due to rigidity at higher energy.

Ferromagnetic boundaries



Flat ferromagnetic magnet poles at distance g from beam I impose perpendicular magnetic fields, $B_\parallel = 0$ as boundary condition. A fictive parallel image current $I_i = I$ at distance g behind ads a field which cancels $B_\parallel$ on boundary and close to the beam, $x,\, y \ll g$ a field $B = \mu_0 I_i/(2\pi r)$

$$B_y = \frac{\mu_0 I}{2\pi(2g-y)} \approx \frac{\mu_0 I}{4\pi g}\left(1+\frac{y}{2g}\right).$$

Upper and lower walls give two primary and many secondary image currents with field

$$\begin{aligned} B_{ix} &= \frac{\mu_0 I}{2\pi}\sum_1^\infty\left(\frac{1}{2ng-y}-\frac{1}{2ng+y}\right) \\ &= \frac{\mu_0 I y}{4\pi g^2}\sum_1^\infty\frac{1}{n^2} = \frac{\mu_0 I y}{4\pi g^2}\frac{\pi^2}{6} \\ F_y &= \frac{2eIy}{2\pi\epsilon\beta c}\left(\frac{1}{2a^2\gamma^2}+\frac{\pi^2}{48h^2}+\frac{\pi^2\beta^2}{24g^2}\right) \\ \Delta Q_y &= -\frac{2r_0 IR\langle\beta_y\rangle}{ec\beta^3\gamma}\left(\frac{1}{2a^2\gamma^2}+\frac{\pi^2}{48h^2}+\frac{\pi^2}{24g^2}\right). \end{aligned}$$

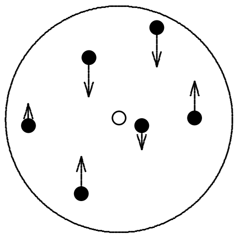
Vertical field and horizontal tune shift are

$$\begin{aligned} \mathrm{div}\vec{B} &= \frac{\partial B_y}{\partial x}+\frac{\partial B_x}{\partial y} = 0 \rightarrow B_{iy} = -\frac{\mu_0 I x}{4\pi g^2}\frac{\pi^2}{6} \\ \Delta Q_x &= -\frac{2r_0 IR\langle\beta_x\rangle}{ec\beta^3\gamma}\left(\frac{1}{2a^2\gamma^2}-\frac{\pi^2}{48h^2}-\frac{\pi^2\beta^2}{24g^2}\right). \end{aligned}$$

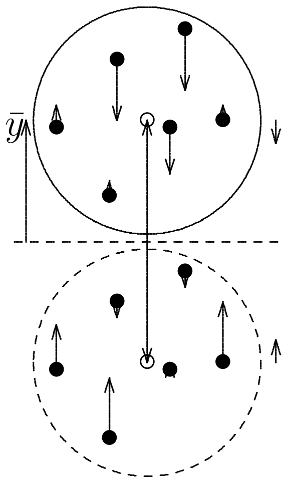
ΔQ has 3 terms: direct effects, indirect by electric boundary at distance h and magnetic boundary at distance g.

Incoherent and coherent motion

Direct space-charge effect



Individual particles move center-of-mass not: **incoherent** motion



Center of mass moves **coherent** motion

For an incoherent motion particles have a space-charge tune shift

$$\Delta Q_{\text{inc.}} = -\frac{r_0 I R \beta_y}{cea^2\beta^3\gamma^3}$$

For a coherent motion space-charge force is intern, moves with beam and does not affect center-of-mass motion $\Delta Q_{\text{coh.}} = 0$

Indirect space-charge effect

Space-charge field with a conducting wall at distance h was obtained by image line charge at h behind wall. A coherent beam motion by $\bar{y}$ moves first images to $\pm 2h - \bar{y}$ with a field at the beam

$$E_{c1y} = \frac{-\lambda}{2\pi\epsilon_0}\left(\frac{1}{2h+2\bar{y}} - \frac{1}{2h-2\bar{y}}\right).$$

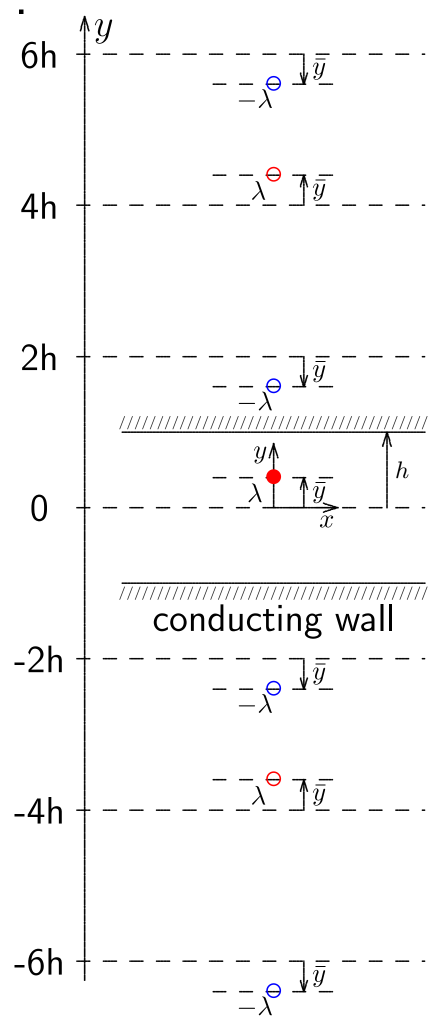
Equidistant 2nd images cancel, general

$$E_{cny} = -\frac{(-1)^n\lambda\bar{y}}{4\pi\epsilon_0 h^2}\left(\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right)$$

$$E_{cy} = \sum_1^\infty E_{cny} = \frac{\lambda\bar{y}}{4\pi\epsilon_0 h^2}\left(\frac{\pi^2}{12} + \frac{\pi^2}{6}\right)$$

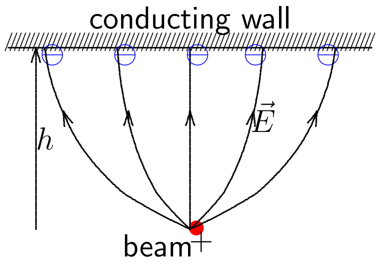
$$Q_{ycoh.} = Q_0 - \frac{\pi^2}{16}\frac{2r_0 I R\langle\beta_y\rangle}{ec\beta^3\gamma h^2}$$

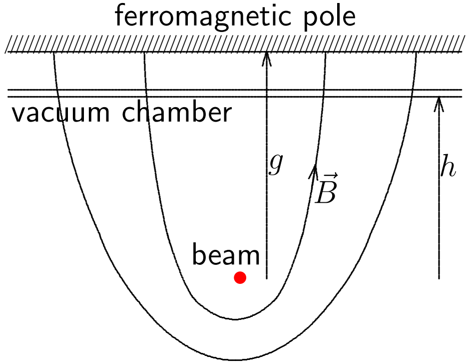
$$Q_{ycoh.} - Q_{yinc.} = \frac{2r_0 I R\langle\beta_y\rangle}{ec\beta^3\gamma}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{24h^2}\right).$$



Review and conclusions

Summary





Direct and indirect, incoherent and coherent ΔQ:

$$\Delta Q_{inc}^{x/y} = -\frac{2r_0 IR\langle\beta_{x/y}\rangle}{ec\beta^3\gamma}\left(\frac{1}{2a^2\gamma^2} + \epsilon_1^{x/y}\frac{1}{h^2} + \epsilon_2^{x/y}\frac{1}{g^2}\right)$$

$$\Delta Q_{coh}^{x/y} = -\frac{2r_0 IR\langle\beta_{x/y}\rangle}{ec\beta^3\gamma}\left(\quad 0 \quad + \xi_1^{x/y}\frac{1}{h^2} + \xi_2^{x/y}\frac{1}{g^2}\right)$$

direct elec. magn.

Laslett coefficients ϵ, ξ for a conducting and a ferromagnetic plate ad distances h and g:

$$\epsilon_1^y = \frac{\pi^2}{48}, \quad \epsilon_1^x = -\frac{\pi^2}{48}, \quad \epsilon_2^y = \frac{\pi^2}{24}, \quad \epsilon_2^x = -\frac{\pi^2}{24}, \quad \xi_1^y = \xi_2^y = \frac{\pi^2}{16}, \quad \xi_1^x = \xi_2^x = 0.$$

Direct space-charge fields:

Reduce transverse focusing and tune Q of individual particles. For an unbunched, circular beam of radius a and uniform density

$$\Delta Q_{x/y} = -\frac{r_0 IR\langle\beta_{x/y}\rangle}{ec\beta^3\gamma^3 a^2} = -\frac{r_0 RI}{ec\mathcal{E}_{x/y}\beta^3\gamma^3}.$$

Cancelation between E and B forces gives $1/\gamma^2$-factor and reduces ΔQ at high energy.

High intensity also changes β-function and beam size, leading to "space-charge dominated beams". ΔQ depends in bunched beam on longitudinal particle position and for non-uniform density on oscillation amplitude giving a tune spread.
Related: Neutralization, beam-beam effect.

Indirect space-charge effect

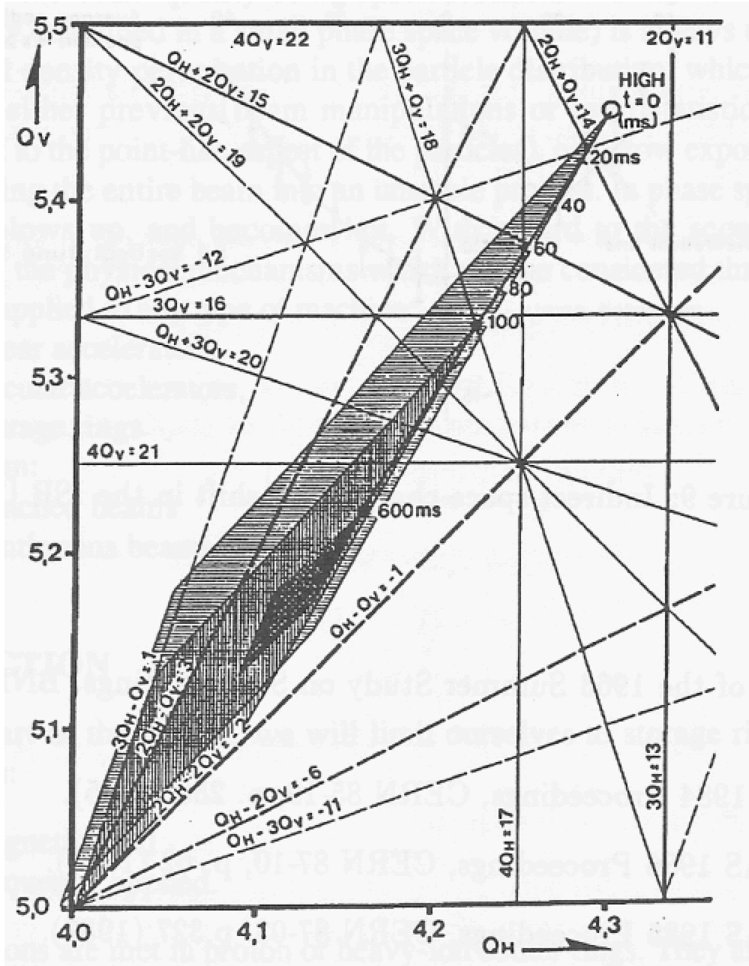
Conducting or ferromagnetic materials of the vacuum chamber modify the fields E and B making their forces different and perturbing their cancelation resulting in important tune shifts also at high beam energies.

Coherent effects

A betatron oscillation of the beam as a whole represents a coherent, or center-of-mass motion. This does not affect the direct space-charge effects based on internal forces. However, image charges and currents move and alter the force on the beam resulting in different shifts $\Delta Q_{inc} \neq \Delta Q_{coh}$, This can affect beam stability to be discussed later.

Problems caused by space-charge tune shifts

Optical imperfections can make the motion of a particle unstable if its tune is on a resonance, i.e. is a simple rational number N/M. With a space-charge tune spread and different coherent and incoherent tune shifts it may be difficult to avoid this for all particles



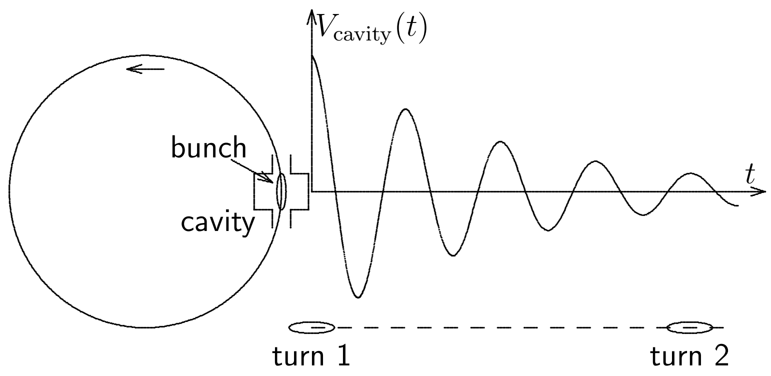
Direct space-charge tune shift, CERN booster, E. Brouzet, K.H. Schindl.

3) MECHANISMS OF INSTABILITIES

Many particles in a beam represent a charge and current and create electromagnetic fields (self fields) which induce charges in the beam surroundings. While for indirect space-charge effects a quasi-static approach could be used for continuous beams or long bunches in most cases high frequencies are involved and the electrical properties of the wall is described by an **impedance**. Beam induced currents create fields in it which act back on the beam leading to:

frequency shift (change of the betatron or synchrotron tunes), **change of the particle distribution**, (bunch lengthening) or an increase of an initial disturbance, **instability**, all due to a **collective** action by many particles.

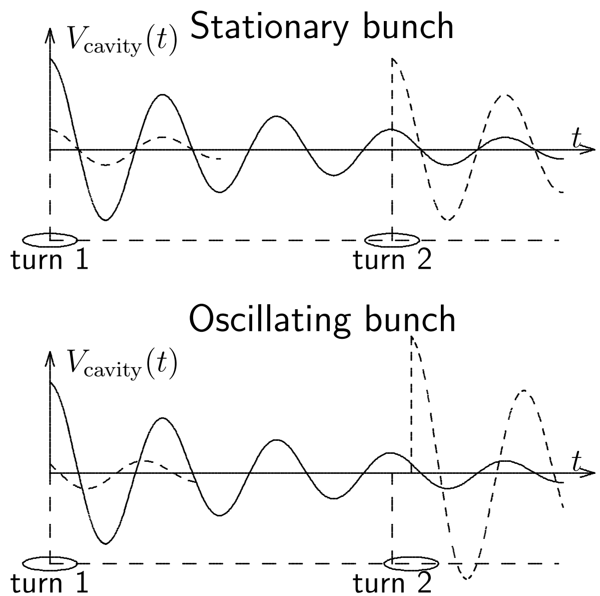
Example: a circulating bunch induces electromagnetic fields in a passive cavity which oscillate and slowly decay away. Next turn they affect the bunch and, depending on their phase, increase or decrease its initial perturbation.



Types of multi-particle effects

Performance reduction or beam loss .

Multi-particle effects can modify beam parameters, e.g. betatron or synchrotron frequency, particle distributions or lead to an energy loss which is compensated by the RF-system. These do not prevent machine operation but may reduce performance. However, others lead to a growing oscillation resulting in a beam loss, i.e. an instability.



Longitudinal and transverse effects

Longitudinal effects involve synchrotron (energy, phase) oscillations and longitudinal impedances. They lead to a shift of synchrotron frequencies, bunch lengthening and longitudinal instabilities,

Transverse effects involve betatron oscillations and transverse impedances. They shift betatron frequency and make transverse instabilities.

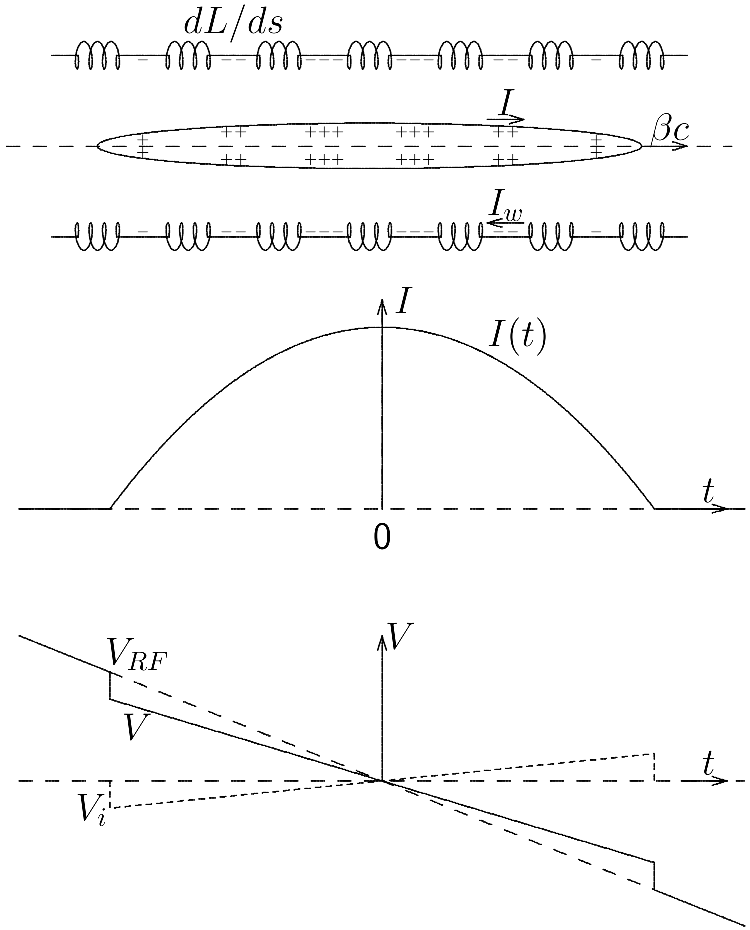
For both the longitudinal distribution (bunch length) is "resolved" by impedance and important while transverse distribution rarely matters.

Single traversal effects

Strong self-fields from broad-band impedances change the stationary distribution and modify oscillation modes which are no longer independent. A self consistent solution is difficult to obtain. The most common such effect is **bunch lengthening**. Small vacuum chamber aperture changes represent at low frequencies an inductive impedance ωL in which the bunch current $I(t)$ induces a voltage

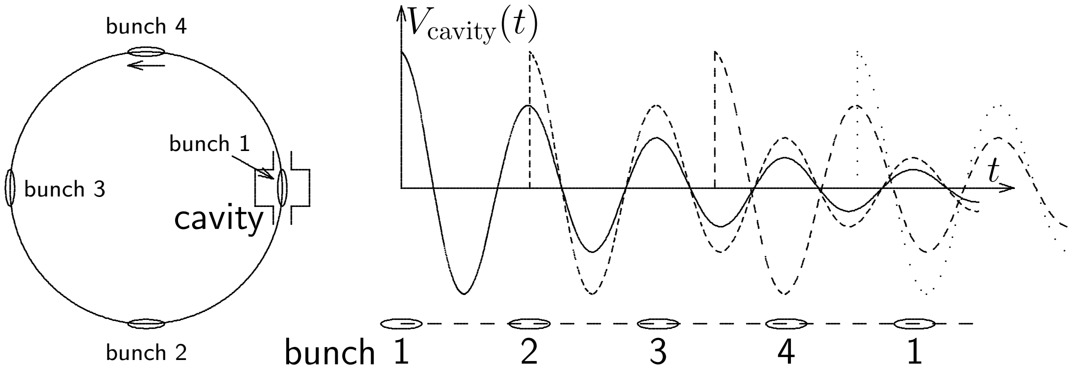
$$V_i(t) = -L\frac{dI}{dt}.$$

It is added to the external RF-voltage, reduces its slope and increases the bunch length, called potential well bunch lengthening.



Multi-bunch effects

With many circulating bunches, their individual oscillations can be coupled by an impedance with a shorter memory bridging just the bunch spacing instead of the revolution time. Multi-turn and multi-bunch instabilities have the same qualitative properties and are called multi-traversal effects. Cures: damp cavity modes, feed-back system.



Calculation methods

For self fields small compared to guide fields we use a **perturbation** in 3 steps.

a) We determine the stationary particle distribution given by the guide field, initial condition and synchrotron radiation.

b) We consider small disturbances and calculate the fields they create including the boundary conditions (impedance).

c) We calculate the effect of these fields to see if the initial disturbance is increased **(instability)** or decreased **(damping)** or the oscillation mode changed **(frequency shift)**.

Strong self-fields change the stationary distribution and modify oscillation modes which are no longer independent. A self consistent solution is difficult to obtain. The most common such effect is bunch lengthening.

4) IMPEDANCE, WAKE FUNCTION

Resonator

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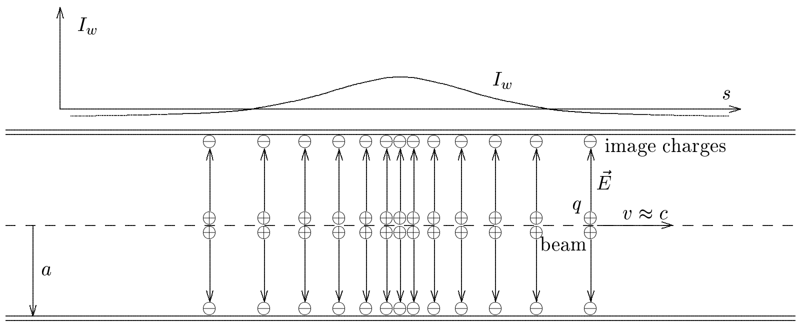
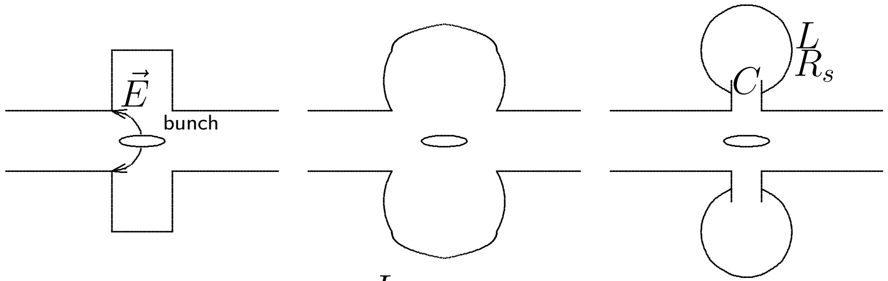
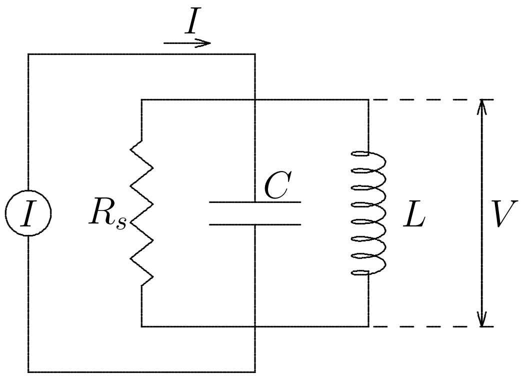


Beam induces wall current $I_w = -(I_b - \langle I_b \rangle)$



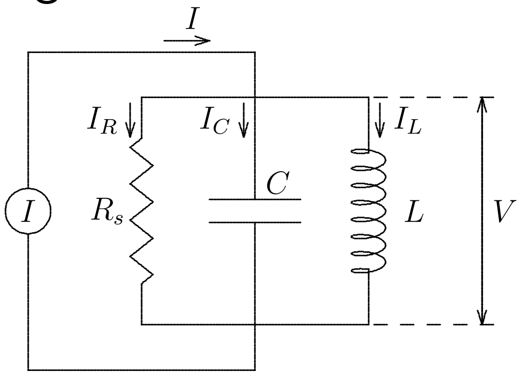


Cavities have narrow band oscillation modes which can drive coupled bunch instabilities. Each resembles an **RCL - circuit** and can, in good approximation, be treated as such. This circuit has a shunt impedance R_s, an inductance L and a capacity C. In a real cavity these parameters cannot easily be separated and we use others which can be measured directly: The **resonance frequency** ω_r, the **quality factor** Q and the **damping rate** α:

$$\omega_r = \frac{1}{\sqrt{LC}}\,, \quad Q = R_s\sqrt{\frac{C}{L}} = \frac{R_s}{L\omega_r} = R_s C \omega_r$$

$$\alpha = \frac{\omega_r}{2Q}\,, \quad L = \frac{R_s}{Q\omega_r}\,, \quad C = \frac{Q}{\omega_r R_s}.$$

Driving this circuit with a current I gives the voltages and currents across the elements



$$
\begin{aligned}
V_R &= I_R R_s \\
V_C &= \frac{1}{C}\int I_C dt \\
V_L &= L\frac{dI_L}{dt}
\end{aligned}
$$

$$V_R = V_C = V_L = V$$

$$I_R + I_C + I_L = I$$

Differentiating with respect to t gives

$$\dot{I} = \dot{I}_R + \dot{I}_C + \dot{I}_L = \frac{\dot{V}}{R_s} + C\ddot{V} + \frac{V}{L}.$$

Using $L = R_s/(\omega_r Q)$ and $C = Q/(\omega_r R_s)$ gives the differential equation

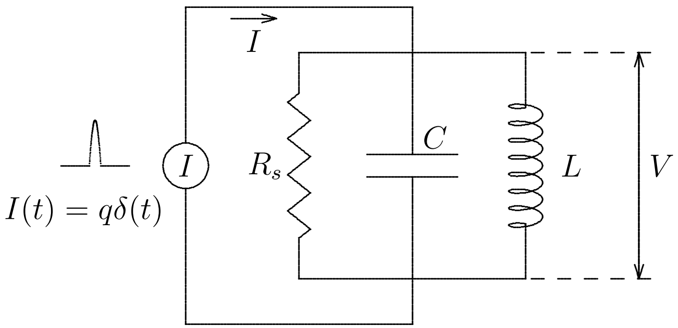
$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

The solution of the homogeneous equation represents a damped oscillation

$$
\begin{aligned}
V(t) &= \hat{V}\mathrm{e}^{-\alpha\mathrm{t}}\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\mathrm{t}+\phi\right) \\
V(t) &= \mathrm{e}^{-\alpha\mathrm{t}}\left(\mathrm{A}\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\mathrm{t}\right)\right. \\
&\qquad \left. +B\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right)\right)
\end{aligned}
$$

Wake function – Green function

Response of RCL circuit to a delta pulse



$$\omega_r = \frac{1}{\sqrt{LC}}$$
$$Q = R_s\sqrt{\frac{C}{L}}$$
$$\alpha = \frac{\omega_r}{2Q}$$

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

Charge q brings the capacity to a voltage

$$V(0^+) = \frac{q}{C} = \frac{\omega_r R_s}{Q} q \text{ using } C = \frac{Q}{\omega_r R_s}$$

Energy stored in C = energy lost by q

$$U = \frac{q^2}{2C} = \frac{\omega_r R_s}{2Q} q^2 = \frac{V(0^+)}{2} q = k_{pm} q^2$$

with the **parasitic mode loss factor** $k_{pm} = \omega_r R_s/(2Q)$, given usually in [V/pC].
Capacitor discharges first through resistor

$$\dot{V}(0^+) = -\frac{\dot{q}}{C} = -\frac{I_R}{C} = -\frac{1}{C}\frac{V(0^+)}{R_s}$$
$$= -\frac{\omega_r^2 R_s}{Q^2} q = -\frac{2\omega_r k_{pm}}{Q} q.$$

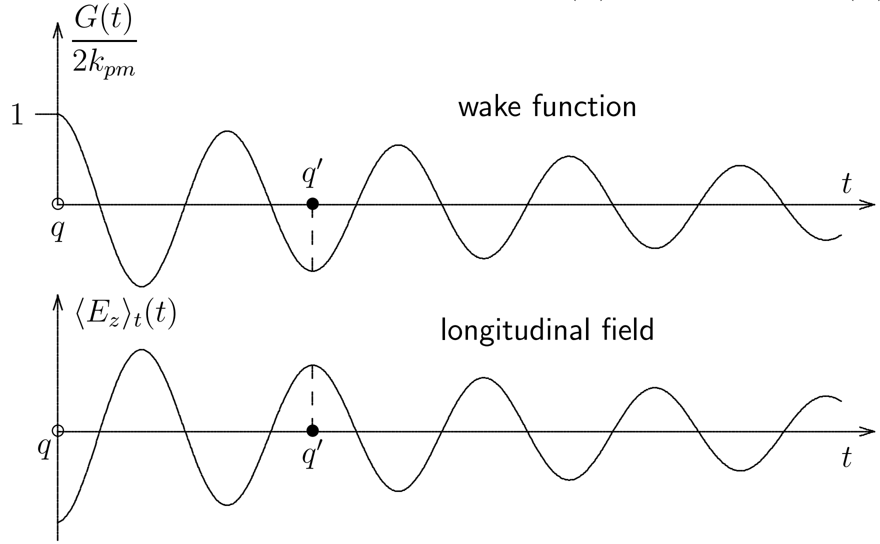
Initial conditions $V(0^+)$, $\dot{V}(0^+)$ give from

$$\text{general solution } V(t) = \mathrm{e}^{-\alpha \mathrm{t}}\left(\mathrm{A}\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) + \mathrm{B}\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)\right)$$

$$\text{pulse response } V(t) = 2qk_{pm}\mathrm{e}^{-\alpha \mathrm{t}}\left(\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) - \frac{\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)}{2\mathrm{Q}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}}\right)$$

$$G(t) = \frac{V(t)}{q} = 2k_{pm} e^{-\alpha t} \left(\cos\left(\omega_r \sqrt{1 - \frac{1}{4Q^2}}\, t \right) - \frac{\sin\left(\omega_r \sqrt{1 - \frac{1}{4Q^2}}\, t \right)}{2Q\sqrt{1 - \frac{1}{4Q^2}}} \right) , \ \omega_r = \frac{1}{\sqrt{LC}}$$

$G(t)$ is called **Green or wake function**.
$G(t) \approx 2k_{pm} e^{-\alpha t} \cos(\omega_r t)$ for $Q \gg 1$
This voltage induced by charge q at $t = 0$ changes energy of a second charge q' traversing cavity at t by $U = -q'V(t) = -qq'G(t)$.



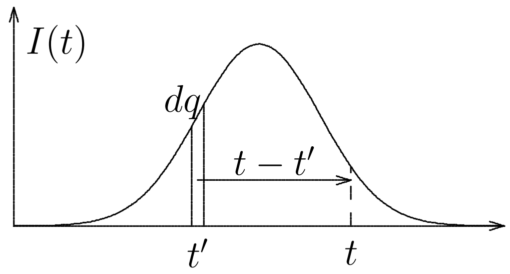
$G(t)$ is related to longitudinal field E_z by an integration following the particle with $v \approx c$ and taking momentary field value

$$V = Gq = -\int_{z_1}^{z_2} E_z(z,t) dz = -f_t \int_{z_1}^{z_2} E_z(z) dz.$$

with "transit time factor" f_t. We use $G(t) > 0$ where energy is lost.
A particle inside a bunch of charge q and current $I(t)$ going through a cavity at time t sees the wake function created by all the particles passing at earlier times $t' < t$ resulting in a voltage

.

$$V(t) = \int_{-\infty}^{t} G(t') dq = \int_{-\infty}^{t} I(t') G(t') dt' = qW(t)$$
$W(t) = V(t)/q$ wake potential .



Impedance

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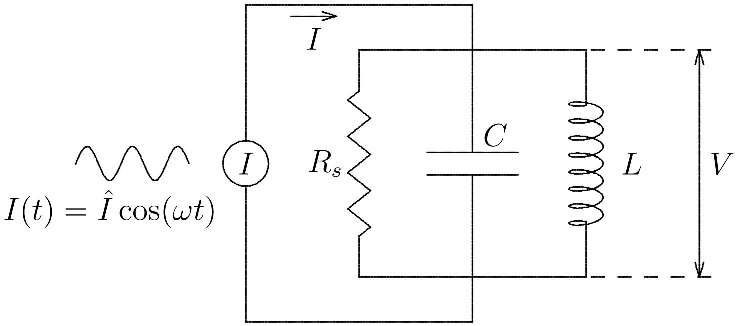


A **harmonic** excitation of circuit with current $I = \hat{I}\cos(\omega t)$ gives differential equation

$$\ddot{V}+\frac{\omega_r}{Q}\dot{V}+\omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I} = -\frac{\omega_r R_s}{Q}\hat{I}\omega\sin(\omega t).$$

Homogeneous solution damps leaving particular one $V(t) = A\cos(\omega t) + B\sin(\omega t)$. Put into diff-equation, separating cosine and sine

$$-(\omega^2-\omega_r^2)A + \frac{\omega_r\omega}{Q}B = 0$$

$$(\omega^2-\omega_r^2)B + \frac{\omega_r\omega}{Q}A = \frac{\omega_r\omega R_s}{Q}\hat{I}.$$

Induced voltage by the harmonic excitation

$$V(t) = \hat{I}R_s\frac{\cos(\omega t) + Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}\sin(\omega t)}{1+Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

has a cosine term **in phase** with exciting current. It absorbs energy, is **resistive**. The sine term is **out of phase**, does not absorb energy, **reactive**. Ratio between voltage and current is **impedance** as **function of frequency** ω

$$Z_r(\omega) = R_s\frac{1}{1+Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2}$$

$$Z_i(\omega) = -R_s\frac{Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1+Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}.$$

Resistive part $Z_r(\omega) \geq 0$, reactive part $Z_i(\omega)$ positive below, negative above ω_r.

Complex notation

We used a harmonic excitation of the form

$$I(t) = \hat{I}\cos(\omega t) = \hat{I}\frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\text{with } 0 \le \omega \le \infty.$$

It is convenient to use a complex notation

$$I(t) = \hat{I}e^{j\omega t} \quad \text{with} \quad -\infty \le \omega \le \infty$$

giving compact expressions. Using the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

with $I(t) = \hat{I}\exp(j\omega t)$ and seeking a solution $V(t) = V_0\exp(j\omega t)$, where V_0 is in general complex, one gets

$$\left(-\omega^2 + j\frac{\omega_r\omega}{Q} + \omega_r^2\right)V_0 e^{j\omega t} = j\frac{\omega_r\omega R_s}{Q}\hat{I}e^{j\omega t}.$$

The impedance, defined as the ratio V/I becomes

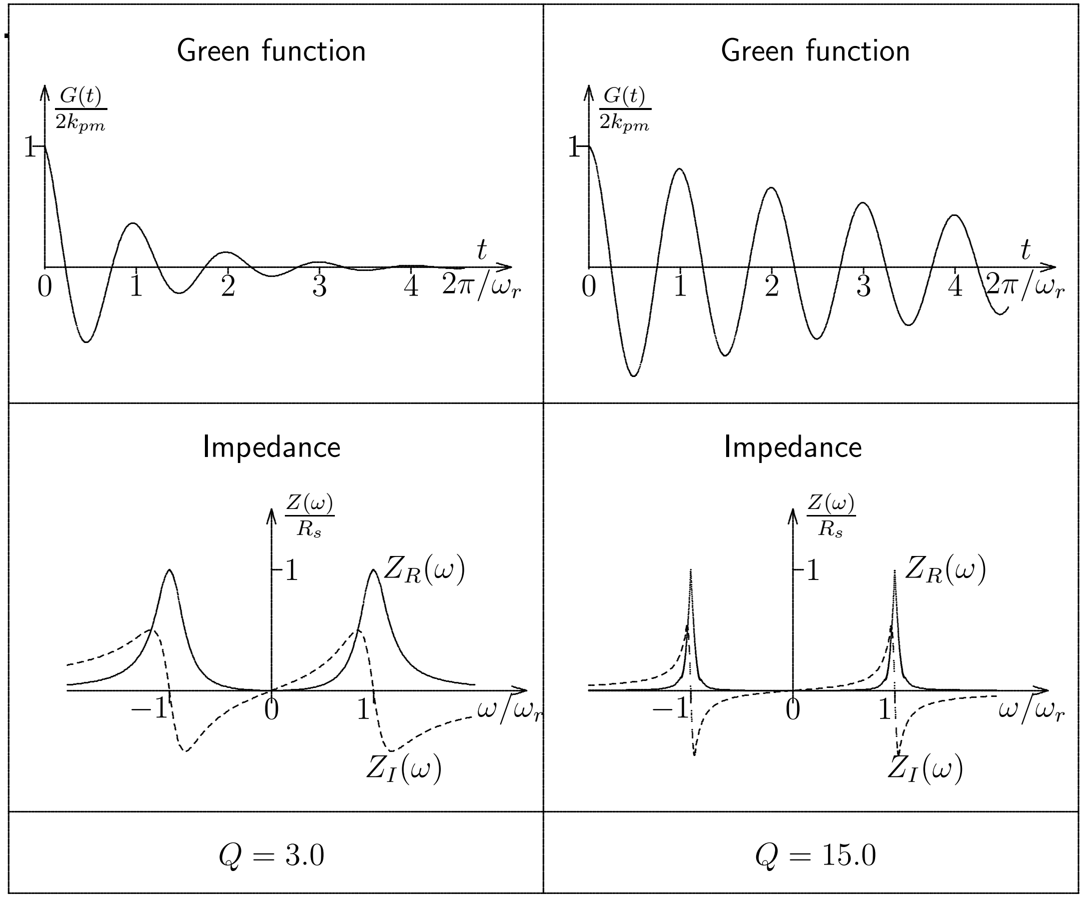
$$\begin{aligned} Z(\omega) &= \frac{V_0}{\hat{I}} = \frac{R_s}{1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)} \\ &= R_s\frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} = Z_r + jZ_i \end{aligned}$$

For $Q \gg 1$ the impedance is only large for $\omega \approx \omega_r$ or $|\omega - \omega_r|/\omega_r = |\Delta\omega|/\omega_r \ll 1$ and can be simplified

$$Z(\omega) \approx R_s\frac{1 - j2Q\frac{\Delta\omega}{\omega_r}}{1 + 4Q^2\left(\frac{\Delta\omega}{\omega_r}\right)^2}.$$

Caution: sometimes $I(t) = \hat{I}e^{-i\omega t}$ instead of $I(t) = \hat{I}e^{j\omega t}$ is used, this reverses the sign $Z_i(\omega)$.

Resonator and general Green function and impedances



$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} = Z_r + jZ_i$$

The resonator impedance has some specific properties:

$$\text{at}\,\omega = \omega_r \;\rightarrow\; Z_r(\omega_r)\ \text{max.},\ Z_i(\omega_r) = 0$$

$$0 < \omega < \omega_r \;\rightarrow\; Z_i(\omega) > 0\ \text{(inductive)}$$

$$\omega > \omega_r \;\rightarrow\; Z_i(\omega) < 0\ \text{(capacitive)}$$

and any impedance or wake potential has the general properties

$$Z_r(\omega) = Z_r(-\omega)\ \ ,\ \ Z_i(\omega) = -Z_i(-\omega)$$

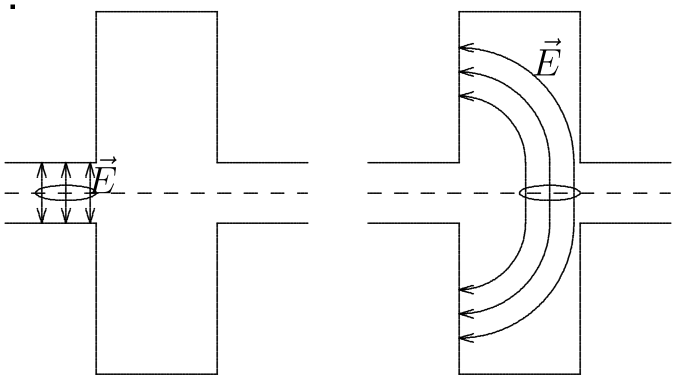
$$Z(\omega) = \int_{-\infty}^{\infty} G(t)\mathrm{e}^{-j\omega t} dt$$

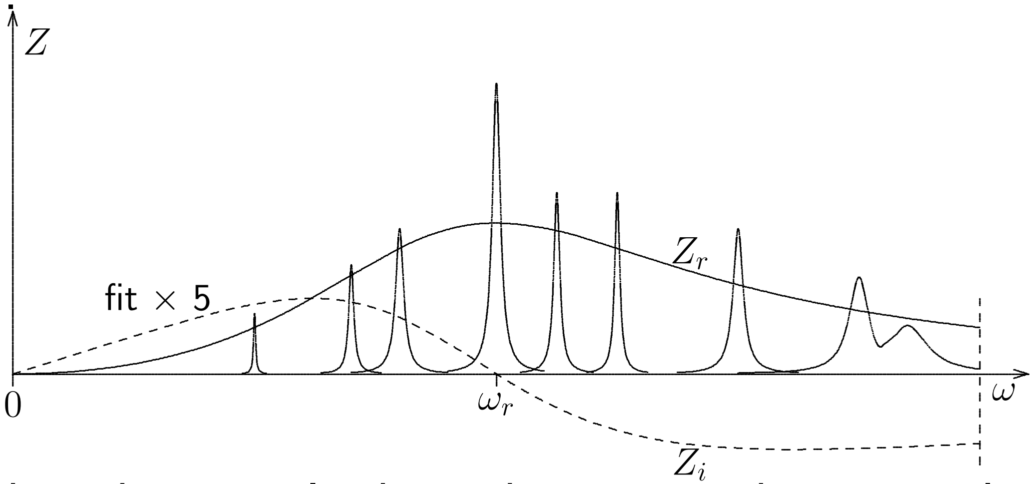
$Z(\omega) \propto$ Fourier transform of $G(t)$

$$\text{for}\, t < 0 \;\rightarrow\; G(t) = 0,$$

no fields before particle arrives, $\beta \approx 1$.

Typical impedance of a ring





Aperture changes form cavity-like objects with $\omega_r,\ R_s$ and Q and impedance $Z(\omega)$ developed for $\omega < \omega_r$, where it is inductive

$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1 + \left(Q\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} \approx j\frac{R_s\omega}{Q\omega_r} + ...$$

Sum impedance at $\omega \ll \omega_{rk}$ divided by mode number $n = \omega/\omega_0$ is with inductance L

$$\left|\frac{Z}{n}\right|_0 = \sum_k \frac{R_{sk}\omega_0}{Q_k\omega_{rk}} = L\omega_0 = L\frac{\beta c}{R}.$$

It depends on impedance per length, $\approx$ 15 Ω in older, 1 Ω in newer rings. The shunt impedances R_{sk} increase with ω up to cut-off frequency where wave propagation starts and become wider and smaller. A broad band resonator fit helps to characterize impedance giving $Z_r,\ Z_i,\ G(t)$ useful for single traversal effects. However, for multi-traversal instabilities narrow resonances at ω_{rk} must be used.

5) LONGITUDINAL DYNAMICS

A particle with momentum deviation Δp has different orbit length L, revolution time T_0 and frequency ω_0

$$\frac{\Delta L}{L} = \alpha_c \frac{\Delta p}{p} = \frac{\alpha_c}{\beta^2}\frac{\Delta E}{E}$$

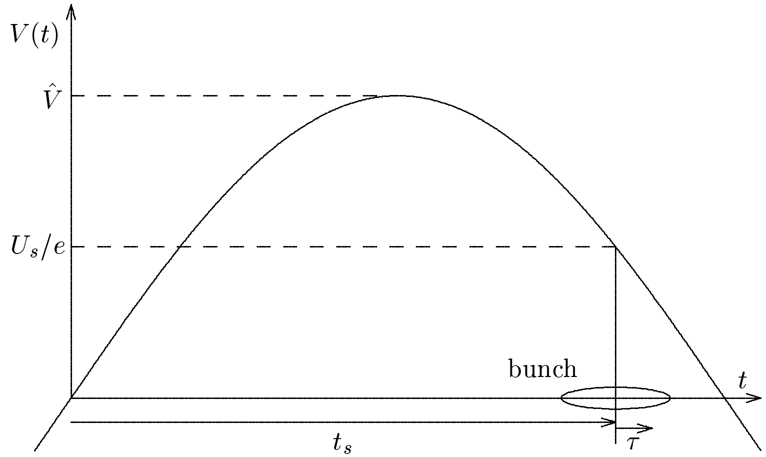
$$\frac{\Delta T}{T} = -\frac{\Delta\omega_0}{\omega_0} = \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} = \eta_c \frac{\Delta p}{p}$$

with momentum compaction $\alpha_c = 1/\gamma_T^2$, slip factor η_c. At transition energy $m_0 c^2 \gamma_T$ the ω_0-dependence on Δp changes sign

$$E > E_T \rightarrow \frac{1}{\gamma^2} < \alpha_c \rightarrow \eta_c > 1,\ \frac{\Delta\omega_0}{\Delta E} < 0$$

$$E < E_T \rightarrow \frac{1}{\gamma^2} > \alpha_c \rightarrow \eta_c < 1,\ \frac{\Delta\omega_0}{\Delta E} > 0\,.$$

For $\gamma \gg 1 \rightarrow \Delta p/p \approx \Delta E/E = \epsilon$, $\eta_c \approx \alpha_c$.



RF-cavity of voltage $\hat{V}$, frequency $\omega_{\text{RF}} = h\omega_0$, SR energy loss U the energy gain or loss of a particle in one turn $\delta\epsilon = \delta E/E$ is

$$\delta E = e\hat{V}\sin(h\omega_0(t_s + \tau)) - U$$

t_s= synchronous arrival time at the cavity, τ= deviation from it, synchronous phase $\phi_s = h\omega_0 t_s$. For $h\omega_0\tau \ll 1$ we develop

$$\delta E = e\hat{V}\sin(\phi_s) + h\omega_0 e\hat{V}\cos\phi_s \tau - U.$$

For $\delta E/E \ll 1$ use smooth approximation $\dot{E} \approx \delta E/T_0$, $\dot{\tau} = \Delta T/T_0 = \eta_c \Delta E/E$

$$\dot{E} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi}\tau - \frac{\omega_0}{2\pi}U.$$

Use $T_0 = 2\pi/\omega_0$, relative energy $\epsilon = \Delta E/E$

$$\dot{\epsilon} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U}{E}.$$

Energy loss U may depend on ϵ and τ

$$U(\epsilon,\tau) \approx U_0 + \frac{\partial U}{\partial E}\Delta E + \frac{\partial U}{\partial t}\tau$$

giving for the derivative of the energy loss

$$\begin{aligned}\dot{\epsilon} &= \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{\partial U}{\partial E}\epsilon - \frac{\omega_0}{2\pi}\frac{\partial U}{\partial t}\tau \\ \dot{\tau} &= \eta_c\epsilon\end{aligned}$$

where we used that for synchronous particle $\epsilon = 0\,,\ \tau = 0$ we have $U_0 = e\hat{V}\sin\phi_s$

Combining these into a second order equation

$$\ddot{\tau} + \frac{\omega_0}{2\pi}\frac{\partial U}{\partial E}\dot{\tau} + \left(\omega_{s0}^2 - \frac{\omega_0\eta_c}{2\pi E}\frac{\partial U}{\partial t}\right)\tau = 0,$$

$$\omega_{s0}^2 = \frac{-\omega_0^2 h\eta_c e\hat{V}\cos\phi_s}{2\pi E}\ ,\ \alpha_s = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{\partial U}{\partial E}$$

$$\omega_{s1}^2 = \omega_{s0}^2 - \alpha_s^2 - \frac{\omega_0\eta_c}{2\pi E}\frac{\partial U}{\partial t}$$

$$\ddot{\tau} + 2\alpha_s\dot{\tau} + \omega_{s0}^2\tau = 0$$

$$\tau = \hat{\tau}\mathrm{e}^{-\alpha_s t}\cos(\omega_{s1}t)\ ,\ \epsilon = \hat{\epsilon}\mathrm{e}^{-\alpha_s t}\sin(\omega_{s1}t)$$

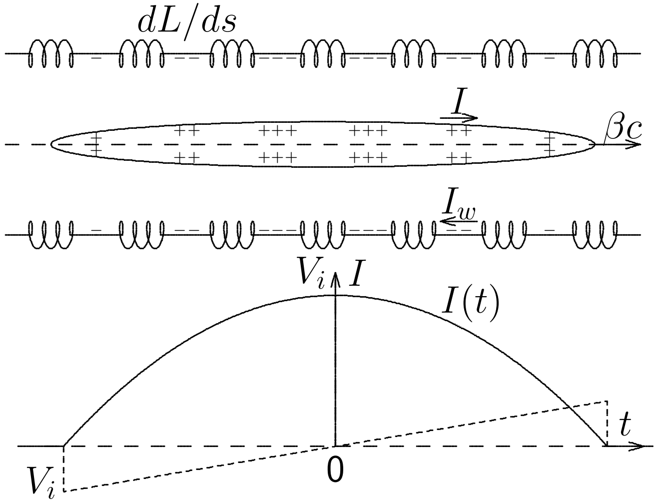
From $\dot{\tau} = \eta_c\epsilon$ we get $\hat{\epsilon} = \omega_{s0}\hat{\tau}/\eta_c$.

To get real ω_{s0} we need $\cos\phi_s \le 0$ above transition where $\eta_c > 0$ and vice versa.

To get a stable (decaying) solution we need an energy loss which increases with E

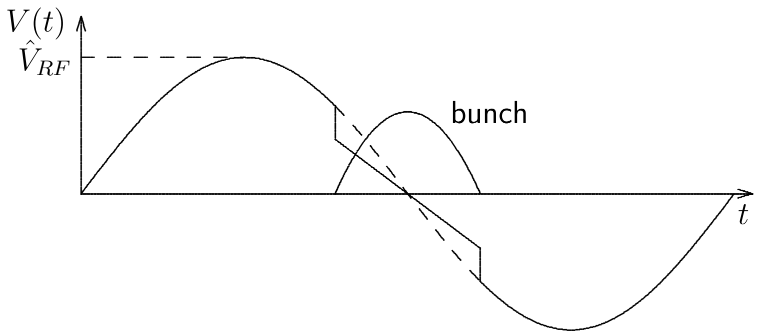
$$\alpha_s = \frac{\omega_0}{4\pi}\frac{\partial U}{\partial E} = \frac{\omega_0}{4\pi E}\frac{\partial U}{\partial \epsilon} > 0.$$

6) POTENTIAL WELL BUNCH LENGTHENING



$$E_z \;=\; -\frac{dL}{dz}\frac{dI_w}{dt} = \frac{dL}{dz}\frac{dI_b}{dt}$$

$$V \;=\; -\int E_z dz = -L\frac{dI_b}{dz}$$



We take a parabolic bunch form

$$I_b(\tau) \;=\; \hat{I}\left(1 - \frac{\tau^2}{\hat{\tau}^2}\right) = \frac{3\pi I_0}{2\omega_0\hat{\tau}}\left(1 - \frac{\tau^2}{\hat{\tau}^2}\right)$$

$$\frac{dI_b}{d\tau} \;=\; -\frac{3\pi I_0 \tau}{\omega_0\hat{\tau}^3},\;\; I_0 = \langle I_b \rangle,$$

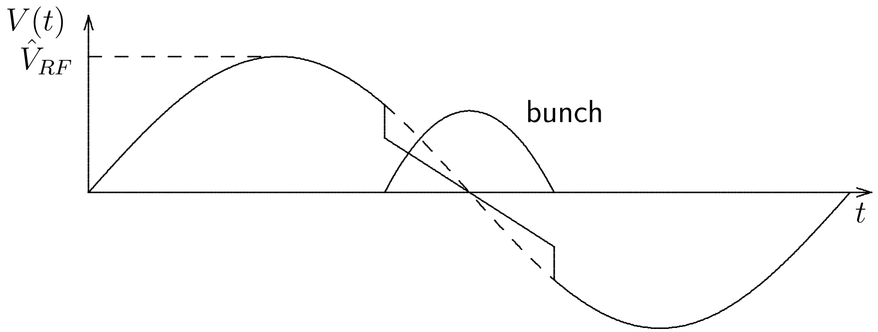
$$V \;=\; \hat{V}(\sin\phi_s + h\omega_0\cos\phi_s\tau) + \frac{3\pi I_0 L\tau}{\omega_0\hat{\tau}^3}$$

$$V \;=\; \hat{V}\left[\sin\phi_s + \cos\phi_s h\omega_0\left(1 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\hat{\tau})^3}\right)\tau\right]$$

$$\omega_{s0}^2 \;=\; -\frac{\omega_0^2 h\eta_c e\hat{V}\cos\phi_s}{2\pi E}$$

$$\omega_s^2 \;=\; \omega_{s0}^2\left[1 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau})^3}\right]$$

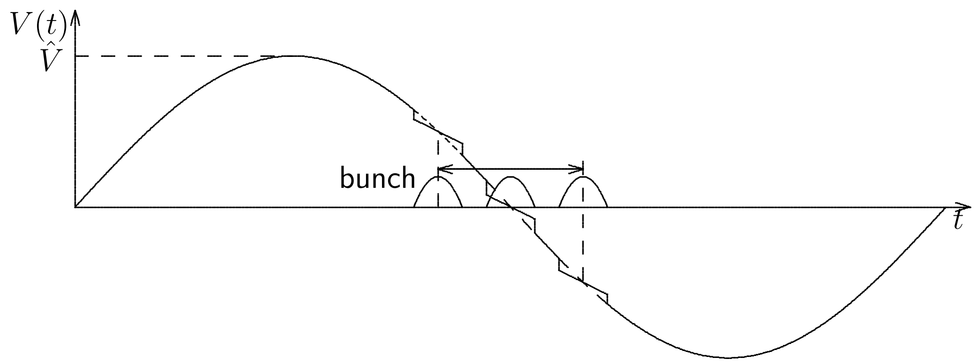
$$\frac{\Delta\omega_s}{\omega_0} \;=\; \frac{\omega_s - \omega_{s0}}{\omega_{s0}} \approx \frac{3\pi|Z/n|_0 I_0}{2h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$$



$$\frac{\omega_s^2}{\omega_{s0}^2} = 1 + \frac{3\pi |Z/n|_0 I_0}{h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau})^3}$$

$$\frac{\omega_s - \omega_{s0}}{\omega_{s0}} = \frac{\Delta\omega_s}{\omega_s} \approx \frac{3\pi |Z/n|_0 I_0}{2h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$$

Only incoherent frequency of single particles is changed (reduced for $\gamma > \gamma_T$, increased for $\gamma < \gamma_T$), but not the coherent dipole (rigid bunch) mode. This separates the two.



Reduction of ω_s reduces longitudinal focusing and increases the bunch length

$$\hat{\tau} = \hat{\epsilon}\eta_c/\omega_s\ ,\ \hat{\tau}^2 = \hat{\tau}\hat{\epsilon}\eta_c/\omega_s = \mathcal{E}_s\eta_c/\omega_s$$

rel. energy spread $\hat{\epsilon}$, long. emitt. $\mathcal{E}_s = \hat{\tau}\hat{\epsilon}$

Protons: $\mathcal{E}_s$= constant, $\tau \propto 1/\sqrt{\omega_s}$

small: $\dfrac{\Delta\hat{\tau}}{\hat{\tau}_0} \approx -\dfrac{\Delta\omega_s}{2\omega_{s0}} \approx -\dfrac{3\pi |Z/n|_0 I_0}{4h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$,

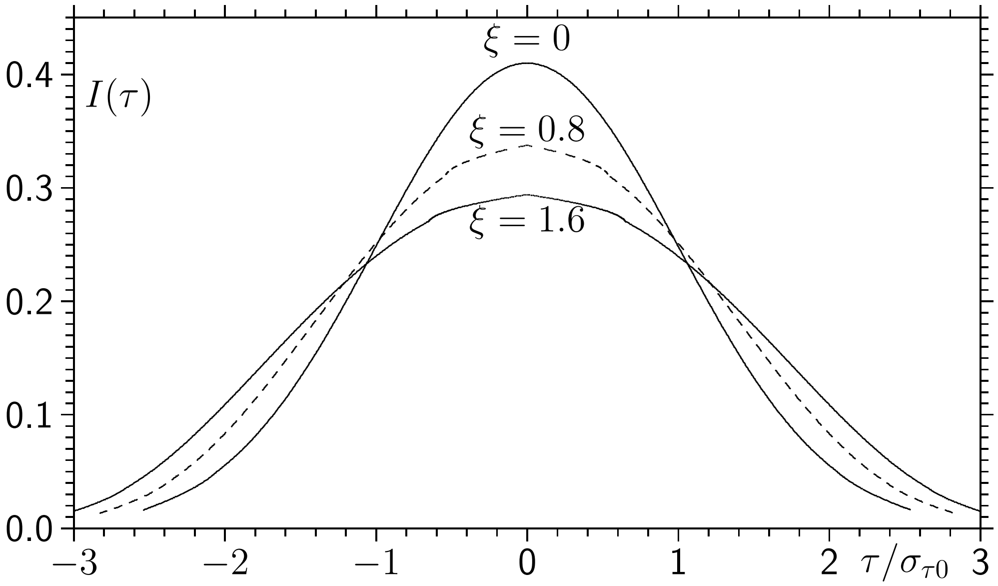
or: $\left(\dfrac{\hat{\tau}}{\hat{\tau}_0}\right)^4 + \dfrac{3\pi |Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}\left(\dfrac{\hat{\tau}}{\hat{\tau}_0}\right) - 1 = 0$

Electrons: $\hat{\epsilon}$= const. by syn. rad. $\hat{\tau} \propto 1/\omega_s$

small: $\dfrac{\Delta\hat{\tau}}{\hat{\tau}_0} \approx -\dfrac{\Delta\omega_s}{\omega_{s0}} \approx -\dfrac{3\pi |Z/n|_0 I_0}{2h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$,

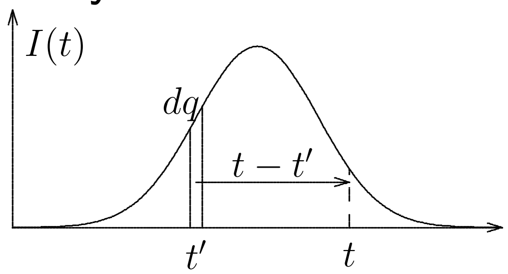
or: $\left(\dfrac{\hat{\tau}}{\hat{\tau}_0}\right)^3 - \dfrac{\hat{\tau}}{\hat{\tau}_0} + \dfrac{3\pi |Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3} = 0$

The parabolic bunch current is the projection of an elliptic phase space distribution. In this case the bunch form is not changed just its length increased. This is more complicated for other distribution like for the Gaussian shown in the figure.



$$\xi = \frac{\sqrt{2\pi} h^2 I_0 |Z/n|_0}{\hat{V} \cos\phi (h\omega_0 \sigma_{\tau 0})^3}$$

We calculated the potential well bunch lengthening in time domain using actually the wake function $G(t)$.



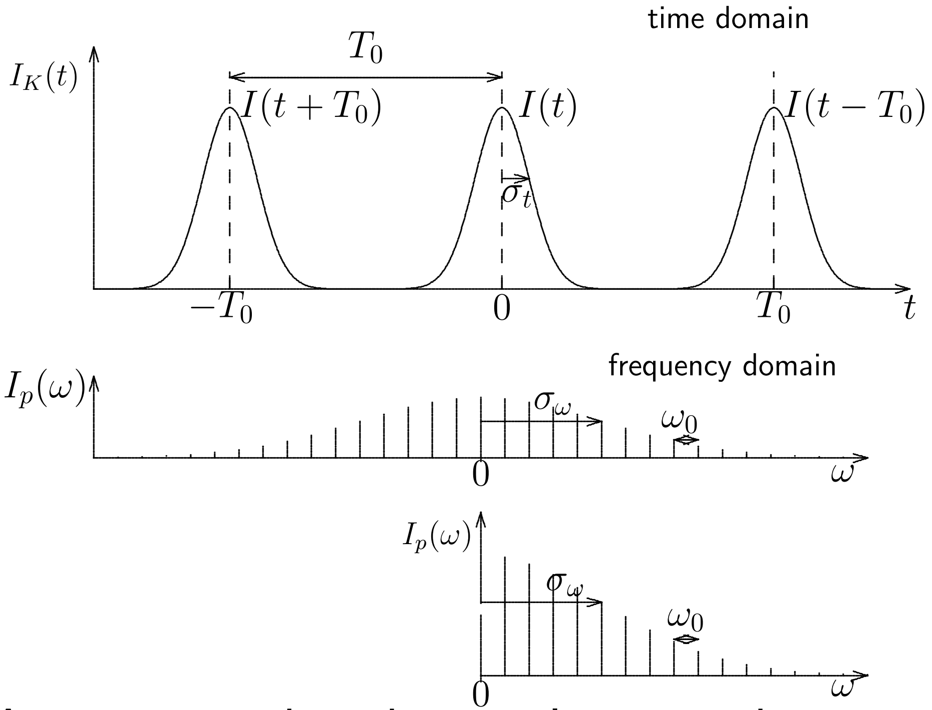
$$V(t) = \int_{-\infty}^{t+} I(t') G(t') dt' = L \frac{dI}{dt}$$
$$G(t) = L \dot{\delta}(t).$$

The wake function is the inductance times the derivative of the δ-function.

7) ROBINSON INSTABILITY

Stationary bunch

Spectrum



A symmetric bunch, circulating with turns k of duration T_0 represents a periodic current and is expressed by a Fourier series

$$
\begin{aligned}
I_K(t) &= \sum_{k=-\infty}^{k=\infty} I(t-kT_0) = \sum_{-\infty}^{\infty} I_p \mathrm{e}^{jp\omega_0 t} \\
&= I_0 + 2\sum_{p=1}^{\infty} I_p \cos(p\omega_0 t) \\
I_p &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} I(t)\mathrm{e}^{jp\omega_0 t} \\
&= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} I(t)\cos(p\omega_0 t) = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(p\omega_0).
\end{aligned}
$$

Assume symmetry $I(t) = I(-t)$, real I_p, cosine terms, at low frequencies $I_p \approx I_0$

Gaussian bunch:

$$
I(t) = \frac{q}{\sqrt{2\pi}\sigma_t}\mathrm{e}^{-\frac{t^2}{2\sigma_t^2}}\ ,\ I_p = \frac{q}{T_0}e^{\frac{-p^2\omega_0^2}{2\sigma_\omega^2}}\ ,\ \sigma_\omega = \frac{1}{\sigma_t}.
$$

Voltage induced by a stationary bunch

In impedance $Z(\omega) = Z_r(\omega) + jZ_i(\omega)$ the stationary bunch induces voltage

$$V_K(t) = \sum_{p=-\infty}^{\infty} Z(p\omega_0) I_p \mathrm{e}^{jp\omega_0 t}.$$

Combining positive and negative frequencies, using $Z_r(-\omega) = Z_r(\omega)$, $Z_i(-\omega) = -Z_i(\omega)$ and $Z(0) = 0$ we get

$$V_K(t) = 2 \sum_{p=1}^{\infty} I_p \left[Z_r(p\omega_0) \cos(p\omega_0 t) - Z_i(p\omega_0) \sin(p\omega_0 t) \right].$$

Energy loss of a stationary bunch

Energy lost by the whole bunch with N_b particles per turn in impedance $Z(\omega)$ is

$$W_b = \int_0^{T_0} I_K(t) V_K(t) dt.$$

This contains expressions and integrals

$$I_K(t) V_K(t) = \sum_{p=-\infty}^{\infty} Z(p\omega_0) I_p \mathrm{e}^{jp\omega_0 t}.$$

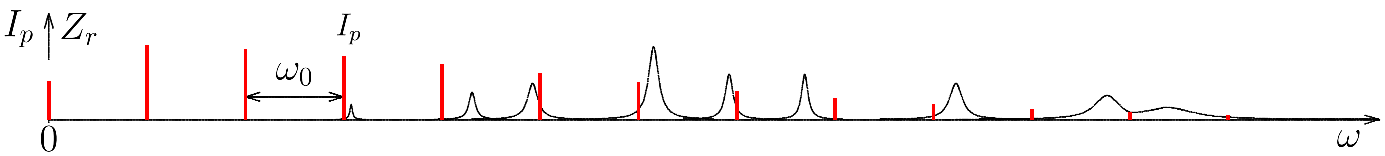
$$\int_0^{T_0} \cos(p'\omega_0 t) \sin(p\omega_0 t) dt = 0.$$

$$\int_0^{T_0} \cos(p'\omega_0 t) \cos(p\omega_0 t) dt = \begin{cases} \frac{T_0}{2} & \text{for } p' = p \\ 0 & \text{for } p' \neq p \end{cases}$$

$$W_b = T_0 \sum_{p=-\infty}^{\infty} I_p^2 Z(p\omega_0) = 2T_0 \sum_1^{\infty} I_p^2 Z_r(p\omega_0)$$

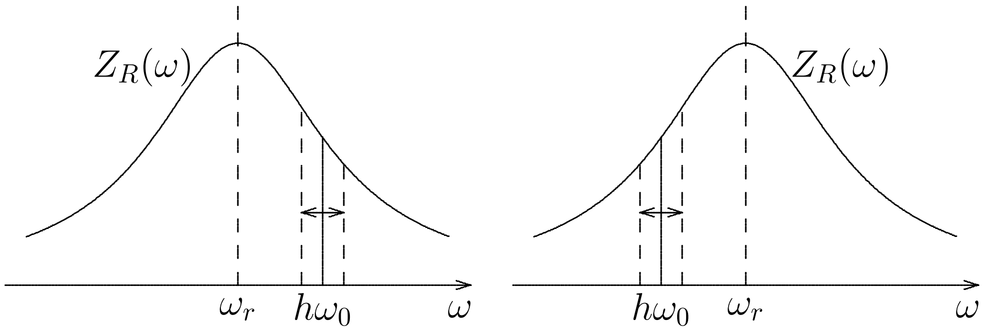
has only Z_r. Loss $U = W_b/N_b$ per particle is

$$U = \frac{2T_0}{N_b} \sum_1^{\infty} I_p^2 Z_r(p\omega_0) = \frac{2e}{I_0} \sum_1^{\infty} I_p^2 Z_r(p\omega_0).$$



Robinson instability

Qualitative treatment

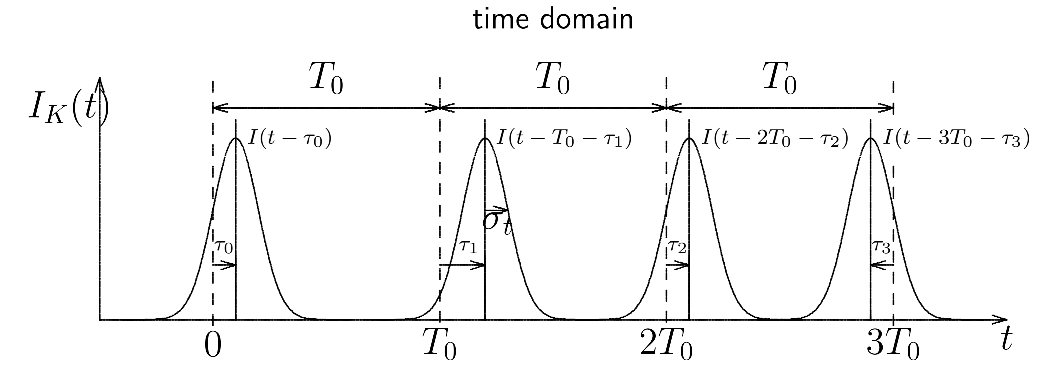


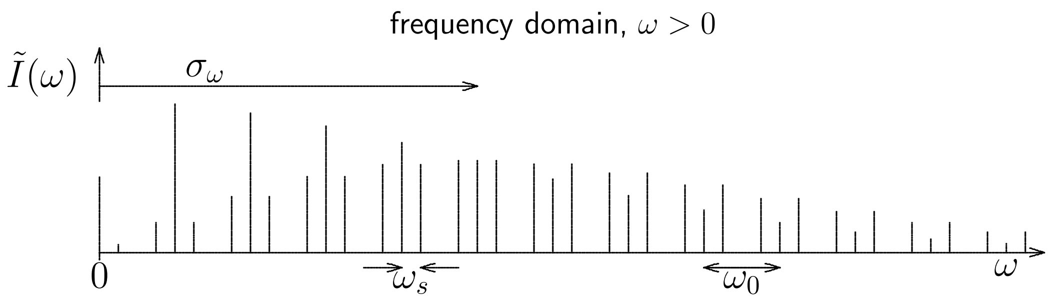
Important longitudinal instability of a bunch interacting with an narrow impedance, called **Robinson** instability. In a qualitative approach we take single bunch and a narrow-band cavity of resonance frequency ω_r and impedance $Z(\omega)$ taking only its resistive part Z_r. The revolution frequency ω_0 depends on energy deviation ΔE

$$\frac{\Delta\omega_0}{\omega_0} = -\eta_c \frac{\Delta E}{E}.$$

While the bunch is executing a coherent dipole mode oscillation $\epsilon(t) = \hat{\epsilon}\cos(\omega_s t)$ its energy and revolution frequency are modulated. **Above transition** ω_0 is **small** when the **energy is high** and ω_0 is **large** when the **energy is small**. If the cavity is tuned to a resonant frequency slightly smaller than the RF-frequency $\omega_r < p\omega_0$ the bunch sees a higher impedance and **loses more energy** when it has an **energy excess** and it **loses less energy** when it has a **lack of energy**. This leads to a **damping** of the oscillation. If $\omega_r > p\omega_0$ this is reversed and leads to an **instability**. Below transition energy the dependence of the revolution frequency is reversed which changes the stability criterion.

Oscillating bunch





Bunch executing synchrotron oscillation with $\omega_s = \omega_0 Q_s$ and amplitude $\hat{\tau}$ modulates passage time t_k at cavity in successive turns k

$$I_K(t) = \sum_{k=-\infty}^{\infty} I(t - kT_0 - \tau_k)$$

with $\tau_k = \hat{\tau}\cos(2\pi Q_s k) \approx \hat{\tau}\cos(\omega_s t)$

giving current without DC-part

$$I_K(t) = 2\sum_{\omega>0} I_p \cos(p\omega_0(t - \hat{\tau}\cos(\omega_s t))).$$

We develop for $p\omega_0\hat{\tau} \ll 1$

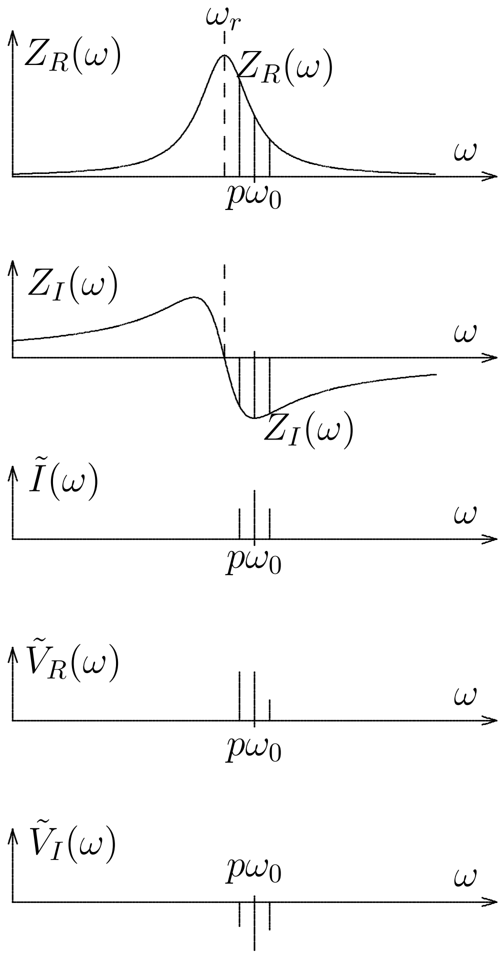
$$\begin{aligned} I_K(t) &\approx 2\sum_{\omega>0} I_p \left[\cos(p\omega_0 t) + p\omega_0\hat{\tau}\sin(p\omega_0 t)\cos(\omega_s t)\right] \\ &= 2\sum_{\omega>0} I_p \left[\cos(p\omega_0 t) + \frac{p\omega_0\hat{\tau}}{2}(\sin((p+Q_s)\omega_0 t) + \sin((p-Q_s)\omega_0 t))\right]. \end{aligned}$$

The modulation by the synchrotron oscillation results in sidebands in the spectrum.
They are out of phase with respect to carriers and increase first with frequency $p\omega_0$.

Voltage induced by oscillating bunch

Abbreviate: $\omega_p^+ = (p + Q_s)\omega_0$, $\omega_p^- = (p - Q_s)\omega_0$

$$I_K(t) = 2 \sum_{\omega>0} I_p \left[\cos(p\omega_0 t) + \frac{p\omega_0\hat{\tau}}{2}(\sin(\omega_p^+ t) + \sin(\omega_p^- t))\right].$$



We restrict on resistive impedance Z_r and get voltage

$$\begin{aligned} V_{Kr}(t) \;&=\; 2 \sum_{\omega>0}^{\infty} I_p \left[Z_r(p\omega_0) \cos(p\omega_0 t) \right. \\ &+\; \left. \frac{p\omega_0\hat{\tau}}{2} \left(Z_r(\omega_p^+) \sin(\omega_p^+ t) + Z_r(\omega_p^-) \sin(\omega_p^- t) \right) \right] \end{aligned}$$

$$\begin{aligned} V_{Kr}(t) &= 2 \sum_{\omega>0}^{\infty} I_p \left[Z_r(p\omega_0) \cos(p\omega_0 t) \right. \\ &+ \frac{p\omega_0\hat{\tau}}{2} \left[Z_r(\omega_p^+) \left(\sin(p\omega_0 t) \cos(\omega_s t) + \cos(p\omega_0 t) \sin(\omega_s t) \right) \right. \\ &\left. \left. + Z_r(\omega_p^-) \left(\sin(p\omega_0 t) \cos(\omega_s t) - \cos(p\omega_0 t) \sin(\omega_s t) \right) \right] \right] \end{aligned}$$

$\cdots$ $\cdots$

Synchr. motion: $\tau_k = \hat{\tau} \cos(2\pi Q_s k) \;\rightarrow\; \tau = \hat{\tau} \cos(\omega_s t)$

$$\begin{aligned} V_{kr}(t) \;&=\; 2 \sum_{\omega>0}^{\infty} I_p \left[Z_r(p\omega_0) \cos(p\omega_0 t) \right. \\ &+\; \left. \frac{p\omega_0}{2} \left[Z_r(\omega_p^+) \left(\sin(p\omega_0 t)\tau - \cos(p\omega_0 t) \frac{\dot{\tau}}{\omega_s} \right) + Z_r(\omega_p^-) \left(\sin(p\omega_0 t)\tau + \cos(p\omega_0 t) \frac{\dot{\tau}}{\omega_s} \right) \right] \right] \end{aligned}$$

Energy exchange

The energy per particle and turn exchanged between bunch and impedance

$$U(\tau,\dot\tau)=\frac{1}{N_b}\int_0^{T_0} I_K(t)V_K(t)dt,\ \ N_b=\frac{2\pi I_0}{e\omega_0}$$

Express factors differently, use $\dot\tau=\eta_c\epsilon$

$$I_K(t)=2\sum_{\omega>0} I_p\left[\cos(p'\omega_0 t)+p'\omega_0\sin(p'\omega_0 t)\tau\right]$$

$$\begin{aligned}V_{kr}(t)=&2\sum_{\omega>0}^{\infty} I_p\left[Z_r(p\omega_0)\cos(p\omega_0 t)\right.\\ &+\frac{p\omega_0}{2}\left[(Z_r(\omega_p^+)+Z_r(\omega_p^-))\sin(p\omega_0 t)\tau\right.\\ &\left.\left.-(Z_r(\omega_p^+)-Z_r(\omega_p^-))\cos(p\omega_0 t)\frac{\eta\epsilon}{\omega_s}\right]\right]\end{aligned}$$

Neglect higher terms in $\tau,\dot\tau$, use integrals

$$\int_0^{T_0}\cos(p'\omega_0 t)\cos(p\omega_0 t)dt=\begin{cases}\frac{T_0}{2} & \text{if } p'=p\\ 0 & \text{if } p'\neq p\end{cases}$$

$$\int_0^{T_0}\cos(p'\omega_0 t)\sin(p\omega_0 t)dt=0.$$

$$\begin{aligned}U=&\frac{2e}{I_0}\sum_{\omega>0}^{\infty} I_p^2 Z_r(p\omega_0)\\ &-\frac{e}{I_0}\sum_{\omega>0}^{\infty} I_p^2 p\omega_0(Z_r(\omega_p^+)-Z_r(\omega_p^-))\frac{\eta\epsilon}{\omega_s}\end{aligned}$$

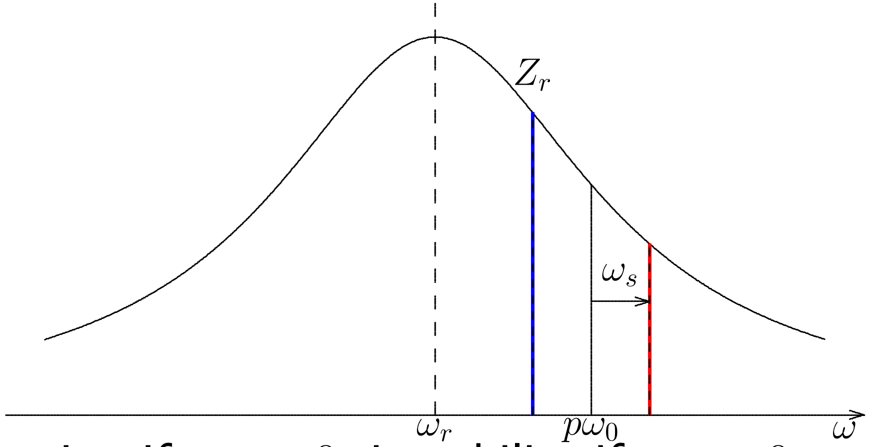
$$\frac{\partial U}{\partial\epsilon}=-\frac{e}{I_0}\sum_{\omega>0}^{\infty} I_p^2 p\omega_0(Z_r(\omega_p^+)-Z_r(\omega_p^-))\frac{\eta}{\omega_s}$$

We discussed stability of a phase oscillation

$$\ddot\tau+2\alpha_s\dot\tau+\omega_{s0}^2\tau=0\ ,\ \ \tau=\hat\tau e^{-\alpha_s t}\cos(\omega_{s1}t)\ ,\ \ \alpha_s=\frac{\omega_0}{4\pi E}\frac{dU}{d\epsilon}.$$

$$\alpha_s=\frac{\omega_{s0}\sum pI_p^2(Z_r(\omega_p^+)-Z_r(\omega_p^-))}{2I_0h\hat V\cos\phi_s}\quad\begin{matrix}>0 & \text{stable}\\ <0 & \text{unstable}\end{matrix}$$

Narrow impedance, only one harmonic p



Damping if $\alpha_s > 0$, instability if $\alpha_s < 0$

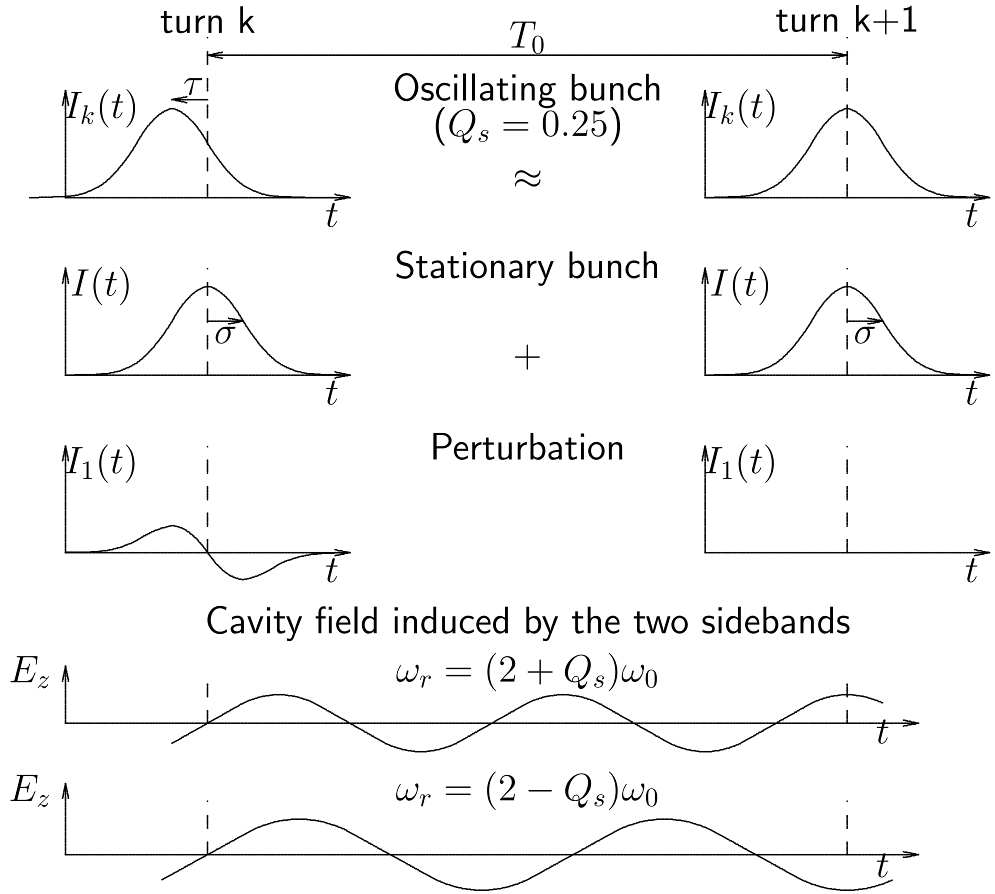
$$\epsilon = \hat{\epsilon} e^{-\alpha_s t} \sin(\omega_s t)$$

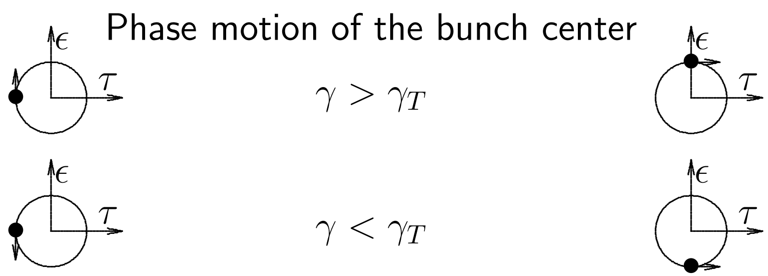
$$\alpha_s = \frac{\omega_{s0} p I_p^2 (Z_r(\omega_p^+) - Z_r(\omega_p^-))}{2 I_0 h \hat{V} \cos\phi_s} > 0$$

Above transition: $\cos\phi_s < 0$, stability if: $Z_r(\omega_p^-) > Z_r(\omega_p^+)$ Damping rate proportional to difference in Z_r between lower and upper sideband. Important:narrow-band impedances. The RF-cavity:$p = h,\ I_p \approx I_0$.

$$\alpha_s \approx \frac{\omega_{s0} I_0 (Z_r(\omega_p^+) - Z_r(\omega_p^-))}{2 \hat{V} \cos\phi_s}.$$

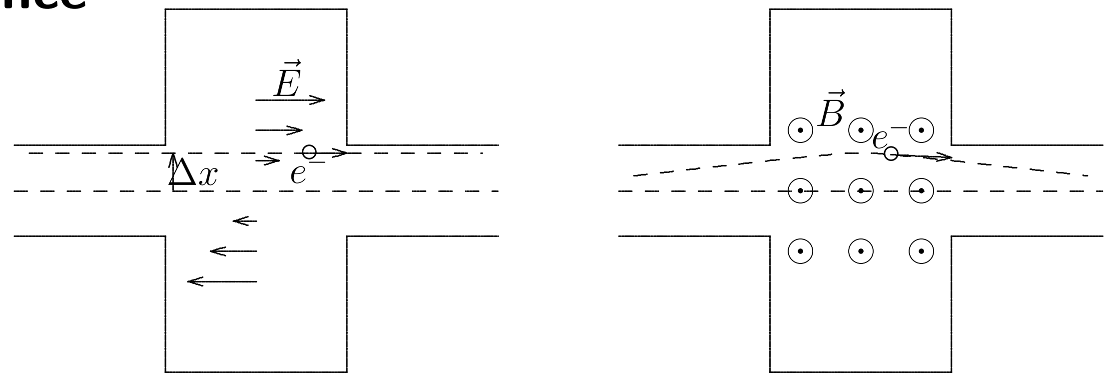
Qualitative understanding





8) TRANSVERSE INSTABILITIES

Transverse impedance



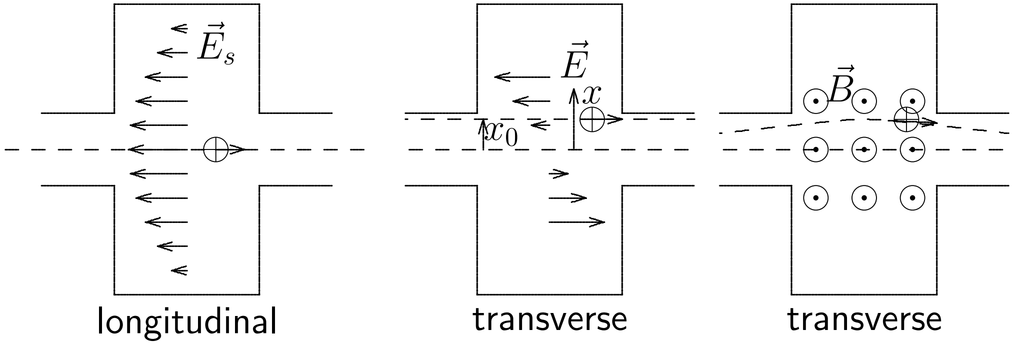
A transverse impedance is excited by longitudinal bunch motion and produces deflection field. Cavity oscillating with ω in dipole-mode with longitudinal E_z-field having transverse gradient $\partial E_z/\partial x$. E_z vanishes on axis and is excited by bunch with dipole moment $I_b \Delta x$. After 1/4 oscillation E_z becomes B_y-field which deflects beam in x-direction. Maxwell's equation $\dot{\vec{B}} = -\text{curl}\vec{E}$, integral form $\oint \dot{\vec{B}} d\vec{a} = -\oint \vec{E} d\vec{s}$, $\dot{B}_y x dz = -(\partial E_x/\partial x) x dz$

$$E_z = \frac{\partial \hat{E}}{\partial x} x \cos(\omega t) \ \rightarrow \ B_y = \frac{1}{\omega} \frac{\partial \hat{E}}{\partial x} \sin(\omega t)$$

For general case we use transverse impedance, Z_T or $Z_\perp$ in analogy to longitudinal

$$Z_T(\omega) = j \frac{\int \left(\vec{E}(\omega) + [\vec{v} \times \vec{B}(\omega)] \right)_T ds}{I x(\omega)}$$
$$= -\frac{\omega \int \left(\vec{E}(\omega) + [\vec{v} \times \vec{B}(\omega)] \right)_T ds}{I \dot{x}(\omega)}$$

using $\mathrm{e}^{j\omega t}$. If deflecting field is in phase with exciting dipole moment there is no energy transfer to transverse motion, factor 'j' on top. If it is in phase with transverse velocity there is energy transfer, real on bottom.



Relation between Z_L and Z_T of same mode:
A dipole moment Ix_0 induces in longitudinal impedance Z_L a gradient $\partial E_z/\partial x = Ix_0$

$$\frac{\partial E_z}{\partial x} = kIx_0, \; E_z(x) = \frac{dE_z}{dx}x = kIx_0 x$$

$$\begin{aligned} E_z(x_0) &= kIx_0^2, \text{ gives long. impedance} \\ Z_L(x_0) &= -\textstyle\int E_z(x_0)dz/I = -kx_0^2\ell \\ d^2Z_L/dx_0^2 &= -2k\ell\ , \;\; (\ell = \text{cavity length}) \end{aligned}$$

Maxwell's equation $\int \dot{\vec{B}} d\vec{a} = -\oint \vec{E} d\vec{s}$ gives the relation in complex notation $I(t) = \hat{I}e^{j\omega t}$

$$\dot{B}_y = \hat{B}_y j\omega e^{j\omega t} = -d\hat{E}_z/dx e^{j\omega t}$$

$$\begin{aligned} B_y &= \frac{j}{\omega}\frac{\partial E_z}{\partial x} = \frac{jkIx_0}{\omega} \\ Z_T &= j\frac{\int [\vec{v}\times\vec{B}]_T ds}{Ix(\omega)} = j\frac{cB_y\ell}{Ix_0} = -\frac{ck\ell}{\omega} \end{aligned}$$

$$Z_T(\omega) = \frac{c}{2\omega}\frac{d^2 Z_L(\omega)}{dx^2}.$$

This gives the ω-symmetry relation for Z_T

$$\begin{aligned} Z_{Lr}(-\omega) &= Z_{Lr}(\omega)\ , \; Z_{Li}(-\omega) = -Z_{Li}(\omega) \\ Z_{Tr}(-\omega) &= -Z_{Tr}(\omega)\ , \; Z_{Ti}(-\omega) = Z_{Ti}(\omega) \end{aligned}$$

Relation Z_L to Z_T of different modes:
In ring of global and vacuum chamber radii R and b the impedances, averaged for **different** modes and objects, have semi-empirical

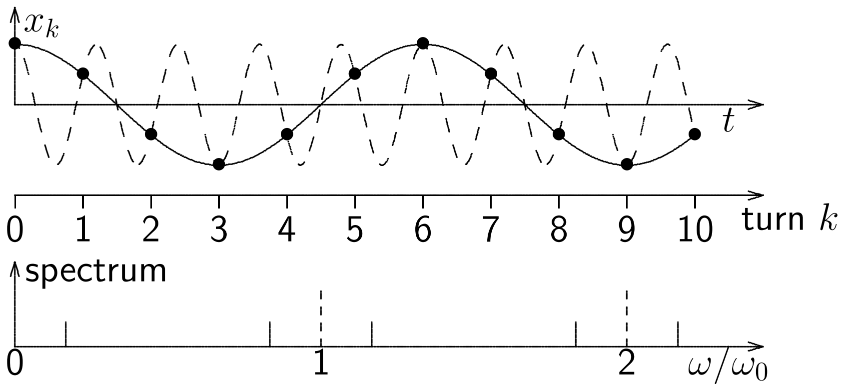
$$\text{ratio:} \quad Z_T(\omega) \approx \frac{2R}{b^2}\frac{Z_L(\omega)}{\omega/\omega_0}.$$

From the area available for the wall current we expect $Z_L \propto 1/b$ and therefore $Z_T \propto 1/b^3$.

Transverse dynamics

Due to transverse focusing particle executes a betatron motion around the orbit with a betatron tune Q_x having a fractional part q and the revolution frequency ω_0. It is locally harmonic but has a complicated phase advance around the ring. A stationary observer, or impedance, samples particle position x_k at one location each turn k without information for the rest of the ring

$$x_k = \hat{x}\cos(2\pi q k)\ , \quad x'_k = -\frac{\hat{x}}{\beta_x}\sin(2\pi q k).$$



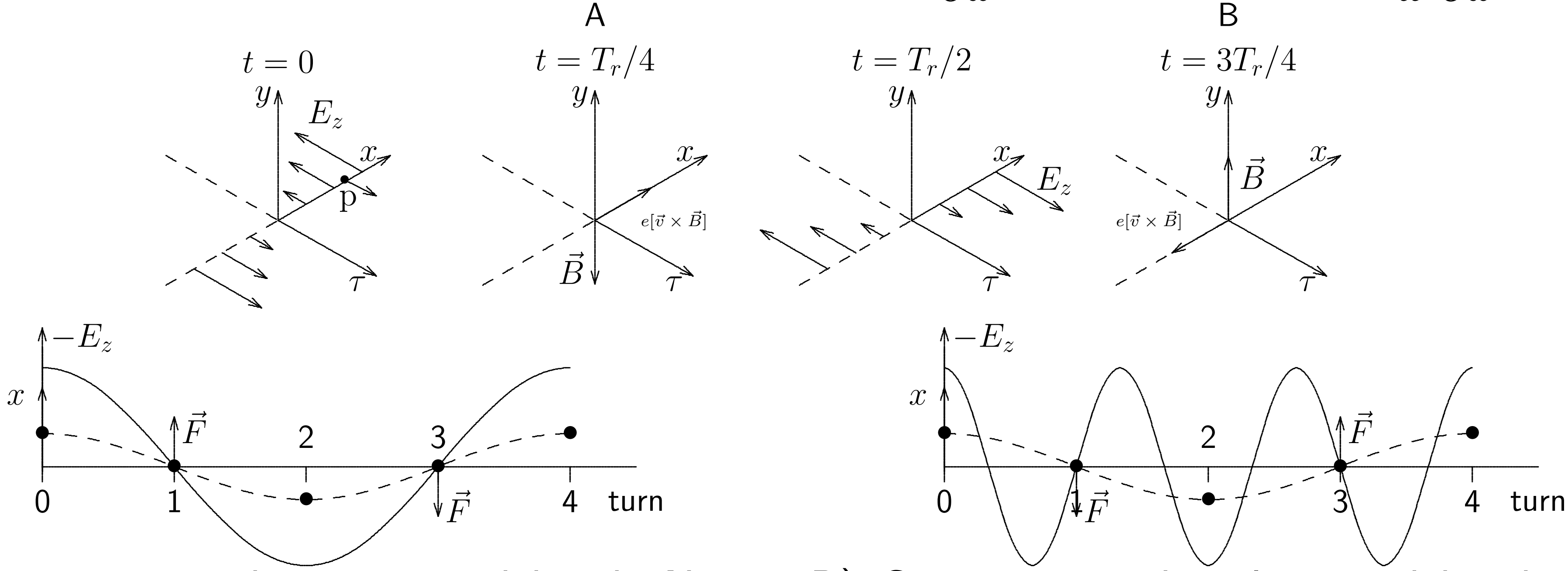
We observe this motion as a function of turn k and make a harmonic fit, i.e. Fourier analysis. For a single bunch we find at a revolution harmonic $p\omega_0$ an upper and lower sideband at distance $\pm q\omega_0$. Only the fractional part q matters since the integer cannot be observed. For a very short bunch these sidebands will extend to very high frequencies, for longer bunches they level off. A transverse impedance (or a position monitor) is sensitive to the dipole moment Ix of the current and does not see the revolution harmonics.

Multi-traversal instability of a single bunch

A bunch P traverses a cavity with off-set x, excites a field $\vec{E}$ which turns after $T_r/4$ into a field $-\vec{B}$, then into $-\vec{E}$ and after into $\vec{B}$.

Fractional tune $q = 1/4$, use $\omega = \omega_0(p \pm q)$.

$$E_z = \frac{\partial \hat{E}}{\partial x} x \cos(\omega t) \;\rightarrow\; B_y = \frac{1}{\omega}\frac{\partial \hat{E}}{\partial x}\sin(\omega t)$$



A) Cavity is tuned to upper sideband. Next turn the bunch traverses it in the situation 'A', $t = T_r(k + 1/4)$ with a velocity in $-x$-direction and gets by B_y a force in $+x$-direction which damps the oscillation.

B) Cavity is tuned to lower sideband. The bunch traverses it next turn in situation 'B', $t = T_r(k' + 3/4) = T_r(k' + 1 - 1/4)$ with negative velocity and a force in same direction increasing its velocity, giving instability.

Transverse instability for $Q' = 0$

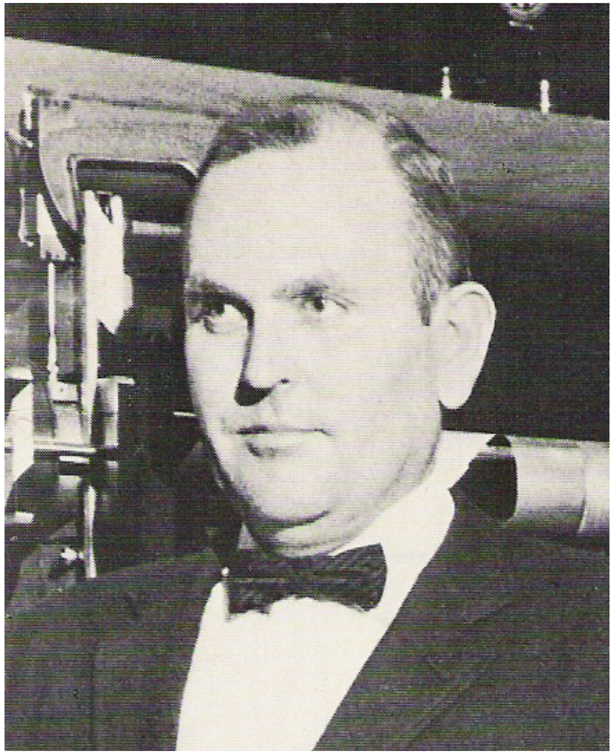
The resistive impedance at the upper sideband damps, the one at the lower sideband excites the oscillation. If we have a more general impedance extending over several sidebands $\omega_0(p+q)$ and $\omega_0(p-q)$ we expect that the growth or damping rate of the oscillation is given by an expression of the form

$$\alpha_s = \frac{1}{\tau_s} \propto \sum_p I_p^2 \left(Z_{Tr}(\omega_{p+}) - I_{p-}^2 Z_{Tr}(\omega_{p-}) \right)$$

with $\omega_{p\pm} = \omega_0 \left(p \pm q \right)$ where I_p is the Fourier component of the beam current at $p\omega_0$. It appears here as the square I_p^2 since the instability is driven by the energy transfer from the longitudinal to the transverse motion.

We have again Robinson-type instability which has been generalized by F. Sacherer.

The transverse instability is more complicated for finite chromaticity $Q' \neq 0$. A particle executes a betatron and a synchrotron (energy) oscillation. In going from head to tail it has an energy deviation which changes the betatron phase due to the chromaticity. This can give a so-called head-tail instability.



9) ILLUSTRATION OF COHERENT AND INCOHERENT MOTION

This difference is difficult to imagine but can be illustrated by a simple set of swings having different length and therefore different frequencies. If the frame is stiff any coherent oscillation will decay quickly. However, a flexible frame can create a difference between coherent and incoherent frequency and couple the individual swings together. This can disturb a stabilization, called Landau damping, where a coherent (center-of-mass) motion decays due to a spread of frequencies.