Advanced Accelerator Physics Course

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Low Emittance Machines

Part 3: Vertical Emittance Generation, Calculation, and Tuning

Andy Wolski

The Cockcroft Institute, and the University of Liverpool, UK

Summary of Lectures 1 and 2

In Lecture 1, we:

- discussed the effect of synchrotron radiation on the (linear) motion of particles in storage rings;
- derived expressions for the damping times of the vertical, horizontal, and longitudinal emittances;
- discussed the effects of quantum excitation, and derived expressions for the equilibrium horizontal and longitudinal emittances in an electron storage ring in terms of the lattice functions and beam energy.

In Lecture 2, we:

- derived expressions for the natural emittance in different types of lattice (FODO, DBA, MBA, TME), of the form $\varepsilon_0 = F C_q \gamma^2 \theta^3$;
- considered how the natural emittance in an achromate could be reduced by "detuning" from the achromat conditions.

In this lecture, we shall:

- discuss how vertical emittance is generated by alignment and tuning errors;

- describe methods for calculating the vertical emittance in the presence of known alignment and tuning errors;

- discuss briefly how a storage ring can be tuned to minimise the vertical emittance (even when the alignment and tuning errors are not well known).

Recall that the natural (horizontal) emittance in a storage ring is given by:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}. \tag{1}$$

If the horizontal and vertical motion are independent of each other (i.e. if there is no betatron coupling) then we can apply the same analysis to the vertical motion as we did to the horizontal.

Then, if we build a ring that is completely flat (i.e. no vertical bending), then there is no vertical dispersion:

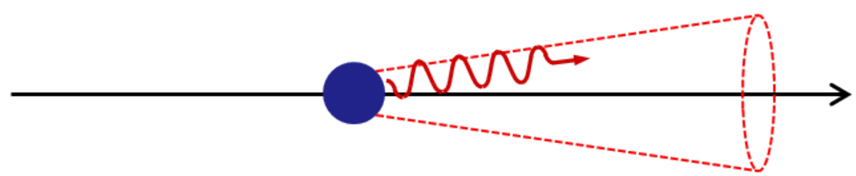
$$\eta_y = \eta_{py} = 0 \qquad \therefore \quad \mathcal{H}_y = 0 \qquad \therefore \quad I_{5y} = 0. \tag{2}$$

This implies that the vertical emittance will damp to zero.

However, in deriving equation (1) for the natural emittance, we assumed that all photons were emitted directly along the instantaneous direction of motion of the electron.

In fact, photons are emitted with a distribution with angular width $1/\gamma$ about the direction of motion of the electron.

This leads to some vertical "recoil" that excited vertical betatron motion, resulting in a non-zero vertical emittance.



A detailed analysis* leads to the following formula for the fundamental lower limit on the vertical emittance:

$$\varepsilon_{y,min} = \frac{13}{55}\frac{C_q}{j_y I_2}\oint \frac{\beta_y}{|\rho|^3}\,ds. \qquad (3)$$

*T. Raubenheimer, SLAC Report 387 (1992)

To estimate a typical value for the lower limit on the vertical emittance, let us write equation (3) in the approximate form:

$$\varepsilon_{y,min} \approx \frac{C_q \langle \beta_y \rangle}{4 j_y I_2} \oint \frac{1}{|\rho|^3} \, ds = \frac{\langle \beta_y \rangle}{4} \frac{j_z}{j_y} \frac{\sigma_\delta^2}{\gamma^2}. \tag{4}$$

Using some typical values ($\langle \beta_y \rangle = 20\,\mathrm{m}$, $j_z = 2$, $j_y = 1$, $\sigma_\delta = 10^{-3}$, $\gamma = 6000$), we find:

$$\varepsilon_{y,min} \approx 0.3\,\mathrm{pm}. \tag{5}$$

The lowest vertical emittance achieved so far in a storage ring is around a picometer, several times larger than the fundamental lower limit.

Practical limits on the vertical emittance

In practice, vertical emittance in a (nominally flat) storage ring is dominated by two effects:

- residual vertical dispersion, which couples longitudinal and vertical motion;
- betatron coupling, which couples horizontal and vertical motion.

The dominant causes of residual vertical dispersion and betatron coupling are magnet alignment errors, in particular:

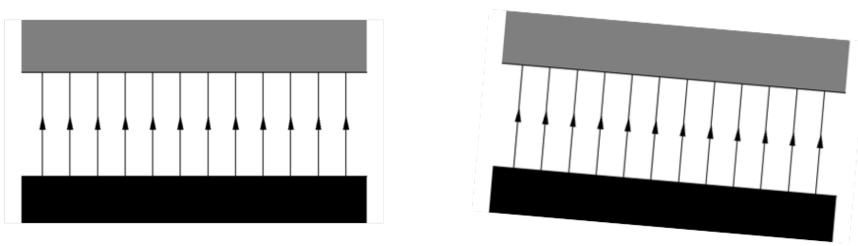
- tilts of the dipoles around the beam axis;
- vertical alignment errors on the quadrupoles;
- tilts of the quadrupoles around the beam axis;
- vertical alignment errors of the sextupoles.

Let us consider these errors in a little more detail...

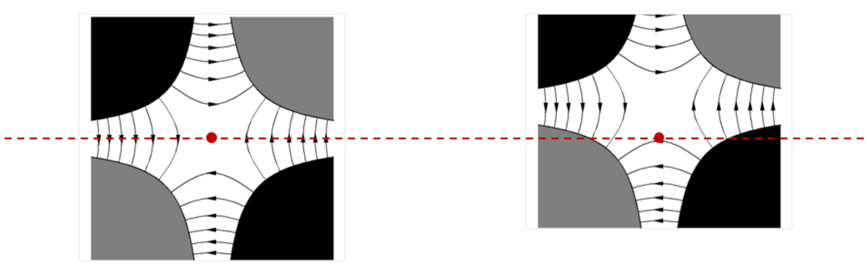
Steering errors lead to a distortion of the closed orbit, which generates vertical dispersion and (through beam offset in the sextupoles) betatron coupling.

A vertical steering error may be generated by:

- rotation of a dipole, so that the field is not exactly vertical:

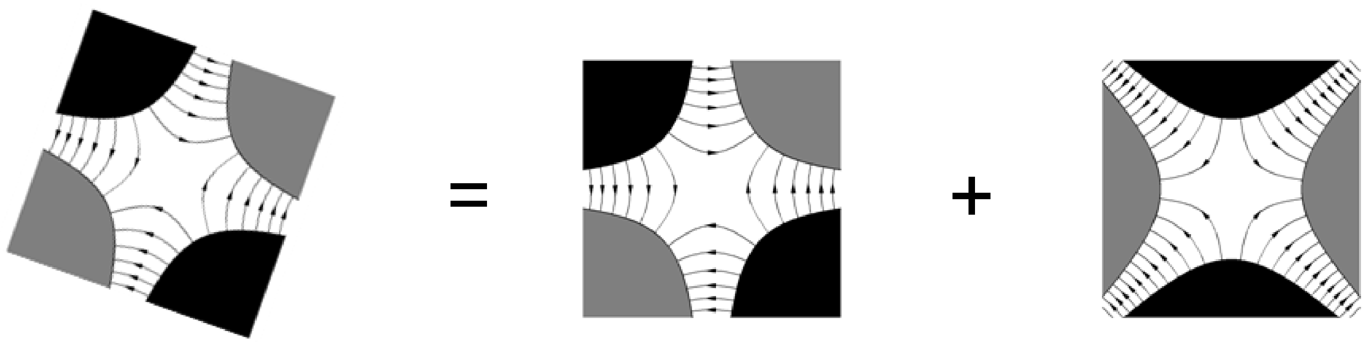


- vertical misalignment of a quadrupole, so that there is a horizontal magnetic field at the location of the reference trajectory:

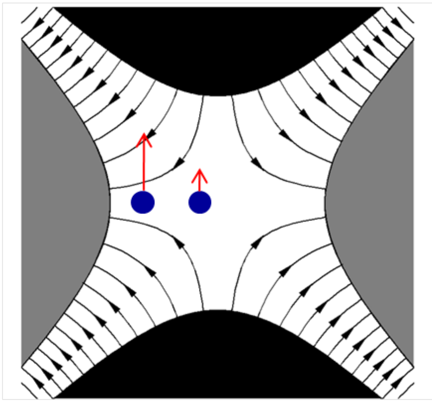


Coupling errors lead to transfer of horizontal betatron motion and dispersion into the vertical plane: in both cases, the result is an increase in vertical emittance.

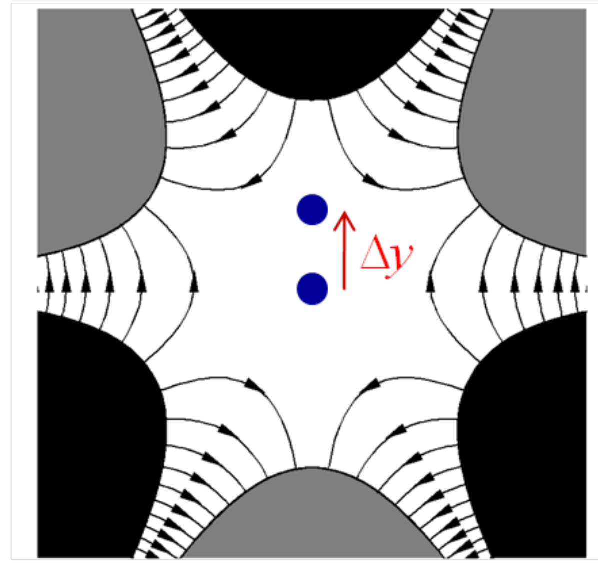
Coupling may result from rotation of a quadrupole, so that the field contains a skew component:



When particles pass through a skew quadrupole, they receive a vertical kick that depends on their horizontal offset. As a result, quantum excitation of the horizontal emittance feeds into the vertical plane.



A vertical beam offset in a sextupole has the same effect as a skew quadrupole. To understand this, recall that a sextupole field is given by:



$$B_x = (B\rho)k_2 xy, \tag{6}$$

$$B_y = \frac{1}{2}(B\rho)k_2\left(x^2 - y^2\right). \tag{7}$$

A vertical offset can be represented by the transformation $y \mapsto y + \Delta y$:

$$B_x \mapsto (B\rho)k_2\Delta y\, x + (B\rho)k_2 xy, \tag{8}$$

$$B_y \mapsto -(B\rho)k_2\Delta y\, y + \frac{1}{2}(B\rho)k_2\left(x^2 - y^2\right) - \frac{1}{2}k_2\Delta y^2. \tag{9}$$

The first terms in the expressions in (8) and (9) constitute a skew quadrupole of strength $(B\rho)k_2\Delta y$.

When designing and building a storage ring, we need to know how accurately the magnets must be aligned, to keep the vertical emittance below some specified limit (though beam-based tuning methods also normally have to be applied).

In the next few slides, we shall see how to derive expressions that relate:

- closed orbit distortion;
- vertical dispersion;
- betatron coupling;

and (ultimately):

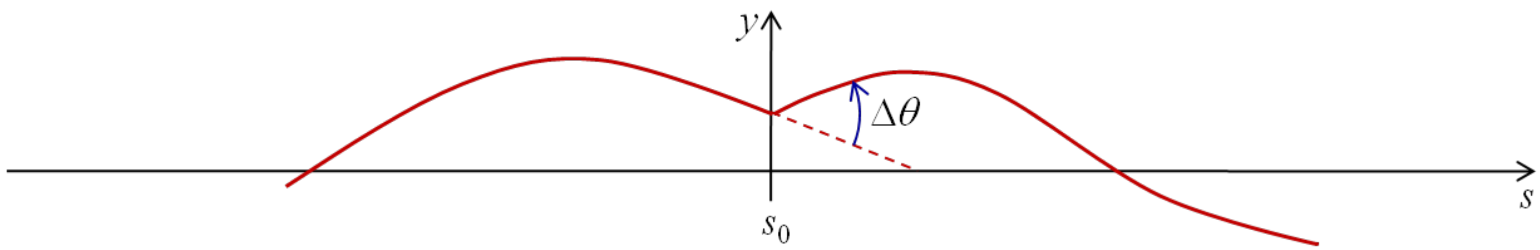
- vertical emittance,

to the alignment errors on the magnets.

In terms of the action-angle variables, we can write the coordinate and momentum of a particle at any point:

$$y = \sqrt{2\beta_y J_y}\cos\phi_y, \qquad p_y = -\sqrt{\frac{2J_y}{\beta_y}}\left(\sin\phi_y + \alpha_y\cos\phi_y\right). \tag{10}$$

Suppose there is a steering error at some location $s = s_0$ which gives a "kick" $\Delta\theta$ to the vertical momentum.



The trajectory of a particle will close on itself if:

$$\sqrt{2\beta_{y0}J_{y0}}\cos\phi_{y1} = \sqrt{2\beta_{y0}J_{y0}}\cos\phi_{y0}, \tag{11}$$

$$-\sqrt{\frac{2J_{y0}}{\beta_{y0}}}\left(\sin\phi_{y1} + \alpha_{y0}\cos\phi_{y1}\right) = -\sqrt{\frac{2J_{y0}}{\beta_{y0}}}\left(\sin\phi_{y0} + \alpha_{y0}\cos\phi_{y0}\right) - \Delta\theta. \tag{12}$$

where $\phi_{y1} = \phi_{y0} + 2\pi\nu_y$, and ν_y is the vertical tune.

Solving equations (11) and (12) for the action and angle at s_0:

$$J_{y0} = \frac{\beta_{y0}\Delta\theta^2}{8\sin^2\pi\nu_y}, \qquad \phi_{y0} = \pi\nu_y, \tag{13}$$

where $\nu_y = \mu_y/2\pi$ is the vertical tune.

Note that if the tune is an integer, there is no solution for the closed orbit: even the smallest steering error will kick the beam out of the ring.

From (13), we can write the coordinate for the closed orbit at any point in the ring:

$$y_{\text{co}}(s) = \frac{\sqrt{\beta_y(s_0)\beta_y(s)}}{2\sin\pi\nu_y}\Delta\theta\cos\left[\pi\nu_y + \mu_y(s;s_0)\right], \tag{14}$$

where $\mu_y(s;s_0)$ is the phase advance from s_0 to s.

In general, there will be many steering errors distributed around a storage ring.

The closed orbit can be found by summing the effects of all the steering errors:

$$y_{\mathrm{co}}(s) = \frac{\sqrt{\beta_y(s)}}{2\sin\pi\nu_y} \oint \sqrt{\beta_y(s')} \frac{d\theta}{ds'} \cos\left[\pi\nu_y + \mu_y(s;s')\right] \, ds'. \tag{15}$$

It is often helpful to be able to estimate the size of the closed orbit distortion that may be expected from random quadrupole misalignments of a given magnitude.

We can derive an expression for this from equation (15).

For a quadrupole of integrated focusing strength k_1L, vertically misaligned from the reference trajectory by ΔY, the steering is:

$$\Delta\theta = (k_1L)\Delta Y. \tag{16}$$

Squaring equation (15), then averaging over many seeds of random alignment errors, we find:

$$\left\langle \frac{y_{\text{co}}^2(s)}{\beta_y(s)} \right\rangle = \frac{\langle \Delta Y^2 \rangle}{8\sin^2 \pi\nu_y} \sum_{\text{quads}} \beta_y (k_1L)^2. \tag{17}$$

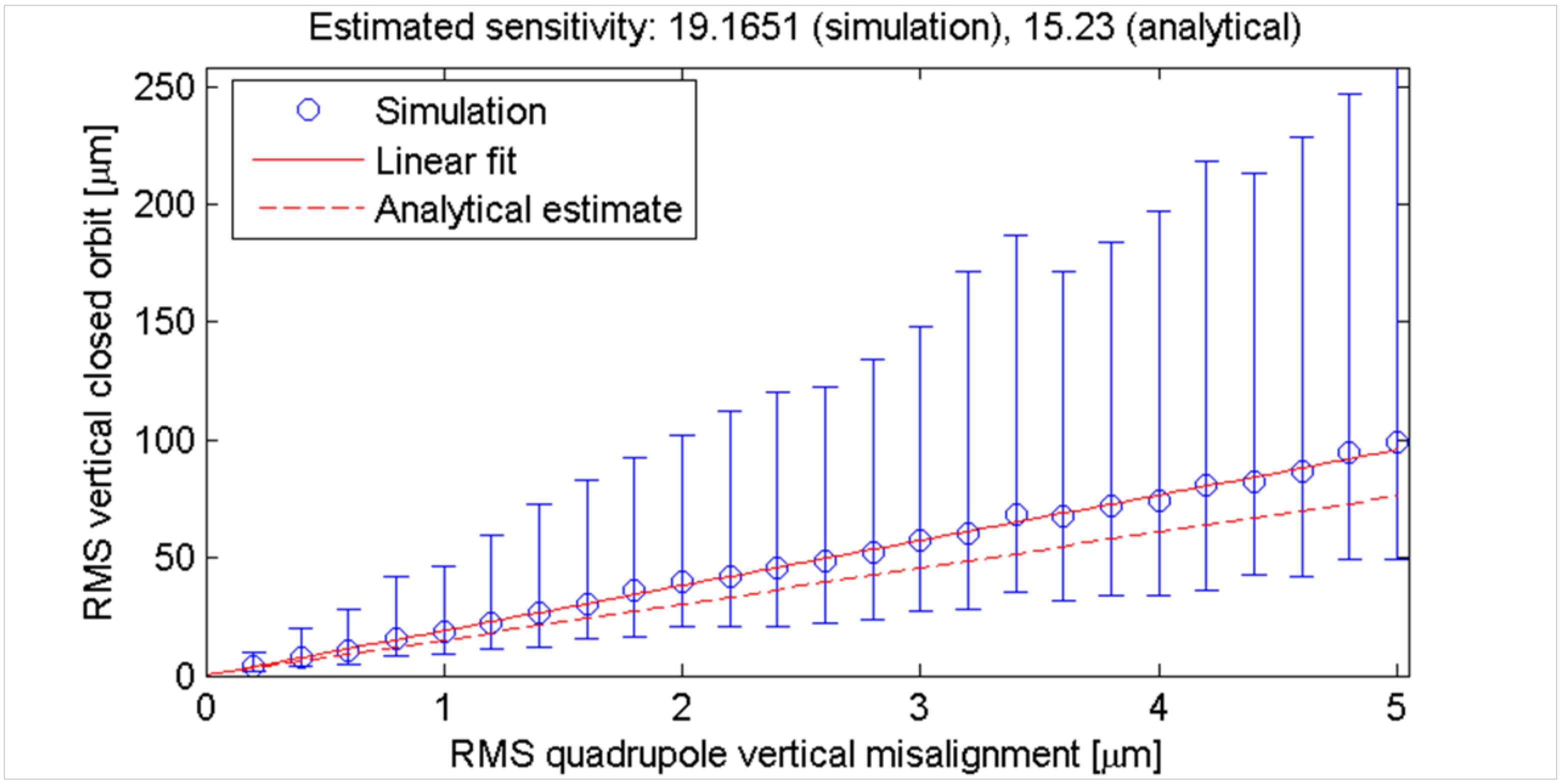
In performing the average, we assume that the alignments of different quadrupoles are not correlated in any way.

Closed orbit distortion from steering errors

The ratio between the closed orbit rms and the magnet misalignment rms is sometimes known as the "orbit amplification factor".

Values for the orbit amplification factor are commonly in the range 10–100.

Of course, the amplification factor is a statistical quantity. The actual rms of the orbit distortion depends on the particular set of alignment errors present.



In the context of low-emittance storage rings, vertical closed orbit errors are of concern for two reasons:

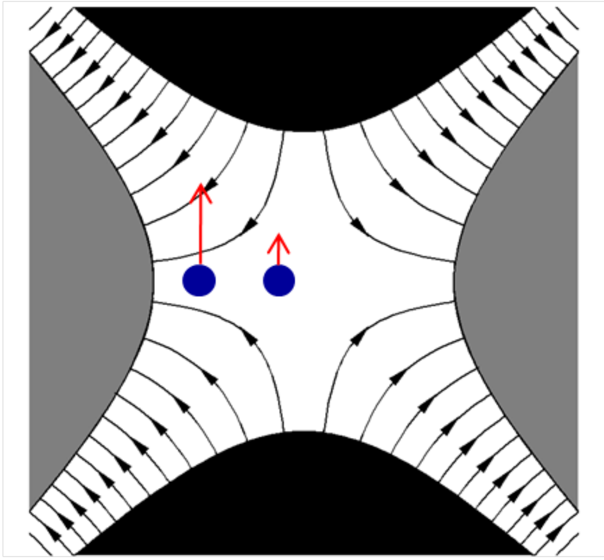
- Vertical steering generates vertical dispersion, which is a source of vertical emittance.
- Vertical orbit errors contribute to vertical beam offset in the sextupoles, which effectively generates skew quadrupole fields, which in turn lead to betatron coupling.

We have seen how to analyse the beam dynamics to understand the closed orbit distortion that arises from quadrupole alignment errors of a given magnitude.

Our goal is to relate quantities such as orbit distortion, vertical dispersion, coupling, and vertical emittance, to the alignment errors on the magnets.

We continue with betatron coupling...

Betatron coupling describes the effects that can arise when the vertical motion of a particle depends on its horizontal motion, and vice-versa. Betatron coupling can arise (for example) from skew quadrupoles, and solenoids.



In a storage ring, skew quadrupole fields ofen arise from quadrupole tilts, and from vertical alignment errors on sextupoles.

A full treatment of betatron coupling can become quite complex. There are many different formalisms that can be used.

However, it is possible to use a simplified model to derive an approximate expression for the equilibrium emittances in the presence of coupling.

The procedure is as follows:

1. Write down the equations of motion for a single particle in a beamline containing coupling.
2. Look for a "steady state" solution to the equations of motion, in which the horizontal and vertical actions are each constants of the motion.
3. Assume that the actions in the steady state solution are the equilibrium emittances (since $\varepsilon = \langle J \rangle$), and that the sum of the horizontal and vertical emittances equals the natural emittance of the "ideal" lattice (i.e. the natural emittance of the lattice in the absence of errors).

This procedure can give some useful results, but because of the approximations involved, the formulae are not always very accurate.

The details of the calculation are given in Appendix A. Here, we simply quote the results.

The horizontal and vertical emittances in the presence of coupling generated by skew quadrupoles in a lattice are:

$$\varepsilon_x = \frac{\varepsilon_0}{2}\left(1 + \frac{1}{\sqrt{1+\kappa^2/\Delta\nu^2}}\right), \tag{18}$$

$$\varepsilon_y = \frac{\varepsilon_0}{2}\left(1 - \frac{1}{\sqrt{1+\kappa^2/\Delta\nu^2}}\right), \tag{19}$$

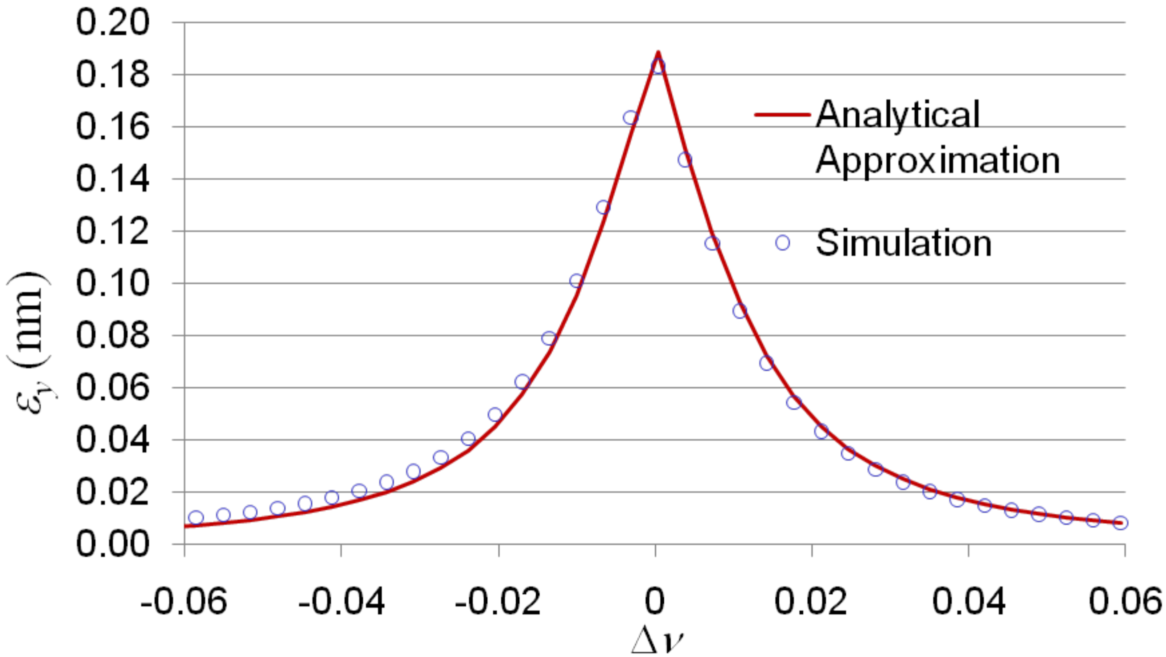
where ε_0 is the natural emittance, and $\Delta\nu$ is the difference in the fractional parts of the betatron tunes.

The "coupling strength" κ is found from:

$$\kappa e^{i\chi} = \frac{1}{2\pi}\oint e^{i(\mu_x-\mu_y)} k_s \sqrt{\beta_x\beta_y}\, ds. \tag{20}$$

As an illustration, we can plot the vertical emittance as a function of the "tune split" $\Delta\nu$, in a model of the ILC damping rings, with a single skew quadrupole (located at a point of zero dispersion – why?).

The tunes are controlled by adjusting the regular (normal) quadrupoles in the lattice:

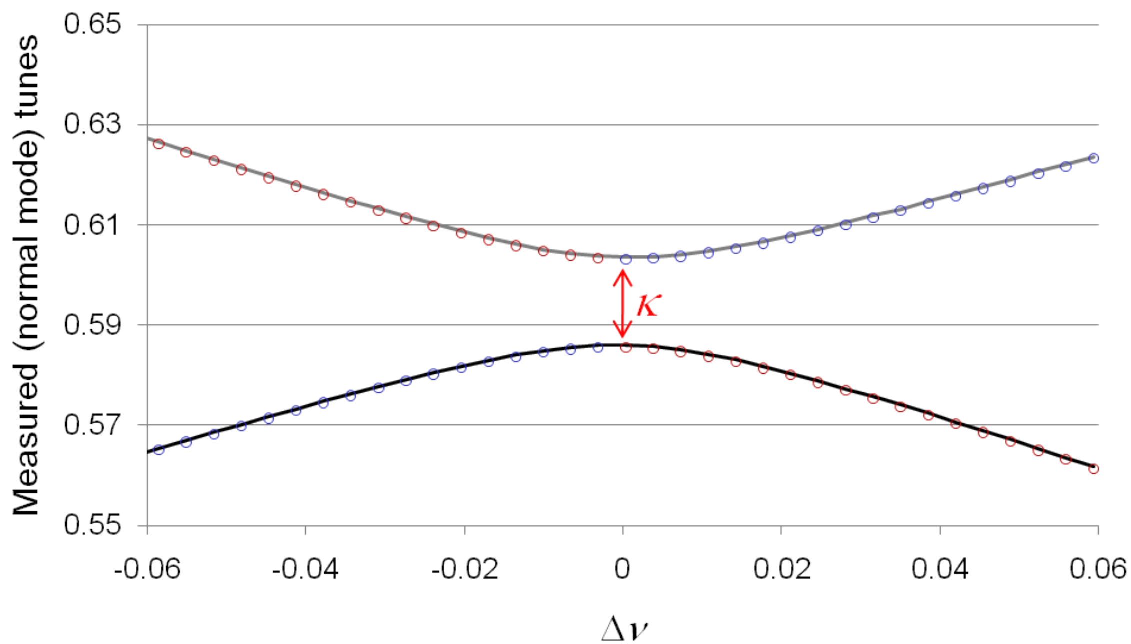


Note: the "simulation" results are based on emittance calculation using Chao's method, which we shall discuss later.

The presence of skew quadrupole errors in a storage ring affects the betatron tunes. In Appendix B, it is shown that the measured betatron tunes $\nu_\pm$ are given (in terms of the tunes ν_x and ν_y in the absence of errors) by:

$$\nu_\pm = \frac{1}{2}\left(\nu_x + \nu_y \pm \sqrt{\kappa^2 + \Delta\nu^2}\right). \tag{21}$$

This provides a useful method for measuring the coupling strength κ in a real lattice:



Major sources of coupling in storage rings are quadrupole tilts and sextupole alignment.

Using the theory just outlined, we can estimate the alignment tolerances on these magnets, for given optics and specified vertical emittance.

Starting with equation (20), we take the modulus squared and use (for sextupoles) $k_s = k_2 \Delta Y_S$.

Including a term to account for quadrupole tilts $\Delta\Theta_Q$, we find:

$$\langle \kappa^2 \rangle \approx \frac{\langle \Delta Y_S^2 \rangle}{4\pi^2} \sum_{\text{sexts}} \beta_x \beta_y (k_2 l)^2 + \frac{\langle \Delta\Theta_Q^2 \rangle}{4\pi^2} \sum_{\text{quads}} \beta_x \beta_y (k_1 l)^2. \quad (22)$$

Note that ΔY_S is the beam offset from the centre of a sextupole: this includes the effects of closed orbit distortion.

Vertical emittance is generated by vertical dispersion as well as by betatron coupling.

Vertical emittance is in turn generated by:

- vertical closed orbit distortion (vertical steering);
- coupling of horizontal dispersion into the vertical plane by skew quadrupole fields.

Our next goal is to estimate the amount of vertical dispersion generated from magnet alignment errors; we can then estimate the contribution to the vertical emittance.

The equation of motion for a particle with momentum P is:

$$\frac{d^2y}{ds^2} = \frac{B_x}{(B\rho)} = \frac{e}{P}B_x. \tag{23}$$

For small energy deviation δ, P is related to the reference momentum P_0 by:

$$P \approx (1+\delta)P_0. \tag{24}$$

We can write for the horizontal field (to first order in the derivatives):

$$B_x \approx B_{0x} + y\frac{\partial B_x}{\partial y} + x\frac{\partial B_x}{\partial x}. \tag{25}$$

If we consider a particle following an off-momentum closed orbit, so that:

$$y = \eta_y\delta, \qquad \text{and} \qquad x = \eta_x\delta, \tag{26}$$

then, combining the above equations, we find to first order in δ:

$$\frac{d^2\eta_y}{ds^2} - k_1\eta_y \approx -k_{0s} + k_{1s}\eta_x. \tag{27}$$

Equation (27) gives the "equation of motion" for the dispersion. It is similar to the equation of motion for the closed orbit:

$$\frac{d^2 y_{\text{co}}}{ds^2} - k_1 y_{\text{co}} \approx -k_{0s} + k_{1s} x_{\text{co}}. \tag{28}$$

We can therefore immediately generalise the relationship (17) between the closed orbit and the quadrupole misalignments, to find for the dispersion:

$$\left\langle \frac{\eta_y^2}{\beta_y} \right\rangle = \frac{\langle \Delta Y_Q^2 \rangle}{8 \sin^2 \pi \nu_y} \sum_{\text{quads}} \beta_y (k_1 L)^2 + \frac{\langle \Delta \Theta_Q^2 \rangle}{8 \sin^2 \pi \nu_y} \sum_{\text{quads}} \eta_x^2 \beta_y (k_1 L)^2 + \frac{\langle \Delta Y_S^2 \rangle}{8 \sin^2 \pi \nu_y} \sum_{\text{sexts}} \eta_x^2 \beta_y (k_2 L)^2. \tag{29}$$

Here, we assume that the skew dipole terms k_{0s} come from vertical alignment errors on the quads with mean square $\langle \Delta Y_Q^2 \rangle$, and the skew quads k_{1s} come from tilts on the quadrupoles with mean square $\langle \Delta \Theta_Q^2 \rangle$ and from vertical alignment errors on the sextupoles, with mean square $\langle \Delta Y_S^2 \rangle$, and that all alignment errors are uncorrelated.

The final step is to relate the vertical dispersion to the vertical emittance. This is not too difficult.

First, we can apply the formula that we derived in Lecture 1 for the natural emittance, to the vertical emittance:

$$\varepsilon_y = C_q \gamma^2 \frac{I_{5y}}{j_y I_2}, \tag{30}$$

where j_y is the vertical damping partition number (usually, $j_y = 1$), and the synchrotron radiation integrals are given by:

$$I_{5y} = \oint \frac{\mathcal{H}_y}{|\rho|^3}\, ds, \qquad \text{where} \qquad \mathcal{H}_y = \gamma_y \eta_y^2 + 2\alpha_y \eta_y \eta_{py} + \beta_y \eta_{py}^2, \tag{31}$$

and

$$I_2 = \oint \frac{1}{\rho^2}\, ds. \tag{32}$$

If the vertical dispersion is generated randomly, then we can assume that it will *not* be correlated with the curvature $1/\rho$ of the reference trajectory†.

Then, we can write:

$$I_{5y} \approx \langle \mathcal{H}_y \rangle \oint \frac{1}{|\rho|^3} \, ds = \langle \mathcal{H}_y \rangle I_3. \tag{33}$$

Hence, we can write for the vertical emittance:

$$\varepsilon_y \approx C_q \gamma^2 \langle \mathcal{H}_y \rangle \frac{I_3}{j_y I_2}. \tag{34}$$

It is convenient to use:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}, \tag{35}$$

which gives:

$$\varepsilon_y \approx \frac{j_z}{j_y} \langle \mathcal{H}_y \rangle \sigma_\delta^2. \tag{36}$$

†This is not the case for the horizontal dispersion!

Now, note the similarity between the action:

$$2J_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2, \tag{37}$$

and the curly-H function:

$$\mathcal{H}_y = \gamma_y \eta_y^2 + 2\alpha_y \eta_y \eta_{py} + \beta_y \eta_{py}^2. \tag{38}$$

This implies that we can write:

$$\eta_y = \sqrt{\beta_y \mathcal{H}_y} \cos \phi_{\eta y}, \qquad \therefore \quad \left\langle \frac{\eta_y^2}{\beta_y} \right\rangle = \frac{1}{2} \langle \mathcal{H}_y \rangle. \tag{39}$$

Combining equations (36) and (39) gives a useful (approximate) relationship, between the vertical dispersion and the vertical emittance:

$$\varepsilon_y \approx 2 \frac{j_z}{j_y} \left\langle \frac{\eta_y^2}{\beta_y} \right\rangle \sigma_\delta^2. \tag{40}$$

Equation (29) tells us how the vertical dispersion depends on the magnet alignment:

$$\left\langle \frac{\eta_y^2}{\beta_y} \right\rangle = \frac{\langle \Delta Y_Q^2 \rangle}{8 \sin^2 \pi\nu_y} \sum_{\mathrm{quads}} \beta_y (k_1 L)^2 + \frac{\langle \Delta \Theta_Q^2 \rangle}{8 \sin^2 \pi\nu_y} \sum_{\mathrm{quads}} \eta_x^2 \beta_y (k_1 L)^2 + \frac{\langle \Delta Y_S^2 \rangle}{8 \sin^2 \pi\nu_y} \sum_{\mathrm{sexts}} \eta_x^2 \beta_y (k_2 L)^2.$$

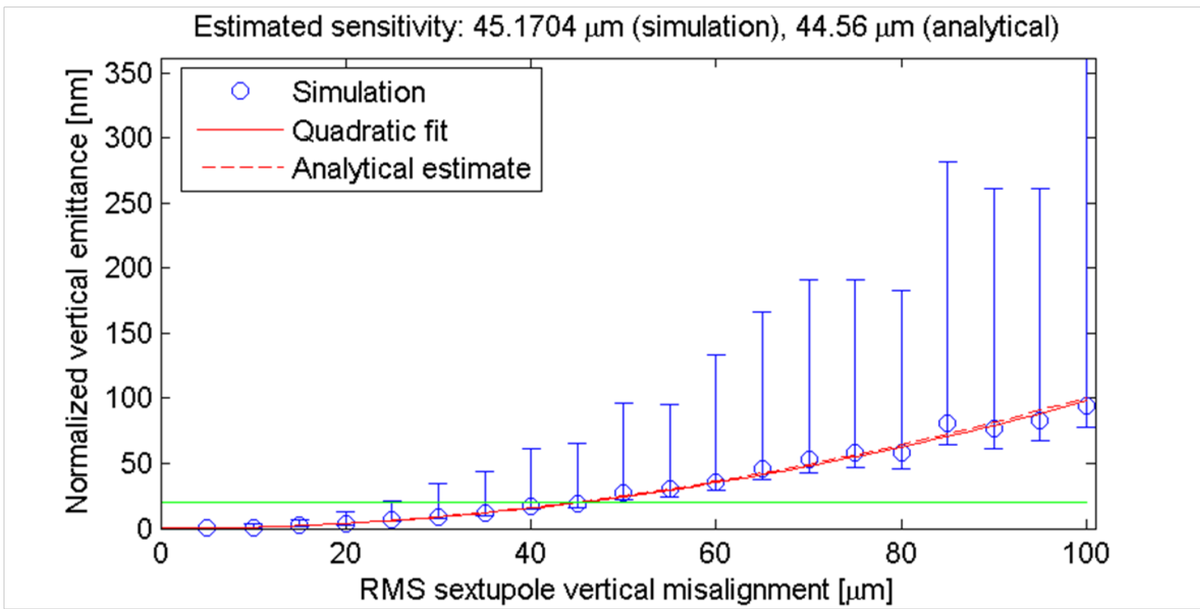
Equation (40) tells us how the vertical emittance depends on the vertical dispersion:

$$\varepsilon_y \approx 2 \frac{j_z}{j_y} \left\langle \frac{\eta_y^2}{\beta_y} \right\rangle \sigma_\delta^2. \tag{41}$$

Simply combining these two equations gives us an equation for the contribution of the vertical dispersion to the emittance, in terms of the magnet alignment errors.

It should be remembered that the total vertical emittance is found by adding together the contributions from betatron coupling (equations (19) and (22)) and vertical dispersion (equations (29) and (40)).

All these expressions involve significant approximations. However, they can give results that agree quite well with more reliable methods:



The formulae we have derived so far are useful for developing a "feel" for how the vertical emittance depends on magnet alignment errors, and for making rough estimates of the sensitivity to particular types of error.

For detailed studies, including modelling and simulations, we need more accurate formulae for computing the vertical emittance in a storage ring with a given set of alignment errors.

The calculations involved then become quite complex, and need to be solved using a computer.

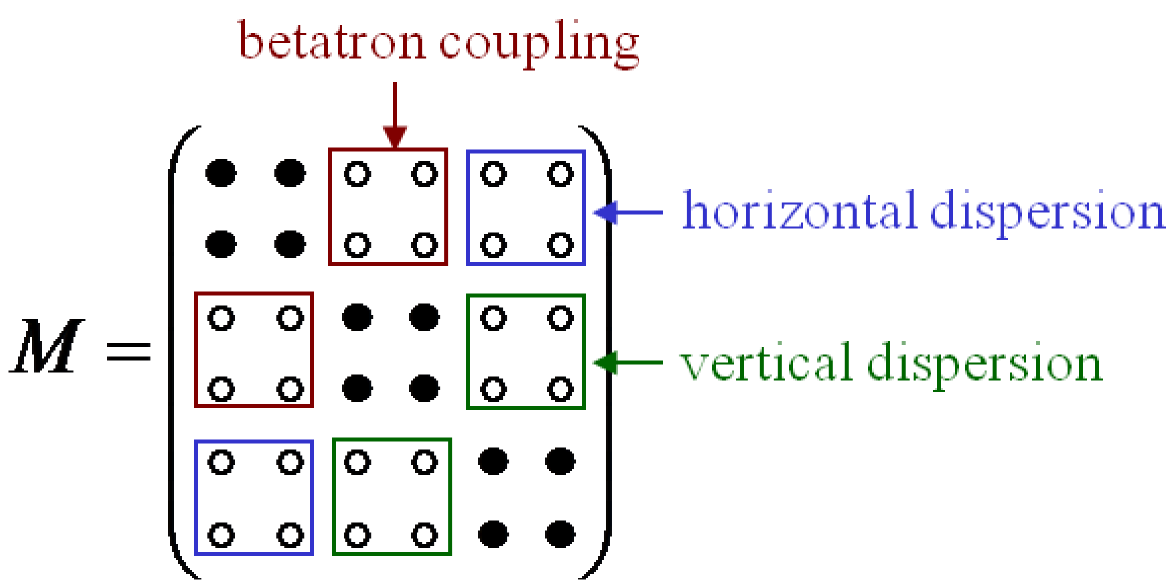
Perhaps the most commonly used methods for computing the equilibrium emittances in complex lattices (including lattices with errors) are:

- Radiation integrals generalised to the normal modes.
- Chao's method: A. Chao, "Evaluation of beam distribution parameters in an electron storage ring," J. Appl. Phys. 50, 595-598 (1979);
- The 'envelope' method: K. Ohmi, K. Hirata, K. Oide, "From the beam-envelope matrix to the synchrotron radiation integrals," Phys. Rev. E 49, 751-765 (1994).

We shall discuss each of these briefly, in turn.

Let us assume that we have a lattice code that will compute the *symplectic* single-turn transfer matrix at any point in a given lattice.

In general, the transfer matrix will have non-zero off-block-diagonal terms, that represent coupling between the horizontal, vertical, and longitudinal motion:



The formula we derived for the natural emittance assumed no betatron coupling, and that the coupling between the horizontal and longitudinal motion was relatively weak.

However, we can generalise the formula to the case that betatron coupling is present.

We still need to assume that the longitudinal motion is weakly coupled to each of the transverse degrees of freedom (i.e. the horizontal and vertical motion).

In that case, we can consider separately the 4×4 matrix $M_\perp$ describing the transverse motion:

$$M = \begin{pmatrix} M_\perp & \bullet \\ \bullet & M_\parallel \end{pmatrix}. \tag{42}$$

$M_\parallel$ is a 2×2 matrix describing the longitudinal motion, and we assume we can neglect the terms represented by the bullets ($\bullet$).

Now we look for a transformation, represented by a 4×4 matrix V, that puts $M_\perp$ into block-diagonal form, i.e. that "decouples" the transverse motion:

$$\tilde{M}_\perp = V\, M_\perp\, V^{-1} = \begin{pmatrix} M_\mathrm{I} & 0 \\ 0 & M_\mathrm{II} \end{pmatrix}. \tag{43}$$

M_I and M_II are 2×2 matrices describing betatron motion in a coordinate system in which the motion appears uncoupled.

There are various recipes for constructing the decoupling transformation V (which is not unique): see for example, Sagan and Rubin, PRST-AB 2, 074001 (1999).

Having obtained the matrices describing the uncoupled motion, we can derive the Courant–Snyder parameters for the normal mode motion in the usual way.

For example, we can write:

$$M_{\mathrm{II}} = \begin{pmatrix} \cos\mu_{\mathrm{II}} + \alpha_{\mathrm{II}} \sin\mu_{\mathrm{II}} & \beta_{\mathrm{II}} \sin\mu_{\mathrm{II}} \\ -\gamma_{\mathrm{II}} \sin\mu_{\mathrm{II}} & \cos\mu_{\mathrm{II}} - \alpha_{\mathrm{II}} \sin\mu_{\mathrm{II}} \end{pmatrix}, \qquad (44)$$

and similarly for mode I.

We can also obtain the normal mode dispersion functions, by applying the transformation V to a vector constructed from the dispersion functions in the original Cartesian coordinates.

Then, we can construct the curly-H function for each mode; for example:

$$\mathcal{H}_{\mathrm{II}} = \gamma_{\mathrm{II}}\eta_{\mathrm{II}}^2 + 2\alpha_{\mathrm{II}}\eta_{\mathrm{II}}\eta_{p,\mathrm{II}} + \beta_{\mathrm{II}}\eta_{p,\mathrm{II}}^2. \qquad (45)$$

Finally, we can write for the *mode II emittance*:

$$\varepsilon_{\mathrm{II}} = C_q \gamma^2 \frac{I_{5,\mathrm{II}}}{I_2 - I_{4,\mathrm{II}}}, \tag{46}$$

and similarly for mode I.

For many storage rings, equation (46) works well, and gives an accurate result.

However, if there is strong coupling between the transverse and the longitudinal motion (which can happen, for example, for large values of the synchrotron tune), then the approximations needed to derive equation (46) start to break down.

Chao's method[‡] for computing the emittances provides a formula that is conveniently expressed (though not always easy to apply). It is again based on the single-turn transfer matrix, but it is more accurate that the "decoupling" method, since it uses the full 6×6 transfer matrix, and does not assume weak coupling between the longitudinal and transverse motion.

We do not attempt to explain the physics behind the formula, but simply quote the result:

$$\varepsilon_k = C_L \frac{\gamma^5}{c\alpha_k} \oint \frac{|E_{k\,5}(s)|^2}{|\rho(s)|^3} \, ds, \tag{47}$$

where $k =$I, II, III indexes the degrees of freedom; the eigenvalues of the single-turn matrix *including radiation damping* are $e^{-\alpha_k \pm 2\pi i \nu_k}$, $E_{k\,5}$ is the fifth component of the k^{th} eigenvector of the *symplectic* single-turn matrix, and:

$$C_L = \frac{55}{48\sqrt{3}} \frac{r_e \hbar}{m_e}. \tag{48}$$

[‡]A. Chao, J. Appl. Phys., Vol. 50, No. 2, 595–598 (1979)

Finally, we mention the envelope method. Like the Chao method, it gives accurate results for the emittances even if there is strong coupling between all three degrees of freedom.

The envelope method is based on finding the equilibrium beam distribution described by the "sigma matrix":

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle & \langle x y \rangle & \langle x p_y \rangle & \langle x z \rangle & \langle x \delta \rangle \\ \langle p_x x \rangle & \langle p_x^2 \rangle & \langle p_x y \rangle & \langle p_x p_y \rangle & \langle p_x z \rangle & \langle p_x \delta \rangle \\ \langle y x \rangle & \langle y p_x \rangle & \langle y^2 \rangle & \langle y p_y \rangle & \langle y z \rangle & \langle y \delta \rangle \\ \langle p_y x \rangle & \langle p_y p_x \rangle & \langle p_y y \rangle & \langle p_y^2 \rangle & \langle p_y z \rangle & \langle p_y \delta \rangle \\ \langle z x \rangle & \langle z p_x \rangle & \langle z y \rangle & \langle z p_y \rangle & \langle z^2 \rangle & \langle z \delta \rangle \\ \langle \delta x \rangle & \langle \delta p_x \rangle & \langle \delta y \rangle & \langle \delta p_y \rangle & \langle \delta z \rangle & \langle \delta^2 \rangle \end{pmatrix} \tag{49}$$

This is a symmetric matrix, constructed from the second-order moments of all possible combinations of the dynamical variables. (We assume that the first order moments are all zero.)

Under a single turn around an accelerator, Σ transforms as:

$$\Sigma \mapsto M \cdot \Sigma \cdot M^{\mathsf{T}} + D, \tag{50}$$

where M is the single-turn transfer matrix (including radiation damping) and D is a constant matrix representing the effects of quantum excitation.

From knowledge of the properties of synchrotron radiation, we can compute the matrices M and D for a given lattice design.

The *equilibrium* distribution Σ_{eq} has the property:

$$\Sigma_{eq} = M \cdot \Sigma_{eq} \cdot M^{\mathsf{T}} + D. \tag{51}$$

For given M and D, we can solve equation (51) to find Σ_{eq}...

...and then from Σ_{eq} we can find the *invariant* emittances i.e. the conserved quantities under symplectic transport.

For any beam distribution Σ, the invariant emittances ε_k are given by:

$$\varepsilon_k = \text{eigenvalues of } \Sigma \cdot S, \tag{52}$$

where S is the antisymmetric block-diagonal "unit" matrix:

$$S = \begin{pmatrix} S_2 & & \\ & S_2 & \\ & & S_2 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{53}$$

For details of the envelope method, see Appendix C.

In practice, tuning a storage ring to achieve a vertical emittance of no more than a few picometres is a considerable challenge.

This cannot be done just by survey alignment of the magnets: beam-based methods are also required. However, precise alignment of the magnets is always the first step.

A variety of beam-based methods for tuning storage rings have been developed over the years.

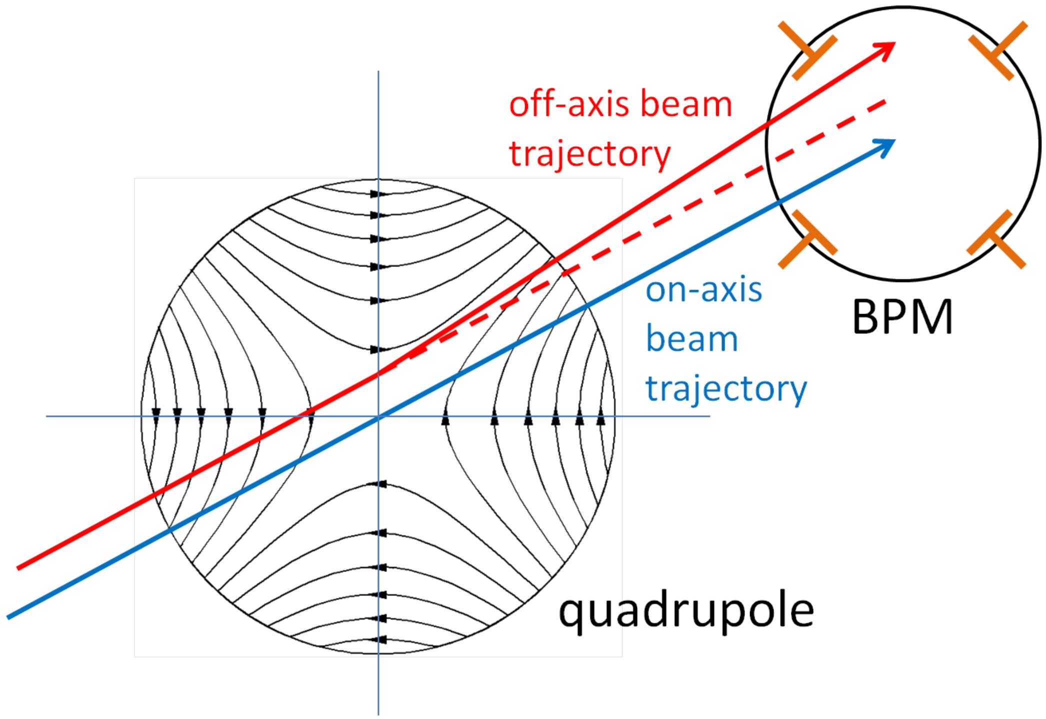
A typical procedure might look as follows...

Step 1: Align the magnets by a survey of the ring.

Typically, quadrupoles need to be aligned to better than a few tens of microns, and sextupoles to better than a couple of hundred microns.

Step 2: Determine the positions of the BPMs relative to the quadrupoles.

This is known as "beam-based alignment", (BBA), and can be achieved by steering the beam to a position in each quadrupole, where changing the quadrupole strength has no effect on the orbit.



Step 3: Correct the orbit (using steering magnets) so that it is as close as possible to the centres of the quadrupoles.

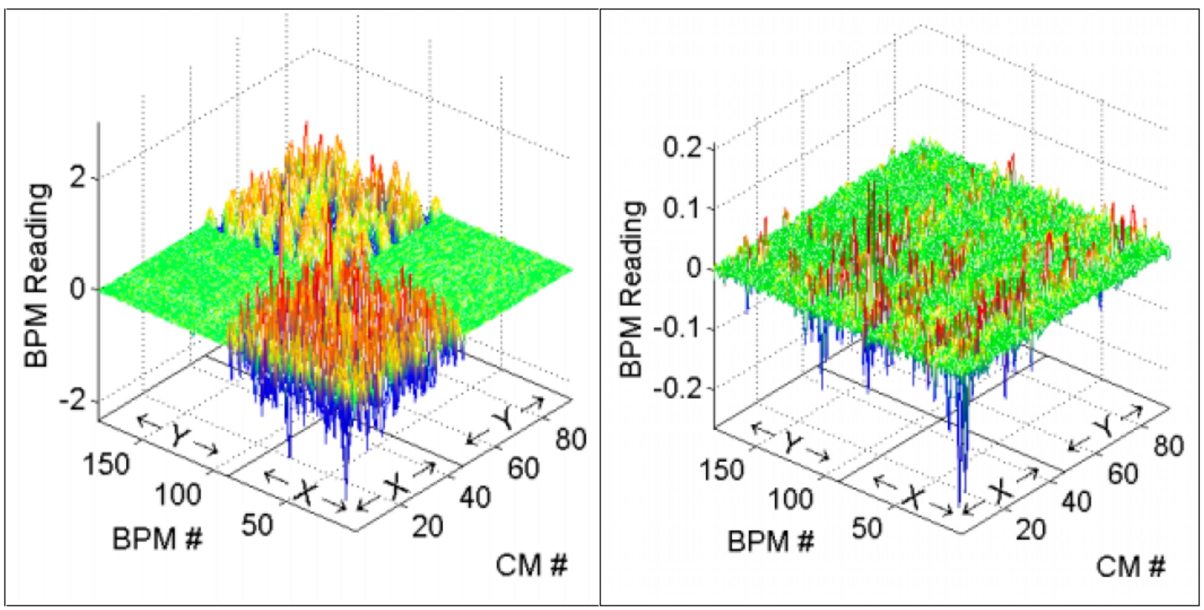
Step 4: Correct the vertical dispersion (using steering magnets and/or skew quadrupoles, and measuring at the BPMs) as close to zero as possible.

Step 5: Correct the coupling, by adjusting skew quadrupoles so that an orbit "kick" in one place (from any orbit corrector) has no effect on the orbit in the other plane.

Usually, these last three steps need to be iterated several (or even many) times.

Results from the tuning procedure described above can be limited by errors on the BPMs, which can affect dispersion and coupling measurements.

A useful technique for overcoming such limitations is to apply Orbit Response Matrix (ORM) analysis. This can be used to determine a wide range of magnet and diagnostics parameters, including coupling errors and BPM tilts.



ORM analysis in KEK ATF.

Left: measured orbit response matrix.

Right: residuals between measured ORM and machine model.

We have:

- derived approximate formulae for estimating the sensitivity of the vertical emittance to a range of magnet alignment errors;

- taken a brief look at methods for accurate emittance computation in storage rings with specified coupling and alignment errors;

- looked briefly at some of the practical techniques used for low-emittance tuning.

Our goal is to find the equations of motion for a particle in a coupled storage ring, and by solving these equations, to show equations (18) and (19).

We will use Hamiltonian mechanics. In this formalism, the equations of motion for the action-angle variables (with path length s as the independent variable) are derived from the Hamiltonian:

$$H = H(\phi_x, J_x, \phi_y, J_y; s), \tag{54}$$

using Hamilton's equations:

$$\frac{dJ_x}{ds} = -\frac{\partial H}{\partial \phi_x}, \qquad \frac{dJ_y}{ds} = -\frac{\partial H}{\partial \phi_y}, \tag{55}$$

$$\frac{d\phi_x}{ds} = \frac{\partial H}{\partial J_x}, \qquad \frac{d\phi_y}{ds} = \frac{\partial H}{\partial J_y}. \tag{56}$$

For a particle moving along a linear, uncoupled beamline, the Hamiltonian is:

$$H = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y}. \tag{57}$$

The first step is to derive an appropriate form for the Hamiltonian in a storage ring with skew quadrupole perturbations.

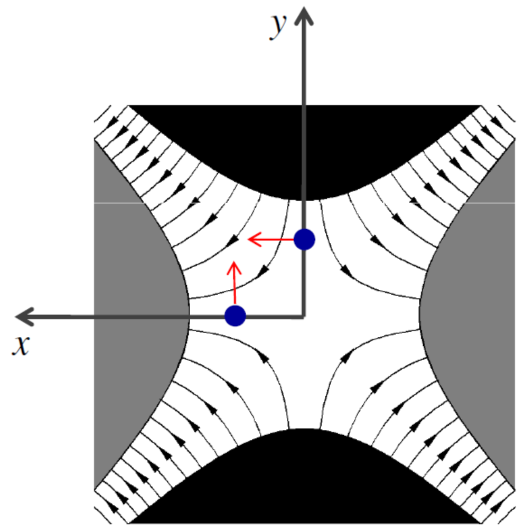
In Cartesian variables, the equations of motion in a skew quadrupole can be written:

$$\frac{dp_x}{ds} = k_s y, \qquad \frac{dp_y}{ds} = k_s x, \tag{58}$$

$$\frac{dx}{ds} = p_x, \qquad \frac{dy}{ds} = p_y, \tag{59}$$

where:

$$k_s = \frac{1}{B\rho}\frac{\partial B_x}{\partial x}. \tag{60}$$



These equations can be derived from the Hamiltonian:

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - k_s xy. \tag{61}$$

We are interested in the case where there are skew quadrupoles distributed around a storage ring.

The "focusing" effect of a skew quadrupole is represented by a term in the Hamiltonian:

$$k_s xy = 2k_s\sqrt{\beta_x\beta_y}\sqrt{J_x J_y}\cos\phi_x\cos\phi_y. \tag{62}$$

This implies that the Hamiltonian for a beam line with distributed skew quadrupoles can be written:

$$H = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y} - 2k_s(s)\sqrt{\beta_x\beta_y}\sqrt{J_x J_y}\cos\phi_x\cos\phi_y. \tag{63}$$

The beta functions and the skew quadrupole strength are functions of the position s. This makes it difficult to solve the equations of motion exactly.

Therefore, we simplify the problem by "averaging" the Hamiltonian:

$$H = \omega_x J_x + \omega_y J_y - 2\bar{\kappa}\sqrt{J_x J_y}\cos\phi_x\cos\phi_y. \tag{64}$$

Here, ω_x, ω_y and $\bar{\kappa}$ are constants.

ω_x and ω_y are the betatron frequencies, given by:

$$\omega_{x,y} = \frac{1}{C}\int_0^C \frac{ds}{\beta_{x,y}}. \tag{65}$$

For reasons that will become clear shortly, we re-write the coupling term, to put the Hamiltonian in the form:

$$H = \omega_x J_x + \omega_y J_y - \bar{\kappa}_- \sqrt{J_x J_y}\cos(\phi_x - \phi_y) - \bar{\kappa}_+ \sqrt{J_x J_y}\cos(\phi_x + \phi_y). \tag{66}$$

The constants $\bar{\kappa}_\pm$ represent the skew quadrupole strength averaged around the ring. However, we need to take into account that the kick from a skew quadrupole depends on the betatron phase. Thus, we write:

$$\bar{\kappa}_\pm e^{i\chi} = \frac{1}{C}\int_0^C e^{i(\mu_x \pm \mu_y)} k_s \sqrt{\beta_x \beta_y}\, ds, \tag{67}$$

where μ_x and μ_y are the betatron phase advances from the start of the ring.

Now suppose that $\bar{\kappa}_{-} \gg \bar{\kappa}_{+}$. (This can occur, for example, if $\omega_x \approx \omega_y$, in which case all the skew quadrupole perturbations will add together in phase.) Then, we can simplify things further by dropping the term in $\bar{\kappa}_{+}$ from the Hamiltonian:

$$H = \omega_x J_x + \omega_y J_y - \bar{\kappa}_{-}\sqrt{J_x J_y}\,\mathrm{cos}(\phi_x - \phi_y). \tag{68}$$

We can now write down the equations of motion:

$$\begin{aligned}
\frac{dJ_x}{ds} &= -\frac{\partial H}{\partial \phi_x} = \bar{\kappa}_{-}\sqrt{J_x J_y}\,\mathrm{sin}(\phi_x - \phi_y), &(69)\\
\frac{dJ_y}{ds} &= -\frac{\partial H}{\partial \phi_y} = -\bar{\kappa}_{-}\sqrt{J_x J_y}\,\mathrm{sin}(\phi_x - \phi_y), &(70)\\
\frac{d\phi_x}{ds} &= \frac{\partial H}{\partial J_x} = \omega_x + \frac{\bar{\kappa}_{-}}{2}\sqrt{\frac{J_x}{J_y}}\,\mathrm{cos}(\phi_x - \phi_y), &(71)\\
\frac{d\phi_y}{ds} &= \frac{\partial H}{\partial J_y} = \omega_y + \frac{\bar{\kappa}_{-}}{2}\sqrt{\frac{J_y}{J_x}}\,\mathrm{cos}(\phi_x - \phi_y). &(72)
\end{aligned}$$

Even after all the simplifications we have made, the equations of motion are still rather difficult to solve. Fortunately, however, we do not require the general solution. In fact, we are only interested in the properties of some special cases.

First of all, we note that the sum of the actions is constant:

$$\frac{dJ_x}{ds} + \frac{dJ_y}{ds} = 0 \qquad \therefore \quad J_x + J_y = \text{constant}. \tag{73}$$

This is true in all cases.

Going further, we notice that if $\phi_x = \phi_y$, then the rate of change of each action falls to zero, i.e.:

$$\text{if} \quad \phi_x = \phi_y \quad \text{then} \quad \frac{dJ_x}{ds} = \frac{dJ_y}{ds} = 0. \tag{74}$$

This implies that if we can find a solution to the equations of motion with $\phi_x = \phi_y$ for all s, then the actions will remain constant.

From the equations of motion, we find that if:

$$\phi_x = \phi_y \quad \text{and} \quad \frac{d\phi_x}{ds} = \frac{d\phi_y}{ds}, \tag{75}$$

then:

$$\frac{J_y}{J_x} = \frac{\sqrt{1+\bar{\kappa}_-^2/\Delta\omega^2} - 1}{\sqrt{1+\bar{\kappa}_-^2/\Delta\omega^2} + 1}, \tag{76}$$

where $\Delta\omega = \omega_x - \omega_y$.

If we further use $J_x + J_y = J_0$, where J_0 is a constant, then we have the *fixed point solution*:

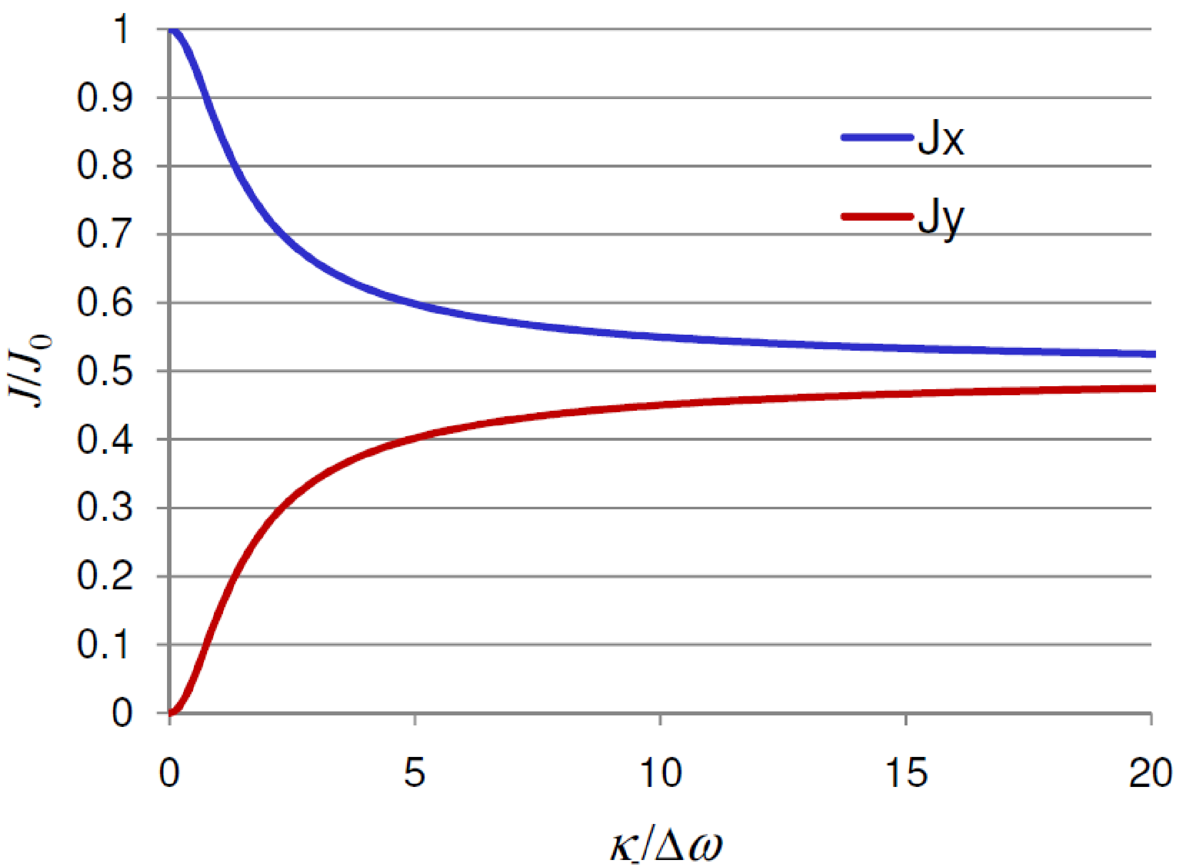
$$J_x = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1+\bar{\kappa}_-^2/\Delta\omega^2}}\right) J_0, \tag{77}$$

$$J_y = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\bar{\kappa}_-^2/\Delta\omega^2}}\right) J_0. \tag{78}$$

Note the behaviour of the fixed-point actions as we vary the "coupling strength" $\bar{\kappa}_-$ and the betatron tunes (betatron frequencies).

The fixed-point actions are well-separated for $\bar{\kappa}_- \ll \Delta\omega$, but approach each other for $\bar{\kappa}_- \gg \Delta\omega$.

The condition at which the tunes are equal (or differ by an exact integer) is known as the *difference coupling resonance*.



Recall that the emittance may be defined as the betatron action averaged over all particles in the beam:

$$\varepsilon_x = \langle J_x \rangle, \qquad \text{and} \qquad \varepsilon_y = \langle J_y \rangle. \tag{79}$$

Now, synchrotron radiation will damp the beam towards an equilibrium distribution. In this equilibrium, we expect the betatron actions of the particles to change only slowly, i.e. on the timescale of the radiation damping, whis is much longer than the timescale of the betatron motion.

In that case, the actions of most particles must be in the correct ratio for a fixed-point solution to the equations of motion. Then, if we assume that $\varepsilon_x + \varepsilon_y = \varepsilon_0$, where ε_0 is the natural emittance of the storage ring, we must have for the equilibrium emittances:

$$\varepsilon_x = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + \bar{\kappa}_-^2/\Delta\omega^2}}\right)\varepsilon_0, \tag{80}$$

$$\varepsilon_y = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + \bar{\kappa}_-^2/\Delta\omega^2}}\right)\varepsilon_0. \tag{81}$$

Hence, we have shown equations (18) and (19).

To estimate the effect of a skew quadrupole perturbation on the betatron tunes, we use the Hamiltonian (68). If we consider a particle close to the fixed point solution, we can assume that $\phi_x = \phi_y$, so that the Hamiltonian becomes:

$$H = \omega_x J_x + \omega_y J_y - \bar{\kappa}_- \sqrt{J_x J_y}. \tag{82}$$

The normal modes describe motion that is periodic with a single well-defined frequency. In the absence of coupling, the transverse normal modes correspond to motion in just the horizontal or vertical plane. When coupling is present, the normal modes involve combination of horizontal and vertical motion.

Let us write the Hamiltonian (82) in the form:

$$H = \begin{pmatrix} \sqrt{J_x} & \sqrt{J_y} \end{pmatrix} \cdot A \cdot \begin{pmatrix} \sqrt{J_x} \\ \sqrt{J_y} \end{pmatrix}, \text{ where } A = \begin{pmatrix} \omega_x & -\frac{1}{2}\bar{\kappa}_- \\ -\frac{1}{2}\bar{\kappa}_- & \omega_y \end{pmatrix}. \tag{83}$$

The normal modes can be constructed from the eigenvectors of the matrix A, and the frequency of each mode is given by the corresponding eigenvalue.

From the eigenvalues of A, we find that the normal mode frequencies are:

$$\omega_\pm = \frac{1}{2}\left(\omega_x + \omega_y \pm \sqrt{\bar{\kappa}_-^2 + \Delta\omega^2}\right). \tag{84}$$

The envelope method for computing the beam emittances is based on finding the equilibrium distribution described by the "Sigma matrix" (the matrix of second order moments of the dynamical variables):

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xp_x \rangle & \langle xy \rangle & \langle xp_y \rangle & \langle xz \rangle & \langle x\delta \rangle \\ \langle p_x x \rangle & \langle p_x^2 \rangle & \langle p_x y \rangle & \langle p_x p_y \rangle & \langle p_x z \rangle & \langle p_x \delta \rangle \\ \langle yx \rangle & \langle yp_x \rangle & \langle y^2 \rangle & \langle yp_y \rangle & \langle yz \rangle & \langle y\delta \rangle \\ \langle p_y x \rangle & \langle p_y p_x \rangle & \langle p_y y \rangle & \langle p_y^2 \rangle & \langle p_y z \rangle & \langle p_y \delta \rangle \\ \langle zx \rangle & \langle zp_x \rangle & \langle zy \rangle & \langle zp_y \rangle & \langle z^2 \rangle & \langle z\delta \rangle \\ \langle \delta x \rangle & \langle \delta p_x \rangle & \langle \delta y \rangle & \langle \delta p_y \rangle & \langle \delta z \rangle & \langle \delta^2 \rangle \end{pmatrix}. \tag{85}$$

This can be conveniently written as:

$$\Sigma_{ij} = \langle x_i x_j \rangle, \tag{86}$$

where Σ_{ij} is the (i, j) component of the Sigma matrix, and the set x_i (for $i = 1 \ldots 6$) are the dynamical variables. The brackets $\langle \cdot \rangle$ indicate an average over all particles in the bunch.

In the absence of coupling, the sigma matrix will be block diagonal. We are interested in the more general case, where coupling is present.

The emittances and the lattice functions can be calculated from the sigma matrix, and vice-versa.

Consider the (simpler) case of motion in one degree of freedom. The sigma matrix in this case is:

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle p_x x \rangle & \langle p_x^2 \rangle \end{pmatrix} = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \varepsilon_x. \tag{87}$$

Note that given a Sigma matrix, we can compute the emittance as follows. First, define the matrix S:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{88}$$

Then, the eigenvalues of $\Sigma \cdot S$ are $\pm i\varepsilon_x$. (The proof of this is left as an exercise.)

Now, we can show that (under certain assumptions) the emittance is conserved as a bunch is transported along a beam line.

The linear transformation in phase space coordinates of a particle in the bunch between two points in the beam line can be represented by a matrix M:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto M \cdot \begin{pmatrix} x \\ p_x \end{pmatrix}. \tag{89}$$

If (for the moment) we neglect radiation and certain other effects, and consider only the Lorentz force on particles from the external electromagnetic fields, then the transport is *symplectic*.

Physically, this means that the phase-space volume of the bunch is conserved as the bunch moves along the beam line.

Mathematically, it means that M is a *symplectic matrix*, i.e. M satisfies:

$$M^{\mathsf{T}} \cdot S \cdot M = S. \tag{90}$$

Now consider how the Sigma matrix transforms. Since it is written as the product of the phase-space coordinates averaged over the bunch, we have:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto M \cdot \begin{pmatrix} x \\ p_x \end{pmatrix}, \qquad \therefore \qquad \Sigma \mapsto M \cdot \Sigma \cdot M^{\mathsf{T}}. \tag{91}$$

Since S is a constant matrix, it immediately follows that:

$$\Sigma \cdot \mapsto M \cdot \Sigma \cdot M^{\mathsf{T}} \cdot S. \tag{92}$$

Then, using the fact that M is symplectic, we have:

$$\Sigma \cdot S \mapsto M \cdot \Sigma \cdot S \cdot M^{-1}. \tag{93}$$

This is a similarity transformation of $\Sigma \cdot S$: the eigenvalues of any matrix are conserved under a similarity transformation. Therefore, since the eigenvalues of $\Sigma \cdot S$ give the emittance of the bunch, it follows that the emittances are conserved under linear, symplectic transport.

The above discussion immediately generalises to three degrees of freedom.

We define the matrix S in three degrees of freedom by:

$$\S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{94}$$

The six eigenvalues of $\Sigma \cdot S$ are then:

$$\pm i\varepsilon_x, \quad \pm i\varepsilon_y, \quad \pm i\varepsilon_z. \tag{95}$$

These quantities are all conserved under linear, symplectic transport.

Even if, as is generally the case, the Sigma matrix is not block-diagonal (i.e. if there is coupling present), then we can still find three invariant emittances using this method, without any modification.

Appendix C: The envelope method for computing emittances

If M is a matrix that represents the linear single-turn transformation at some point in a storage ring, then an invariant or "matched" distribution is one that satisfies:

$$\Sigma \mapsto M \cdot \Sigma \cdot M^{\mathsf{T}} = \Sigma. \tag{96}$$

(In general, all the particles in the bunch change position in phase space after one turn around the ring: but for a matched distribution, the second order moments remain the same.)

This is not sufficient to determine the beam emittances – though this condition will determine the lattice functions (which can be found from the *eigenvectors* of $\Sigma \cdot S$).

In other words, the matched distribution condition determines the *shape* of the bunch, but not the *size* of the bunch. This makes sense: after all, in a proton storage ring, we can have a matched bunch of any emittance.

However, in an electron storage ring, we know that radiation effects will damp the emittances to some equilibrium values.

We shall now show how to apply the concept of a matched distribution, when radiation effects are included, to find the equilibrium emittances in an electron storage ring.

In an electron storage ring, we must make two modifications to the single-turn transformation to account for radiation effects:

1. The matrix M will no longer be symplectic: this accounts for radiation damping.

2. As well as first-order terms in the transformation (represented by the matrix M), there will be zeroth-order terms: these will turn out to correspond to the quantum excitation.

The condition for a matched distribution should then be written:

$$\Sigma = M \cdot \Sigma \cdot M^{\mathsf{T}} + D, \tag{97}$$

where M and D are constant, non-symplectic matrices that represent the first-order and zeroth-order terms in the single-turn transformation, respectively.

This equation is sufficient to determine the Sigma matrix uniquely - in other words, using just this equation (with known M and D) we can find the bunch emittances and the matched lattice functions.

Appendix C: The envelope method for computing emittances

The envelope method for finding the equilibrium emittances in a storage ring consists of three steps:

1. Find the first-order terms M and zeroth-order terms D in the single-turn transformation:

$$\Sigma \mapsto M \cdot \Sigma \cdot M^{\mathsf{T}} + D. \tag{98}$$

2. Use the matching condition:

$$\Sigma = M \cdot \Sigma \cdot M^{\mathsf{T}} + D, \tag{99}$$

   to determine the Sigma matrix.

3. Find the equilibrium emittances from the eigenvalues of $\Sigma \cdot S$.

Note: strictly speaking, since M is not symplectic, the emittances are not invariant as the bunch moves around the ring. Therefore, we may expect to find a different emittance at each point around the ring. However, if radiation effects are fairly small, then the variations in the emittances will also be small.

As an illustration of the transformation matrices M and D, we shall consider a thin "slice" of a dipole.

The thin slice of dipole is an important case:

- in most storage rings, radiation effects are only significant in dipoles;
- "complete" dipoles can be constructed by concatenating the maps for a number of slices.

Once we have a map for a thin slice of dipole, we simply need to concatenate the maps for all the elements in the ring, to construct the map for a complete turn starting at any point.

Recall that the transformation for the phase space variables in the emission of radiation carrying momentum dp is:

$$
\begin{array}{lll}
x \mapsto x & y \mapsto y & z \mapsto z \\
p_x \mapsto \left(1 - \frac{dp}{P_0}\right) p_x & p_y \mapsto \left(1 - \frac{dp}{P_0}\right) p_y & \delta \mapsto \delta - \frac{dp}{P_0}
\end{array}
\tag{100}
$$

where P_0 is the reference momentum. In general, dp is a function of the coordinates.

To find the transformation matrices M and D, we find an explicit expression for dp/P_0, and then write down the above transformations to first order.