Lattice Design in Particle Accelerators

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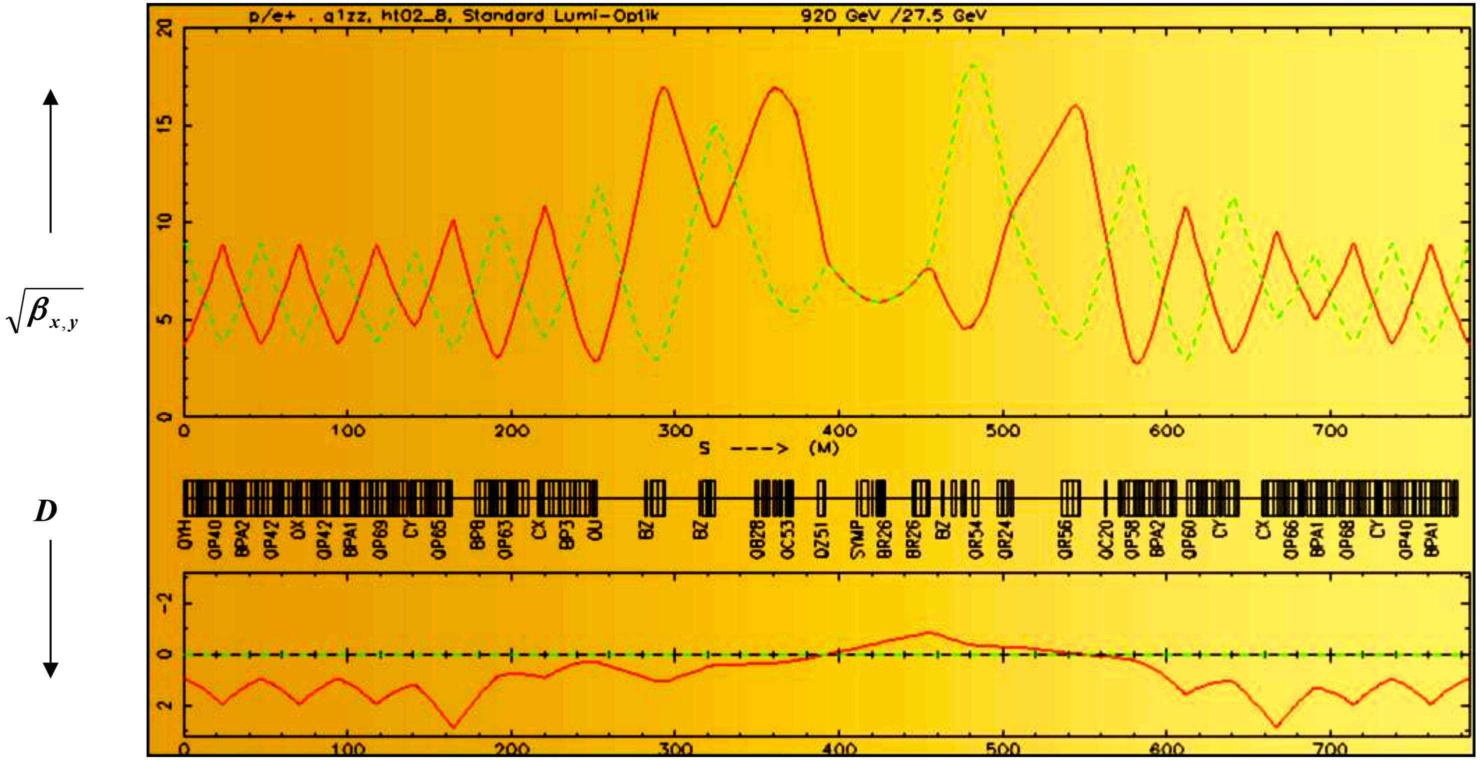


1952: Courant, Livingston, Snyder:

Theory of strong focusing in particle beams

Lattice Design: ***„… how to build a storage ring“***

High energy accelerators → circular machines

somewhere in the lattice we need a number of dipole magnets, that are bending the design orbit to a closed ring

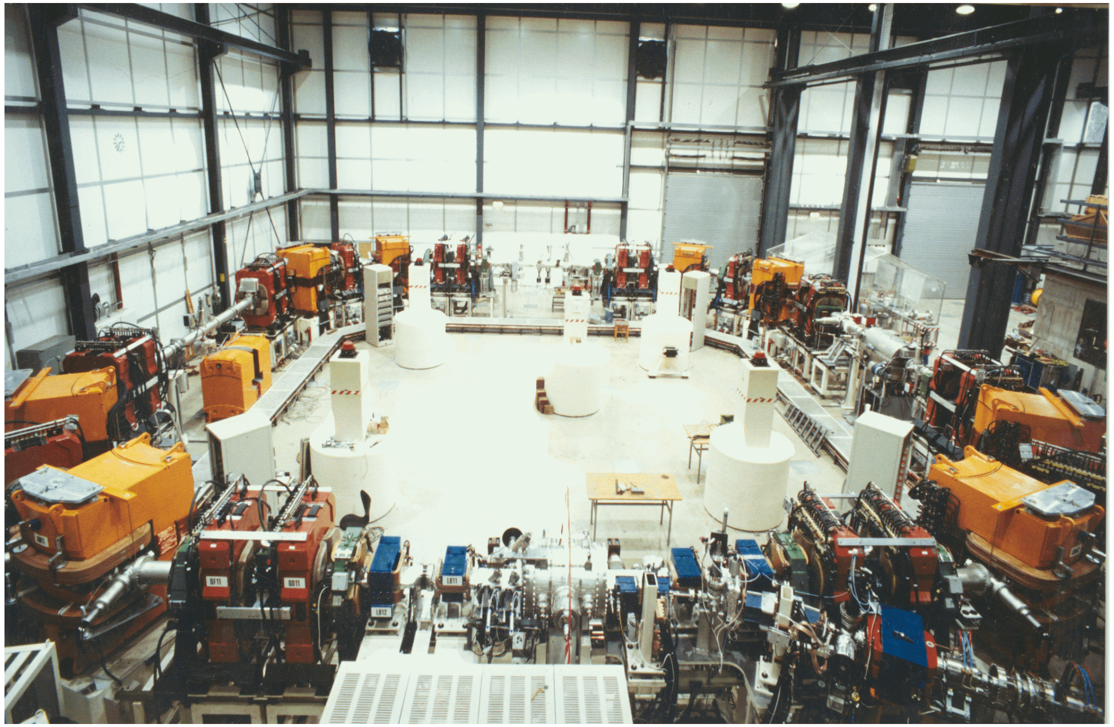
Geometry of the ring:

centrifugal force = Lorentz force

$$e * v * B = \frac{mv^2}{\rho}$$

$$\rightarrow e * B = \frac{mv}{\rho} = p / \rho$$

$$\boxed{\rightarrow B * \rho = p / e}$$



Example: heavy ion storage ring TSR
8 dipole magnets of equal bending strength

p = momentum of the particle,
ρ = curvature radius

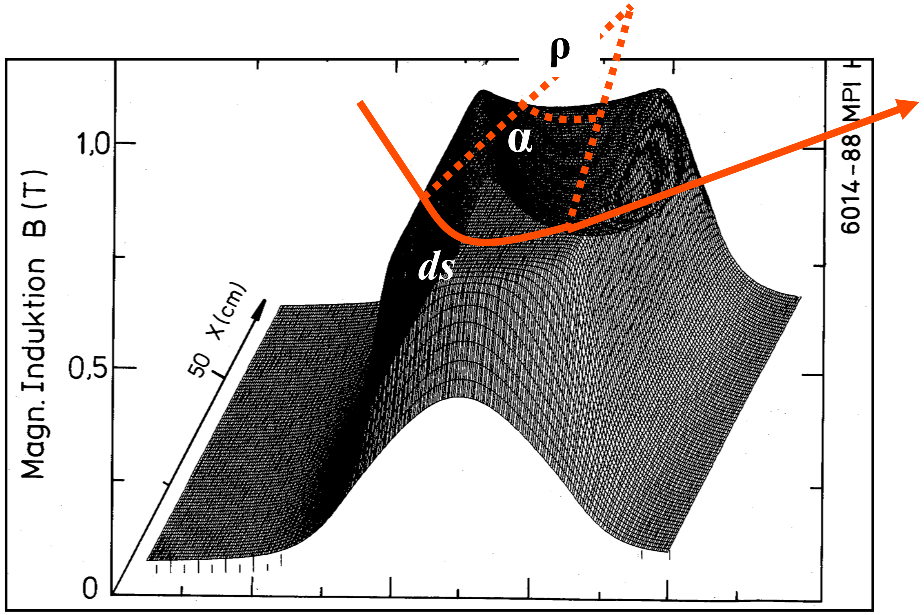
Bρ= beam rigidity

1.) Circular Orbit:

„… defining the geometry"

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho}$$

$$\alpha = \frac{B * dl}{B * \rho}$$



field map of a storage ring dipole magnet

The angle swept out in one revolution must be 2π, so

$$\alpha = \frac{\int B dl}{B * \rho} = 2\pi \qquad \rightarrow \qquad \int B dl = 2\pi * \frac{p}{q}$$

… for a full circle

Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ ***is usually required !!***

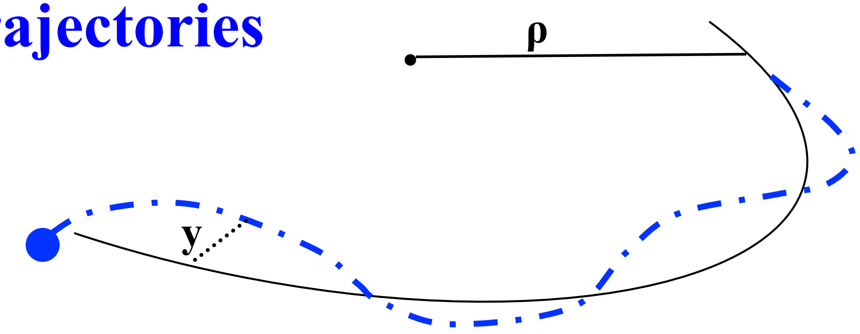
Example LHC:

7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\, dl \approx N\, l\, B = 2\pi\ p / e$$

$$B \approx \frac{2\pi\ \ 7000\ \ 10^{9}\, eV}{1232\ \ 15\, m\ \ 3\ 10^{8}\, \frac{m}{s}\ \ e} = 8.3\ Tesla$$

Focusing Forces: single particle trajectories



$$y'' + K * y = 0$$

$K = -k + 1/\rho^2$ **hor. plane**

$K = k$ **vert. plane**

dipole magnet $\dfrac{1}{\rho} = \dfrac{B}{p/q}$

quadrupole magnet $k = \dfrac{g}{p/q}$

Example: HERA Ring:
Bending radius: $\rho = 580$ m
Quadrupol Gradient: $g = 110$ T/m

$k = 33.64 * 10^{-3}\ /m^2$
$1/\rho^2 = 2.97 * 10^{-6}\ /m^2$

For estimates in large accelerators the weak focusing term $1/\rho^2$ can in general be neglected

Solution for a focusing magnet

$$y(s) = y_0 * \cos(\sqrt{K} * s) + \frac{y'_0}{\sqrt{K}} * \sin(\sqrt{K} * s)$$

$$y'(s) = -y_0 * \sqrt{K} * \sin(\sqrt{K} * s) + y'_0 * \cos(\sqrt{K} * s)$$

Or written more convenient in matrix form:

$$\begin{pmatrix} y \\ y' \end{pmatrix}_s = M * \begin{pmatrix} y \\ y' \end{pmatrix}_0$$

Hor. focusing Quadrupole Magnet

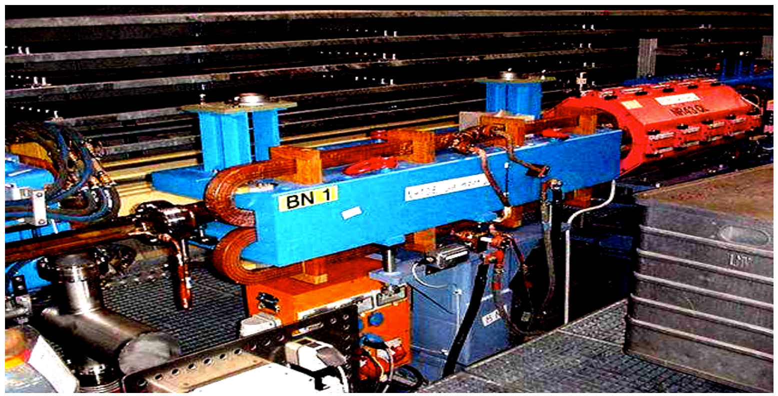
$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

Hor. defocusing Quadrupole Magnet

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sinh(\sqrt{K} * l) \\ \sqrt{K} \sinh(\sqrt{K} * l) & \cosh(\sqrt{K} * l) \end{pmatrix}$$

Drift space

$$M_{Drift} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$



$$M_{lattice} = M_{QF1} * M_{D1} * M_{QD} * M_{D1} * M_{QF2} \ldots$$

8.) Transfer Matrix M … yes we had the topic already

general solution of Hill´s equation

$$\left\{\begin{aligned} x(s) &= \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \\ x'(s) &= \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right] \end{aligned}\right.$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point s(0) = s_0, where we put Ψ(0) = 0

$$\left.\begin{aligned} \cos\phi &= \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \\ \sin\phi &= -\frac{1}{\sqrt{\varepsilon}}\left(x'_0\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{aligned}\right\}$$

inserting above …

$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s + \alpha_0 \sin\psi_s\right\} x_0 + \left\{\sqrt{\beta_s \beta_0}\,\sin\psi_s\right\} x_0'$$

$$x'(s) = \frac{1}{\sqrt{\beta_s \beta_0}}\left\{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s\right\} x_0 + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s - \alpha_s \sin\psi_s\right\} x_0'$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

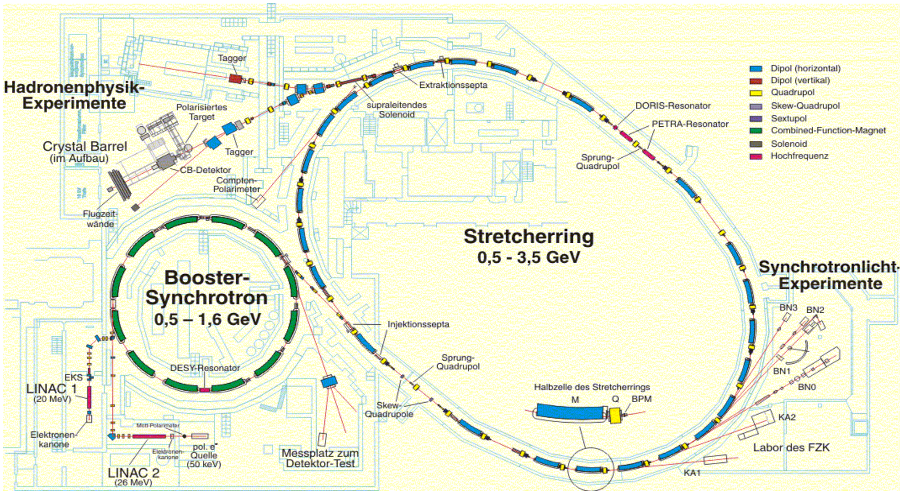
*** we can calculate the single particle trajectories between two locations in the ring,**
if we know the α β γ at these positions.
*** and nothing but the α β γ at these positions.**

*** ... !**

*** Äquivalenz der Matrizen**

9.) Periodic Lattices

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$



DELTA Electron Storage Ring

„This **rather formidable looking matrix** simplifies considerably if we **consider one complete revolution** …“

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

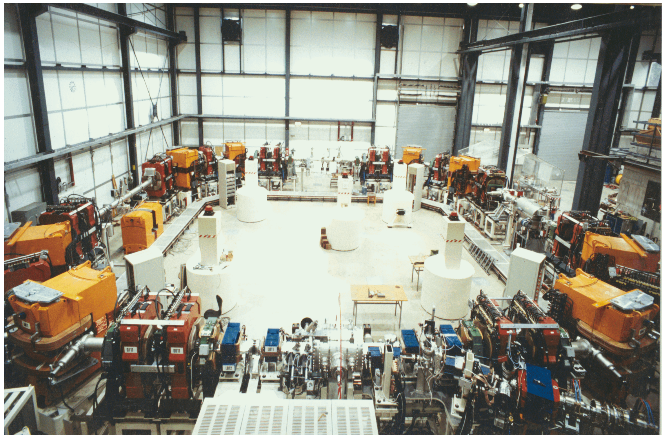
ψ_{turn} = phase advance per period

Tune: Phase advance per turn in units of 2π

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

Stability Criterion:

Question: **what will happen, if we do not make too many mistakes and your particle performs one complete turn ?**



Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{1}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Matrix for N turns:

$$M^N = \left(1 \cdot \cos\psi + J \cdot \sin\psi\right)^N = 1 \cdot \cos N\psi + J \cdot \sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

$$\psi = real \quad \leftrightarrow \quad |\cos\psi| \le 1 \quad \leftrightarrow \quad Tr(M) \le 2$$

stability criterion …. proof for the disbelieving collegues !!

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{I}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Matrix for 2 turns:

$$\boldsymbol{M}^2 = (\boldsymbol{I}\cos\psi_1 + \boldsymbol{J}\sin\psi_1)(\boldsymbol{I}\cos\psi_2 + \boldsymbol{J}\sin\psi_2)$$

$$= \boldsymbol{I}^2 \cos\psi_1 \cos\psi_2 + \boldsymbol{IJ}\cos\psi_1 \sin\psi_2 + \boldsymbol{JI}\sin\psi_1 \cos\psi_2 + \boldsymbol{J}^2 \sin\psi_1 \sin\psi_2$$

now …

$$\boldsymbol{I}^2 = \boldsymbol{I}$$

$$\boldsymbol{I}\,\boldsymbol{J} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\boldsymbol{J}\,\boldsymbol{I} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\boldsymbol{I}\,\boldsymbol{J} = \boldsymbol{J}\,\boldsymbol{I}$$

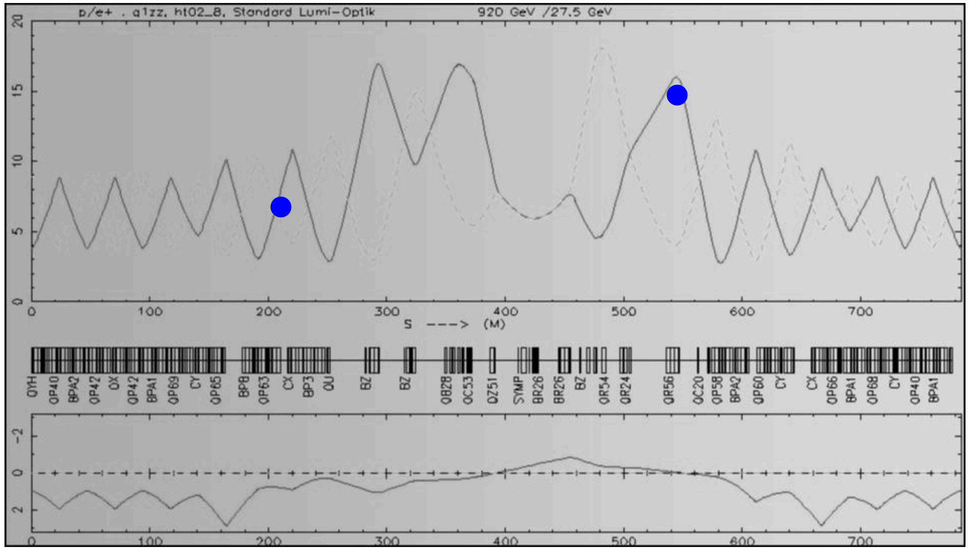
$$\boldsymbol{J}^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\boldsymbol{I}$$

$$\boldsymbol{M}^2 = \boldsymbol{I}\cos(\psi_1 + \psi_2) + \boldsymbol{J}\sin(\psi_1 + \psi_2)$$

$$\boldsymbol{M}^2 = \boldsymbol{I}\cos(2\psi) + \boldsymbol{J}\sin(2\psi)$$

10.) Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s



Betafunction in a Storage Ring

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

since ε = const (Liouville):

$$\varepsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0{}^2$$

... remember W = CS´-SC´ = 1

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} * \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

$$\rightarrow \quad x_0 = m_{22}x - m_{12}x'$$

$$x_0' = -m_{21}x + m_{11}x'$$

... inserting into ε

$$\varepsilon = \beta_0 (m_{11}x' - m_{21}x)^2 + 2\alpha_0 (m_{22}x - m_{12}x')(m_{11}x' - m_{21}x) + \gamma_0 (m_{22}x - m_{12}x')^2$$

sort via x, x´and compare the coefficients to get ....

The Twiss parameters α, β, γ can be transformed through the lattice via the matrix elements defined above.

$$\beta(s) = m_{11}^2\,\beta_0 - 2m_{11}m_{12}\,\alpha_0 + m_{12}^2\,\gamma_0$$

$$\alpha(s) = -m_{11}m_{21}\,\beta_0 + (m_{12}m_{21} + m_{11}m_{22})\,\alpha_0 - m_{12}m_{22}\,\gamma_0$$

$$\gamma(s) = m_{21}^2\,\beta_0 - 2m_{21}m_{22}\,\alpha_0 + m_{22}^2\,\gamma_0$$

in matrix notation:

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_{s1}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

... and here starts the lattice design !!!

Most simple example: drift space

$$M_{drift} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

particle coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 \quad \rightarrow \quad \boxed{\begin{aligned} x(l) &= x_0 + l * x_0' \\ x'(l) &= x_0' \end{aligned}}$$

transformation of twiss parameters:

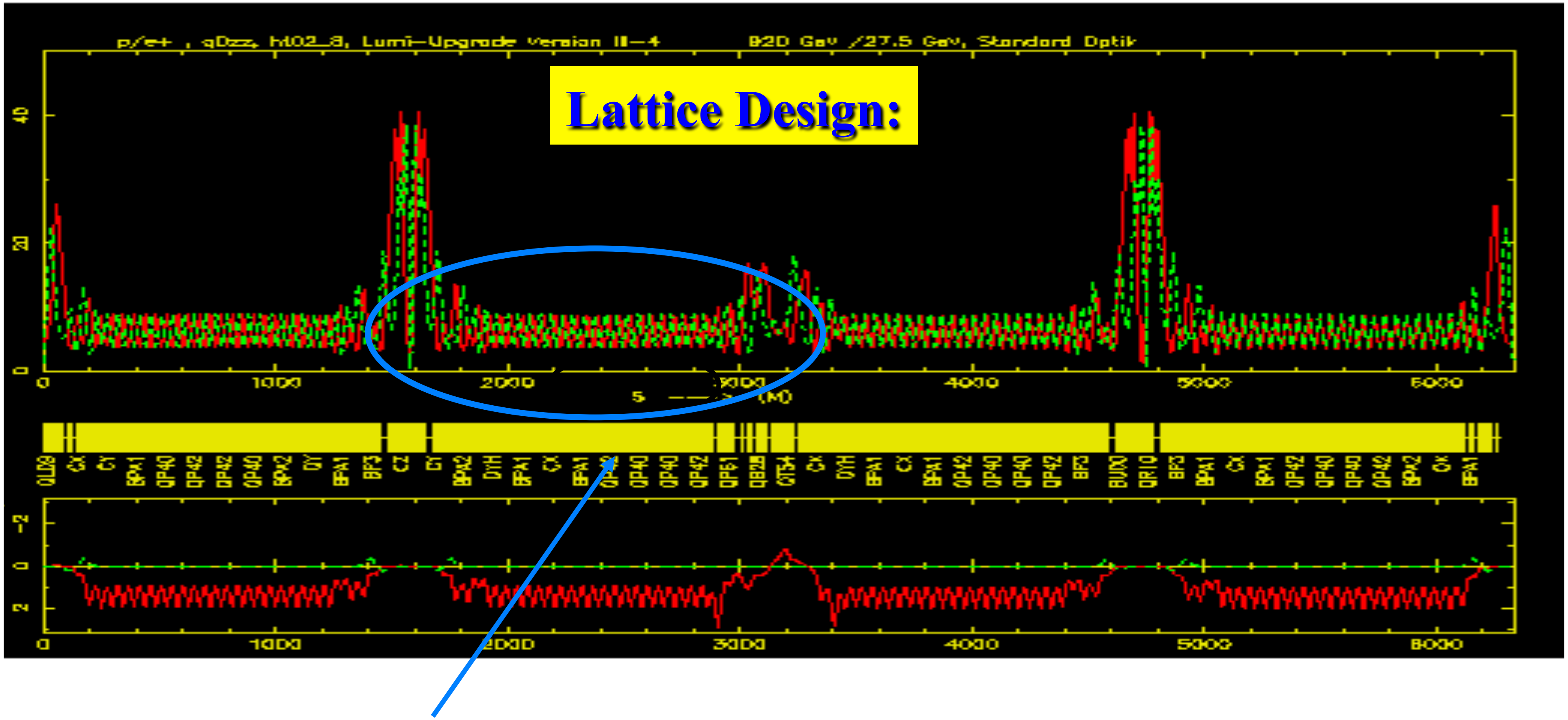
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

$$\boxed{\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0}$$

Stability ...?

$$trace(M) = 1 + 1 = 2$$

→A periodic solution doesn't exist in a lattice built exclusively out of drift spaces.



Arc: **regular** **(periodic) magnet structure:**

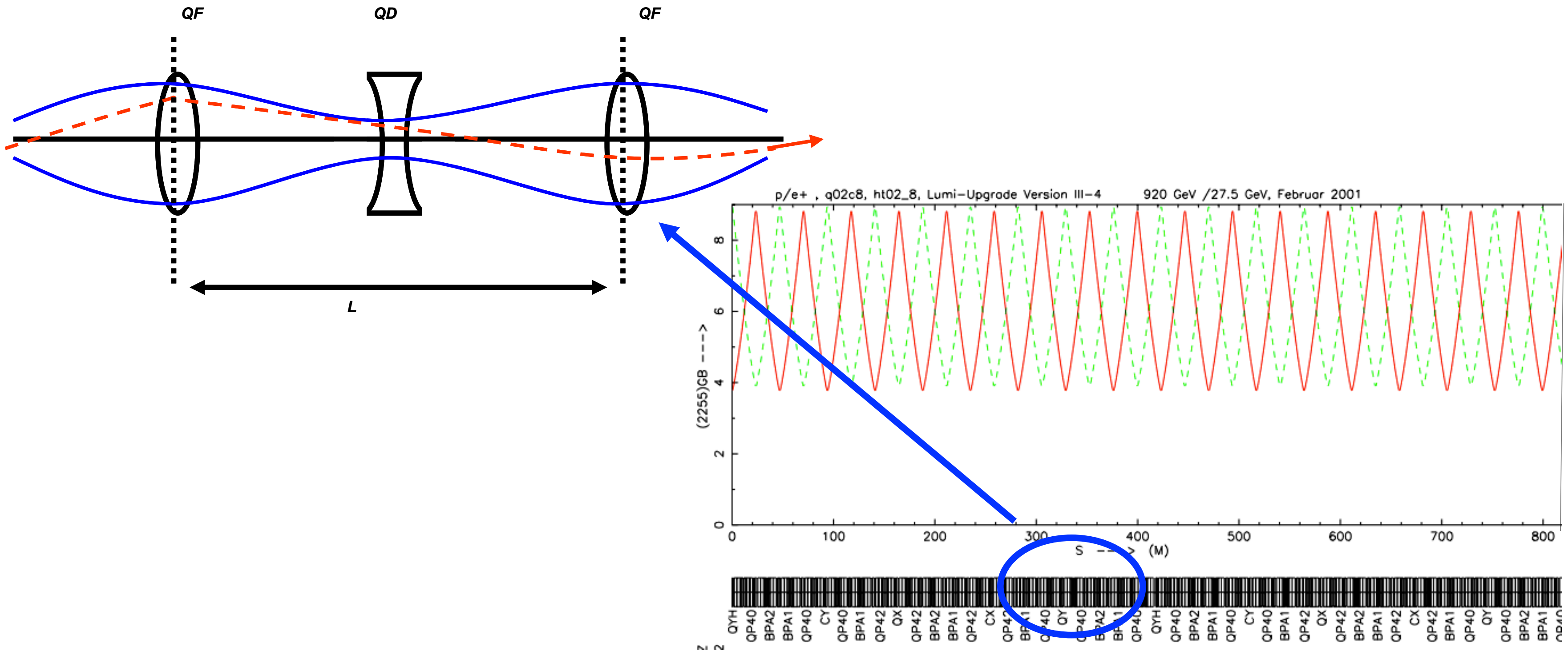
bending magnets → define the energy of the ring
main focusing & tune control, chromaticity correction,
multipoles for higher order corrections

Straight sections: **drift spaces for injection, dispersion suppressors,**
low beta insertions, RF cavities, etc....

... and the high energy experiments if they cannot be avoided

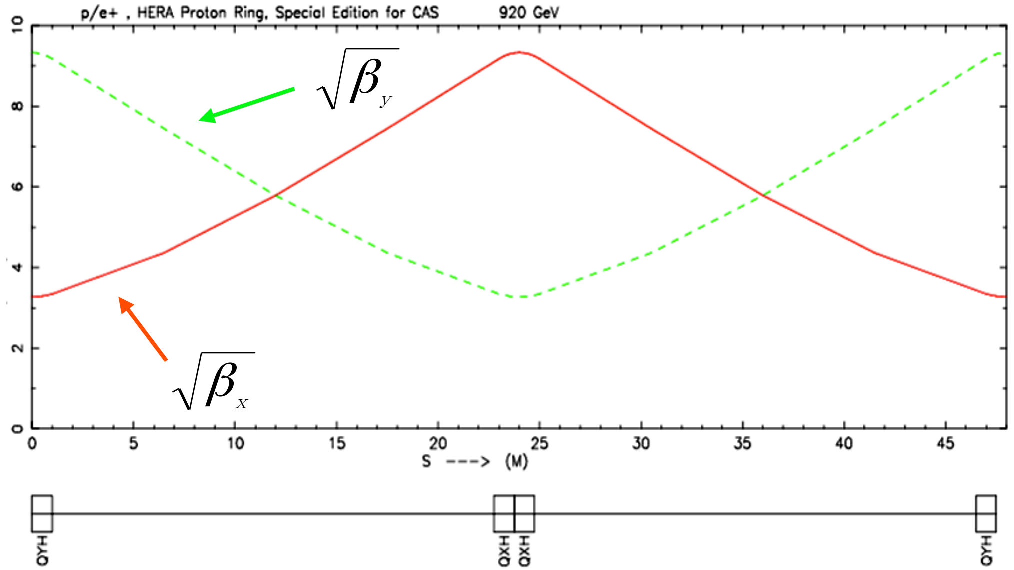
3.) The FoDo-Lattice

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell μ = 45°,
→ calculate the twiss parameters for a periodic solution

Periodic Solution of a FoDo Cell



Output of the optics program:

Nr	*Type*	*Length*	*Strength*	β_x	α_x	φ_x	β_z	α_z	φ_z
		m	*1/m2*	*m*		*1/2π*	*m*		*1/2π*
0	*IP*	*0,000*	*0,000*	*11,611*	*0,000*	*0,000*	*5,295*	*0,000*	*0,000*
1	*QFH*	*0,250*	*-0,541*	*11,228*	*1,514*	*0,004*	*5,488*	*-0,781*	*0,007*
2	*QD*	*3,251*	*0,541*	*5,488*	*-0,781*	*0,070*	*11,228*	*1,514*	*0,066*
3	*QFH*	*6,002*	*-0,541*	*11,611*	*0,000*	*0,125*	*5,295*	*0,000*	*0,125*
4	*IP*	*6,002*	*0,000*	*11,611*	*0,000*	*0,125*	*5,295*	*0,000*	*0,125*

QX= 0,125 QZ= 0,125

0.125 * 2π = 45⁰

Can we understand what the optics code is doing ?

matrices

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l_q) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l_q) \\ -\sqrt{K} \sin(\sqrt{K} * l_q) & \cos(\sqrt{K} * l_q) \end{pmatrix}, \qquad M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix}$$

strength and length of the FoDo elements

K = +/- 0.54102 m^{-2}
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qfh}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for 1 period gives us all the information that we need !

1.) is the motion stable?

$$trace(M_{FoDo}) = 1.415 \rightarrow \quad < 2$$

2.) Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_s \sin\psi_{cell} & \beta_s \sin\psi_{cell} \\ -\gamma_s \sin\psi_{cell} & \cos\psi_{cell} - \alpha_s \sin\psi_{cell} \end{pmatrix}$$

$$\cos\psi_{cell} = \frac{1}{2} trace(M) = 0.707$$

$$\psi_{cell} = \cos^{-1}\left(\frac{1}{2} trace(M)\right) = 45$$

3.) hor β-function

$$\beta = \frac{m_{12}}{\sin\psi_{cell}} = 11.611\, m$$

4.) hor α-function

$$\alpha = \frac{m_{11} - \cos\psi_{cell}}{\sin\psi_{cell}} = 0$$

Can we do a bit easier ?

We can ... in thin lens approximation !

Matrix of a focusing quadrupole magnet:

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

If the focal length *f* is much larger than the length of the quadrupole magnet,

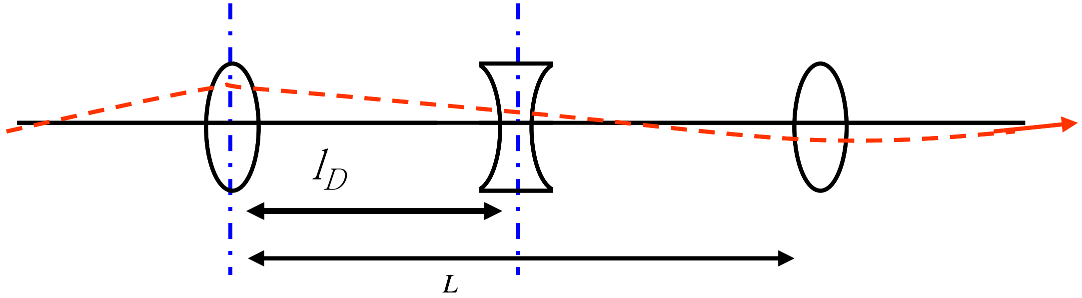
$$f = 1/kl_Q >> l_Q$$

the transfer matrix can be aproximated using ♪

$$kl_q = const, \; l_q \to 0$$

$$M = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

4.) FoDo in thin lens approximation



$$l_D = L/2$$
$$\tilde{f} = 2f$$

Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

$$M_{halfCell} = M_{QD/2} * M_{lD} * M_{QF/2}$$

$$M_{halfCell} = \begin{pmatrix} 1 & 0 \\ 1/\tilde{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix}$$

note: $\tilde{f}$ denotes the focusing strength of half a quadrupole, so $\tilde{f} = 2f$

$$M_{halfCell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

for the second half cell set f → -f

FoDo in thin lens approximation

Matrix for the complete FoDo cell

$$M = \begin{pmatrix} 1 + {l_D}/{\tilde f} & l_D \\ -{l_D}/{\tilde f^2} & 1 - {l_D}/{\tilde f} \end{pmatrix} * \begin{pmatrix} 1 - {l_D}/{\tilde f} & l_D \\ -{l_D}/{\tilde f^2} & 1 + {l_D}/{\tilde f} \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{2 l_D^2}{\tilde f^2} & 2 l_D (1 + \frac{l_D}{\tilde f}) \\ 2(\frac{l_D^2}{\tilde f^3} - \frac{l_D}{\tilde f^2}) & 1 - 2\frac{l_D^2}{\tilde f^2} \end{pmatrix}$$

Now we know, that the phase advance is related to the transfer matrix by

$$\cos\psi_{cell} = \frac{1}{2} trace(M) = \frac{1}{2} * (2 - \frac{4 l_d^2}{\tilde f^2}) = 1 - \frac{2 l_d^2}{\tilde f^2}$$

After some beer and with a little bit of trigonometric gymnastics

$$\cos(x) = \cos^2({x}/{2}) - \sin^2(x/2) = 1 - 2\sin^2({x}/{2})$$

we can calculate the phase advance as a function of the FoDo parameter …

$$\cos\psi_{cell} = 1 - 2\sin^2(\psi_{cell}/2) = 1 - \frac{2l_d^2}{\tilde{f}^2}$$

$$\sin(\psi_{cell}/2) = l_d/\tilde{f} = \frac{L_{cell}}{2\tilde{f}}$$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f}$$

Example:
45-degree Cell

$$L_{Cell} = l_{QF} + l_D + l_{QD} + l_D = 0.5m+2.5m+0.5m+2.5m = 6m$$

$$1/f = k*l_Q = 0.5m*0.541\ m^{-2} = 0.27\ m^{-1}$$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f} = 0.405$$

$$\rightarrow \quad \psi_{cell} = 47.8^\circ$$

$$\rightarrow \quad \beta = 11.4\ m$$

Remember:
Exact calculation yields:

$$\rightarrow \quad \psi_{cell} = 45^\circ$$

$$\rightarrow \quad \beta = 11.6\ m$$

Stability in a FoDo structure

SPS Lattice

$$M_{FoDo} = \begin{pmatrix} 1 - \dfrac{2l_D^2}{\tilde{f}^2} & 2l_D(1 + \dfrac{l_D}{\tilde{f}}) \\ 2(\dfrac{l_D^2}{\tilde{f}^3} - \dfrac{l_D}{\tilde{f}^2}) & 1 - 2\dfrac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Stability requires:

$$|\boldsymbol{Trace}(\boldsymbol{M})| < 2$$

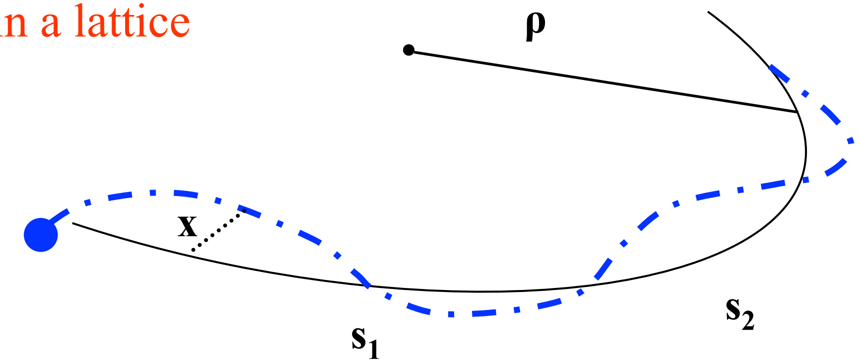
$$|\boldsymbol{Trace}(\boldsymbol{M})| = \left| 2 - \frac{4\boldsymbol{l}_d^2}{\tilde{\boldsymbol{f}}^2} \right| < 2$$

$$\boxed{\rightarrow f > \frac{L_{cell}}{4}}$$

For stability the focal length has to be larger than a quarter of the cell length
... don't focus to strong !

Transformation Matrix in Terms of the Twiss Parameters

Transformation of the coordinate vector (x,x′) in a lattice



$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M_{s1,s2} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

General solution of the equation of motion

$$x(s) = \sqrt{\varepsilon * \beta(s)} * \cos(\psi(s) + \varphi)$$

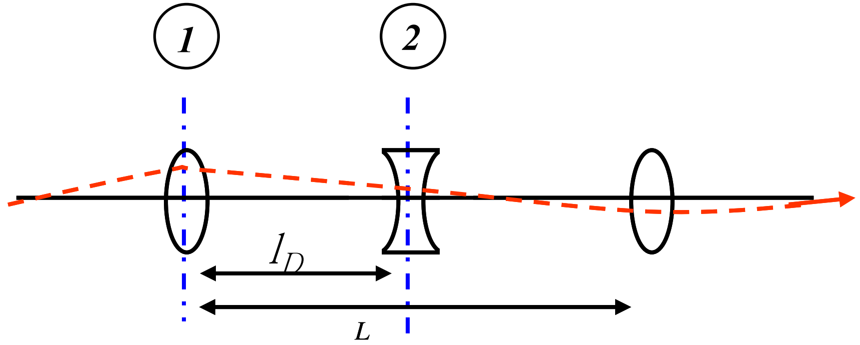
$$x'(s) = \sqrt{\varepsilon / \beta(s)} * \{ \alpha(s)\cos(\psi(s) + \varphi) + \sin(\psi(s) + \varphi) \}$$

Transformation of the coordinate vector (x,x′)
expressed as a function of the twiss parameters

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

Transfer Matrix for half a FoDo cell:

$$M_{halfcell} = \begin{pmatrix} 1 - \frac{l_D}{\tilde{f}} & l_D \\ -\frac{l_D}{\tilde{f}^2} & 1 + \frac{l_D}{\tilde{f}} \end{pmatrix}$$



Compare to the twiss parameter form of M

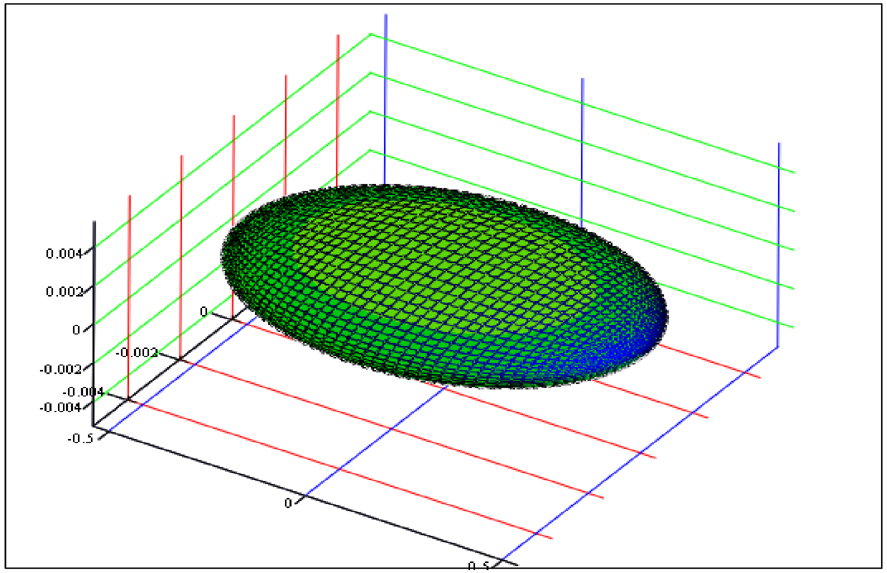
$$M_{1 \to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1 \beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

In the middle of a foc (defoc) quadrupole of the FoDo we allways have $\alpha = 0$, and the half cell will lead us from β_{max} to β_{min}

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}}\cos\frac{\psi_{cell}}{2} & \sqrt{\check{\beta}\hat{\beta}}\sin\frac{\psi_{cell}}{2} \\ \frac{-1}{\sqrt{\hat{\beta}\check{\beta}}}\sin\frac{\psi_{cell}}{2} & \sqrt{\frac{\hat{\beta}}{\check{\beta}}}\cos\frac{\psi_{cell}}{2} \end{pmatrix}$$

Solving for β_{max} and β_{min} and remembering that …. $\sin\frac{\psi_{cell}}{2} = \frac{l_d}{\tilde{f}} = \frac{L}{4f}$

$$\left.\begin{aligned} &\frac{m_{22}}{m_{11}} = \frac{\hat{\beta}}{\breve{\beta}} = \frac{1 + l_d/\tilde{f}}{1 - l_d/\tilde{f}} = \frac{1+\sin(\psi_{cell}/2)}{1-\sin(\psi_{cell}/2)} \\ &\frac{m_{12}}{m_{21}} = \hat{\beta}\breve{\beta} = \tilde{f}^{\,2} = \frac{l_d^2}{\sin^2(\psi_{cell}/2)} \end{aligned}\right\} \rightarrow \quad \hat{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \; ! \qquad \breve{\beta} = \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \; !$$



(Z, X, Y)

The maximum and minimum values of the β-function are solely determined by the phase advance and the length of the cell.

Longer cells lead to larger β

typical shape of a proton bunch in a FoDo Cell

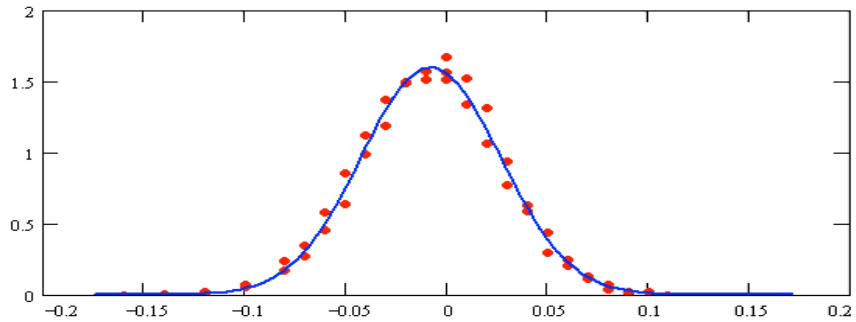
5.) Beam dimension:

Optimisation of the FoDo Phase advance

In both planes a gaussian particle distribution is assumed, given by the beam emittance ε and the β-function

$$\sigma = \sqrt{\varepsilon\beta}$$

HERA beam size

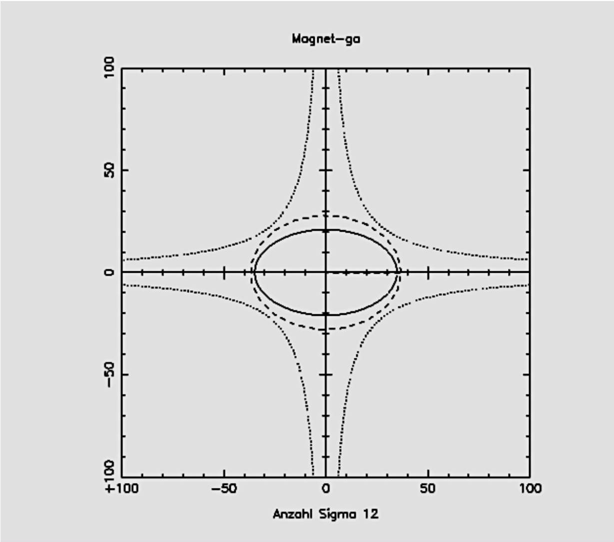


In general proton beams are *„round"* in the sense that

$$\varepsilon_x \approx \varepsilon_y$$

So for highest aperture we have to minimise the β-function in both planes:

$$r^2 = \varepsilon_x\beta_x + \varepsilon_y\beta_y$$



typical beam envelope, vacuum chamber and pole shape in a foc. Quadrupole lens in HERA

Optimisation of the FoDo Phase advance

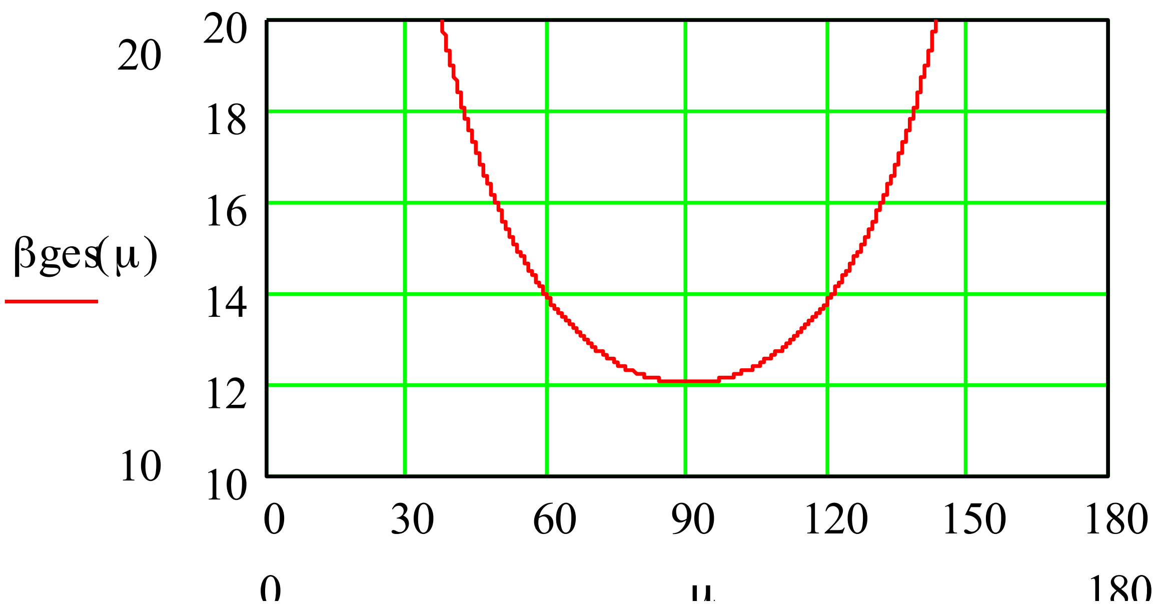
$$r^2 = \varepsilon_x \beta_x + \varepsilon_y \beta_y$$

search for the phase advance μ that results in a minimum of the sum of the beta's

$$\hat{\beta} + \breve{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} + \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}}$$

$$\hat{\beta} + \breve{\beta} = \frac{2L}{\sin\psi_{cell}}$$

$$\frac{d}{d\psi_{cell}}\left(\frac{2L}{\sin\psi_{cell}}\right) = 0$$



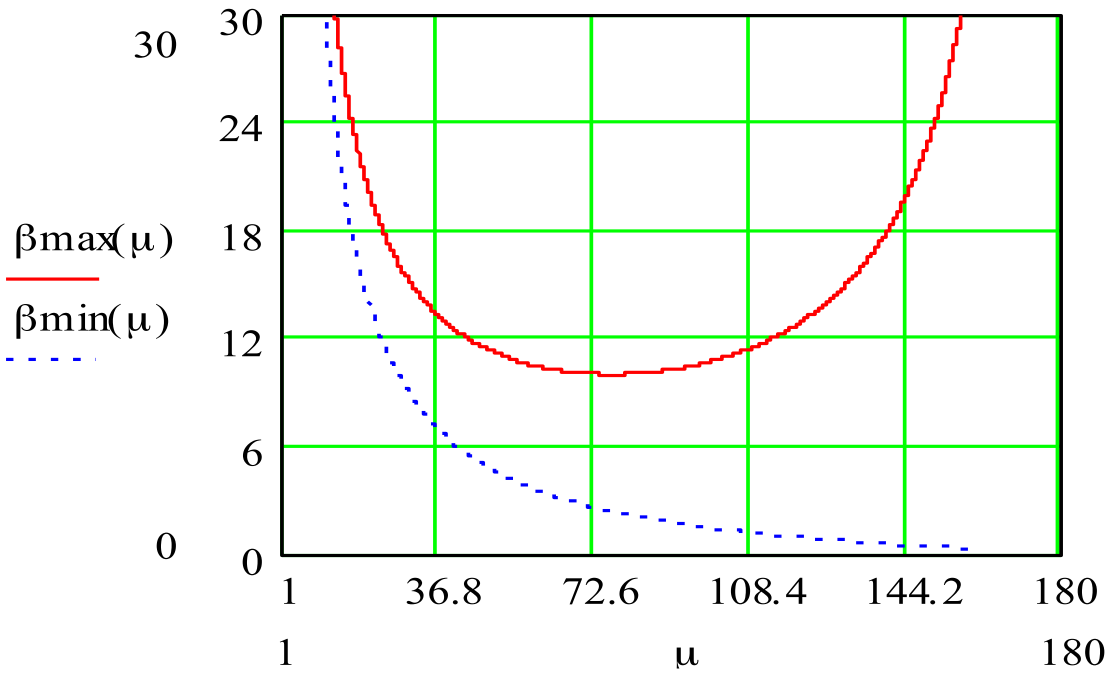
$$\frac{L}{\sin^2\psi_{cell}} * \cos\psi_{cell} = 0 \quad \rightarrow \quad \psi_{cell} = 90^\circ$$

Electrons are different

electron beams are usually flat, $\varepsilon_y \approx 2 - 10\ \%\ \varepsilon_x$

→ optimise only β_{hor}

$$\frac{d}{d\psi_{cell}}(\hat{\beta}) = \frac{d}{d\psi_{cell}} \frac{L(1+\sin\frac{\psi_{cell}}{2})}{\sin\psi_{cell}} = 0 \quad \rightarrow \quad \psi_{cell} = 76^\circ$$

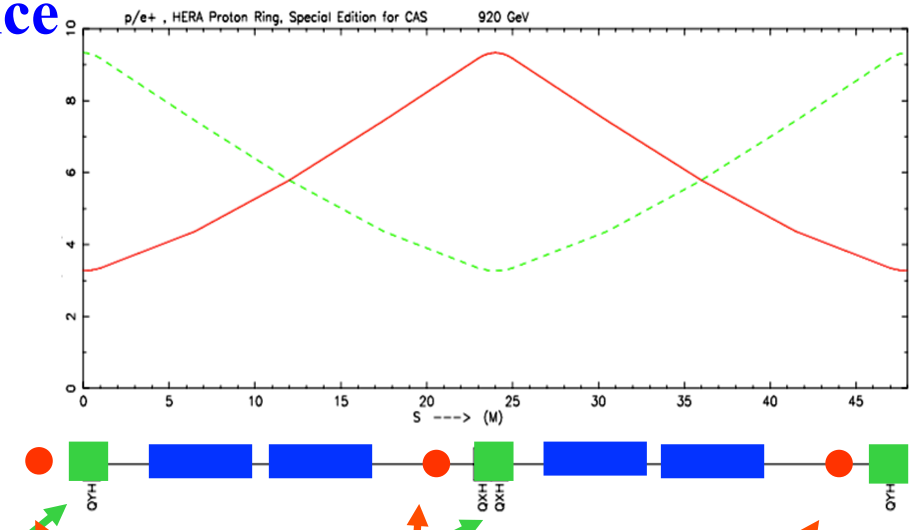


red curve: $\boldsymbol{\beta_{max}}$

blue curve: β_{min}

as a function of the phase advance ψ

Orbit distortions in a periodic lattice



field error of a dipole/distorted quadrupole

$$\rightarrow \delta(mrad) = \frac{ds}{\rho} = \frac{\int B ds}{p/e}$$

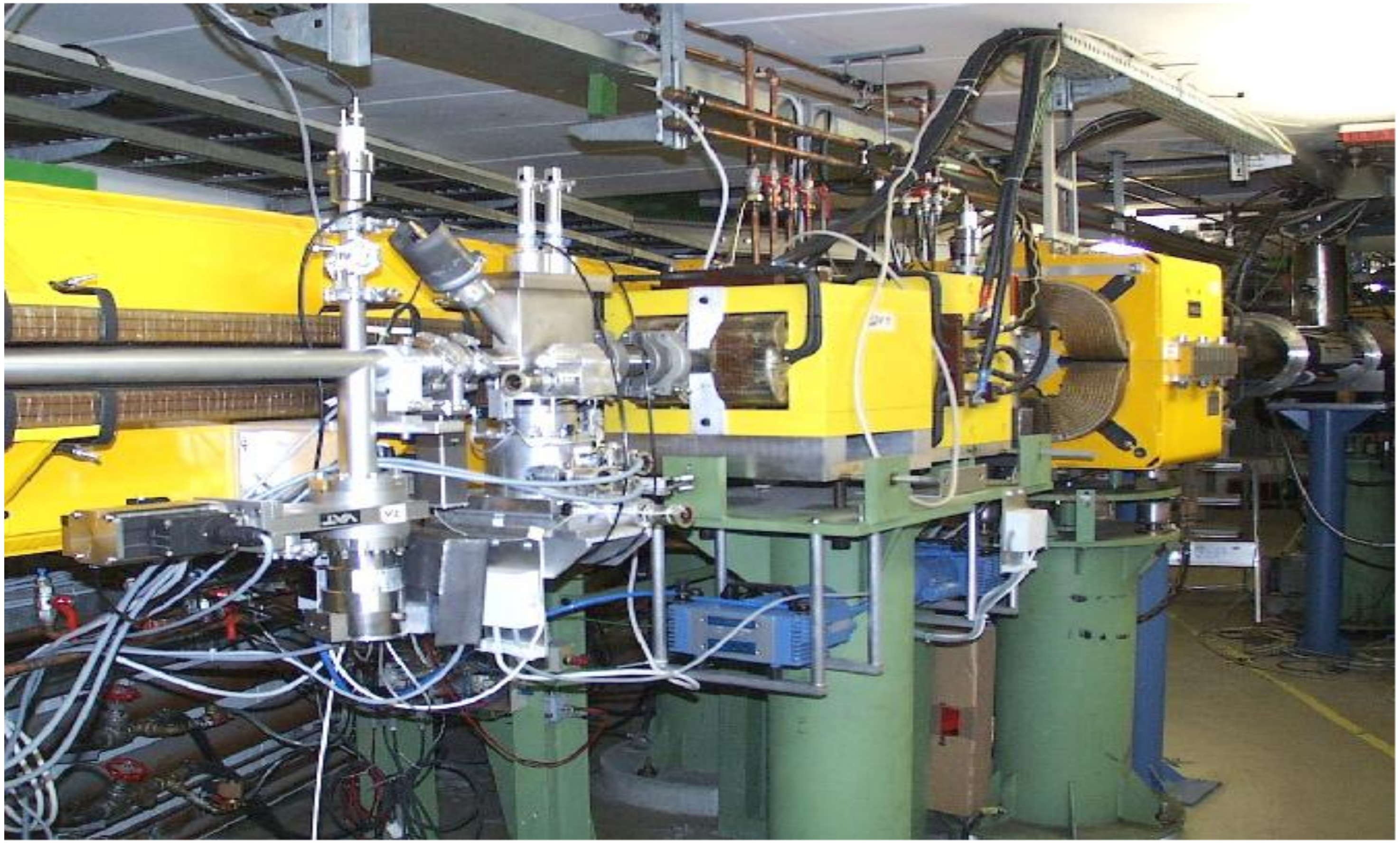
the particle will follow a new closed trajectory, the distorted orbit:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} * \oint \sqrt{\beta(\tilde{s})} \frac{1}{\rho(\tilde{s})} \cos(|\psi(\tilde{s}) - \psi(s)| - \pi Q) d\tilde{s}$$

* **the orbit amplitude will be large if the β function at the location of the kick is large**
$\beta(\tilde{s})$ **indicates the sensitivity of the beam → here orbit correctors should be placed in the lattice**

* **the orbit amplitude will be large at places where in the lattice β(s) is large → here beam position monitors should be installed**

Orbit Correctors and Beam Instrumentation in a Storage Ring



Elsa ring, Bonn

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Résumé

1.) Dipole strength

$$\int B ds = N * B_0 * l_{eff} = 2\pi \frac{p}{q}$$

l_{eff} effective magnet length, N number of magnets

2.) Stability condition

$$Trace(M) < 2$$

for periodic structures within the lattice / at least for the transfer matrix of the complete circular machine

3.) Transfer matrix for periodic cell

$$M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_s \sin\psi_{cell} & \beta_s \sin\psi_{cell} \\ -\gamma_s \sin\psi_{cell} & \cos\psi_{cell} - \alpha_s \sin\psi_{cell} \end{pmatrix}$$

α,β,γ depend on the position s in the ring, μ (phase advance) is independent of s

4.) Thin lens approximation

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_Q} & 1 \end{pmatrix} \qquad f_Q = \frac{1}{k_Q l_Q}$$

focal length of the quadrupole magnet $f_Q = 1/(k_Q l_{Q)} >> l_Q$

5.) Tune (rough estimate)

$$\psi_{period} = \int_{s}^{s+L} \frac{ds}{\beta(s)}$$

Tune = phase advance in units of 2π

$$Q = N * \frac{\psi_{period}}{2\pi} = \frac{1}{2\pi} * \oint \frac{ds}{\beta(s)} \approx \frac{1}{2\pi} * \frac{2\pi\overline{R}}{\overline{\beta}} = \frac{\overline{R}}{\overline{\beta}}$$

$\overline{R}$, $\overline{\beta}$ average radius and β-function

$$Q \approx \frac{\overline{R}}{\overline{\beta}}$$

6.) Phase advance per FoDo cell

(thin lens appro

$$\sin\frac{\psi_{cell}}{2} = \frac{l_d}{\tilde{f}} = \frac{L_{cell}}{4 f_Q}$$

L_{Cell} length of the complete FoDo cell, f_Q focal length of the quadrupole, μ phase advance per cell

7.) Stability in a FoDo cell

(thin lens approx)

$$f_Q > \frac{L_{Cell}}{4}$$

8.) Beta functions in a FoDo cell

(thin lens approx)

$$\hat{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L_{cell}}{\sin\psi_{cell}} \qquad \breve{\beta} = \frac{(1-\sin\frac{\psi_{cell}}{2})L_{cell}}{\sin\psi_{cell}}$$

L_{Cell} length of the complete FoDo cell, μ phase advance per cell