Mathematical and Numerical Methods for Non-linear Beam Dynamics in Rings

(an introduction)

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http://cern.ch/Werner.Herr/CAS2013/lectures/Trondheim_methods.pdf

For many more details:

http://cern.ch/Werner.Herr/METHODS

Primary purpose of this lectures

- **Assumption: familiar with linear, transverse dynamics**
- **Need to introduce new tools for non-linear dynamics**
- **Avoid mathematical derivations and proofs rather give** "*raison d'etre*" **and** "*mode d'emploi*"
- **Give an overview of the** *modern*[*)] **tools used in accelerator physics**
- **Necessarily brief and incomplete**

→ **An invitation to further studies ...**

[*)] *modern*: **"contemporary", not "fashionable" !**

Recommended Bibliography:

[EF1] E. Forest, *Beam Dynamics - A New Attitude and Framework*, Harwood Academic Publishers, 1998.

[AC1] A. Chao, *Lecture Notes on Topics in Accelerator Physics* SLAC, 2001.

[AD] A. Dragt, *Lie Methods for Non-linear Dynamics with Applications to Accelerator Physics*

[AC2] A. Chao and M. Tigner, *Handbook of Accelerator Physics and Engineering*, World Scientific Publishing, 1998.

[MB] M. Berz, *Modern Map Methods in Particle Beam Physics*, Academic Press, 1999.

[AW] A. Wolski, *Lecture Notes on non-linear single particle dynamics*, University of Liverpool.

[HF] W. Herr and E. Forest, *Non-linear Dynamics in Accelerators*, Landolt-Börnstein, Vol.21C, "Accelerators and Colliders", edited by .S. Myers, H. Schopper.

Why Beam Dynamics in Rings ?

- Most lectures deal with rings
- Rings are periodic systems
- Implies stability (at least for some time) and confinement

➞ This restricts the methods and tools applicable to study of beam dynamics

➞ Applicable to other machine and beam lines !

Outline of this lectures

- **Motivation, introduction and classical concepts**
- **New concepts and modern techniques**
  - **Maps**
  - **Computation: maps, symplectic integration**
  - **Hamiltonian theory (for our purpose)**
  - **Analysis: Lie transforms, normal forms**
  - **Analysis: Differential algebra**
- **Identify possible traps and pitfalls ...**

Treatment of LINEAR dynamics in rings

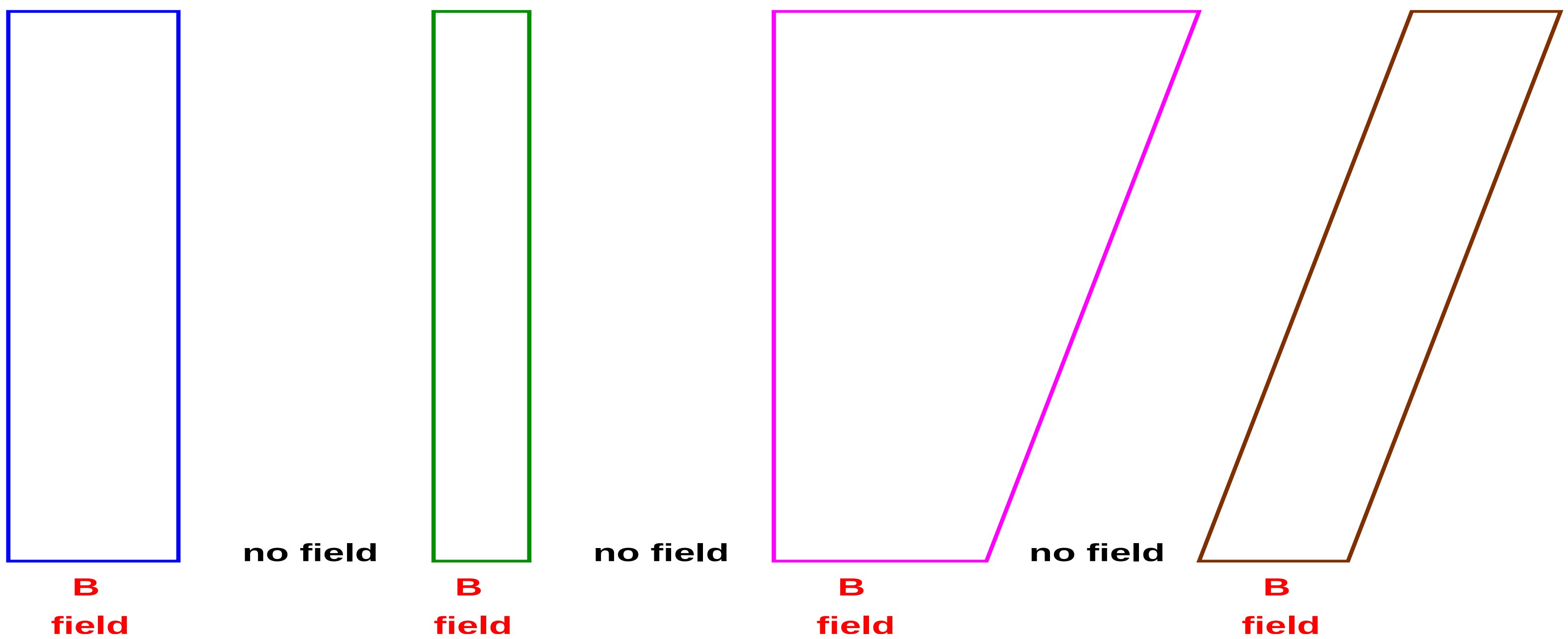
- **Standard introduction using Hill's equation**

  (for simplicity: show for one dimension first):

$$\frac{d^2x(s)}{ds^2} + K(s)x(s) = 0$$

- $K(s)$ **periodic, smooth function**
- **Is that true ?**
- **No, normally not**

Arrangement of beam line elements



- Cannot be described by Hill's equation
- Not smooth, not periodic

Treatment of LINEAR dynamics in rings

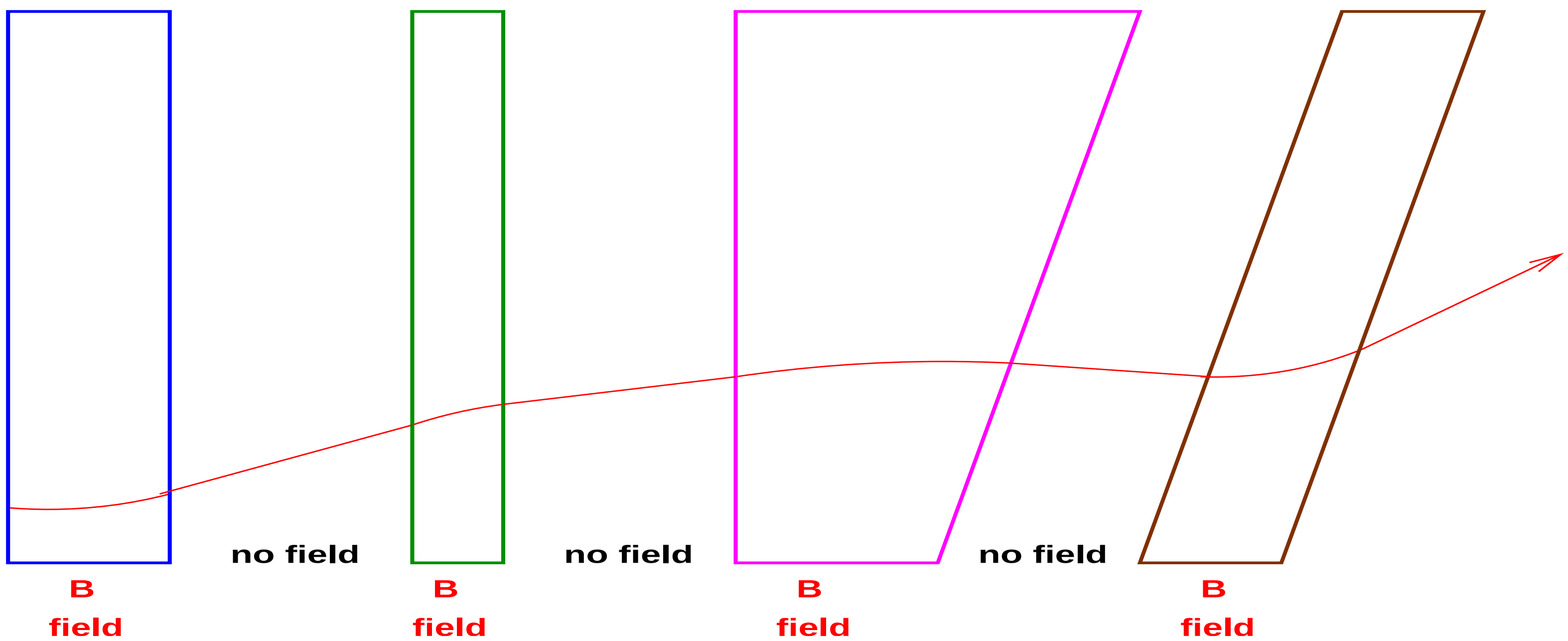
- **Used to "derive" Courant-Snyder ansatz:**

$$x(s) = \sqrt{\beta(s) \cdot \epsilon} \; \cdot \; cos(\mu(s) + \mu_0)$$

$$x'(s) = \sqrt{\frac{\epsilon}{\beta(s)}} \; \cdot \; (sin(\mu(s) + \mu_0) + \alpha \cdot cos(\mu(s) + \mu_0))$$

- **Is the solution to any system that is: confined and periodic !**
- **Do particles really move like this ?**

Trajectories in beam line elements



- Not a solution of the above .....
- What if we put additional elements (distortions ?)

Treatment of DISTORTED dynamics

- **Hill's equation with distortions, we have to re-write (similar for the other plane):**

$$\frac{d^2x(s)}{ds^2} + K(s)x(s) = -\frac{B_y(x,y,s)}{p}$$

- **or in general as (any order) multipoles:**

$$\frac{d^2x(s)}{ds^2} + K(s)x(s) = \sum_{i,j,k,l\geq 0} p_{ijkl}(s)x^i x'^j y^k y'^l$$

- **Very non-linear differential equation to solve ...**

→ **Enter the field of non-linear dynamics**

Can we deal with that ?

- **Under certain circumstances (see lecture by Oliver Brüning):**
  - **All $p_{ijkl}(s)$ are perturbations, i.e. (very) small**
  - **Only a few $p_{ijkl}(s)$ are non-zero**
  - **You can avoid resonances**
  - **Perturbations are smooth or possibly periodic**
  - **Perturbation treatment to leading order is sufficient**
- **Would you build a 3 billion Euro machine on these assumptions and approximations ?**

What is normally not said ....

- **Hill's equation, β-function, ...etc.:**
  - **All concepts developped for synchrotrons !**
    (Courant and Snyder, 1957)
- **Strictly speaking, not applicable to:**
  - **Beam lines, LINACs, cyclotrons, ...**
- **Computer programs do not use Hill's equation**
- **Can we find a better framework ?**

A disclaimer ...

- **Traditional treatment requires many approximations**
- **Useful to understand and demonstrate concepts**
- **See Oliver Brüning's, Bernhard Holzer's lectures**
- **For practical work on realistic machine:**
  - **New tools required**
  - **Should exploit modern computing techniques to the maximum**
  - **It is much easier that you think (.. and other people tell you !)**

A better framework

Start with the differential equation:

$$\frac{d^2x(s)}{ds^2} + K(s)x(s) = \sum_{i,j,k,l\geq 0} p_{ijkl}(s)x^i x'^j y^k y'^l$$

- **Bad news:**

  Description not very realistic (see above)

  We have no global analytical solution

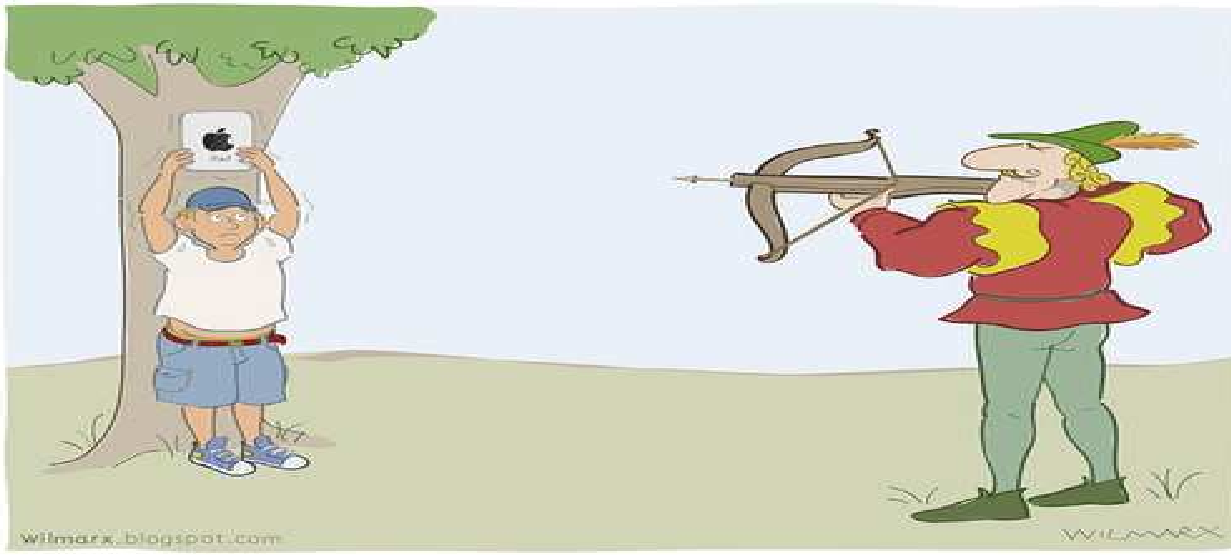
- **Good news: An analytical solution is not needed !**

A better framework

- **Why not ?**
- **We do not want to know:**
  - **The particle's position and momentum at 2h 45min 22.3s ?**

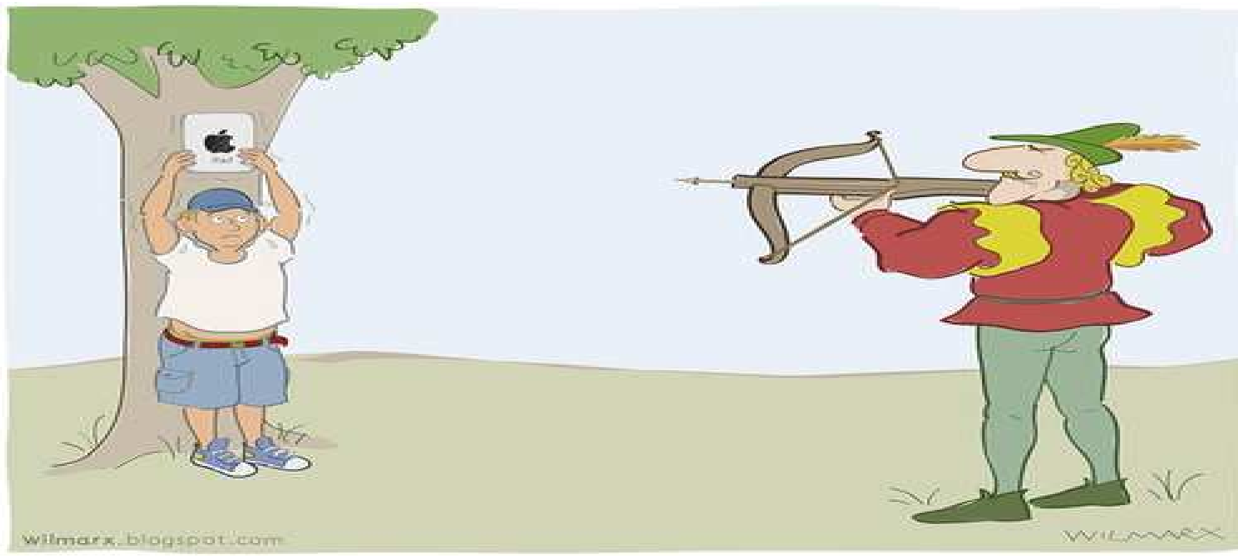
    (Remember Thermodynamics !)
- **We do want to know:**
  - **Is the beam stable for a long time ?**
  - **Is the motion confined ?**
  - **Does the beam hit the target ?**

An every day example ...



- **Not important to know trajectory as function of time**
- **Very important to know trajectory at end of flight**
- **Can we get a framework to get that (easily) ?**

An every day example ...

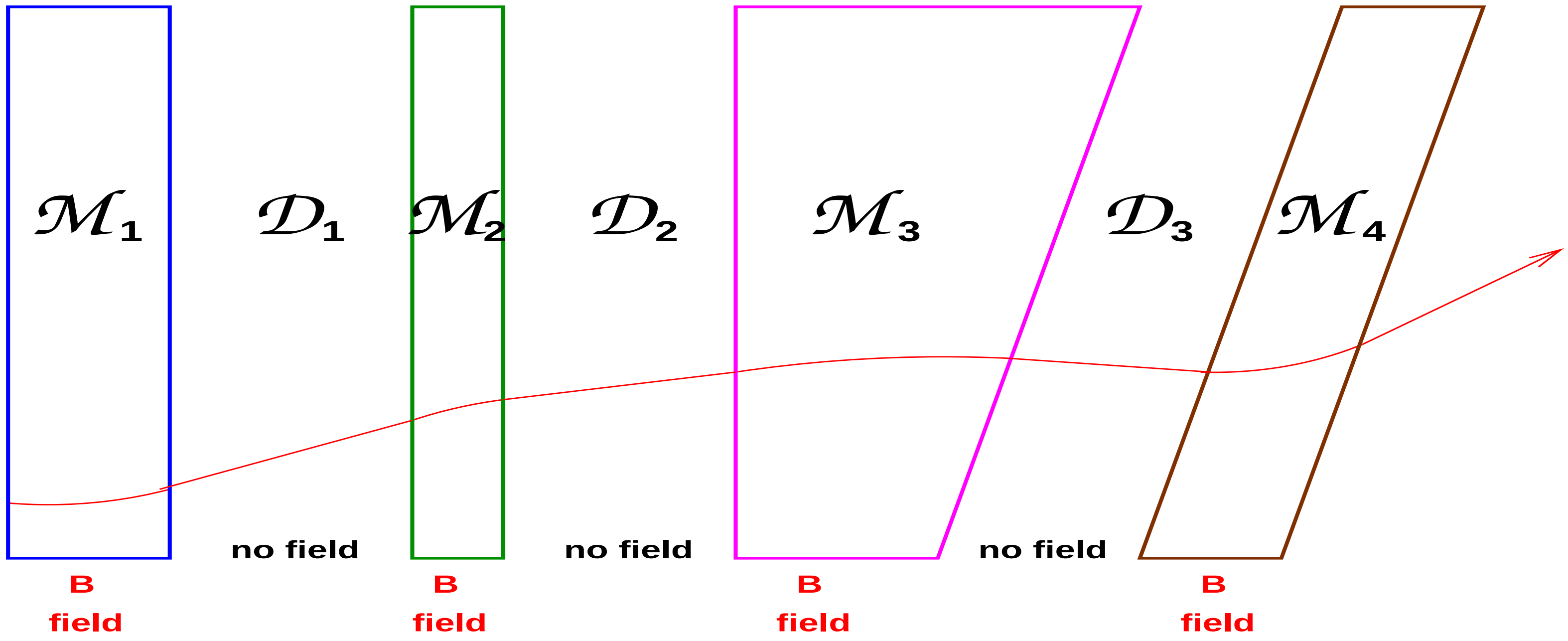


- Not important to know trajectory as function of time
- Very important to know trajectory at end of flight
- Can we get a framework to get that (easily) ?
- Yes we can ! Should not go back 50 years !

A better framework - go back 100 years ...

- "Old" to "New" classical dynamics:
  - Topology and properties of phase space (see Oliver's lecture)
  - Chaotic motion, non-integrable systems
  - Sensitivity to initial conditions

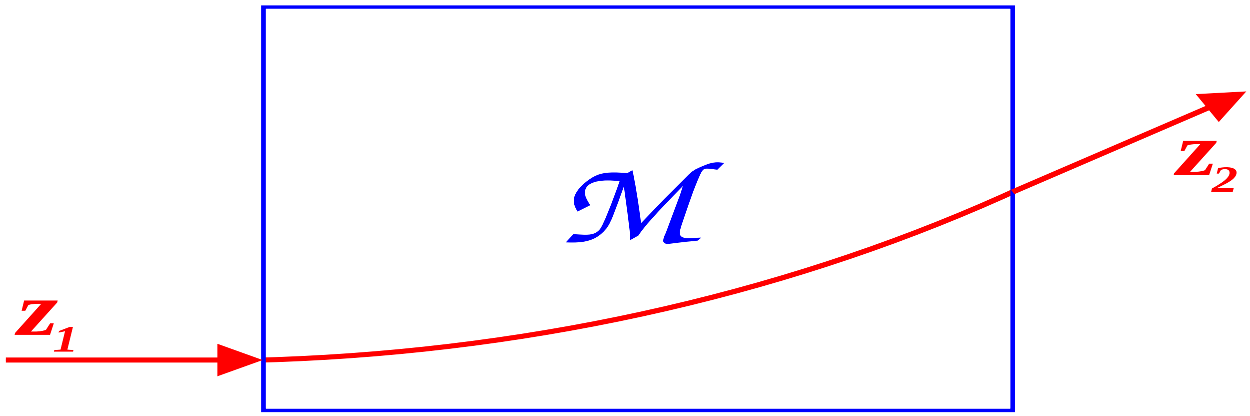
How is an beam line described ?



- **Beam line (or ring) made of machine elements and drifts**
- **Described by maps for magnets ($\mathcal{M}$) and drifts ($\mathcal{D}$)**

How can an element really be described ?

- You need to describe what happens to the particle in M and in the drifts D



- In general: $\vec{z}_2 \neq \vec{z}_1$

How is an element described ?

- **Let $\vec{z_1}, \vec{z_2}$ describe a quantity (coordinates, beam sizes ...) before and after the element**
- **Take an machine element (e.g. magnet) and build a mathematical model $\mathcal{M}$**
  - **In general:** $\vec{z_2} \; = \; \mathcal{M} \; (\vec{z_1})$
  - $\mathcal{M}$ **is a so-called map**
  - **Very important: no need to know what happens in the rest of the machine !!**
- **The complete sequence of MAPS connects the pieces together to make a ring (or beam line)**

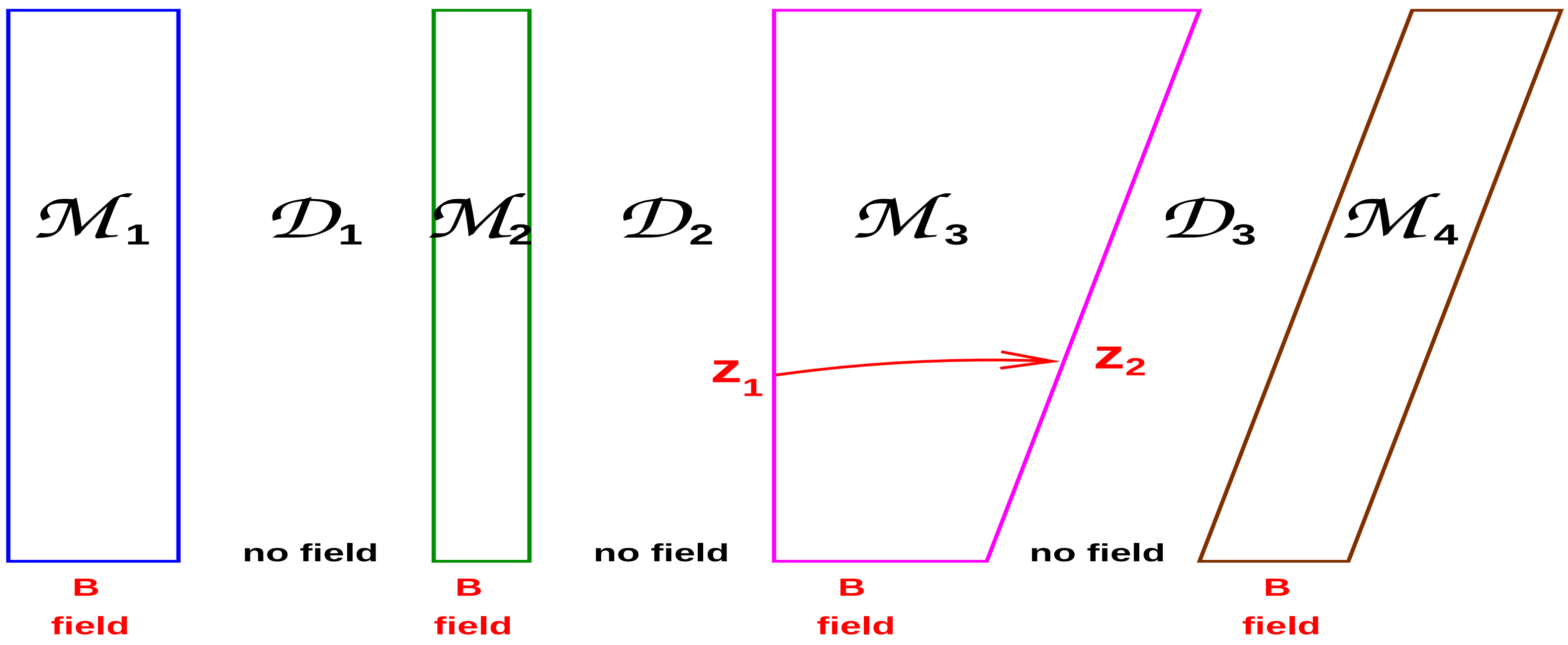
MAPS transform coordinates through an element

- **Use coordinate vector:** $\vec{z} \; = \; (x, \quad x' = \frac{\partial x}{\partial s}, \quad y, \quad y' = \frac{\partial y}{\partial s})$ $^{*)}$
- $\mathcal{M}_3$ **transforms the coordinates** $\vec{z_1}(s_1)$ **through the magnet** M_3 **at position** s_1 **to new coordinates** $\vec{z_2}(s_2)$ **at position** s_2**:**

$$\vec{z_2}(s_2) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \mathcal{M}_3 \circ \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} = \mathcal{M}_3 \circ \vec{z_1}(s_1)$$

$^{*)}$ **not unique, see later**

MAPS transform coordinates through an element



- **The MAP fully describes what happens inside the magnet**

What can $\mathcal{M}$ be ?

- **Any "description" to go from $\vec{z}_1$ to $\vec{z}_2$**
- **This "description" can be:**
  - **A simple linear matrix or transformation**
  - **A non-linear transformation (Taylor series, Lie Transform ...)**
  - **High order integration algorithm**
  - **A computer program, subroutine etc.**
- **Let us look at linear theory first !**

  Then generalize to non-linear theory

Simple examples (one dimensional)

First a drift space of length L

Two possible descriptions are (there are more):

- **1. Go straight from s_1 to s_2!!**
- **2. More formal:**

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

Simple examples (one dimensional)

Focusing quadrupole of length L and strength k:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} \cos(L \cdot k) & \frac{1}{k} \cdot \sin(L \cdot k) \\ -k \cdot \sin(L \cdot k) & \cos(L \cdot k) \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

Quadrupole with short length L (i.e.: $1 \gg L \cdot k^2$)

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} 1 & 0 \\ -k^2 \cdot L(= \frac{1}{f}) & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

They are $\mathcal{M}$aps, describe the movement in an element (quadrupole)

Interlude: there was already a trap ... !

According to B. Holzer (lectures) or K. Wille (textbook):

$$k = \frac{1}{B\rho}\frac{dB_y}{dx}$$

According to "Handbook for Accelerator Physics" ([AC2]):

$$k^2 = \frac{1}{B\rho}\frac{dB_y}{dx}$$

→ **The lesson: check what people use !!**

(remember Air Canada 143)

Interlude: it can be worse ... !

You also find (and it may even be useful ...):

$$K^2 \;=\; k \;=\; \frac{1}{B\rho}\frac{dB_y}{dx}$$

Often different conventions in simulation programs !

Some programs want fields, not gradients !
Found this construction:

$$B_y \;=\; \frac{1}{0.1} \cdot k \cdot x \cdot B \cdot \rho$$

Interlude: what about 3D ... ?

- **Formally extended by adding more variables:**
  - **(x, x', y, y',** Δs**,** $\frac{\Delta p}{p}$**)**
  - $\Delta s = c\Delta t$**: longitudinal displacement with respect to reference particle**
  - $\frac{\Delta p}{p}$**: relative momentum difference with respect to reference particle**

⚠ **Not all programs use this, but rather canonical variables**

- **(x,** p_x/p_s**, y,** p_y/p_s**,** $-c\Delta t$**,** $p_t = \frac{\Delta E}{p_s c}$**)**

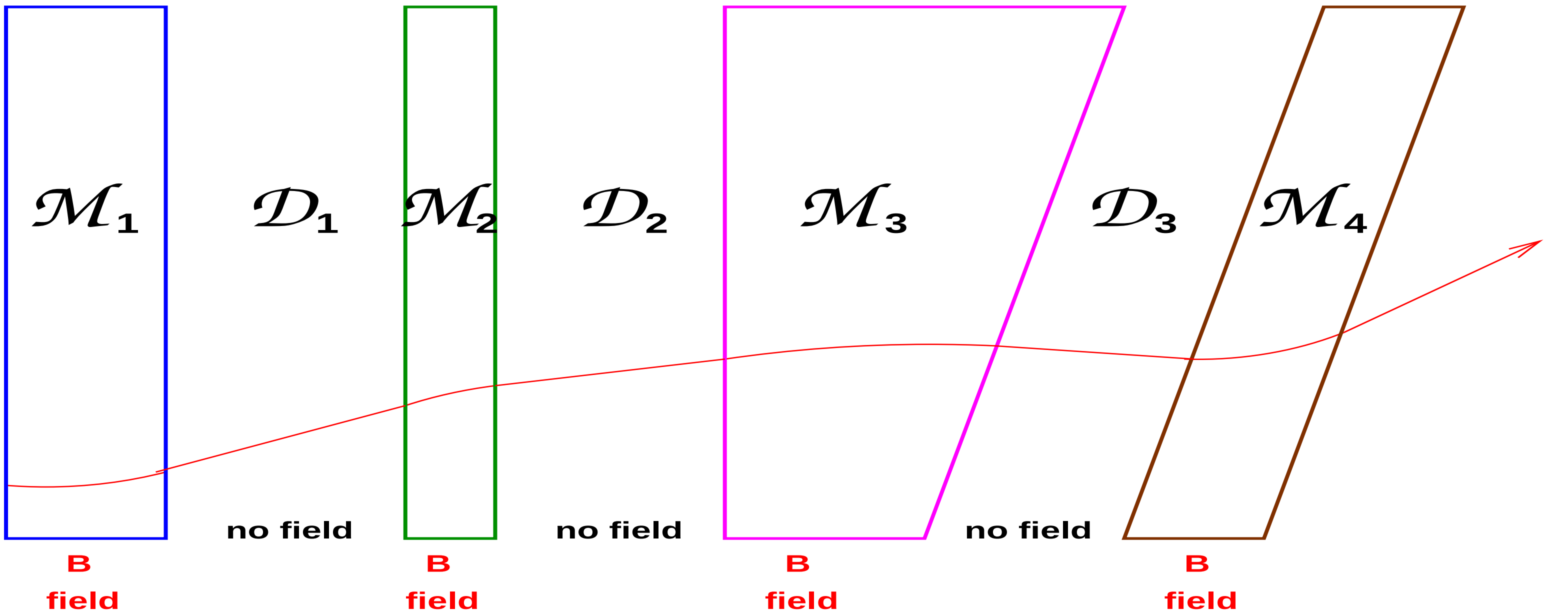
⚠⚠ p_s **may be:** $p_s = p_0$ **or** $p_s = p_0(1 + \delta_s) = m\beta_s\gamma_s$

δ_s**: difference between reference momentum and design momentum**

Putting the "pieces" together

- **We have to deal with many elements in our machines**
- **To make a ring or beam line:**
  - **Combine all elements maps together**
  - **Concatenated maps are a map again**
  - **Represents a bigger part of the machine (or the whole machine ...)**

How is an beam line described ?



$$\mathcal{M}_{\text{all}} = \mathcal{M}_4 \circ \mathcal{D}_3 \circ \mathcal{M}_3 \circ \mathcal{D}_2 \circ \mathcal{M}_2 \circ \mathcal{D}_1 \circ \mathcal{M}_1$$

- Beam line (or ring) is combination of all elements

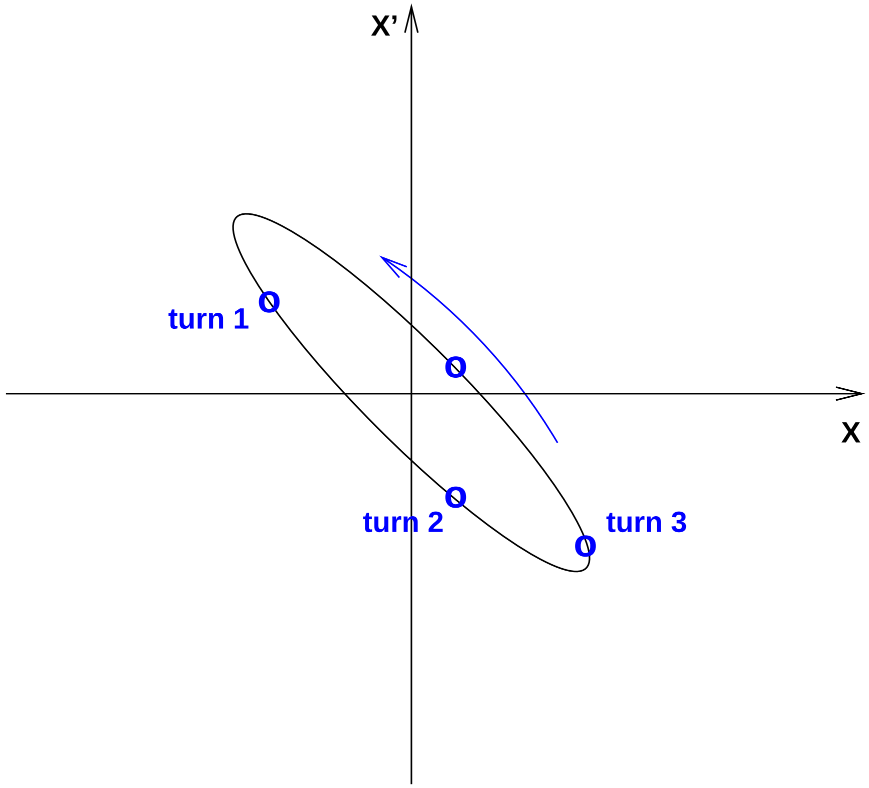
Putting the "pieces" together

Starting from a position s_0 and applying all maps (for N elements) in sequence around a ring with circumference C to get the One-Turn-Map (OTM) for the position s_0 (for one dimension only):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 + C} = \mathcal{M}_1 \circ \mathcal{M}_2 \circ \ldots \circ \mathcal{M}_N \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\Longrightarrow \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 + C} = \mathcal{M}_{ring}(s_0) \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

What does $\mathcal{M}_{ring}$ do ?



- **Transforms coordinates in phase space once per turn**

Analysis of the One-Turn-Map

- **We have obtained a map for the whole ring**
- **In simplest (linear) case: multiply matrices to get a One-Turn-Matrix**
- **Have to get now the information we want:**
  - **Optics parameters (Tune, Twiss functions, ..)**
  - **Closed orbit**
  - **Stability**
  - **etc. ...**
- **How to analyse a MAP (first: a matrix) ???**

  (see also B. Holzer lecture, but practice comes here)

Normal forms

- **Maps can be transformed into (Jordan) Normal Forms**
- **Original maps and normal form are equivalent, but ...**
- **Easily used to analyse the maps:**
  - **Get parameters (Q, Q', Twiss function, ..)**
  - **Study invariants, etc.**
  - **Stability**
  - **For resonance analysis**
  - **etc. ...**
- **Idea is to make a transformation to get a simpler form for the map**

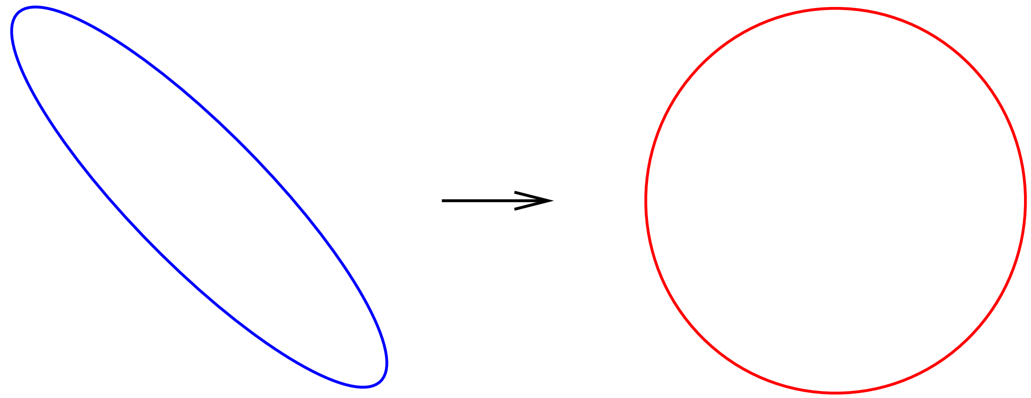
Normal forms

Assume the map $\mathcal{M}_{12}$ propagates the variables from location 1 to location 2, we try to find transformations $\mathcal{A}_1, \mathcal{A}_2$ such that:

$$\mathcal{A}_1 \mathcal{M}_{12} \mathcal{A}_2^{-1} \;=\; \mathcal{R}_{12}$$

- **The map $\mathcal{R}_{12}$ is:**
  - **A "Jordan Normal Form", (or at least a very simplified form of the map)**
  - **Example: $\mathcal{R}_{12}$ becomes a pure rotation**
- **The map $\mathcal{R}_{12}$ describes the same dynamics as $\mathcal{M}_{12}$, but:**
  - **All coordinates are transformed**
  - **The transformations $\mathcal{A}_1, \mathcal{A}_2$ "analyse" the motion**

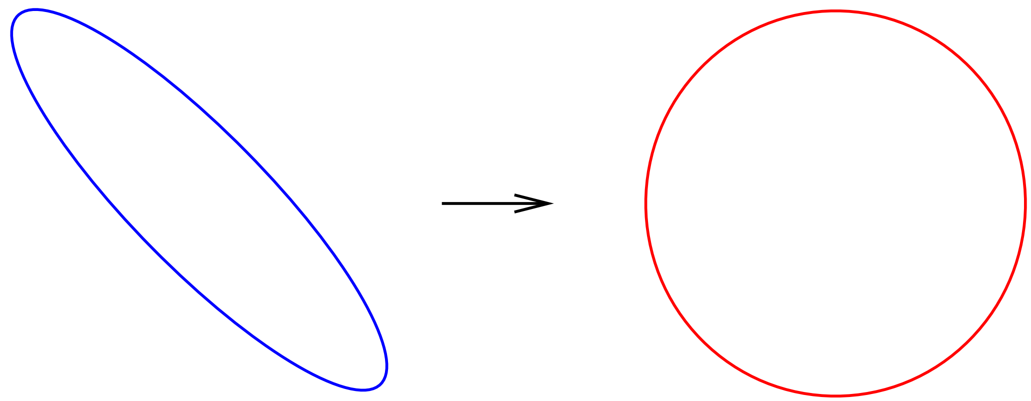
Normal forms - linear case



$$\mathcal{M} \rightarrow \mathcal{R}$$

- **Pictorial form of the transformation**
- **Motion on a complicated ellipse becomes motion on a circle (i.e. a pure rotation)**

Normal forms - linear case



$$M = \mathcal{A} \circ \mathcal{R}(\Delta\mu) \circ \mathcal{A}^{-1} \qquad \text{or :} \qquad \mathcal{R}(\Delta\mu) = \mathcal{A}^{-1} \circ M \circ \mathcal{A}$$

Normal forms - linear case (1D)

Assume the one-turn-map (here a matrix) $\mathcal{M}(s)$ at the position s is (e.g. lecture on transverse dynamics):

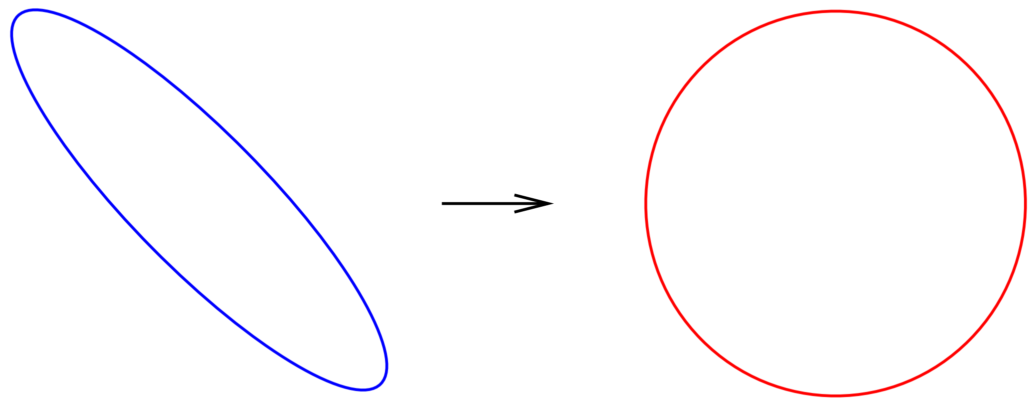
$$\mathcal{M}(s) = \begin{pmatrix} cos(\Delta\mu) + \alpha(s)sin(\Delta\mu) & \beta(s)sin(\Delta\mu) \\ -\gamma(s)sin(\Delta\mu) & cos(\Delta\mu) - \alpha(s)sin(\Delta\mu) \end{pmatrix}$$

- **Describes the motion on a phase space ellipse**
- **Re-write $\mathcal{M}$ such that one part $\mathcal{R}$ becomes a pure rotation (a circle), i.e.:**

$$\mathcal{A}\mathcal{R}\mathcal{A}^{-1} \ = \ \mathcal{M}$$

- **How ? Remember lectures on Linear Algebra (Eigenvectors, Eigenvalues ...)**

Normal forms - linear case (1D)



$$M = \mathcal{A} \circ \mathcal{R}(\Delta\mu) \circ \mathcal{A}^{-1} \qquad \text{or :} \qquad \mathcal{R}(\Delta\mu) = \mathcal{A}^{-1} \circ M \circ \mathcal{A}$$

with

$$\mathcal{A} = \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta(s)}} \end{pmatrix} \quad \text{and} \quad \mathcal{R} = \begin{pmatrix} cos(\Delta\mu) & sin(\Delta\mu) \\ -sin(\Delta\mu) & cos(\Delta\mu) \end{pmatrix}$$

Normal forms - linear case (1D)

We had:

$$M = \mathcal{A} \circ \mathcal{R}(\Delta\mu) \circ \mathcal{A}^{-1} \qquad \text{or :} \qquad \mathcal{R}(\Delta\mu) = \mathcal{A}^{-1} \circ M \circ \mathcal{A}$$

with

$$\mathcal{A} = \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta(s)}} \end{pmatrix} \quad \text{and} \quad \mathcal{R} = \begin{pmatrix} cos(\Delta\mu) & sin(\Delta\mu) \\ -sin(\Delta\mu) & cos(\Delta\mu) \end{pmatrix}$$

- **This is just the Courant-Snyder transformation to get $\beta, \alpha, \ldots$ etc., $\Delta\mu$ is the tune !**
- **That is: the Courant-Snyder analysis is just a normal form transform of the linear one turn matrix**
- **Works in more than one dimension**

Normalized variables:

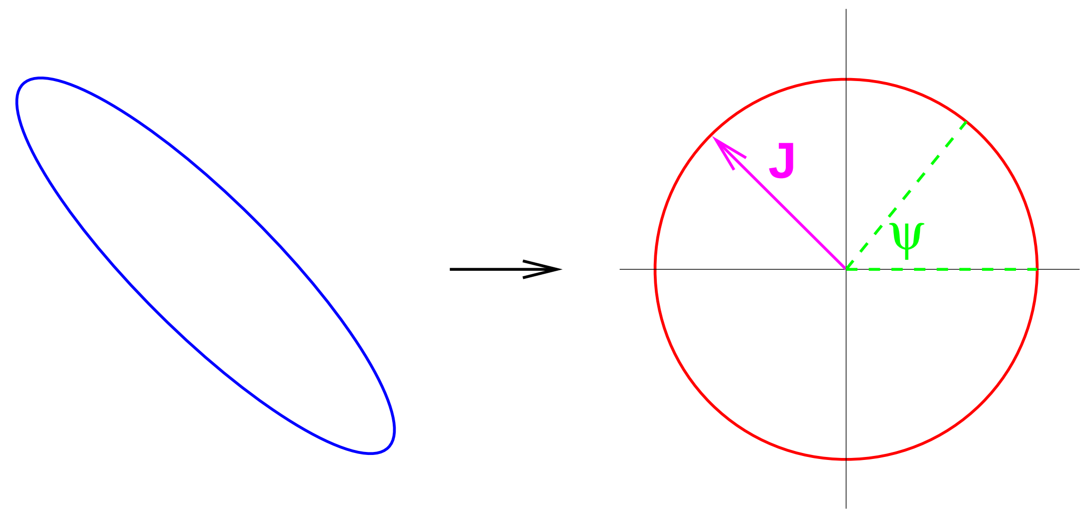
Please note that:

$$\begin{pmatrix} x_n \\ x'_n \end{pmatrix} = \mathcal{A}^{-1} \quad \circ \quad \begin{pmatrix} x \\ x' \end{pmatrix}$$

is just a variable transformation to new, normalized variables.

- **Tune** $(\Delta\mu)$ **in the** **<u>normalized map</u>,** **stability for real values of phase advance** $(\Delta\mu)$
- **Optical functions** $(\beta, \alpha, ...)$ **in the** **<u>normalizing map</u>**
- **No need to make any assumptions, ansatz, approximation, ...**

Interlude: action - angle variables



- **Once the particles "travel" on a circle:**
  - **Radius (say: $\sqrt{2J}$, with $J = \frac{x_n^2 + x_n'^2}{2}$) is constant (invariant of motion): action J**
  - **Phase advances by constant amount: angle Ψ**

Another example: coupling (2D)

Assume a one-turn-matrix in 2D:

$$T = \begin{pmatrix} M & n \\ m & N \end{pmatrix}$$

M,m,N,n are 2-by-2 matrices. In case of coupling: $m \neq 0, n \neq 0$**.**
we can try to re-write as:

$$T = \begin{pmatrix} M & n \\ m & N \end{pmatrix} = V\,R\,V^{-1}$$

with:

$$R = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \gamma I & C \\ -C^t & \gamma I \end{pmatrix}$$

What have we obtained ?

- **The matrix R is our simple rotation:**
  - A **and** B **are the one-turn-matrices for the ”normal modes”**
  - **The matrix** C **contains the ”coupling coefficients”**
  - **The matrix** V **transforms from the coordinates** (x, x', y, y') **into the ”normal mode” coordinates** (w, w', v, v') **via the expression:**

    $(x, x', y, y') = V(w, w', v, v')$
- **The last 2 slides: normally 1 hour lecture**

Normal forms - linear case

This is extremely useful when map is applied k times (e.g. k turns):

$$M^k(x, x') = AR^k\ A^{-1}(x, x') = AR^k(X, X')$$

- **For multi-turns: study effect of map in normalized coordinates**
- **Multiplying a matrix k (e.g. 4x4) can be quite a job !**
- **Easier to apply k times using the simple map (e.g. a rotation of μ becomes just a rotation $k \cdot \mu$)**
- **The A just transforms back to physical coordinates at the end (once !)**

The general philosophy (linear systems):

- **Describe your elements by a linear map**
- **Combine all maps into a ring or beam line to get the linear one turn matrix**
- **Normal form analysis of the linear one turn matrix will give all the information**

No need for any assumptions !

No need for any approximations !

Works in more than 1D and with coupling !

The general philosophy (non-linear systems):

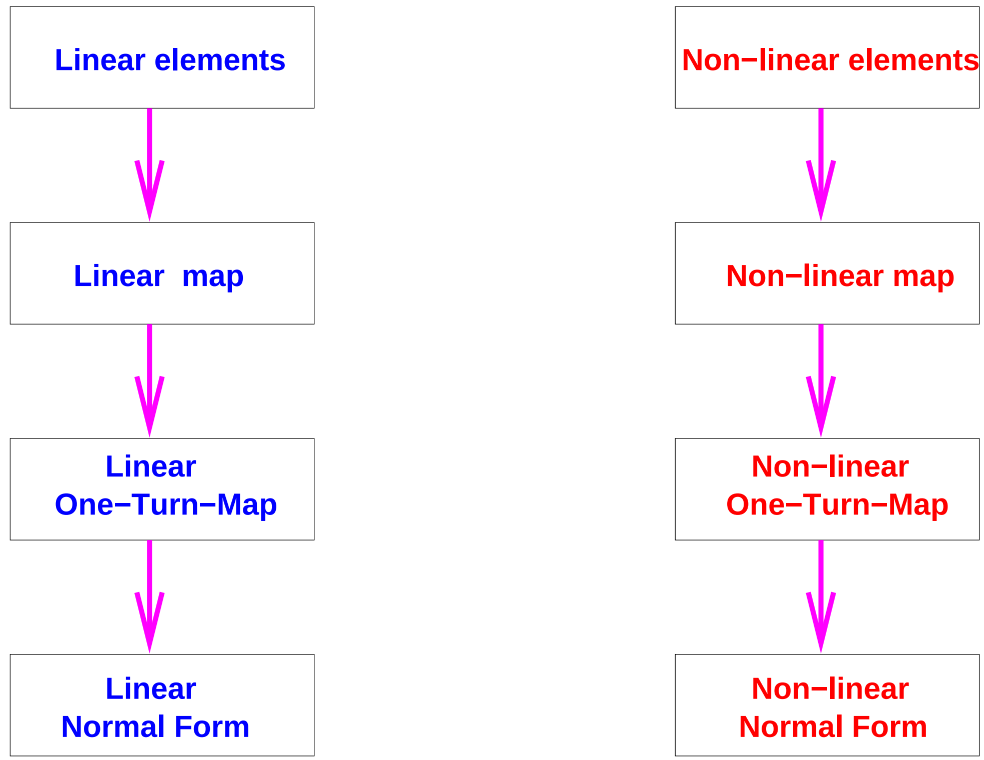
- **Describe your elements by a non-linear map**
- **Combine all maps into a ring or beam line to get the non-linear one turn map**
- **Normal form analysis of the non-linear one turn map will give all the information**

No need for any assumptions !

No need for any approximations !

Works in more than 1D and non-linearities !

The general philosophy



General formalism for all cases !

A small complication ...

Non-linear maps are not matrices !

Various types of non-linear MAPS

- Choice depends on the application
  - Taylor maps
  - Symplectic integration techniques
  - Lie transformations
  - Truncated power series algebra (TPSA), can also generate Taylor map from tracking
  - ...

(A key concept: Symplecticity)

- **Not all possible maps are allowed !**
- **Requires for a matrix** $\mathcal{M} \rightarrow \mathcal{M}^T \cdot S \cdot \mathcal{M} = \mathcal{S}$

  with:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

- **It basically means:** $\mathcal{M}$ **is area preserving and**

$$\lim_{n \to \infty} \mathcal{M}^n = \text{finite} \qquad \Longrightarrow \qquad \det\, \mathcal{M} = 1$$

Introducing non-linear elements

Effect of a (short) quadrupole depends linearly on amplitude (re-written from the matrix form):

$$\vec{z}(s_2) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} + \begin{pmatrix} 0 \\ k_1 \cdot x_{s_1} \\ 0 \\ k_1 \cdot y_{s_1} \end{pmatrix}$$

➡ $\vec{z}(s_2) \;=\; \mathbf{M} \;\cdot\; \vec{z}(s_1)$

➡ **M is a matrix**

Non-linear elements (e.g. sextupole)

Effect of a (thin) sextupole with strength k_2 is:

$$\vec{z}(s_2) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} + \begin{pmatrix} 0 \\ \frac{1}{2}k_2 \cdot (x_{s_1}^2 - y_{s_1}^2) \\ 0 \\ k_2 \cdot (x_{s_1} \cdot y_{s_1}) \end{pmatrix}$$

➡ $\vec{z}(s_2) = \mathcal{M} \circ \vec{z}(s_1)$

➡ $\mathcal{M}$ **is not a matrix, i.e. cannot be expressed by matrix multiplication**

Non-linear elements

Cannot be written in linear matrix form !

We need something like:

$$
\begin{aligned}
z_1(s_2) = x(s_2) \quad = \quad & R_{11}\cdot x \quad + R_{12}\cdot x' \quad + R_{13}\cdot y + ... \\
& + T_{111}\cdot x^2 + T_{112}\cdot xx' + T_{122}\cdot x'^2 + \\
& + T_{113}\cdot xy + T_{114}\cdot xy' + ... \\
& + U_{1111}\cdot x^3 + U_{1112}\cdot x^2x' + ...
\end{aligned}
$$

and the equivalent for all other variables ...

Higher order (Taylor -) MAPS:

We have (for: j = 1...4):

$$z_j(s_2) \;=\; \sum_{k=1}^{4} R_{jk} z_k(s_1) \;+ \sum_{k=1}^{4}\sum_{l=1}^{4} T_{jkl} z_k(s_1) z_l(s_1)$$

$$\text{Let's call it :} \quad \mathcal{A}_2 \;=\; [R, T] \quad (\text{second order map } \mathcal{A}_2)$$

Higher orders can be defined as needed ...

$$\mathcal{A}_3 \;=\; [R, T, U] \quad \Longrightarrow \quad + \sum_{k=1}^{4}\sum_{l=1}^{4}\sum_{m=1}^{4} U_{jklm} z_k(s_1) z_l(s_1) z_m(s_1)$$

Higher order (Taylor -) MAPS:

Example: complete second order map for a (thick) sextupole with length L and strength K (in 4D):

$$
\begin{aligned}
x_2 &= x_1 + Lx_1' & -K\left(\tfrac{L^2}{4}(x_1^2 - y_1^2) + \tfrac{L^3}{6}(x_1 x_1' - y_1 y_1') + \tfrac{L^4}{24}(x_1'^2 - y_1'^2)\right)\\
x_2' &= x_1' & -K\left(\tfrac{L}{2}(x_1^2 - y_1^2) + \tfrac{L^2}{2}(x_1 x_1' - y_1 y_1') + \tfrac{L^3}{6}(x_1'^2 - y_1'^2)\right)\\
y_2 &= y_1 + Ly_1' & +K\left(\tfrac{L^2}{4} x_1 y_1 + \tfrac{L^3}{6}(x_1 y_1' + y_1 x_1') + \tfrac{L^4}{24}(x_1' y_1')\right)\\
y_2' &= y_1' & +K\left(\tfrac{L}{2} x_1 y_1 + \tfrac{L^2}{2}(x_1 y_1' + y_1 x_1') + \tfrac{L^3}{6}(x_1' y_1')\right)
\end{aligned}
$$

⚠ **Definition of K not unique, can differ by some factor !!**

e.g. $$\left(\frac{\partial^2 x}{\partial t^2} = S \cdot x^2 \quad \text{versus} \quad \frac{\partial^2 x}{\partial t^2} = \frac{k}{2} \cdot x^2\right)$$

Symplecticity for higher order MAPS

- **Truncated Taylor expansions are not matrices !!**
- **It is the associated Jacobian matrix $\mathcal{J}$ which must fulfil the symplecticity condition:**

$$\mathcal{J}_{ik} = \frac{\partial z_2^i}{\partial z_1^k} \quad \left(\text{e.g. } \mathcal{J}_{xy} = \frac{\partial z_2^x}{\partial z_1^y}\right)$$

$$\mathcal{J} \quad \textbf{must fulfil:} \quad \mathcal{J}^t \cdot S \cdot \mathcal{J} = \mathcal{S}$$

- **In general:** $\mathcal{J}_{ik} \neq$ **const** $\rightarrow$ **for truncated Taylor map can be difficult to fulfil for all** z

Symplecticity for higher order MAPS

Take the sextupole map (for simplicity in one dimension):

$$\begin{aligned} x_2 &= x_1 + Lx_1' & -K\left(\tfrac{L^2}{4}x_1^2 + \tfrac{L^3}{6}x_1x_1' + \tfrac{L^4}{24}x_1'^2 + \mathcal{O}(3)\right) \\ x_2' &= x_1' & -K\left(\tfrac{L}{2}x_1^2 + \tfrac{L^2}{2}x_1x_1' + \tfrac{L^3}{6}x_1'^2 + \mathcal{O}(3)\right) \end{aligned}$$

we compute:

$$\mathcal{J}^T \cdot S \cdot \mathcal{J} = \begin{pmatrix} 0 & 1+\Delta S \\ -1-\Delta S & 0 \end{pmatrix} \quad \neq \quad S$$

is non-symplectic with error:

$$\Delta S = \frac{K^2}{72} L^4 (L^2 x'^2 + 6Lxx' + 6x^2)$$

Symplecticity for higher order MAPS

Take the sextupole map (for simplicity in one dimension):

$$\begin{aligned} x_2 &= x_1 + Lx_1' & -K\left(\tfrac{L^2}{4}x_1^2 + \tfrac{L^3}{6}x_1x_1' + \tfrac{L^4}{24}x_1'^2 + \mathcal{O}(3)\right) \\ x_2' &= x_1' & -K\left(\tfrac{L}{2}x_1^2 + \tfrac{L^2}{2}x_1x_1' + \tfrac{L^3}{6}x_1'^2 + \mathcal{O}(3)\right) \end{aligned}$$

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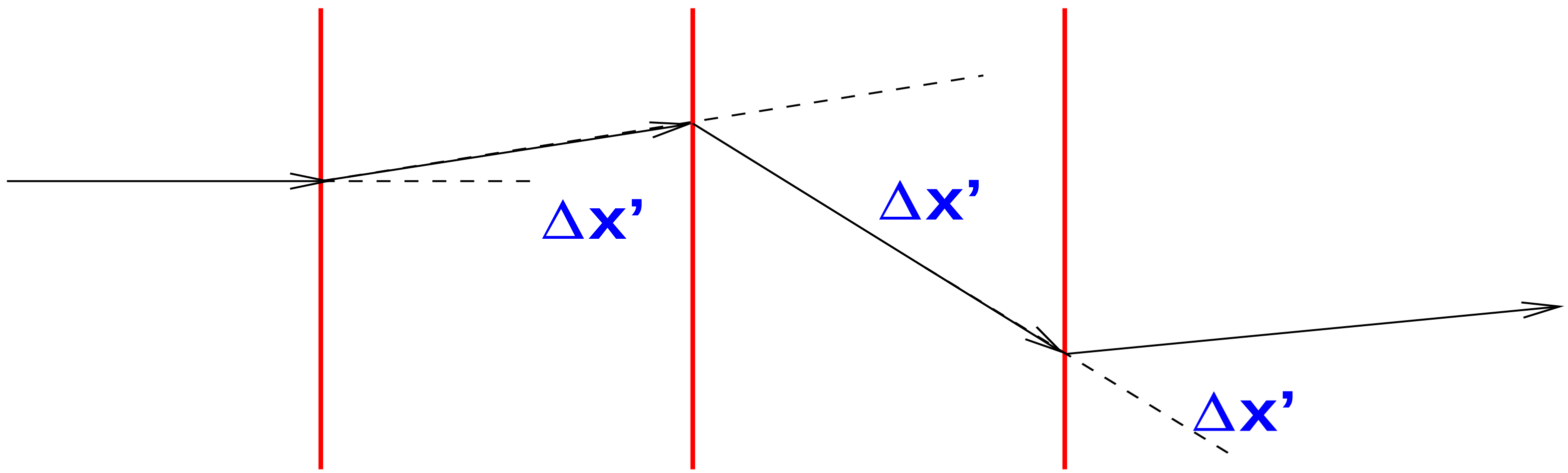
The way out: thin magnets

- **Real magnets have a finite length, i.e. thick magnets**
- **Thick magnet: field and length used to compute effect, i.e. the map**
- **Consequence: they are not always linear elements (even not dipoles, quadrupoles)**
- **For thick, non-linear magnets closed solution for maps often does not exist**

Thick versus thin magnets

- Thin ”magnet”: let the length go to zero, but keep field integral finite (constant)
- Thin dipoles and quadrupoles are linear elements
- Thin elements are much easier to use ...

Moving through thin elements



- **No change of amplitudes x and y**
- **The momenta x' and y' receive an amplitude dependent deflection (kick)**

→ $x' \rightarrow x' + \Delta x'$ **and** $y' \rightarrow y' + \Delta y'$

Using thin elements

- Can we approximate a thick element by thin element(s) ?
  - Yes, when the length is small or does not matter
  - Yes, when we can model the thick magnet correctly
  - What about accuracy, symplecticity etc. ??
  - Demonstrate with some simple examples

Thick → thin quadrupole

$$\mathcal{M}_{s\rightarrow s+L} = \begin{pmatrix} \cos(L\cdot K) & \frac{1}{K}\cdot\sin(L\cdot K) \\ -K\cdot\sin(L\cdot K) & \cos(L\cdot K) \end{pmatrix}$$

- **Exact map (matrix) for quadrupole**
- **What happens when we make it thin ?**
  - **Accuracy ?**
  - **Symplecticity ?**
- **(What follows is valid for all elements)**

Accuracy of thin lenses

$$\mathcal{M}_{s\rightarrow s+L} = \begin{pmatrix} \cos(L\cdot K) & \frac{1}{K}\cdot\sin(L\cdot K) \\ -K\cdot\sin(L\cdot K) & \cos(L\cdot K) \end{pmatrix}$$

- **Start with exact map**
- **Taylor expansion in "small" length L:**

$$L^0\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + L^1\cdot\begin{pmatrix} 0 & 1 \\ -K^2 & 0 \end{pmatrix} + L^2\cdot\begin{pmatrix} -\frac{K^2}{2} & 0 \\ 0 & -\frac{K^2}{2} \end{pmatrix} + \ldots$$

Accuracy of thin lenses (B)

- **Keep up to <u>first order</u> term in** L

$$\mathcal{M}_{s\rightarrow s+L} = L^0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + L^1 \cdot \begin{pmatrix} 0 & 1 \\ -K^2 & 0 \end{pmatrix}$$

$$\mathcal{M}_{s\rightarrow s+L} = \begin{pmatrix} 1 & L \\ -K^2 \cdot L & 1 \end{pmatrix} + \mathcal{O}(L^2)$$

- **Precise to first order** $\mathcal{O}(L^1)$
- **det** $\mathcal{M} \neq$ **1, non-symplectic**

Accuracy of thin lenses (C)

$$\mathcal{M}_{s\rightarrow s+L} = \begin{pmatrix} 1 & L \\ -K^2 \cdot L & 1 \end{pmatrix} + \mathcal{O}(L^2)$$

$\Rightarrow$

$$\mathcal{M}_{s\rightarrow s+L} = \begin{pmatrix} 1 & L \\ -K^2 \cdot L & 1 - K^2 L^2 \end{pmatrix}$$

- **Precise to first order** $\mathcal{O}(L^1)$
- **"symplectified" by adding term** $-K^2 L^2$

  (it is wrong to $\mathcal{O}(L^2)$ **anyway ...)**

Accuracy of thin lenses

- Keep up to <u>second order</u> term in L

$$\mathcal{M}_{s\rightarrow s+L} = \begin{pmatrix} 1-\frac{1}{2}K^2L^2 & L \\ -K^2\cdot L & 1-\frac{1}{2}K^2L^2 \end{pmatrix} + \mathcal{O}(L^3)$$

- Precise to second order $\mathcal{O}(L^2)$
- More accurate than (C), but not symplectic

Accuracy of thin lenses (D)

- **Symplectification like:**

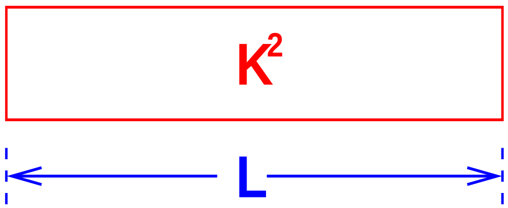
$$\mathcal{M}_{s\rightarrow s+L} = \begin{pmatrix} 1-\frac{1}{2}K^2L^2 & L-\frac{1}{4}K^2L^3 \\ -K^2\cdot L & 1-\frac{1}{2}K^2L^2 \end{pmatrix} + \mathcal{O}(L^3)$$

- **Precise to second order** $\mathcal{O}(L^2)$
- **Fully symplectic**

Accuracy of thin lenses

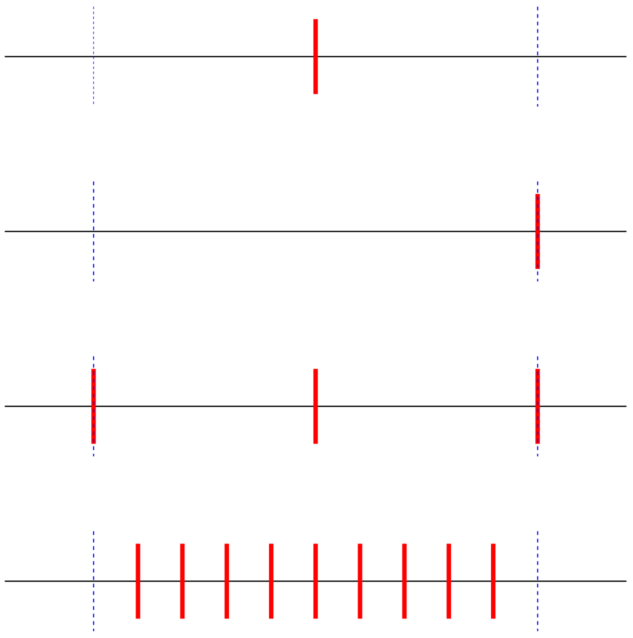
- Looks like we made some arbitrary changes and called it "symplectification"
- Is there a physical picture behind the approximations ?
- Yes, geometry of thin lens kicks ...
- A thick element is split into thin elements with drifts between them

Thick → thin quadrupole



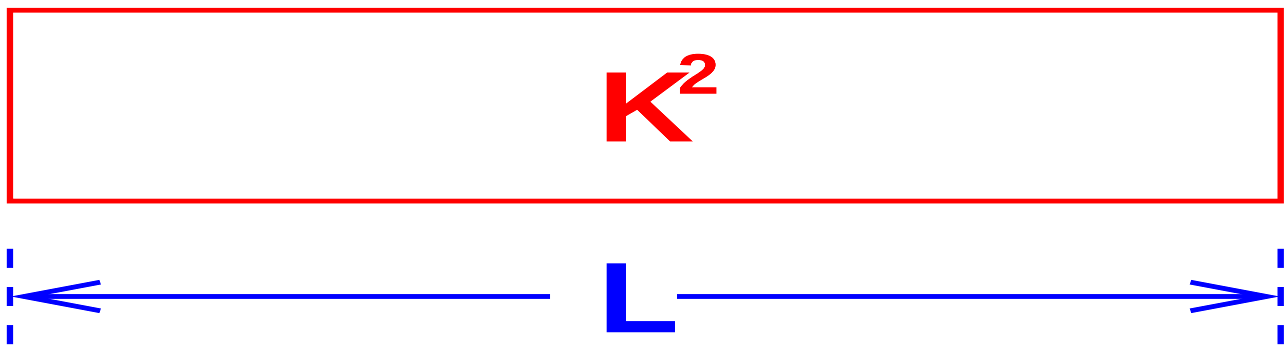
quadrupole of
finite length

options:

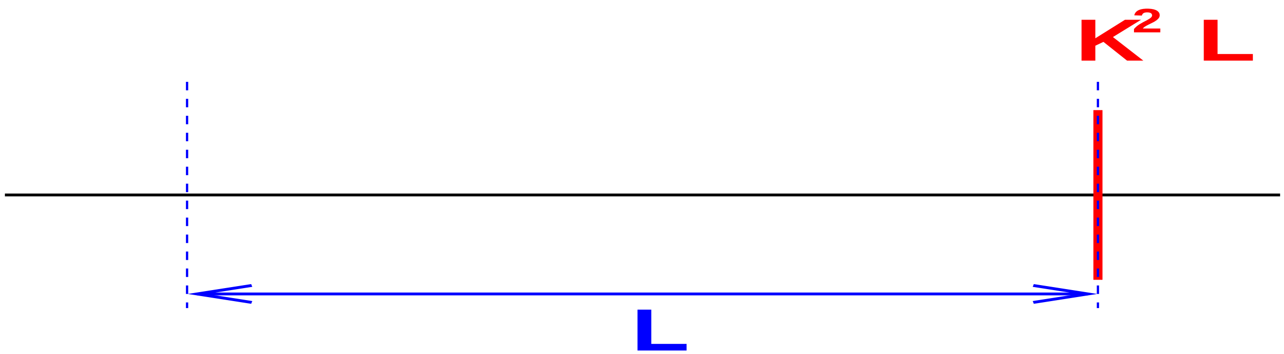


Which is a good strategy ? → accuracy and speed

Thick quadrupole ..



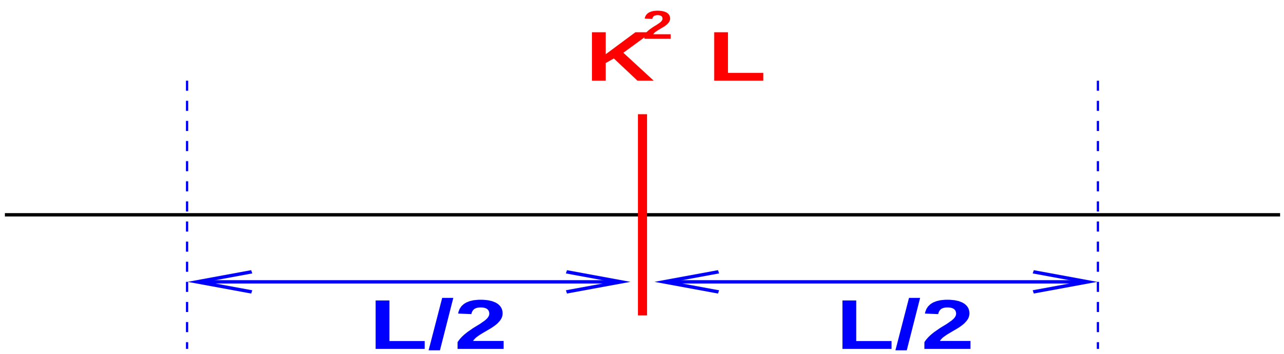
First order ..



- **One thin quadrupole "kick" and one drift combined**
- **Resembles "symplectification" of type (C)**

$$\begin{pmatrix} 1 & 0 \\ -K^2 \cdot L & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & L \\ -K^2 \cdot L & 1 - K^2 L^2 \end{pmatrix}$$

Second order ..



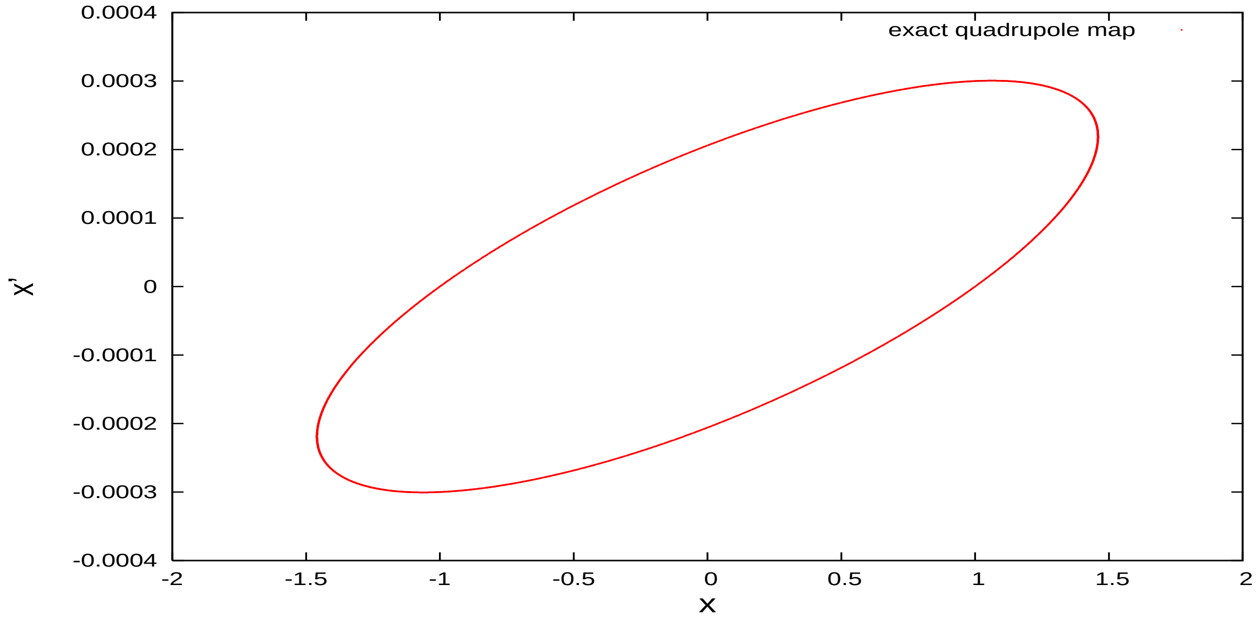
- **One thin quadrupole "kick" between two drifts**
- **Resembles more accurate, symplectic model of type (D)**

$$\begin{pmatrix} 1 & \frac{1}{2}L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -K^2 \cdot L & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2}L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}K^2L^2 & L - \frac{1}{4}K^2L^3 \\ -K^2 \cdot L & 1 - \frac{1}{2}K^2L^2 \end{pmatrix}$$

Accuracy of thin lenses

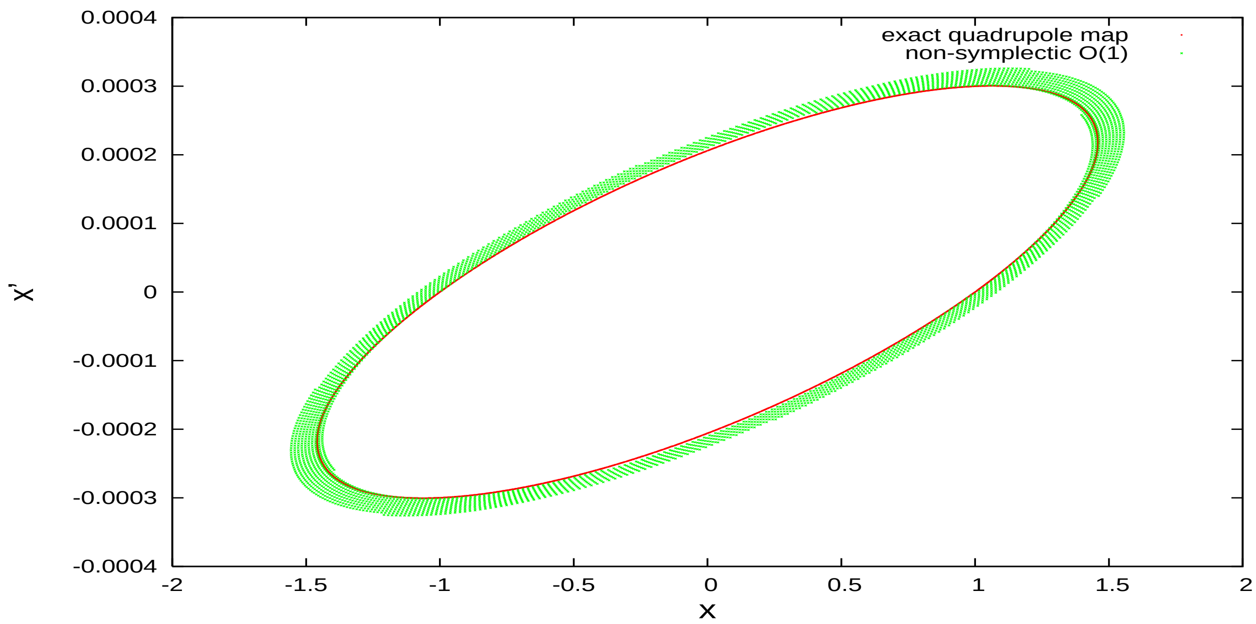
- **One kick at the end (or beginning):**
  - **Error (inaccuracy) of first order** $\mathcal{O}(L^1)$
- **One kick in the centre:**
  - **Error (inaccuracy) of second order** $\mathcal{O}(L^2)$
- **It is very relevant how to apply thin lenses**
- **Aim should be to be precise and fast (and simple to implement)**

What is the point ???



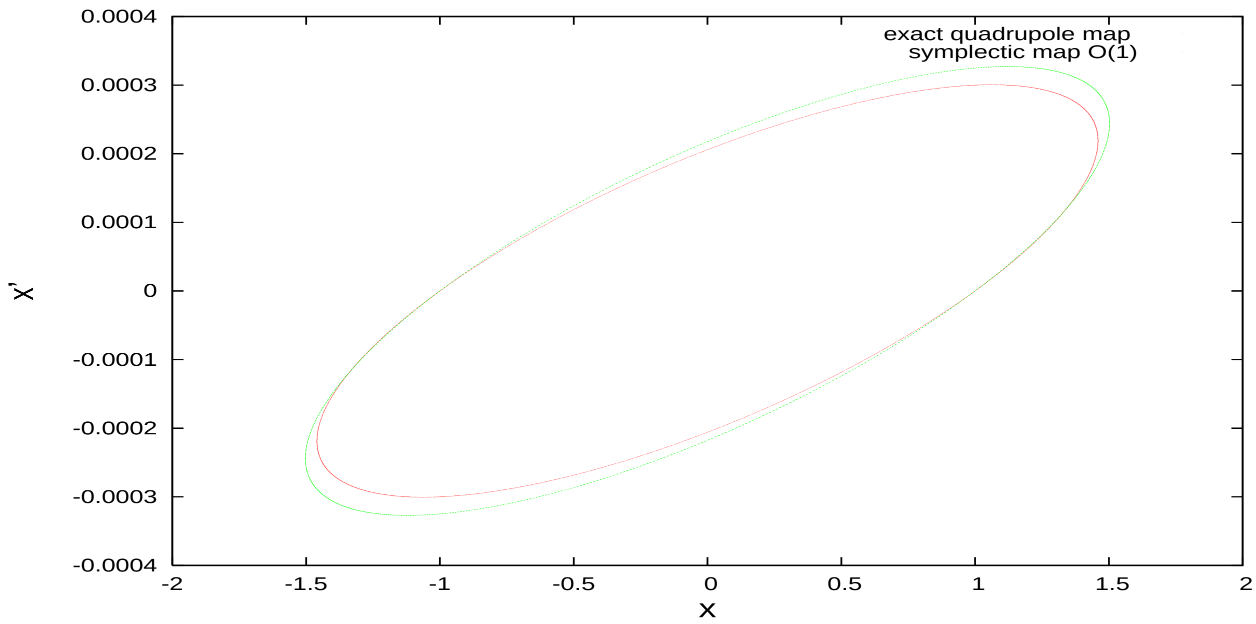
- **Phase ellipse - quadrupole exact solution**

Quadrupole non-symplectic solution



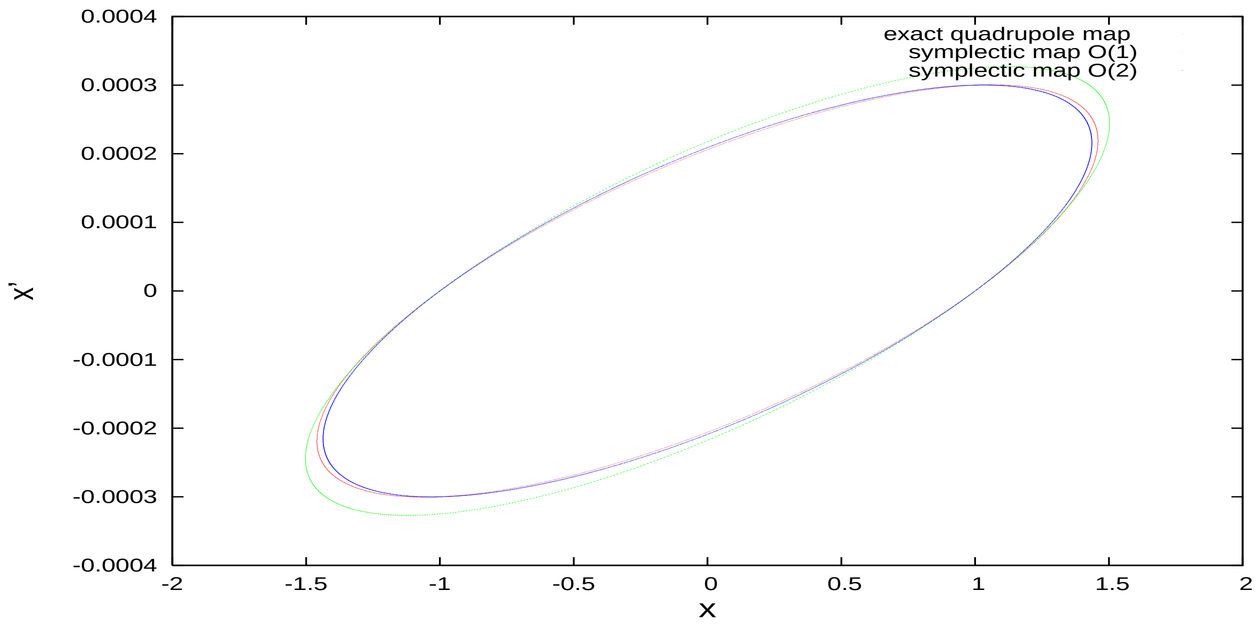
- **Non-symplecticity: particles spiral towards outside**

Quadrupole symplectic $\mathcal{O}(L^1)$ solution



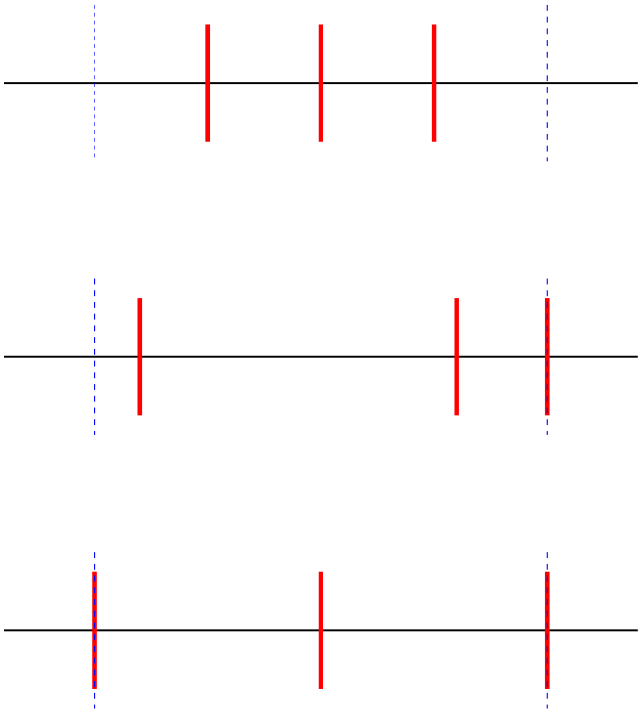
- **symplecticity: but phase space ellipse not accurate**

Quadrupole symplectic $\mathcal{O}(L^2)$ solution



- **symplecticity: phase space ellipse accurate enough**

Can we do better ?

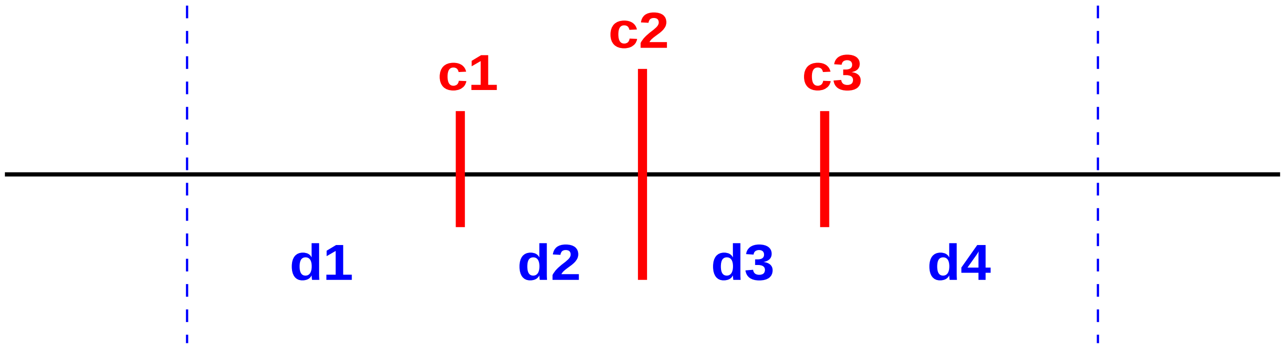


- Try more slices, e.g. 3 kicks:
- How to put them ?
- Hope you are already alerted ...
- Allow that they are at different positions and have different strengths
- Minimize the inaccuracy

Question: is one of the options obviously wrong ? If yes, why ?

Can we do better ?

- **Try a model with 3 kicks:**

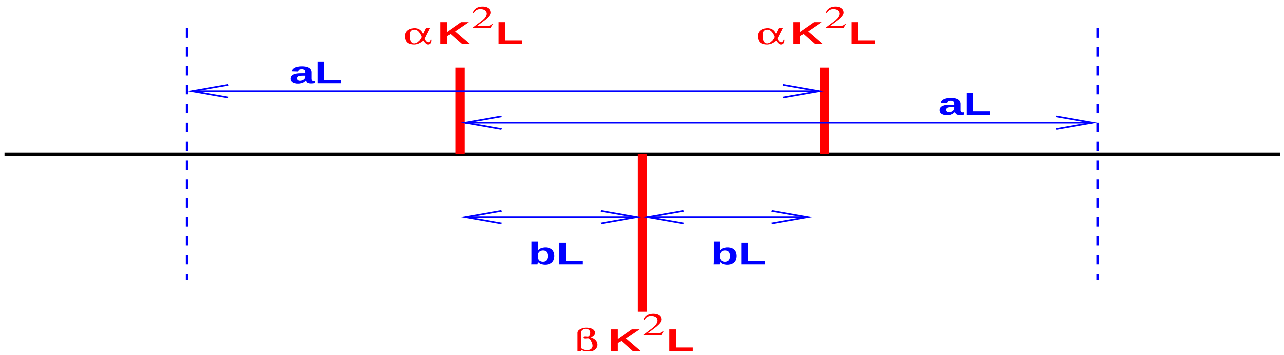


→ **To get best accuracy (i.e. deviation from exact solution):**

- **Optimize kicks c1, c2, c3**
- **Optimize drifts d1, d2, d3, d4**

Can we do better ?

- **Try a model with 3 kicks:**



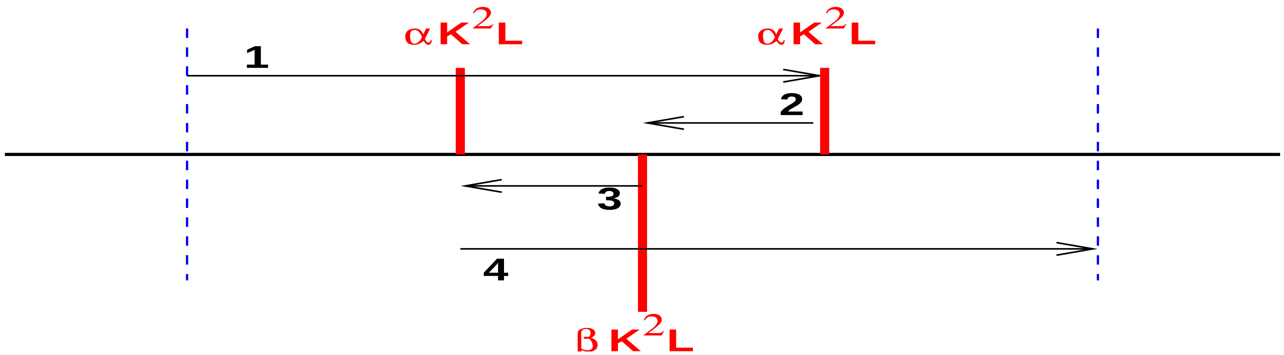
- **with:**

$$\mathbf{a} \approx \mathbf{0.6756},\ \mathbf{b} \approx \mathbf{-0.1756},\ \alpha \approx \mathbf{1.3512},\ \beta \approx \mathbf{-1.7024}$$

- **we have a $\mathcal{O}(4)$ integrator ...**
- **(a $\mathcal{O}(6)$ integrator would require 9 kicks (!) ...)**

Can we do better ?

- **Try a model with 3 kicks:**



- **Must track backwards ! Change interpretation !**
- **Thin lenses not a new sequence of magnets (a la MAD)**
- **What about space charge calculations ?**

Symplectic integration

- **What we do is Symplectic Integration**
- **From a lower order integration scheme (1 kick), construct higher order scheme**
- **Formally (for the formulation of $S_k(t)$ see later):**
  - **From a 2nd order scheme (1 kick) $S_2(t)$ we construct a 4th order scheme (3 kicks = 3 x 1 kick) like:**

    $S_4(t) = S_2(x_1 t) \circ S_2(x_0 t) \circ S_2(x_1 t)$ **with:**

$$x_0 = \frac{-2^{1/3}}{2 - 2^{1/3}} \approx -1.7024 \qquad x_1 = \frac{1}{2 - 2^{1/3}} \approx 1.3512$$

Symplectic integration

- **Can be considered as an iterative scheme (see e.g. H. Yoshida, 1990, E. Forest, 1998[2]):**
  - **If $S_{2k}(t)$ is a symmetric integrator of order $2k$, then:**

    $S_{2k+2}(t) = S_{2k}(x_1 t) \circ S_{2k}(x_0 t) \circ S_{2k}(x_1 t)$ **with:**

$$x_0 = \frac{-\sqrt[2k+1]{2}}{2 - \sqrt[2k+1]{2}} \qquad x_1 = \frac{1}{2 - \sqrt[2k+1]{2}}$$

- **Higher order integrators can be obtained in a similar way**

[2] E. Forest, "Beam Dynamics, A New Attitude and Framework", 1998

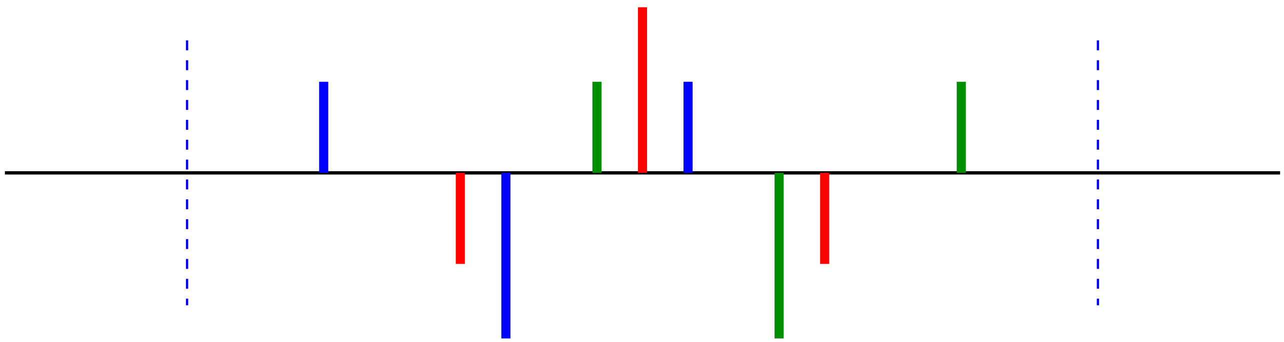
Symplectic integration

- **Example: From a 4th order to 6th order**

  $$S_6(t) = S_4(x_1 t) \circ S_4(x_0 t) \circ S_4(x_1 t)$$

- **We get 3 times 4th order with 3 kicks each, we have the 9 kick, 6th order integrator mentioned earlier**

Integrator of order 6



- **Requires 9 kicks**
- **We have 3 interleaved 4th order integrators**
- **Can be used in iterative scheme**

Some remarks:

- **We have used a linear map (quadrupole) to demonstrate the integration**
- **Can that be applied for other maps (solenoids, higher order, non-linear maps) ?**
  - **Yes !!**
  - **We get the same integrators !**
  - **Proof and systematic (and easy) extension in the form of Lie operators[2] (see later)**

→ **Best accuracy for thin lenses !**

[2] H. Yoshida, Physics Letters A, Volume 150 (1990) 262.

Accuracy of thin lenses

What about accuracy of non-linear elements ?

assume a general case:

$$x'' = f(x)$$

- **Disadvantage : usually a closed solution through the element does not exist, integration necessary**
- **Advantage : They are usually thin (thinner than dipoles, quadrupoles ...)**
  - **Dipoles: $\approx$ 14.3 m**
  - **Quadrupole: $\approx$ 2 - 5 m**
  - **Sextupoles, Octupoles: $\approx$ 0.30 m**

→ **Can try our simplest thin lens approximation first ...**

Accuracy of thin lenses - our $\mathcal{O}(2)$ model

$$1.Step \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L/2} = \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

$$2.Step \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L/2} = \begin{pmatrix} x \\ x' + \Delta x' \end{pmatrix}_{s_1+L/2}$$

$$3.Step \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L} = \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L/2}$$

Accuracy of thin lenses

Assume the general case:

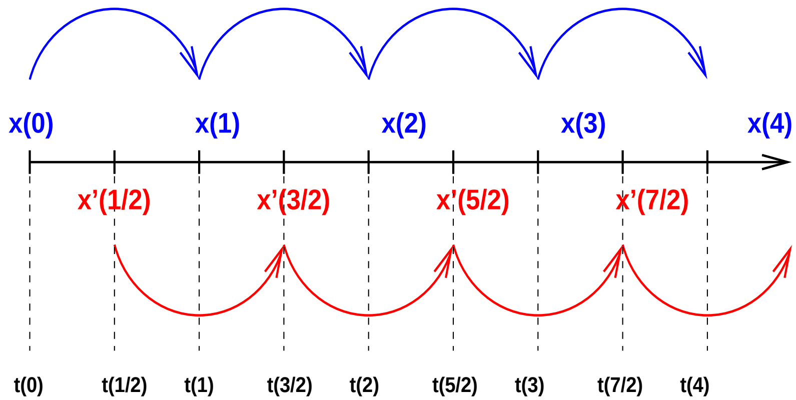
$$x'' \quad = \quad f(x) \;\; (= \Delta x')$$

- **Using this thin lens approximation (type D, $\mathcal{O}(2)$) gives:**

$$\begin{aligned} x'(L) &\approx x'_0 + Lf(x_0 + \frac{L}{2}x'_0) \\ x(L) &\approx x_0 + \frac{L}{2}(x'_0 + x'(L)) \end{aligned}$$

- **This is also called "leap frog" algorithm/integration**
- **It is symplectic (... and time reversible) !!**

Interlude ...



For any: $x'' = f(x, x', t)$ **we can solve it by:**

$$
\begin{aligned}
x'_{i+3/2} &\approx x'_{i+1/2} + f(x_{i+1})\Delta t \\
x_{i+1} &\approx x_i + x'_{i+1/2}\Delta t
\end{aligned}
$$

Accuracy of thin lenses

Accuracy of "leap frog" algorithm/integration"

the (exact) Taylor expansion gives:

$$x(L) \quad = \quad x_0 + x'_0 L + \frac{1}{2} f(x_0) L^2 + \frac{1}{6} x'_0 f'(x_0) L^3 + \ldots$$

the (approximate) "leap frog" algorithm gives:

$$x(L) \quad = \quad x_0 + x'_0 L + \frac{1}{2} f(x_0) L^2 + \frac{1}{4} x'_0 f'(x_0) L^3 + \ldots$$

- **Errors are $\mathcal{O}(L^3)$ (of course)**
- **For small L acceptable, and symplectic, extend to our symplectic integration**

Accuracy of thin lenses

For bar/coffee discussions:

why did I write:

$$x'' = f(x)$$

and not:

$$x'' = f(x, x')$$

Accuracy of thin lenses

An application, assume a (1D) sextupole with (definition of k not unique !):

$$x'' = k \cdot x^2 = f(x)$$

using the thin lens approximation (type D) gives:

$$\begin{aligned} x(L) &= x_0 + x_0' L + \frac{1}{2} k x_0^2 L^2 + \frac{1}{2} k x_0 x_0' L^3 + \frac{1}{8} k x_0'^2 L^4 \\ x'(L) &= x_0' + k x_0^2 L + k x_0 x_0' L^2 + \frac{1}{4} k x_0'^2 L^3 \end{aligned}$$

Map for thick sextupole of length L in thin lens approximation, accurate to $\mathcal{O}(L^2)$

Accuracy of thin lenses

An application, assume a (1D) sextupole with (definition of k **not unique !):**

$$x'' = k \cdot x^2 = f(x)$$

using the thin lens approximation (type D) gives:

$$x(L) = x_0 + x_0' L + \frac{1}{2} k x_0^2 L^2 + \frac{1}{2} k x_0 x_0' L^3 + \frac{1}{8} k x_0'^2 L^4$$

$$x'(L) = x_0' + k x_0^2 L + k x_0 x_0' L^2 + \frac{1}{4} k x_0'^2 L^3$$

Map for thick sextupole of length L **in thin lens approximation, accurate to** $\mathcal{O}(L^2)$

Some comments:

- **We have interleaved kicks with drifts**
- **Is that always necessary ?**
  - **No !**
  - **Can be any map with an exact expression**
  - **In most cases the exact map is a linear map (matrix)**
- **We have derived element maps for tracking from the equation of motion using this technique → can track now**

Simulation and tracking

We have now sufficient tools for a simulation code

- **Main purpose of such a code: Propagate particles around a ring or along a beam line**
- **Results (amongst others):**
  - **Phase space topology (Poincare sections,..)**
  - **Global properties (after some analysis), e.g. stability, detuning, invariants, frequency map analysis ....**
- **In our terminology: Tracking codes produce maps (i.e. relate output to the input)!**
- **Can we extract more "analytical" maps ?**

So far ...

- Concept and representation by MAPS
- Computation and analysis of One-Turn-Maps
- Normal form analysis of LINEAR MAPS
- Introduction of Taylor maps
- Introduction of symplectic integration

Mathematical and Numerical Methods for Non-linear Beam Dynamics in Rings

(an introduction)

Part 2

Werner Herr, CERN

http://cern.ch/Werner.Herr/CAS2013/lectures/Trondheim_methods.pdf

For many more details:

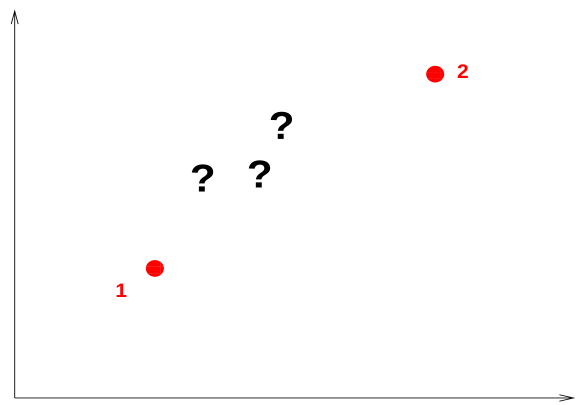
http://cern.ch/Werner.Herr/METHODS

The plan now ...

- **Extend all concepts to non-linear dynamics**
  - **Lagrangian and Hamiltonian dynamics**
  - **How to use that $\Rightarrow$ Lie transforms**
  - **How to analyse that $\Rightarrow$ Non-linear normal forms**
  - **How to analyse that better $\Rightarrow$ Differential Algebra (DA)**
- **Avoid abstract definitions and formulation, but:**
  - **Intuitive (but correct !) treatment**
  - **Useful formulae and examples**
  - **Real life examples and demonstration (DA)**

Hamilton principle

- **Problem: describe the motion of a system (e.g. 1 or more particles) between times t_1 and t_2**



- **Describe by coordinates q_i** $\qquad (i = 1, n)$

  n are degrees of freedom of the system

  (Goldstein convention)

Hamilton principle

- **Describe motion by a function L**

$$L(q_1, ...q_n, \dot{q}_1, ...\dot{q}_n, t)$$

$(q_1, ...q_n)$ **... generalized coordinates**

$(\dot{q}_1, ...\dot{q}_n)$ **... generalized velocities**

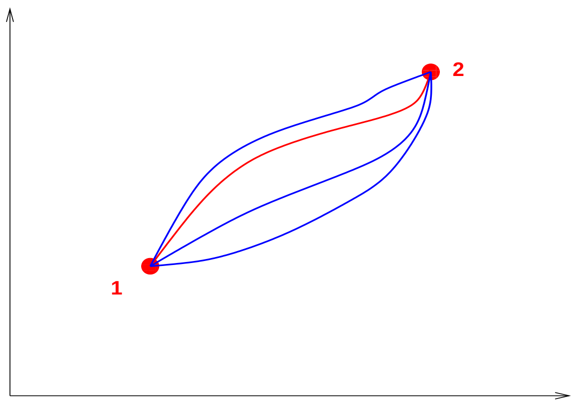
- **The function L defines the Lagrange function**
- **The integral $I = \int L(q_i, \dot{q}_i, t)dt$ defines the action**

Without proof or derivation:

$L = T - V$ **= kinetic energy - potential energy**

Hamilton principle

$$I = \int_1^2 L(q_i, \dot{q}_i, t)dt = extremum$$



- **Hamiltonian principle: system moves such that the action I becomes an extremum**

Extremum principle ?

- **Not new:**
  - **Used in optics: Fermat principle**
  - **Quantum mechanics (path integrals)**
  - **General relativity**
  - **...**

Lagrange formalism

Without proof:

$$I = \int_1^2 L(q_i, \dot{q}_i, t)dt = extremum$$

is fulfilled when:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

(Euler - Lagrange equation)

From Lagrangian to Hamiltonian ..

- **Lagrangian $L(q_1, ...q_n, \dot{q_1}, ...\dot{q_n}, t)$ in generalized coordinates and velocities**
- **Provides (n) second order differential equations**
- **Try to get:**
  - **Generalized momenta instead of velocities**
  - **First order differential equations (always solvable)**

Corresponding (so-called conjugate) momenta p_j are:

$$\boxed{p_j = \frac{\partial L}{\partial \dot{q_j}}}$$

From Lagrangian to Hamiltonian ..

- **Lagrangian:**
  - n **second order equations**
  - n**-dimensional coordinate space**
- **Hamiltonian:**
  - $2n$ **first order equations**
  - $2n$**-dimensional phase space**

From Lagrangian to Hamiltonian ..

Once we know what the canonical momenta p_i are: the Hamiltonian is a transformation of the Lagrangian:

$$H(q_j, p_j, t) = \sum_i \dot{q}_i p_i - L(q_j, \dot{q}_j, t)$$

without proof:

$H = T + V$ **= kinetic energy + potential energy**

we obtain 2 first order equation of motion:

$$\boxed{\frac{\partial H}{\partial q_j} = -\dot{p}_j = -\frac{dp_j}{dt}}, \qquad \boxed{\frac{\partial H}{\partial p_j} = \dot{q}_j = \frac{dq_j}{dt}}$$

Hamiltonian of particle in EM fields

For the Hamiltonian of a (relativistic) particle in a electro-magnetic field we have ($q \rightarrow x$):

$$\mathcal{H}(\vec{x},\vec{p},t) = c\sqrt{(\vec{p} - e\vec{A}(\vec{x},t))^2 + m_0^2c^2} + e\Phi(\vec{x},t)$$

where $\vec{A}(\vec{x},t)$, $\Phi(\vec{x},t)$ the vector and scalar scalar potential

Using canonical variables and the design path length s as independent variable (bending in x-plane):

$$\mathcal{H} = -(1 + \frac{x}{\rho}) \cdot \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} + \frac{x}{\rho} + \frac{x^2}{2\rho^2} - \frac{A_s(x,y)}{B_0\rho}$$

where $\delta = (p - p_0)/p$ is relative momentum deviation and $A_s(x,y)$ longitudinal component of the vectorpotential [MB].

Hamiltonian of particle in EM fields

The magnetic fields can be described with the multipole expansion:

$$B_y + iB_x = \sum_{n=1} (b_n + ia_n)(x + iy)^{n-1}$$

and since $\vec{B} = \nabla \times \vec{A}$**:**

$$A_s = \sum_{n=1} \frac{1}{n} [(b_n + ia_n)(x + iy)^n]$$

- $n = 1$ **refers to dipole (not always true !)**
- **For a large machine (**$x \ll \rho$**) we expand the root and sort the variables** ➡

Hamiltonian (for large machine) ..

$$\mathcal{H} = \overbrace{\frac{p_x^2 + p_y^2}{2(1+\delta)}}^{kinematic} - \overbrace{\underbrace{\frac{x\delta}{\rho}}_{dispersive} + \underbrace{\frac{x^2}{2\rho^2}}_{focusing}}^{dipole} + \overbrace{\frac{k_1}{2}(x^2 - y^2)}^{quadrupole} + \overbrace{\frac{k_2}{6}(x^3 - 3xy^2)}^{sextupole}$$

$$\left(\text{using (MAD convention)}: \quad k_n = \frac{1}{B\rho}\frac{\partial^n B_y}{\partial x^n} \right)$$

- **The Hamiltonian describes exactly the motion of a particle through a magnet**
- **Basis to extend the linear to a non-linear formalism**

But how do we use it ??

Poisson brackets

Introduce Poisson bracket for a differential operator:

$$[f, g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right)$$

Here the variables x_i, p_i are canonical variables, f and g are functions of x_i and p_i.
We can now write (using the Hamiltonian H for g):

$$f(x_i, p_i) = x_i \Rightarrow \quad [x_i, H] = \frac{\partial H}{\partial p_i} = \frac{dx_i}{dt}$$

$$f(x_i, p_i) = p_i \Rightarrow \quad [p_i, H] = -\frac{\partial H}{\partial x_i} = \frac{dp_i}{dt}$$

Poisson brackets encode Hamiltons's equations

Poisson brackets

- **Poisson bracket $[f, H]$ describes the time evolution of the system (the function f)**

It is a special case of:

$$\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t}$$

If H does not explicitly depend on time and:

$$[f, H] = 0$$

implies that f is an invariant of the motion !

Poisson brackets determine invariants

Lie transformations

We can define:

$$: f : g = [f, g]$$

where $: f :$ **is an operator acting on the function** g**:**

$$: f :=[f, \,]$$

The operator $: f :$ **is called a Lie Operator**

It acts on functions $g(x,p)$**, special cases:**

$$g(x,p) = x \qquad \rightarrow \qquad : f : x$$

$$g(x,p) = p \qquad \rightarrow \qquad : f : p$$

Lie operators are Poisson brackets in waiting

Useful formulae for calculations

With x coordinate, p momentum, try special cases for f:

$$:x: \; = \; \frac{\partial}{\partial p} \qquad :p: \; = \; -\frac{\partial}{\partial x}$$

$$:x:^2 \; = \; \frac{\partial^2}{\partial p^2} \qquad :p:^2 \; = \; \frac{\partial^2}{\partial x^2}$$

$$:x^2: \; = \; 2x\frac{\partial}{\partial p} \qquad :p^2: \; = \; -2p\frac{\partial}{\partial x}$$

$$:xp: \; = \; p\frac{\partial}{\partial p} - x\frac{\partial}{\partial x} \qquad :x::p: \; = \; :p::x: \; = \; -\frac{\partial^2}{\partial x \partial p}$$

More useful formulae for calculations

With x coordinate, p momentum, as usual:

$$: p^2 : x \;=\; \frac{\partial p^2}{\partial x}\frac{\partial x}{\partial p} - \frac{\partial p^2}{\partial p}\frac{\partial x}{\partial x} \;=\; -2p$$

$$: p^2 : p \;=\; \frac{\partial p^2}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial p^2}{\partial p}\frac{\partial p}{\partial x} \;=\; 0$$

$$(: p^2 :)^2 x =: p^2 : (: p^2 : x) =: p^2 : (-2p) = 0$$

$$(: p^2 :)^2 p =: p^2 : (: p^2 : p) =: p^2 : (0) = 0$$

Lie transformations

We can define powers as:

$$(:f:)^2 g =: f : (:f:g) = [f,[f,g]] \qquad \text{etc.}$$

in particular:

$$e^{:f:} = \sum_{i=0}^{\infty} \frac{1}{i!}(:f:)^i$$

$$e^{:f:} = 1 \;+\; :f: \;+\; \frac{1}{2!}(:f:)^2 \;+\; \frac{1}{3!}(:f:)^3 + ...$$

The operator $e^{:f:}$ is call an Lie Transformation

Lie transformations - example

Lie operators act on functions like x, p (canonical momentum, instead of x'), for example:

$$:p^2: x = -2p \qquad :p^2: p = 0$$

or as a Lie transformation with $f = -Lp^2/2$:

$$
\begin{aligned}
e^{:-Lp^2/2:}x &= x - \frac{1}{2}L\underbrace{:p^2:x}_{=-2p} + \frac{1}{8}L^2\underbrace{(:p^2:)^2x}_{=0} + .. \\
&= \boxed{\mathbf{x + Lp}} \\
e^{:-Lp^2/2:}p &= p - \frac{1}{2}L\underbrace{:p^2:p}_{=0} + ... \\
&= \boxed{\mathbf{p}}
\end{aligned}
$$

Lie transformations - example

Lie operators act on functions like x, p (canonical momentum, instead of x'), for example:

$$:p^2:x = -2p \qquad :p^2:p = 0$$

or as a Lie transformation with $f = -Lp^2/2$:

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&= \boxed{\mathbf{x + Lp}} \\
e^{:-Lp^2/2:}p &= p - \frac{1}{2}L\underbrace{:p^2:p}_{=0} + ... \\
&= \boxed{\mathbf{p}}
\end{aligned}$$

This is the transformation of a drift space of length L !!

Lie transformations - general

Acting on the phase space coordinates:

$$e^{:f:}(x,p)_0 = (x,p)_1$$

that is:

$$e^{:f:}x_0 = x_1$$

$$e^{:f:}p_0 = p_1$$

- **Lie transforms describe how to go from one point to another [AC1, AD].**
- **Through a machine element (drift, magnet ...) fully described by f**
- **But what is f ?**

Lie transformations

- The generator f is the Hamiltonian of the element !
- We use the Hamiltonian to describe the motion through an individual element
- Inside a single element the motion is "smooth" (in the full machine it is not !)
- Can track "thick" elements (and still symplectic !)

Some formulae for Lie transforms

With a constant, f, g, k arbitrary functions:

$$:a: \;=\; 0 \qquad\qquad e^{:a:} \;=\; 1$$

$$:f:a \;=\; 0 \qquad\qquad e^{:f:}a \;=\; a$$

$$e^{:f:}g(x) \;=\; g(e^{:f:}x)$$

$$e^{:f:}G(:g:)e^{-:f:} \;=\; G(:e^{:f:}g:)$$

$$e^{:f:}[g, h] = [e^{:f:}g, e^{:f:}h]$$

$$(e^{:f:})^{-1} \;=\; e^{-:f:}$$

and very important:

$$e^{:f:}e^{:g:}e^{-:f:} \;=\; e^{:e^{:f:}g:}$$

More examples (1D):

For:

$$f = -\frac{L}{2}p^2$$

we obtained:

$$\begin{aligned} e^{:f:}x &= x + Lp \\ e^{:f:}p &= p \end{aligned}$$

- **Drift space, seen that already**

More examples (1D):

For:

$$f = -\frac{L}{2}(k^2x^2 + p^2)$$

we would get (try it !):

$$\begin{aligned} e^{:f:}x &= e^{:-\frac{L}{2}(k^2x^2+p^2):}x \\ e^{:f:}p &= e^{:-\frac{L}{2}(k^2x^2+p^2):}p \end{aligned}$$

Remember:

$$e^{:f:}g = \sum_{n=0}^{\infty} \frac{:f:^n}{n!} g$$

More examples (1D):

For:

$$f = -\frac{L}{2}(k^2x^2 + p^2)$$

we would get (try it !):

$$e^{:-\frac{L}{2}(k^2x^2+p^2):}x = \sum_{n=0}^{\infty}\left(\frac{(-k^2L^2)^{2n}}{(2n)!}\cdot x + L\frac{(-k^2L^2)^{2n+1}}{(2n+1)!}\cdot p\right)$$

$$e^{:-\frac{L}{2}(k^2x^2+p^2):}p = \sum_{n=0}^{\infty}\left(\frac{(-k^2L^2)^{2n}}{(2n)!}\cdot p - k\frac{(-k^2L^2)^{2n+1}}{(2n+1)!}\cdot x\right)$$

More examples (1D):

For:

$$f = -\frac{L}{2}(k^2x^2 + p^2)$$

we would get (try it !):

$$\begin{aligned} e^{:f:}x &= x\ \cos(kL) + \frac{p}{k}\ \sin(kL) \\ e^{:f:}p &= -kx\ \sin(kL) + p\ \cos(kL) \end{aligned}$$

- **Thick, focusing quadrupole !**

Hamiltonians of some thick machine elements (3D)

dipole:

$$H = -\frac{-x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

quadrupole:

$$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

sextupole:

$$H = \frac{1}{6}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

octupole:

$$H = \frac{1}{4}k_3(x^4 - 6x^2y^2 + y^4) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

Remark:

In many cases the non-linear effects by the kinematic term is negligible and

$$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

is written as:

$$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2}$$

In 1D it reduces to previous example

Why all that ???

If we know the Hamiltonian of a machine elements (magnet) then:

$$e^{:H:}x_0 = x_1$$

$$e^{:H:}p_0 = p_1$$

This is also true for functions of x and p:

$$e^{:H:}f_0(x,p) = f_1(x,p)$$

- **The miracles:**
  - **Poisson brackets create symplectic maps**
  - **Exponential form $e^{:h:}$ is always symplectic**
  - **Better: the exponent is directly connected to the invariant of the transfer map !!**

Many machine elements

- **We can combine many machine elements f_n by applying one transformation after the other:**

$$e^{:h:} = e^{:f_1:}e^{:f_2:}....e^{:f_N:}$$

- **Not restricted to matrices, i.e. linear elements ...**
- **And arrive at a transformation for the full ring**
  → a one turn map
- **The one turn map is the exponential of the effective Hamiltonian:**

$$\mathcal{M}_{ring} = e^{:-C\mathcal{H}_{eff}:}$$

Why all that ???

concatenation very easy:

$$e^{:h:} = e^{:f:}e^{:g:} = e^{:f+g:}$$

when f **and** g **commute (i.e.** $[f, g] = [g, f] = 0$**)**

- **Otherwise formalism exist**

Concatenation

To combine:

$$e^{:h:} = e^{:f:}e^{:g:}$$

We can use the formula (Baker-Campbell-Hausdorff (BCH)):

$$\begin{aligned}
h = f \quad & + \; g \; + \; \tfrac{1}{2}[f,g] \; + \; \tfrac{1}{12}[f,[f,g]] \; + \; \tfrac{1}{12}[g,[g,f]] \\
& + \; \tfrac{1}{24}[f,[g,[g,f]]] \; - \; \tfrac{1}{720}[g,[g,[g,[g,f]]]] \\
& - \; \tfrac{1}{720}[f,[f,[f,[f,g]]]] \; + \; \tfrac{1}{360}[g,[f,[f,[f,g]]]] \; + \; \ldots
\end{aligned}$$

or :

$$\begin{aligned}
h = f \quad & + \; g \; + \; \tfrac{1}{2} :f: g \; + \; \tfrac{1}{12} :f:^2 g \; + \; \tfrac{1}{12} :g:^2 f \\
& + \; \tfrac{1}{24} :f::g:^2 f \; - \; \tfrac{1}{720} :g:^4 f \\
& - \; \tfrac{1}{720} :f:^4 g \; + \; \tfrac{1}{360} :g::f:^3 g \; + \; \ldots
\end{aligned}$$

Concatenation

To combine:

$$e^{:h:} = e^{:f:} e^{:g:}$$

if one of them (f or g) is small, can truncate the series and get a very useful formula. Assume g is small:

$$e^{:f:}\ e^{:g:} \ = \ e^{:h:} \ = \ \exp\left[: f + \left(\frac{:f:}{1 - e^{-:f:}}\right) g + \mathcal{O}(g^2) :\right]$$

Non-linear kicks

General thin lens kick $f(x)$:

$$e^{:\int_0^x f(x')dx':}$$

gives for the map:

$$\begin{aligned} x &= x_0 \\ p &= p_0 + f(x) \end{aligned}$$

Example: thin lens multipole of order n ($f(x) = a \cdot x^n$):

$$e^{:\frac{a}{n+1} \cdot x^{n+1}:}$$

gives for the map:

$$\begin{aligned} x &= x_0 \\ p &= p_0 + a x^n \end{aligned}$$

Extension: general monomials

Monomials in x and p of orders n and m $(x^n p^m)$

$$e^{:ax^n p^m:}$$

gives for the map (for $n \neq m$):

$$\begin{aligned} e^{:ax^n p^m:}x &= x \cdot [1 + a(n-m)x^{n-1}p^{m-1}]^{m/(m-n)} \\ e^{:ax^n p^m:}p &= p \cdot [1 + a(n-m)x^{n-1}p^{m-1}]^{n/(n-m)} \end{aligned}$$

gives for the map (for $n = m$):

$$\begin{aligned} e^{:ax^n p^n:}x &= x \cdot e^{-anx^{n-1}p^{n-1}} \\ e^{:ax^n p^n:}p &= p \cdot e^{anx^{n-1}p^{n-1}} \end{aligned}$$

From the Hamiltonian to the map

We have seen that given the Hamiltonian f of a machine element is known, the Lie operator becomes:

$$f \quad \rightarrow \quad :f:$$

the corresponding map is than:

$$e^{:f:} \qquad (e^{:-Lf:})$$

This map is always symplectic and we have (in 1D):

$$e^{:f:}x_0 = x_1$$

$$e^{:f:}p_0 = p_1$$

or using $Z = (x, p_x, y, p_y, ...)$ **(in 2D):**

$$e^{:f:}Z_0 = Z_1$$

From the map to the Hamiltonian

The other question ➡ assuming we do not have the Hamiltonian, but a matrix M (from somewhere):

$$\mathcal{M} \equiv \begin{pmatrix} \cos(\mu) + \alpha sin(\mu) & \beta \sin(\mu) \\ -\gamma \sin(\mu) & \cos(\mu) - \alpha sin(\mu) \end{pmatrix}$$

i.e.:

$$\mathcal{M} Z_0 = Z_1$$

how do we find the corresponding form for f ?

$$\mathcal{M} \quad \leftrightarrow \quad e^{:f:} \qquad (e^{:-\mu f:})$$

From the map to the Hamiltonian

For the linear matrix we know that f must be a quadratic form in (x, p, ...).

Any quadratic form can be written as:

$$f = -\frac{1}{2} Z^* F Z \qquad [\ = \ -\frac{1}{2}(a \cdot x^2 + b \cdot xp + c \cdot p^2)\ ]$$

where F is a symmetric, positive definite (why ?) matrix.

Then we can write (without proof, see e.g. Dragt):

$$: f : Z = SFZ$$

where S is the "symplecticity" matrix.

Therefore we get for the Lie transformation:

$$e^{:f:} Z \quad \leftrightarrow \quad e^{SF} Z$$

From the map to the Hamiltonian

Since we have $n = 2$, we get (using $Hamilton - Cayley$ theorem):

$$e^{SF} = \exp\begin{pmatrix} b & c \\ -a & -b \end{pmatrix} = a_0 + a_1 \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}$$

We now have to find a_0 and a_1 !

The eigenvalues of SF are:

$$\lambda_\pm = \pm\, i\sqrt{ac - b^2}$$

From the map to the Hamiltonian

This tells us for the coefficients the conditions:

$$e^{\lambda_+} = a_0 + a_1 \cdot \lambda_+$$

$$e^{\lambda_-} = a_0 + a_1 \cdot \lambda_-$$

and therefore:

$$a_0 = \cos(\sqrt{ac - b^2})$$

$$a_1 = \frac{\sin(\sqrt{ac - b^2})}{\sqrt{ac - b^2}}$$

and

$$e^{SF} \;=\; \cos(\sqrt{ac - b^2}) + \frac{\sin(\sqrt{ac - b^2})}{\sqrt{ac - b^2}} \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}$$

From the map to the Hamiltonian

For a general 2×2 matrix:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

we get by comparison:

$$\cos(\sqrt{ac - b^2}) = \frac{1}{2} tr(M)$$

and

$$\frac{a}{-m_{21}} = \frac{2b}{m_{11} - m_{22}} = \frac{c}{m_{12}} = \frac{\sqrt{ac - b^2}}{\sin(\sqrt{ac - b^2})}$$

for the quadratic form of f:

$$f = -\frac{1}{2}(a \cdot x^2 + b \cdot xp + c \cdot p^2)$$

From the map to the Hamiltonian

For the example of a drift:

$$\mathcal{M} \equiv \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

we find:

$$a = 0, \quad b = 0, \quad c = L$$

and for the generator:

$$f = -\frac{1}{2}(Lp^2)$$

From the map to the Hamiltonian

For the example of a thin quadrupole:

$$\mathcal{M} \equiv \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

we find:

$$a = \frac{1}{f}, \quad b = 0, \quad c = 0$$

and for the generator:

$$f = -\frac{1}{2f}(x^2)$$

A very important example ...

$$\mathcal{M} \equiv \begin{pmatrix} \cos\mu + \alpha sin(\mu) & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha sin(\mu) \end{pmatrix}$$

corresponds to:

$$e^{:h:} = e^{:f_2:} = e^{:-\mu \frac{1}{2}(\gamma x^2 + 2\alpha x p + \beta p^2):}$$

In this form f is: $-\mu \cdot$ **(Courant-Snyder invariant)**

$$e^{:h:} = e^{:f_2:} = e^{:-\mu\epsilon:}$$

- **We have standard ($e^{:f_2:}$) for the linear one-turn-matrix (a rotation)...**

A very important example ...

With our linear transformation to normalized variables:

$$\begin{pmatrix} \cos\mu + \alpha sin(\mu) & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha sin(\mu) \end{pmatrix} \Rightarrow \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}$$

therefore:

$$e^{:-\mu\frac{1}{2}(\gamma x^2 + 2\alpha x p + \beta p^2):} \quad \Rightarrow \quad e^{:-\mu\frac{1}{2}(x^2+p^2):}$$

and for a 3D linear system we have for f_2:

$$f_2 = -\frac{\mu_x}{2}(x^2 + p_x^2) - \frac{\mu_y}{2}(y^2 + p_y^2) - \frac{1}{2}\alpha_c \delta^2$$

or in action variables J:

$$f_2 = -\mu_x J_x - \mu_y J_y - \frac{1}{2}\alpha_c \delta^2$$

- **A standard ($e^{:f_2:}$) transformation in 3D**

First summary: Lie transforms and integrators

- **We have powerful tools to decribe non-linear elements**
- **They are always symplectic !**
- **Can be combined to form a ring (and therefore a non-linear One-Turn-Map)**
- **Tools and programs are available for their manipulation and computation**
- **How do we analyse the maps ? Guess: Normal Forms**

Normal forms non-linear case

Normal form transformations can be generalized for non-linear maps (i.e. not matrices). If $\mathcal{M}$ is our usual one-turn-map, we try to find a transformation:

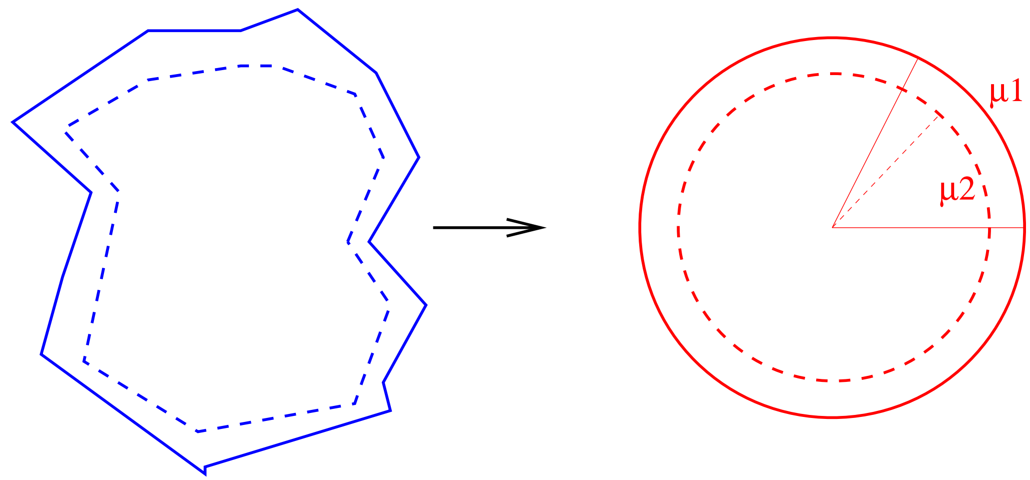
$$\mathcal{N} = \mathcal{A}\mathcal{M}\mathcal{A}^{-1}$$

- **where $\mathcal{N}$ is a simple form (like the rotation we had before)**

Of course we now do not have matrices, we use a Lie transform F to decribe the transform $\mathcal{A}$:

$$\mathcal{N} = e^{-:h:} = \mathcal{A}\mathcal{M}\mathcal{A}^{-1} = e^{:F:}\mathcal{M}e^{-:F:}$$

Normal forms - non-linear case



- **More complicated transformation F required**
- **Transform to coordinates where map is just a rotation**
- **In general better done in action angle variables: J, Ψ**
- **Rotation angle may be amplitude dependent: $\mu \rightarrow \mu(J)$**

Normal forms - non-linear case

The canonical transformation A:

$$\mathcal{N} = \mathcal{A}\mathcal{M}\mathcal{A}^{-1} \quad \Rightarrow \quad A = e^{:F:}$$

should be the transformation to produce our simple form

- $h(J_x, \Psi_x, J_y, \Psi_y, z, \delta) \quad \Rightarrow \quad h(J_x, J_y, \delta) = h_{eff}(J_x, J_y, \delta)$
- **Should work for any kind of local perturbation**
- **Formalism and software tools exist to find F (see e.g. Chao[1)] or E.Forest, M. Berz, J. Irwin, SSC-166)**
- **Once we know $h_{eff}(J_x, J_y, \delta)$ we can derive everything !**

[1)] A. Chao, Lecture Notes on Topics in Accelerator Physics, 2001

Normal forms - non-linear case

Once we can write the map as (now example in 3D):

$$\mathcal{N} = e^{-:h_{eff}(J_x,J_y,\delta):}$$

where h_{eff} depends only on $J_x, J_x,$ and δ, then we have the tunes:

$$Q_x(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_x}$$

$$Q_y(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_y}$$

and the change of path length:

$$\Delta z = -\frac{\partial h_{eff}}{\partial \delta}$$

Particles with different J_x, J_y and δ have different tunes:

→ **Dependence on J is amplitude detuning, dependence on δ are the chromaticities !**

How does h_{eff} look like ?

The effective Hamiltonian can be written (here to 3rd order) (see e.g. E. Forest, M. Berz, J. Irwin, SSC-166) as:

$$
\begin{aligned}
h_{eff} \;=\; & + \quad \mu_x J_x + \mu_y J_y + \frac{1}{2}\alpha_c \delta^2 \\
& + \quad c_{x1} J_x \delta + c_{y1} J_y \delta + c_3 \delta^3 \\
& + \quad c_{xx} J_x^2 + c_{xy} J_x J_y + c_{yy} J_y^2 + c_{x2} J_x \delta^2 + c_{y2} J_y \delta^2 + c_4 \delta^4
\end{aligned}
$$

and then:

$$Q_x(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_x} = \frac{1}{2\pi}\left(\mu_x + 2c_{xx}J_x + c_{xy}J_y + c_{x1}\delta + c_{x2}\delta^2\right)$$

$$Q_y(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_y} = \frac{1}{2\pi}\left(\mu_y + 2c_{yy}J_y + c_{xy}J_x + c_{y1}\delta + c_{y2}\delta^2\right)$$

$$\Delta z \;=\; -\frac{\partial h_{eff}}{\partial \delta} \;=\; \alpha_c \delta + 3c_3\delta^2 + 4c_4\delta^3 + c_{x1}J_x + c_{y1}J_y + 2c_{x2}J_x\delta + 2c_{y2}J_y\delta$$

What's the meaning of it ?

- μ_x, μ_y: **tunes**
- $\frac{1}{2}\alpha_c, c_3, c_4$: **linear and non-linear "momentum compaction"**
- c_{x1}, c_{y1}: **first order chromaticities**
- c_{x2}, c_{y2}: **second order chromaticities**
- c_{xx}, c_{xy}, c_{yy}: **detuning with amplitude**

Example: sextupole

A linear map followed by a single (weak) sextupole:

$$\mathcal{M} = e^{-:\frac{\mu}{2}x^2+p^2 \ + \ \frac{1}{2}\alpha_c\delta^2:} \ e^{:f_3:} = e^{-:\mu J_x \ + \ \frac{1}{2}\alpha_c\delta^2:} \ e^{:kx^3 \ + \ \frac{p^2}{2(1+\delta)}:}$$

we get for h_{eff} **(see e.g. [AC1, EF]):**

$$h_{eff} = \mu_x J_x + \frac{1}{2}\alpha_c\delta^2 - kD^3\delta^3 - 3k\beta_x J_x D\delta$$

or in 3D:

$$h_{eff} = \mu_x J_x + \mu_y J_y + \frac{1}{2}\alpha_c\delta^2 - kD^3\delta^3 - 3k\beta_x J_x D\delta + 3k\beta_y J_y D\delta$$

Example: sextupole

When we have h_{eff} in 3D we obtain:

$$Q_x(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_x} = \frac{1}{2\pi}(\mu_x - 3k\beta_x D\delta)$$

$$Q_y(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_y} = \frac{1}{2\pi}(\mu_y + 3k\beta_y D\delta)$$

and the change of path length:

$$\Delta s = -\frac{\partial h_{eff}}{\partial \delta} = \alpha_c \delta - 3kD^3\delta^2 - 3kD(\beta_x J_x - \beta_y J_y)$$

Normal forms - non-linear case

Assume a linear rotation (as always) followed by an octupole, the Hamiltonian is (1D to keep it simple):

$$\mathcal{H} = \frac{\mu}{2}(x^2 + p^2) + k_3 \cdot \frac{x^4}{4} \qquad (p = p_x)$$

- **With the Hamilton's equation leading to:**

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \mu p$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -\mu x - k_3 \cdot x^3$$

Normal forms - non-linear case

The map, written in Lie representation is:

$$\mathcal{M} = e^{-\frac{\mu}{2}:x^2+p^2:}\ e^{:k_3\cdot\frac{x^4}{4}:} = Re^{:k_3\cdot\frac{x^4}{4}:}$$

we transform by applying:

$$\mathcal{N} \ = \ \mathcal{A}\mathcal{M}\mathcal{A}^{-1} \ = \ e^{:F:}\ Re^{:k_3\cdot\frac{x^4}{4}:}\ e^{-:F:} = RR^{-1}\ e^{:F:}\ Re^{:k_3\cdot\frac{x^4}{4}:}\ e^{-:F:}$$

$$= Re^{:R^{-1}F+k_3\cdot\frac{x^4}{4}-F:+O(\epsilon^2)} = Re^{:(R^{-1}-1)F+k_3\cdot\frac{x^4}{4}:+O(\epsilon^2)}$$

we have now to choose F to simplify the expression:

$$= (R^{-1}-1)F + k_3\cdot\frac{x^4}{4}$$

and get [EF1, AW]:

$$F = -\frac{1}{64}\{-5x^4+3p^4+6x^2p^2+x^3p\cdot(8cot(\mu)+4cot(2\mu))+xp^3(8cot(\mu)-4cot(2\mu))\}$$

Normal forms - non-linear case

We go back to x and p coordinates and with:

$$J = (x^2 + p^2)/2$$

we can write the map:

$$M = e^{-:F:} \quad e^{:-\mu J + \frac{3}{8} k_3 \cdot J^2:} \quad e^{:F:}$$

the term $\frac{3}{8} k_3 \cdot J^2$ produces the tune shift with amplitude we know for an octupole ($\cdot < \beta^2 >$ in real space)

Note: the normalized map (our most simple map):

$$R = e^{:-\mu J + \frac{3}{8} k_3 \cdot J^2:}$$

is again a rotation in phase space, but the rotation angle now depends on the amplitude J

For the tune shift: octupole (1D)

When we have h_{eff} in 1D for a single octupole (see before):

$$h_{eff} = -\mu J + \frac{3}{8} k_3 \cdot J^2$$

$$Q_x(J_x, J_y) = \frac{1}{2\pi} \frac{\partial h_{eff}}{\partial J_x} = -\frac{1}{2\pi} \mu_x + \frac{3}{8 \cdot 2\pi} k_3 J_x$$

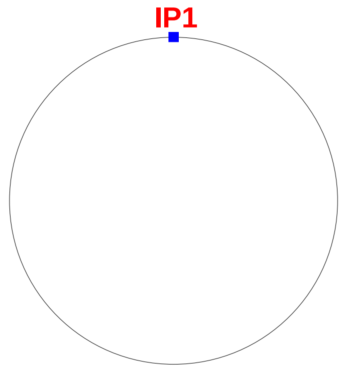
and with normalization in real space (i.e. $\beta \neq 1$):

$$\Delta Q_x(J_x, J_y) = \frac{3}{8 \cdot 2\pi} k_3 < \beta^2 > J_x$$

Example: $\beta = 300$**m,** k_3 **= 0.01**

$$\Delta Q_x(J_x, J_y) = 53.7 \cdot J_x$$

A real life example: beam-beam interaction



- **Localized distortion, very strong non-linearity**
- **Standard perturbation theory not appropriate**

Effect on invariants - start with single IP

- **Look for invariant h**
- **Linear transfer $e^{:f_2:}$ and beam-beam interaction $e^{:F:}$, i.e.:**

$$e^{:f_2:} \; \cdot \; e^{:F:} \; = \; e^{:h:}$$

with (see before):

$$f_2 \; = \; -\frac{\mu}{2}(\frac{x^2}{\beta} + \beta p_x^2)$$

and (see before):

$$F = \int_0^x dx' f(x')$$

Effect on invariants

For a Gaussian beam we have for f(x) (see lecture on beam-beam effects):

$$f(x) = \frac{2}{x}\left(1 - e^{\frac{-x^2}{2\sigma^2}}\right)$$

as usual go to action angle variables Ψ, J:

$$x = \sqrt{2J\beta}\sin\Psi, \qquad p = \sqrt{\frac{2J}{\beta}}\cos\Psi$$

and write F(x) as Fourier series:

$$F(x) = \sum_{n=-\infty}^{\infty} c_n(J) e^{in\Psi}$$

We need:

REMEMBER: with this transform f_2 becomes very simple:

$$f_2 = -\mu J$$

and useful properties of Lie operators (any textbook[2]):

$$: f_2 : g(J) = 0, \qquad : f_2 : e^{in\Psi} = in\mu e^{in\Psi}, \qquad g(: f_2 :)e^{in\Psi} = g(in\mu)e^{in\Psi}$$

and the formula (any textbook[2]):

$$e^{:f_2:}\, e^{:F:} \;=\; e^{:h:} \;=\; \exp\left[: f_2 + \left(\frac{: f_2 :}{1 - e^{-:f_2:}}\right) F + \mathcal{O}(F^2) :\right]$$

[2] E. Forest, "Beam Dynamics, A New Attitude and Framework", 1998

Single IP

gives immediately for h:

$$h = -\mu J + \sum_n c_n(J) \frac{in\mu}{1 - e^{-in\mu}} e^{in\Psi}$$

$$h = -\mu J + \sum_n c_n(J) \frac{n\mu}{2\sin(\frac{n\mu}{2})} e^{(in\Psi + i\frac{n\mu}{2})}$$

away from resonance normal form transformation gives:

$$h_n \;=\; -\,\mu J + c_0(J) \;= const.$$

$$\left[homework: \qquad \frac{dc_0(J)}{dJ} \right]$$

Single IP - analysis of h

$$h = -\mu J + \sum_n c_n(J) \frac{n\mu}{2\sin(\frac{n\mu}{2})} e^{(in\Psi + i\frac{n\mu}{2})}$$

On resonance:

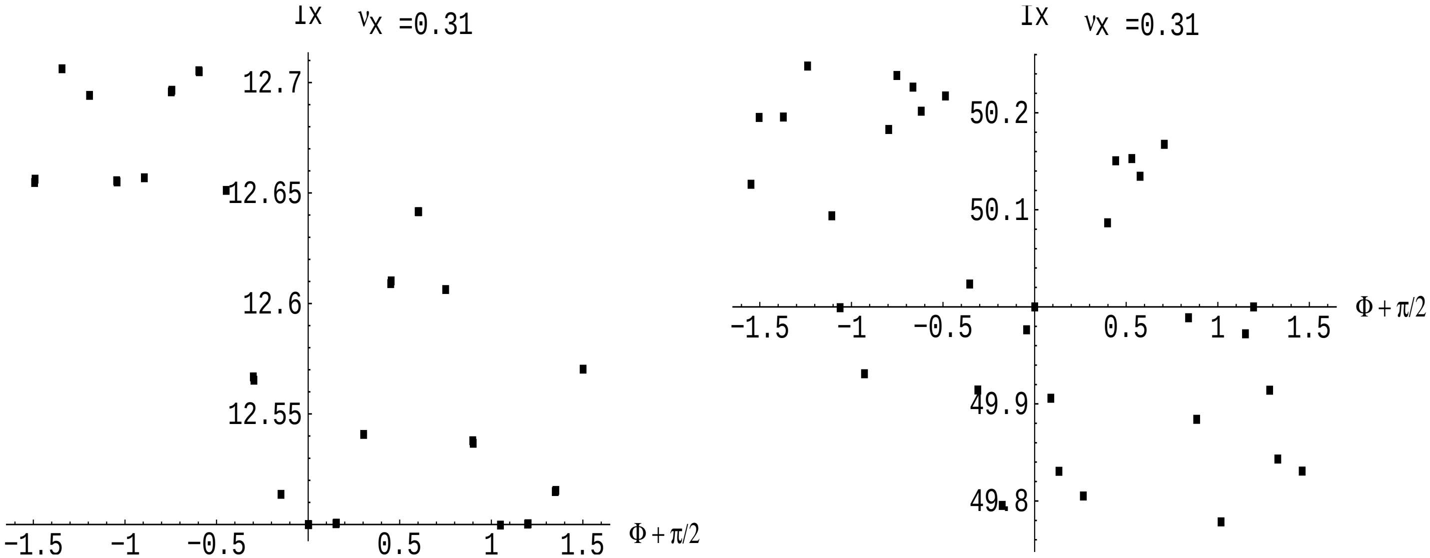
$$Q = \frac{p}{n} = \frac{\mu}{2\pi}$$

with $c_n \neq 0$**:**

$$\sin(\frac{n\pi p}{n}) = \sin(p\pi) \equiv 0 \quad \forall \quad integer \quad p$$

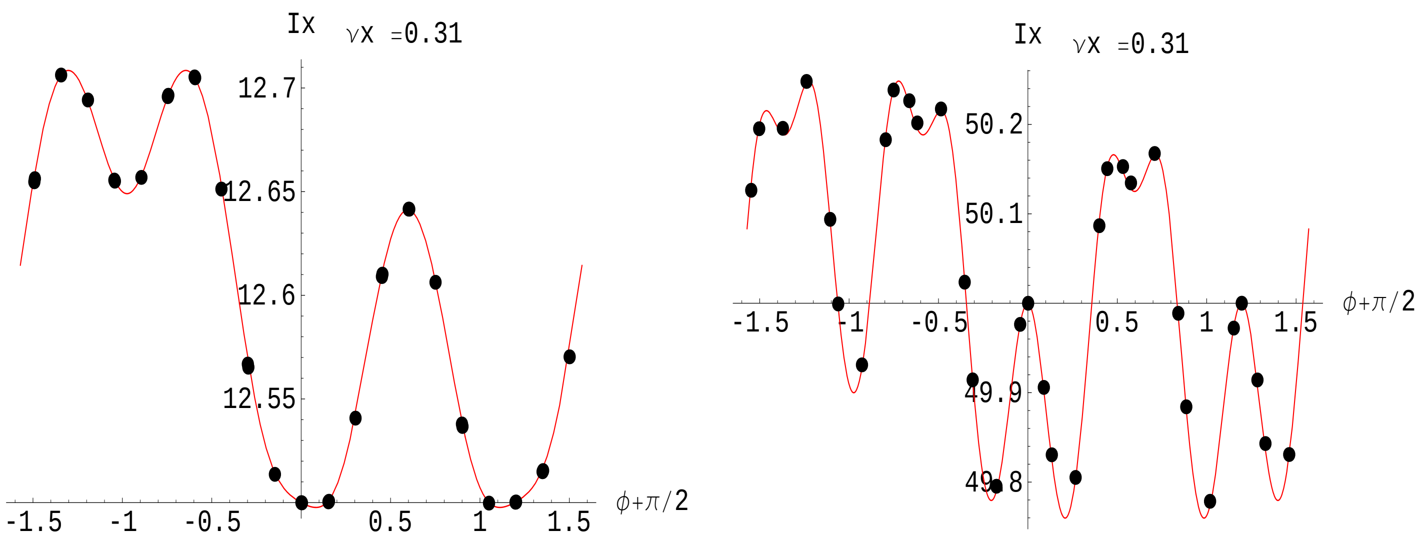
and h **diverges, find automatically all resonance conditions**

Invariant from tracking: one IP



- Shown for $5\sigma_x$ and $10\sigma_x$

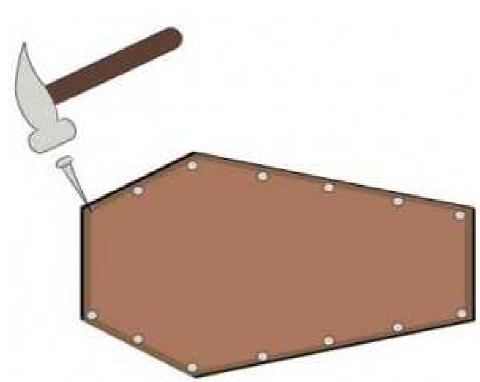
Invariant versus tracking: one IP



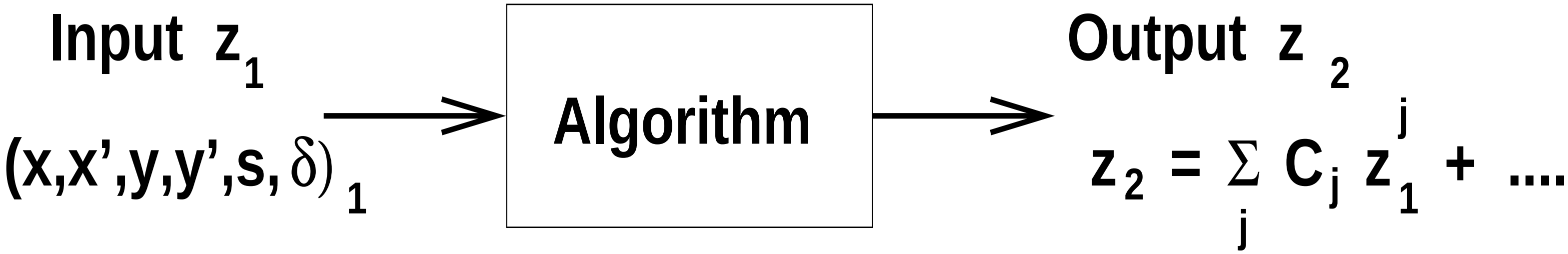
- Shown for $5\sigma_x$ and $10\sigma_x$

Truncated Power Series Algebra (TPSA)

- **Tracking particles is very reliable method**
- **Simulation can produce maps for complicated configurations**
- **How can we analyse the map produced by a tracking code ?**
- **Now we put the final nail into the coffin of any other approach ...**



Truncated Power Series Algebra

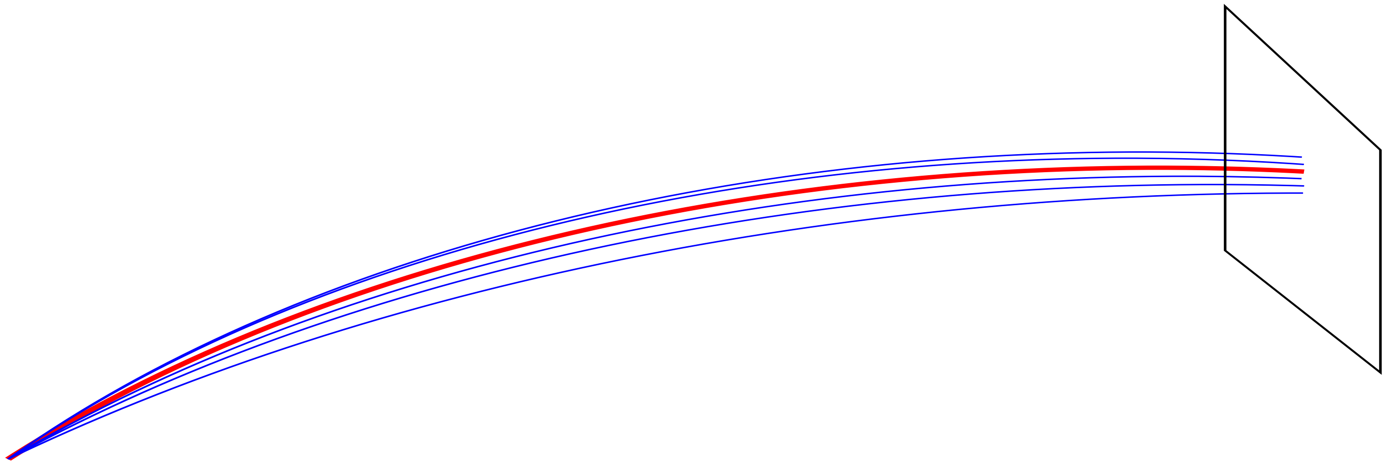


- The tracking of a complicated system relates the output numerically to the input
- Could we imagine something that relates the output algebraically to the input ?
- For example a Taylor series ?

$$z_2 = \sum C_j z_1^j = \sum f^{(n)} z_1^j$$

Why are Taylor series useful ?

Let us study the paraxial behaviour:



- **Red line is the ideal orbit**
- **Blue lines are small deviations**
- **If we understand how small deviations behave, we understand the system much better**

Why are Taylor series useful ?

Now remember the definition of the Taylor series:

$$f(a+\Delta x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} \Delta x^n$$

$$f(a+\Delta x) = f(a) + \frac{f'(a)}{1!}\Delta x^1 + \frac{f''(a)}{2!}\Delta x^2 + \frac{f'''(a)}{3!}\Delta x^3 + ...$$

- **The coefficients determine the behaviour of small deviations Δx from the ideal orbit x**

- **The Taylor expansion does a paraxial analysis of the system**

Why are Taylor series useful ?

If the function $f(x)$ is represented by a Taylor series:

$$f(a+\Delta x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} \Delta x^n$$

if it is truncated to the m-th order:

$$f(a+\Delta x) = f(a) + \sum_{n=1}^{m} \frac{f^{(n)}(a)}{n!} \Delta x^n$$

- **There is a equivalence between the function $f(x)$ and the vector** $(\ f(a),\ \ f'(a),\ \ f''(a),\ \ ..., f^{(m)}(a))$
- **This vector is a Truncated Power Series Algebra (TPSA) representation of $f(x)$ around a**
- **How to get these coefficients without extra work ?**

Numerical differentiation

The problem getting the derivatives $f^{(n)}(a)$ of $f(x)$ at a:

$$f'(a) = \frac{f(a+\epsilon) - f(a)}{\epsilon}$$

- **Need to subtract almost equal numbers and divide by small number.**
- **For higher orders f'', f'''.., accuracy hopeless !**
- **We can use Differential Algebra (DA) (M. Berz, 1988 and [MB])**

Differential Algebra

1. **Define a pair** (q_0, q_1)**, with** q_0, q_1 **real numbers**

Differential Algebra

1. **Define a pair (q_0, q_1), with q_0, q_1 real numbers**
2. **Define operations on a pair like:**

$$(q_0, q_1) + (r_0, r_1) = (q_0 + r_0, q_1 + r_1)$$

$$c \cdot (q_0, q_1) = (c \cdot q_0, c \cdot q_1)$$

$$(q_0, q_1) \cdot (r_0, r_1) = (q_0 \cdot r_0,\ q_0 \cdot r_1 + q_1 \cdot r_0)$$

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$$(q_0, q_1) \cdot (r_0, r_1) = (q_0 \cdot r_0,\; q_0 \cdot r_1 + q_1 \cdot r_0)$$

3. **And some ordering:**

$$(q_0, q_1) < (r_0, r_1) \quad \text{if} \quad q_0 < r_0 \quad \text{or} \quad (q_0 = r_0 \quad \text{and} \quad q_1 < r_1)$$

$$(q_0, q_1) > (r_0, r_1) \quad \text{if} \quad q_0 > r_0 \quad \text{or} \quad (q_0 = r_0 \quad \text{and} \quad q_1 > r_1)$$

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4. **This implies something strange:**

$$(0,0) < (0,1) < (r,0) \ \text{(for any pos. } r\text{)}$$

$$(0,1) \cdot (0,1) \quad = \quad (0,0) \quad \Rightarrow \quad (0,1) \quad = \quad \sqrt{(0,0)}!!$$

Differential Algebra

This means that (0,1) is between 0 and ANY real number

→ infinitely small !!!

We call this therefore "differential unit" $d = (0, 1) = \delta$**.**

Of course $(q, 0)$ **is just the real number** q **and we define "real part" and "differential part" (a bit like complex numbers..):**

$$q_0 = \mathcal{R}(q_0, q_1) \quad \text{and} \quad q_1 = \mathcal{D}(q_0, q_1)$$

With our rules we can further see that:

$$(1, 0) \ \cdot \ (q_0, q_1) = (q_0, q_1)$$

$$(q_0, q_1)^{\ -1} \ = \ \left(\frac{1}{q_0}, -\frac{q_1}{q_0^2}\right)$$

Differential Algebra

Of course can let a function f act on the pair (or vector) using our rules.

For example:

$$f(x) \to f(x, 0)$$

acts like the function f on the real variable x:

$$f(x) = \mathcal{R}[f(x, 0)]$$

What about the differential part $\mathcal{D}$?

Differential Algebra

For a function $f(x)$ without proof:

$$\mathcal{D}[f(x+d)] = \mathcal{D}[f((x,0)+(0,1))] = \mathcal{D}[f(x,1)] = f'(x)$$

An example instead:

$$f(x) = x^2 + \frac{1}{x}$$

then using school calculus:

$$f'(x) = 2x - \frac{1}{x^2}$$

For $x = 2$ we get then:

$$f(2) = \frac{9}{2}, f'(2) = \frac{15}{4}$$

Differential Algebra

For x in:

$$f(x) = x^2 + \frac{1}{x}$$

we substitute: x $\rightarrow$ (x, 1) = (2, 1) and use our rules:

$$\begin{aligned} f[(2,1)] &= (2,1)^2 + (2,1)^{-1} \\ &= (4,4) + (\frac{1}{2}, -\frac{1}{4}) \\ &= (\frac{9}{2},\ \frac{15}{4}) = (f(2),\ f'(2)) \quad !!! \end{aligned}$$

The computation of derivatives becomes an algebraic problem, no need for small numbers, exact !

Differential Algebra - higher orders

1. **The pair $(q_0, 1)$, becomes a vector of order N:**

   $(q_0, 1) \;\rightarrow\; (q_0, 1, \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}) \qquad \delta = (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \ldots)$

2. $(q_0, q_1, q_2, \ldots q_N) + (r_0, r_1, r_2, \ldots r_N) = (s_0, s_1, s_2, \ldots s_N)$

   with: $s_i = q_i + r_i$

3. $c \cdot (q_0, q_1, q_2, \ldots q_N) = (c \cdot q_0, c \cdot q_1, c \cdot q_2, \ldots c \cdot q_N)$

4. $(q_0, q_1, q_2, \ldots q_N) \cdot (r_0, r_1, r_2, \ldots r_N) = (s_0, s_1, s_2, \ldots s_N)$

   with:

$$s_i = \sum_{k=0}^{i} \frac{i!}{k!(i-k)!} q_k r_{i-k}$$

Differential Algebra

If we had started with:

$$x = (a, 1, 0, 0, 0...)$$

we would get:

$$f(x) = (\ f(a),\ \ f'(a),\ \ f''(a),\ \ f'''(a), ...\ f^{(n)}(a)\ \ )$$

can be extended to more variables x, y:

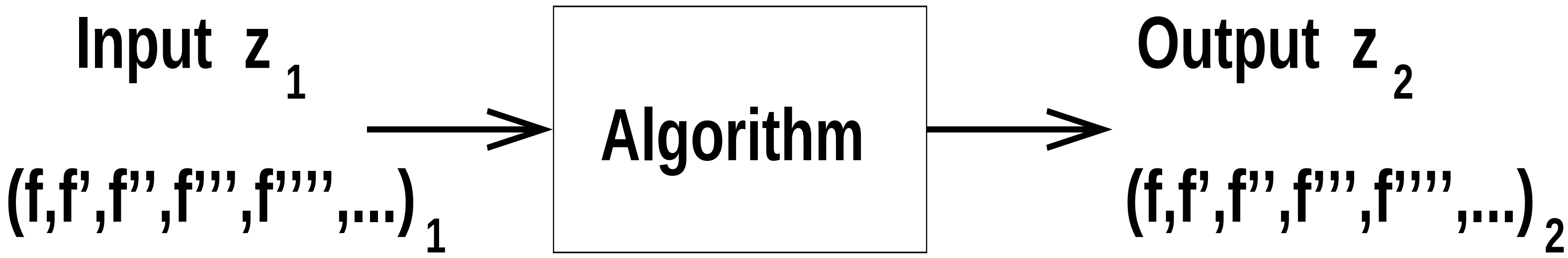
$$x = (a, 1, 0, 0, 0...) \qquad dx = (0, 1, 0, 0, 0, ..)$$

$$y = (b, 0, 1, 0, 0...) \qquad dy = (0, 0, 1, 0, 0, ..)$$

and get (with more complicated multiplication rules):

$$f((x + dx), y + dy)) = \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, ...\right)(x, y)$$

What is the use of that:



Can extract a truncated Taylor map of a beam line or ring by pushing the identity map $f(x) = (a, 1, 0, 0, 0...)$ through the algorithm as if it is a vector in phase space !
The maps are provided with the desired accuracy and to any order.

What is the use of that:

- "Algorithm" can be a mathematical function
- "Algorithm" can be a complex computer code
- Easy using polymorphism of modern languages (see example)
- Normal form analysis on Taylor series is much easier !!

→ We get a Taylor map for a computer code !!!

What is the use of that:

- **Demonstrate with simple examples (FORTRAN 95):**
  - **First show the concept**
  - **Simple FODO cell**
  - **Normal form analysis of the FODO cell with octupoles**
- **All examples and all source code in:**

  Website: http://cern.ch/Werner.Herr/CAS2013/DA

  Small DA package provided by E. Forest

Look at this small example:

```
PROGRAM DATEST1
use my_own_da
real(8) x,z, dx
my_order=3
dx=0.0
x=3.141592653/6.0 + dx
call track(x, z)
call print(z,6)
END PROGRAM DATEST1

SUBROUTINE TRACK(a, b)
use my_own_da
real(8) a,b
b = sin(a)
END SUBROUTINE TRACK
```

```
PROGRAM DATEST2
use my_own_da
type(my_taylor) x,z, dx
my_order=3
dx=1.0.mono.1     ! this is our (0,1)
x=3.141592653/6.0 + dx
call track(x, z)
call print(z,6)
END PROGRAM DATEST2

SUBROUTINE TRACK(a, b)
use my_own_da
type(my_taylor) a,b
b = sin(a)
END SUBROUTINE TRACK
```

[2] Courtesy E. Forest for the small DA package used here ...

Look at the results:

(0,0) 0.500000000000E+00

(0,0) 0.500000000000E+00
(1,0) 0.866025403784E+00
(0,1) 0.000000000000E+00
(2,0) -0.250000000000E+00
(0,2) 0.000000000000E+00
(1,1) 0.000000000000E+00
(3,0) -0.144337567297E+00
(0,3) 0.000000000000E+00
(2,1) 0.000000000000E+00
(1,2) 0.000000000000E+00

We have $sin(\frac{\pi}{6}) \; = \; 0.5$ **all right, but what is the rest ??**

Look at the results:

(0,0) 0.500000000000E+00

(0,0) 0.500000000000E+00
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(3,0) -0.144337567297E+00
(0,3) 0.000000000000E+00
(2,1) 0.000000000000E+00
(1,2) 0.000000000000E+00

$$sin(\frac{\pi}{6}+\Delta x) = sin(\frac{\pi}{6}) + cos(\frac{\pi}{6})\Delta x^1 - \frac{1}{2}sin(\frac{\pi}{6})\Delta x^2 - \frac{1}{6}cos(\frac{\pi}{6})\Delta x^3$$

What is the use of that:

- **We have used a simple algorithm here (sin) but it can be anything very complex**
- **We can compute nonlinear maps as a Taylor expansion of anything the program computes**
- **Simply by:**
  - **Replacing regular (e.g. REAL) types by TPSA types (my_taylor) i.e. variables x, p are automatically replaced by $(x, 1, 0, ..)$ and $(p, 0, 1, 0, ..)$ etc.**
  - **Operators and functions ($+, -, *, =, ..., exp, sin$, ...) automatically overloaded, i.e. behave according to new type**

What is the use of that:

Assume the $Algorithm$ **describes one turn, then:**

- **Normal tracking:**
  - $X_n = (x, p_x, y, p_y, s, \delta)_n \quad \rightarrow \quad X_{n+1} = (x, p_x, y, p_y, s, \delta)_{n+1}$
  - **Coordinates after one completed turn**
- **TPSA tracking:**
  - $X_n = (x, p_x, y, p_y, s, \delta)_n \quad \rightarrow \quad X_{n+1} = \sum C_j X_n^j$
  - **Taylor expansion after one completed turn**
  - **Automatically all** X_{n+1} **where it converges**
  - **The** C_j **contain useful information about behaviour**
  - **Taylor map directly used for normal form analysis**

Another example:

- **Track through a FODO lattice:**

  QF - DRIFT - QD

  Integrate 100 steps in the quadrupoles

  Now we use three variables:

  $$\mathbf{x},\ \mathbf{p},\ \Delta\mathbf{p} = (\mathbf{z}(1), \mathbf{z}(2), \mathbf{z}(3))$$

[2] Courtesy E. Forest for the small DA package used here ...

Another example:

```
program fodo1
use my_own_da
use my_analysis
type(my_taylor) z(3)
type(normalform) NORMAL
type(my_map) M,id

real(dp) L,DL,k1,k3,fix(3)
integer i,nstep


my_order=4  ! maximum order 4
fix=0.0 ! fixed point
id=1
z=fix+id

LC=62.5     ! half cell length
L=3.0     ! quadrupole length
nstep=100
DL=L/nstep
k1=0.003     ! quadrupole strength
```

```
do i=1,nstep     ! track through quadrupole
z(1)=z(1)+DL/2*z(2)
z(2)=z(2)-k1*DL*z(1)/(1 + z(3))
z(1)=z(1)+DL/2*z(2)
enddo
z(1)=z(1)+LC*z(2)        ! drift of half cell
length

do i=1,nstep     ! track through quadrupole
z(1)=z(1)+DL/2*z(2)
z(2)=z(2)-k1*DL*z(1)/(1 + z(3))
z(1)=z(1)+DL/2*z(2)
enddo

z(1)=z(1)+LC*z(2)        ! drift of half cell
length

call print(z(1),6)
call print(z(2),6)
M=z
NORMAL=M
write(6,*) normal%tune, normal%dtune_da
end program fodo1
```

[2] Courtesy E. Forest for the small DA package used here ...

The result is:

(0,0,0) 0.9442511679729E-01
(0,0,1) -0.9729519276183E-01

(1,0,0) 0.6972061935061E-01
(0,1,0) 0.1677727932585E+03
(1,0,1) 0.1266775134236E+01
(0,1,1)-0.3643444875882E+02
(1,0,2)-0.1603248617779E+01
(0,1,2) 0.3609522079691E+02
(1,0,3) 0.1939697138318E+01
(0,1,3)-0.3575511053483E+02

(1,0,0)-0.5300319873866E-02
(0,1,0) 0.1588490329398E+01
(1,0,1) 0.1060055415702E-01
(0,1,1)-0.5832024543075E+00
(1,0,2)-0.1590066005419E-01
(0,1,2) 0.5779004431627E+00
(1,0,3) 0.2120059477024E-01
(0,1,3)-0.5725843143370E+00

Only linear elements in the Taylor expansion, the result for the matrix per cell:

$$\Delta x_f = 0.06972 \Delta x_i + 167.77 \Delta p_i$$

$$\Delta p_f = -0.00530 \Delta x_i + 1.5885 \Delta p_i$$

The output from the normal form analysis are (per cell !):
Tune = 0.094425
Chromaticity= -0.097295

2) Courtesy E. Forest for the small DA package used here ...

Modified previous example (with one octupole):

```
program fodo3
use my_own_da
use my_analysis
type(my_taylor) z(3)
type(normalform) NORMAL
type(my_map) M,id

real(dp) L,DL,k1,k3,fix(3)
integer i,nstep


my_order=4  ! maximum order 4
fix=0.0 ! fixed point
id=1
z=fix+id

LC=62.5     ! half cell length
L=3.0     ! quadrupole length
nstep=100
DL=L/nstep
k1=0.003     ! quadrupole strength
k3=0.01     ! octupole strength
```

```
do i=1,nstep     ! track through quadrupole
z(1)=z(1)+DL/2*z(2)
z(2)=z(2)-k1*DL*z(1)/(1 + z(3))
z(1)=z(1)+DL/2*z(2)
enddo
z(2)=z(2)-k3*z(1)**3/(1 + z(3))  ! octupole
kick !!!
z(1)=z(1)+LC*z(2)       !  drift of half cell
length

do i=1,nstep     ! track through quadrupole
z(1)=z(1)+DL/2*z(2)
z(2)=z(2)-k1*DL*z(1)/(1 + z(3))
z(1)=z(1)+DL/2*z(2)
enddo

z(1)=z(1)+LC*z(2)       !  drift of half cell
length

call print(z(1),6)
call print(z(2),6)
M=z
NORMAL=M
write(6,*) normal%tune, normal%dtune_da
end program fodo3
```

The result is:

(0,0,0) 0.9442511679729E-01
(0,0,1) -0.9729519276183E-01

(2,0,0) 0.5374370086899E+02
(0,2,0) 0.5374370086899E+02
(0,0,2) 0.1018391758451E+00
(2,0,1) 0.2035776281196E+02
....

(1,0,0) 0.6972061935061E-01
(0,1,0) 0.1677727932585E+03
(1,0,1) 0.1266775134236E+01
(0,1,1)-0.3643444875882E+02
(3,0,0)-0.1586519461687E+01
(2,1,0)-0.1440953324752E+02
(1,2,0)-0.4362477179879E+02

....
(1,0,0)-0.5300319873866E-02
(0,1,0) 0.1588490329398E+01
(1,0,1) 0.1060055415702E-01
(0,1,1)-0.5832024543075E+00
(3,0,0)-0.1519218878892E-01

Now non-linear elements in the Taylor expansion,
The output from the normal form analysis are (per cell !):
Tune = 0.094425
Chromaticity= -0.097295
The detuning with amplitude is 53.74 !

[2] Courtesy E. Forest for the small DA package used here ...

Modified previous example (with octupole):

Remember the normal form transformation:

$$\mathcal{A}\mathcal{M}\mathcal{A}^{-1} = \mathcal{R}$$

The type normalform in the demonstration package also contains the maps $\mathcal{A}$ and $\mathcal{R}$!

j2=(x2+p**2)*NORMAL%A**(-1)**

(remember: x2+p**2 is the tilted ellipse ....**
Can get the optical functions out because

- **β: coefficient of p**2 of invariant j2**
- **α: coefficient of x*p of invariant j2**
- **γ: coefficient of x**2 of invariant j2**

Modified previous example (with octupole):

In our code use like :

β = **j2.sub.beta**

α = **0.5*j2.sub.twoalpha**

γ = **j2.sub.gamma**

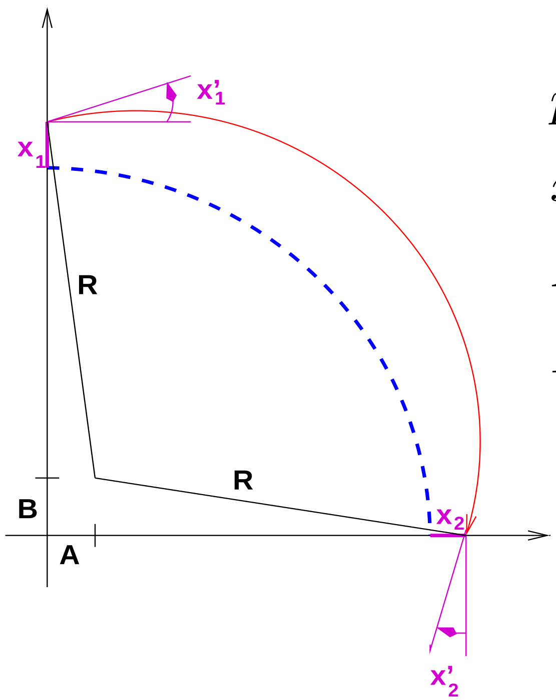
we obtain (here at the end of the cell):

beta, alpha, gamma

300.080714 -1.358246 9.480224E-003

This was trivial - now a (normally) hard one

The exact map:



$$
\begin{aligned}
p_2 = &\quad \sin(x_2') = -\tfrac{B}{R} \\
x_2 = &\quad A - R(1 - \cos(x_2')) = A - R(1 - \sqrt{1 - p_2^2}) \\
A = &\quad R \cdot p_1 = R \cdot \sin(x_1') \\
B = &\quad R(1 - \cos(x_1')) + x_1 = R(1 - \sqrt{1 - p_1^2}) + x_1
\end{aligned}
$$

A 90^0 bending magnet ..

How to apply Differential Algebra here ...

- **Start with initial coordinates in DA style:**

$$x_1 = (0, 1, 0, ...)$$

$$p_1 = (0, 0, 1, ...) \qquad \textbf{and have:}$$

$$A = (0, 0, R, 0, ...)$$

$$B = (0, 1, 0, 0, 0, R, 0, ...)$$

- **After pushing them through the algorithm:**

→ $x_2 = (0, 0, R, -\frac{1}{R}, 0, 0, 0...) = (0, \frac{\partial x_2}{\partial x_1}, \frac{\partial x_2}{\partial p_1}, \frac{\partial^2 x_2}{\partial x_1^2}, \frac{\partial^2 x_2}{\partial x_1 \partial p_1}, ...)$

→ $p_2 = (0, -\frac{1}{R}, 0, 0, 0, -1, 0...) = (0, \frac{\partial p_2}{\partial x_1}, \frac{\partial p_2}{\partial p_1}, \frac{\partial^2 p_2}{\partial x_1^2}, \frac{\partial^2 p_2}{\partial x_1 \partial p_1}, ...)$

- **Automatically evaluates all non-linearities to any desired order ..**

How to apply Differential Algebra here ...

- **Start with initial coordinates in DA style:**

$$x_1 = (0, 1, 0, ...)$$

$$p_1 = (0, 0, 1, ...) \qquad \textbf{and have:}$$

$$A = (0, 0, R, 0, ...)$$

$$B = (0, 1, 0, 0, 0, R, 0, ...)$$

- **After pushing them through the algorithm:**

$$\Rightarrow \quad x_2 = (0, 0, R, -\tfrac{1}{R}, 0, 0, 0...) = (0, \frac{\partial x_2}{\partial x_1}, \frac{\partial x_2}{\partial p_1}, \frac{\partial^2 x_2}{\partial x_1^2}, \frac{\partial^2 x_2}{\partial x_1 \partial p_1}, ...)$$

$$\Rightarrow \quad p_2 = (0, -\tfrac{1}{R}, 0, 0, 0, -1, 0...) = (0, \frac{\partial p_2}{\partial x_1}, \frac{\partial p_2}{\partial p_1}, \frac{\partial^2 p_2}{\partial x_1^2}, \frac{\partial^2 p_2}{\partial x_1 \partial p_1}, ...)$$

- **Automatically evaluates all non-linearities to any desired order ..**

Some we know ...

Transfer matrix of a dipole:

$$M_{dipole} = \begin{pmatrix} cos(\frac{L}{R}) & Rsin(\frac{L}{R}) \\ -\frac{1}{R}sin(\frac{L}{R}) & cos(\frac{L}{R}) \end{pmatrix} = \begin{pmatrix} \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial p_1} \\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial p_1} \end{pmatrix}$$

For a 90^0 bending angle we get:

$$M_{dipole} = \begin{pmatrix} 0 & R \\ -\frac{1}{R} & 0 \end{pmatrix}$$

as computed, but we also have all derivatives and non-linear effects !

What is the use of that:

- **Although not strictly an analytic method in the traditional sense:**
  - **TPSA provide analytic expression (Taylor series) for the one turn map**
  - **Can be used for tracking**
  - **Can be analysed for dynamic behaviour of the system**
  - **Typical use: Normal Form Analysis discussed earlier, rather straightforward from a Taylor expansion**

Is there a summary ?

$$\mathbf{m} = \mathbf{z}$$

$$\mathbf{NORMAL} = \mathbf{m}$$

- **Get the map m somehow (no matter how)**
- **Analyse this map (Normal form)**

And another summary

- **Perturbation treatment limited to:**
  - **Small perturbations (not in real machines)**
  - **Pedagogical purpose**
- **For realistic machines symplectic, iterative mapping is appropriate, using:**
  - **Symplectic integration**
  - **Lie transformations and normal form analysis**
  - **Differential algebra**

- backup slides -

Example: sextupole (1D)

Given the Hamiltonian h:

$$h = -\mu J - \frac{3}{8}k(2\beta J)^{3/2} \cdot \left(\frac{sin(3\Psi + \frac{3\mu}{2})}{sin\frac{3\mu}{2}} - \frac{sin(\Psi + \frac{\mu}{2})}{sin\frac{\mu}{2}} \right)$$

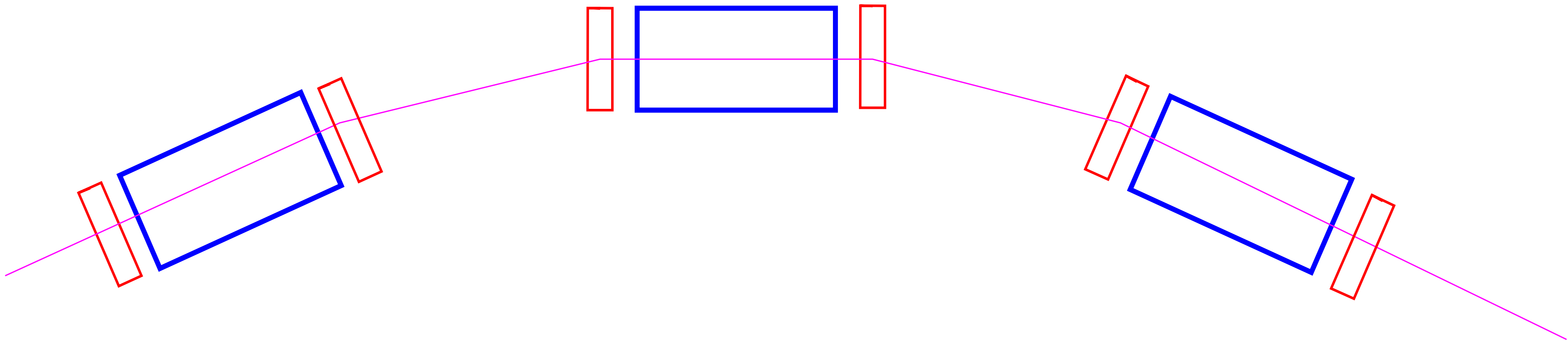
particles move in phase space along constant h.

Back to Cartesian coordinates we get for h:

$$h = -\frac{\mu}{2}(x^2 + x'^2)\frac{3}{8}\mu\beta^{3/2}x[(3x'^2 - x^2)cot\frac{3\mu}{2} - (x^2 + x'^2)cot\frac{\mu}{2} - 4xx']$$

Constant h defines the trajectory in phase space !

Where to put the elements in an accelerator ?



$$\frac{d^2x}{ds^2} + K(s)\,x = 0$$

- **Usually use s (pathlength) along "reference path"**
- **"Reference path" defined geometrically by straight sections and bending magnets**

Second order MAPS concatenation

Assume now 2 maps of second order:

$\mathcal{A}_2 = [R^A, T^A]$ **and** $\mathcal{B}_2 = [R^B, T^B]$

the combined second order map

$\mathcal{C}_2 \;=\; \mathcal{A}_2 \circ \mathcal{B}_2$ **is** $\mathcal{C}_2 \;=\; [R^C, T^C]$ **with:**

$$R^C \;=\; R^A \cdot \; R^B$$

and (after truncation of higher order terms !!):

$$T^C_{ijk} \;=\; \sum_{l=1}^{4} R^B_{il} T^A_{ljk} \;+\; \sum_{l=1}^{4} \sum_{m=1}^{4} T^B_{ilm} R^A_{lj} R^A_{mk}$$

Symplecticity for higher order MAPS

try truncated Taylor map in 2D, second order:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} R_{11}x_0 + R_{12}x'_0 + T_{111}x_0^2 + T_{112}x_0x'_0 + T_{122}x_0'^2 \\ R_{21}x_0 + R_{22}x'_0 + T_{211}x_0^2 + T_{212}x_0x'_0 + T_{222}x_0'^2 \end{pmatrix}$$

The Jacobian becomes:

$$\mathcal{J} = \begin{bmatrix} R_{11} + 2T_{111}x_0 + T_{112}x'_0 & R_{12} + T_{112}x_0 + 2T_{122}x'_0 \\ R_{21} + 2T_{211}x_0 + T_{212}x'_0 & R_{22} + T_{212}x_0 + 2T_{222}x'_0 \end{bmatrix}$$

symplecticity condition requires that:

det $\mathcal{J}$ = 1 for all x_0 and all x'_0

Symplecticity for higher order MAPS

This is only possible for the conditions:

$$\left(\begin{array}{c} R_{11}R_{22} - R_{12}R_{21} = 1 \\ R_{11}T_{212} + 2R_{22}T_{111} - 2R_{12}T_{211} - R_{21}T_{112} = 0 \\ 2R_{11}T_{222} + R_{22}T_{112} - R_{12}T_{212} - 2R_{21}T_{122} = 0 \end{array} \right)$$

- **10 coefficients, but 3 conditions**
- **number of independent coefficients only 7 !**
- **Taylor map requires more coefficients than necessary**
- **e.g. 4D, order 4: coefficients 276 instead of 121**

Canonical transformations

- **With Hamiltonian's equations, still have to solve (2n) differential equations**
- **Not necessarily easy, but:**
  - **More freedom to choose the variables q and p (because they have now "equal" status)**
  - **Try to find variables where they are easy to solve**
- **Change of variables through "canonical transformations"**

Why canonical transformations ?

- **Hamiltonian have one advantage over Lagrangians:**
  - **If the system has a symmetry, i.e. a coordinate q_i does not occur in H (i.e. $\frac{\partial H}{\partial q_i} = 0 \quad \rightarrow \quad \frac{dp_i}{dt} = 0$) ➡ the corresponding momentum p_i is conserved (and the coordinate q_i can be ignored in the other equations of the set).**
  - **Comes also from Lagrangian, but the velocities still occur in $\mathcal{L}$!**

Canonical transformations

Starting with $H(q, p, t)$ get new coordinates:

$$Q_i = Q_i(q, p, t)$$

$$P_i = P_i(q, p, t)$$

and new Hamiltonian $K(Q, P, t)$ with:

$$\frac{\partial K}{\partial Q_j} = -\dot{P}_j = -\frac{dP_j}{dt}, \qquad \frac{\partial K}{\partial P_j} = \dot{Q}_j = \frac{dQ_j}{dt}$$

- **We can two types of canonical transformations**

Canonical transformations - type 1

- **Ideally one would like a Hamiltonian H and coordinates with:**

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j = -\frac{dp_j}{dt} = 0$$

- **Coordinate q_j not explicit in H**
- **p_j is a constant of the motion (!) and:**

$$\frac{dq_j}{dt} = \frac{\partial H(p_1, p_2, ..p_n)}{\partial p_j} = F_j(p_1, p_2, ..p_n)$$

which can be directly integrated to get $q_j(t)$

Canonical transformations - type 1, example

Harmonic oscillator:

$$H = T + V \quad = \quad \frac{1}{2} m v^2 + \frac{m\omega^2}{2} x^2 \quad = \quad \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2$$

try: $x = \sqrt{\frac{2P}{m\omega}} \cdot sin(X)$ **and** $p = \sqrt{2m\omega P} \cdot cos(X)$ **and we get:**

$$K = \omega P cos^2(X) + \omega P sin^2(X) = \omega P$$

then:

$$\frac{dX}{dt} = \frac{\partial K}{\partial P} = \omega \quad \Rightarrow \quad X = \omega t + \alpha$$

back transformation to x,p:

$$x = \sqrt{\frac{2E}{m\omega^2}} sin(\omega t + \alpha)$$

Canonical transformations - type 2

- **Find a transformation of q, p at time t to values q_0, p_0 at time $t = 0$.**

$$q = q(q_0, p_0, t)$$

$$p = p(q_0, p_0, t)$$

- **The transformations ARE the solution of the problem !**

For both types: how to find the transformation ?

- **Without details: Hamilton-Jacobi equation ...**

Extension: general monomials

Monomials in x and p of orders n and m ($x^n p^m$)

$$e^{:ax^n p^m:}$$

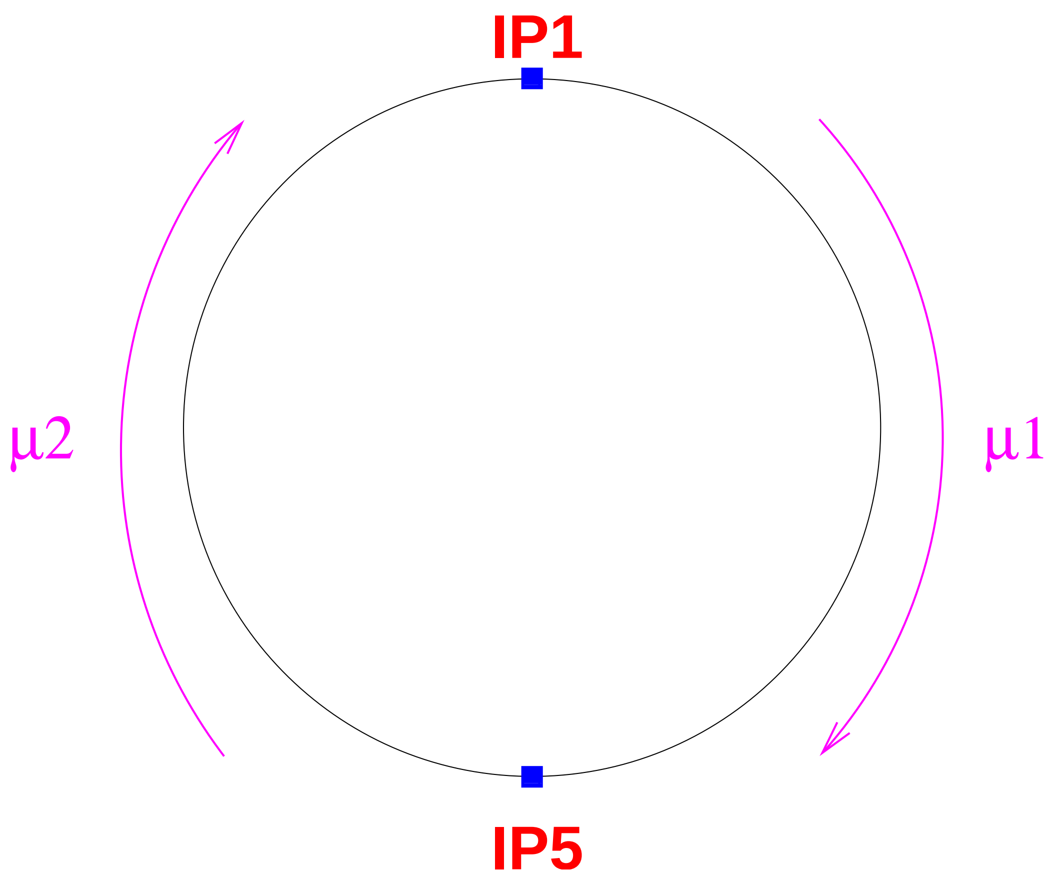
gives for the map (for $n \neq m$):

$$\begin{aligned} e^{:ax^n p^m:} x &= x \cdot [1 + a(n-m)x^{n-1}p^{m-1}]^{m/(m-n)} \\ e^{:ax^n p^m:} p &= p \cdot [1 + a(n-m)x^{n-1}p^{m-1}]^{n/(n-m)} \end{aligned}$$

gives for the map (for $n = m$):

$$\begin{aligned} e^{:ax^n p^n:} x &= x \cdot e^{-anx^{n-1}p^{n-1}} \\ e^{:ax^n p^n:} p &= p \cdot e^{anx^{n-1}p^{n-1}} \end{aligned}$$

Collision scheme - two IPs



$$\mu = \mu 1 + \mu 2$$

Two IPs

→ **two transfers f_2^1, f_2^2 and two beam-beam kicks F^1, F^2, first IP at μ_1, second IP at μ:**

$$= \quad e^{:f_2^1:}\ e^{:F^1:}\ e^{:f_2^2:}\ e^{:F^2:} \ = \ e^{:h_2:}$$

$$= \quad e^{:f_2^1:}\ e^{:F^1:}\ e^{-:f_2^1:}\ e^{:f_2^1:}\ e^{:f_2^2:}\ e^{:F^2:} \ = \ e^{:h_2:}$$

$$= \quad e^{:f_2^1:}\ e^{:F^1:}\ e^{-:f_2^1:}\ e^{:f_2:}\ e^{:F^2:}\ e^{-:f_2:}\ e^{:f_2:} = \ e^{:h_2:}$$

$$= \quad e^{:e^{-:f_2^1:}F^1:}\ e^{:e^{-:f_2:}F^2:}\ e^{:f_2:} = \ e^{:h_2:}$$

$$f_2 \ = \ -\mu A, \quad f_2^1 \ = \ -\mu_1 A, \quad \textit{and} \quad f_2^2 \ = \ -\mu_2 A$$

Two IPs

here a miracle occurs (remember $g(:f_2:)e^{in\Psi} = g(in\mu)e^{in\Psi}$**):**

$$e^{:f_2^1:}e^{in\Psi} \;=\; e^{in\mu_1}e^{in\Psi} \;=\; e^{in(\mu_1+\Psi)}$$

i.e. the Lie transforms of the perturbations are phase shifted[2]**. Therefore:**

$$e^{:e^{-:f_2^1:}F^1:}\; e^{:e^{-:f_2:}F^2:}\; e^{:f_2:} = \; e^{:h_2:}$$

becomes simpler with substitutions of $\Psi_1 \;=\; \Psi + \mu_1$ **and** $\Psi \;=\; \Psi + \mu$ **in** F^1 **and** F**:**

$$e^{:F^1(\Psi_1):}e^{:F(\Psi):}e^{:f_2:} \quad \Rightarrow \qquad e^{:F^1(\Psi_1)+F(\Psi):}e^{:f_2:}$$

[2] E. Forest, "Beam Dynamics, A New Attitude and Framework", 1998

Two IPs

gives for h_2:

$$h_2 = -\mu A + \sum_{n=-\infty}^{\infty} \frac{n\mu c_n(A)}{2\sin(n\frac{\mu}{2})} e^{-in(\Psi+\mu/2+\mu_1)} + e^{-in(\Psi+\mu/2)}$$

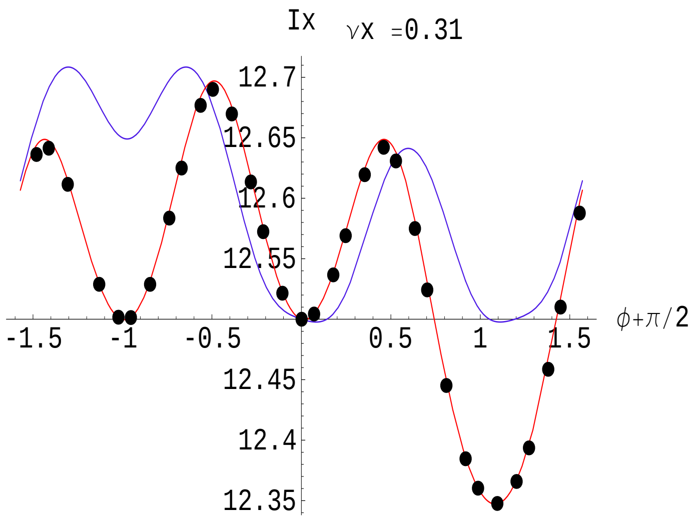
$$h_2 = -\mu A + 2c_0(A) + \underbrace{\sum_{n=1}^{\infty} \frac{2n\mu c_n(A)}{2\sin(n\frac{\mu}{2})} \cos(n(\Psi + \frac{\mu}{2} + \frac{\mu_1}{2}))\cos(n\frac{\mu_1}{2})}_{\text{interesting part}}$$

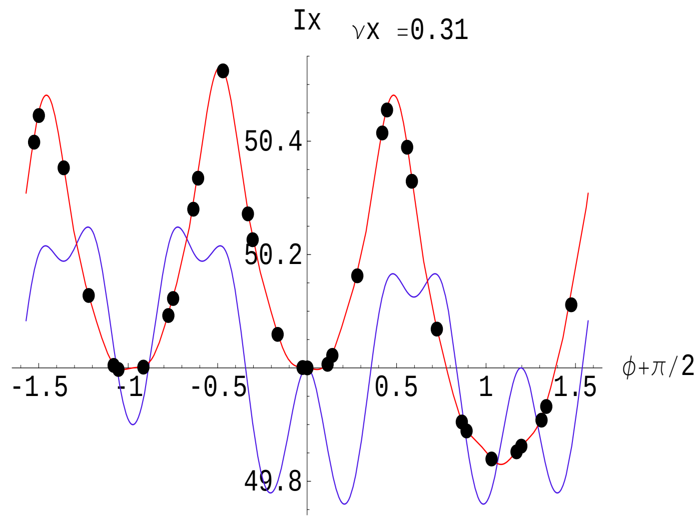
Nota bene, because of:

$$e^{:F(\Psi):} e^{:f_2:} \quad \Rightarrow \quad e^{:F^1(\Psi_1)+F(\Psi):} e^{:f_2:}$$

can be generalized to more interaction points ...

Invariant versus tracking: two IPs





➡ **Shown for $5\sigma_x$ and $10\sigma_x$**

Recap: Hamiltonian for a finite length element

We have from the Hamiltonian equations for the motion through an element with the Hamiltonian H for the element of length L:

$$\frac{dq}{dt} = [q, H] =: -H : q \quad \text{(from lecture 5)}$$

$$\rightarrow \frac{d^k q}{dt^k} = (: -H :)^k q$$

$$\rightarrow q(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\frac{d^k q}{dt^k} \right) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (- : H :)^k = e^{:-tH:}$$

with independent variable s instead of t (nota bene: $s_0 = 0, t_0 = 0$):

$$\rightarrow q(s) = e^{:-LH:}$$

Lie transformations on moments:

We have used Lie transformations mainly to propagate coordinates and momenta, i.e. like:

$$e^{:f:}x_0 = x_1$$

$$e^{:f:}p_0 = p_1$$

or using $Z = (x, p_x, y, p_y, ...)$**:**

$$e^{:f:}Z_0 = Z_1$$

- **Remember: can be applied to any function of** x **and** p **!!**
- **In particular to moments like** $x^2, xp, p^2, ...$

Lie transformations on moments

Assume a matrix M of the type:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

described by a generator f, we have for the Lie transformation on the moment:

$$e^{:f:}x^2 = (e^{:f:}x)^2 \qquad \text{(see lecture 5)}$$

therefore:

$$(e^{:f:}x)^2 = (m_{11}x + m_{12}p)^2$$

$$(e^{:f:}x)^2 = m_{11}^2 x^2 \ + \ 2\ m_{11}m_{12}xp \ + \ m_{12}^2 p^2$$

More on moments

To summarize the moments:

$$\begin{pmatrix} x^2 \\ xp \\ p^2 \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11}^2 & 2m_{11}m_{12} & m_{12}^2 \\ m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & m_{12}m_{22} \\ m_{21}^2 & 2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \circ \begin{pmatrix} x^2 \\ xp \\ p^2 \end{pmatrix}_{s_1}$$

This is the well known transfer matrix for optical parameters

A real life example: beam-beam interaction*)

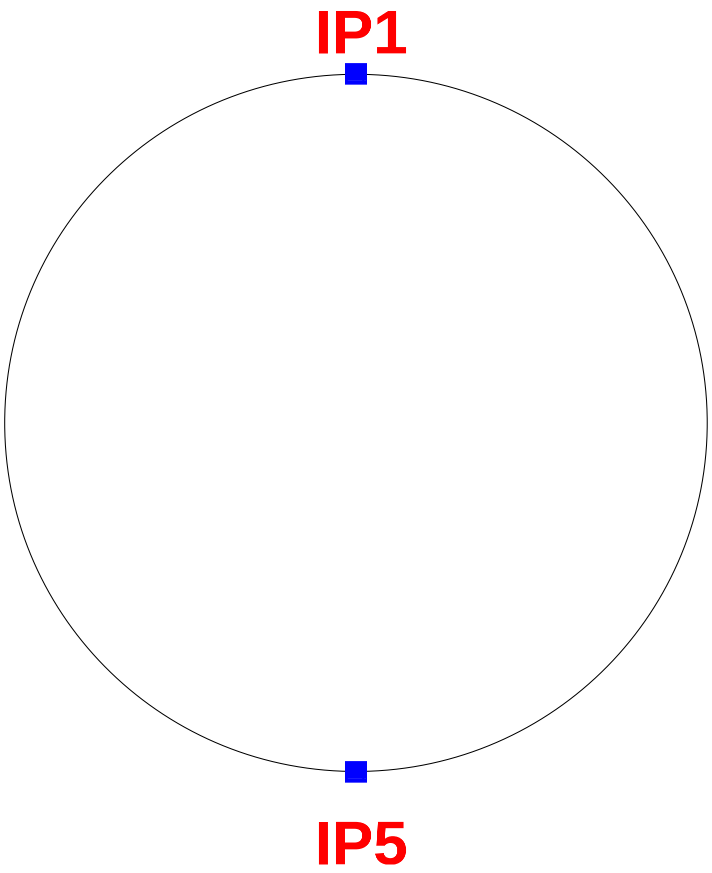
- Beam-beam interaction very non-linear
- Important to understand stability
- Non-linear effects such as amplitude detuning very important

Our questions ?

- How does the particles behave in phase space ?
- Do we have an invariant ?
- Can we calculate the invariant ?

*) From: W. Herr, D. Kaltchev, LHC Project Report 1082, (2008).

Collision scheme - two IPs



Start with single IP

"Classic" (B.C.) approach:

- **Interaction point at beginning (end) of the ring (very local interactions, δ-functions)**
- **s-dependent Hamiltonian and perturbation theory:**

$$\mathcal{H} \; = \; .... + \delta(s)\epsilon V$$

- **Disadvantages:**
  - **for several IPs endless mathematics**
  - **conceptually and computationally easier method**

Effect on invariants - start with single IP

Look for invariants h, (see e.g. Dragt[1)]), and evaluate for different number of interactions and phase advance.
Very well suited for local distortions (e.g. beam-beam kick)
Linear transfer $e^{:f_2:}$ and beam-beam interaction $e^{:F:}$, i.e.:

$$e^{:f_2:} \cdot e^{:F:} = e^{:h:}$$

with

$$f_2 = -\frac{\mu}{2}\left(\frac{x^2}{\beta} + \beta p_x^2\right)$$

and

$$F = \int_0^x dx' f(x')$$

1) A. Dragt, AIP Conference proceedings, Number 57 (1979)

Effect on invariants

using for a Gaussian beam f(x):

$$f(x) = \frac{2}{x}(1 - e^{\frac{-x^2}{2\sigma^2}})$$

as usual go to action angle variables Ψ, A:

$$x = \sqrt{2A\beta}\sin\Psi, \qquad p = \sqrt{\frac{2A}{\beta}}\cos\Psi$$

and write F(x) as Fourier series:

$$F(x) = \sum_{n=-\infty}^{\infty} c_n(A) e^{in\Psi} \quad \text{with :} \quad c_n(A) \ = \ \frac{1}{2\pi}\int_0^{2\pi} e^{-in\Psi} F(x) d\Psi$$

We need:

REMEMBER: with this transform:

$$f_2 = -\mu A$$

and useful properties of Lie operators (any textbook[2]):

$$: f_2 : g(A) = 0, \qquad : f_2 : e^{in\Psi} = in\mu e^{in\Psi}, \qquad g(: f_2 :) e^{in\Psi} = g(in\mu) e^{in\Psi}$$

and the formula (because the beam-beam perturbation is small !):

$$e^{:f_2:}\; e^{:F:} \;=\; e^{:h:} \;=\; \exp\left[: f_2 + \left(\frac{: f_2 :}{1 - e^{-:f_2:}} \right) F + \mathcal{O}(F^2) : \right]$$

[2] E. Forest, "Beam Dynamics, A New Attitude and Framework", 1998

Single IP

gives immediately for h:

$$h = -\mu A + \sum_n c_n(A) \frac{in\mu}{1 - e^{-in\mu}} e^{in\Psi}$$

$$h = -\mu A + \sum_n c_n(A) \frac{n\mu}{2\sin(\frac{n\mu}{2})} e^{(in\Psi + i\frac{n\mu}{2})}$$

away from resonance, a normal form transformation takes away the pure oscillatory part and we have only:

$$h \;=\; -\mu A + c_0(A) \;= const.$$

$$\left[homework: \qquad \frac{dc_0(A)}{dA} \right]$$

Single IP

If you are too lazy or too busy:

$$\Delta Q = \frac{-1}{2\pi} \frac{dc_0(A)}{dA}$$

is the detuning with amplitude, i.e. the amplitude dependent frequency change of the transformation we had before ...

We get:

$$\Delta Q = \frac{-1}{2\pi} \frac{N r_0}{\gamma A} [1 - e^{-A\beta/2\sigma^2} I_0(A\beta/2\sigma^2)]$$

Single IP - analysis of h

$$h = -\mu A + \sum_n c_n(A) \frac{n\mu}{2\sin(\frac{n\mu}{2})} e^{(in\Psi + i\frac{n\mu}{2})}$$

On resonance:

$$Q = \frac{p}{n} = \frac{\mu}{2\pi}$$

with $c_n \neq 0$**:**

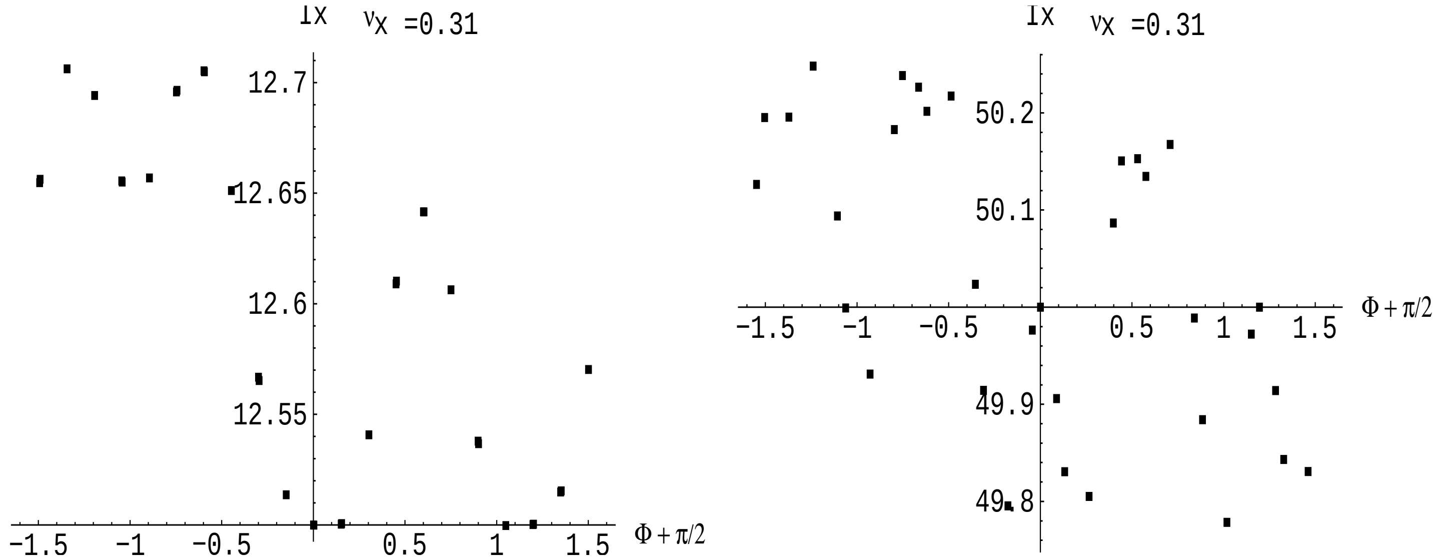
$$\sin(\frac{n\pi p}{n}) = \sin(p\pi) \equiv 0 \quad \forall \quad integer \quad p$$

and h **diverges**

Invariant versus tracking

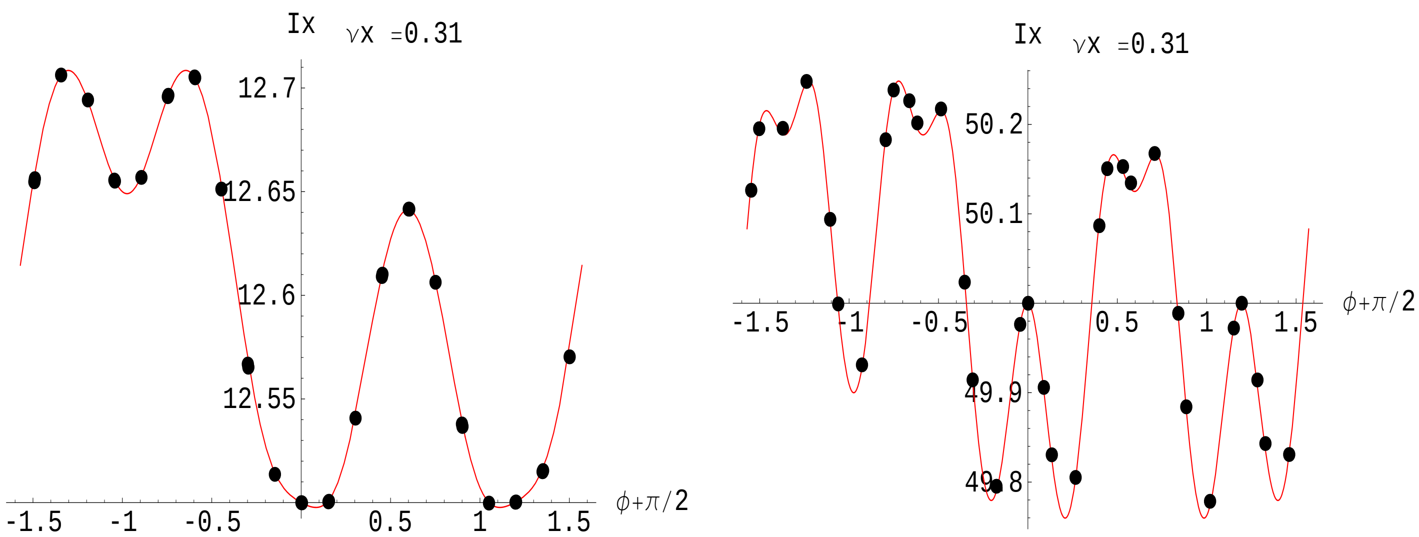
- **Is it useful what we obtained ?**
  - **Debug and compare ("benchmark")**
- **Compare to very simple tracking program:**
  - **linear transfer between interactions**
  - **beam-beam kick for round beam**
  - **compute action** $I = \frac{\beta^*}{2\sigma^2}(\frac{x^2}{\beta^*} + p_x^2\beta^*)$
  - **and phase** $\Psi = \arctan(\frac{p_x}{x})$
  - **compare** I **with** h

Invariant from tracking: one IP



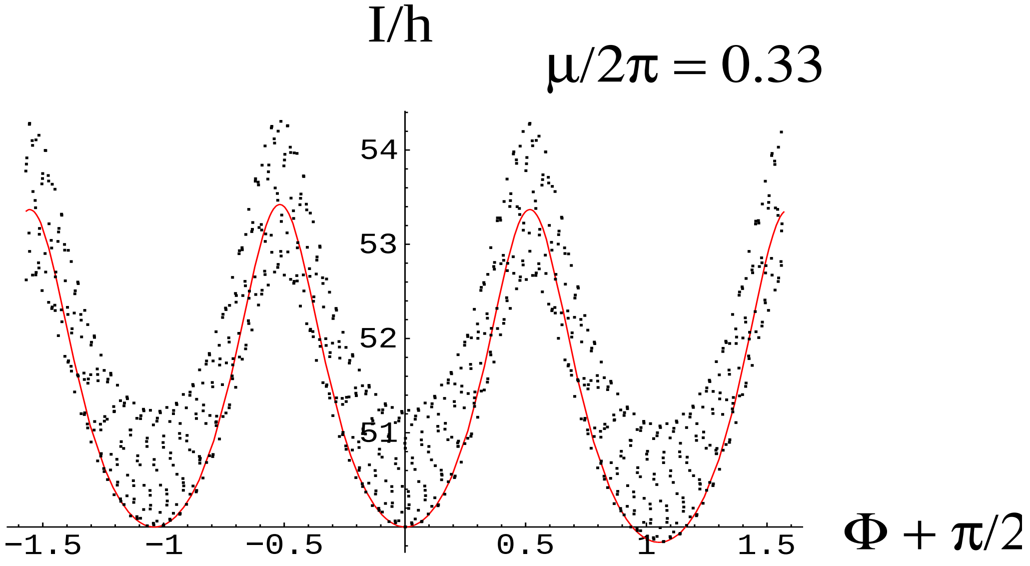
➡ **Shown for $5\sigma_x$ and $10\sigma_x$**

Invariant versus tracking: one IP



➡ **Shown for $5\sigma_x$ and $10\sigma_x$**

Invariant versus tracking:



- **Behaviour near a resonances: no more invariant possible**
- **Envelope of tracking well described**

What about close to resonance ?

If we have $Q = \frac{\mu}{2\pi} \approx \frac{m}{3}$ (3rd order resonance). Using a "distance to resonance d" as:

$$Q = \frac{m+d}{3} \qquad \text{where}: \quad d \ll 1$$

The trick is to observe the motion every 3 turns:

$$\mathcal{M}^3 = (e^{:-\mu J:} e^{:kx^3:})^3 = e^{:3h:}$$

We get a factor:

$$e^{:-3\mu J:} \;=\; e^{:-2\pi d J:} \quad (\text{because}: \quad e^{:-2\pi m J:} \equiv 1)$$

$$d = \frac{3\mu}{2\pi}$$

What about close to resonance ?

Without proof (but like before, see e.g. Chao), we get:

$$h = -\frac{2\pi}{3}dJ - \frac{\pi}{12}dk(2J)^{3/2} \cdot \left(\frac{sin(3\Psi + \frac{3\mu}{2})}{sin\frac{3\mu}{2}} - \frac{sin(\Psi + \frac{\mu}{2})}{sin\frac{\mu}{2}} \right)$$

For small d ($sin\frac{3\mu}{2} \approx -\pi d$) **we can simplify:**

$$h \approx -\frac{2\pi}{3}dJ - \frac{1}{\sqrt{2}}k(\beta J)^{3/2} sin(3\Psi)$$

Analysis give fixed points, i.e. (back in Cartesian again):

$$\frac{\partial h}{\partial x} = -\frac{2\pi}{3}dx - \frac{1}{4}\beta^{3/2}(3x'^2 - 3x^2) = 0$$

$$\frac{\partial h}{\partial x'} = -\frac{2\pi}{3}dx' - \frac{1}{4}\beta^{3/2}3xx' = 0$$