SPACE CHARGE EFFECTS

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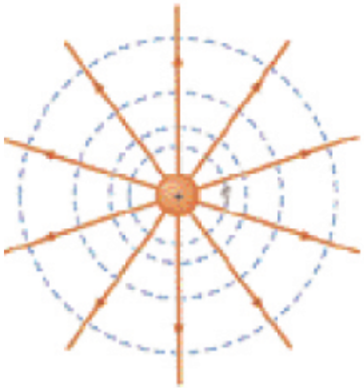
INFN-LNF



THE CERN ACCELERATOR SCHOOL

Trondheim, 24 August, 2013

EQUATION OF MOTION

The motion of charged particles is governed by the Lorentz force :

$$\frac{d(m\gamma \boldsymbol{v})}{dt} = \boldsymbol{F}_{e.m.}^{ext} = e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Where ***m*** is the rest mass, **γ** the relativistic factor and ***v*** the particle velocity

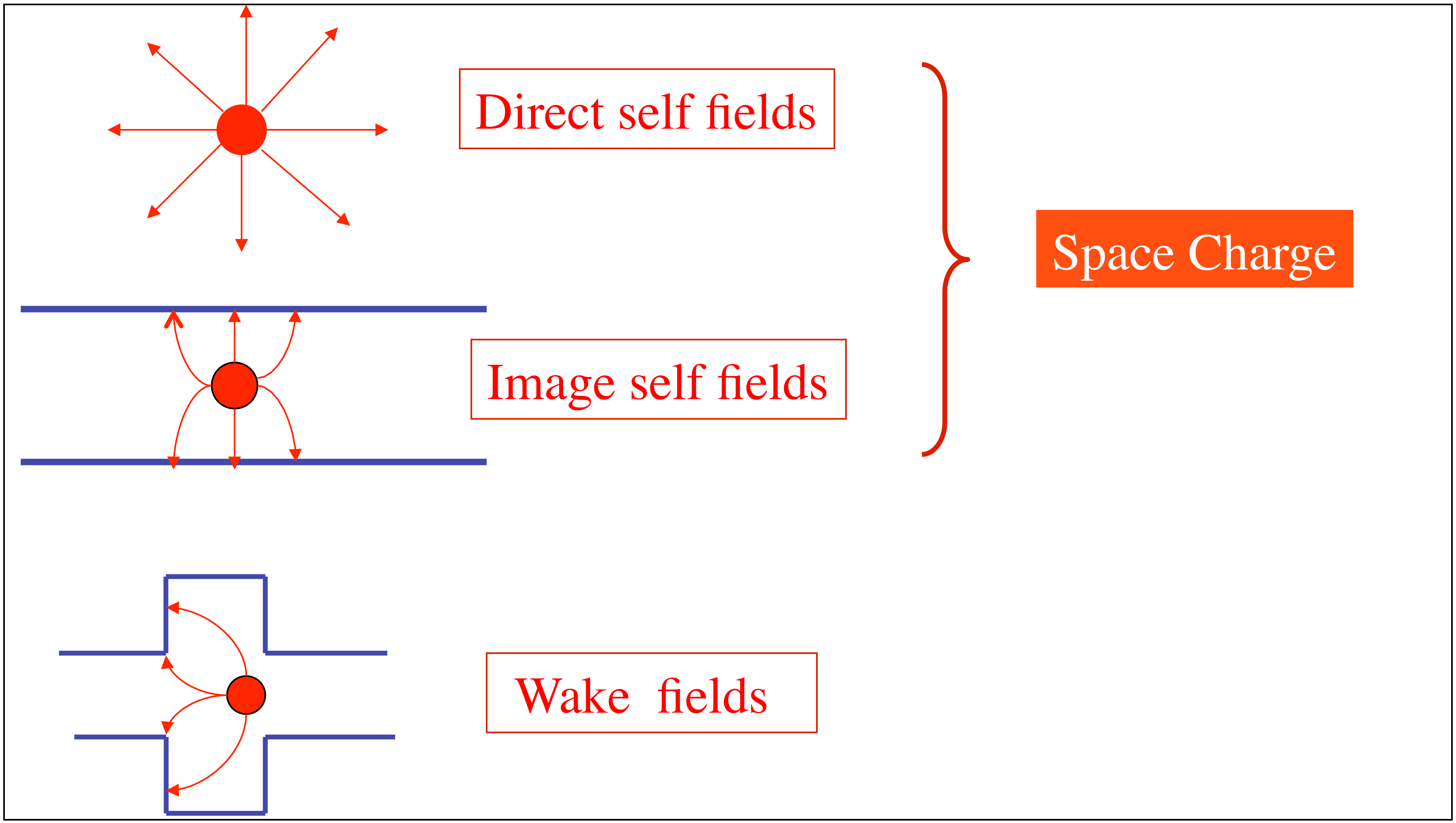
Charged particles are accelerated, guided and confined by external electromagnetic fields.

Acceleration is provided by the electric field of the RF cavity

Magnetic fields are produced in the bending magnets for guiding the charges on the reference trajectory (orbit), in the quadrupoles for the transverse confinement, in the sextupoles for the chromaticity correction.

SELF FIELDS AND WAKE FIELDS

There is another important source of e.m. fields : **the beam itself**



These fields depend on the current and on the charges velocity.

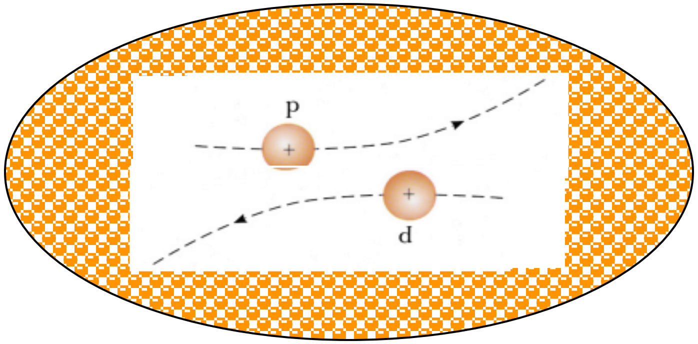
They are responsible of many phenomena of beam dynamics:

- energy loss *(wake-fields)*
- energy spread and emittance degradation
- shift of the synchronous phase and frequency (tune)
- shift of the betatron frequencies (tunes)
- instabilities.

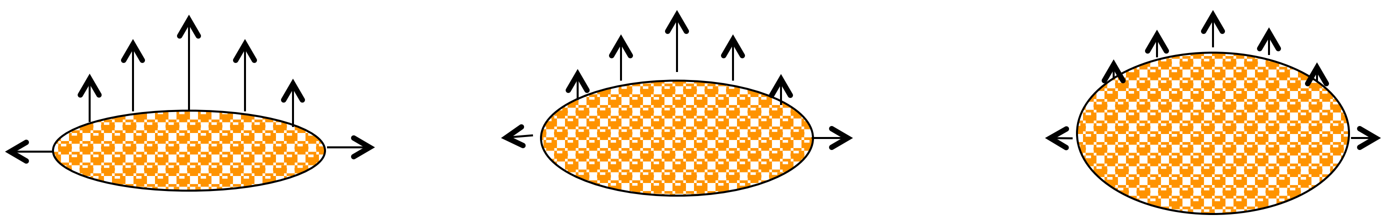
Space Charge: What does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**



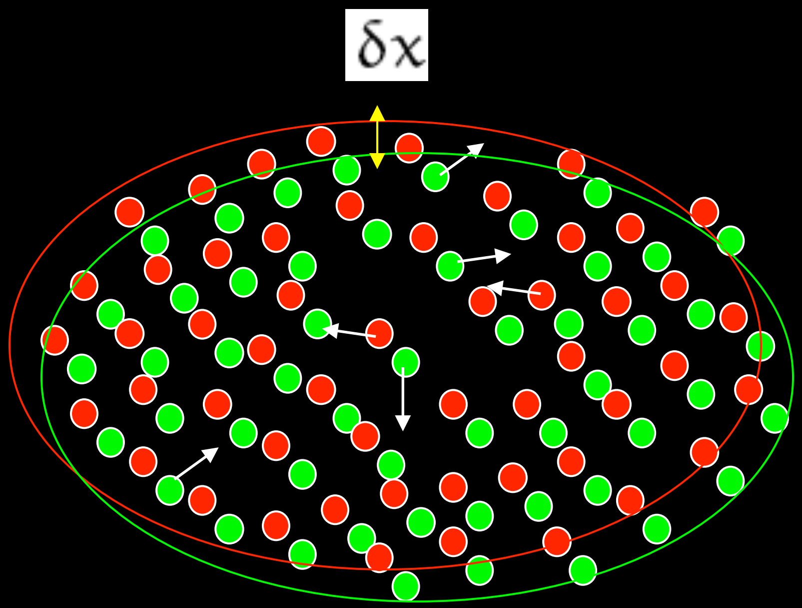
2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects**



Neutral Plasma

Surface charge density

$$\sigma = e\,n\,\delta x$$



Surface electric field

$$E_x = -\sigma/\epsilon_0 = -e\,n\,\delta x/\epsilon_0$$

Restoring force

$$m\,\frac{d^2\delta x}{dt^2} = e\,E_x = -m\,\omega_p^{\,2}\,\delta x$$

Plasma frequency

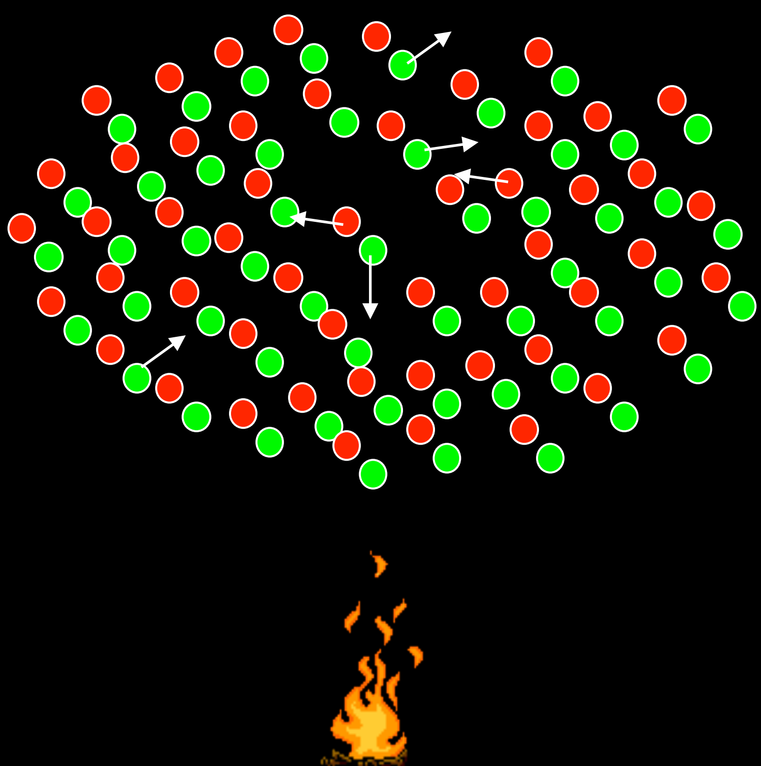
$$\omega_p^{\,2} = \frac{n\,e^2}{\epsilon_0\,m}$$

Plasma oscillations

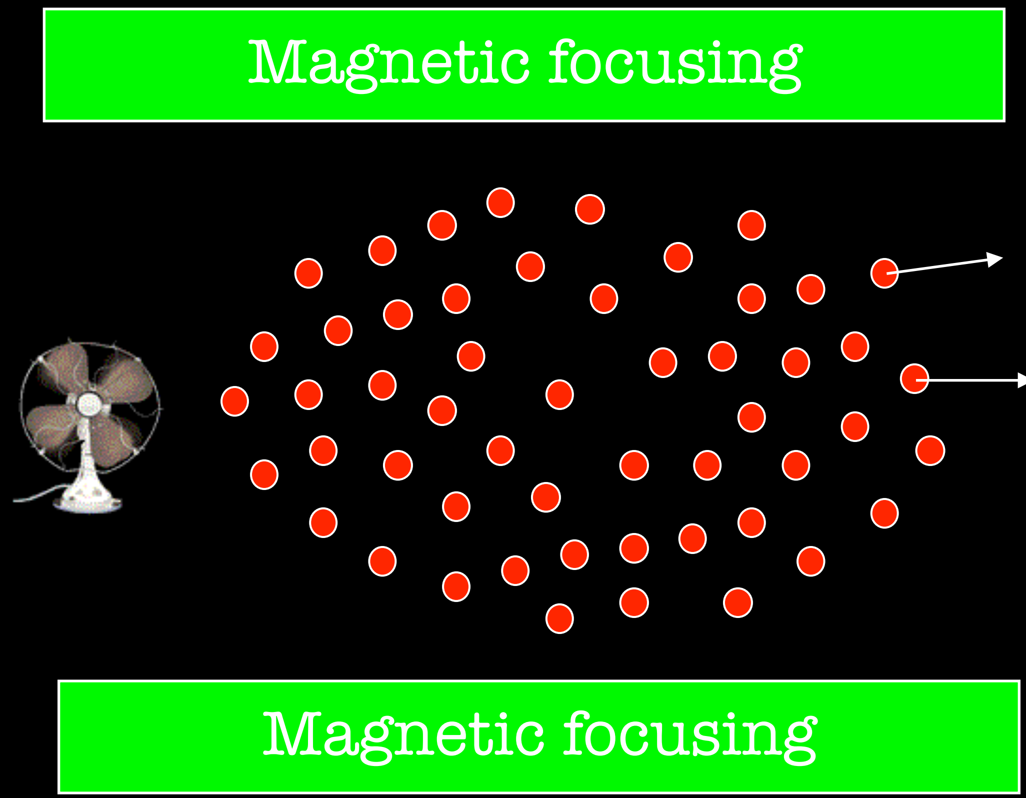
$$\delta x = (\delta x)_0\,\cos\left(\omega_p\,t\right)$$

Neutral Plasma

- Oscillations
- Instabilities
- EM Wave propagation



Single Component Cold Relativistic Plasma



A measure for the relative importance of collisional versus collective effects is the

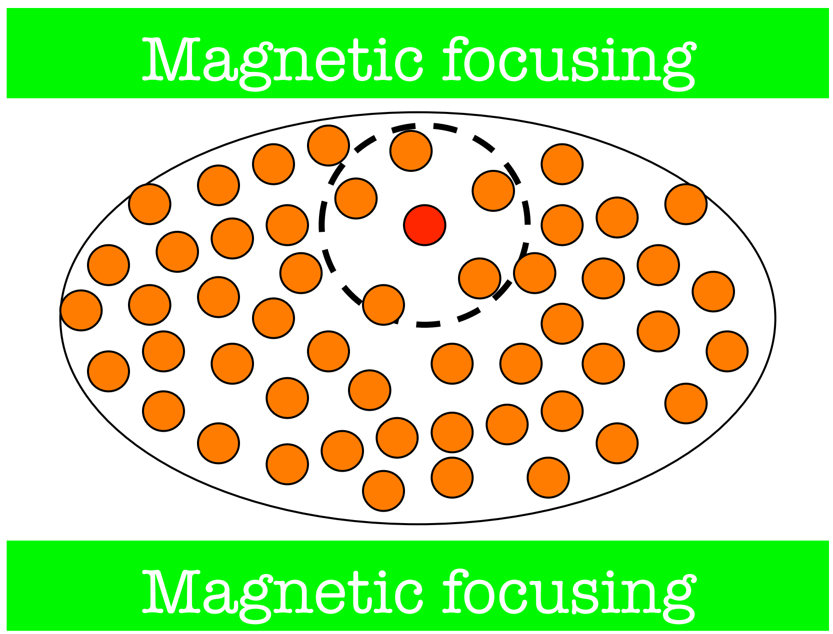
Debye Length λ_D

Let consider a **non-neutralized** system of **identical charged particles**

We wish to calculate the effective potential of a fixed test charged particle surrounded by other particles that are statistically distributed.

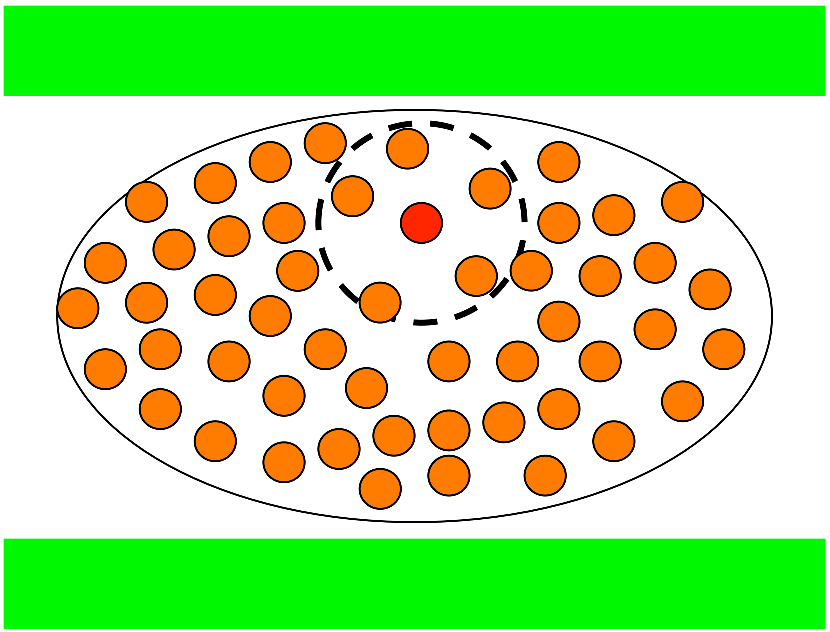
$$\Phi(\vec{r}) = \frac{C}{r}$$

$$C = \frac{e}{4\pi\varepsilon_o}$$



$$\Phi_D(\vec{r}) = ?$$

The effective potential of a test charge can be defined as the sum of the potential of the single particle δ and a "perturbed" term Δn.



From Poisson Equation:

$$\nabla^2\Phi_D(\vec{r}) = \frac{e}{\varepsilon_o}\delta(\vec{r}) + \frac{e}{\varepsilon_o}\Delta n(\vec{r})$$

$$\Delta n = ne^{-e\Phi_D/k_BT} - n \approx -\frac{ne}{k_BT}\Phi_D$$

$$\nabla^2\Phi_D(\vec{r}) + \lambda_D\Phi_D(\vec{r}) = \frac{e}{\varepsilon_o}\delta(\vec{r})$$

$$\lambda_D = \sqrt{\frac{\varepsilon_o k_B T}{e^2 n}}$$

$$\boxed{\Phi_D(\vec{r}) = \frac{C}{r}e^{-r/\lambda_D}}$$

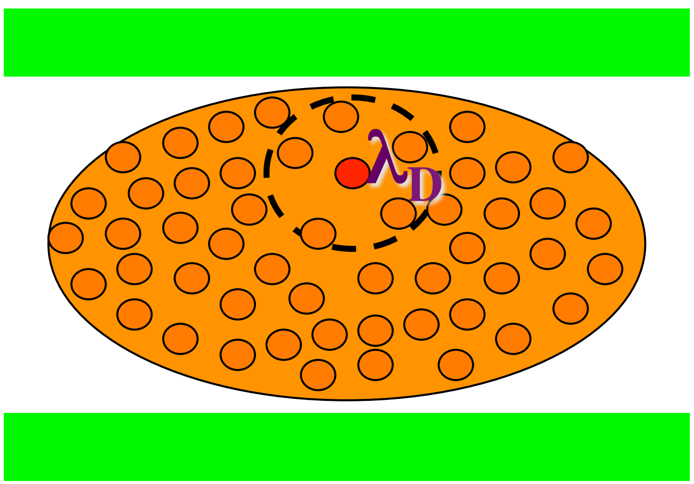
N => total number of particles
n => particle density (N/V)
k_B=> Boltzman constant
T => Temperature
k_B T => average kinetic energy of the particles

the effective interaction range of the test particle is limited to the

Debye length

The charges sourrounding the test particles have a screening effect

$$\Phi_D(\vec{r}) = \frac{C}{r} e^{-r/\lambda_D} \quad \Rightarrow \quad \begin{cases} \Phi_D(\vec{r}) \approx \Phi(\vec{r}) & \text{for} \quad r \ll \lambda_D \\ \Phi_D(\vec{r}) \ll \Phi(\vec{r}) & \text{for} \quad r \geq \lambda_D \end{cases}$$



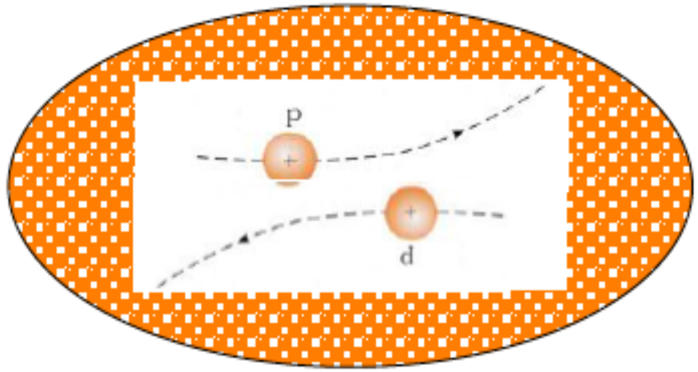
$$\boxed{\Phi_{SC}(\vec{r})} \gg \Phi_D(\vec{r})$$

<u>Smooth functions for the charge and field distributions can be used as long as the Debye length remains small compared to the particle bunch size</u>

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**

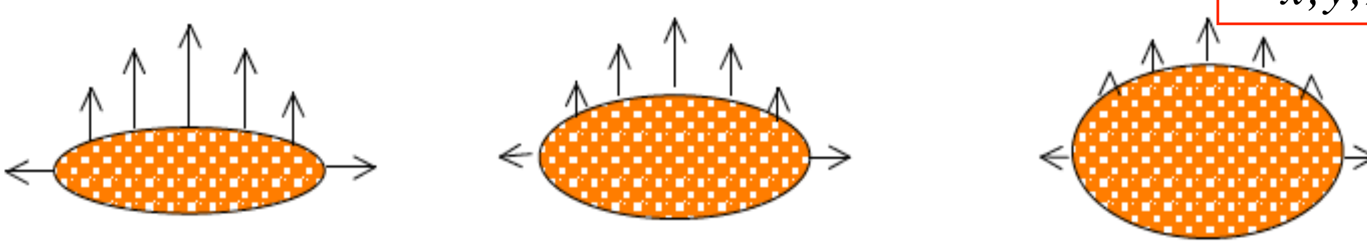
$$\boxed{\lambda_D = \sqrt{\frac{\varepsilon_o k_B T}{e^2 n}}}$$



$$\sigma_{x,y,z} \ll \lambda_D$$

2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects, Single Component Cold Plasma**

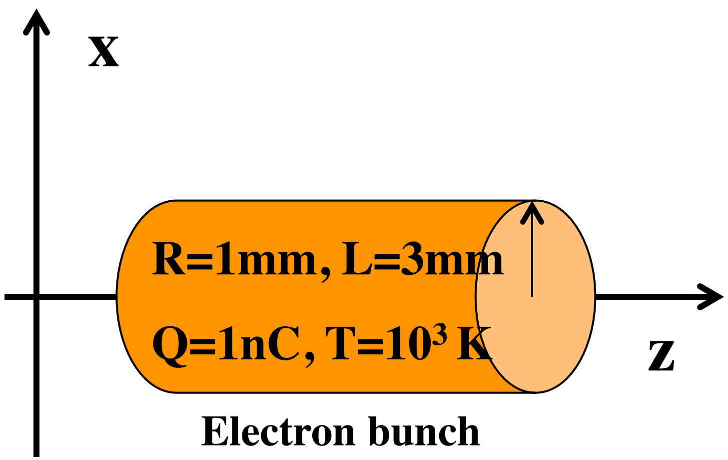
$$\sigma_{x,y,z} \gg \lambda_D$$



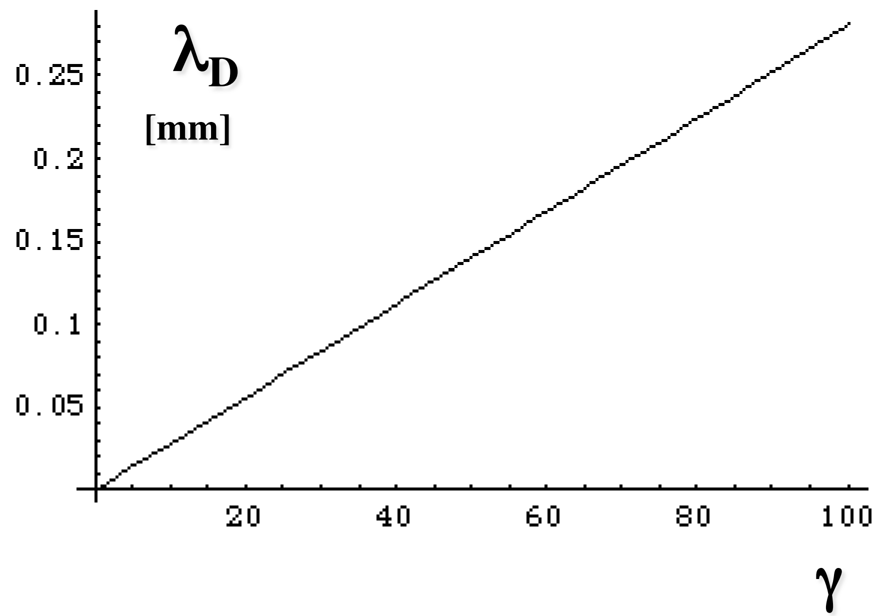
In a charged particle beam moving at a longitudinal relativistic velocity, assuming that the random transverse motion in the beam is non-relativistic, the Debye length has the following form:

$$\lambda_D = \sqrt{\frac{\varepsilon_o \gamma^2 k_B T}{e^2 n}}$$

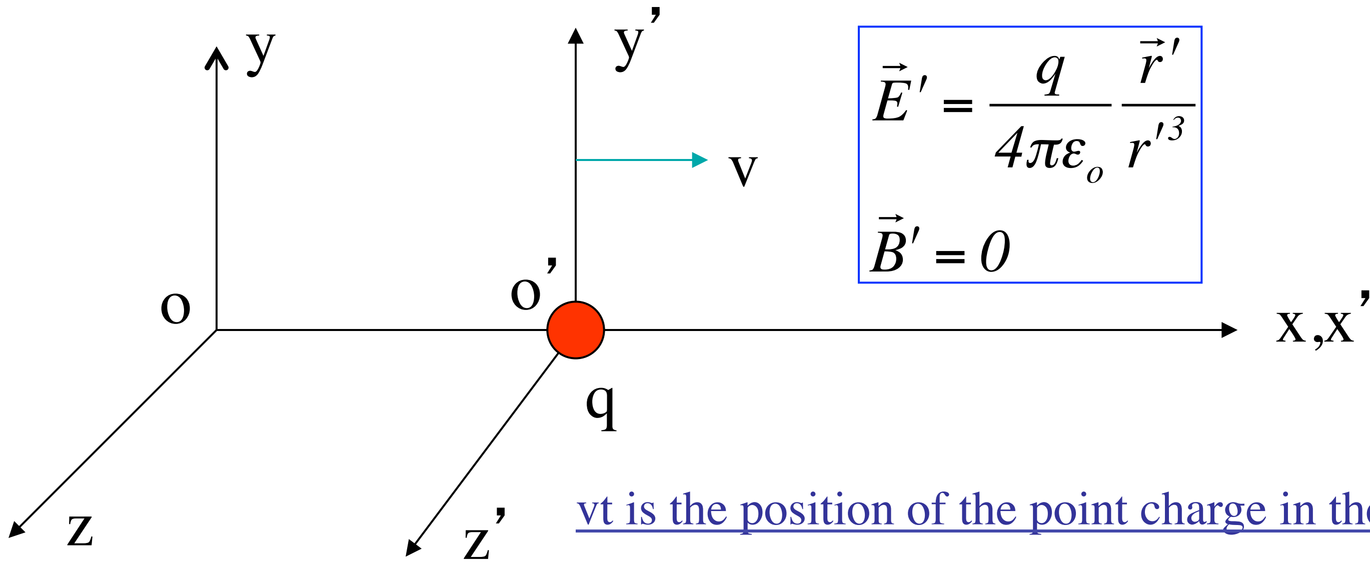
$$\langle v_x \rangle = \sqrt{\frac{k_B T}{\gamma m}} << c$$



$$v_z \approx c$$



Fields of a point charge with uniform motion



$$\vec{E}' = \frac{q}{4\pi\varepsilon_o}\frac{\vec{r}'}{r'^3}$$

$$\vec{B}' = 0$$

vt is the position of the point charge in the lab. frame O.

- In the moving frame O' the charge is at rest
- The electric field is radial with spherical symmetry
- The magnetic field is zero

$$E'_x = \frac{q}{4\pi\varepsilon_o}\frac{x'}{r'^3} \qquad E'_y = \frac{q}{4\pi\varepsilon_o}\frac{y'}{r'^3} \qquad E'_z = \frac{q}{4\pi\varepsilon_o}\frac{z'}{r'^3}$$

Relativistic transforms of the fields from O' to O

$$\boxed{\gamma = \frac{1}{\sqrt{1-\beta^2}}}$$

$$\begin{cases} E_x = E'_x \\ E_y = \gamma(E'_y + vB'_z) \\ E_z = \gamma(E'_z - vB'_y) \end{cases} \qquad \begin{cases} B_x = B'_x \\ B_y = \gamma(B'_y - vE'_z/c^2) \\ B_z = \gamma(B'_z + vE'_y/c^2) \end{cases}$$

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ ct' = \gamma\left(ct - \dfrac{v}{c}x\right) \end{cases}$$

$$r' = \left(x'^2 + y'^2 + z'^2\right)^{1/2}$$

$$r' = \left[\gamma^2(x - vt)^2 + y^2 + z^2\right]^{1/2}$$

$$E_x = E'_x = \frac{q}{4\pi\varepsilon_o}\frac{x'}{r'^3} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma(x - vt)}{\left[\gamma^2(x - vt)^2 + y^2 + z^2\right]^{3/2}}$$

$$E_y = \gamma E'_y = \frac{q}{4\pi\varepsilon_o}\frac{y'}{r'^3} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma y}{\left[\gamma^2(x - vt)^2 + y^2 + z^2\right]^{3/2}}$$

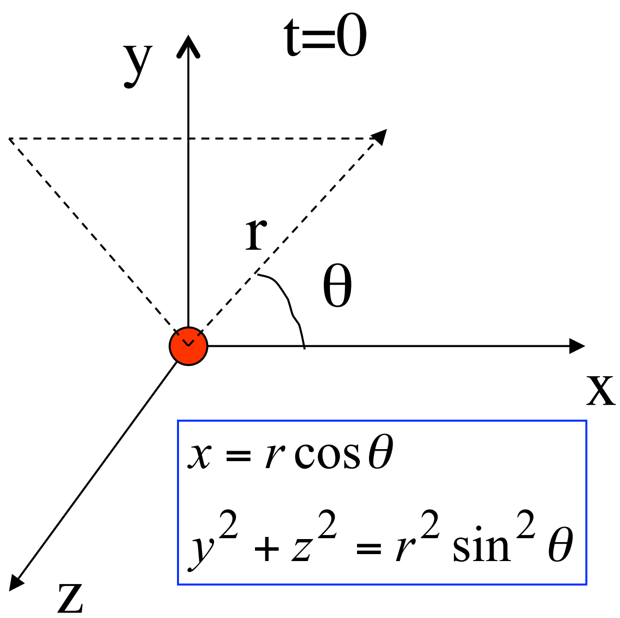
$$E_z = \gamma E'_z = \frac{q}{4\pi\varepsilon_o}\frac{z'}{r'^3} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma z}{\left[\gamma^2(x - vt)^2 + y^2 + z^2\right]^{3/2}}$$

The field pattern is moving with the charge and it can be observed at t=0:

$$\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma\vec{r}}{\left[\gamma^2 x^2 + y^2 + z^2\right]^{3/2}}$$

The fields have lost the spherical symmetry

$$\vec{E} = \frac{q}{4\pi\varepsilon_o} \frac{\gamma\,\vec{r}}{\left[\gamma^2 x^2 + y^2 + z^2\right]^{3/2}}$$



$$x = r\cos\theta$$

$$y^2 + z^2 = r^2 \sin^2\theta$$

$$\gamma^2 x^2 + y^2 + z^2 = r^2\gamma^2\left(1 - \beta^2 \sin^2\theta\right)$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_o} \frac{\left(1-\beta^2\right)}{r^2\left(1-\beta^2 \sin^2\theta\right)^{3/2}} \frac{\vec{r}}{r}$$

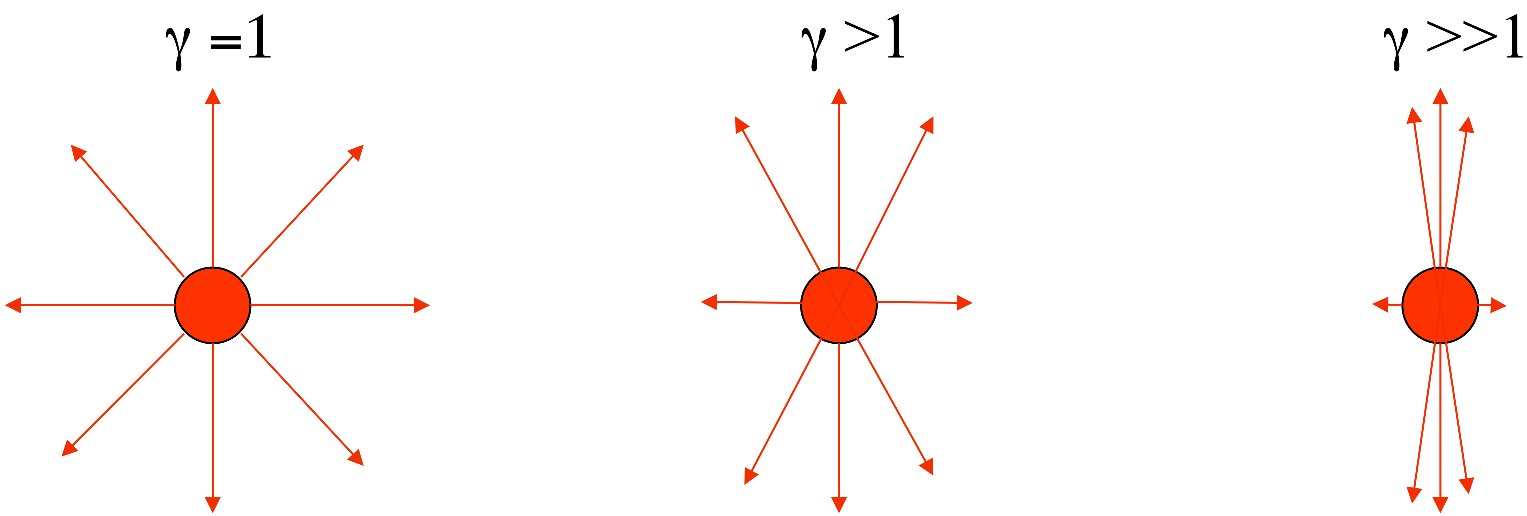
$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_o} \frac{\left(1-\beta^2\right)}{r^2\left(1-\beta^2 \sin^2\theta\right)^{3/2}} \frac{\vec{r}}{r}$$

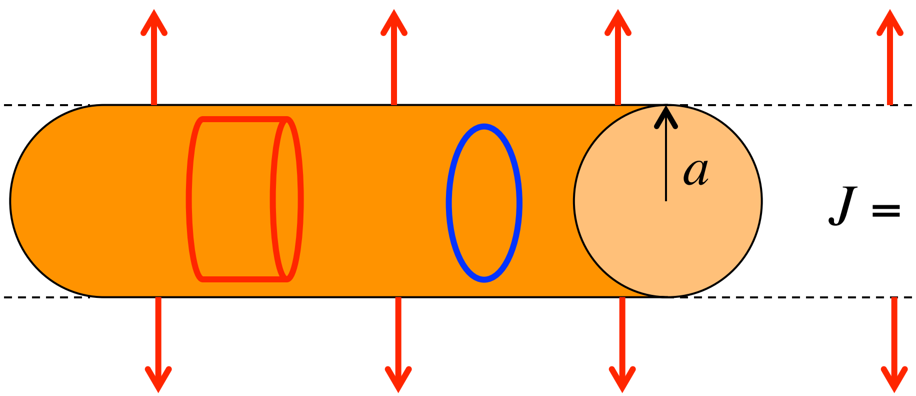
$$\beta = 0 \Rightarrow \vec{E} = \frac{q}{4\pi\varepsilon_o} \frac{1}{r^2} \frac{\vec{r}}{r}$$

$$\theta = 0 \Rightarrow E_{//} = \frac{q}{4\pi\varepsilon_o} \frac{1}{\gamma^2 r^2} \frac{\vec{r}}{r} \xrightarrow{\gamma\to\infty} 0$$

$$\theta = \frac{\pi}{2} \Rightarrow E_{\perp} = \frac{q}{4\pi\varepsilon_o} \frac{\gamma}{r^2} \frac{\vec{r}}{r} \xrightarrow{\gamma\to\infty} \infty$$



Continuous Uniform Cylindrical Beam Model



$$\rho = \frac{I}{\pi a^2 v}$$

$$J = \frac{I}{\pi a^2}$$

Gauss's law

$$\int \varepsilon_o E \cdot dS = \int \rho dV$$

$$\boxed{\begin{aligned} E_r &= \frac{I}{2\pi\varepsilon_o a^2 v} r \quad \textbf{\textit{for}} \quad \boldsymbol{r \le a} \\ E_r &= \frac{I}{2\pi\varepsilon_o v}\frac{1}{r} \quad \textbf{\textit{for}} \quad \boldsymbol{r > a} \end{aligned}}$$

$$B_\vartheta = \frac{\beta}{c} E_r$$

Ampere's law

$$\int B \cdot dl = \mu_o \int J \cdot dS$$

$$\boxed{\begin{aligned} B_\vartheta &= \mu_o \frac{Ir}{2\pi a^2} \quad \textbf{\textit{for}} \quad \boldsymbol{r \le a} \\ B_\vartheta &= \mu_o \frac{I}{2\pi r} \quad \textbf{\textit{for}} \quad \boldsymbol{r > a} \end{aligned}}$$

Lorentz Force

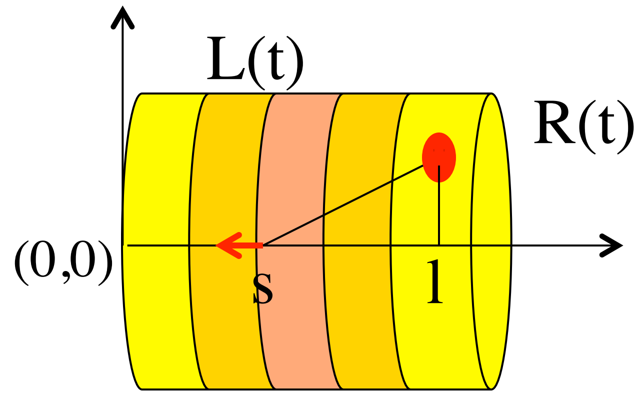
$$\boxed{F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1 - \beta^2\right)E_r = \frac{eE_r}{\gamma^2}}$$

has only **radial** component and

is a **linear** function of the transverse coordinate

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force.

Bunched Uniform Cylindrical Beam Model



$$v_z = \beta c$$

Longitudinal Space Charge field in the bunch moving frame:

$$\tilde{\rho} = \frac{Q}{\pi R^2 \tilde{L}}$$

$$\tilde{E}_z(\tilde{s}, r=0) = \frac{\tilde{\rho}}{4\pi\varepsilon_o} \int_0^R \int_0^{2\pi} \int_0^{\tilde{L}} \frac{(\tilde{l}-\tilde{s})}{\left[(\tilde{l}-\tilde{s})^2 + r^2\right]^{3/2}} \, r dr d\varphi d\tilde{l}$$

$$\tilde{E}_z(\tilde{s}, r=0) = \frac{\tilde{\rho}}{2\varepsilon_0} \left[\sqrt{R^2 + (\tilde{L}-\tilde{s})^2} - \sqrt{R^2 + \tilde{s}^2} + \left(2\tilde{s} - \tilde{L}\right)\right]$$

Radial Space Charge field in the bunch moving frame

by series representation of axisymmetric field:

$$\tilde{E}_r(r,\tilde{s}) \cong \left[\frac{\tilde{\rho}}{\varepsilon_0} - \frac{\partial}{\partial \tilde{s}}\tilde{E}_z(0,\tilde{s})\right]\frac{r}{2} + \left[\cdots\right]\frac{r^3}{16} +$$

$$\tilde{E}_r(r,\tilde{s}) = \frac{\tilde{\rho}}{2\varepsilon_0}\left[\frac{(\tilde{L}-\tilde{s})}{\sqrt{R^2+(\tilde{L}-\tilde{s})^2}} + \frac{\tilde{s}}{\sqrt{R^2+\tilde{s}^2}}\right]\frac{r}{2}$$

Lorentz Transformation to the Lab frame

$$E_z = \tilde{E}_z$$

$$E_r = \gamma \tilde{E}_r$$

$$\tilde{L} = \gamma L \quad \Rightarrow \quad \tilde{\rho} = \frac{\rho}{\gamma}$$

$$\tilde{s} = \gamma s$$

$$E_z(0,s) = \frac{\rho}{\gamma 2\varepsilon_0}\left[\sqrt{R^2 + \gamma^2 (L-s)^2} - \sqrt{R^2 + \gamma^2 s^2} + \gamma\left(2s - L\right)\right]$$

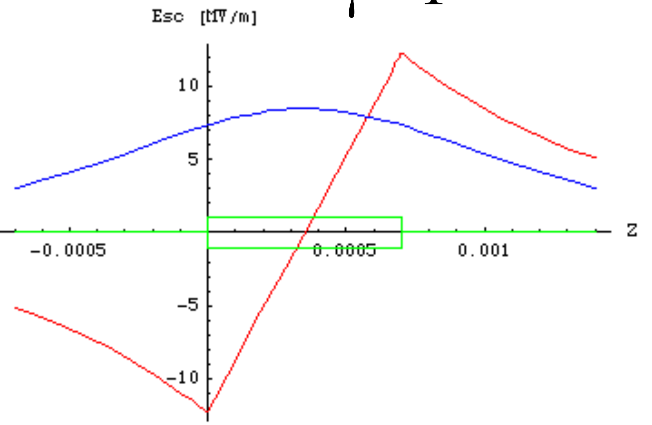
$$E_r(r,s) = \frac{\gamma\rho}{2\varepsilon_0}\left[\frac{(L-s)}{\sqrt{R^2 + \gamma^2 (L-s)^2}} + \frac{s}{\sqrt{R^2 + \gamma^2 s^2}}\right]\frac{r}{2}$$

It is still a linear field with r but with a longitudinal correlation s

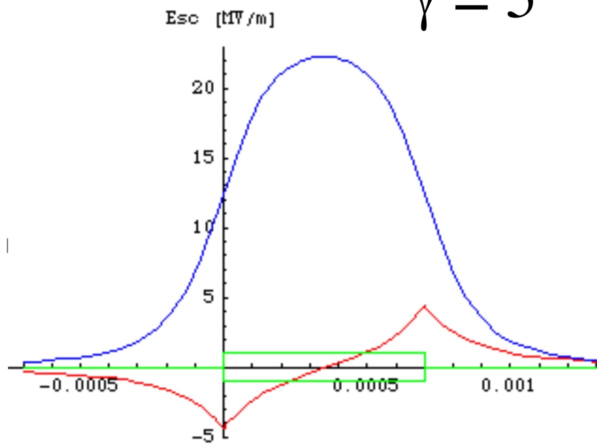
$$E_z(0,s,\gamma) = \frac{I}{2\pi\gamma\varepsilon_0 R^2 \beta c} h(s,\gamma)$$

$$E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$

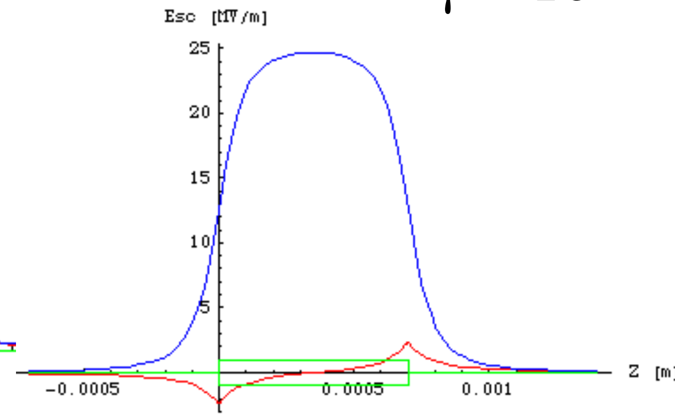
γ= 1



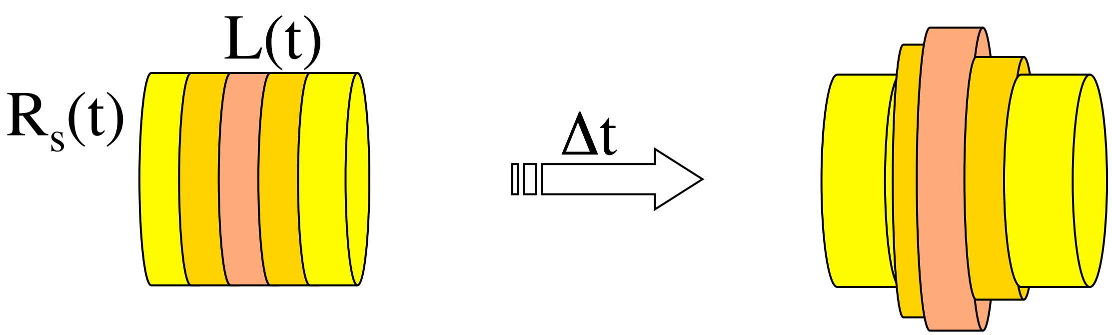
γ = 5



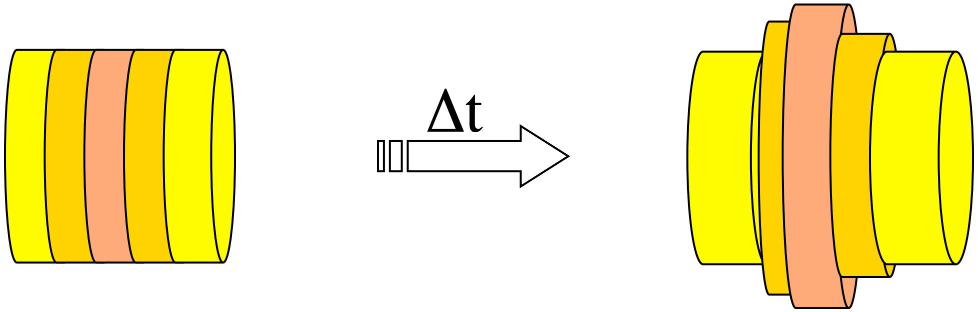
γ = 10



$$F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g(s,\gamma)$$



rms Envelope equation for a beam slice



$$\boxed{\sigma'' + k_s^2\sigma = \frac{k_{sc}(s,\gamma)}{\sigma}}$$

$$k_{sc}(s,\gamma) = \frac{2I}{I_A(\beta\gamma)^3}\, g(s,\gamma)$$

$$I_A = \frac{4\pi\varepsilon_o m_o c^3}{e} = 17kA$$

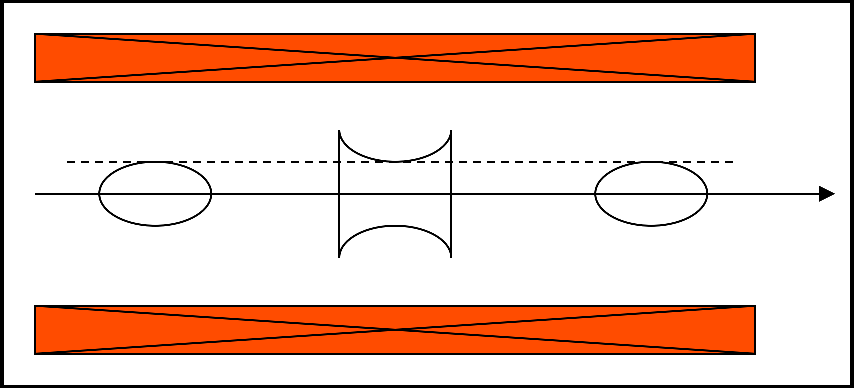
$$\sigma'' + k_s^2 \sigma = \frac{k_{sc}(s,\gamma)}{\sigma}$$

Single Component Relativistic Plasma

$$k_s = \frac{qB}{2mc\beta\gamma}$$

Equilibrium solution:

$$\sigma_{eq}(s,\gamma) = \frac{\sqrt{k_{sc}(s,\gamma)}}{k_s}$$



Small perturbation:

$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma(s)$$

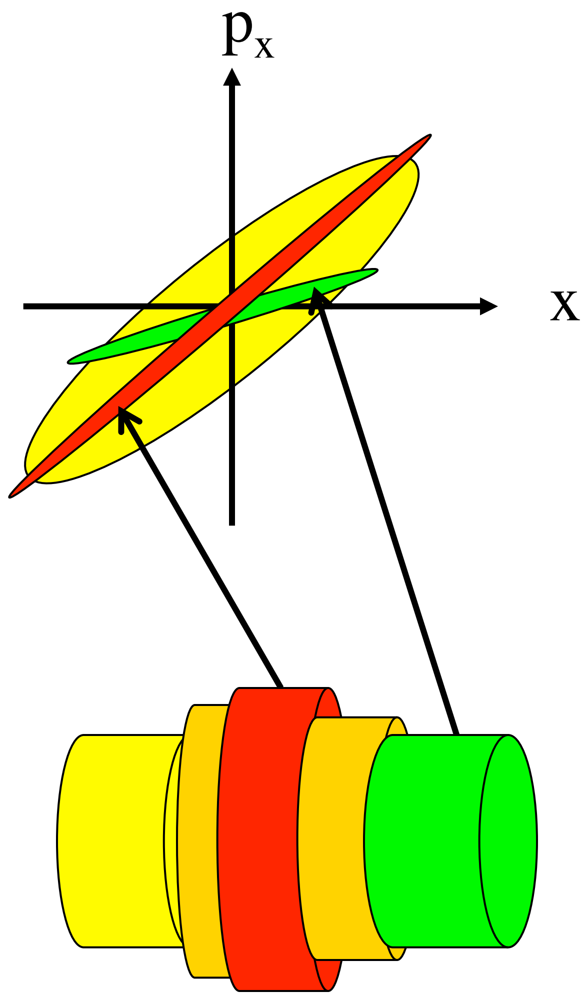
$$\delta\sigma''(s) + 2k_s^2 \delta\sigma(s) = 0$$

$$\delta\sigma(s) = \delta\sigma_o(s)\cos\left(\sqrt{2}k_s z\right)$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

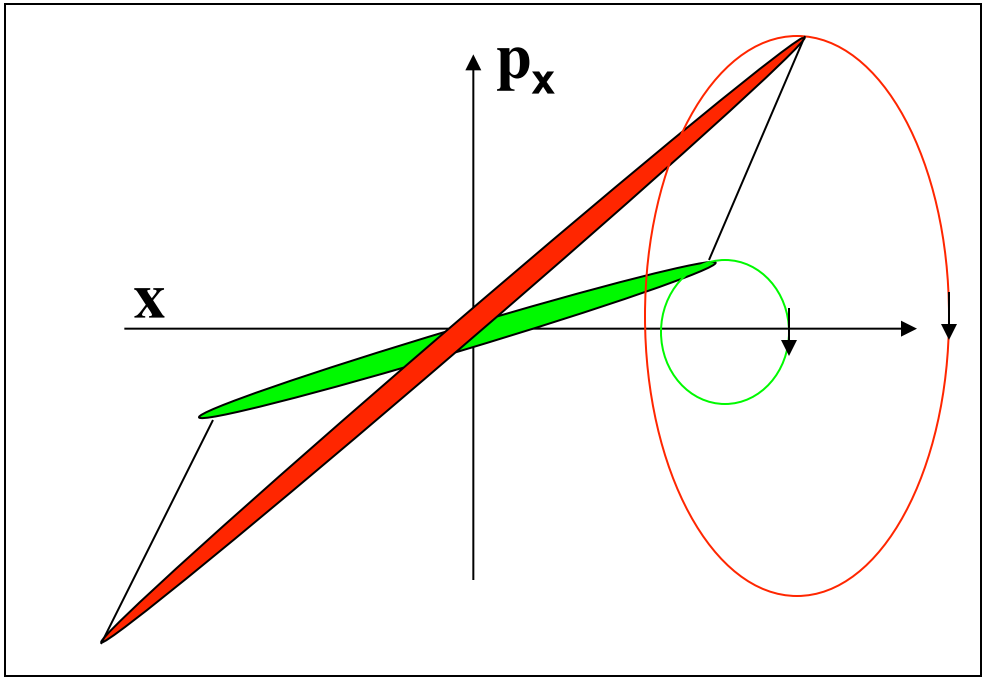
$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma_o(s)\cos\left(\sqrt{2}k_s z\right)$$

Space Charge differential defocusing in core and tails of the beam drive Reversible Emittance Oscillations



$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}{}^2} = \sqrt{\left(\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2\right)} \approx \left| \sin\left(\sqrt{2} k_s z\right) \right|$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes



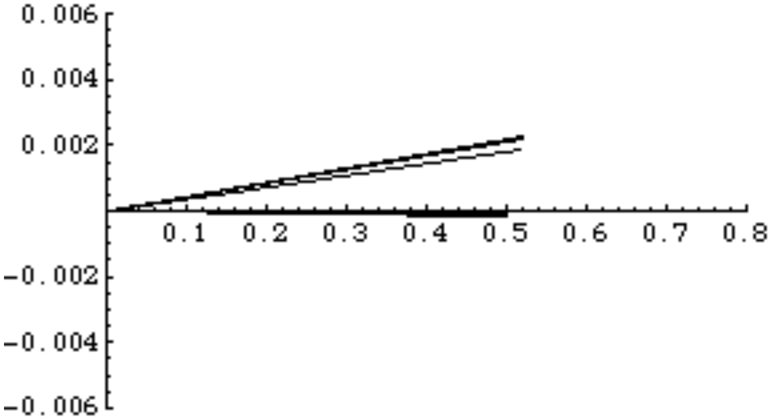
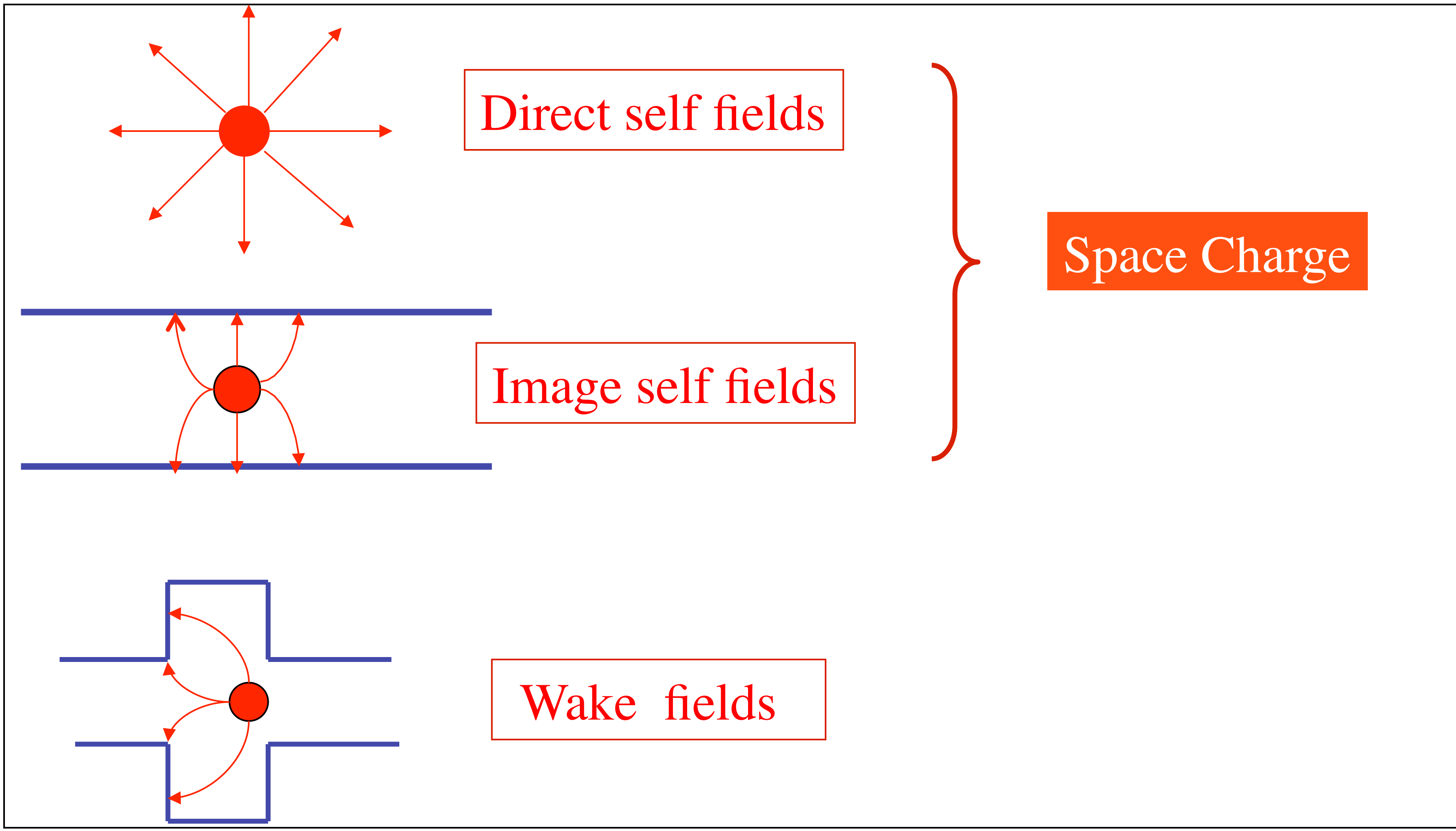


IMAGE SELF FIELDS

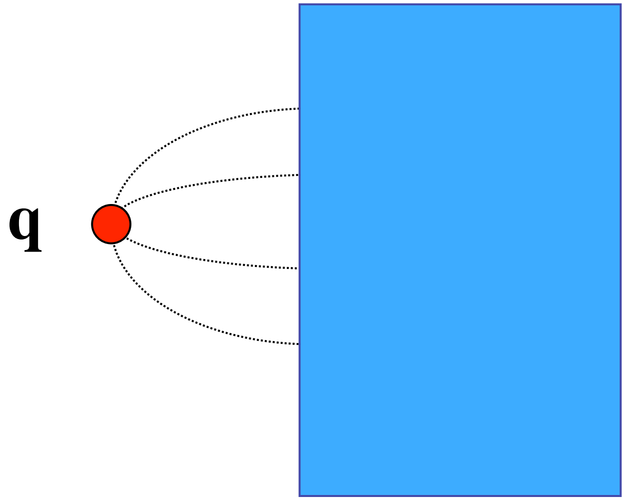
There is another important source of e.m. fields : **the beam itself**

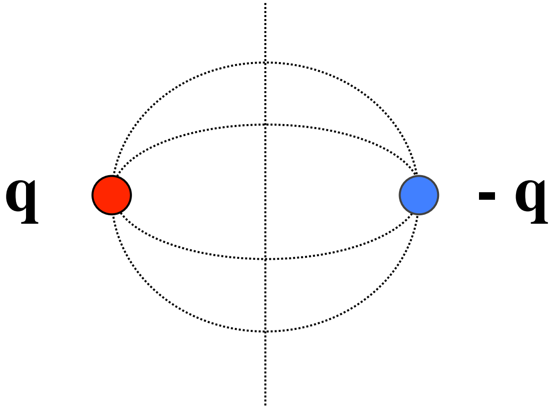


Static Fields: conducting or magnetic screens

Let us consider a point charge q close to a <u>conducting screen</u>.

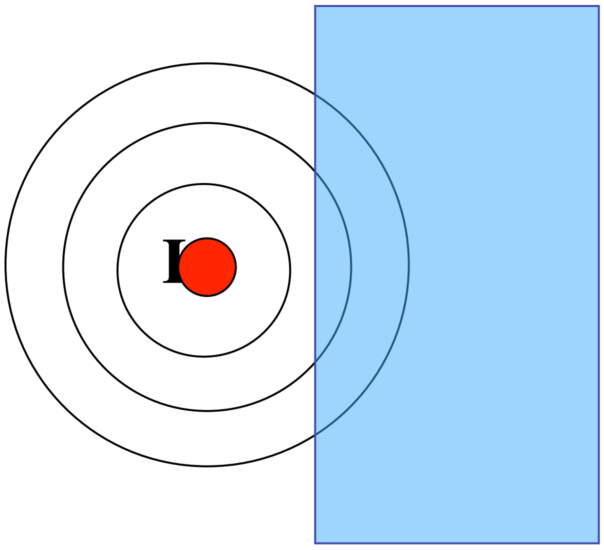
The electrostatic field can be derived through the "image method". Since the metallic screen is an equi-potential plane, it can be removed provided that a "virtual" charge is introduced such that the potential is constant at the position of the screen



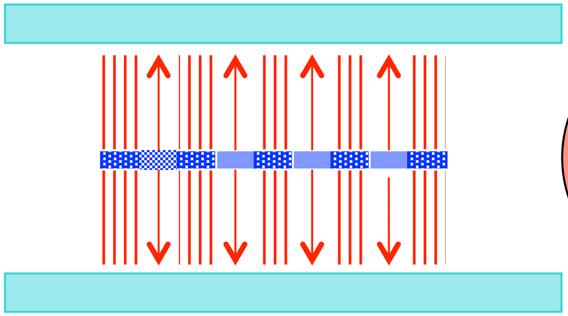


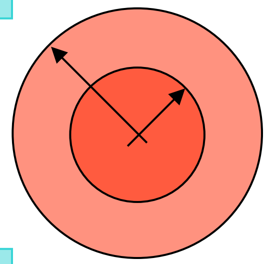
A constant current in the free space produces circular magnetic field.

If $\mu_r \approx 1$, the material, even in the case of a good conductor, does not affect the field lines.



Circular Perfectly Conducting Pipe (Beam at Center)





In the case of charge distribution, and $\gamma \rightarrow \infty$, the electric field lines are perpendicular to the direction of motion. The transverse fields intensity can be computed like in the static case, applying the Gauss and Ampere laws.

$$\lambda_o = \rho \pi a^2$$

$$\lambda(r) = \lambda_o (r/a)^2$$

$$J = \beta c \rho$$

$$I = J \pi a^2 = \beta c \lambda_0 (A)$$

for $r < a$

$$E_r(r) = \frac{\lambda_o r}{2\pi\varepsilon_o a^2}$$

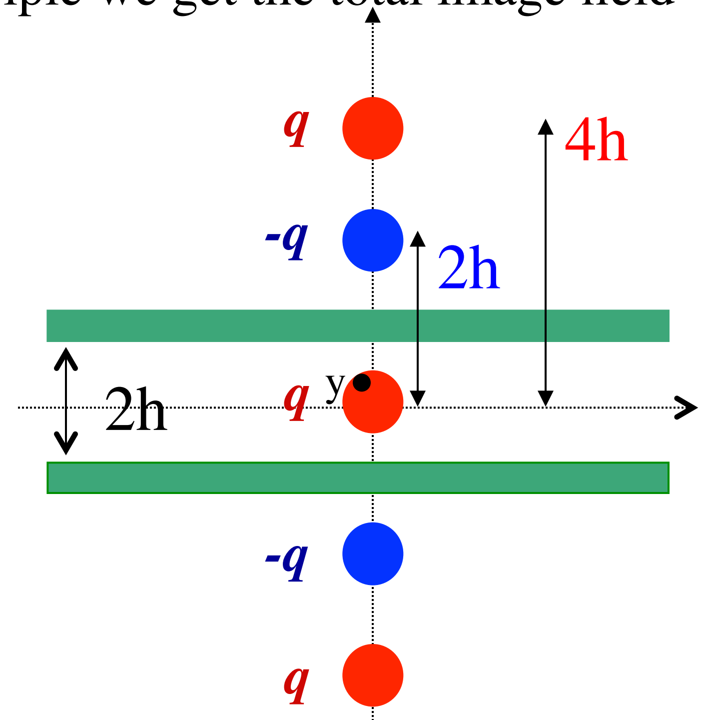
$$\mathrm{B}_\theta(r) = \frac{\beta}{c} E_r(r) = \frac{\lambda_o \beta}{2\pi\varepsilon_o c} \frac{r}{a^2}$$

$$F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1 - \beta^2\right) E_r = \frac{eE_r}{\gamma^2} = \frac{e\lambda_o r}{2\pi\varepsilon_o \gamma^2 a^2}$$

there is a cancellation of the electric and magnetic forces

Parallel Plates (beam at center)

In some cases, the beam pipe cross section is such that we can consider only the surfaces closer to the beam, which behave like two parallel plates. In this case, we use the image method to a charge distribution of radius *a* between two conducting plates *2h* apart. By applying the superposition principle we get the total image field at a position *y* inside the beam.



$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\varepsilon_o}\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{2nh+y} - \frac{1}{2nh-y}\right]$$

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\varepsilon_o}\sum_{n=1}^{\infty}(-1)^n\frac{-2y}{(2nh)^2 - y^2} \cong \frac{\lambda(z)}{4\pi\varepsilon_o h^2}\frac{\pi^2}{12}y$$

Where we have assumed: $h >> a > y$.

For d.c. or slowly varying currents, the boundary condition imposed by the conducting plates does not affect the magnetic field. We do not need "image currents "As a consequence there is no cancellation effect for the fields produced by the "image" charges.

From the divergence equation we derive also the other transverse component, notice the opposite sign:

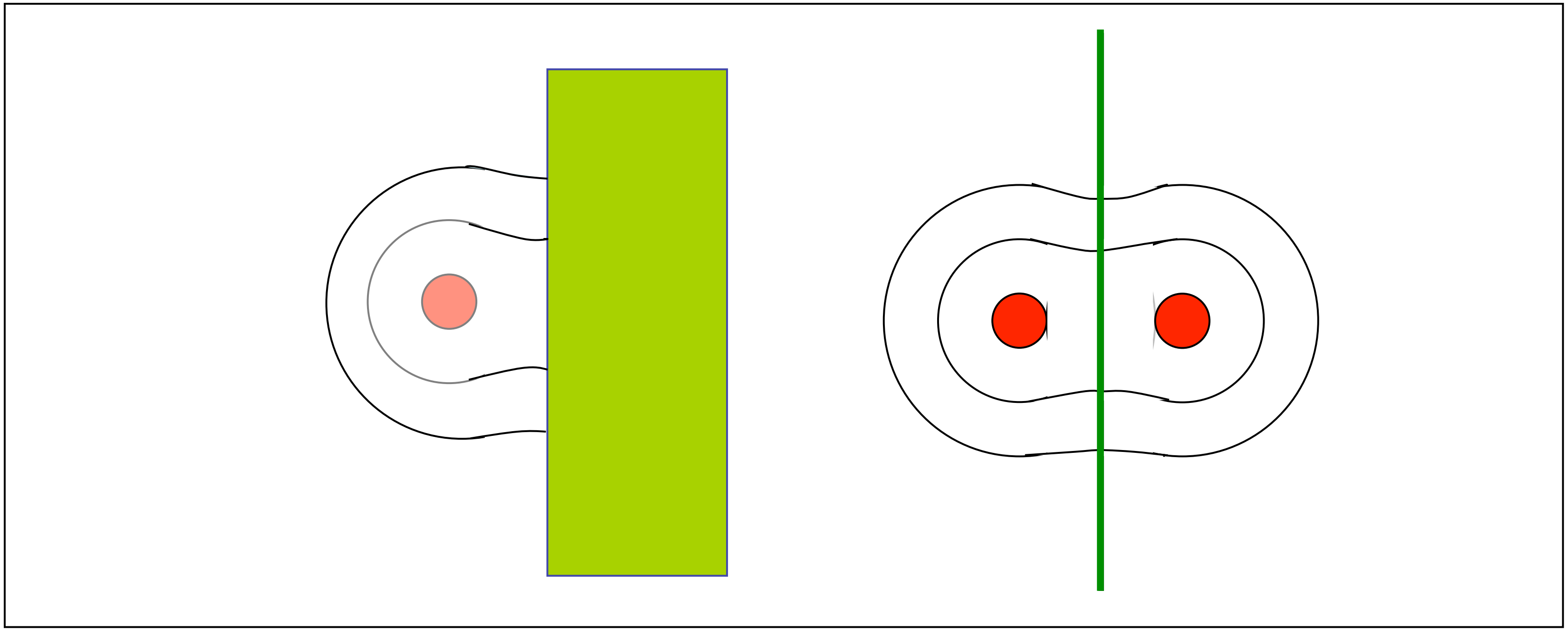
$$\frac{\partial}{\partial x}E_x^{im} = -\frac{\partial}{\partial y}E_y^{im} \Rightarrow E_x^{im}(z,x) = \frac{-\lambda(z)}{4\pi\,\varepsilon_o h^2}\frac{\pi^2}{12}x$$

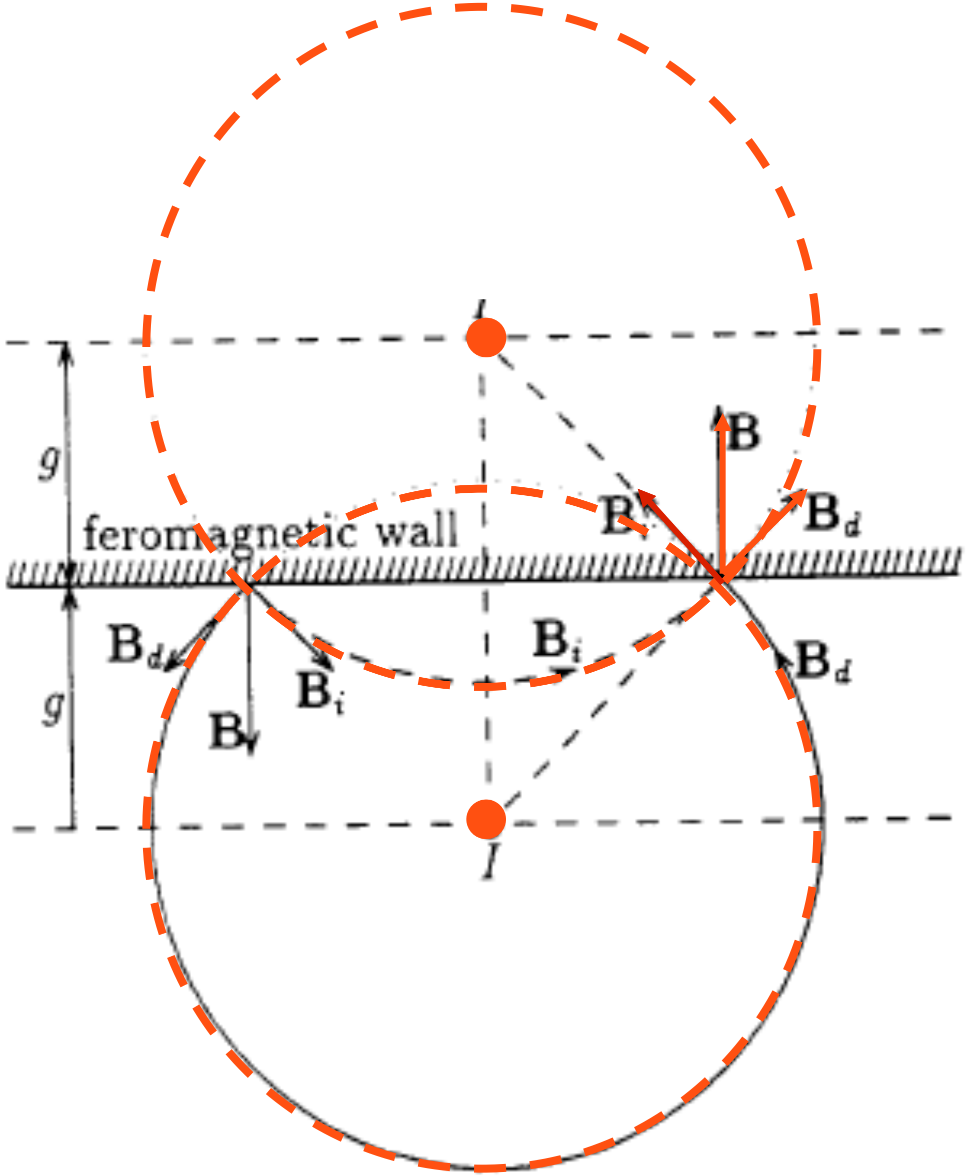
Including also the direct space charge force, we get:

$$\begin{cases} F_x(z,x) = \dfrac{e\lambda(z)x}{\pi\ \varepsilon_o}\left(\dfrac{1}{2a^2\gamma^2} - \dfrac{\pi^2}{48h^2}\right) \\ F_y(z,x) = \dfrac{e\lambda(z)y}{\pi\ \varepsilon_o}\left(\dfrac{1}{2a^2\gamma^2} + \dfrac{\pi^2}{48h^2}\right) \end{cases}$$

Therefore, for $\gamma \gg 1$, and for d.c. or slowly varying currents the cancellation effect applies only for the direct space charge forces. There is no cancellation of the electric and magnetic forces due to the "image" charges.

For ferromagnetic type, with $\mu_r >> 1$, the very high magnetic permeability makes the tangential magnetic field zero at the boundary so that the magnetic field is perpendicular to the surface, just like the electric field lines close to a conductor.

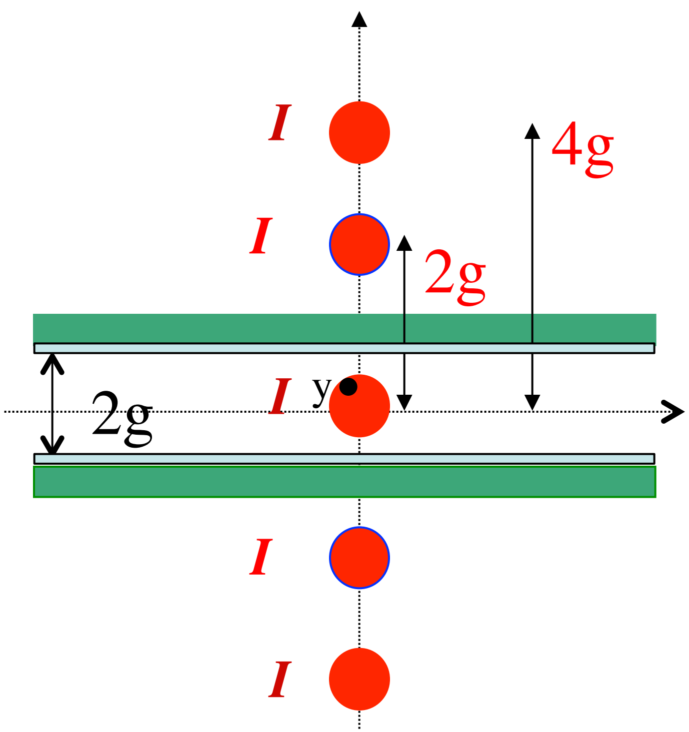




Satisfying a magnetic boundary condition by an image current

A. Hofmann

In analogy with the image method we get the magnetic field, in the region outside the material, as superposition of the fields due to two symmetric equal currents flowing in the same direction.



$$B_x^{im}(z,y) = \frac{\mu_o \beta c \lambda(z)}{2\pi} \sum_{n=1}^{\infty} \left[\frac{1}{2ng - y} - \frac{1}{2ng + y} \right]$$

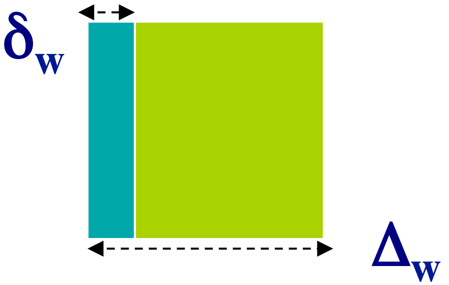
$$B_x^{im}(z,y) \cong \frac{\mu_o \beta c \lambda(z) y}{4\pi g^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\mu_o \beta c \lambda(z) \pi^2 y}{24\pi g^2}$$

$$F_x^{im}(z,x) \cong \frac{\beta^2 \lambda(z) \pi^2}{24\pi \varepsilon_o g^2} x$$

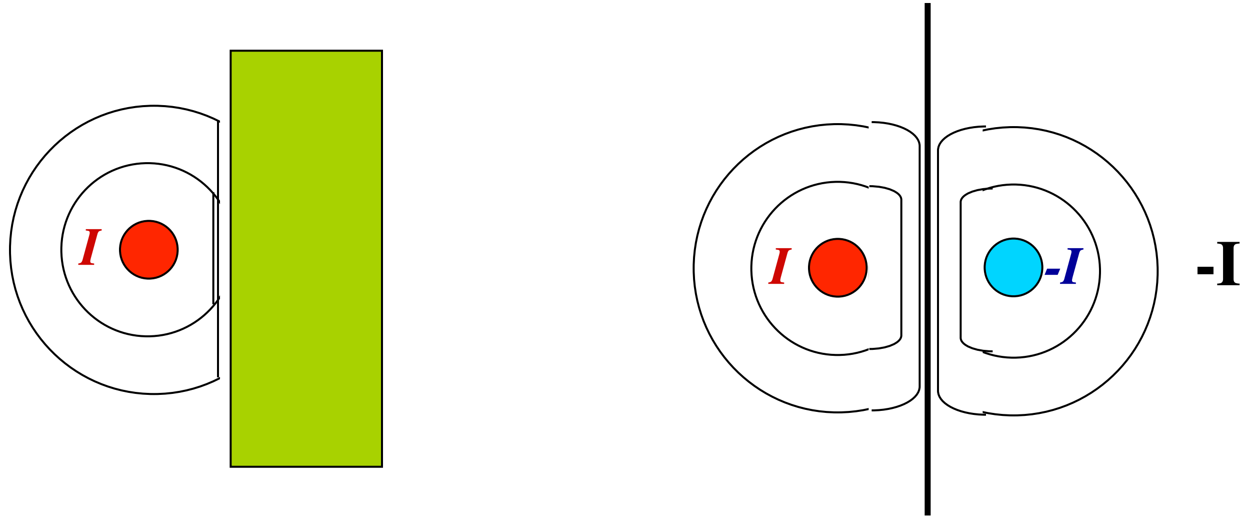
Time-varying fields

It is necessary to compare the wall thickness and the skin depth (region of penetration of the e.m. fields) in the conductor.

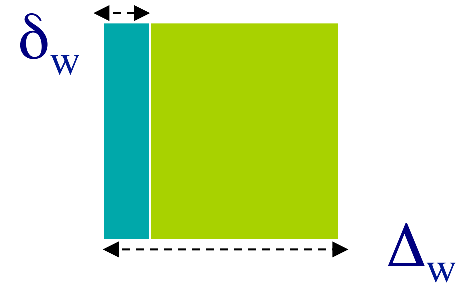
$$\delta_w \cong \sqrt{\frac{2}{\omega\sigma\mu}}$$



If the fields penetrate and pass through the material, we are practically in the static boundary conditions case. Conversely, if the skin depth is very small, fields do not penetrate, the electric filed lines are perpendicular to the wall, as in the static case, while the magnetic field line are tangent to the surface.



Parallel Plates (Beam at Center) a.c. currents

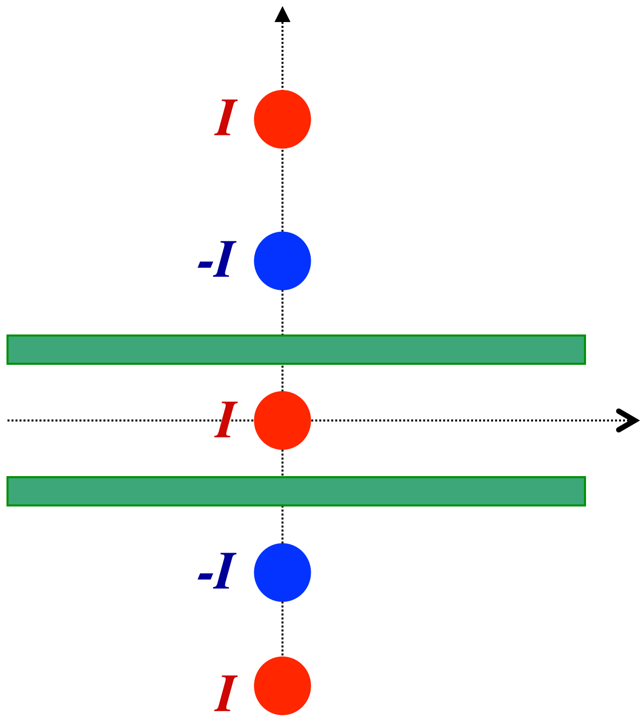


Usually, the frequency beam spectrum is quite rich of harmonics, especially for bunched beams.

It is convenient to decompose the current into a d.c. component, I, for which $\delta_w >> \Delta_w$, and an a.c. component, $\hat{I}$, for which $\delta_w << \Delta_w$.

While the d.c. component of the magnetic field does not perceives the presence of the material, its a.c. component is obliged to be tangent at the wall. For a charge density λ we have $I=\lambda v$.

We can see that this current produces a magnetic field able to cancel the effect of the electrostatic force.



$$\begin{cases} \tilde{E}_y(z,x) = \dfrac{\tilde{\lambda}(z)y}{\pi\ \varepsilon_o}\dfrac{\pi^2}{48h^2} \\ \tilde{B}_x(z,x) = \dfrac{\beta}{c}\tilde{E}_y(z,x) \end{cases}$$

$$\tilde{F}_y(z,x) = e\left(1-\beta^2\right)E_y = \frac{1}{\gamma^2}\frac{e\tilde{\lambda}(z)y}{\pi\ \varepsilon_o}\frac{\pi^2}{48h^2}$$

$$\begin{pmatrix} \tilde{F}_x(z,x) = \dfrac{e\tilde{\lambda}(z)x}{2\pi\ \varepsilon_o\gamma^2}\left(\dfrac{1}{a^2} - \dfrac{\pi^2}{24h^2}\right) \\ \tilde{F}_y(z,x) = \dfrac{e\tilde{\lambda}(z)y}{2\pi\ \varepsilon_o\gamma^2}\left(\dfrac{1}{a^2} + \dfrac{\pi^2}{24h^2}\right) \end{pmatrix}$$

There is cancellation of the electric and magnetic forces !!

Parallel Plates - General expression of the force

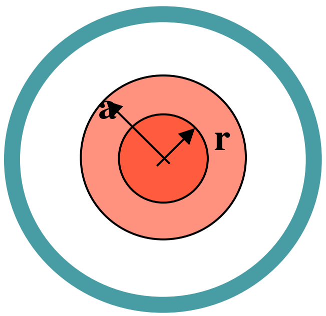
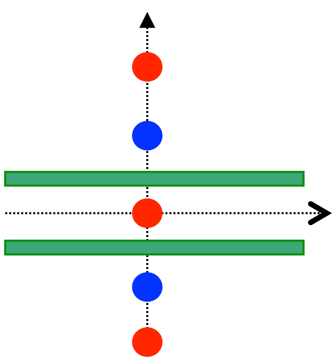
Taking into account all the boundary conditions for d.c. and a.c. currents, we can write the expression of the force as:

$$F_u = \frac{e}{2\pi\,\varepsilon_o}\left[\frac{1}{\gamma^2}\left(\frac{1}{a^2} \mp \frac{\pi^2}{24h^2}\right)\lambda \mp \beta^2\left(\frac{\pi^2}{24h^2} + \frac{\pi^2}{12g^2}\right)\bar{\lambda}\right]u$$

where λ is the total current, and $\bar{\lambda}$ its d.c. part. We take the sign (+) if u=y, and the sign (–) if u=x.

-L. J. Laslett, LBL Document PUB-6161, 1987, vol III

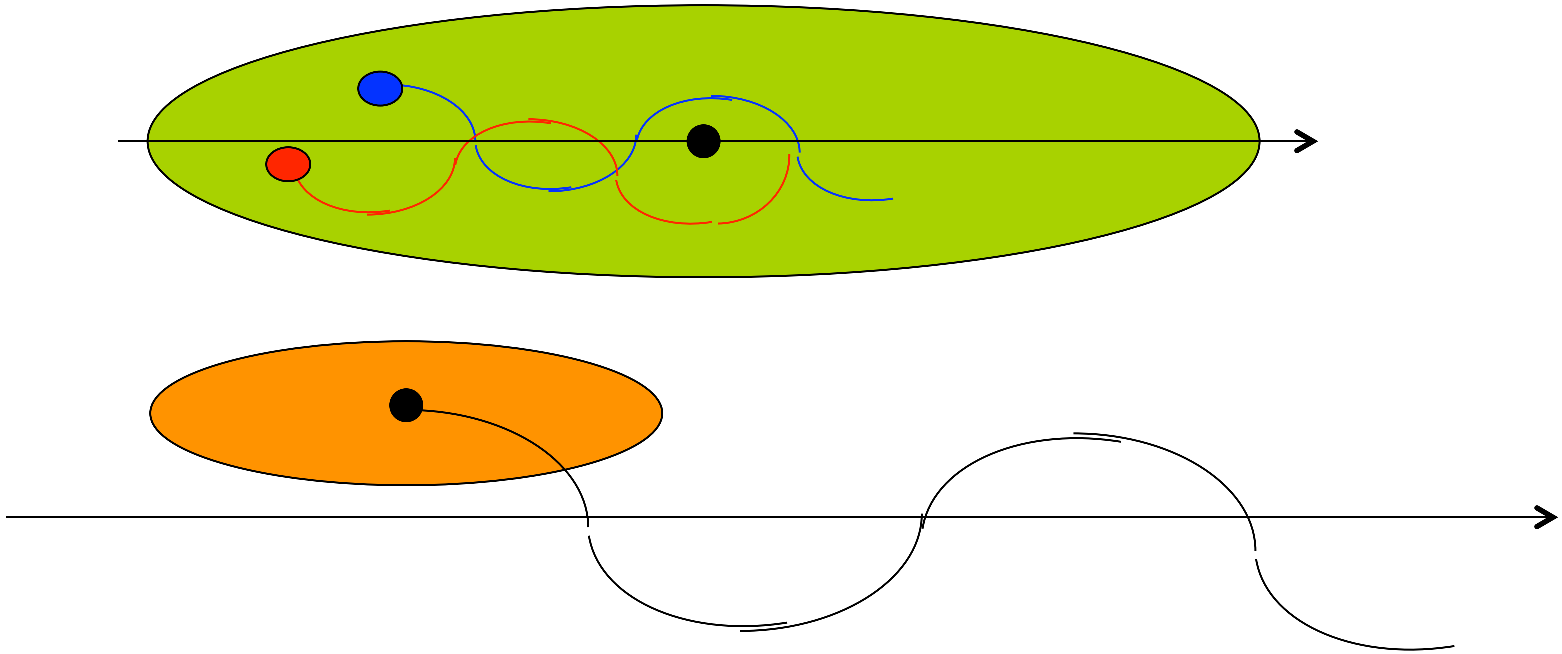
$$\lambda(z) = \lambda_o + \tilde{\lambda}\cos(k_z z) \quad ; \quad k_z = 2\pi / l_w$$

	D.C.	**A.C. $(\delta_w \ll \Delta_w)$**
	$F_\perp(r) = \dfrac{e}{\gamma^2}\dfrac{\lambda(z)}{2\pi\varepsilon_0}\dfrac{r}{a^2}$	
	$F_x(z,x) = \dfrac{e\lambda_0 x}{\pi\varepsilon_0}\left(\dfrac{1}{2a^2\gamma^2} - \dfrac{\pi^2}{48h^2}\right)$ $F_y(z,x) = \dfrac{e\lambda_0 y}{\pi\varepsilon_0}\left(\dfrac{1}{2a^2\gamma^2} + \dfrac{\pi^2}{48h^2}\right)$	$\tilde{F}_x(z,x) = \dfrac{e\tilde{\lambda}(z)x}{\pi\ \varepsilon_0\gamma^2}\left(\dfrac{1}{2a^2} - \dfrac{\pi^2}{48h^2}\right)$ $\tilde{F}_y(z,x) = \dfrac{e\tilde{\lambda}(z)y}{\pi\ \varepsilon_0\gamma^2}\left(\dfrac{1}{2a^2} + \dfrac{\pi^2}{48h^2}\right)$

Space charge effects in storage rings

Incoherent and Coherent Transverse Effects

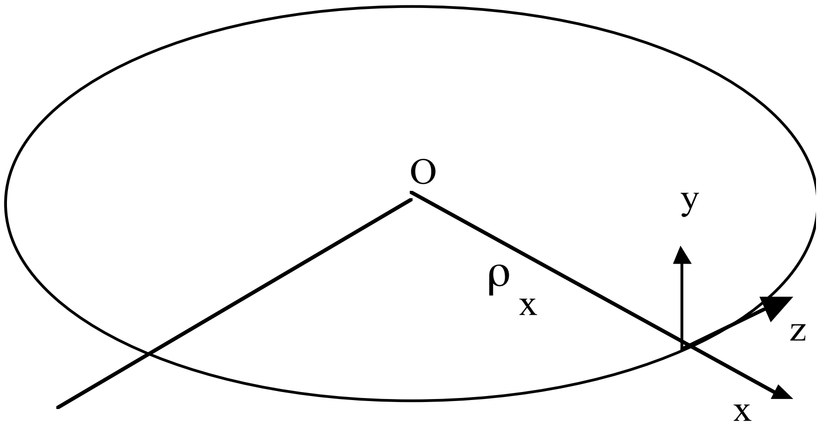
When the beam is located at the centre of symmetry of the pipe, the e.m. forces due to space charge and images cannot affect the motion of the centre of mass **(coherent),** but change the trajectory of individual charges in the beam **(incoherent).**



These force may have a complicate dependence on the charge position. A simple analysis is done considering only the linear expansion of the self-fields forces around the equilibrium trajectory.

Self Fields and betatron motion

Consider a perfectly circular accelerator with radius ρ_x. The beam circulates inside the beam pipe. The transverse single particle motion in the linear regime, is derived from the equation of motion. Including the self field forces in the motion equation, we have



$$\frac{d(m\gamma\,\boldsymbol{v})}{dt} = \boldsymbol{F}^{ext}(\vec{r}) + \boldsymbol{F}^{self}(\vec{r}) \qquad \frac{d\boldsymbol{v}}{dt} = \frac{\boldsymbol{F}^{ext}(\vec{r}) + \boldsymbol{F}^{self}(\vec{r})}{m\gamma}$$

In the analysis of the motion of the particles in presence of the self field, we will adopt a simplified model where particles execute simple harmonic oscillations around the reference orbit.
This is the case where the focussing term is constant. Although this condition in never fulfilled in a real accelerator, it provides a reliable model for the description of the beam instabilities

$$x''(s) + K_x x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x)$$

Q_x Betatron tune: n. of betatron oscillations per turn $\quad K_x = \left(\frac{Q_x}{\rho_x}\right)^2$

$$\boxed{x''(s) + \left(\frac{Q_x}{\rho_x}\right)^2 x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x,s)}$$

Transverse Incoherent Effects

We take the linear term of the transverse force in the betatron equation:

$$\left.\begin{aligned} F_x^{s.c.}(x,z) &\cong \left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0} x \\ x'' + \left(\frac{Q_x}{\rho_x}\right)^2 x &= \frac{1}{\beta^2 E_o}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0} x \end{aligned}\right\}$$

$$\boxed{x'' + \left(\left(\frac{Q_x}{\rho_x}\right)^2 - \frac{1}{\beta^2 E_o}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0}\right) x = 0}$$

$$\left(Q_x + \Delta Q_x\right)^2 \cong Q_x^2 + 2Q_x \Delta Q_x \Rightarrow \Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_o Q_x}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)$$

The betatron shift is negative since the space charge forces are defocusing on both planes. Notice that the tune shift is in general function of “z”, therefore there is a tune spread inside the beam.

Example: Incoherent betatron tune shift for an uniform electron beam of radius *a*, length l_o, inside circular perfectly conducting pipe

$$\left(\frac{\partial F_x^{s.c.}}{\partial x}\right) = \frac{\partial}{\partial x}\frac{e\lambda_o x}{2\pi\varepsilon_o\gamma^2 a^2} = \frac{e\lambda_o}{2\pi\varepsilon_o\gamma^2 a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 N e^2}{4\pi\varepsilon_o a^2\beta^2\gamma^2 E_o Q_{xo} l_o}$$

$$r_{e,p} = \frac{e^2}{4\pi\varepsilon_o m_o c^2} \quad (electrons: 2.82\ 10^{-15} m,\ protons: 1.53\ 10^{-18} m)$$

$$\Delta Q_x = -\frac{\rho_x^2 N r_{e,p}}{a^2\beta^2\gamma^3 Q_{xo} l_o}$$

For a real bunched beams the space charge forces, and the tune shift depend on the longitudinal and radial position of the charge.

Consequences of the space charge tune shifts

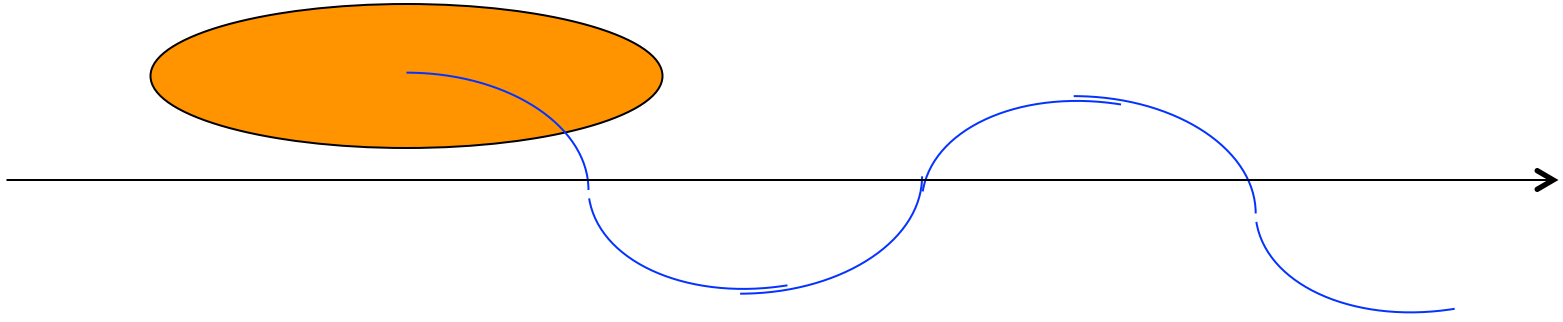
In circular accelerators the values of the betatron tunes should not be close to rational numbers in order to avoid the crossing of linear and non-linear resonances where the beam becomes unstable.

The tune spread induced by the space charge force can make hard to satisfy this basic requirement. Typically, in order to avoid major resonances the stability requires

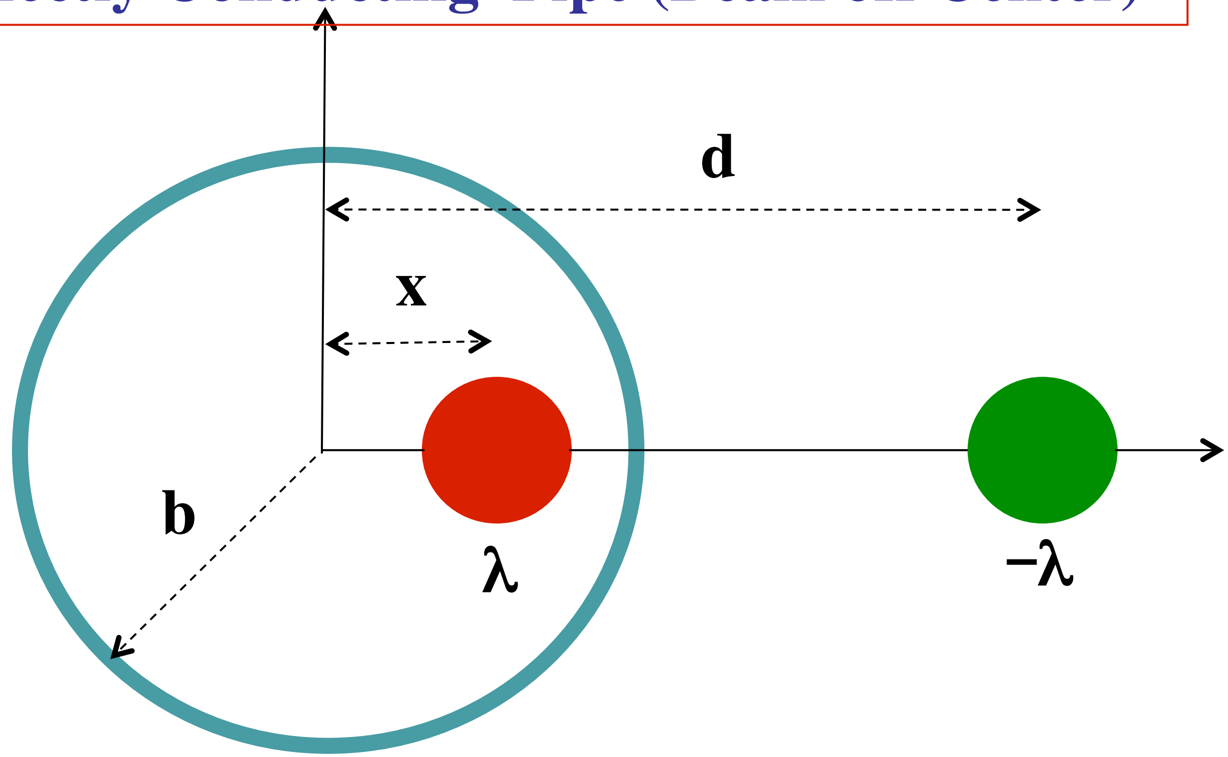
$$\left|\Delta Q_u\right| < 0.3$$

Transverse Coherent Effects

If the beam experiences a transverse deflection kick, it starts to perform betatron oscillations as a whole. The beam, source of the space charge fields moves transversely inside the pipe, while individual particles still continue their incoherent motion around the common coherent trajectory.



Circular Perfectly Conducting Pipe (Beam off Center)



$$\boxed{d = \frac{b^2}{x}}$$

The image charge is at a distance "d" such that the pipe surface is at constant voltage, and pulls the beam away from the center of the pipe.

The effect is defocusing: the horizontal electric image field E and the horizontal force F are:

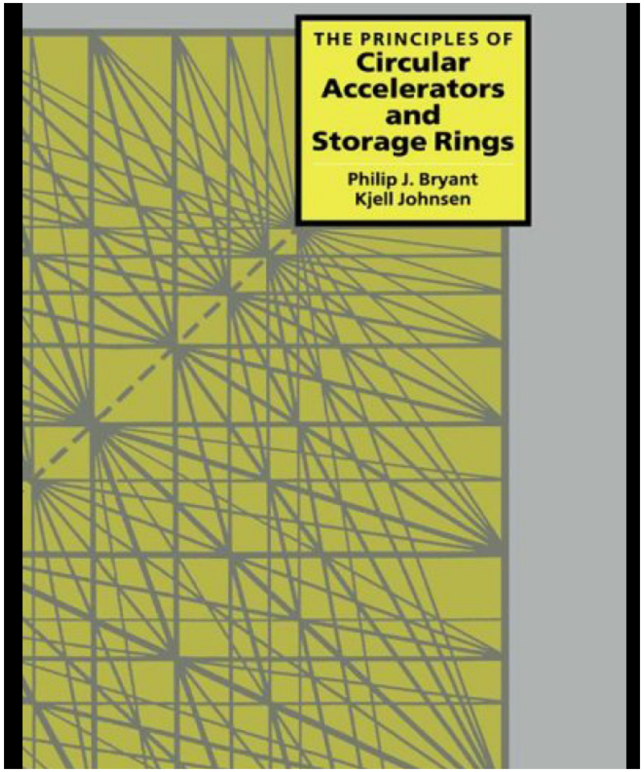
$$E_{xc}(\boldsymbol{x}) = \frac{\lambda(z)}{2\pi\varepsilon_0}\frac{1}{d-x} \approx \frac{\lambda(z)}{2\pi\varepsilon_0}\frac{1}{d} = \frac{\lambda(z)}{2\pi\varepsilon_0}\frac{x}{b^2}$$

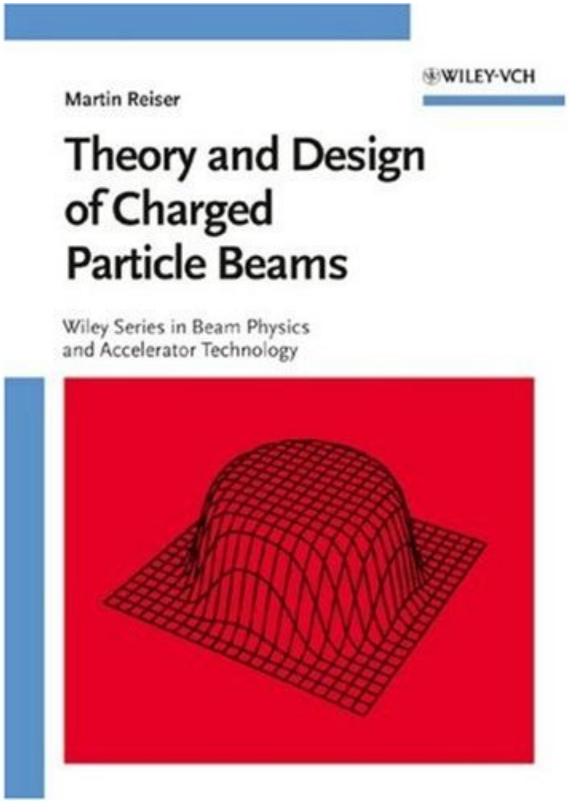
$$F_{xc}(r) \approx \frac{e\lambda(z)}{2\pi\varepsilon_0}\frac{x}{b^2}$$

$$\Delta Q_{xc} = -\frac{\rho_x^2}{2\beta^2 E_o Q_{x0}}\left(\frac{\partial F_{xc}}{\partial x}\right) = -\frac{\rho_x^2}{2\beta^2 E_o Q_{x0}}\frac{e\lambda(z)}{2\pi\,\varepsilon_o b^2}$$

$$\Delta Q_{xc} = -\frac{r_e\rho_x^2}{\beta^2\gamma Q_{x0}}\frac{N}{b^2 l_0}$$

$$r_{e,p} = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2}$$


THE PRINCIPLES OF
Circular
Accelerators
and
Storage Rings
Philip J. Bryant
Kjell Johnsen


WILEY-VCH
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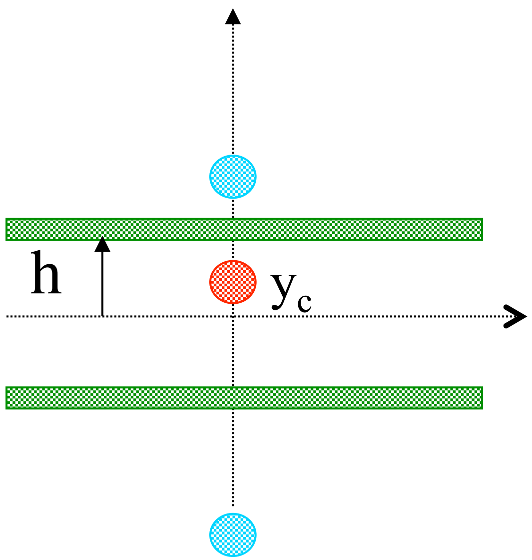
Theory and Design
of Charged
Particle Beams
Wiley Series in Beam Physics
and Accelerator Technology

THE END

Parallel Plates (Beam Off- Center)

If the whole beam is displaced from the axis by y_c, the image charges produce a transverse field which leads to coherent effects:



$$E_y^{im-c}(z, y_c) = \frac{\lambda(z)}{4\pi\varepsilon_o h^2}\frac{\pi^2}{4} y_c$$

$$\Delta Q_y^{c.} = \frac{-\rho_x^2}{2Q_u\beta^2 E_o}\left(\frac{\partial F_u^{s.c.}}{\partial u}\right)^{c.} = \frac{-\rho_x^2}{2\pi Q_y \beta^2 E_o}\frac{\lambda_0}{\varepsilon_o h^2}\frac{\pi^2}{16}$$

Time-varying Fields

Static electric fields vanish inside a conductor for any finite conductivity, while magnetic fields pass through unless of an high permeability.
This is no longer true for time changing fields, which can penetrate inside the material only in a region δ_w called skin depth. Inside the conducting material we write the following Maxwell equations:

$$\begin{cases} \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial \mathbf{t}} \\ \nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial \mathbf{t}} \end{cases} \qquad \begin{cases} \mathbf{B} = \mu \mathbf{H} \\ \mathbf{D} = \varepsilon \mathbf{E} \\ \mathbf{J} = \sigma \mathbf{E} \end{cases}$$

Copper $\sigma = 5.8\ 10^7\ (\Omega m^{)-1}$
Aluminium $\sigma = 3.5\ 10^7\ (\Omega m^{)-1}$
Stainless steel $\sigma = 1.4\ 10^6\ (\Omega m)$-1 .

Consider a plane wave (H_y,E_x) propagating in the material

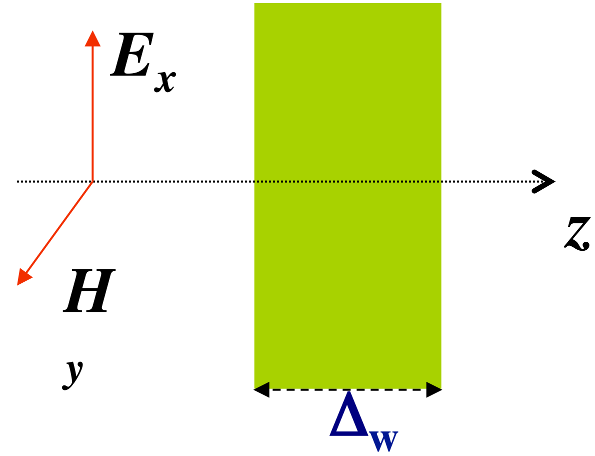
$$\frac{\partial^2 E_x}{\partial z^2} - \varepsilon\mu\frac{\partial^2 E_x}{\partial t^2} - \sigma\mu\frac{\partial E_x}{\partial t} = 0$$

(the same equation holds for H_y). Assuming that fields propagate in the z-direction with the law:

$$H_y = \tilde{H}_0 e^{i(\omega t - \kappa z)}$$

$$E_x = \tilde{E}_0 e^{i(\omega t - \kappa z)}$$

$$(\kappa^2 + \varepsilon\mu\omega^2 - i\omega\mu\sigma)\tilde{E}_o e^{i(\omega t - \kappa z)} = 0$$

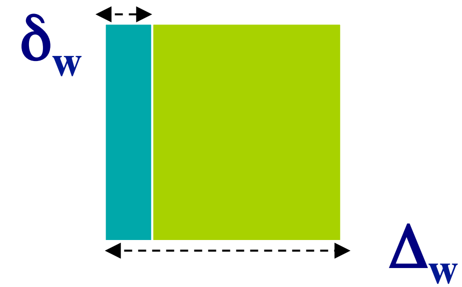


We say that the material behaves like a conductor if $\sigma \gg \omega\varepsilon$ thus:

$$\kappa \cong (1+i)\sqrt{\frac{\sigma\mu\omega}{2}}$$

Fields propagating along “z” are attenuated.
The attenuation constant is called skin depth δ_w:

$$\delta_w \cong \frac{1}{\Re(\kappa)} = \sqrt{\frac{2}{\omega\sigma\mu}}$$



The skin depth depends on the material properties and on the frequency. Fields pass through the conductor wall if $\delta_w > \Delta_w$. This happens at relatively low frequencies.

At higher frequencies, for a good conductor $\delta_w \ll \Delta_w$ and both electric and magnetic fields vanish inside the wall.

For the copper $\delta_w \cong \frac{6.66}{\sqrt{f}}(cm)$; $\omega = 2\pi f$

For a pipe 2mm thick, the fields pass through the wall up to 1 kHz. (Skin depth of Aluminium is larger by a factor 1.27)

•Compare the wall thickness and the skin depth (region of penetration of the e.m. fields) in the conductor.

• If the fields penetrate and pass through the material, they can interact with bodies in the outer region.

• If the skin depth is very small, fields do not penetrate, the electric filed lines are perpendicular to the wall, as in the static case, while the magnetic field line are tangent to the surface.