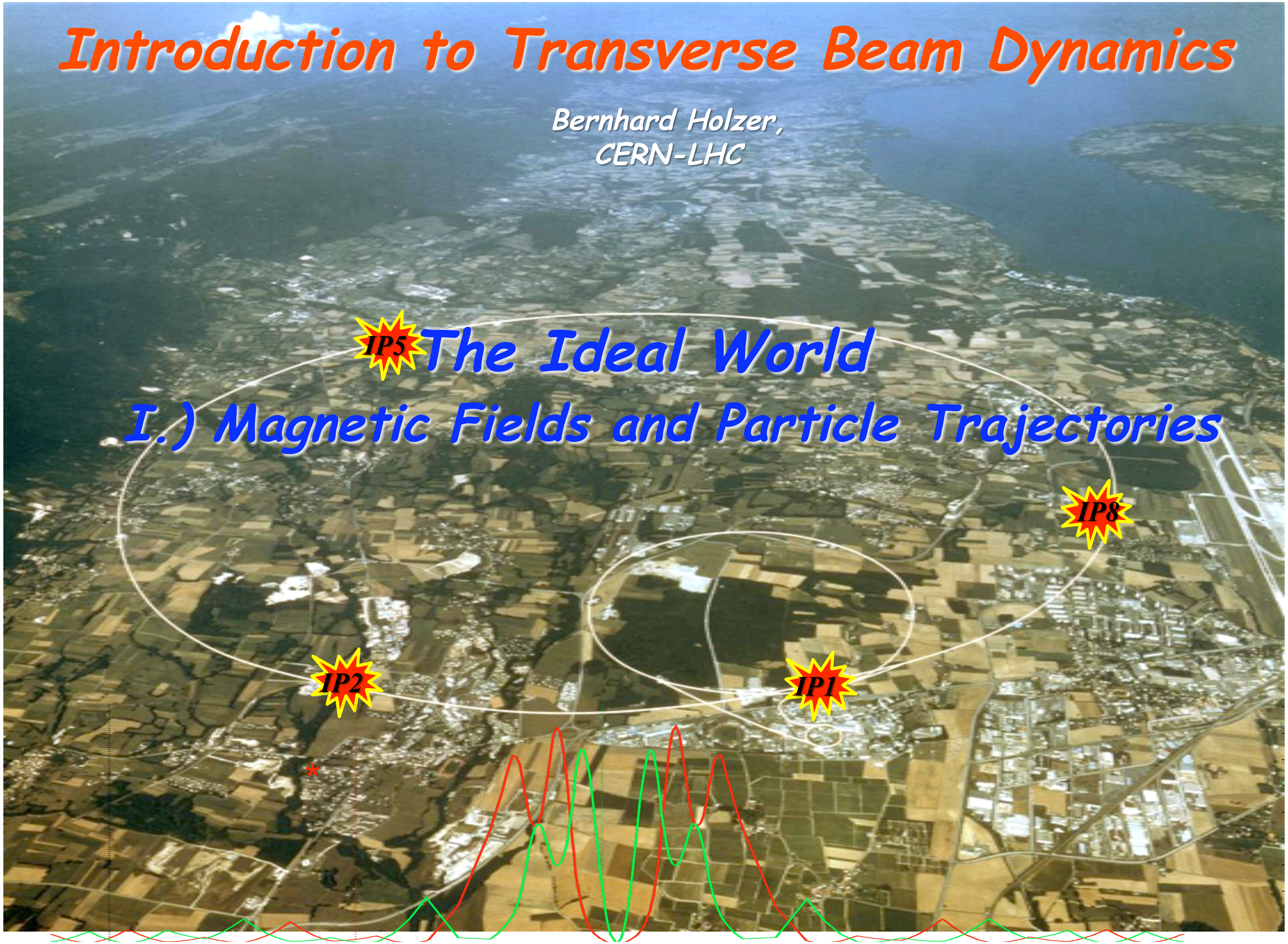
Introduction to Transverse Beam Dynamics

Bernhard Holzer,
CERN-LHC

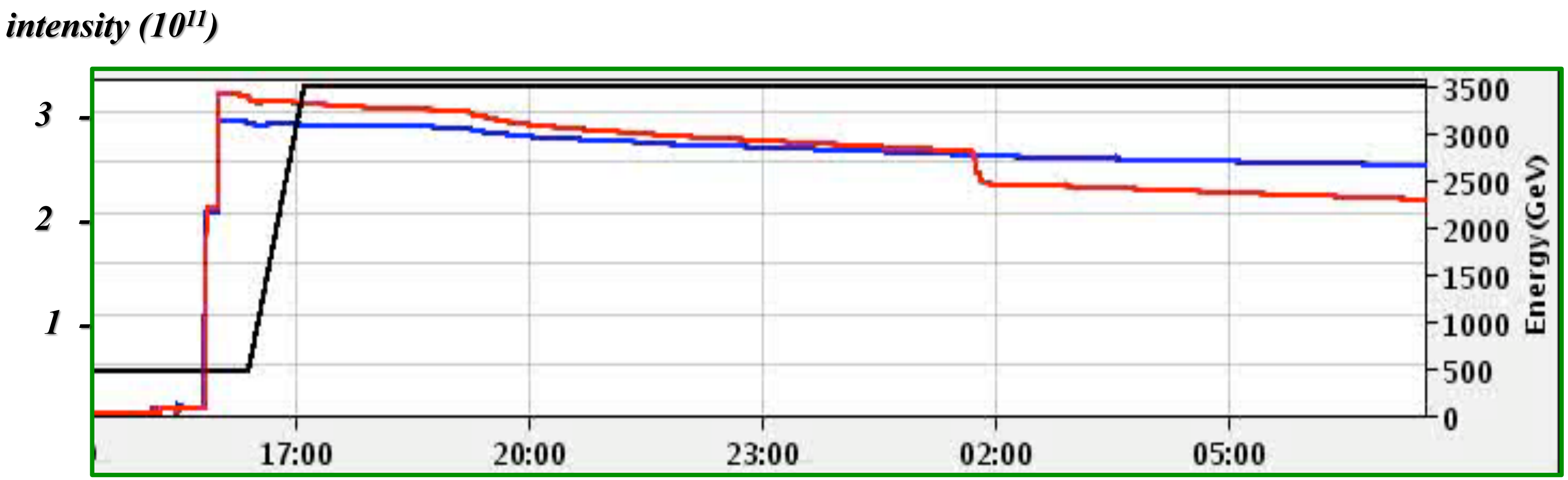
The Ideal World

I.) Magnetic Fields and Particle Trajectories



Luminosity Run of a typical storage ring:

LHC Storage Ring: Protons accelerated and stored for 12 hours
distance of particles travelling at about v ≈ c
$L = 10^{10}$-10^{11} km
... several times Sun - Pluto and back



→ ***guide the particles on a well defined orbit („design orbit“)***
→ ***focus the particles to keep each single particle trajectory***
within the vacuum chamber of the storage ring, i.e. close to the design orbit.

1.) Introduction and Basic Ideas

„ ... in the end and after all it should be a kind of circular machine“

→ need transverse deflecting force

Lorentz force $$\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$$

typical velocity in high energy machines: $$v \approx c \approx 3 * 10^8 \, {}^{m}\!/\!{}_{s}$$

Example:

$$B = 1\,T \quad \rightarrow \quad F = q * 3 * 10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}$$

$$F = q * \underbrace{300 \frac{MV}{m}}_{E}$$

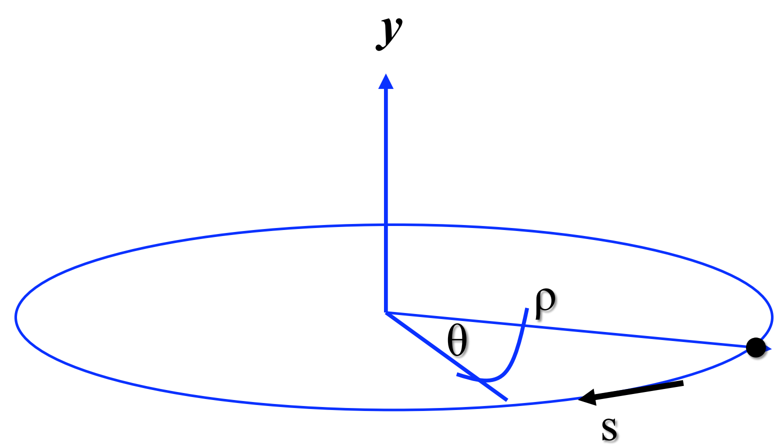
equivalent el. field ... E

technical limit for el. field:

$$E \leq 1 \frac{MV}{m}$$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $$F_L = e\,v\,B$$

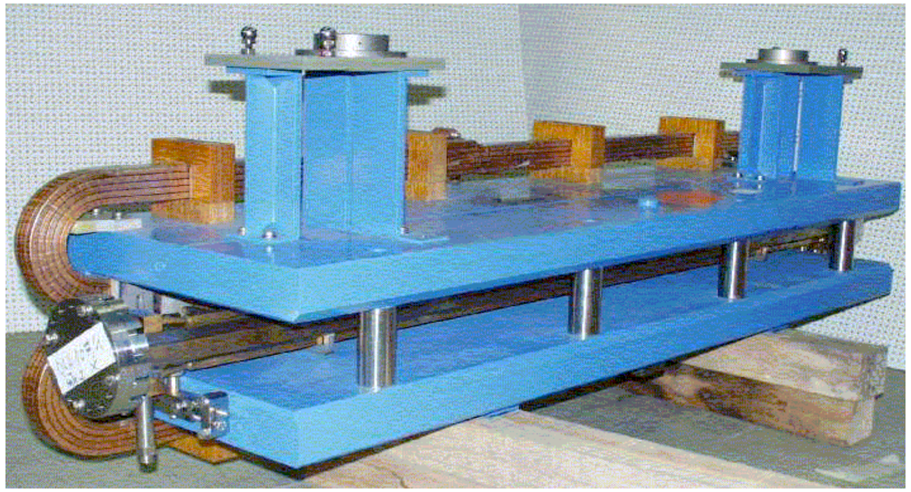
centrifugal force $$F_{centr} = \frac{\gamma\, m_0\, v^2}{\rho}$$

$$\frac{\gamma\, m_0\, v^{\cancel{2}}}{\rho} = e\,\cancel{v}\,B$$

$$\boxed{\frac{p}{e} = B\,\rho}$$

$B\,\rho$ = "beam rigidity"

1.) The Magnetic Guide Field



Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes

$$B = \frac{\mu_0 n I}{h}$$

Normalise magnetic field to momentum:

$$\frac{p}{e} = B\rho \longrightarrow \frac{1}{\rho} = \frac{eB}{p}$$

convenient units:

$$B = [T] = \left[\frac{Vs}{m^2}\right] \qquad p = \left[\frac{GeV}{c}\right]$$

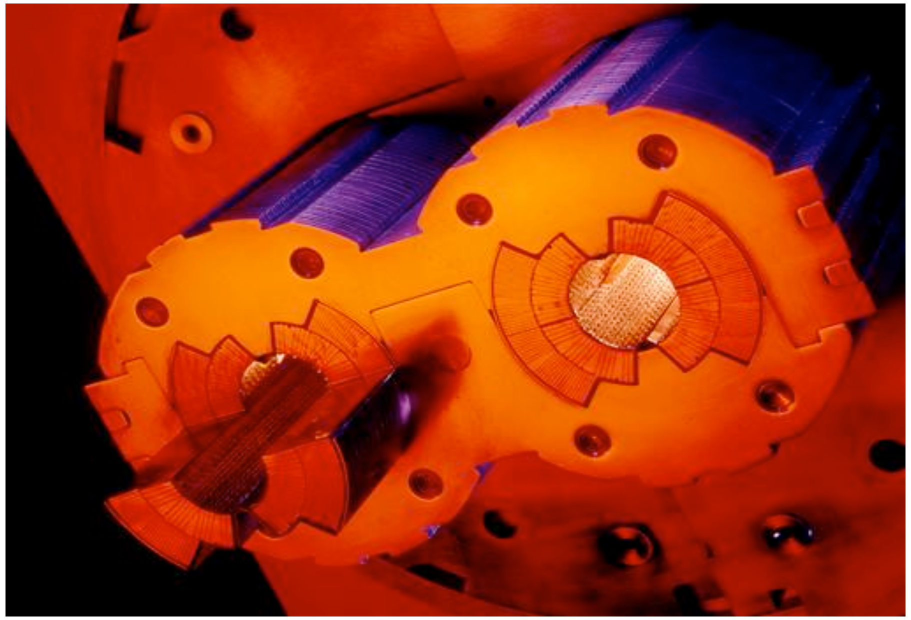
Example LHC:

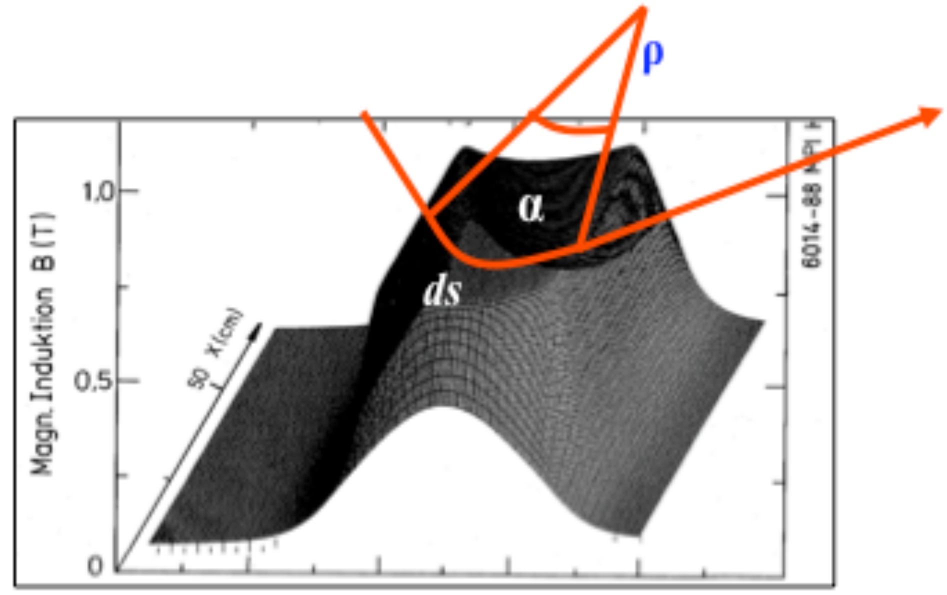
$$\left.\begin{aligned} B &= 8.3\,T \\ p &= 7000\,\frac{GeV}{c} \end{aligned}\right\}$$

$$\frac{1}{\rho} = e\,\frac{8.3\,Vs/m^2}{7000 * 10^9\,eV/c} = \frac{8.3\,s\,3 * 10^8\,m/s}{7000 * 10^9\,m^2}$$

$$\frac{1}{\rho} = 0.333\,\frac{8.3}{7000}\,1/m$$

The Magnetic Guide Field





field map of a storage ring dipole magnet

$\rho = 2.53\ \boldsymbol{km}$ → $2\pi\rho = 17.6\ km$
$\approx 66\%$

$\boldsymbol{B} \approx 1 \ldots 8\ \boldsymbol{T}$

rule of thumb:

$$\frac{1}{\rho} \approx 0.3 \frac{\boldsymbol{B}[\boldsymbol{T}]}{\boldsymbol{p}[\boldsymbol{GeV/c}]}$$

„normalised bending strength“

2.) Quadrupole Magnets:

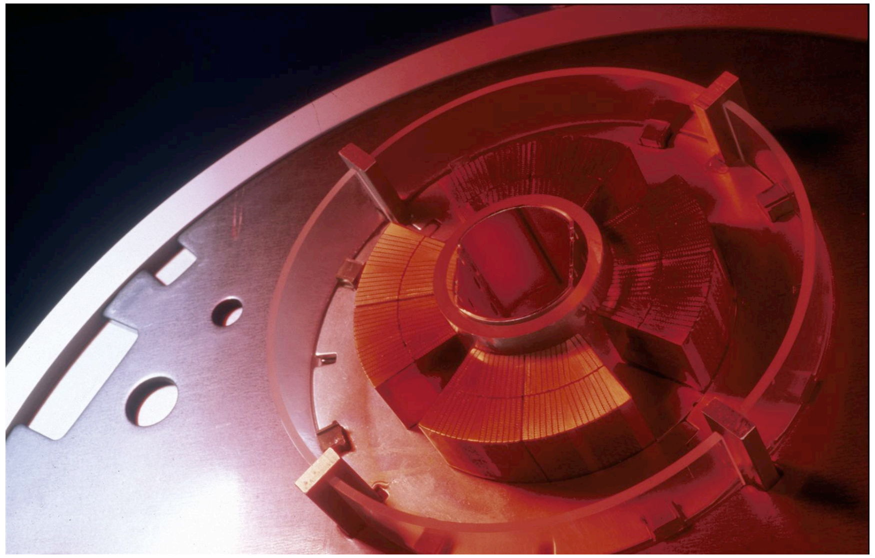
required: *focusing forces* ***to keep trajectories in vicinity of the ideal orbit***

linear increasing Lorentz force

linear increasing magnetic field $B_y = g\,x \qquad B_x = g\,y$

normalised quadrupole field:

gradient of a quadrupole magnet: $g = \dfrac{2\mu_0 n I}{r^2}$



LHC main quadrupole magnet

$g \approx 25 \ldots 220\ \ T/m$

$\longrightarrow \quad k = \dfrac{g}{p/e}$

simple rule: $\boxed{k = 0.3\,\dfrac{g(T/m)}{p(GeV/c)}}$

what about the vertical plane:
... Maxwell

$\vec{\nabla} \times \vec{B} = \cancel{\vec{j}} + \cancel{\dfrac{\partial \vec{E}}{\partial t}} = 0 \qquad \Rightarrow \qquad \dfrac{\partial B_y}{\partial x} = \dfrac{\partial B_x}{\partial y}$

3.) The equation of motion:

Linear approximation:

* *ideal particle* → *design orbit*

* *any other particle* → *coordinates x, y small quantities*
$x, y \ll \rho$

→ *magnetic guide field: only linear terms in x & y of B have to be taken into account*

Taylor Expansion of the B field:

$$B_y(x) = B_{y0} + \frac{dB_y}{dx}x + \frac{1}{2!}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{eg''}{dx^3} + \ldots$$

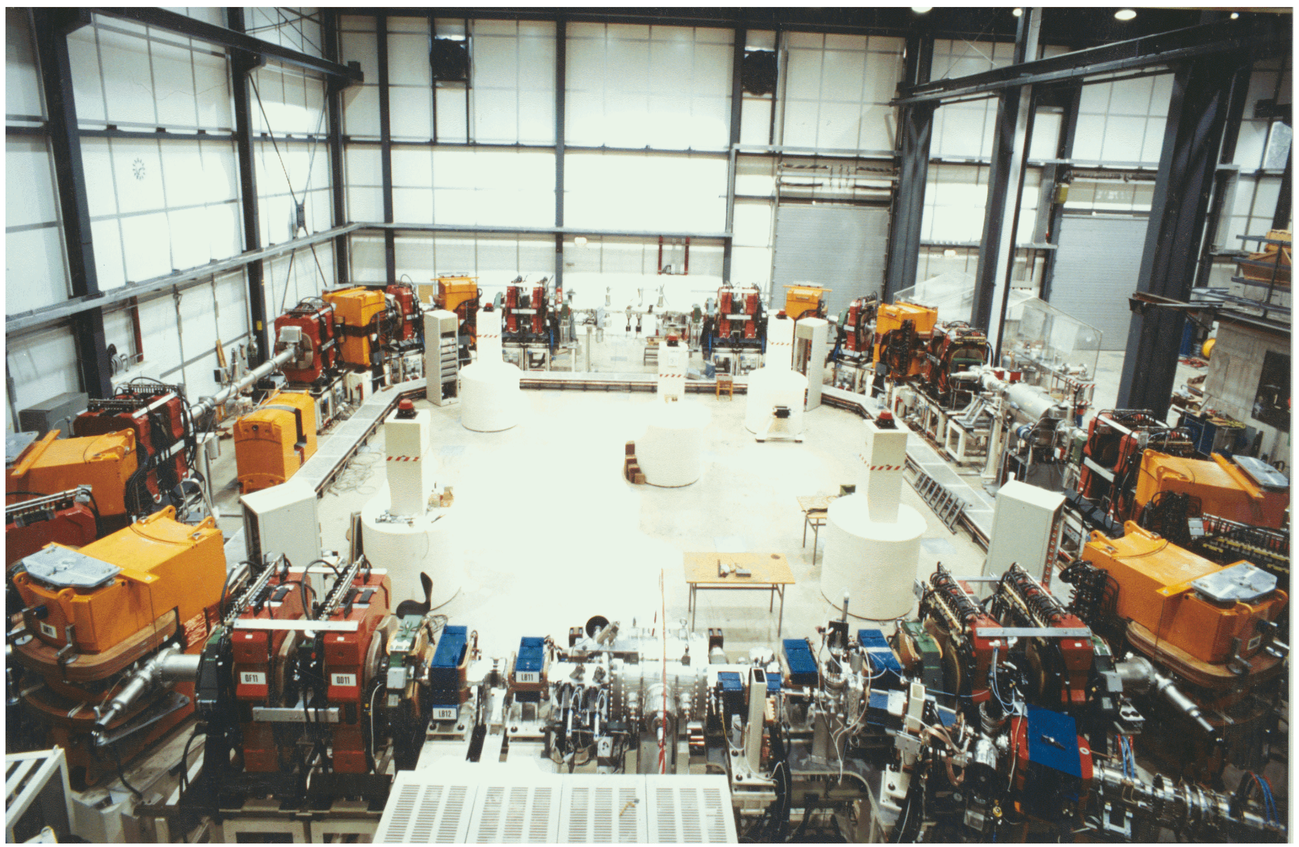
normalise to momentum
$p/e = B\rho$

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0\rho} + \frac{g * x}{p/e} + \frac{1}{2!}\frac{eg'}{p/e} + \frac{1}{3!}\frac{eg''}{p/e} + \ldots$$

The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \cancel{\frac{1}{2!}\,m}\,x^2 + \cancel{\frac{1}{3!}\,n}\,x^3 + \ldots$$

only terms linear in x, y taken into account *dipole fields*
quadrupole fields



Separate Function Machines:

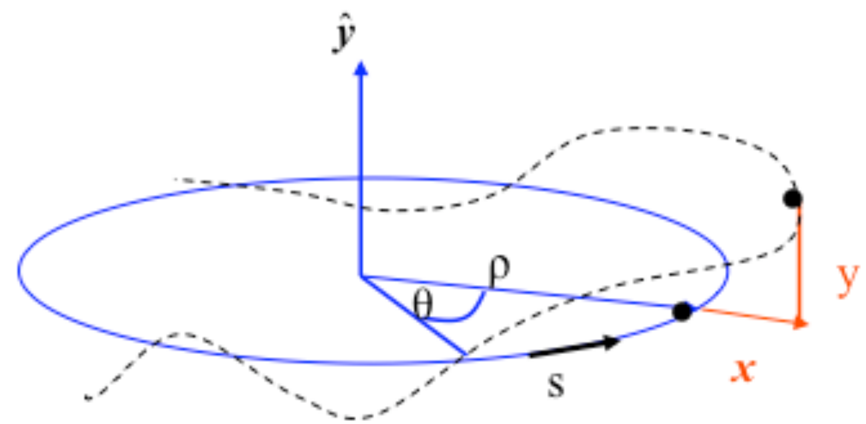
Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

* ***man sieht nur dipole und quads → linear***

Equation of Motion:



Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

general trajectory: $\rho \rightarrow \rho + x$

$$\boldsymbol{F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v}$$

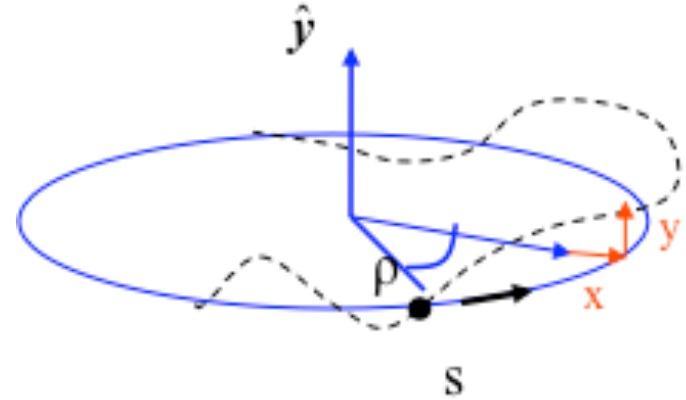
Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

Force: $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

$$F = mv^2/\rho$$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

1 2



1 $$\frac{d^2}{dt^2}(x+\rho) = \frac{d^2}{dt^2}x$$ *… as ρ = const*

2 ***remember:*** ***$x \approx mm$, $\rho \approx m$ …*** → ***develop for small x***

$$\frac{1}{x+\rho} \approx \frac{1}{\rho}\left(1-\frac{x}{\rho}\right)$$

Taylor Expansion

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \quad \dots$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_y v$$

guide field in linear approx.

$$B_y = B_0 + x\frac{\partial B_y}{\partial x} \qquad\qquad m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = ev\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \quad \Big| \;:\; m$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m}$$

independent variable: $t \to s$

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\frac{ds}{dt}\right) = \frac{d}{ds}\left(\underbrace{\frac{dx}{ds}}_{x'}\underbrace{\frac{ds}{dt}}_{v}\right)\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = x''\,v^2 + \cancel{\frac{dx}{ds}\frac{dv}{ds}}\,v$$

$$x''v^2 - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m} \quad \Big| \;:\; v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{e\;B_0}{mv} + \frac{e\;x\,g}{mv}$$

$$m\,v = p$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{x\,g}{p/e}$$

normalize to momentum of particle

$$\frac{B_0}{p/e} = -\frac{1}{\rho}$$

$$\frac{g}{p/e} = k$$

$$x'' - \cancel{\frac{1}{\rho}} + \frac{x}{\rho^2} = -\cancel{\frac{1}{\rho}} + k\,x$$

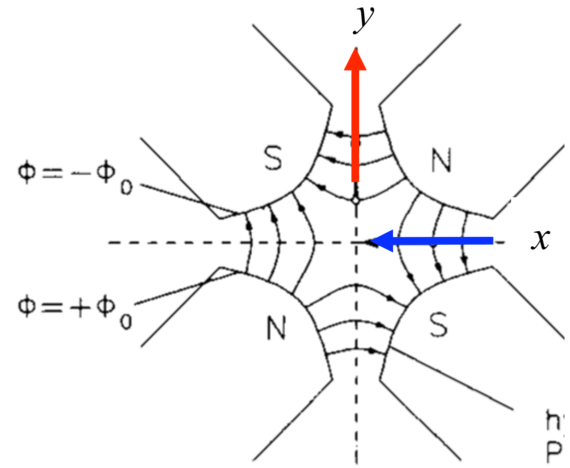
$$\boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = 0}$$

* *Equation for the vertical motion:*

$$\frac{1}{\rho^2} = 0$$ *no dipoles … in general …*

$$k \quad \leftrightarrow \quad -k$$ *quadrupole field changes sign*

$$y'' + k\;y = 0$$



Remarks:

* $$x'' + \left(\frac{1}{\rho^2} - k\right) \cdot x = 0$$

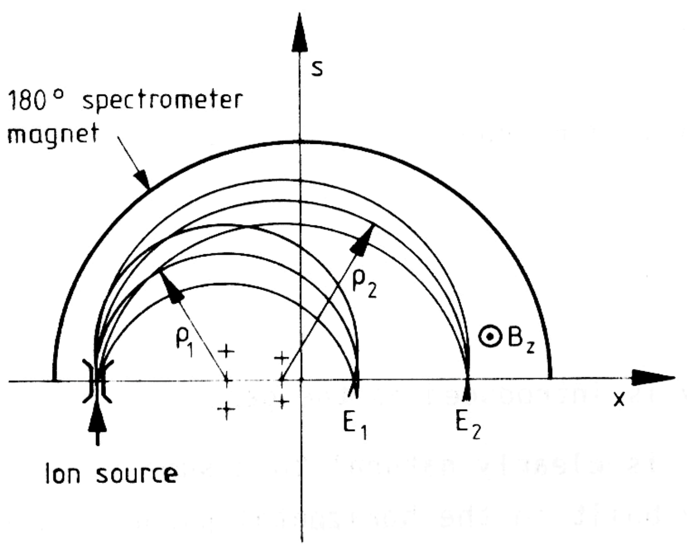
… there seems to be a focusing even without a quadrupole gradient

„weak focusing of dipole magnets“

$$\boldsymbol{k} = 0 \qquad \Rightarrow \qquad \boldsymbol{x}'' = -\frac{1}{\rho^2}\,\boldsymbol{x}$$

even without quadrupoles there is a retriving force (i.e. focusing) in the bending plane of the dipole magnets

… in large machines it is weak. *(!)*



Mass spectrometer: particles are separated according to their energy and focused due to the 1/ρ effect of the dipole

* Hard Edge Model:

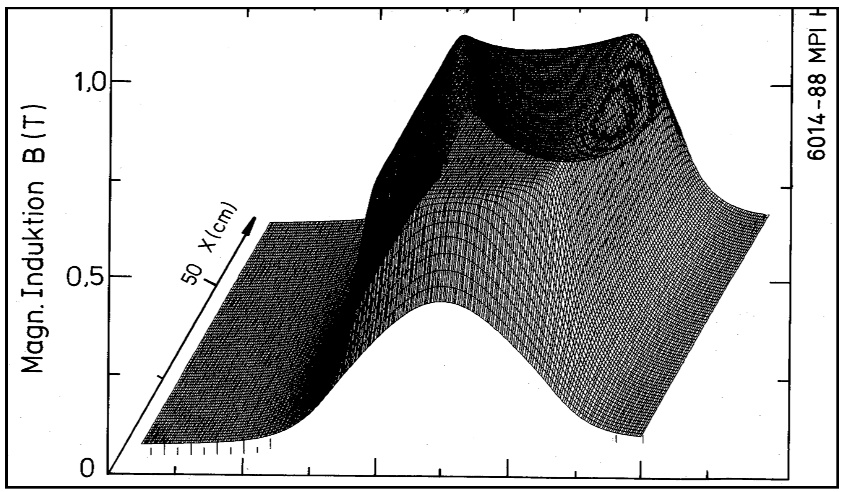
$$x'' + \left\{ \frac{1}{\rho^2} - k \right\} x = 0$$

... this equation is not correct !!!

$$x''(s) + \left\{ \frac{1}{\rho^2(s)} - k(s) \right\} x(s) = 0$$

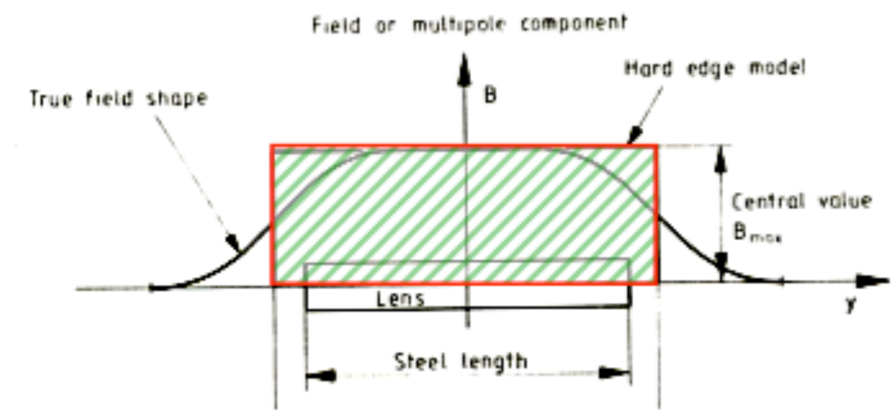
bending and focusing fields ... are functions of the independent variable „s“

!



Inside a magnet we assume constant focusing properties !

$$\frac{1}{\rho} = const \qquad k = const$$



$$B\, l_{eff} = \int_0^{l_{mag}} B\, ds$$

4.) Solution of Trajectory Equations

Define … hor. plane: $K = 1/\rho^2 - k$

… vert. Plane: $K = k$

$$\boldsymbol{x'' + K\ x = 0}$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz:

$$x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)$$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1\omega \sin(\omega s) + a_2\omega \cos(\omega s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) - a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \longrightarrow \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K} s) + a_2 \sin(\sqrt{K} s)$$

determine a_1, a_2 by boundary conditions:

$$s = 0 \longrightarrow \begin{cases} x(0) = x_0 \quad , \quad a_1 = x_0 \\ x'(0) = x'_0 \quad , \quad a_2 = \dfrac{x'_0}{\sqrt{K}} \end{cases}$$

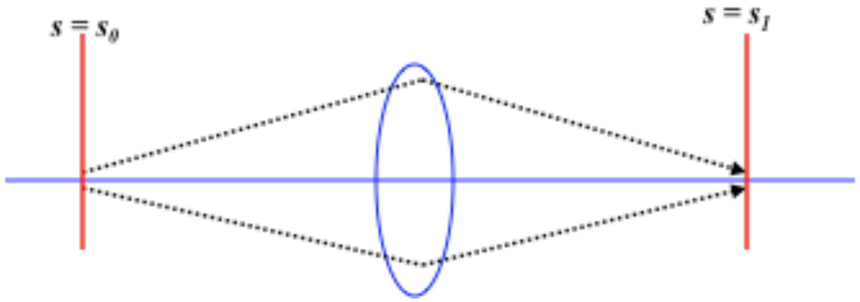
Hor. Focusing Quadrupole $K > 0$:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

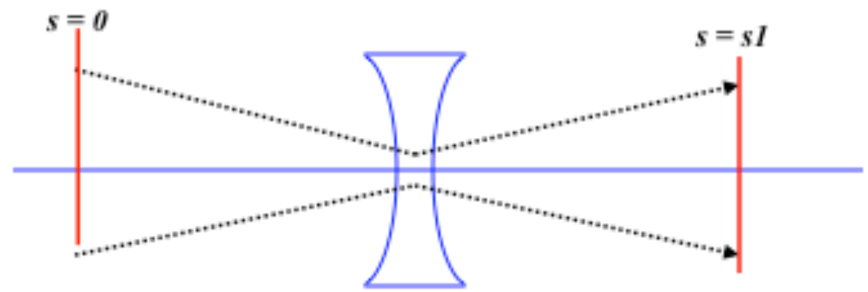
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \dfrac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole:

$$x'' - K\,x = 0$$



Remember from school:

$$f(s) = \cosh(s) \quad , \qquad f'(s) = \sinh(s)$$

Ansatz: $$x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\,l \\ \sqrt{|K|}\sinh\sqrt{|K|}\,l & \cosh\sqrt{|K|}\,l \end{pmatrix}$$

drift space:

K = 0

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“

Thin Lens Approximation:

matrix of a quadrupole lens

$$M = \begin{pmatrix} \cos\sqrt{|k|}l & \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}l \\ -\sqrt{|k|}\sin\sqrt{|k|}l & \cos\sqrt{|k|}l \end{pmatrix}$$

in many practical cases we have the situation:

$$f = \frac{1}{kl_q} >> l_q$$

... focal length of the lens is much bigger than the length of the magnet

limes: $l_q \;\rightarrow\; 0$ ***while keeping*** $k\, l_q = const$

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \qquad M_z = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}$$

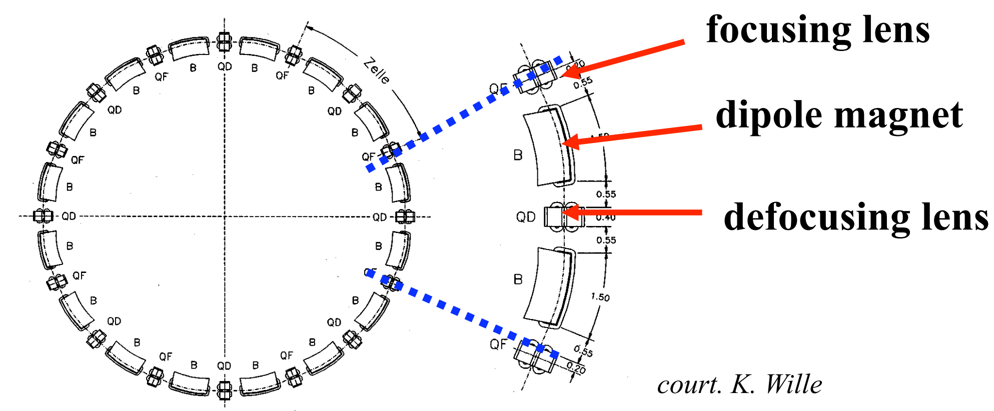
... useful for fast (and in large machines still quite accurate) „back on the envelope calculations“ ... and for the guided studies !

Transformation through a system of lattice elements

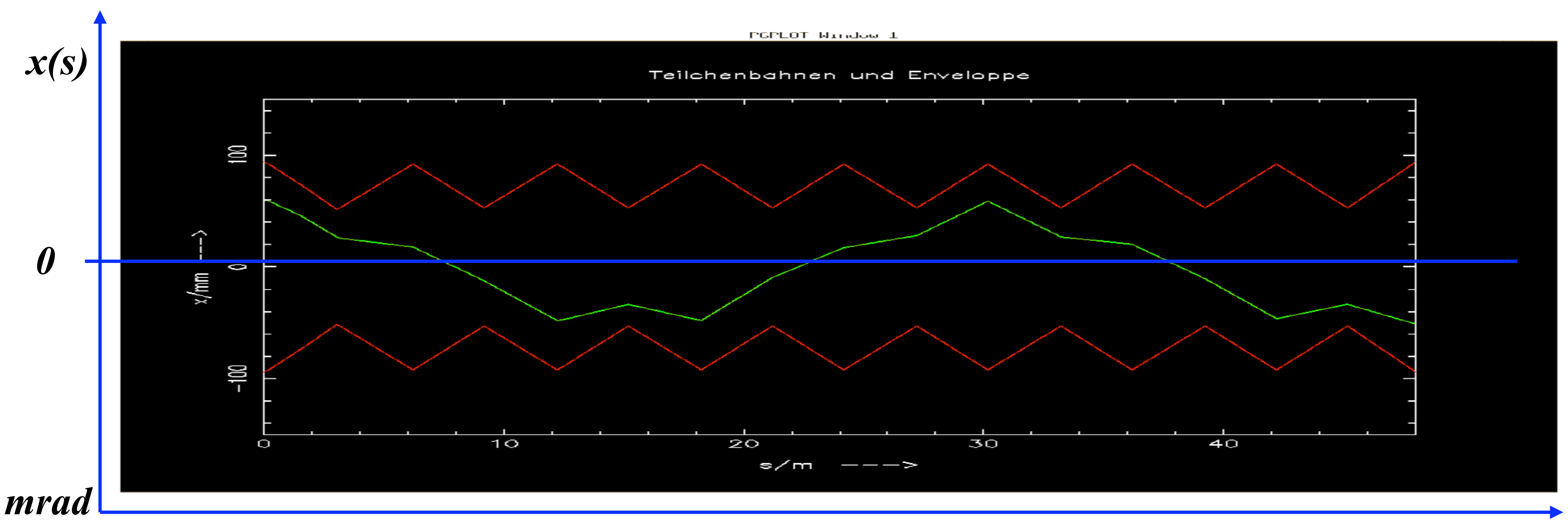
combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$

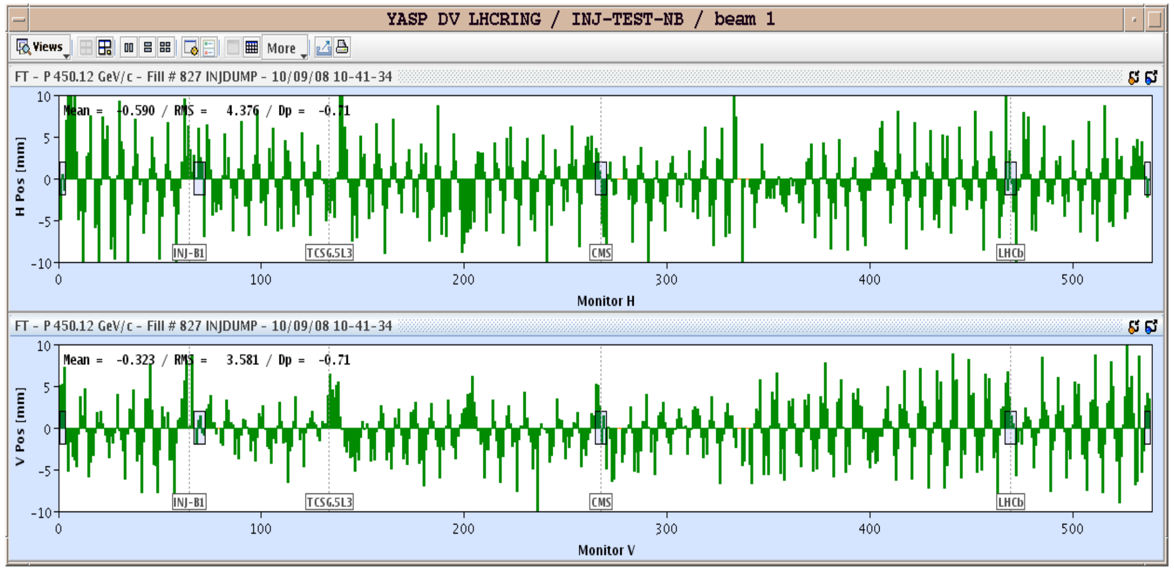


in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



typical values
in a strong
foc. machine:
x ≈ mm, x´ ≤ mrad

5.) Orbit & Tune:



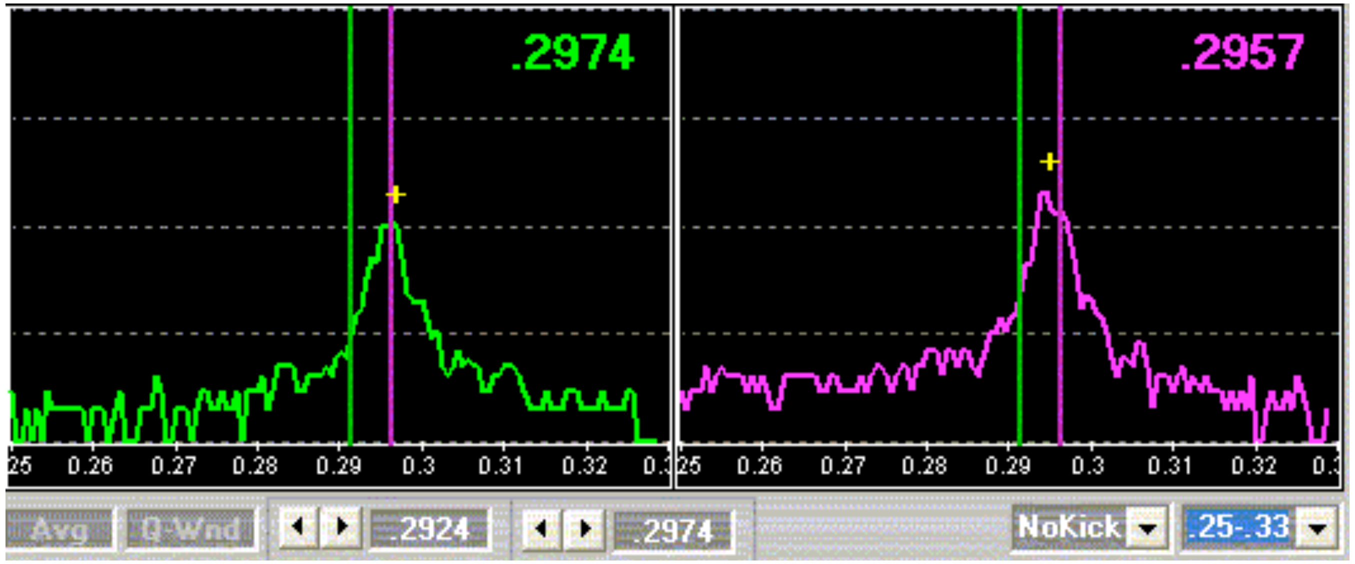
Tune: number of oscillations per turn

64.31
59.32

Relevant for beam stability:
non integer part

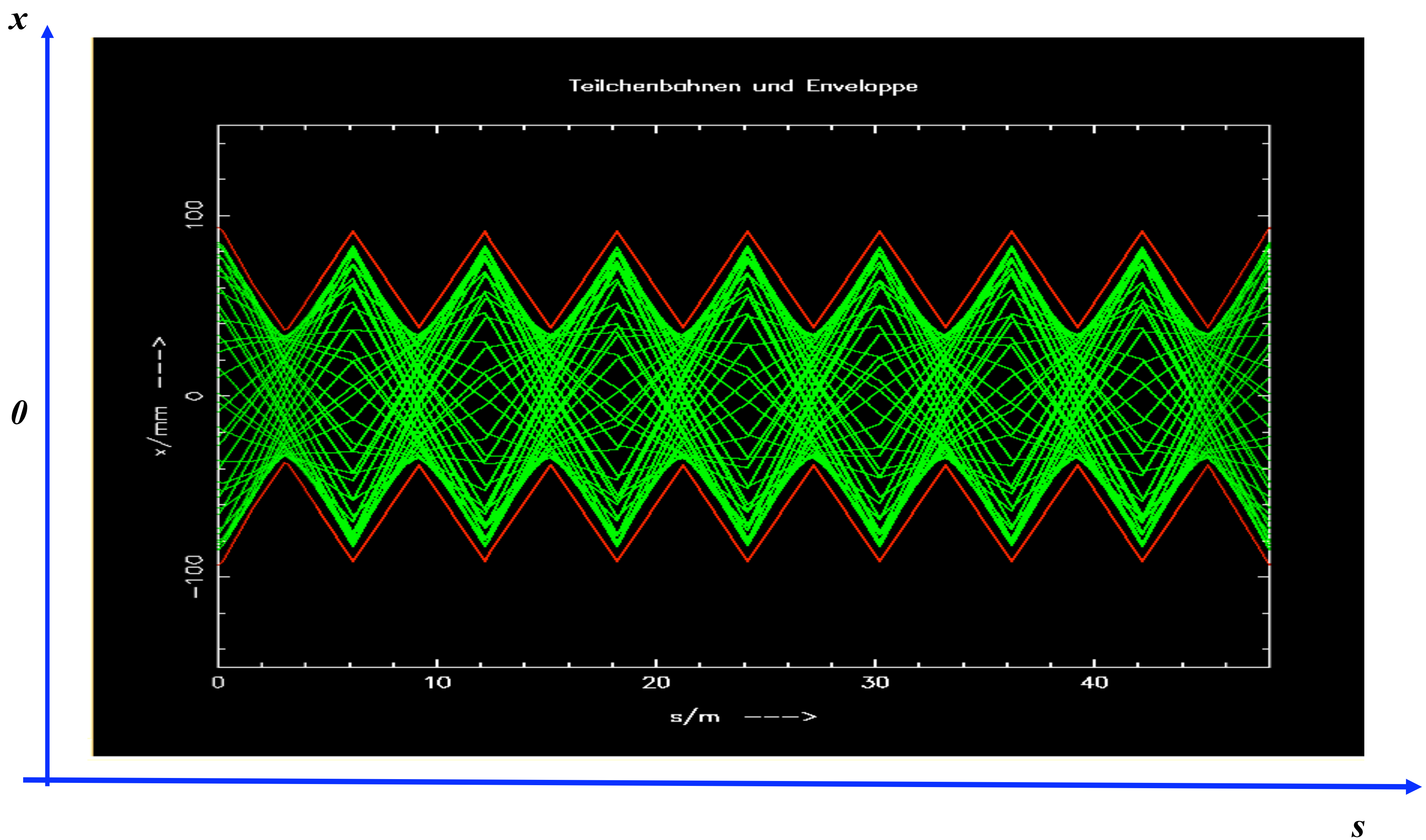
LHC revolution frequency: 11.3 kHz

$$0.31*11.3=3.5kHz$$



Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $$x''(s) - k(s)x(s) = 0$$

restoring force $\neq$ const,
$k(s)$ = depending on the position s
$k(s+L) = k(s)$, periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

6.) The Beta Function

General solution of Hill´s equation:

(i)

$$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s + L) = \beta(s)$$

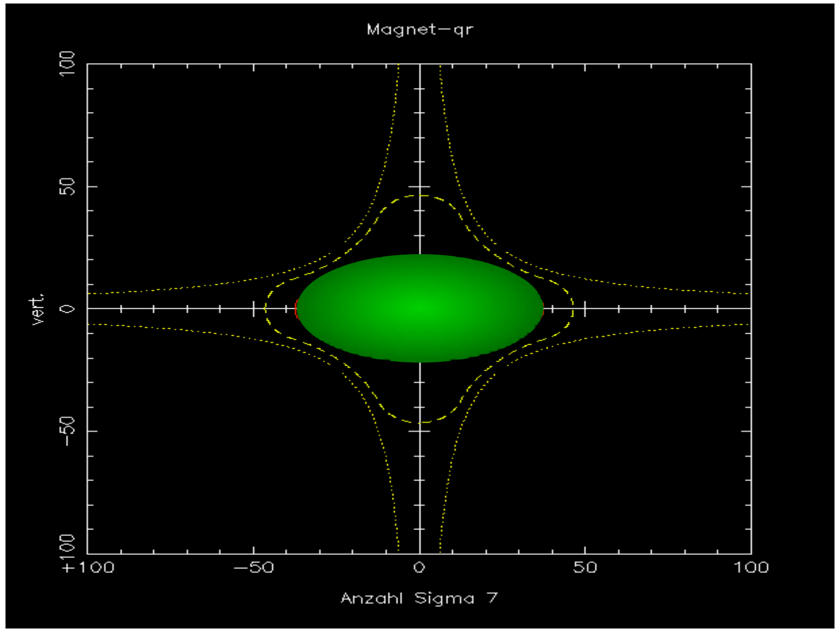
Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.
For one complete revolution: number of oscillations per turn „Tune“

$$\boldsymbol{Q_y = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}}$$

The Beta Function

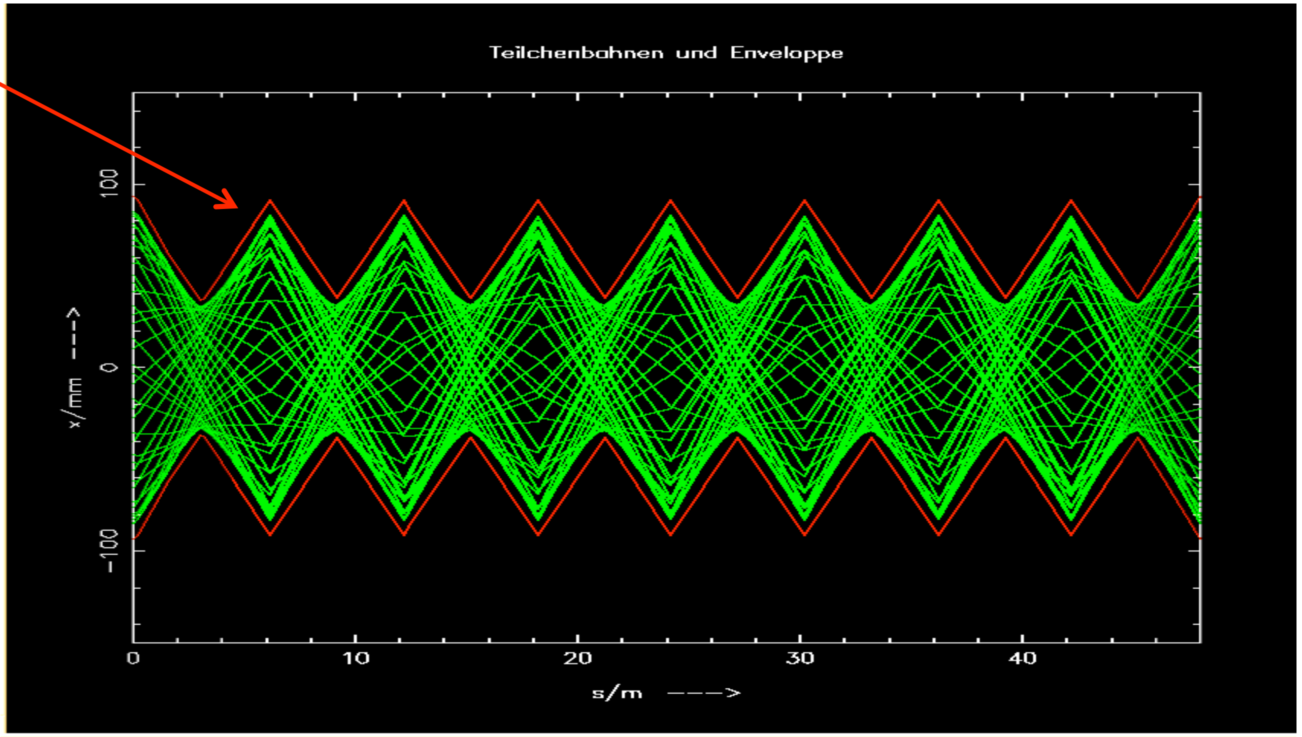


Amplitude of a particle trajectory:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$

Maximum size of a particle amplitude

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

7.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$(1) \qquad \boldsymbol{x}(\boldsymbol{s}) = \sqrt{\varepsilon}\,\sqrt{\beta(\boldsymbol{s})}\,\cos(\psi(\boldsymbol{s})+\phi)$$

$$(2) \qquad \boldsymbol{x}'(\boldsymbol{s}) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(\boldsymbol{s})}}\left\{\alpha(\boldsymbol{s})\cos(\psi(\boldsymbol{s})+\phi)+\sin(\psi(\boldsymbol{s})+\phi)\right\}$$

from (1) we get

$$\cos(\psi(\boldsymbol{s})+\phi) = \frac{\boldsymbol{x}(\boldsymbol{s})}{\sqrt{\varepsilon}\,\sqrt{\beta(\boldsymbol{s})}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

Insert into (2) and solve for ε

$$\varepsilon = \gamma(\boldsymbol{s})\,\boldsymbol{x}^2(\boldsymbol{s}) + 2\alpha(\boldsymbol{s})\boldsymbol{x}(\boldsymbol{s})\boldsymbol{x}'(\boldsymbol{s}) + \beta(\boldsymbol{s})\,\boldsymbol{x}'^2(\boldsymbol{s})$$

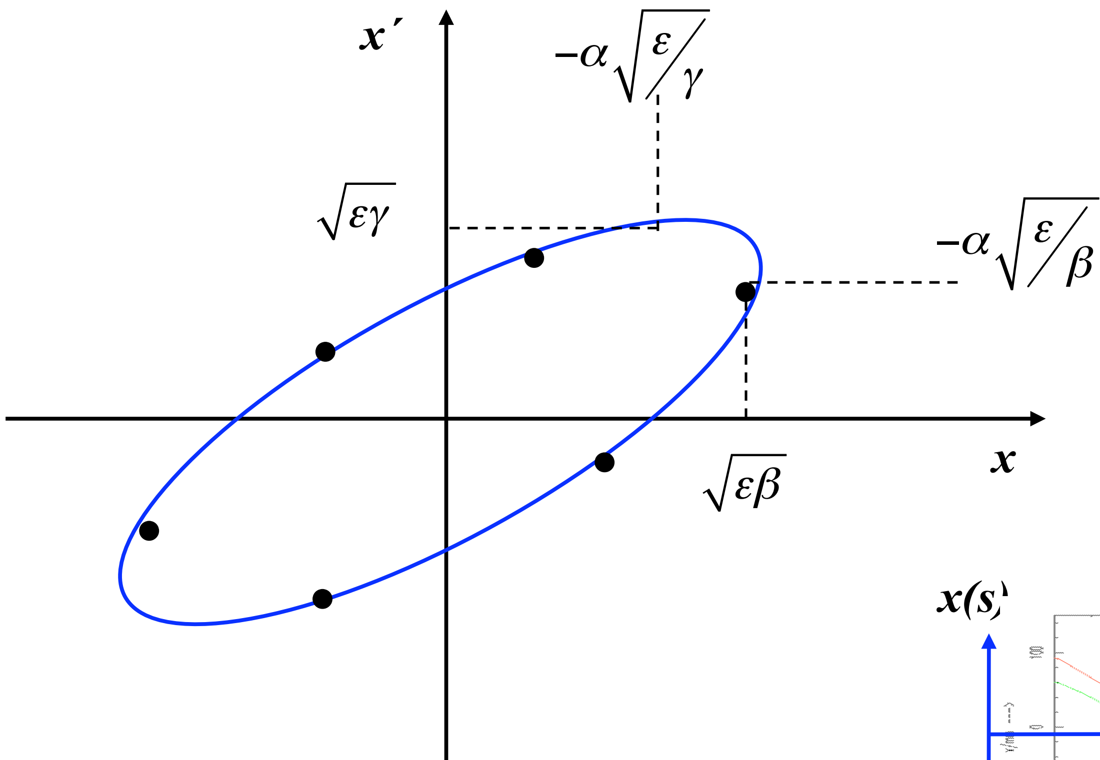
***** ***ε is a constant of the motion … it is independent of „s“***

***** ***parametric representation of an ellipse in the x x‘ space***

***** ***shape and orientation of ellipse are given by α, β, γ***

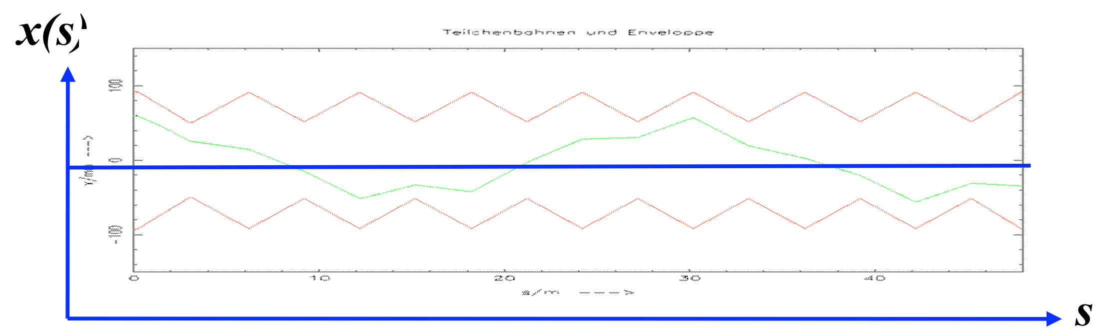
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

$$A = \pi * \varepsilon = const$$

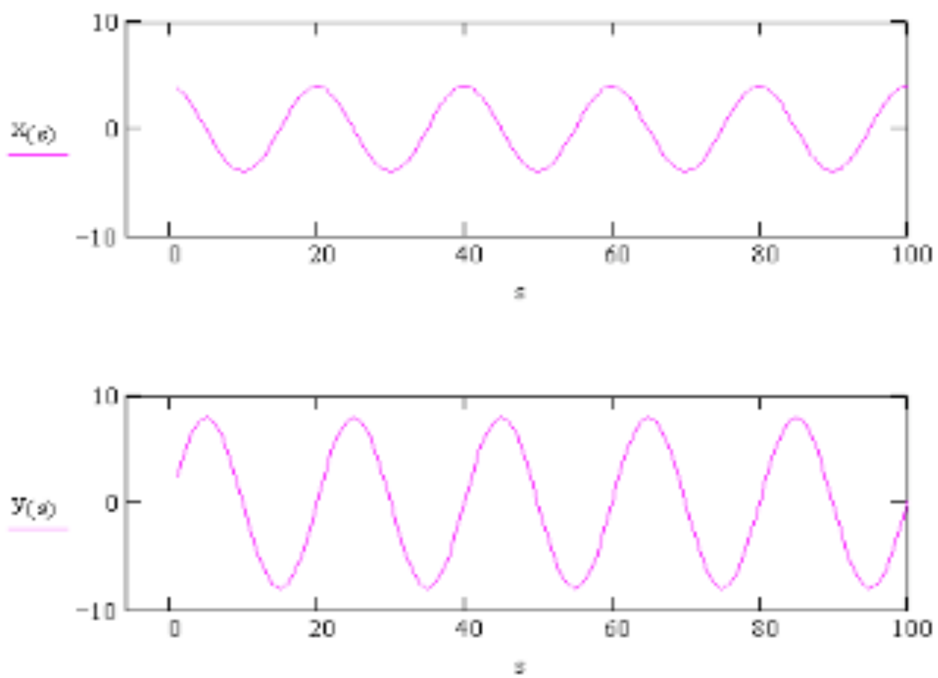


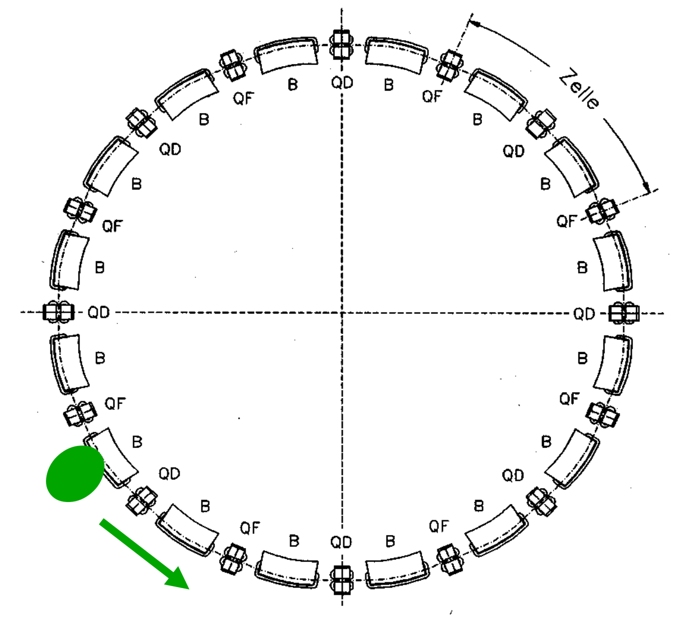
***ε beam emittance* = *woozilycity* of the particle ensemble, *intrinsic beam parameter*,**
cannot be changed by the foc. properties.
***Scientifiquely speaking:* area covered *in transverse x, x´ phase space* … and it is constant !!!**

Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

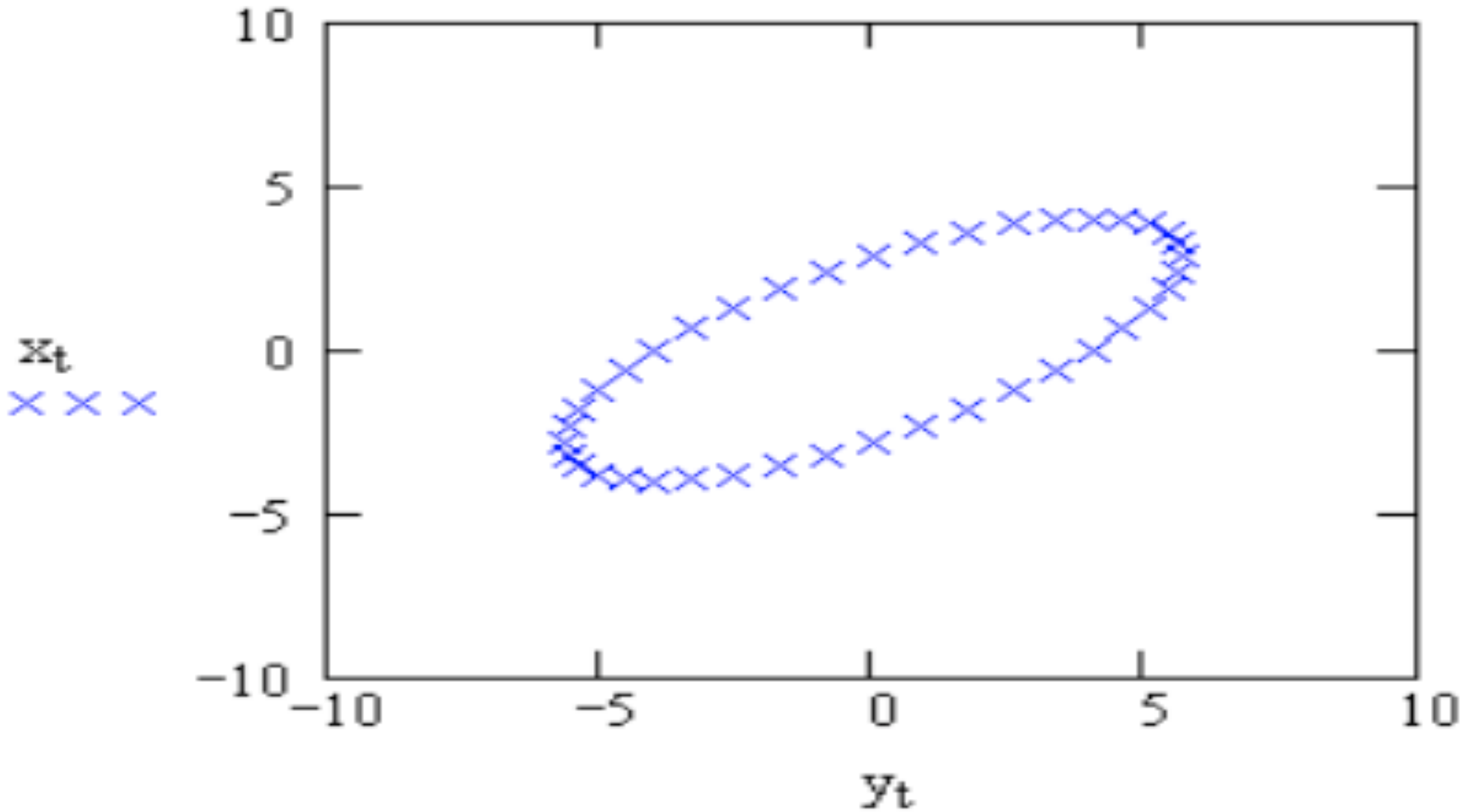
plot x, x´as a function of „s"





… and now the ellipse:

note for each turn x, x´at a given position „s_1" and plot in the phase space diagram



II.) Particle Trajectories, Beams & Bunch

Emittance and Beta-Function

Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\boxed{\hat{x}(s) = \sqrt{\varepsilon\beta}}$ $\longrightarrow$ *x´ at that position …?*

… put $\hat{x}(s)$ *into* $\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$ *and solve for x´*

$$\varepsilon = \gamma\cdot\varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

$$\longrightarrow \quad \boxed{x' = -\alpha\cdot\sqrt{\varepsilon/\beta}}$$

* *A high β-function means a large beam size and a small beam divergence.* **!**
  … et vice versa !!!

* *In the middle of a quadrupole β = maximum,*
  α = zero $\Big\}$ $x' = 0$

… and the ellipse is flat

Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$

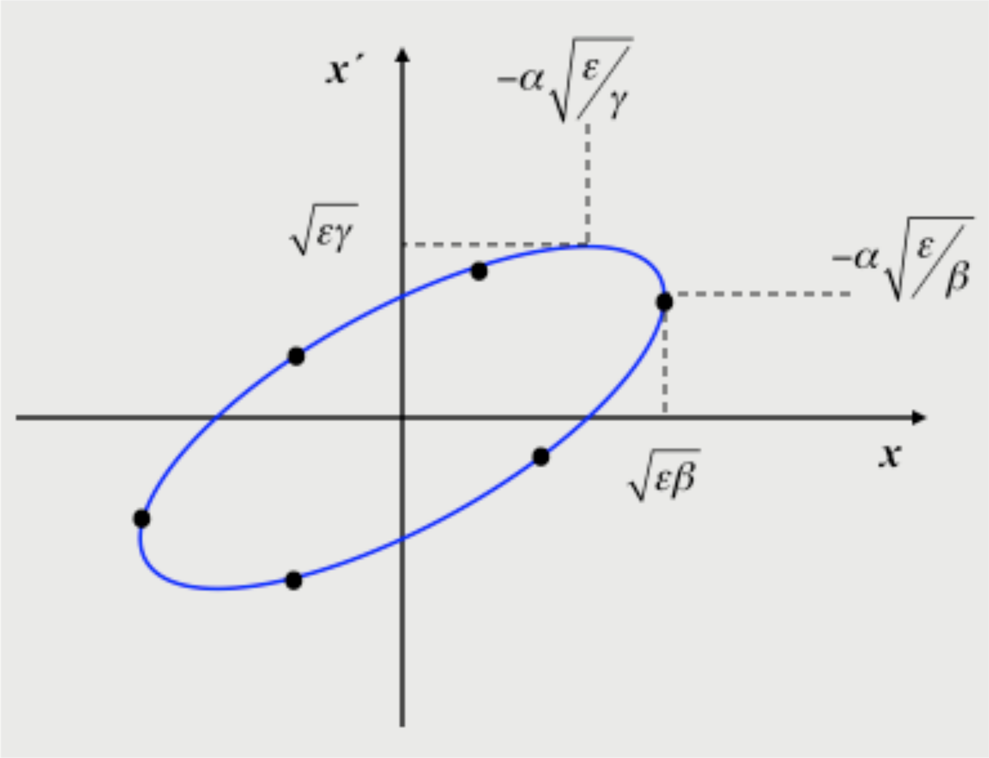
$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\longrightarrow \quad \varepsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot x x' + \beta \cdot x'^2$$

… solve for x´ $\quad x'_{1,2} = \dfrac{-\alpha \cdot x \pm \sqrt{\varepsilon\beta - x^2}}{\beta}$

… and determine $\hat{x}'$ ***via:*** $\quad \dfrac{dx'}{dx} = 0$



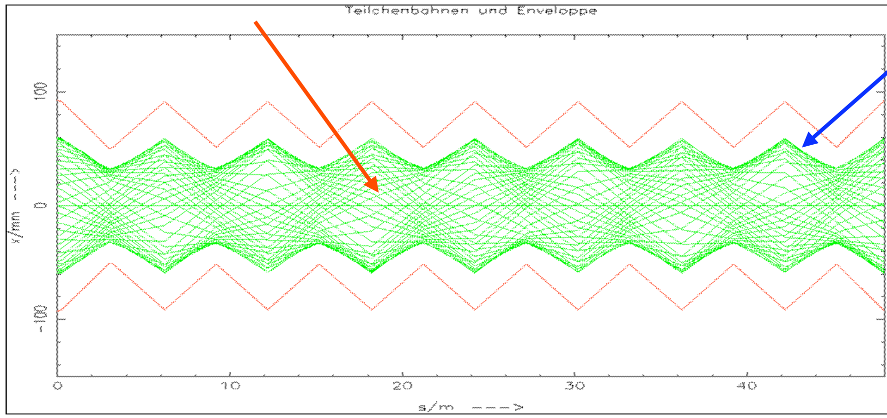
$$\longrightarrow \quad \hat{x}' = \sqrt{\varepsilon\gamma}$$

$$\longrightarrow \quad \hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}$$

shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\cdot\cos(\Psi(s)+\phi) \qquad\qquad \hat{x}(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}$$



single particle trajectories, N ≈ 10 ^{11} per bunch

Gauß
Particle Distribution:

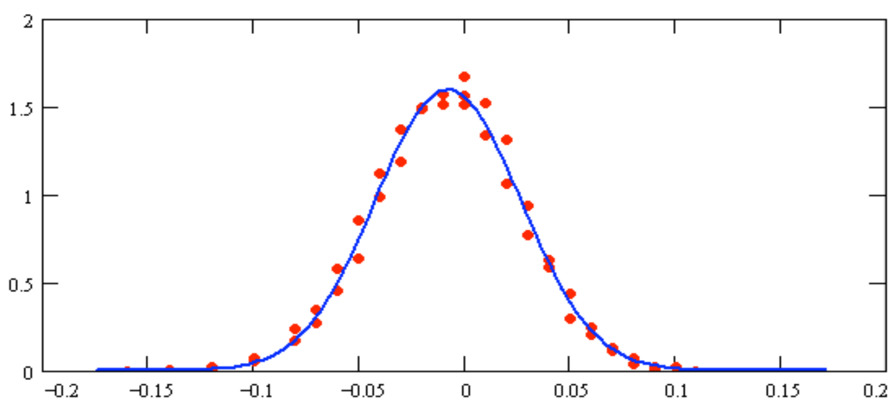
$$\rho(x) = \frac{N\cdot e}{\sqrt{2\pi}\,\sigma_x}\cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x{}^2}}$$

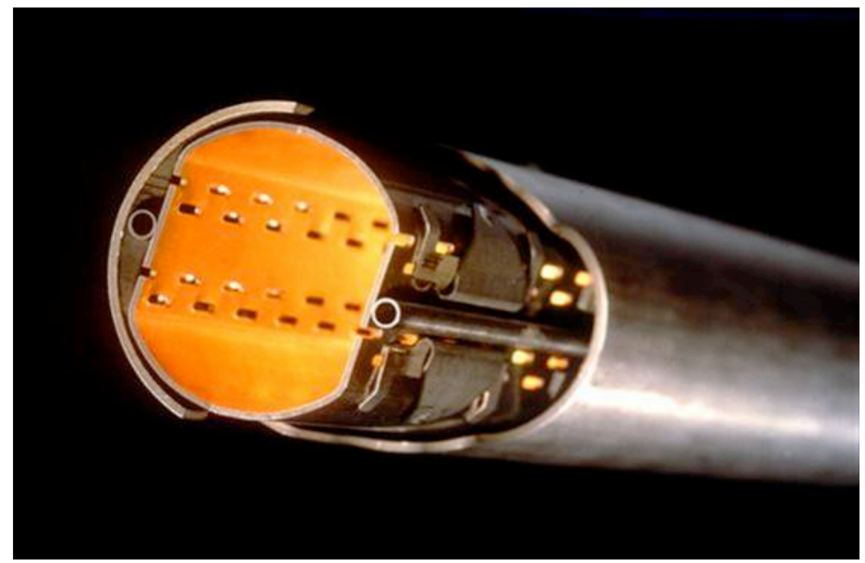
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

LHC: $\beta = 180\,m$

$\varepsilon = 5*10^{-10}\,m\,rad$

$$\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5*10^{-10}\,m * 180\,m} = 0.3\,mm$$





aperture requirements: $r_0 = 12 * \sigma$

9.) Transfer Matrix M

… yes we had the topic already

general solution of Hill´s equation

$$\begin{cases} x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \\ x'(s) = \dfrac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right] \end{cases}$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point s(0) = s_0, where we put Ψ(0) = 0

$$\left.\begin{aligned} \cos\phi &= \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \\ \sin\phi &= -\frac{1}{\sqrt{\varepsilon}}\left(x'_0\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{aligned}\right\}$$

inserting above …

$$x(s)=\sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s+\alpha_0\sin\psi_s\right\}x_0+\left\{\sqrt{\beta_s\beta_0}\sin\psi_s\right\}x_0'$$

$$x'(s)=\frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s\right\}x_0+\sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s-\alpha_s\sin\psi_s\right\}x_0'$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

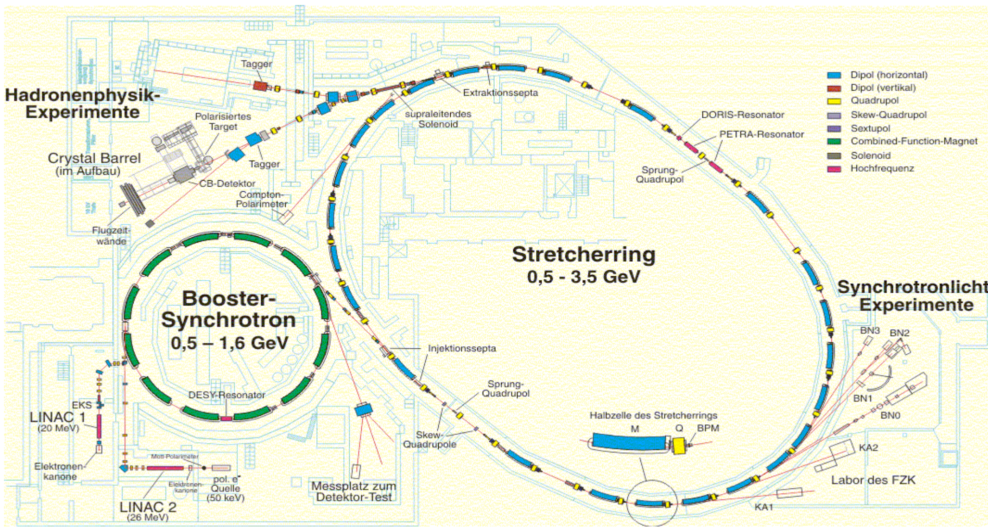
$$M=\begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\psi_s+\alpha_0\sin\psi_s) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}(\cos\psi_s-\alpha_s\sin\psi_s) \end{pmatrix}$$

* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***
* ***and nothing but the α β γ at these positions.***
* ***... !***

** Äquivalenz der Matrizen*

10.) Periodic Lattices

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}(\cos\psi_s - \alpha_s \sin\psi_s) \end{pmatrix}$$



ELSA Electron Storage Ring

*„This **rather formidable looking matrix** simplifies considerably if we **consider one complete revolution** …"*

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

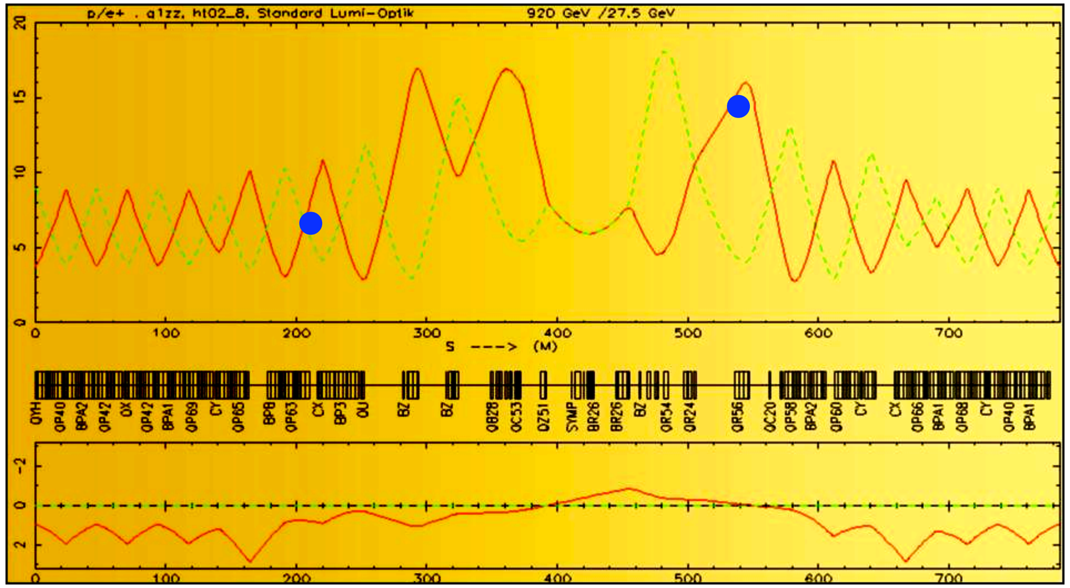
$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

ψ_{turn} = phase advance per period

***Tune:** Phase advance per turn in units of 2π*

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

11.) Transformation of α, β, γ



Betafunction in a storage ring

consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

since ε = const (Liouville):

$$\varepsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

... remember W = CS′-SC′ = 1

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} * \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

$$\rightarrow \quad x_0 = S'x - Sx'$$

$$x_0' = -C'x + Cx'$$

... inserting into ε

$$\varepsilon = \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

sort via x, x′ and compare the coefficients to get ....

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

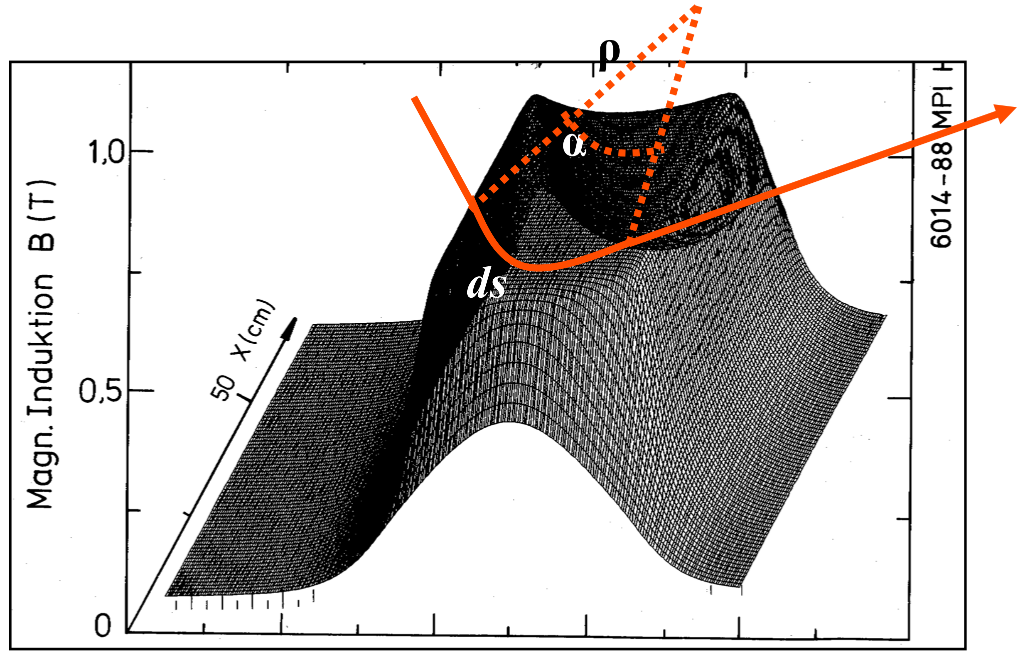
3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)

12.) Lattice Design:

„… how to build a storage ring“

$$\boxed{B\,\rho = p / q}$$



field map of a storage ring dipole magnet

Circular Orbit: dipole magnets to define the geometry

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$

The angle run out in one revolution must be 2π, so

… for a full circle

$$\alpha = \frac{\int Bdl}{B\,\rho} = 2\pi \quad \rightarrow \quad \int Bdl = 2\pi\,\frac{p}{q}$$

… defines the integrated dipole field around the machine.

Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ *is usually required !!*

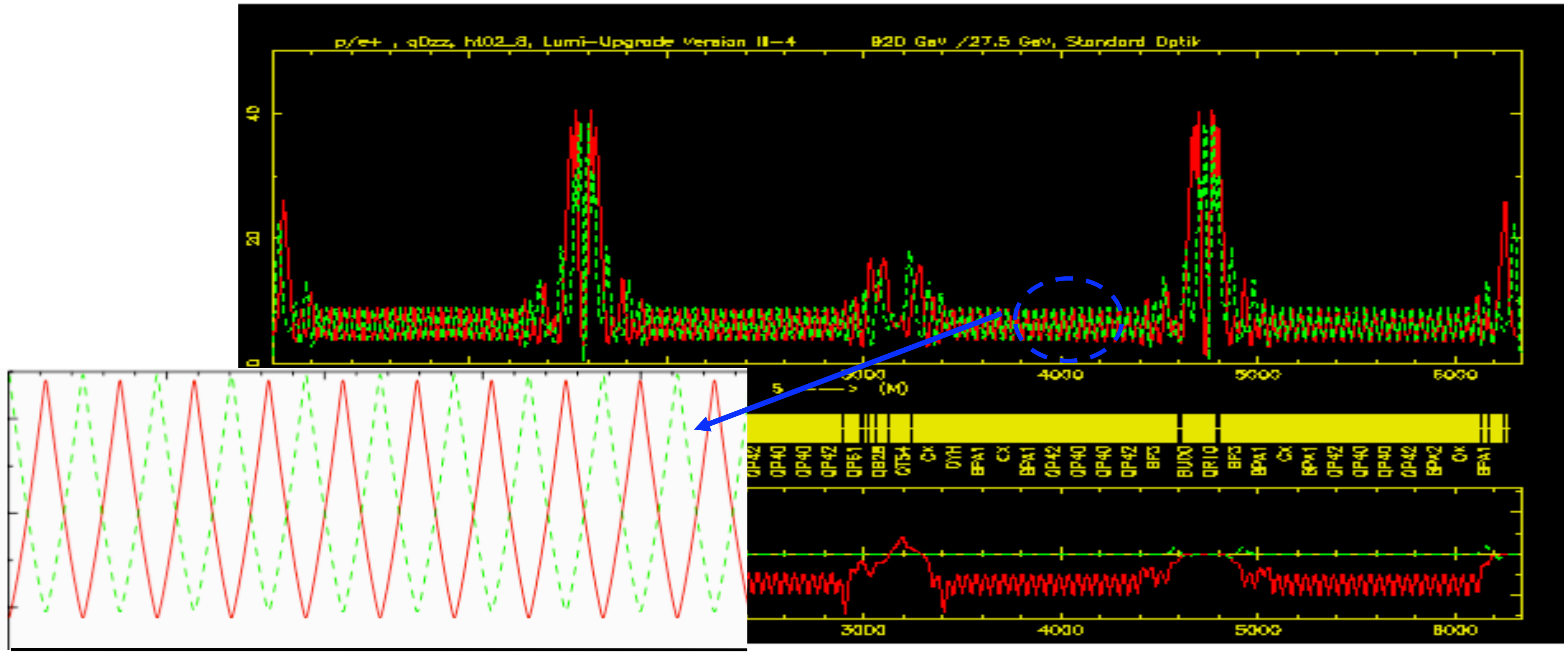
7000 GeV Proton storage ring
dipole magnets $N = 1232$
$l = 15$ m
$q = +1$ e

$$\int B\, dl \approx N\, l\, B = 2\pi\ p/e$$

$$B \approx \frac{2\pi\ 7000\ 10^{9}\, eV}{1232\ \ 15\, m\ \ 3\ 10^{8}\, \frac{m}{s}\ \ e} = 8.3\ Tesla$$

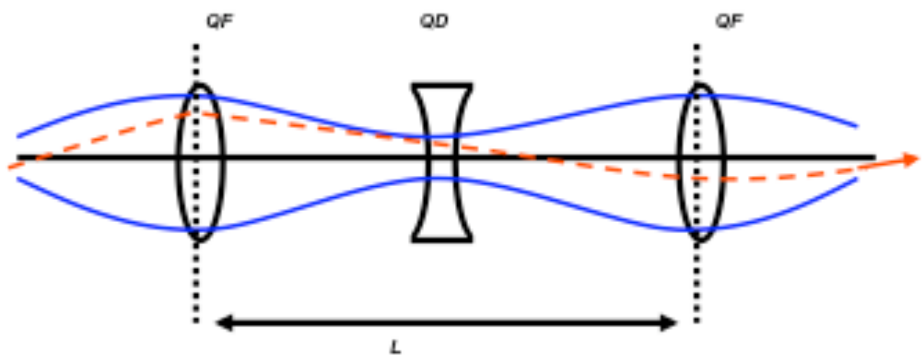
The FoDo-Lattice

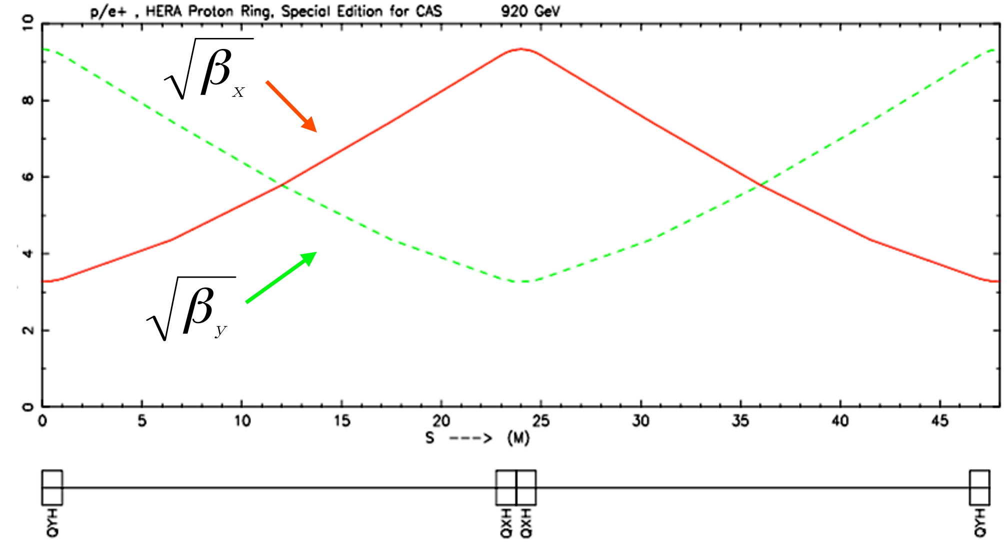
A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell μ = 45°,
→ calculate the twiss parameters for a periodic solution

Periodic solution of a FoDo Cell





Output of the optics program:

Nr	Type	Length	Strength	β_x	α_x	ψ_x	β_y	α_y	ψ_y
		m	1/m2	m		1/2π	m		1/2π
0	IP	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	IP	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

$Q_X=$ 0,125 $Q_Y=$ 0,125

$0.125 * 2\pi = 45°$

Can we understand, what the optics code is doing?

matrices $$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}\,l_q) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}\,l_q) \\ -\sqrt{|K|}\sin(\sqrt{|K|}\,l_q) & \cos(\sqrt{|K|}\,l_q) \end{pmatrix} \qquad M_{drift} = \begin{pmatrix} 1 & l_d \\ 0 & 1 \end{pmatrix}$$

strength and length of the FoDo elements ***K = +/- 0.54102 m^{-2}***
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qf}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for one period gives us all the information that we need !

Phase advance per cell

$$\boldsymbol{M}(\boldsymbol{s}) = \begin{pmatrix} \cos\ \psi + \alpha \sin \psi & \beta \sin \psi \\ -\gamma \sin \psi & \cos\ \psi - \alpha \sin \psi \end{pmatrix} \rightarrow$$

$$\cos(\psi) = \frac{1}{2}\boldsymbol{Trace}(\boldsymbol{M}) = 0.707$$

$$\psi = \boldsymbol{arc}\cos\left(\frac{1}{2}\boldsymbol{Trace}(\boldsymbol{M})\right) = 45°$$

hor β-function

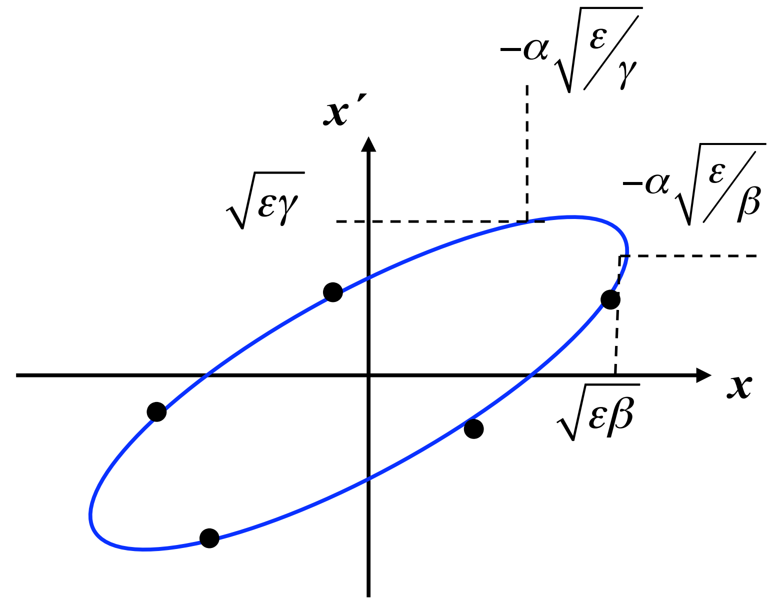
hor α-function

$$\beta = \frac{\boldsymbol{M}_{1,2}}{\sin\psi} = 11.611\ \boldsymbol{m}$$

$$\alpha = \frac{\boldsymbol{M}_{1,1} - \cos\psi}{\sin\psi} = 0$$

13.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$



***Beam Emittance** corresponds to the area covered in the x, x´ Phase Space Ellipse*

***Liouville:** Area in phase space is constant.*

But so sorry ... $\varepsilon \neq$ const !

Classical Mechanics:

phase space = diagram of the two canonical variables

position & momentum

x $\qquad$ p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:

phase space diagram relates the variables q and p

$$q = position = x$$
$$p = momentum = \gamma m v = m c \gamma \beta_x$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $\int p\, dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\beta_x = v_x / c$

$$\int p\, dq = mc \int \gamma \beta_x dx$$

$$\int p\, dq = mc\gamma\beta \underbrace{\int x' dx}_{\varepsilon}$$

$$\Rightarrow \quad \boxed{\varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

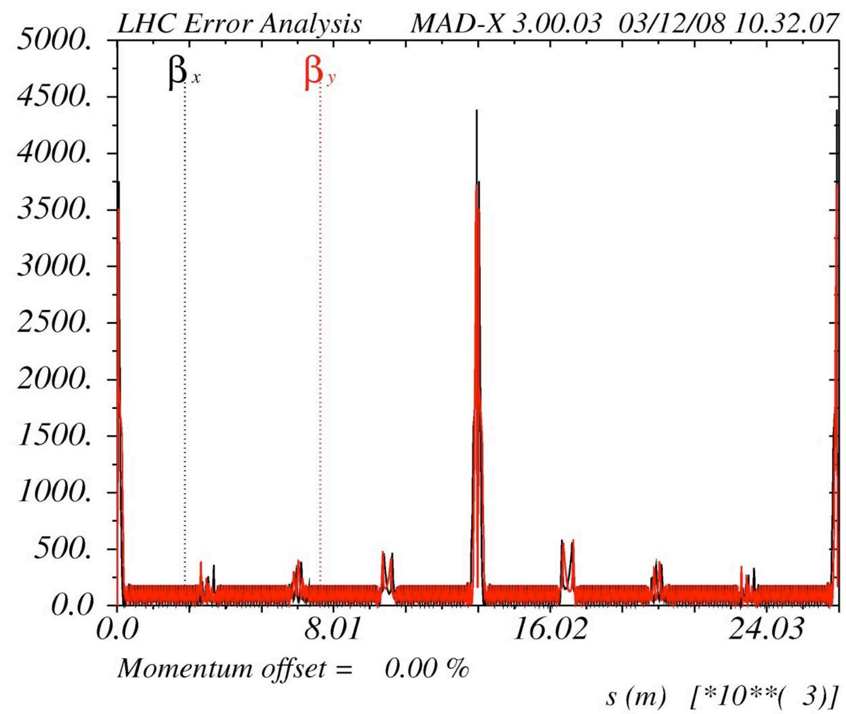
Nota bene:

1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

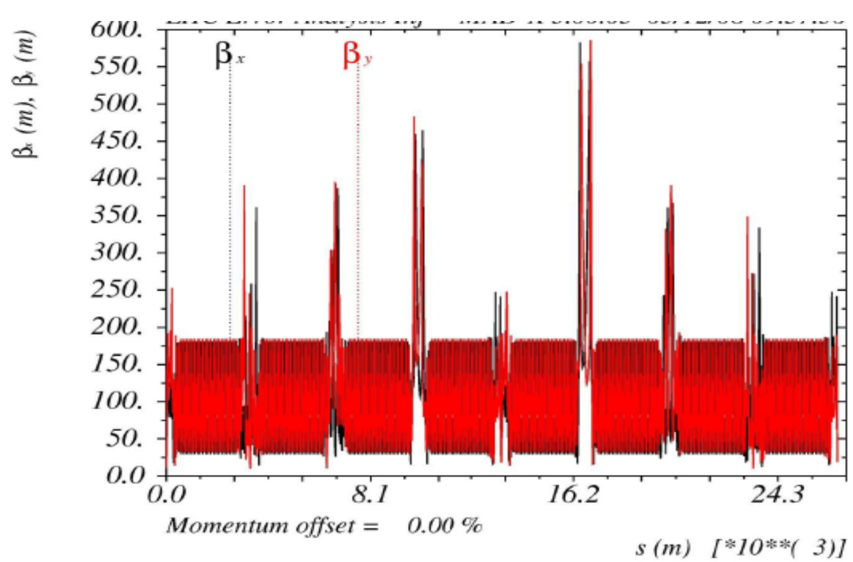
$$\sigma = \sqrt{\varepsilon \beta}$$

2.) At lowest energy the machine will have the major aperture problems,
→ here we have to minimise $\hat{\beta}$

3.) we need different beam optics adopted to the energy:
A Mini Beta concept will only be adequate at flat top.



LHC mini beta optics at 7000 GeV

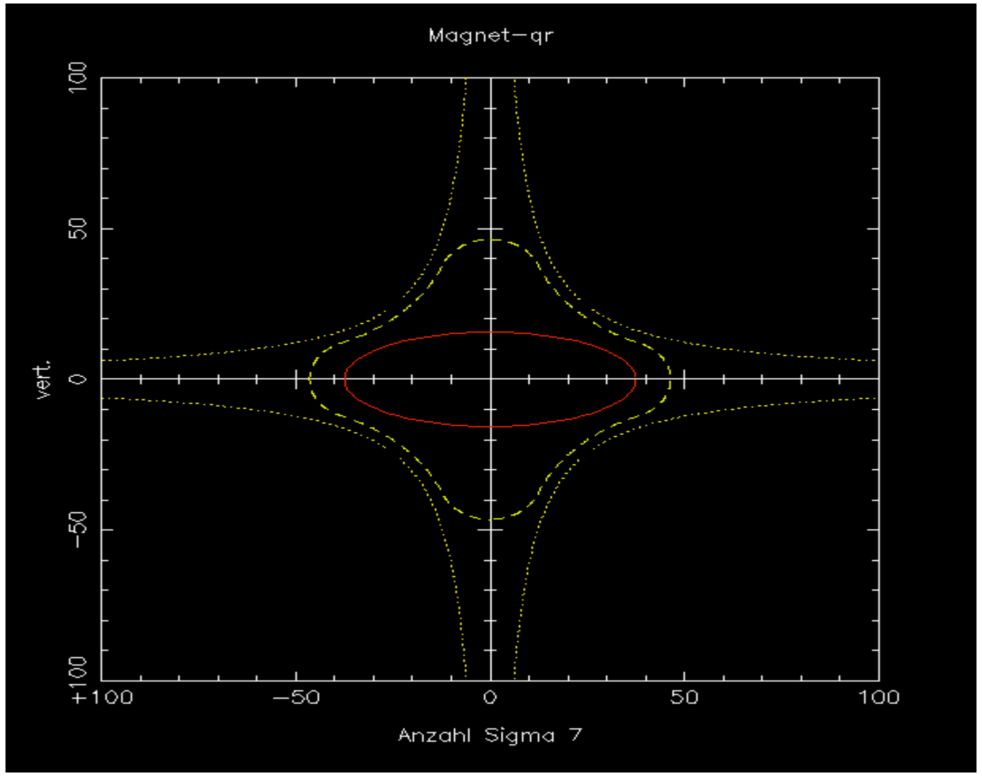


LHC injection optics at 450 GeV

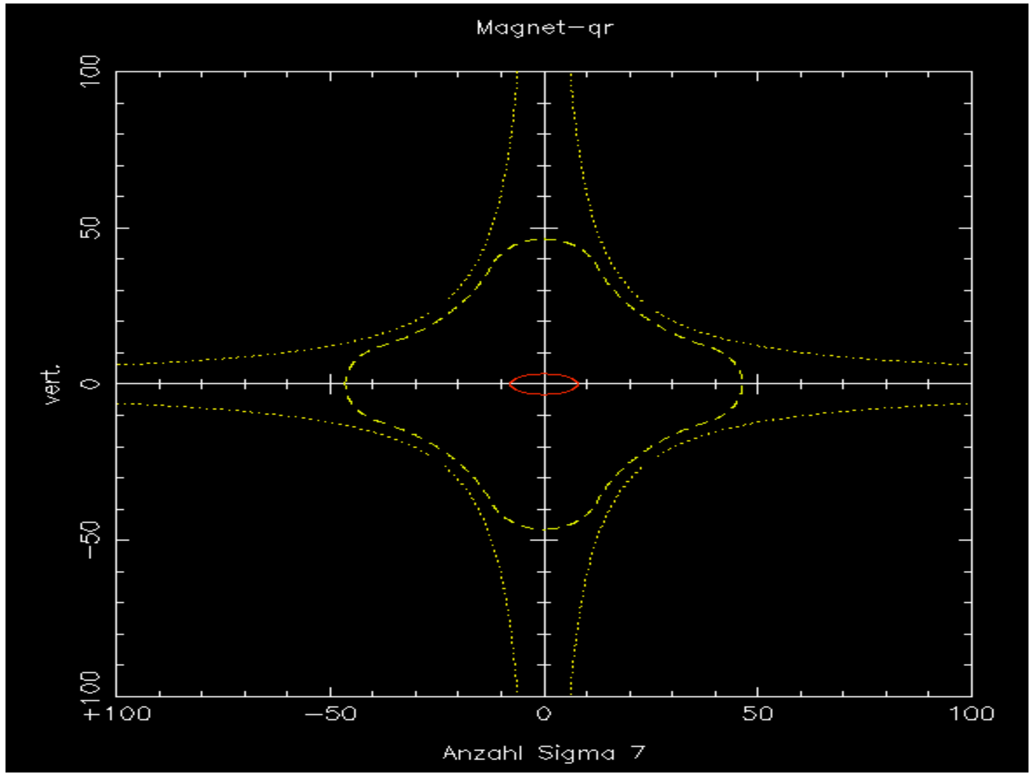
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance $\varepsilon\ (40GeV) = 1.2 * 10^{-7}$
$\varepsilon\ (920GeV) = 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

The „ not so ideal world "

14.) The „ Δp / $p \neq 0$" Problem

ideal accelerator: *all particles will see the* *same accelerating voltage.*

$\rightarrow \Delta p / p = 0$

„nearly ideal" accelerator: *Cockroft Walton or van de Graaf*

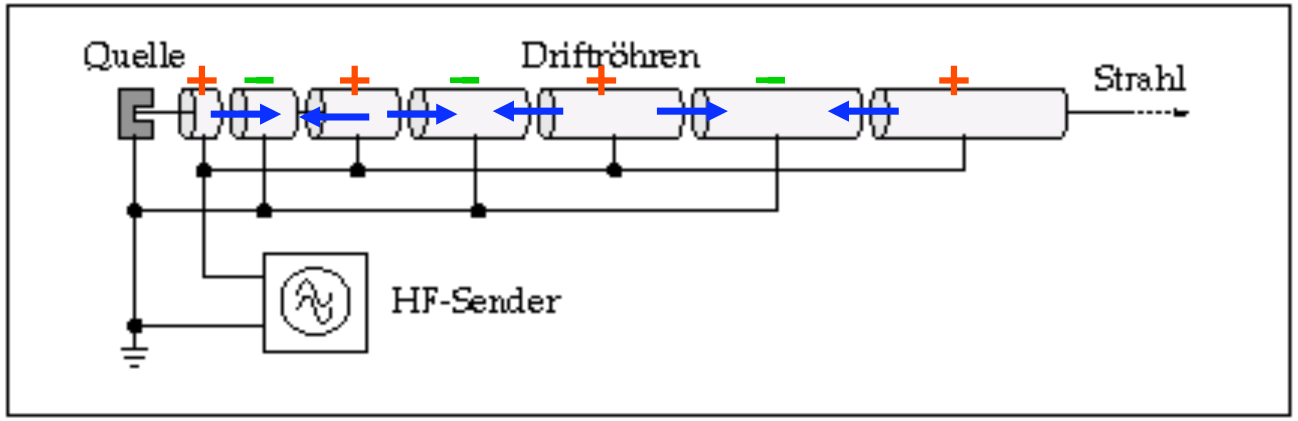
$\Delta p / p \approx 10^{-5}$

Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

RF Acceleration

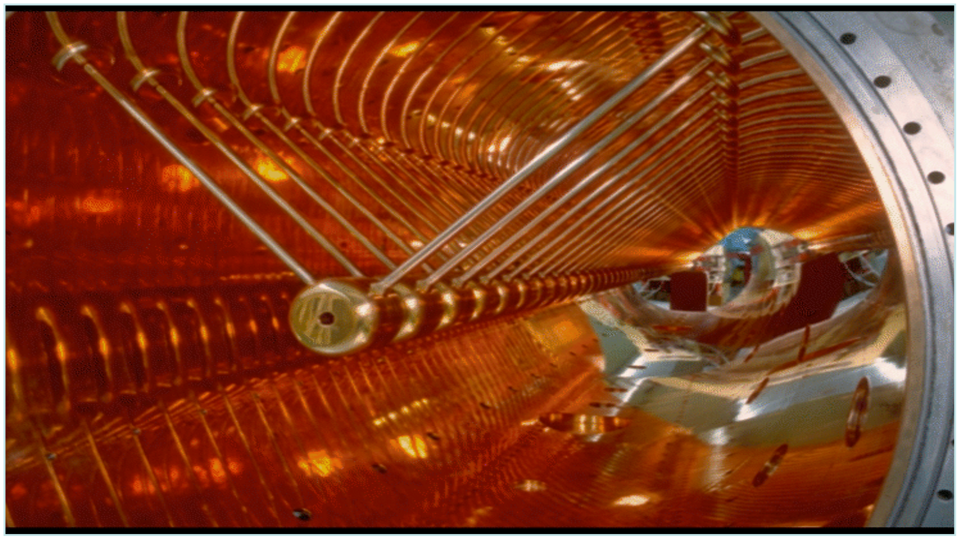
1928, Wideroe



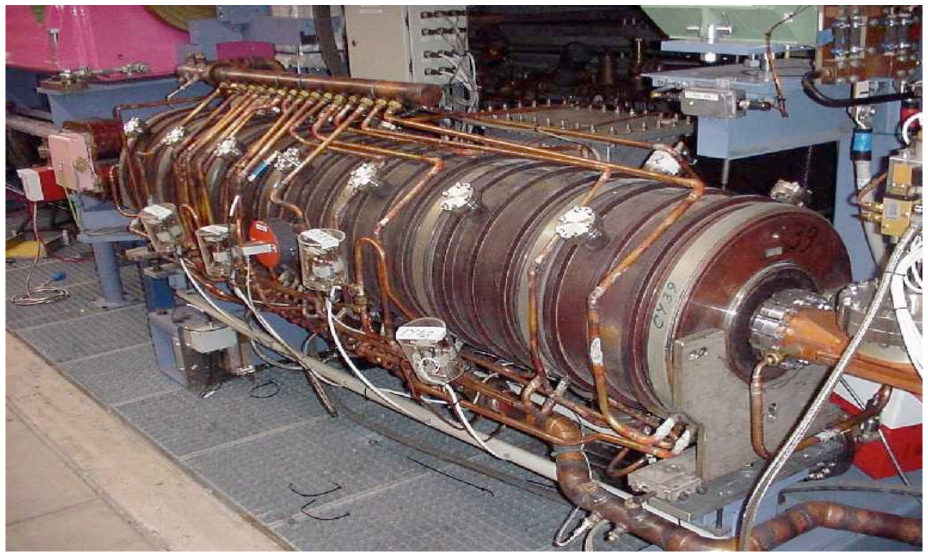
Energy Gain per „Gap":

$$\boxed{W = q\,U_0 \sin \omega_{RF} t}$$

drift tube structure at a proton linac (GSI Unilac)



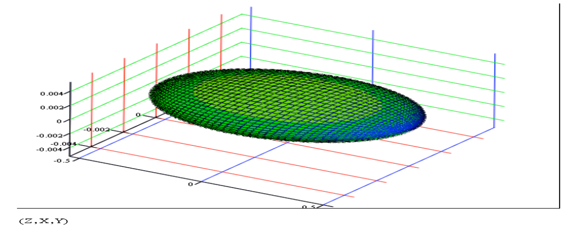
500 MHz cavities in an electron storage ring



* *RF Acceleration:* multiple application of the same acceleration voltage; brillant idea to gain higher energies

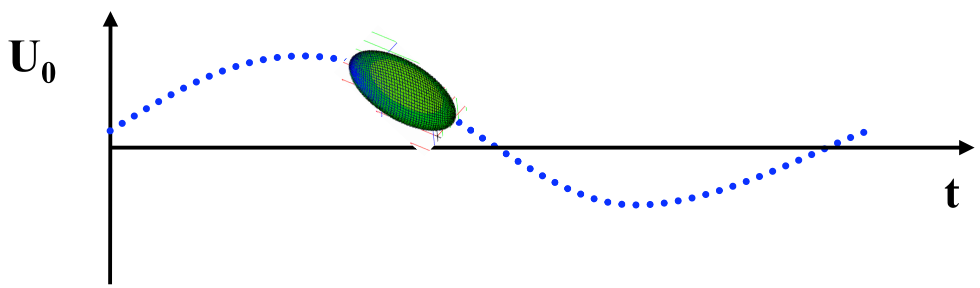
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons ≈ 1cm

Example: HERA RF:



$$\lambda = 60\ cm$$

$$\begin{rcases} \nu = 500\,MHz \\ c = \lambda\,\nu \end{rcases} \quad \lambda = 60\ cm$$

$$\sin(90^o) = 1$$

$$\sin(84^o) = 0.994$$

$$\frac{\Delta U}{U} = 6.0\ \ 10^{-3}$$

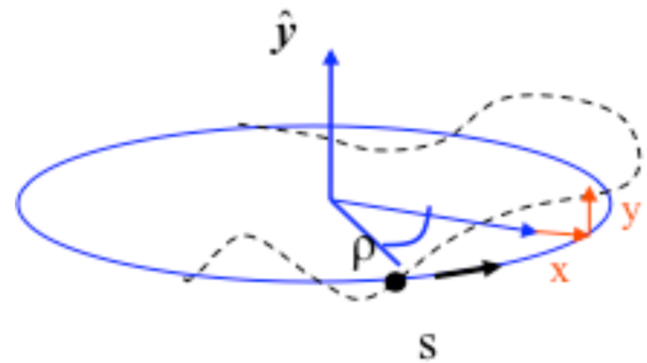
typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0\ \ 10^{-3}$$

16.) Dispersion: trajectories for $\Delta p / p \neq 0$

***Question:* do you remember last session, page 12 ? … sure you do**

***Force* acting on the particle**



$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

remember:* $x \approx mm$, $\rho \approx m$ … → *develop for small x

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = eB_y v$$

consider only linear fields, and change independent variable: $t \to s$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}(1-\frac{x}{\rho}) = \frac{e\ B_0}{mv} + \frac{e\ x\ g}{mv}$$

$p=p_0+\Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error

$$\Delta p << p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\ B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{xeg}{p_0}}_{k*x} - \underbrace{xeg\frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k*x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k*x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0}\frac{1}{\rho} \quad \longrightarrow \quad \boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0}\frac{1}{\rho}}$$

Momentum spread ***of the beam adds a term on the r.h.s. of the equation of motion.***
→ ***inhomogeneous differential equation.***

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

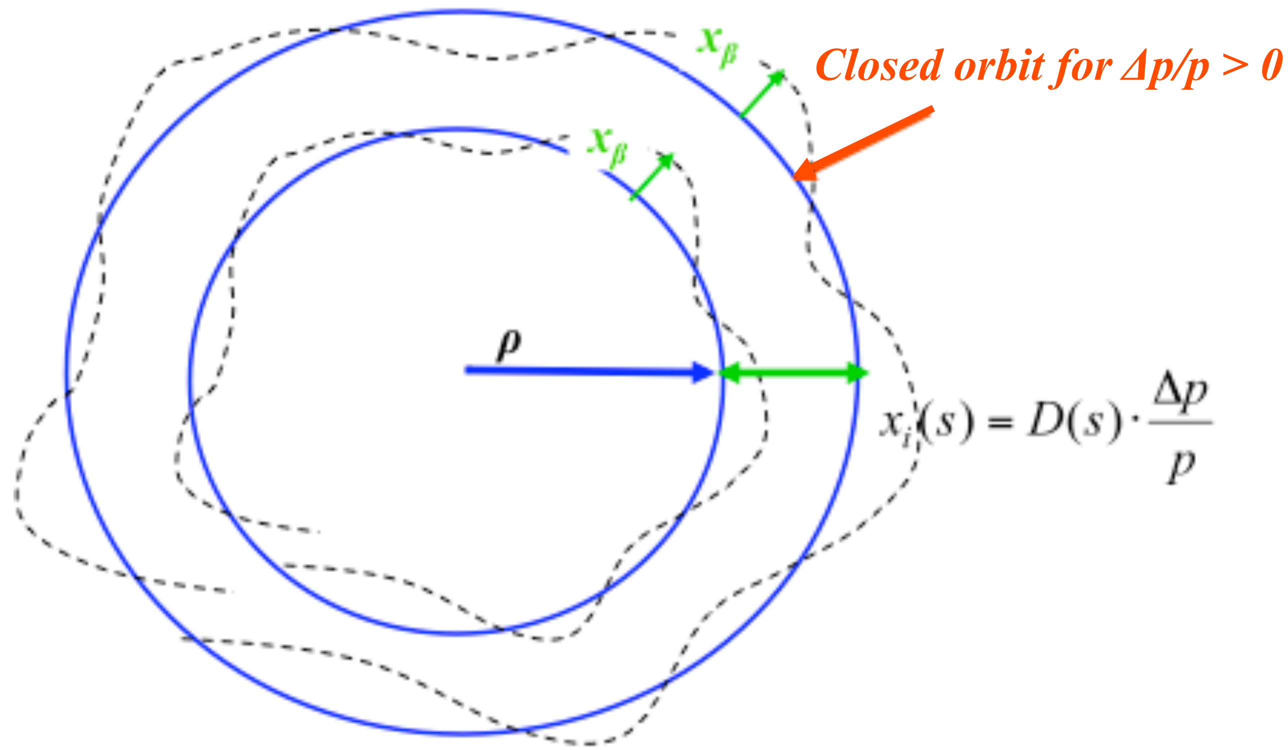
**** is that special orbit, an ideal particle would have for Δp/p = 1***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogeneous dipole field



Matrix formalism:

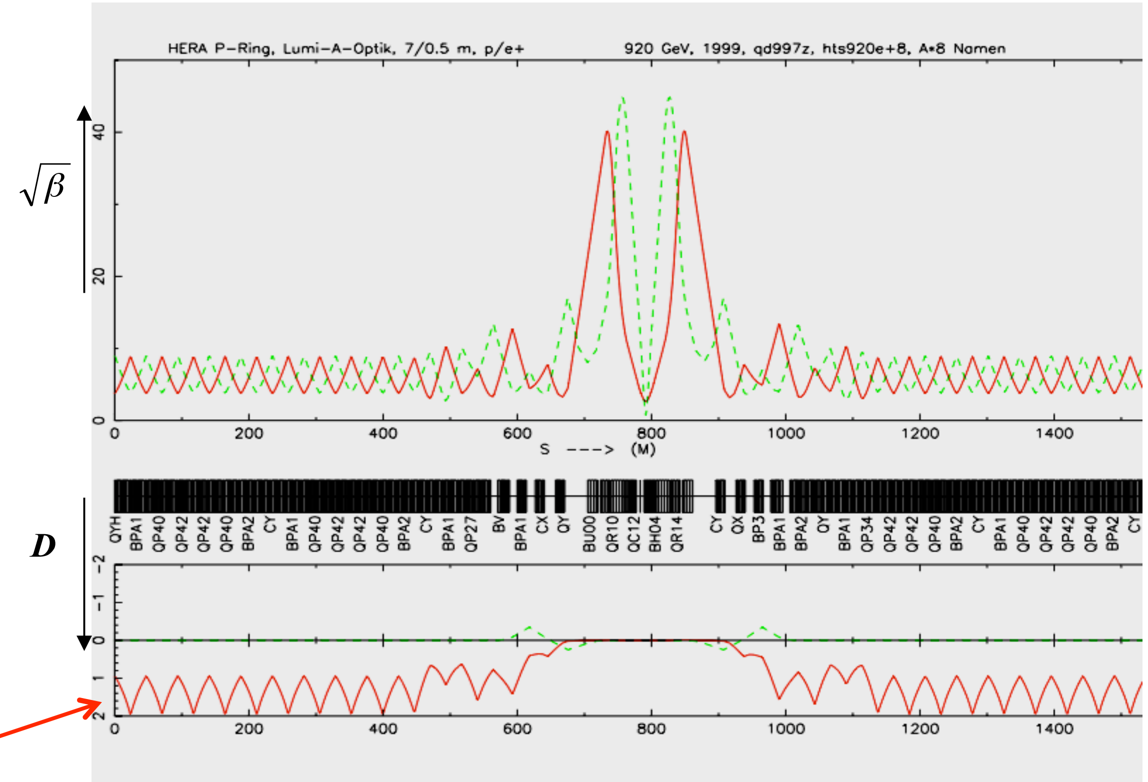
$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example

$$x_\beta = 1 \ldots 2\ mm$$

$$D(s) \approx 1 \ldots 2\ m$$

$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation

contribution due to Dispersion ≈ beam size

→ Dispersion must vanish at the collision point

!

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\,d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\,d\tilde{s}$$

Example: Drift

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$D(s) = S(s)\int_{s0}^{s1} \underbrace{\frac{1}{\rho}}_{=0} C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1} \underbrace{\frac{1}{\rho}}_{=0} S(\tilde{s})d\tilde{s}$$

Example: Dipole

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

$$\boldsymbol{K} = \frac{1}{\rho^2} - \cancel{k}$$

$$\boldsymbol{s} = \boldsymbol{l_B}$$

$$M_{Dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix} \quad \rightarrow$$

$$D(s) = \rho\cdot(1-\cos\frac{l}{\rho})$$

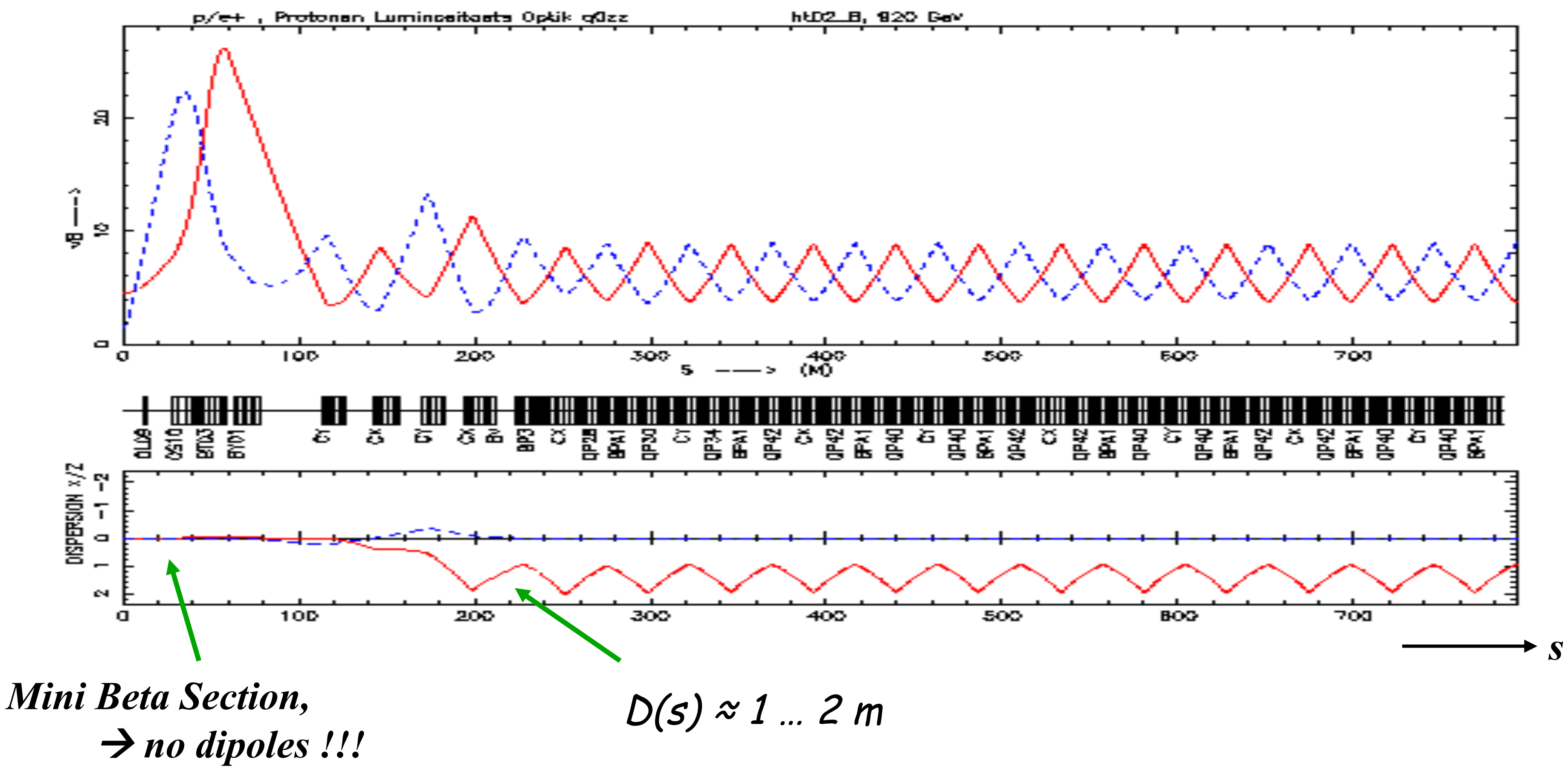
$$D'(s) = \sin\frac{l}{\rho}$$

Example: Dispersion, calculated by an optics code for a real machine

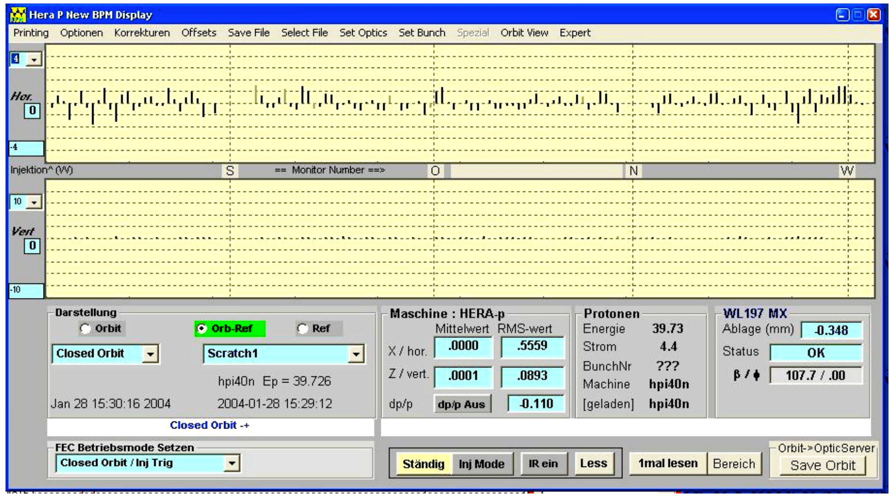
$$x_D = D(s)\,\frac{\Delta p}{p}$$

* D(s) is created by the dipole magnets

… and afterwards focused by the quadrupole fields



Dispersion is visible

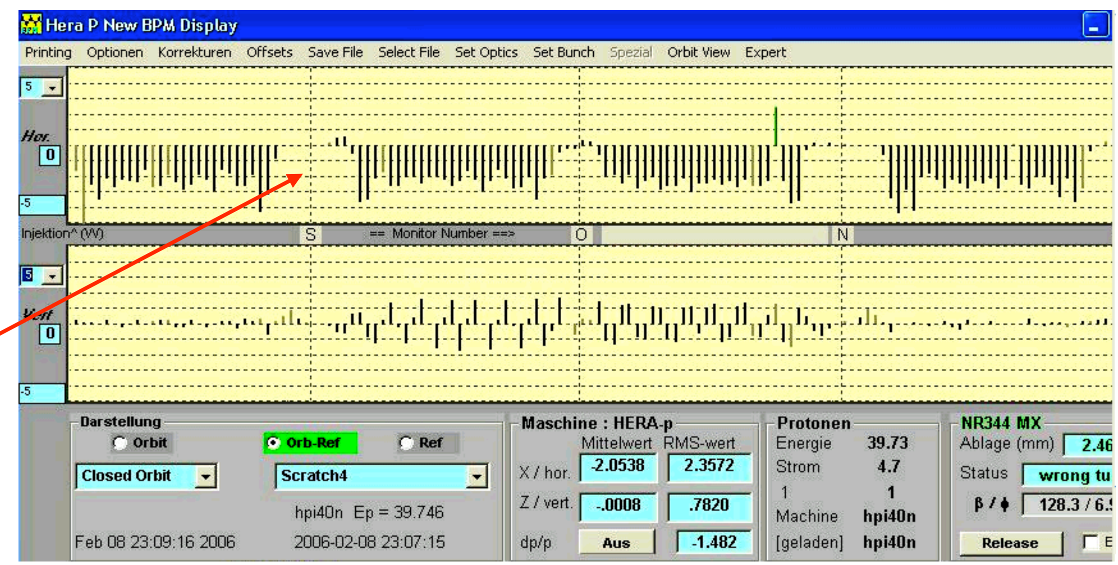


HERA Standard Orbit

HERA Dispersion Orbit

dedicated energy change of the stored beam
→ closed orbit is moved to a
dispersions trajectory

$$x_D = D(s) * \frac{\Delta p}{p}$$



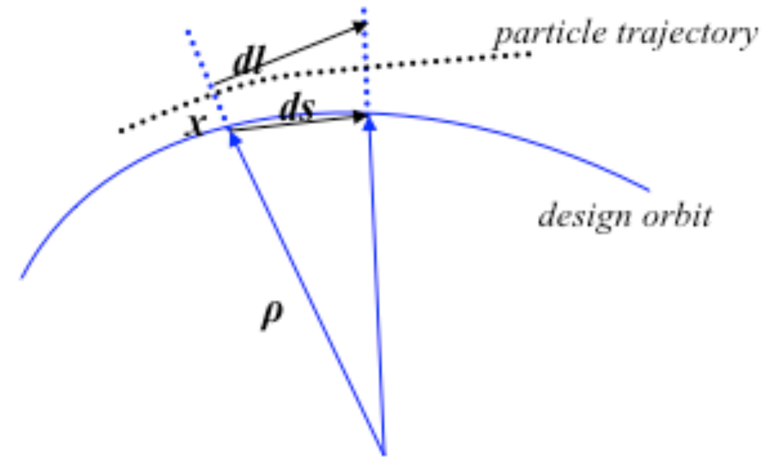
Attention: at the Interaction Points
we require D=D´= 0

17.) Momentum Compaction Factor: α_p

particle with a displacement x to the design orbit
→ path length dl ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \; dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$



circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s)\frac{\Delta p}{p}$$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

*** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.***

Definition:

$$\frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

$$\rightarrow \quad \alpha_p = \frac{1}{L}\oint\left(\frac{D(s)}{\rho(s)}\right) ds$$

For first estimates assume:

$$\frac{1}{\rho} = const.$$

$$\int_{dipoles} D(s)\, ds \approx l_{\Sigma(dipoles)} \cdot \langle D \rangle_{dipole}$$

$$\alpha_p = \frac{1}{L}\, l_{\Sigma(dipoles)} \cdot \langle D \rangle \frac{1}{\rho} = \frac{1}{L}\, 2\pi\rho \cdot \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \alpha_p \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Assume: $v \approx c$

$$\rightarrow \quad \frac{\delta T}{T} = \frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

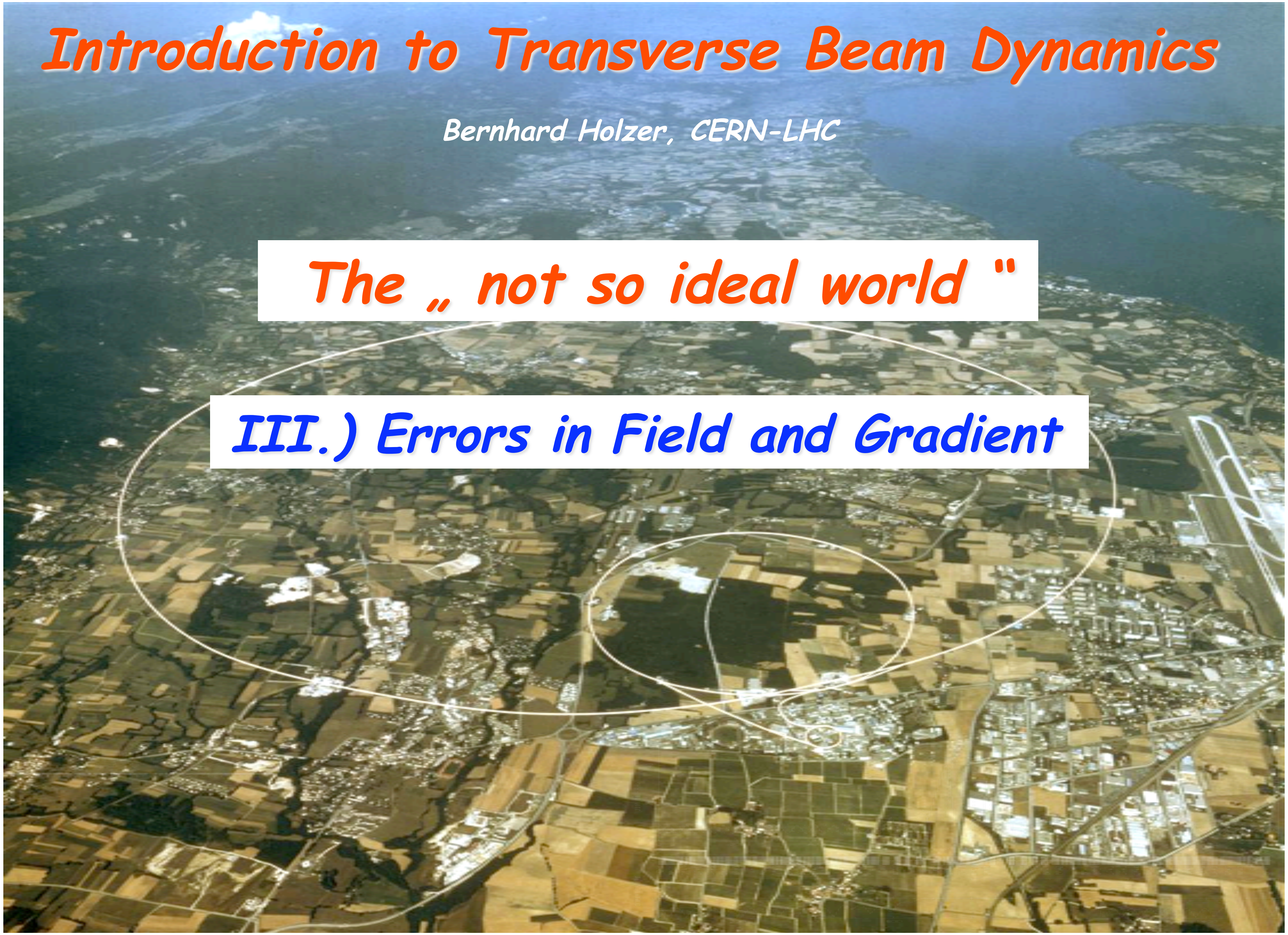
α_p combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

Introduction to Transverse Beam Dynamics

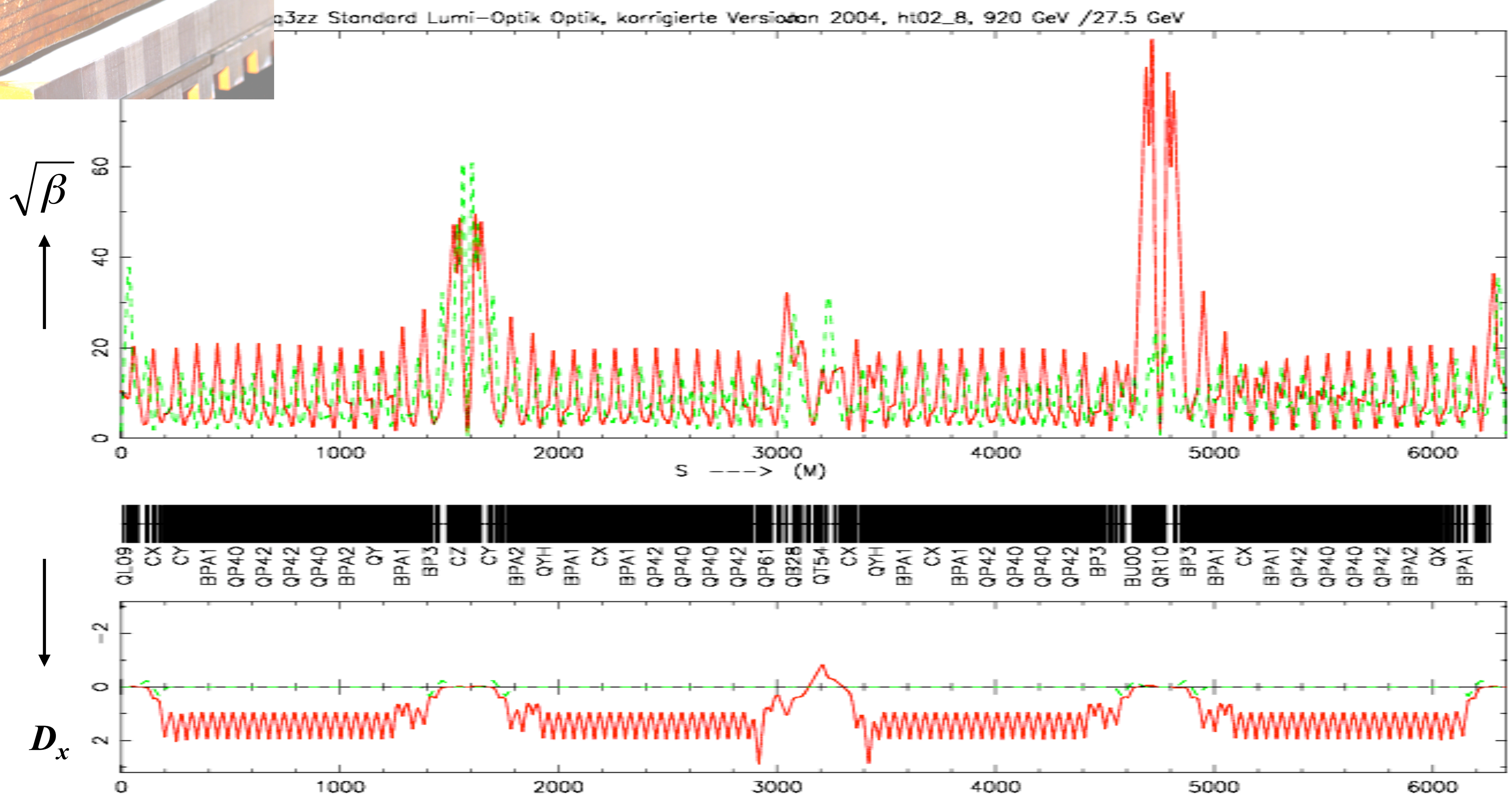
Bernhard Holzer, CERN-LHC

The „ not so ideal world "

III.) Errors in Field and Gradient



18.) Quadrupole Errors



Quadrupole Errors

go back to Lecture I, page 1

single particle trajectory

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{QF} * \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

Solution of equation of motion

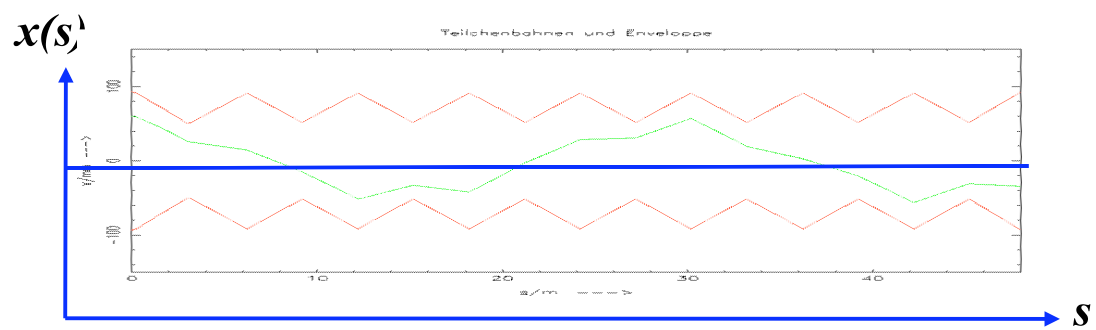
$$x = x_0 \cos(\sqrt{k}\, l_q) + x_0' \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q)$$

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{k}\, l_q) & \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q) \\ -\sqrt{k} \sin(\sqrt{k}\, l_q) & \cos(\sqrt{k}\, l_q) \end{pmatrix} \quad , \quad M_{thinlens} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$$M_{turn} = M_{QF} * M_{D1} * M_{QD} * M_{D2} * M_{QF} \; \ldots$$

Definition: *phase advance of the particle oscillation per revolution in units of 2π is called* *tune*

$$Q = \frac{\psi_{turn}}{2\pi}$$



Matrix in Twiss Form

Transfer Matrix from point „0“ in the lattice to point „s“:

$$M(s) = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos(\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos(\psi_s - \alpha_0 \sin\psi_s) \end{pmatrix}$$

For one complete turn the Twiss parameters have to obey periodic bundary conditions:

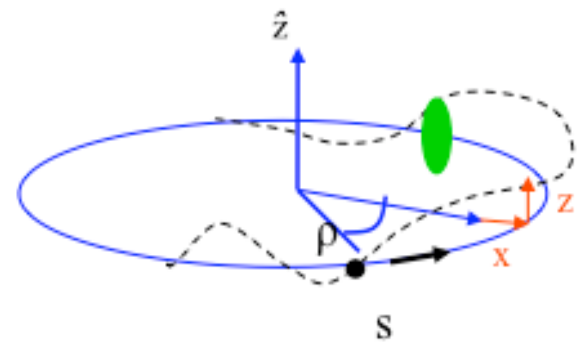
$$\beta(s+L) = \beta(s)$$
$$\alpha(s+L) = \alpha(s)$$
$$\gamma(s+L) = \gamma(s)$$

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_s & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

$$\boldsymbol{M}_{dist} = \boldsymbol{M}_{\Delta k} \cdot \boldsymbol{M}_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta \boldsymbol{kds} & 1 \end{pmatrix}}_{\textit{quad error}} \cdot \underbrace{\begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}}_{\textit{ideal storage ring}}$$



$$\boldsymbol{M}_{dist} = \begin{pmatrix} \cos\psi_0 + \alpha \sin\psi_0 & \beta \sin\psi_0 \\ \Delta \boldsymbol{kds}(\cos\psi_0 + \alpha \sin\psi_0) - \gamma \sin\psi_0 & \Delta \boldsymbol{kds}\, \beta \sin\psi_0 + \cos\psi_0 - \alpha \sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$\boldsymbol{Trace}(\boldsymbol{M}) = 2\cos\psi = 2\cos\psi_0 + \Delta \boldsymbol{kds}\beta \sin\psi_0$$

Quadrupole error → Tune Shift

$$\boxed{\psi = \psi_0 + \Delta\psi} \longrightarrow \cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta \boldsymbol{kds}\,\beta\sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos\psi_0 \underbrace{\cos\Delta\psi}_{\approx 1} - \sin\psi_0 \underbrace{\sin\Delta\psi}_{\approx \Delta\psi} = \cos\psi_0 + \frac{\boldsymbol{kds}\,\beta\sin\psi_0}{2}$$

$$\Delta\psi = \frac{\boldsymbol{kds}\,\beta}{2}$$

and referring to Q instead of ψ:

$$\psi = 2\pi \boldsymbol{Q}$$

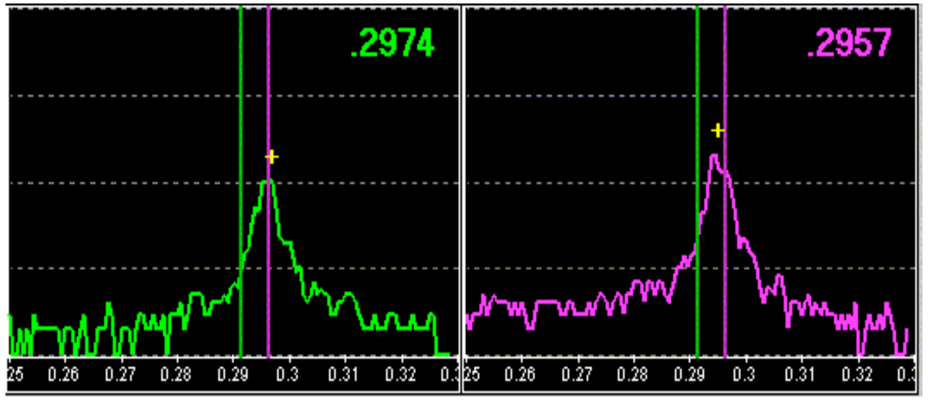
$$\boxed{\Delta \boldsymbol{Q} = \int_{s0}^{s0+l} \frac{\Delta \boldsymbol{k}(\boldsymbol{s})\beta(\boldsymbol{s})\boldsymbol{ds}}{4\pi}}$$

! the tune shift is proportional to the β-function at the quadrupole

!! field quality, power supply tolerances etc are much tighter at places where β is large

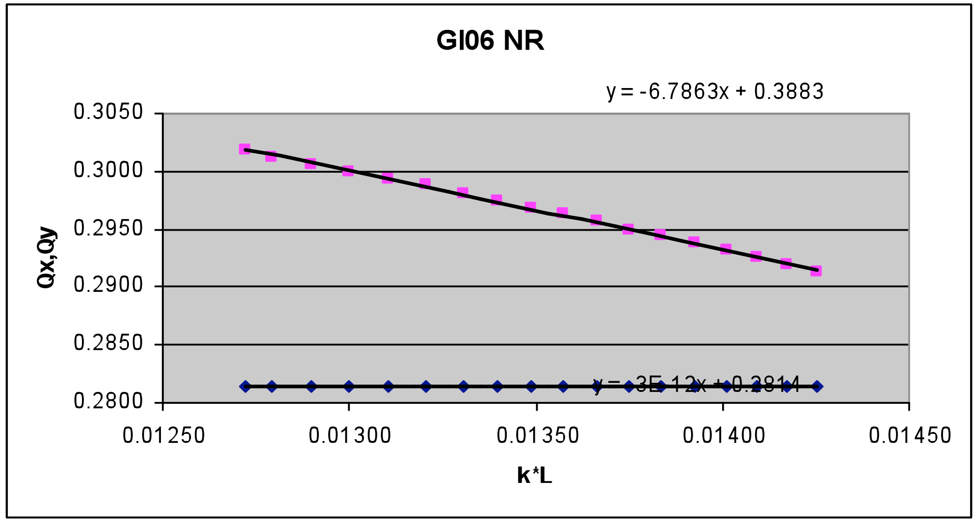
!!! mini beta quads: β ≈ 1900 m
arc quads: β ≈ 80 m

!!!! β is a measure for the sensitivity of the beam *a quadrupol error leads to a shift of the tune:*



$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k \beta(s)}{4\pi} ds \approx \frac{\Delta k l_{quad} \bar{\beta}}{4\pi}$$

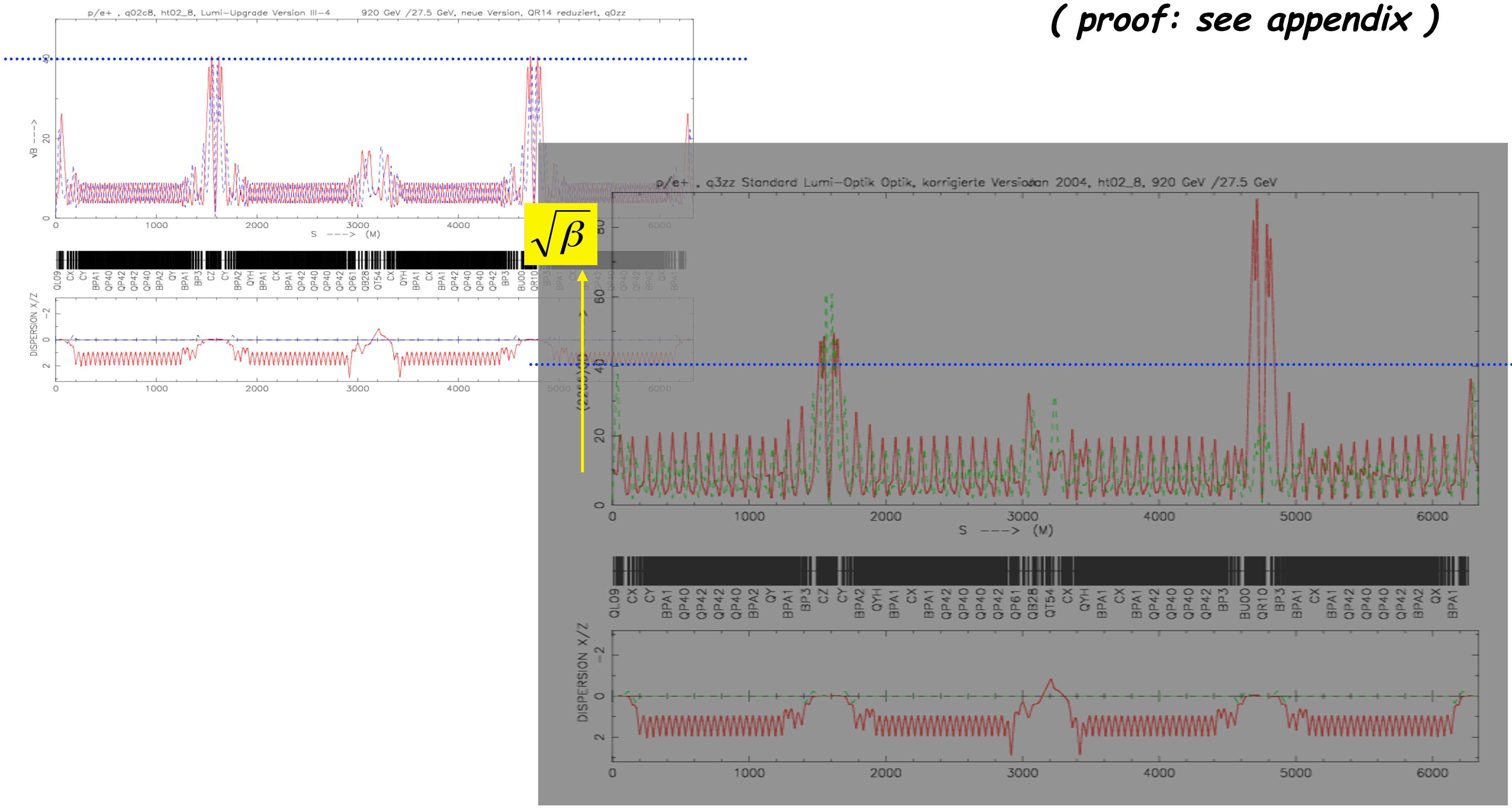
Example: measurement of β in a storage ring: tune spectrum



Quadrupole error: Beta Beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta K \cos\left(2\left|\psi_{s1}-\psi_{s0}\right| - 2\pi Q\right)ds$$

(proof: see appendix)

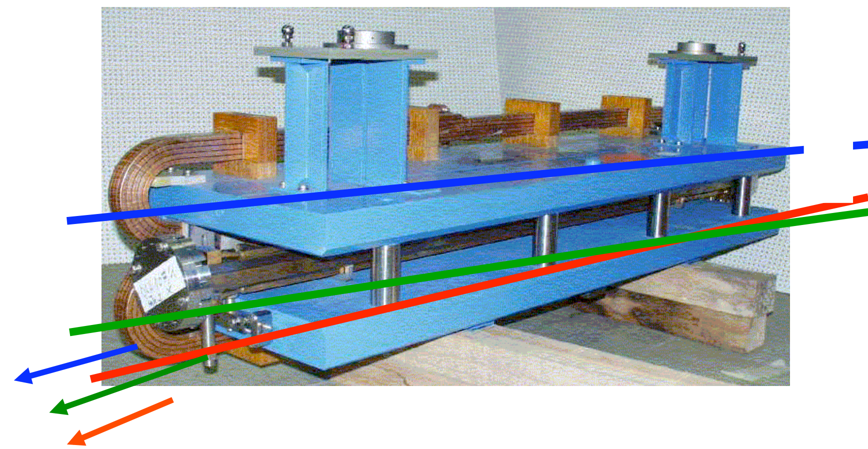


19.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

dipole magnet

$$\alpha = \frac{\int B\,dl}{p/e}$$



$$x_D(s) = D(s)\,\frac{\Delta p}{p}$$

focusing lens

$$k = \frac{g}{p/e}$$

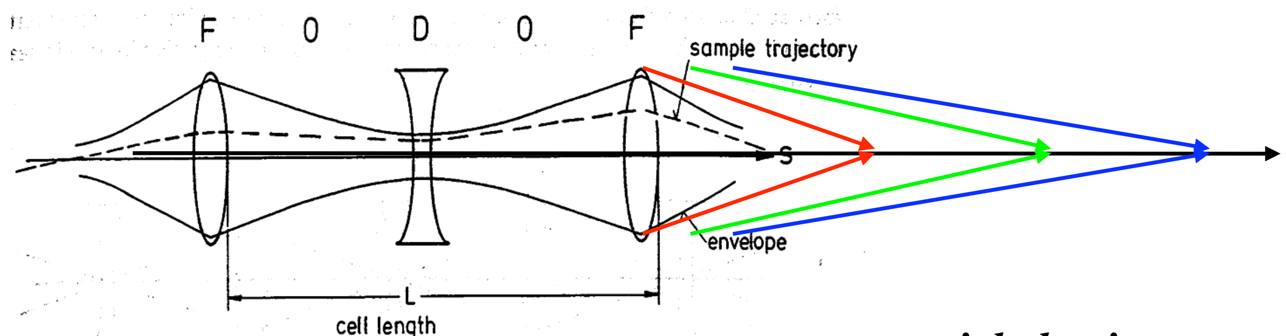


Figure 29: FODO cell

particle having ...

to high energy

to low energy

ideal energy

Chromaticity: Q'

$$k = \frac{g}{p/e} \qquad\qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$\boldsymbol{k} = \frac{\boldsymbol{eg}}{\boldsymbol{p}_0 + \Delta \boldsymbol{p}} \approx \frac{\boldsymbol{e}}{\boldsymbol{p}_0}\left(1 - \frac{\Delta \boldsymbol{p}}{\boldsymbol{p}_0}\right) \boldsymbol{g} = \boldsymbol{k}_0 + \Delta \boldsymbol{k}$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

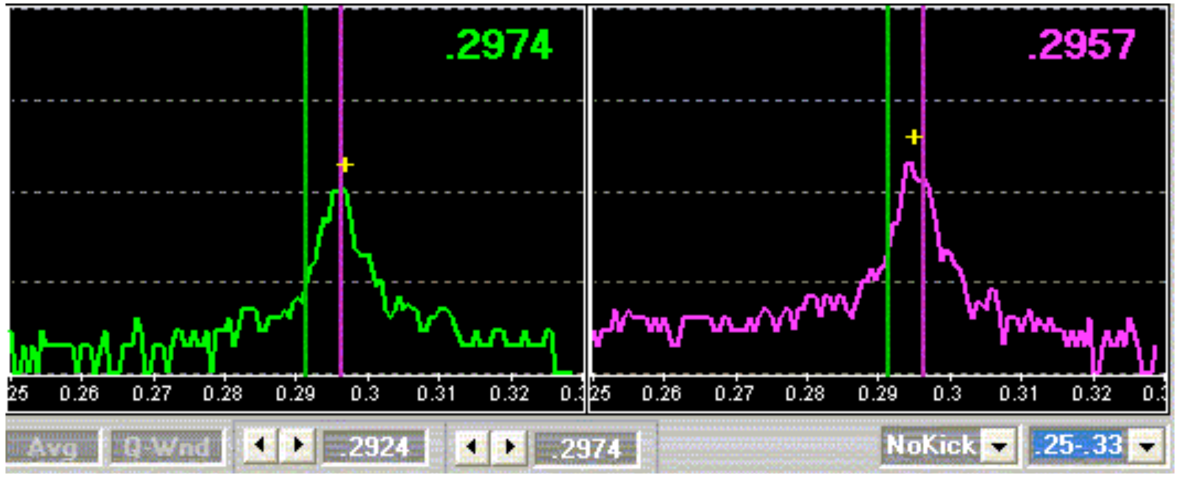
… which acts like a quadrupole error in the machine and leads to a tune spread:

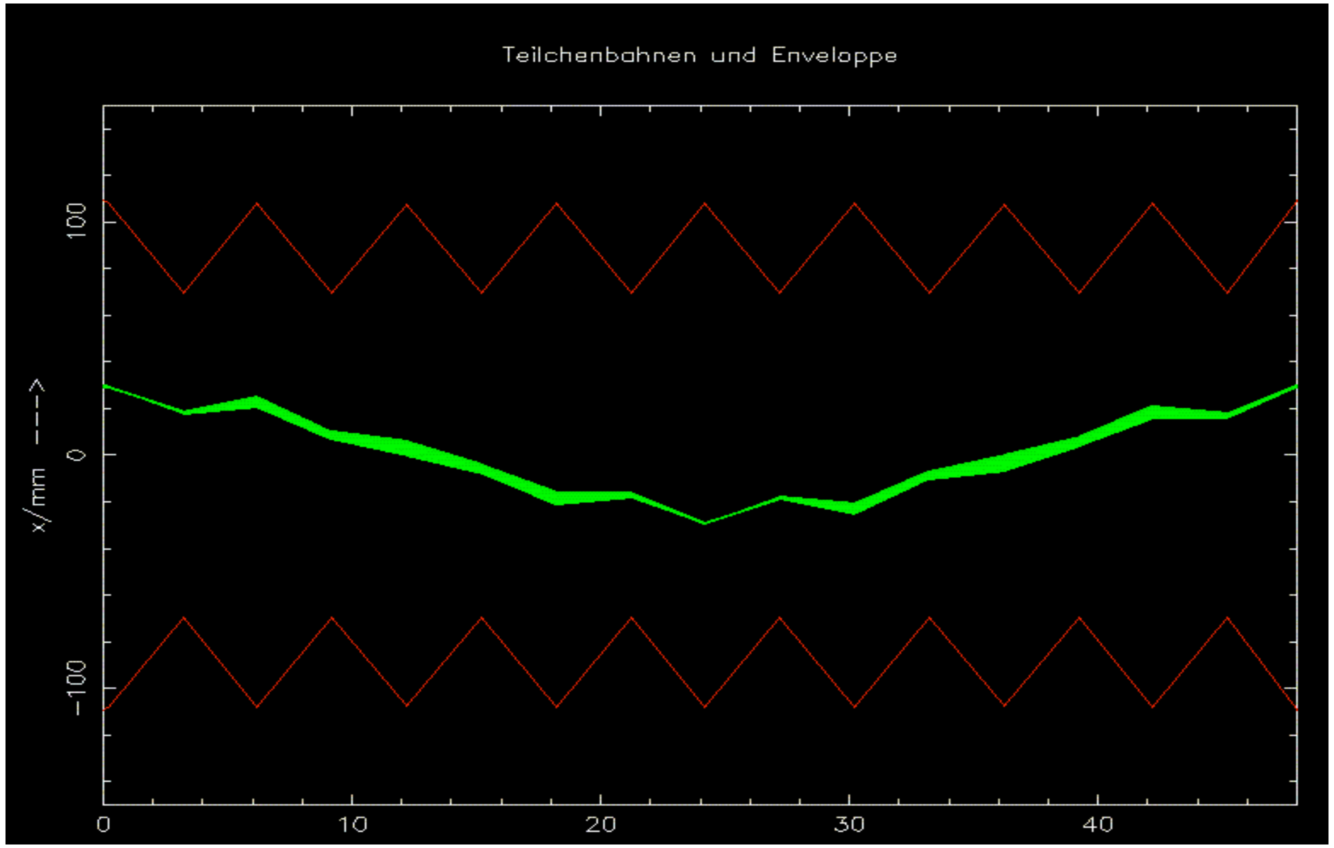
$$\Delta \boldsymbol{Q} = -\frac{1}{4\pi}\frac{\Delta \boldsymbol{p}}{\boldsymbol{p}_0} \boldsymbol{k}_0 \beta(\boldsymbol{s}) \boldsymbol{ds}$$

definition of chromaticity:

$$\Delta \boldsymbol{Q} = \boldsymbol{Q}' \frac{\Delta \boldsymbol{p}}{\boldsymbol{p}} \quad ; \qquad \boldsymbol{Q}' = -\frac{1}{4\pi} \oint \boldsymbol{k}(\boldsymbol{s}) \beta(\boldsymbol{s}) \boldsymbol{ds}$$

Tunes and Resonances





avoid resonance conditions:

$$m\,Q_x + n\,Q_y + l\,Q_s = \text{integer}$$

… for example: $1\,Q_x = 1$

… and now again about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

Q' is a number indicating the size of the tune spot in the working diagram,
Q' is always created if the beam is focussed
→ it is determined by the focusing strength k of all quadrupoles

$$Q' = -\frac{1}{4\pi} \oint k(s)\beta(s)ds$$

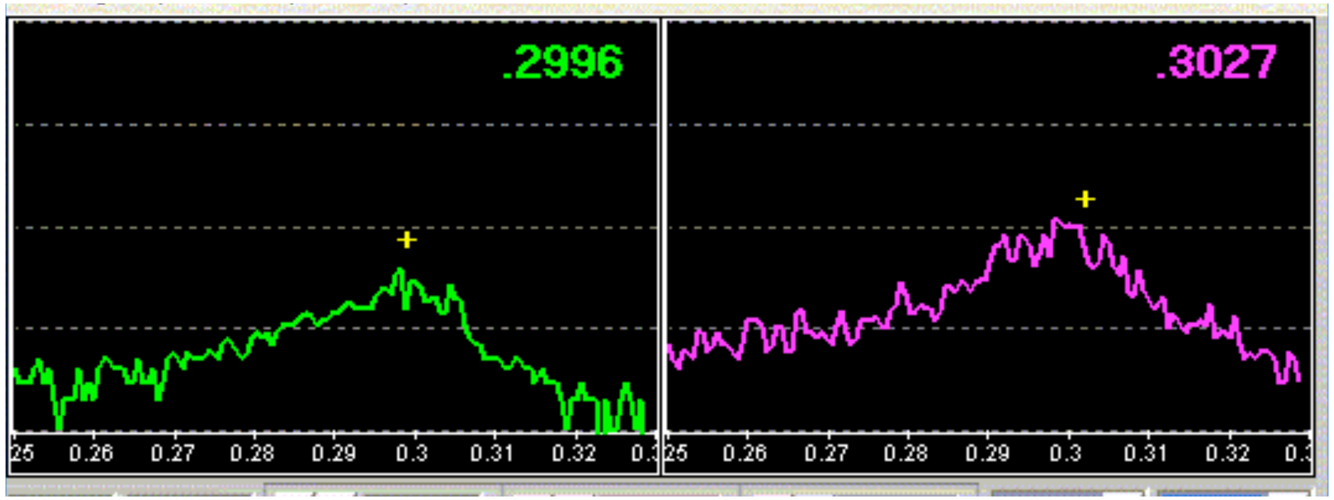
k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: LHC

$Q' = 250$
$\Delta p/p = \pm 0.2 \cdot 10^{-3}$
$\Delta Q = 0.256 \ldots 0.36$

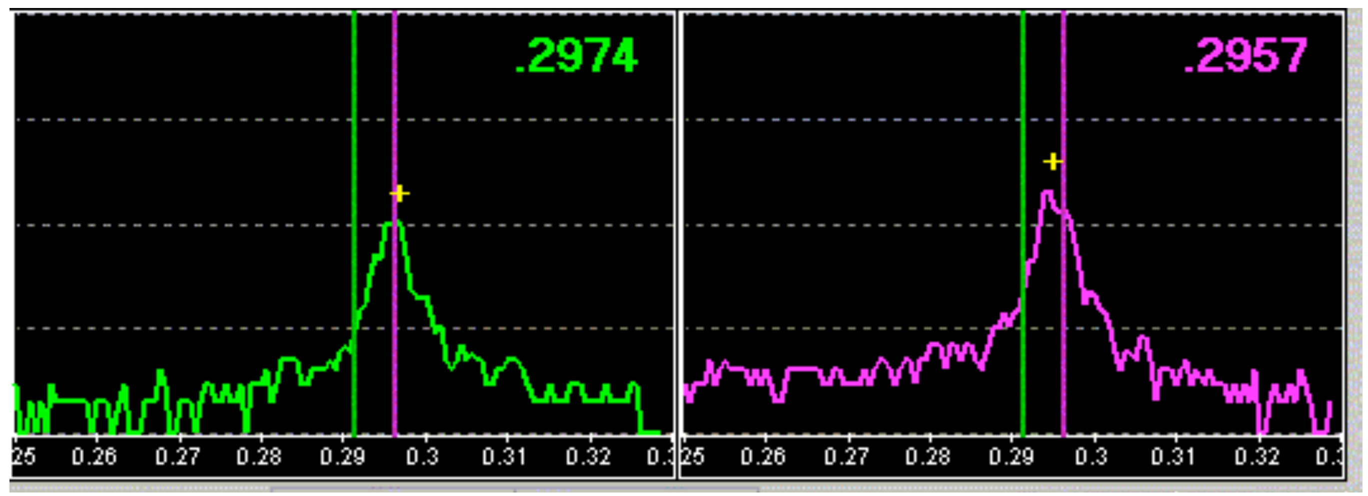
→ Some particles get very close to resonances and are lost

in other words: the tune is not a point it is a pancake



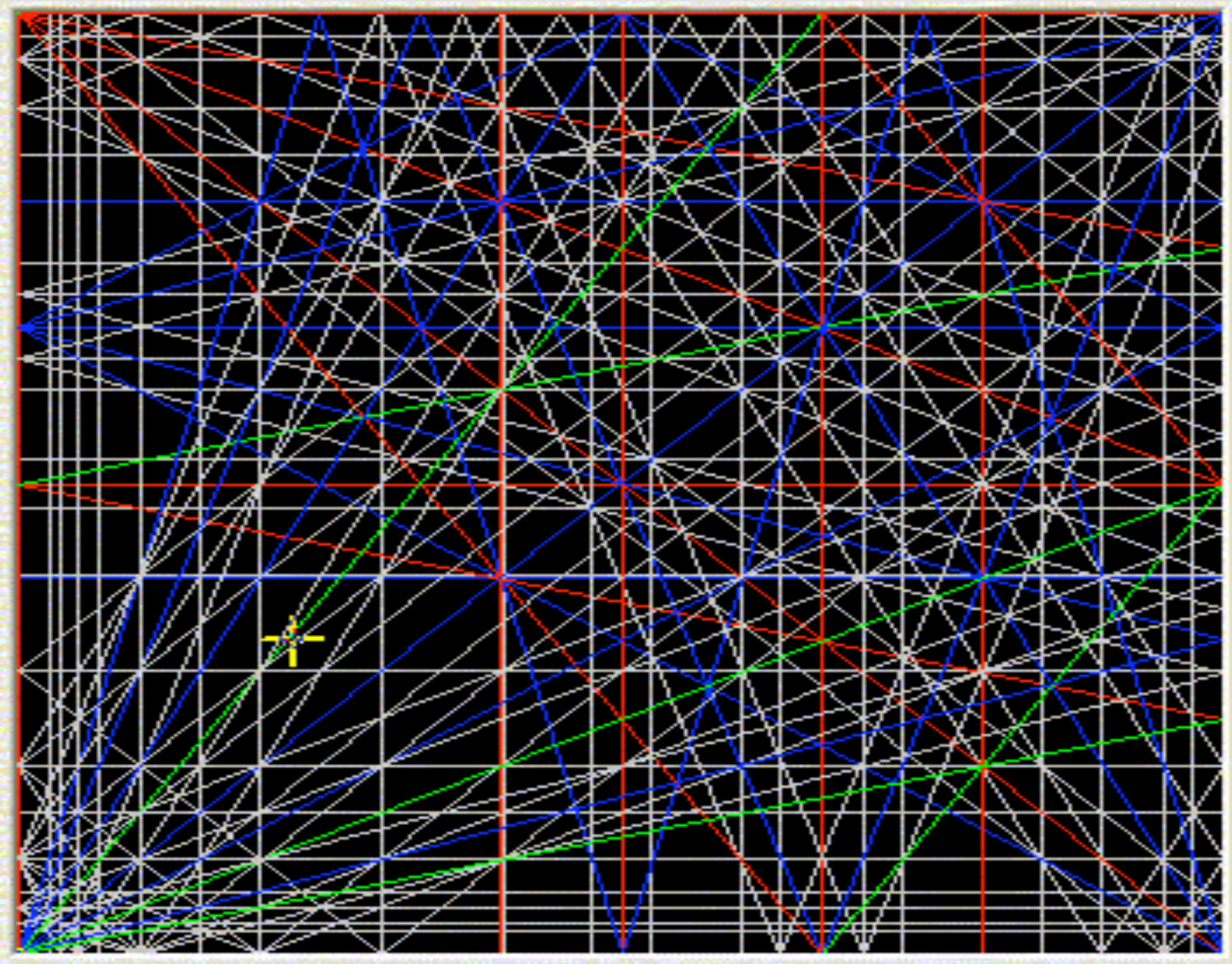
Tune signal for a nearly uncompensated cromaticity (Q' ≈ 20)

Ideal situation: cromaticity well corrected, (Q' ≈ 1)



Tune and Resonances

$$m*Q_x+n*Q_y+l*Q_s = integer$$



ꓤA e Tune diagram up to 3rd order

… and up to 7th order

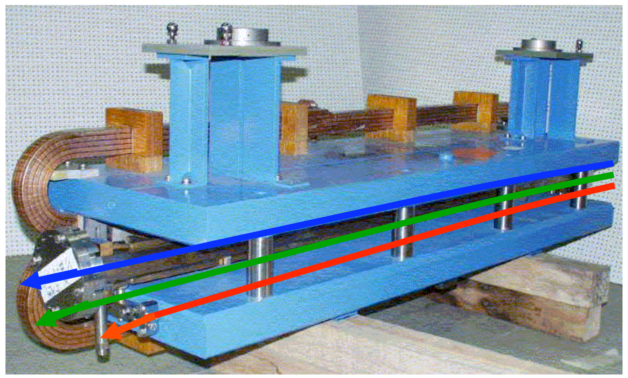
Homework for the operateurs:
find a nice place for the tune
where against all probability
the beam will survive

Correction of Q':

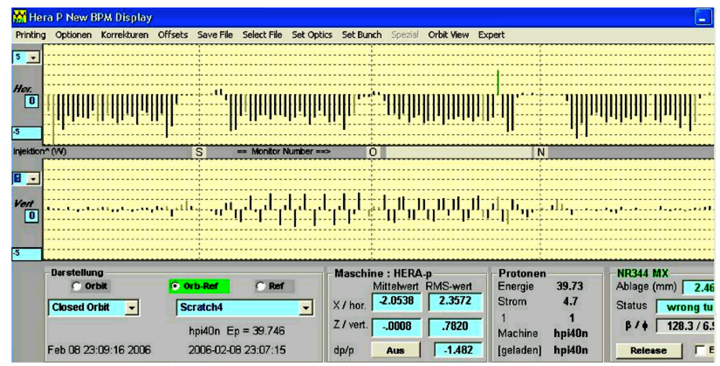
Need: additional quadrupole strength for each momentum deviation Δp/p

1.) sort the particles acording to their momentum

$$x_D(s) = D(s)\frac{\Delta p}{p}$$



… using the dispersion function



2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$B_x = \tilde{g}xz$$

$$B_z = \frac{1}{2}\tilde{g}(x^2 - z^2)$$

$$\frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

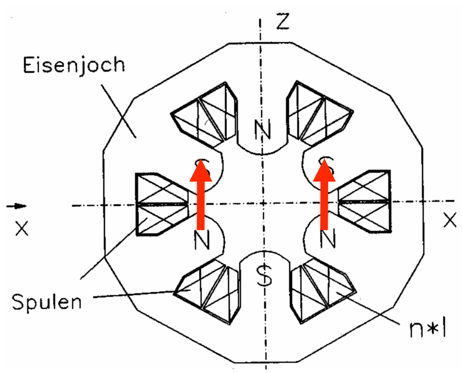
linear rising „gradient":

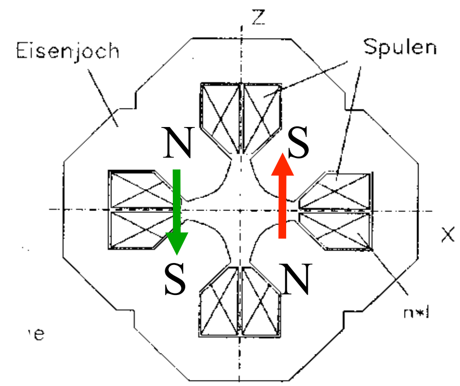
Correction of Q':

k_1 normalised quadrupole strength

k_2 normalised sextupole strength

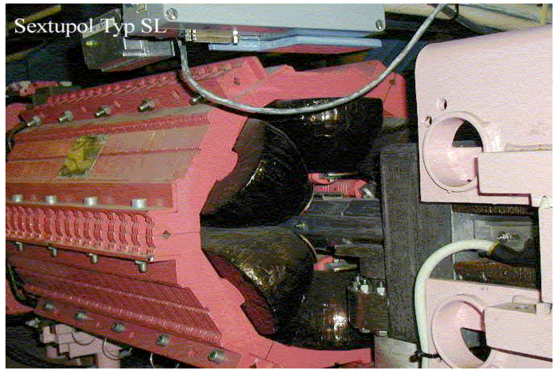
Sextupole Magnets:





$$k_1(sext) = \frac{\tilde{g}\, x}{p/e} = k_2 * x$$

$$k_1(sext) = k_2 * D * \frac{\Delta p}{p}$$



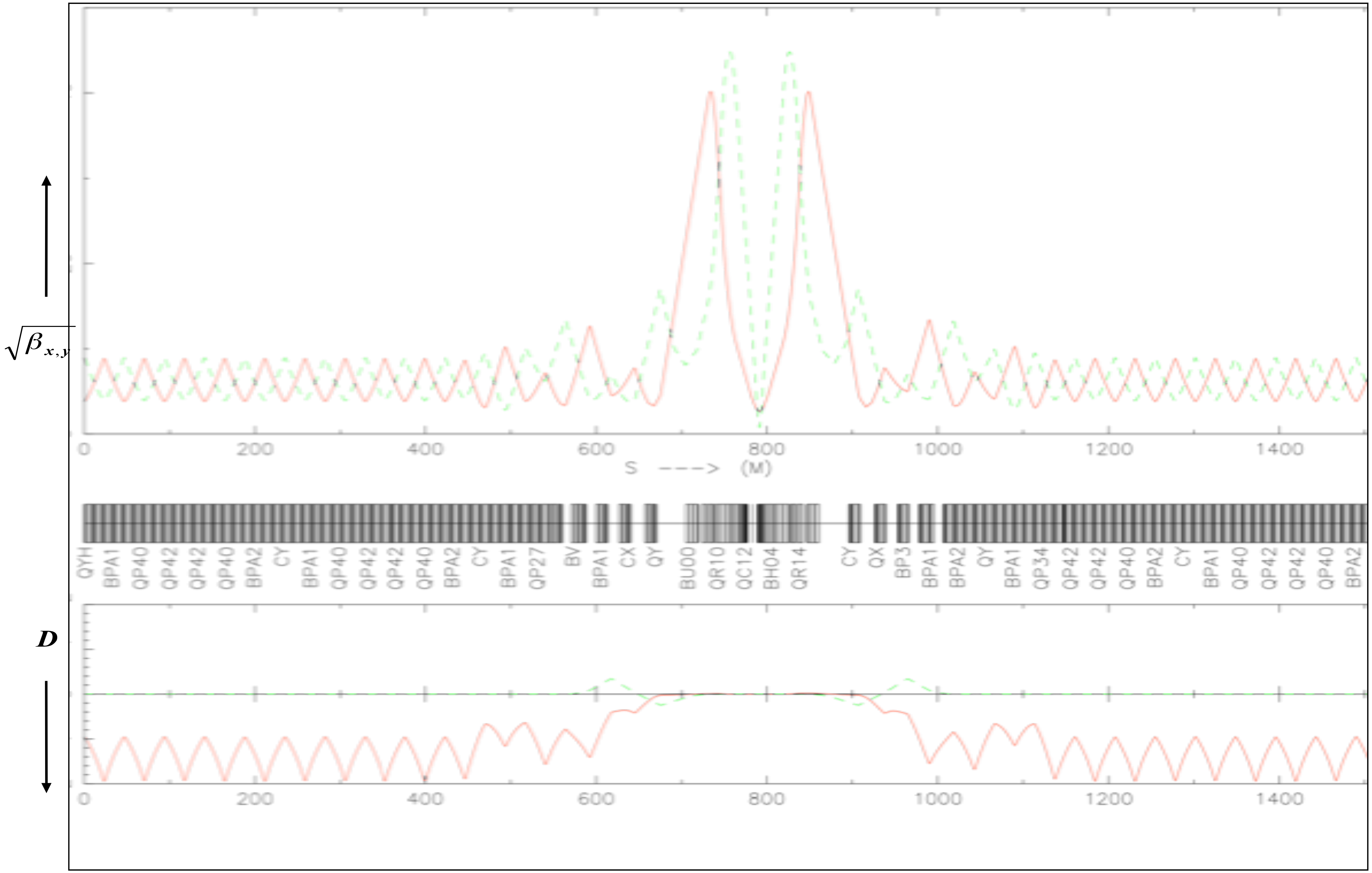
corrected chromaticity

considering a single cell:

$$Q'_{cell\_x} = -\frac{1}{4\pi}\left\{k_{qf}\hat{\beta}_x l_{qf} - k_{qd}\breve{\beta}_x l_{qd}\right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

$$Q'_{cell\_y} = -\frac{1}{4\pi}\left\{-k_{qf}\breve{\beta}_y l_{qf} + k_{qd}\hat{\beta}_y l_{qd}\right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

20.) Insertions



Insertions

... the most complicated one: **the drift space**

Question to the audience: what will happen to the beam parameters α, β, γ if we stop focusing for a while ...?

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_S = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

transfer matrix for a drift:

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \longrightarrow$$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$
$$\alpha(s) = \alpha_0 - \gamma_0 s$$
$$\gamma(s) = \gamma_0$$

β-Function in a Drift:

let's assume we are at a symmetry point in the center of a drift.

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

as $\alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$

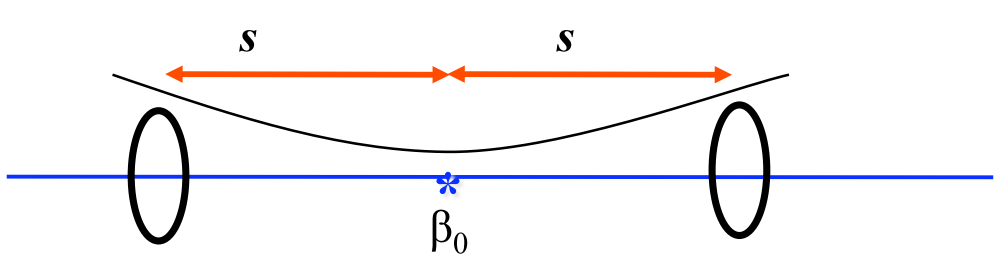
and we get for the β function in the neighborhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

! ! !

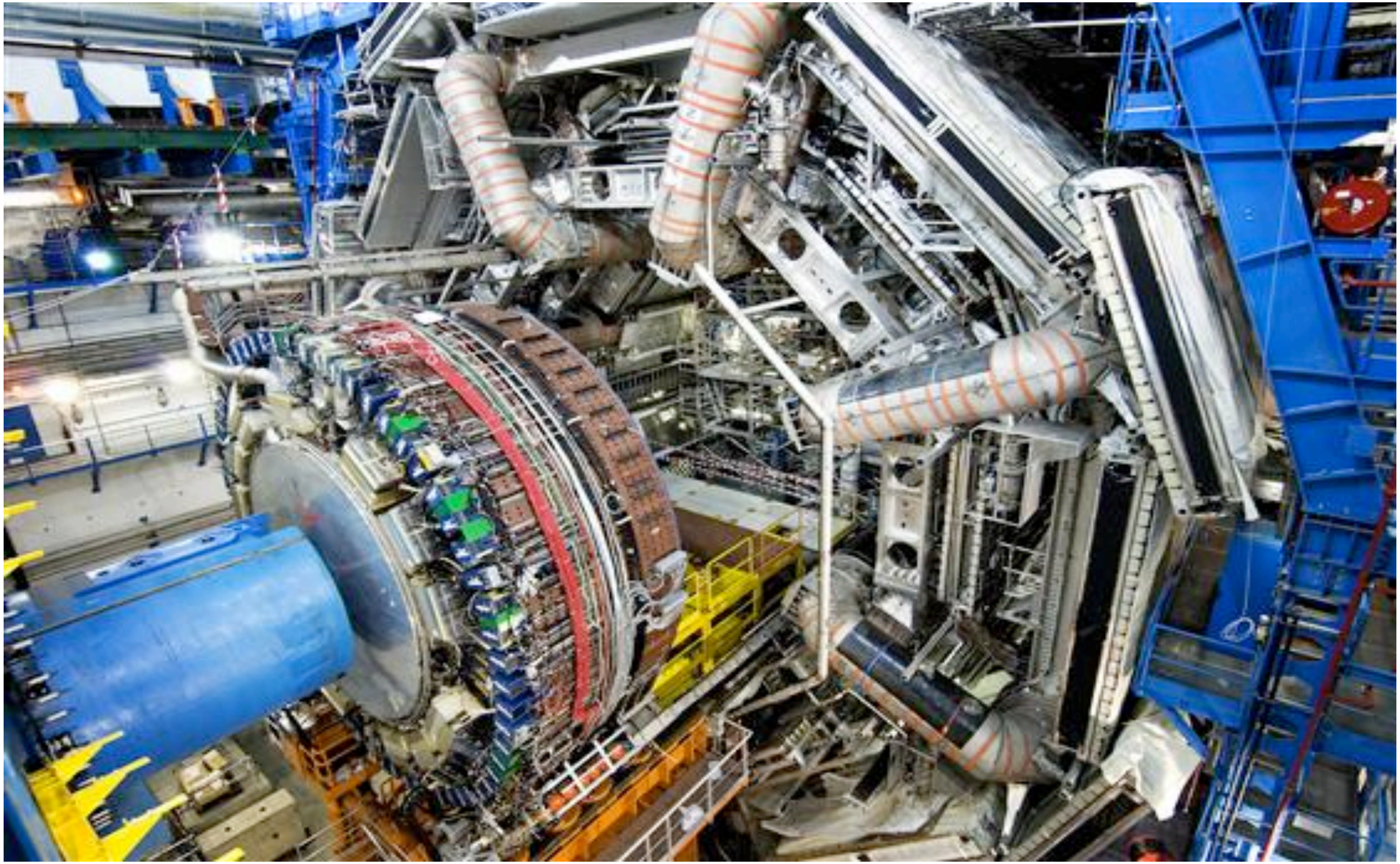
At the end of a long symmetric drift space
the beta function reaches its maximum value
in the complete lattice.
-> here we get the largest beam dimension.

-> keep l as small as possible

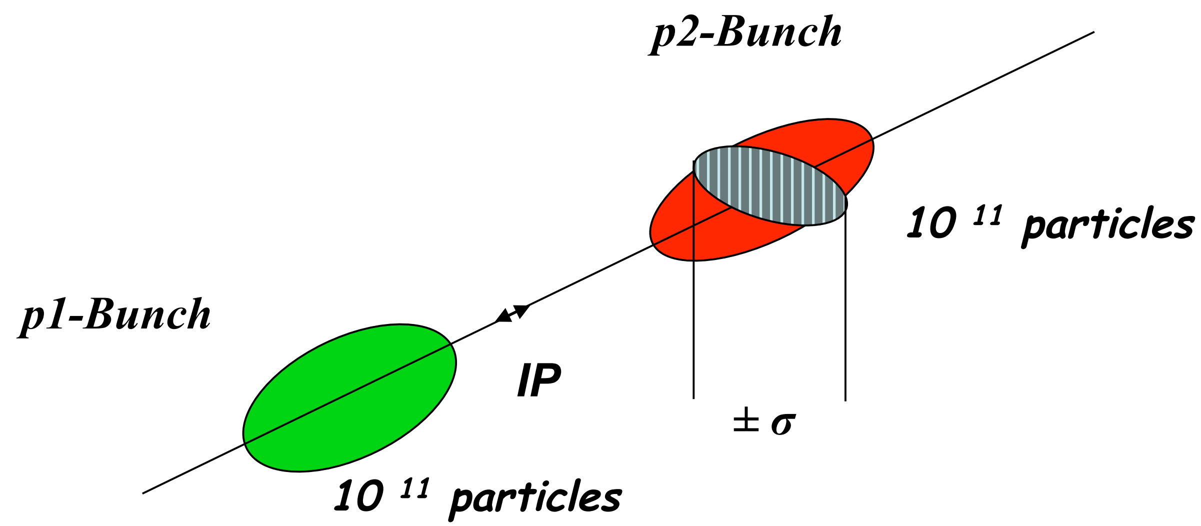


... clearly there is another problem !!!

Example: Luminosity optics at LHC: $\beta^* = 55\ cm$



21.) Luminosity



Example: Luminosity run at LHC

$$\beta_{x,y} = 0.55\ m \qquad f_0 = 11.245\ kHz$$

$$\varepsilon_{x,y} = 5 * 10^{-10}\ rad\ m \qquad n_b = 2808$$

$$\sigma_{x,y} = 17\ \mu m$$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

$$I_p = 584\ mA$$

$$L = 1.0 * 10^{34}\ {}^{1}\!/_{cm^2 s}$$

Mini-β Insertions: some guide lines

* calculate the *periodic solution in the arc*

* *introduce the drift space* needed for the insertion device (detector ...)

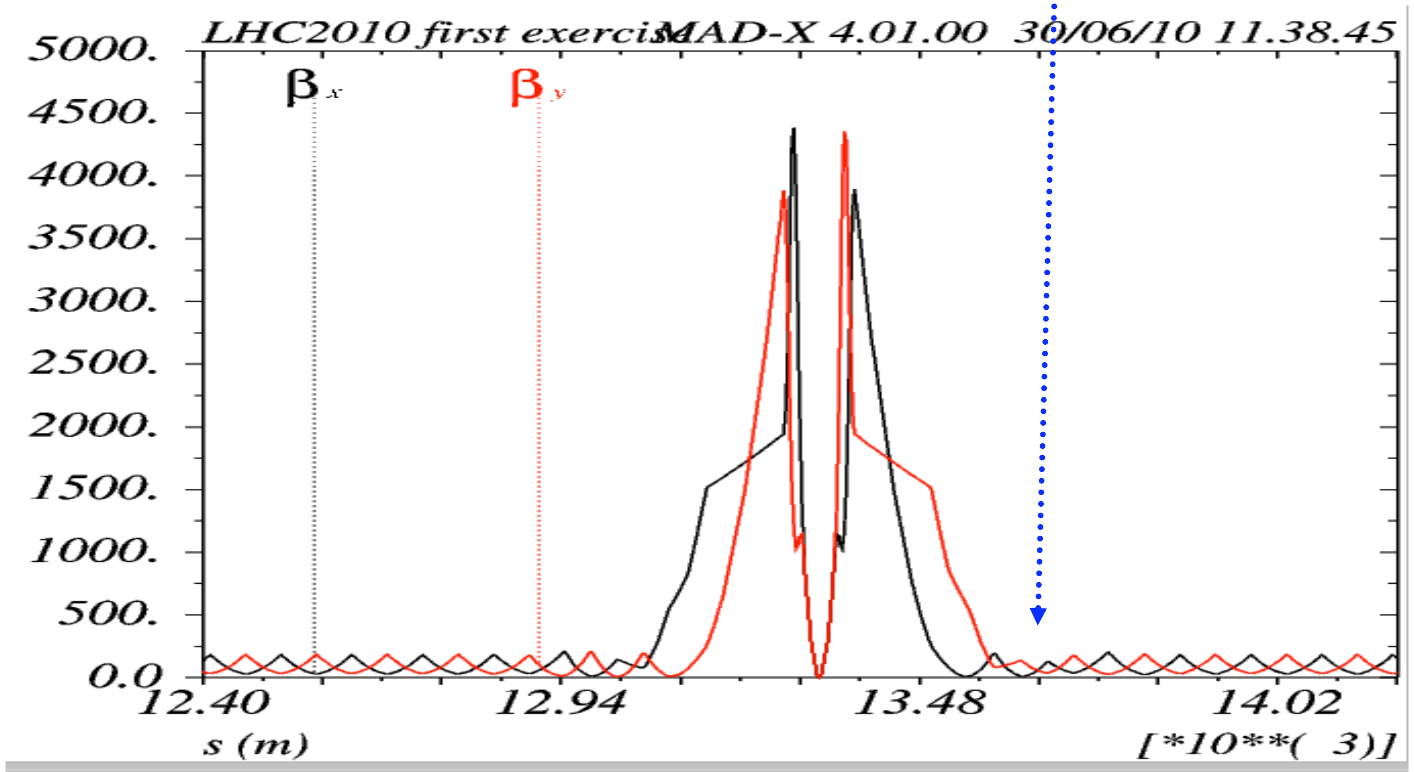
* put a *quadrupole doublet* (triplet ?) *as close as possible*

* introduce *additional quadrupole lenses* to match the beam parameters to the values at the beginning of the arc structure

parameters to be optimised & matched to the periodic solution:

$$\alpha_x,\ \beta_x \qquad D_x,\ D_x'$$
$$\alpha_y,\ \beta_y \qquad Q_x,\ Q_y$$

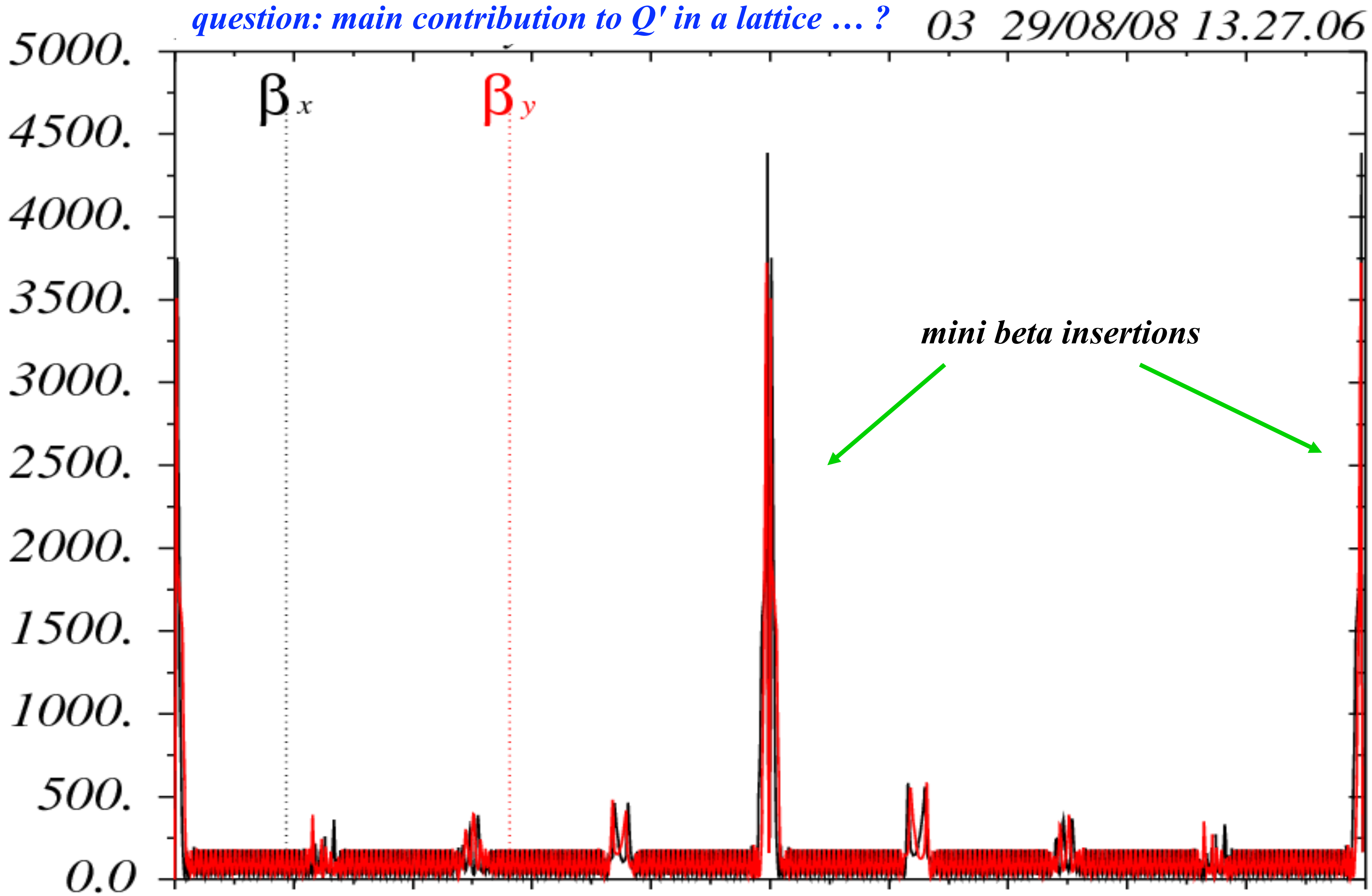
8 individually powered quad magnets are needed to match the insertion
(... at least)



… and now back to the Chromaticity

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

question: main contribution to Q' in a lattice … ?



Resumé:

beam rigidity: $B \cdot \rho = p/q$

bending strength of a dipole: $\frac{1}{\rho}\left[m^{-1}\right] = \frac{0.2998 \cdot B_0(T)}{p(GeV/c)}$

focusing strength of a quadrupole: $k\left[m^{-2}\right] = \frac{0.2998 \cdot g}{p(GeV/c)}$

focal length of a quadrupole: $f = \frac{1}{k \cdot l_q}$

equation of motion: $x'' + Kx = \frac{1}{\rho}\frac{\Delta p}{p}$

matrix of a foc. quadrupole: $x_{s2} = M \cdot x_{s1}$

$$M = \begin{pmatrix} \cos\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}l \\ -\sqrt{|K|}\sin\sqrt{|K|}l & \cos\sqrt{|K|}l \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

Resume´:

transfer matrix in Twiss form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

... and for the periodic case

$$\boldsymbol{M}(\boldsymbol{s}) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

beam emittance during acceleration

$$\varepsilon \propto \frac{1}{\beta\gamma}$$

dispersion

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Resume´:

quadrupole error: *tune shift*

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s)\; l_{quad}\;\overline{\beta}}{4\pi}$$

beta beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q} \int_{s1}^{s1+l} \beta(s_1)\Delta k \cos\bigl(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\bigr)ds$$

chromaticity

$$\Delta Q = Q'\,\frac{\Delta p}{p}$$

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

momentum compaction

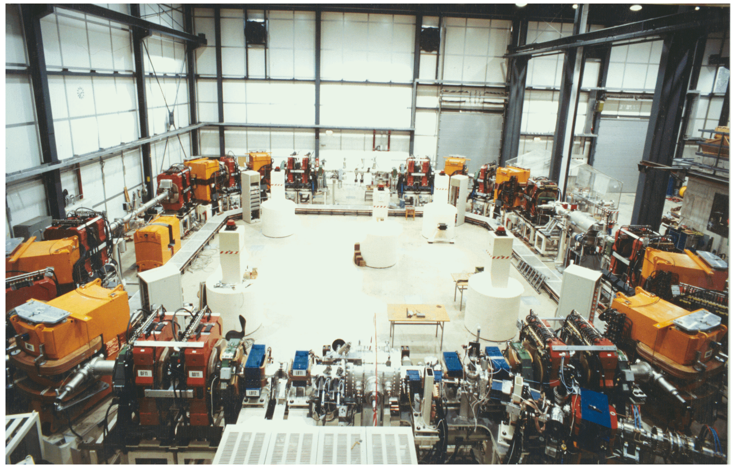
$$\frac{\delta\, l_{\varepsilon}}{L} = \alpha_p\,\frac{\Delta p}{p}$$

$$\alpha_p \approx \frac{2\pi}{L}\,\langle D\rangle \approx \frac{\langle D\rangle}{R}$$

beta function in a symmateric drift

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

APPENDIX



I.) Stability Criterion:

Question: ***what will happen, if we do not make too many mistakes and your particle performs one complete turn ?***

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{1}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\boldsymbol{J}}$$

Matrix for N turns:

$$M^N = \left(1 \cdot \cos\psi + J \cdot \sin\psi\right)^N = 1 \cdot \cos N\psi + J \cdot \sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

$$\psi = real \quad \leftrightarrow \quad \left|\cos\psi\right| \le 1 \quad \leftrightarrow \quad Tr(M) \le 2$$

stability criterion …. proof for the disbelieving collegues !!

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\boldsymbol{I}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\boldsymbol{J}}$$

Matrix for 2 turns:

$$\boldsymbol{M}^2 = (\boldsymbol{I}\cos\psi_1 + \boldsymbol{J}\sin\psi_1)(\boldsymbol{I}\cos\psi_2 + \boldsymbol{J}\sin\psi_2)$$

$$= \boldsymbol{I}^2 \cos\psi_1 \cos\psi_2 + \boldsymbol{IJ}\cos\psi_1 \sin\psi_2 + \boldsymbol{JI}\sin\psi_1 \cos\psi_2 + \boldsymbol{J}^2 \sin\psi_1 \sin\psi_2$$

now …

$$\boldsymbol{I}^2 = \boldsymbol{I}$$

$$\left.\begin{aligned} \boldsymbol{I}\,\boldsymbol{J} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \\ \boldsymbol{J}\,\boldsymbol{I} &= \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \end{aligned}\right\} \quad \boldsymbol{I}\,\boldsymbol{J} = \boldsymbol{J}\,\boldsymbol{I}$$

$$\boldsymbol{J}^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\boldsymbol{I}$$

$$\boldsymbol{M}^2 = \boldsymbol{I}\cos(\psi_1 + \psi_2) + \boldsymbol{J}\sin(\psi_1 + \psi_2)$$

$$\boldsymbol{M}^2 = \boldsymbol{I}\cos(2\psi) + \boldsymbol{J}\sin(2\psi)$$

II.) Dispersion: Solution of the inhomogenious equation of motion

Ansatz:

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\, d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\, d\tilde{s}$$

$$D'(s) = S' * \int \frac{1}{\rho} C\, dt + \cancel{S \frac{1}{\rho} C} - C' * \int \frac{1}{\rho} S\, dt - \cancel{C \frac{1}{\rho} S}$$

$$D'(s) = S' * \int \frac{C}{\rho} dt \; - \; C' * \int \frac{S}{\rho} dt$$

$$D''(s) = S'' * \int \frac{C}{\rho} d\tilde{s} + S' \frac{C}{\rho} \; - \; C'' * \int \frac{S}{\rho} d\tilde{s} \; - \; C' \frac{S}{\rho}$$

$$= S'' * \int \frac{C}{\rho} d\tilde{s} - \; C'' * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho} \underbrace{\left(CS' \quad - S\, C' \right)}_{= \det M = 1}$$

---

remember: for Cs) and S(s) to be independent solutions the Wronski determinant has to meet the condition

$$W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \neq 0$$

and as it is independent of the variable „s“

$$\frac{dW}{ds} = \frac{d}{ds}(CS' - SC') = CS'' - SC'' = -K(CS - SC) = 0$$

we get for the initital conditions that we had chosen ...

$$\left.\begin{array}{ll} C_0 = 1, & C_0' = 0 \\ S_0 = 0, & S_0' = 1 \end{array}\right\} \quad W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} = 1$$

$$D'' = S'' * \int \frac{C}{\rho} d\tilde{s} - C'' * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

remember: S & C are solutions of the homog. equation of motion:

$$S'' + K * S = 0$$
$$C'' + K * C = 0$$

$$D'' = -K * S * \int \frac{C}{\rho} d\tilde{s} + K * C * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

$$D'' = -K * \underbrace{\left\{ S \int \frac{C}{\rho} d\tilde{s} + C \int \frac{S}{\rho} d\tilde{s} \right\}}_{=D(s)} + \frac{1}{\rho}$$

$$D'' = -K * D + \frac{1}{\rho}$$

... or

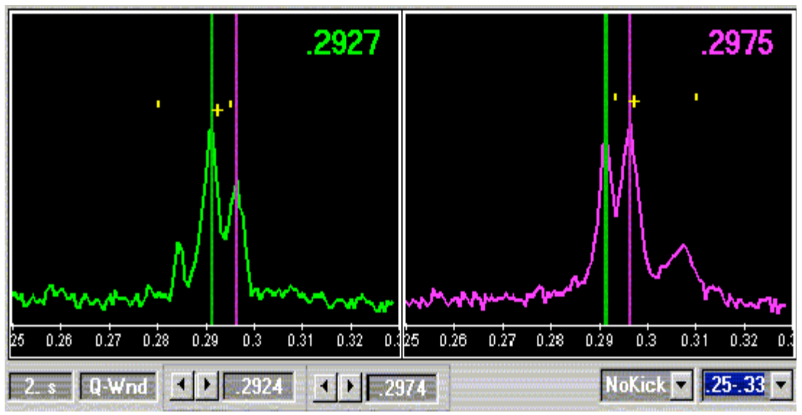
$$D'' + K * D = \frac{1}{\rho}$$

qed

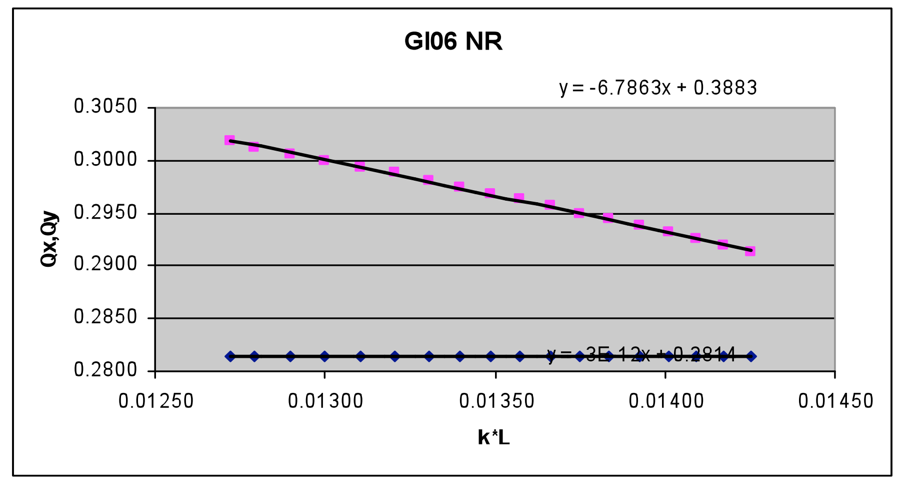
III.) Quadrupole Error and Beta Function

a change of quadrupole strength in a synchrotron leads to tune sift:

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s) * l_{quad} * \overline{\beta}}{4\pi}$$



tune spectrum …



tune shift as a function of a gradient change

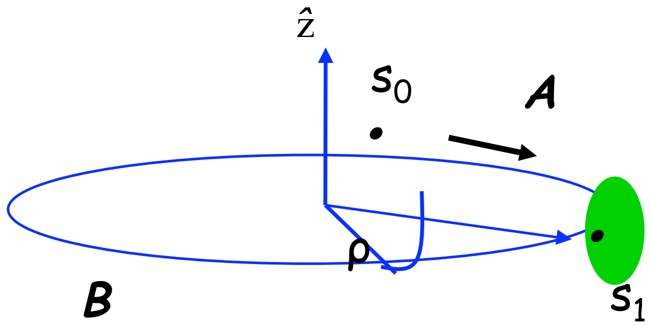
But we should expect an error in the β-function as well …

… shouldn´t we ???

Quadrupole Errors and Beta Function

a quadrupole error will not only influence the oscillation frequency … „tune"
… but also the amplitude … „beta function"

split the ring into 2 parts, described by two matrices A and B



$$M_{turn} = B * A$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

matrix of a quad error between A and B

$$M_{dist} = \begin{pmatrix} m_{11}^{*} & m_{12}^{*} \\ m_{21}^{*} & m_{22}^{*} \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ -\Delta k ds & 1 \end{pmatrix} A$$

$$M_{dist} = B \begin{pmatrix} a_{11} & a_{12} \\ -\Delta k ds a_{11} + a_{12} & -\Delta k ds a_{12} + a_{22} \end{pmatrix}$$

$$M_{dist} = \begin{pmatrix} \sim & b_{11} a_{12} + b_{12}(-\Delta k ds a_{12} + a_{22}) \\ \sim & \sim \end{pmatrix}$$

the beta function is usually obtained via the matrix element „m12", which is in Twiss form for the undistorted case

$$m_{12} = \beta_0 \sin 2\pi Q$$

and including the error:

$$m_{12}^* = \underbrace{b_{11} a_{12} + b_{12} a_{22}}_{m_{12} = \beta_0 \sin 2\pi Q} - b_{12} a_{12} \Delta k ds$$

$$(1) \quad m_{12}^* = \beta_0 \sin 2\pi Q - a_{12} b_{12} \Delta k ds$$

As M is still a matrix for one complete turn we still can express the element m_{12} in twiss form:*

$$(2) \quad m_{12}^* = (\beta_0 + d\beta) * \sin 2\pi (Q + dQ)$$

Equalising (1) and (2) and assuming a small error

$$\beta_0 \sin 2\pi Q - a_{12} b_{12} \Delta k ds = (\beta_0 + d\beta) * \sin 2\pi (Q + dQ)$$

$$\beta_0 \sin 2\pi Q - a_{12} b_{12} \Delta k ds = (\beta_0 + d\beta) * \sin 2\pi Q \underbrace{\cos 2\pi dQ}_{\approx 1} + \cos 2\pi Q \underbrace{\sin 2\pi dQ}_{\approx 2\pi dQ}$$

$$\cancel{\beta_0 \sin 2\pi Q} - a_{12} b_{12} \Delta k ds = \cancel{\beta_0 \sin 2\pi Q} + \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q + \cancel{d\beta_0 2\pi dQ} \cos 2\pi Q$$

ignoring second order terms

$$-a_{12} b_{12} \Delta k ds = \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

remember: tune shift dQ due to quadrupole error:
(index „1" refers to location of the error)

$$dQ = \frac{\Delta k \beta_1 ds}{4\pi}$$

$$-a_{12} b_{12} \Delta k ds = \frac{\beta_0 \Delta k \beta_1 ds}{2} \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

solve for dβ

$$d\beta_0 = \frac{-1}{2 \sin 2\pi Q} \left\{ 2 a_{12} b_{12} + \beta_0 \beta_1 \cos 2\pi Q \right\} \Delta k ds$$

express the matrix elements a_{12}, b_{12} in Twiss form

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} \left(\cos\psi_s + \alpha_0 \sin\psi_s \right) & \sqrt{\beta_s \beta_0} \sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta s}} \left(\cos\psi_s - \alpha_s \sin\psi_s \right) \end{pmatrix}$$

$$d\beta_0 = \frac{-1}{2\sin 2\pi Q}\left\{2a_{12}b_{12} + \beta_0\beta_1 \cos 2\pi Q\right\}\Delta k ds$$

$$\boldsymbol{a}_{12} = \sqrt{\beta_0\beta_1}\,\sin\Delta\psi_{0\to1}$$

$$\boldsymbol{b}_{12} = \sqrt{\beta_1\beta_0}\,\sin(2\pi \boldsymbol{Q} - \Delta\psi_{0\to1})$$

$$d\beta_0 = \frac{-\beta_0\beta_1}{2\sin 2\pi Q}\left\{\underbrace{2\sin\Delta\psi_{12}\sin(2\pi Q - \Delta\psi_{12}) + \cos 2\pi Q}\right\}\Delta k ds$$

... after some TLC transformations ... $= \cos(2\Delta\psi_{01} - 2\pi Q)$

$$\Delta\beta(s_0) = \frac{-\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta k\cos\big(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\big)ds$$

Nota bene: ! the beta beat is proportional to the strength of the error Δk

!! and to the β function at the place of the error ,

!!! and to the β function at the observation point,
(... remember orbit distortion !!!)

!!!! there is a resonance denominator

Bibliography:

1.) P. Bryant, K. Johnsen: *The Principles of Circular Accelerators and Storage Rings Cambridge Univ. Press*

2.) Klaus Wille: *Physics of Particle Accelerators and Synchrotron Radiation Facilicties, Teubner, Stuttgart 1992*

3.) Peter Schmüser: *Basic Course on Accelerator Optics, CERN Acc. School: 5th general acc. phys. course CERN 94-01*

4.) Bernhard Holzer: *Lattice Design, CERN Acc. School: Interm.Acc.phys course, http://cas.web.cern.ch/cas/ZEUTHEN/lectures-zeuthen.htm cern report: CERN-2006-002*

5.) A.Chao, M.Tigner: *Handbook of Accelerator Physics and Engineering, Singapore : World Scientific, 1999.*

6.) Martin Reiser: *Theory and Design of Chargged Particle Beams Wiley-VCH, 2008*

7.) Frank Hinterberger: *Physik der Teilchenbeschleuniger, Springer Verlag 1997*

8.) Mathew Sands: *The Physics of e+ e- Storage Rings, SLAC report 121, 1970*

9.) D. Edwards, M. Syphers : *An Introduction to the Physics of Particle Accelerators, SSC Lab 1990*