Emittance Preservation in Electron Accelerators

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What's so special about 'emittance preservation in electron machines' ?

- Theoretically, very little
  - it's all basically classical mechanics
- Fundamental difference is Synchrotron Radiation
  - Careful choice of lattice for high-energy arcs can mitigate excessive horizontal emittance growth
  - Storage rings: Theoretical Minimum Emittance (TME) lattices
- Electrons are quickly relativistic ($v/c \approx 1$)
  - I will not discuss non-relativistic beams
- Another practical difference: e± emittances tend to be significantly smaller (than protons)
  - High-brightness RF guns for linac based light sources
  - SR damping (storage ring light sources, HEP colliders) generate very small vertical emittances (flat beams)

Critical Emittance

- HEP colliders
  - Luminosity

$$L \propto \frac{1}{\sqrt{\beta_x^* \beta_y^* \varepsilon_x \varepsilon_y}}$$

- Light sources (storage rings)
  - Brightness

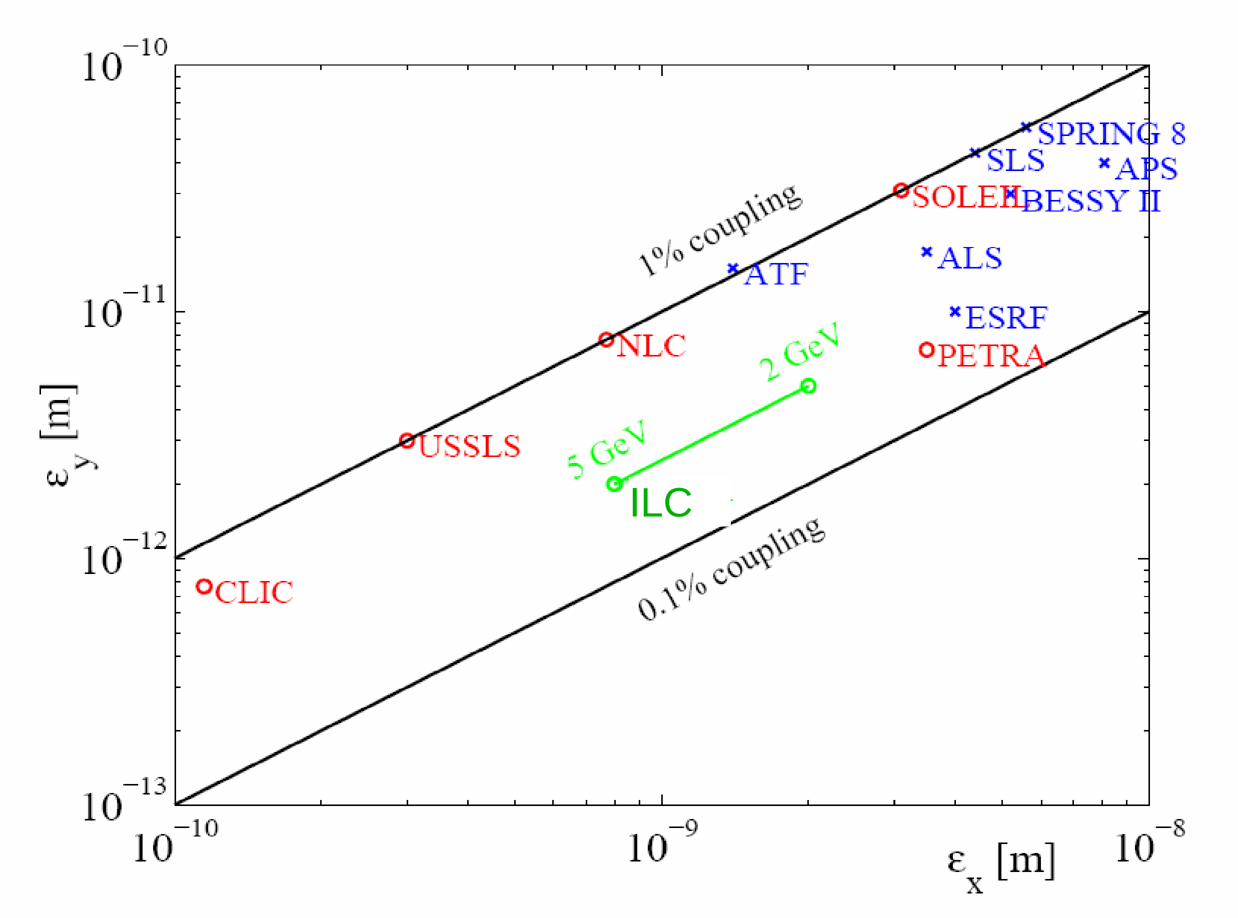
$$B \propto \frac{1}{\sqrt{\left(\varepsilon_x \beta_x + \sigma_r^2\right)\left(\varepsilon_x / \beta_x + \sigma_{r'}^2\right)\left(\varepsilon_y \beta_y + \sigma_r^2\right)\left(\varepsilon_y / \beta_y + \sigma_{r'}^2\right)}}$$

radiation emittance: $\sigma_r \sigma_{r'} = \dfrac{\lambda}{4\pi}$

- SASE XFEL
  - e.g. e-beam/photon beam overlap condition

$$\varepsilon < \frac{\lambda}{4\pi}$$

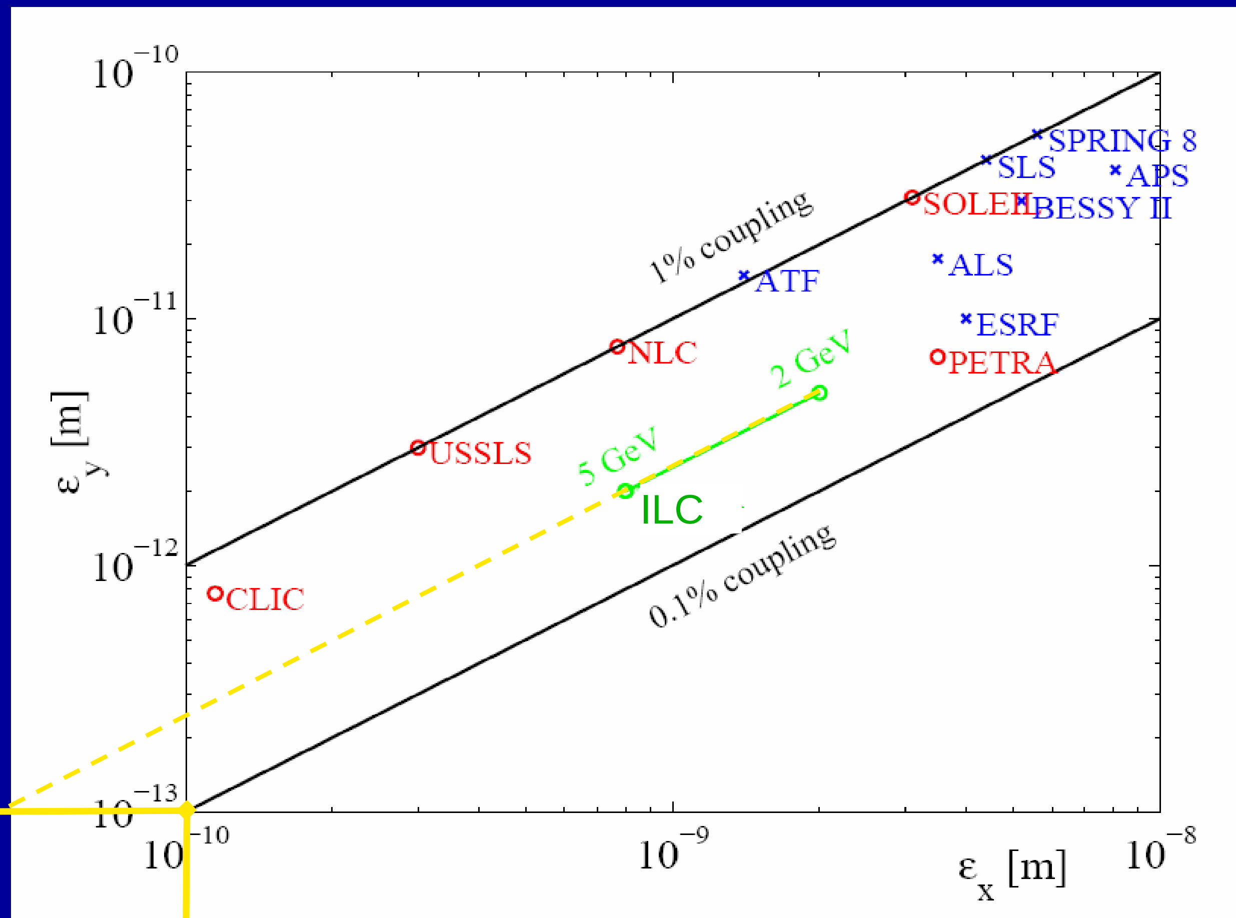
Typical Emittance Numbers



Typical Emittance Numbers

EURO XFEL
@20 GeV

$\varepsilon_r = 2.5\times10^{-11}\mathrm{m}$



10^{-11}

10^{-14}

ILC @250GeV

$$\varepsilon_x = 2\times10^{-11}\mathrm{m} \quad \left(\gamma\varepsilon_x = 10^{-5}\mathrm{m}\right.$$

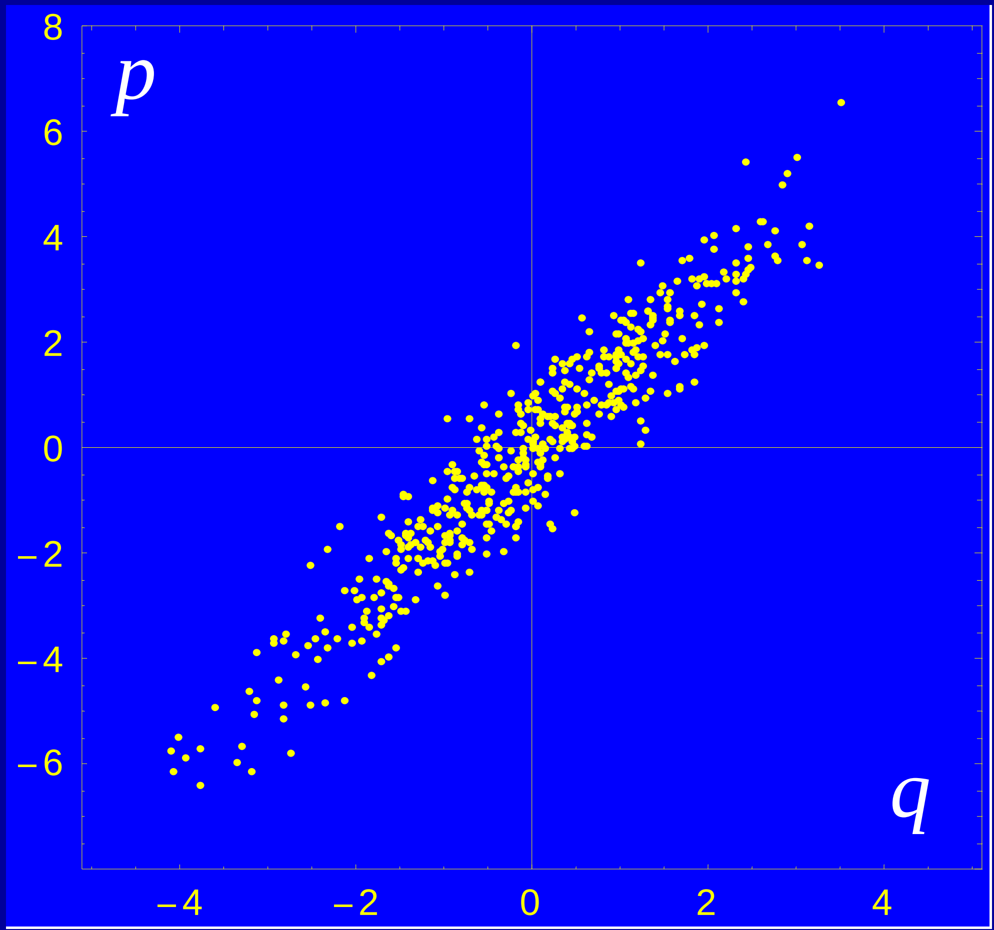
$$\varepsilon_y = 6\times10^{-14}\mathrm{m} \quad \left.\gamma\varepsilon_y = 3\times10^{-8}\mathrm{m}\right)$$

Back to Basics: Emittance Definition

$$\varepsilon = \oint p_i dq_i = cnt$$

<u>Liouville's theorem:</u>

Density in phase space is conserved (under conservative forces)



Back to Basics:
Emittance Definition

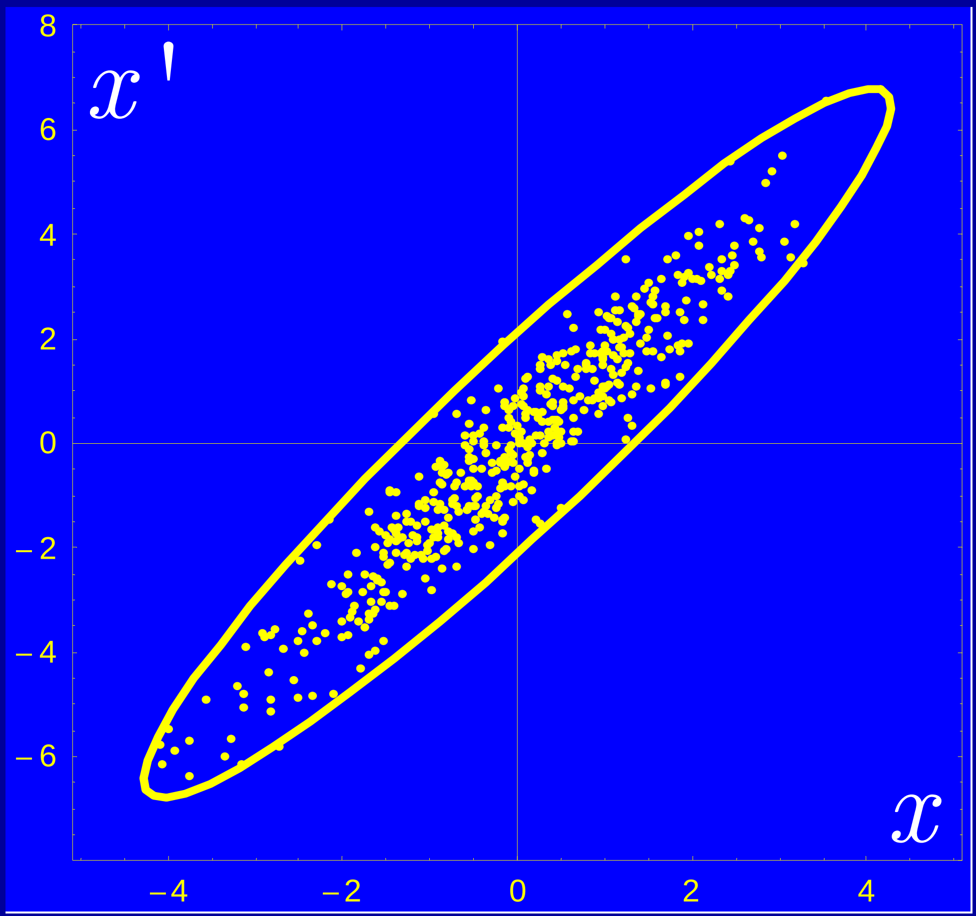
<u>Statistical Definition</u>

2^{nd}-order moments:

$$\sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix}$$

$$\varepsilon = \sqrt{|\sigma|}$$

$$= \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$



RMS emittance is not conserved!

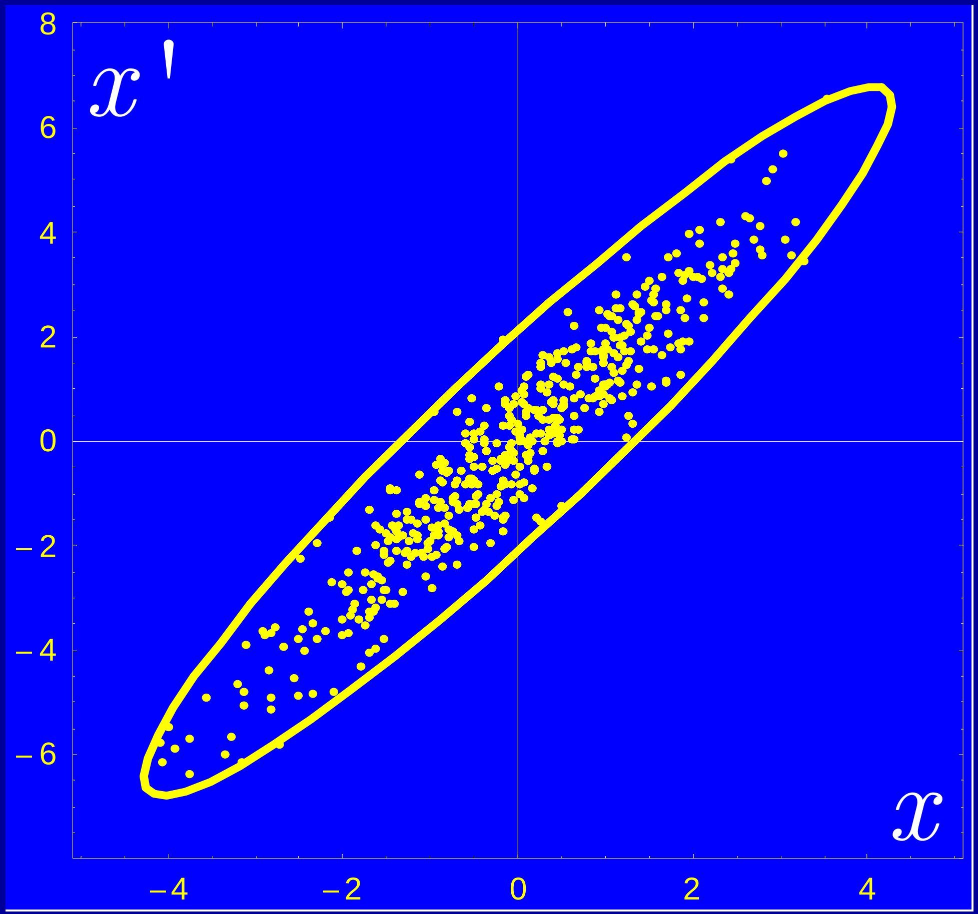
Back to Basics: Emittance Definition

<u>Statistical Definition</u>

Connection to TWISS parameters

$$\sigma = \begin{pmatrix} \varepsilon\beta & -\varepsilon\alpha \\ -\varepsilon\alpha & \varepsilon(1+\alpha^2)/\beta \end{pmatrix}$$

$$\sqrt{|\sigma|} = \varepsilon$$



Some Sources of (RMS) Emittance Degradation

- Synchrotron Radiation
- Collective effects
  - Space charge
  - Wakefields (impedance)
- Residual gas scattering
- Accelerator errors:
  - Beam mismatch
    - field errors
  - Spurious dispersion, x-y coupling
    - magnet alignment errors

Which of these mechanisms result in 'true' emittance growth?

High-Energy Linac

- Simple regular FODO lattice
  - No dipoles
- ‘drifts’ between quadrupoles filled with accelerating structures
- In the following discussions:
  - assume relativistic electrons
    - No space charge
    - No longitudinal motion within the bunch (‘synchrotron’ motion)

Linearised Equation of Motion in a LINAC

Remember Hill's equation: $$y''(s) + K(s)y(s) = 0$$

Must now include effects of acceleration:

$$y''(s) + \frac{\gamma'(s)}{\gamma(s)}y'(s) + K(s)y(s) = 0$$

adiabatic damping

Include lattice chromaticity (first-order in $\delta \equiv \Delta p/p_0$):

$$y''(s) + \frac{\gamma'(s)}{\gamma(s)}y'(s) + [1-\delta(s)]K(s)y(s) = 0$$

And now we add the errors…

(RMS) Emittance Growth Driving Terms

“Dispersive” effect from quadrupole offsets

$$y''(s) + \frac{\gamma'(s)}{\gamma(s)} y'(s) + [1 - \delta(s)] K(s) \big(y(s) - y_q(s) \big) = 0$$

quadrupole offsets

put error source on RHS (driving terms)

$$y''(s) + \frac{\gamma'(s)}{\gamma(s)} y'(s) + K(s) y(s)$$

$$= -K(s) y_q(s) + \delta(s) K(s) y_q(s) + \delta(s) K(s) y(s)$$

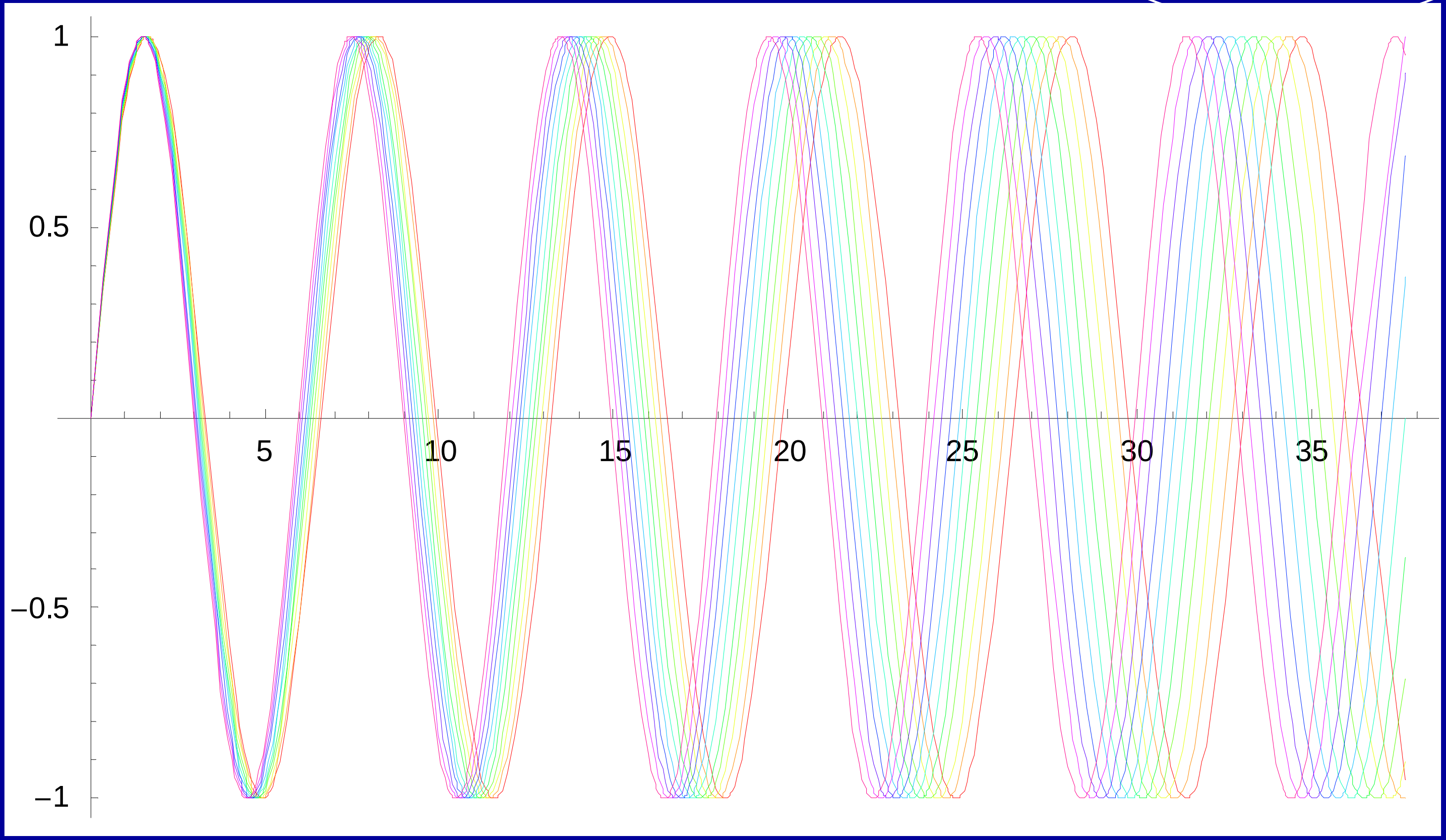
trajectory kicks from offset quads

dispersive kicks from offset quads

dispersive kicks from coherent β-oscillation

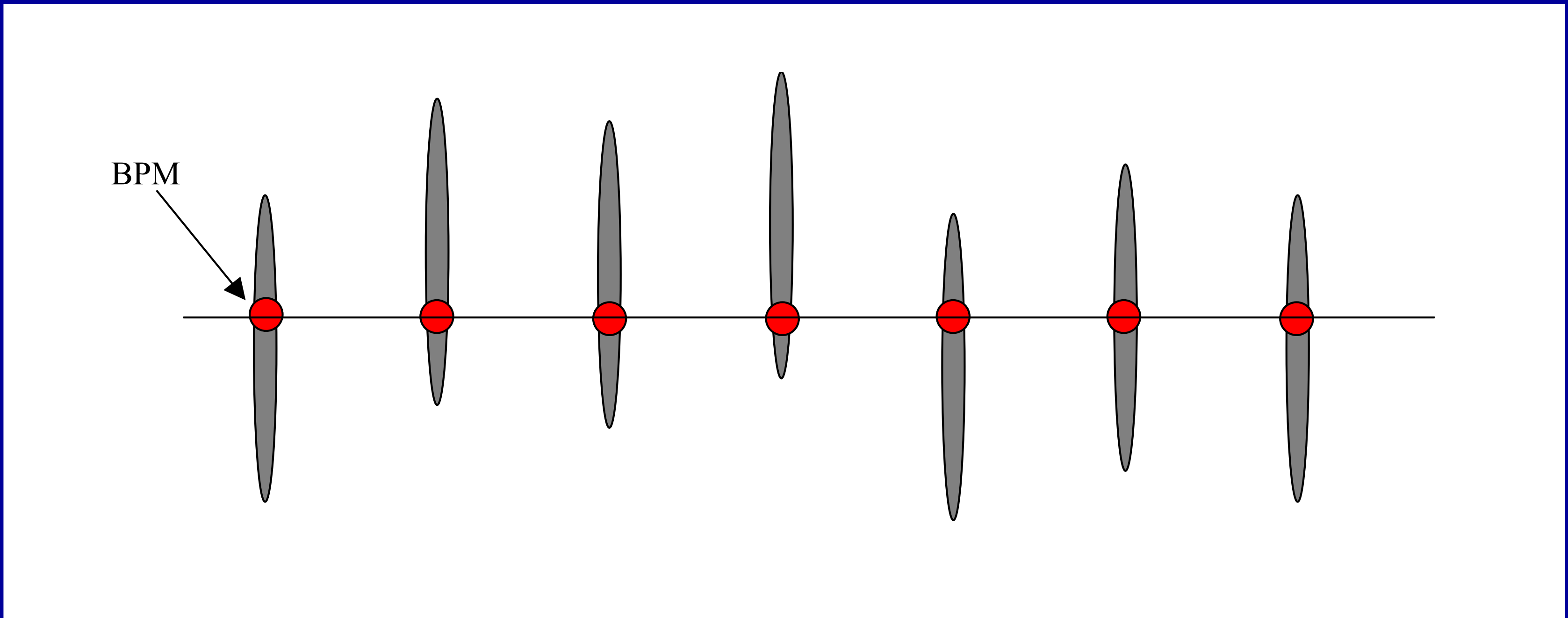
Coherent Oscillation

$$y''(s) + \frac{\gamma'(s)}{\gamma(s)} y'(s) + K(s)y(s)$$
$$= -K(s)y_q(s) + \delta(s)K(s)y_q(s) + \delta(s)K(s)y(s)$$



Just chromaticity repackaged

Scenario 1: Quad offsets, but BPMs aligned



Assuming:

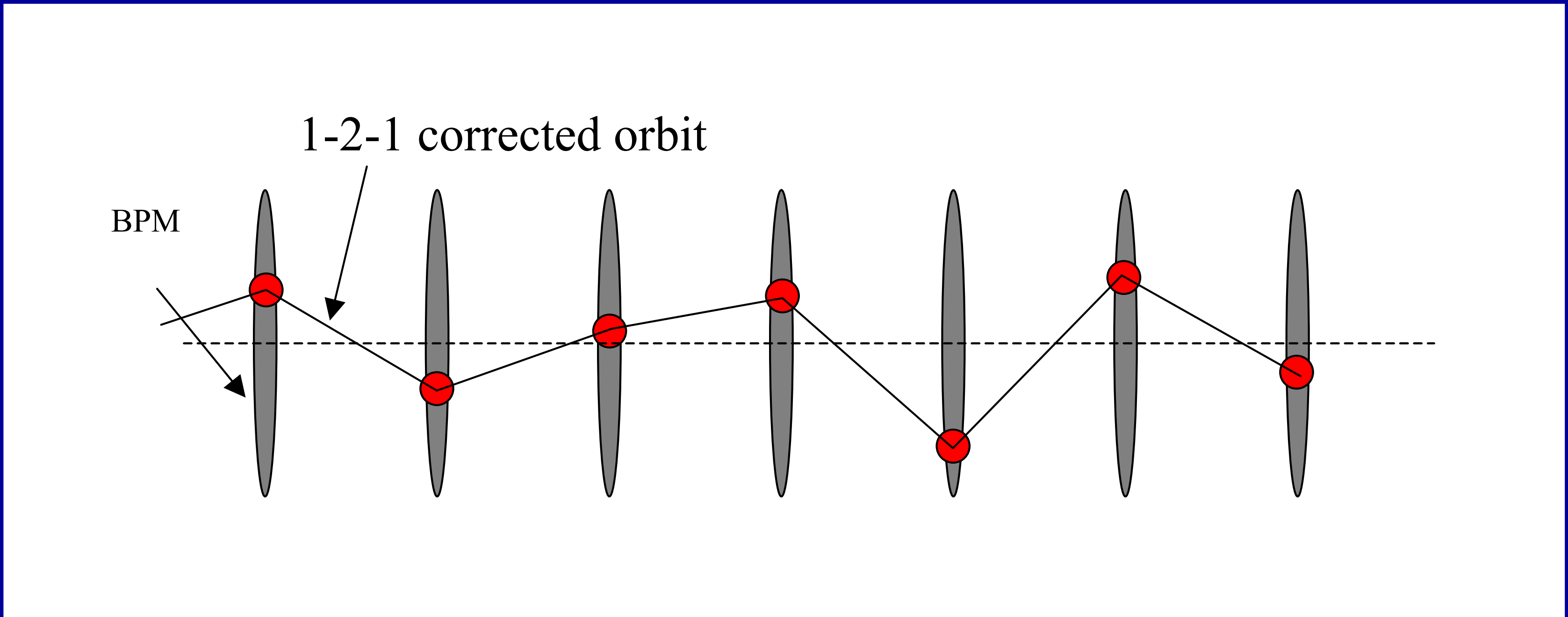
- a BPM adjacent to each quad

- a ‘steerer’ at each quad

steerer $\begin{cases} \text{quad mover} \\ \text{dipole corrector} \end{cases}$

simply apply one to one steering to orbit

Scenario 2: Quads aligned, BPMs offset



one-to-one correction BAD!

Resulting orbit not <u>Dispersion Free</u> ⇒ <u>emittance growth</u>

Need to find a steering algorithm which effectively puts BPMs on (some) reference line

real world scenario: some mix of scenarios 1 and 2

Dispersive Emittance Growth

After trajectory correction (one-to-one steering)

$$\frac{\Delta\varepsilon}{\varepsilon} \propto \frac{\delta_{RMS}^2}{\varepsilon}\frac{1}{\beta_0^2}\frac{1}{1-\alpha}\left[\left(\frac{\gamma_f}{\gamma_i}\right)^{2-2\alpha} - 1\right]\left\langle y_{BPM}^2 \right\rangle \qquad \beta(s) \propto \gamma^{\alpha}(s)$$

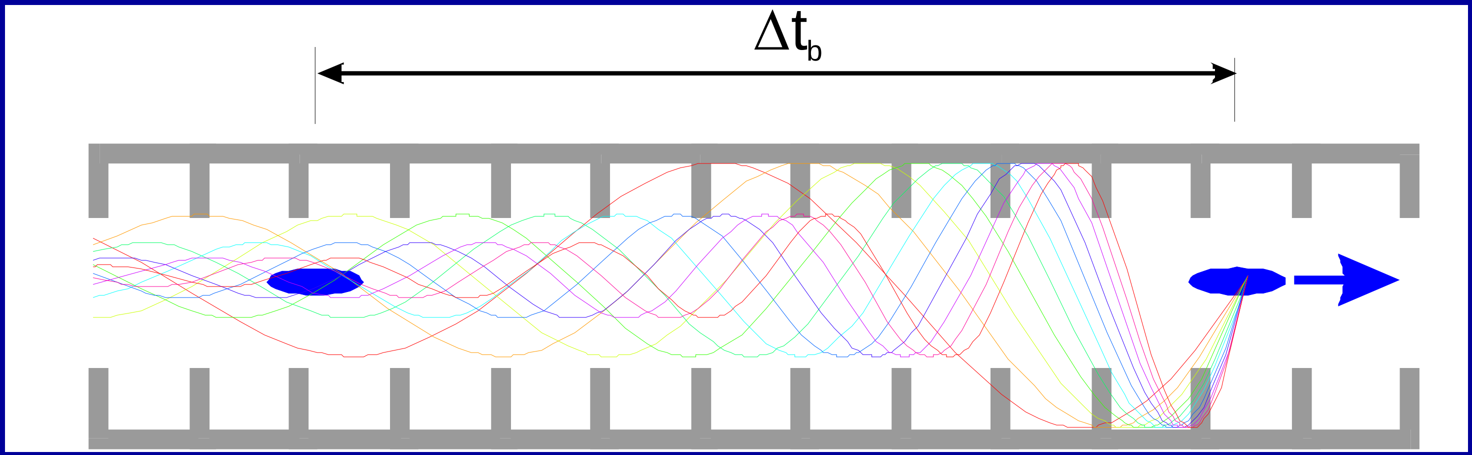
scaling of lattice along linac

Reduction of dispersive emittance growth favours weaker lattice (i.e. larger β functions)

Wakefields and Beam Dynamics

- bunches traversing cavities generate many RF modes.
- higher-order (higher-frequency) modes (HOMs) can act back on the beam and adversely affect it.
- Separate into two time (frequency) domains:
  - long-range, bunch-to-bunch
  - short-range, single bunch effects (head-tail effects)

Long Range Wakefields



$$V(\omega,t) = I(\omega,t)Z(\omega,t)$$

Bunch 'current' generates wake that decelerates trailing bunches.

Bunch current generates transverse deflecting modes when bunches are not on cavity axis

Fields build up resonantly: latter bunches are kicked transversely

$\Rightarrow$ multi- and single-bunch beam break-up (MBBU, SBBU)

wakefield is the time-domain description of impedance

Transverse HOMs

wake is sum over modes: $$W_{\perp}(t) = \sum_n \frac{2k_n c}{\omega_n} e^{-\omega_n t/2Q_n} \sin(\omega_n t)$$

k_n is the *loss parameter* (units $V/pC/m^2$) for the n^{th} mode

Transverse kick of j^{th} bunch after traversing one cavity:

$$\Delta y'_j = \sum_{i=1}^{j-1} \frac{y_i q_i}{E_i} \frac{2k_i c}{\omega_n} e^{-\omega_n i \Delta t/2Q_n} \sin\left(\omega_i i \Delta t_b\right)$$

where y_i, q_i, and $E_{i,}$ are the offset *wrt* the cavity axis, the charge and the energy of the i^{th} bunch respectively.

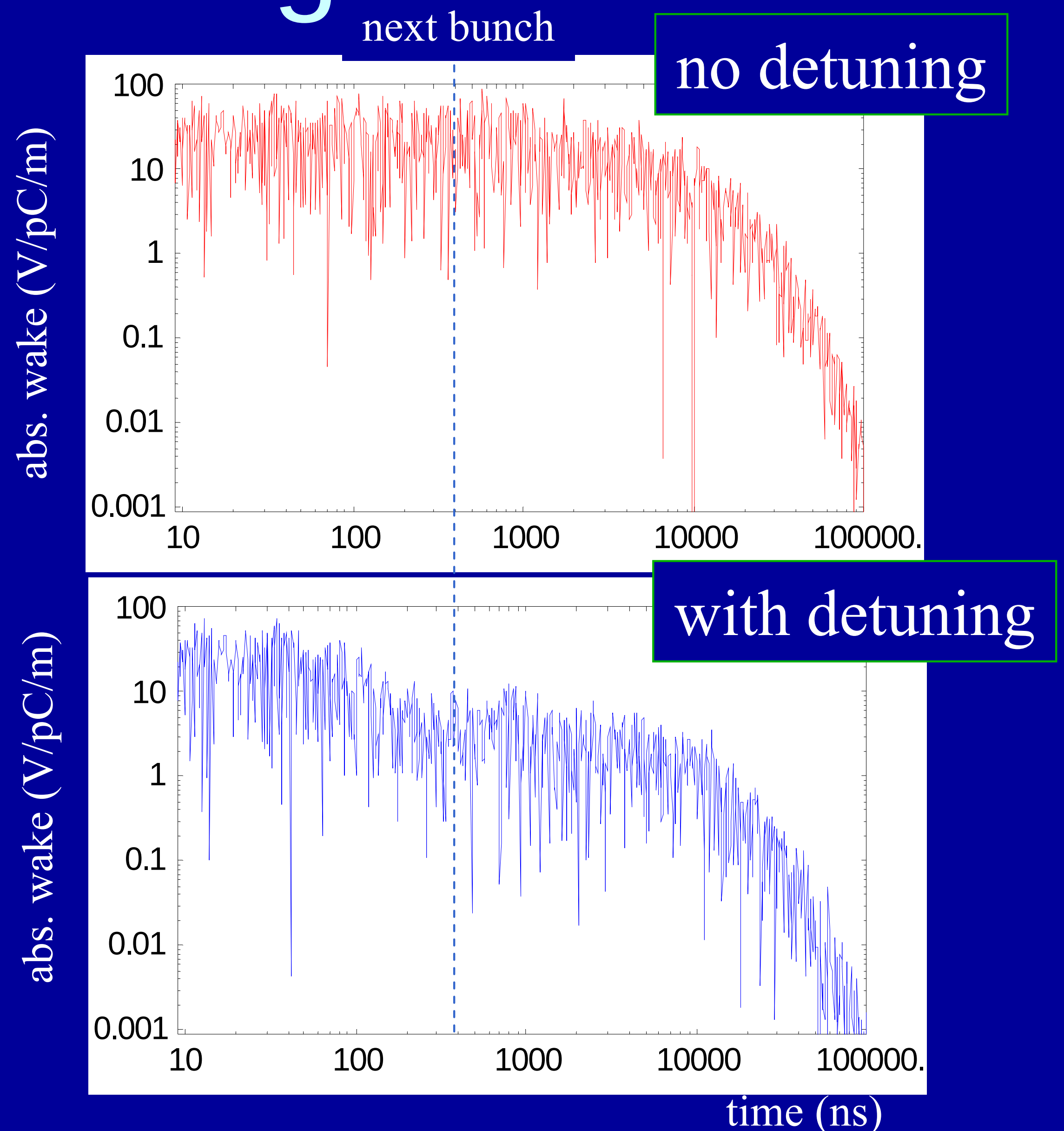
Detuning

HOMs cane be randomly detuned by a small amount.

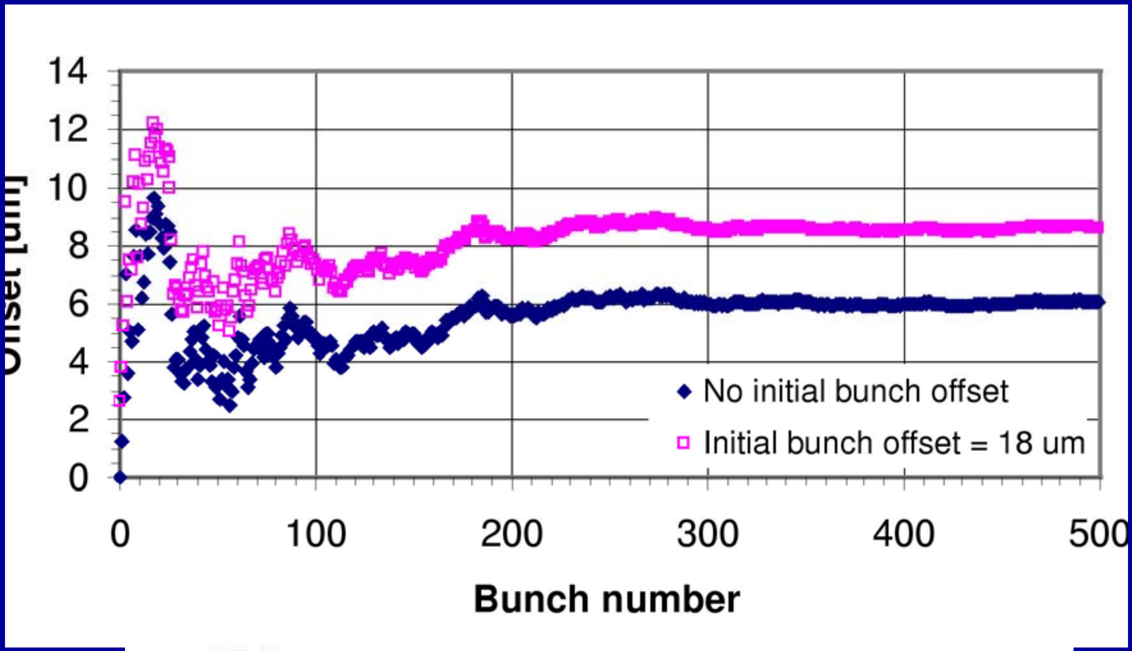
Over several cavities, wake ‘decoheres’.

Effect of random 0.1% detuning
(averaged over 36 cavities).

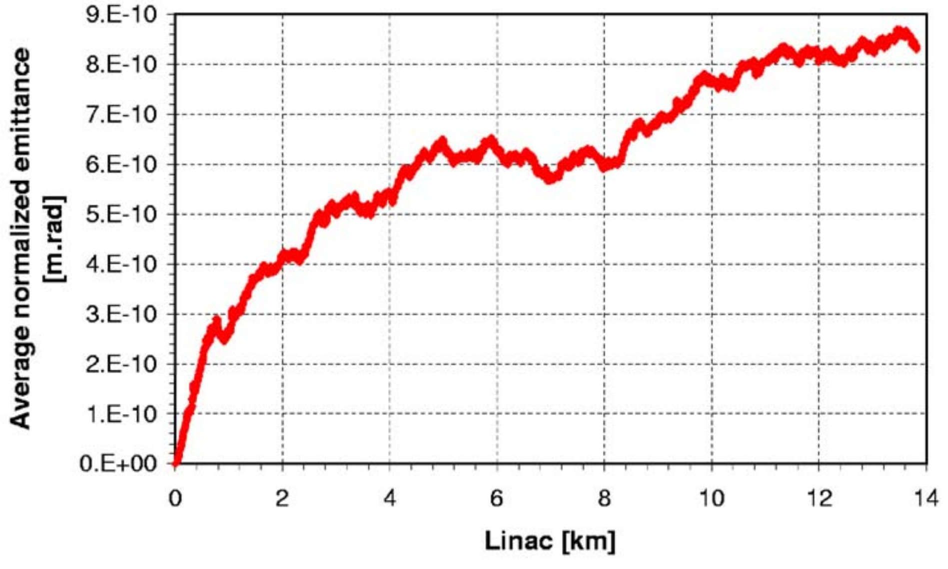
Still require HOM dampers



Effect of Emittance



vertical beam offset along bunch train (n_b = 2920)



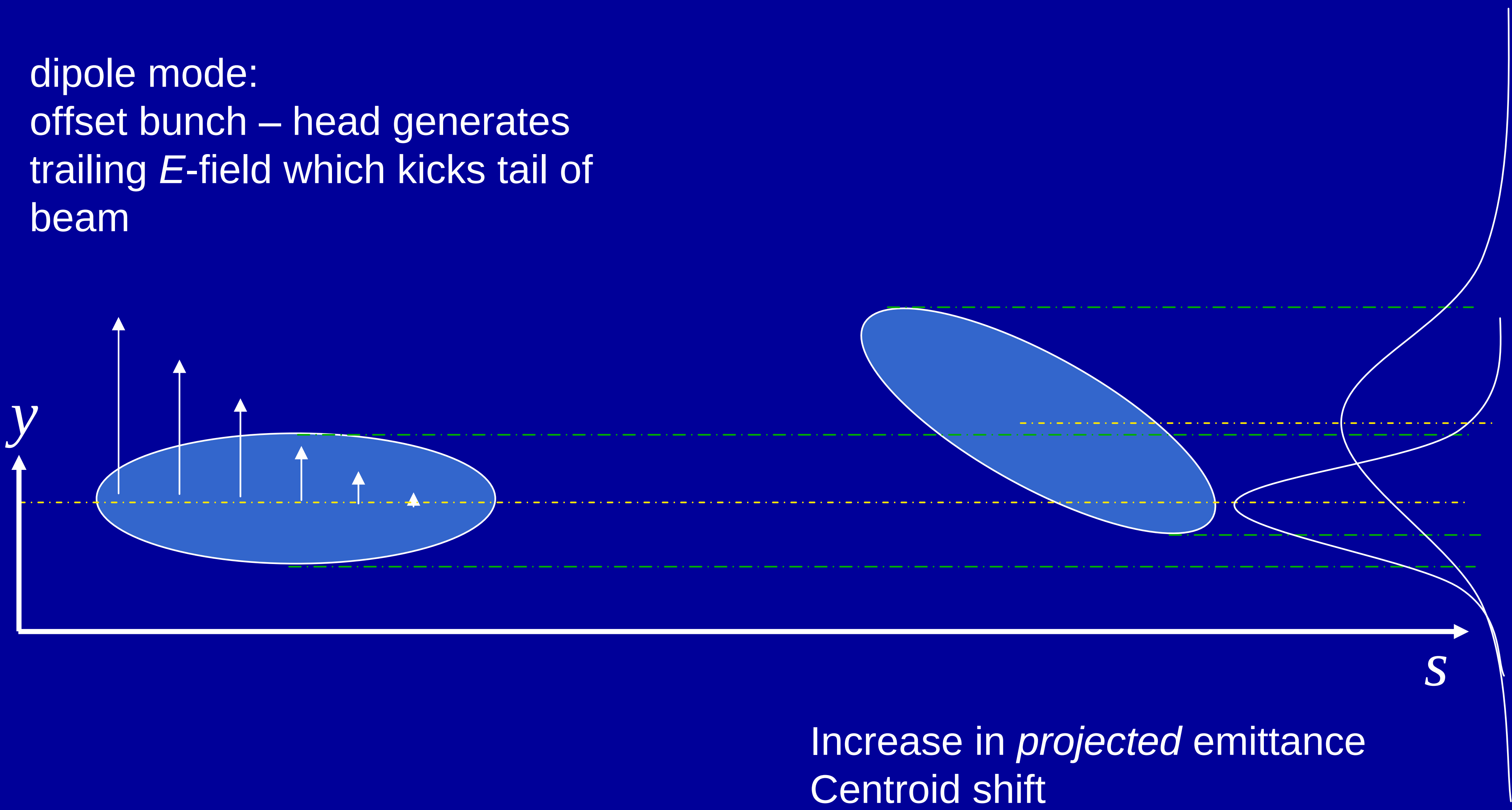
Multibunch emittance growth for cavities with 500µm RMS misalignment

Single Bunch Effects

- Completely analogous to low-range wakes
- wake over a single bunch
- causality (relativistic bunch): head of bunch affects the tail
- Again must consider
  - longitudinal: effects energy spread along bunch
  - transverse: the emittance killer!
- For short-range wakes, tend to consider wake potentials (Greens functions) rather than ‘modes

Transverse Wakefields

dipole mode:
offset bunch – head generates trailing *E*-field which kicks tail of beam



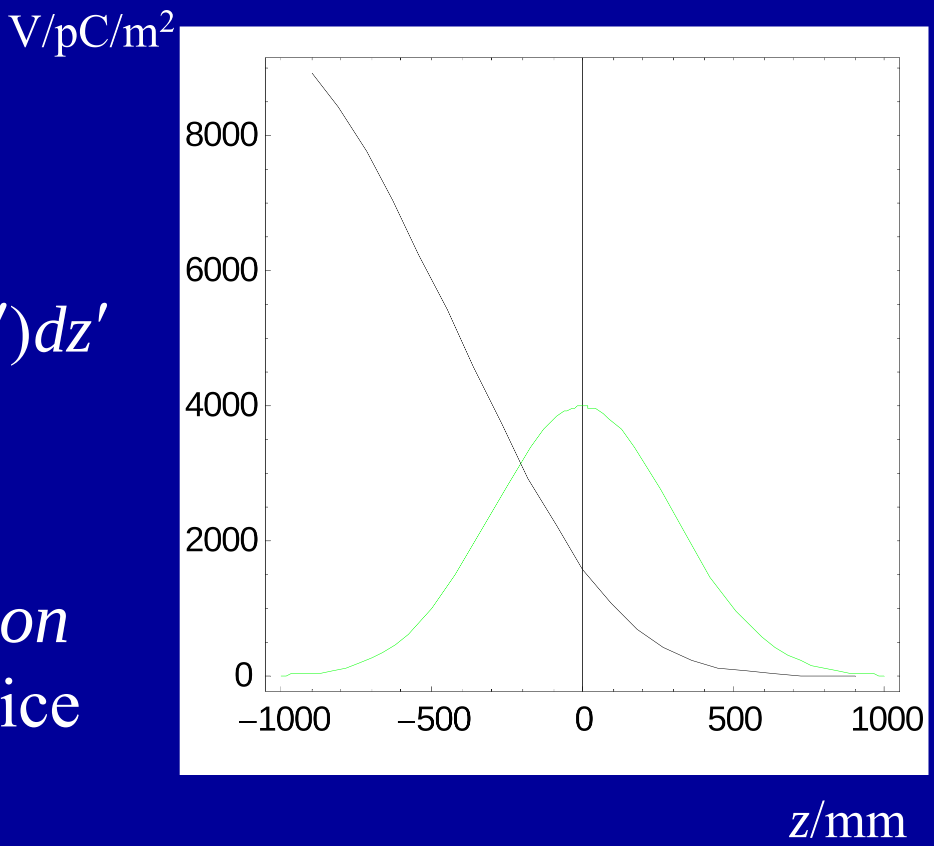
Increase in *projected* emittance
Centroid shift

Transverse Single-Bunch Wakes

When bunch is offset wrt cavity axis, transverse (dipole) wake is excited.

'kick' along bunch:

$$\Delta y'(z) = \frac{q_b}{E(z)} \int_{z'=z}^{\infty} W_{\perp}(z'-z)\rho(z')y(s;z')dz'$$

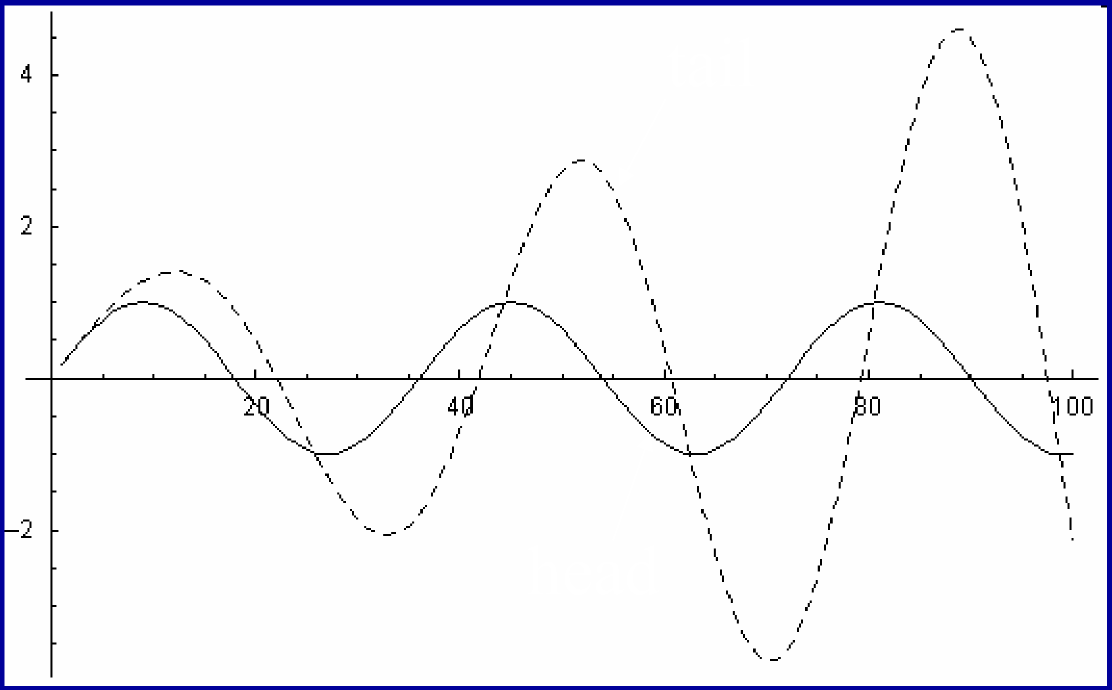


Note: $y(s;z)$ describes a free *betatron* oscillation along linac (FODO) lattice (as a function of s)

2 particle model

Effect of coherent betatron oscillation

- head *resonantly* drives the tail



head eom (Hill's equation):

$$y_1'' + k_\beta^2 y_1 = 0$$

solution:

$$y_1(s) = \sqrt{a\beta(s)}\,\sin\left(\varphi(s) + \varphi_0\right)$$

tail eom:

$$y_2'' + k^2 y_2 = y_1 \frac{W'_\perp \frac{q}{2} 2\sigma_z}{E_{beam}}$$

resonantly driven oscillator

BNS Damping

If both macroparticles have an initial offset y_0 then particle 1 undergoes a sinusoidal oscillation, $y_1 = y_0 \cos(k_\beta s)$. What happens to particle 2?

$$y_2 = y_0 \left[\cos\left(k_\beta s\right) + s \sin\left(k_\beta s\right) \frac{W'_\perp q \sigma_z}{2 k_\beta E_{beam}} \right]$$

Qualitatively: an additional oscillation out-of-phase with the betatron term which grows monotonically with s.

How do we beat it? Higher beam energy, stronger focusing, lower charge, shorter bunches, or a damping technique recommended by Balakin, Novokhatski, and Smirnov *(BNS Damping)*

curtesy: P. Tenenbaum (SLAC)

BNS Damping

Imagine that the two macroparticles have different betatron frequencies, represented by different focusing constants $k_{\beta 1}$ and $k_{\beta 2}$

The second particle now acts like an undamped oscillator driven off its resonant frequency by the wakefield of the first. The difference in trajectory between the two macroparticles is given by:

$$y_2 - y_1 = y_0 \left(1 - \frac{W'_{\perp} q \sigma_z}{E_{beam}} \frac{1}{k_{\beta 2}^2 - k_{\beta 1}^2} \right) \left[\cos\left(k_{\beta 2} s\right) - \cos\left(k_{\beta 1} s\right) \right]$$

curtesy: P. Tenenbaum (SLAC)

BNS Damping

The wakefield can be locally cancelled (ie, cancelled at all points down the linac) if:

$$\frac{W'_{\perp} q \sigma_z}{E_{beam}} \frac{1}{k_{\beta 2}^2 - k_{\beta 1}^2} = 1$$

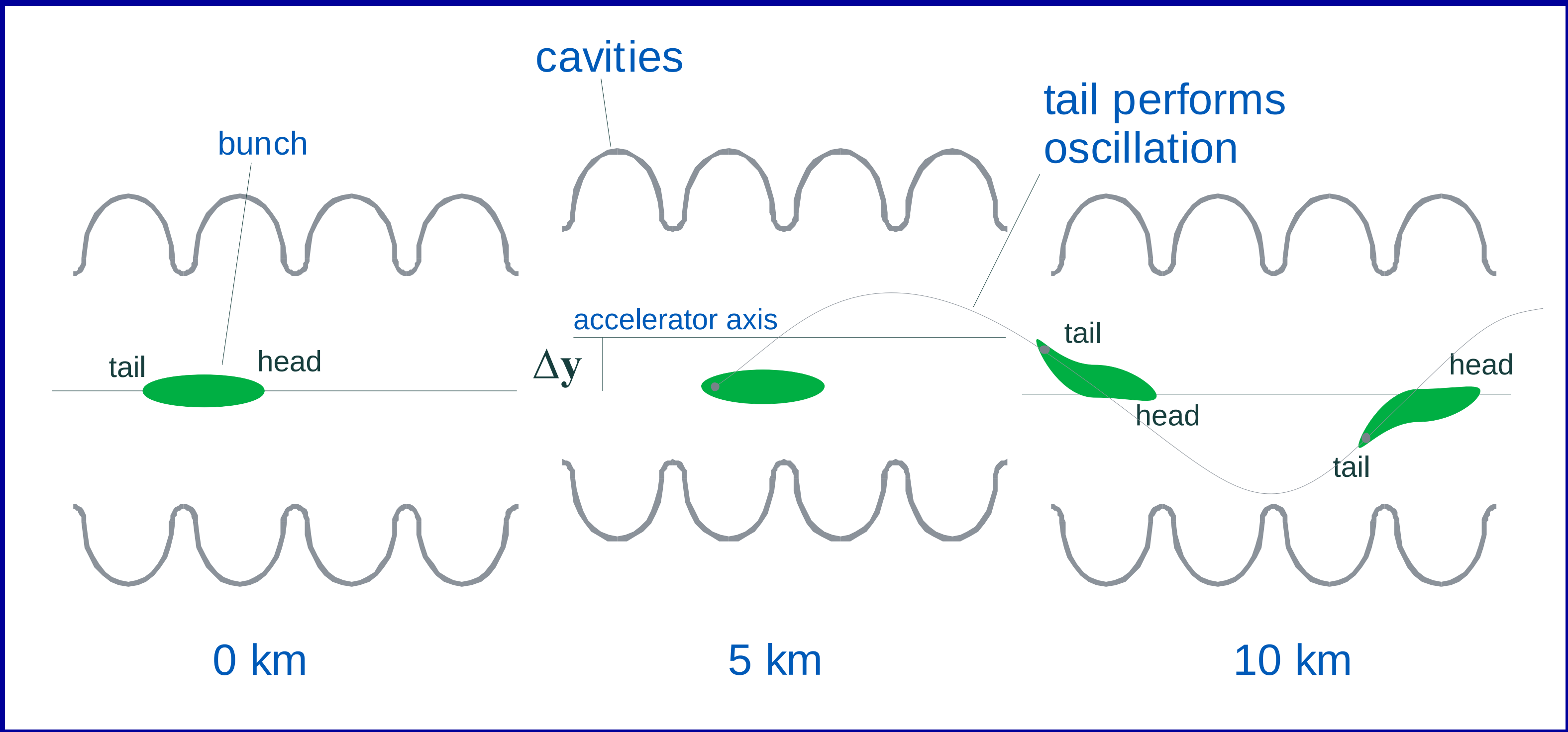
This condition is often known as “autophasing.”

It can be achieved by introducing an energy difference between the head and tail of the bunch. When the requirements of discrete focusing (ie, FODO lattices) are included, the autophasing RMS energy spread is given by:

$$\frac{\sigma_E}{E_{beam}} = \frac{1}{16} \frac{W'_{\perp} q \sigma_z}{E_{beam}} \frac{L_{cell}^2}{\sin^2\left(\pi \nu_\beta\right)}$$

curtesy: P. Tenenbaum (SLAC)

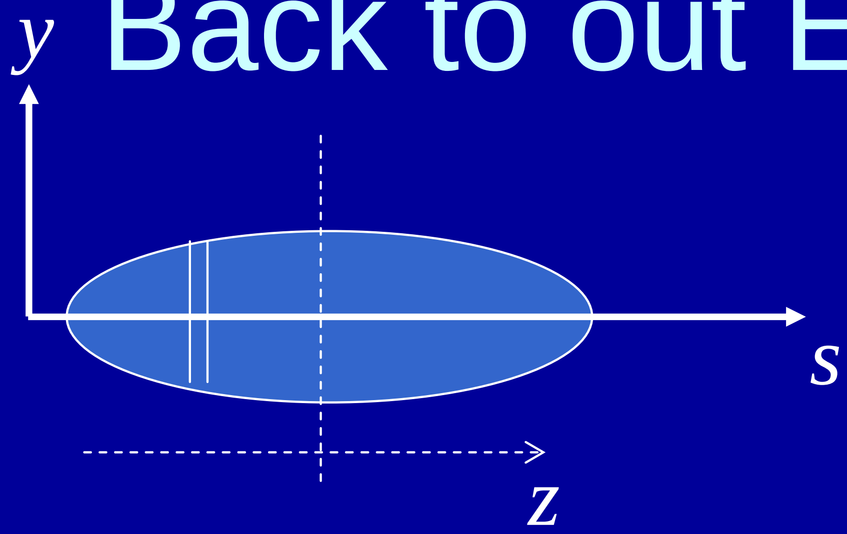
Wakefields (alignment tolerances)



$$\frac{\Delta\varepsilon}{\varepsilon} \propto \frac{Q^2 W_\perp^2}{\varepsilon} \beta_{lattice} \frac{1}{\alpha}\left[\left(\frac{\gamma_f}{\gamma_i}\right)^\alpha - 1\right]\left\langle \Delta y_{cav}^2 \right\rangle \qquad \beta(s) \propto \gamma^\alpha(s)$$

for wakefield control, prefer stronger focusing (small β)

Back to out EoM (Summary)



$$y''(s;z)+\frac{\gamma'(s)}{\gamma(s)}y'(s;z)+K(s)y(s;z)= \left.\begin{aligned}&\delta(s;z)K(s)y(s;z)\\&+K(s)y_q(s)\\&-\delta(s;z)K(s)y_q(s)\end{aligned}\right\}\text{dispersive errors}$$

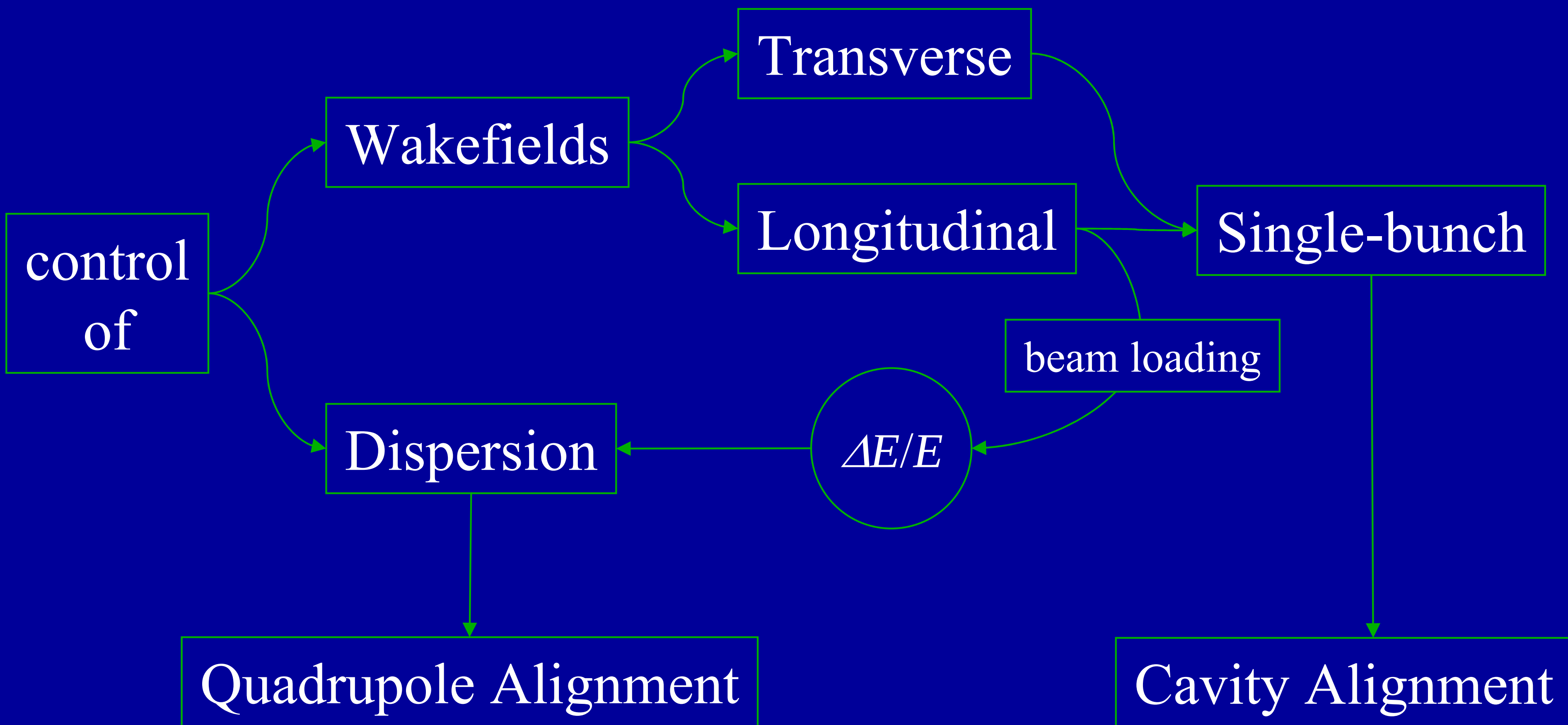
long. distribution

$$+\frac{Q}{\gamma(s)m_0}\left[1-\delta(s;z)\right]\int_{z'=z}^{\infty}W_{\perp}(s;z'-z)\lambda(z')y(s;z')dz'$$

transverse wake potential

(V C^{-1} m^{-2})

Preservation of RMS emittance



Some Number for ILC

RMS random misalignments to produce 5% vertical emittance growth

BPM offsets	11 μm
RF cavity offsets	300 μm
RF cavity tilts	240 μrad

Impossible to achieve with conventional mechanical alignment and survey techniques

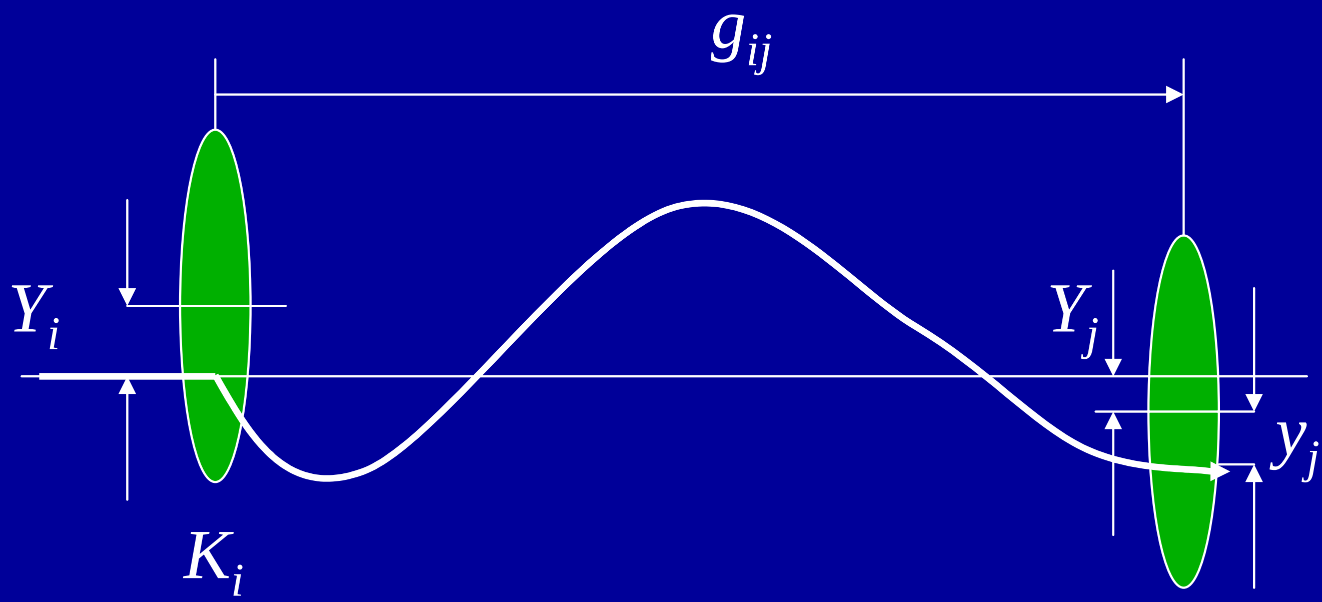
Typical 'installation' tolerance: 300 μm RMS $\Rightarrow$ $5\% \times \left(\frac{300}{11}\right)^2 \approx 3800\%!!$

Use of Beam Based Alignment mandatory

Basics Linear Optics Revisited

thin-lens quad approximation: $\Delta y' = -KY$

$$g_{ij} = \left.\frac{\partial y_i}{\partial y'_j}\right|_{y'_j=0} = R_{34}(i, j)$$



$$y_j = \left(-\sum_{i=1}^{j} g_{ij} K_i Y_i\right) - Y_j$$

linear system: just superimpose oscillations caused by quad kicks.

Introduce matrix notation

Original Equation $$y_j = \left(-\sum_{i=1}^{j} g_{ij} K_i Y_i \right) - Y_j$$

Defining *Response Matrix* $\mathbf{Q}$: $$\mathbf{Q} = \mathbf{G} \cdot \mathbf{diag}(\mathbf{K}) + \mathbf{I}$$

Hence beam offset becomes $$\mathbf{y} = -\mathbf{Q} \cdot \mathbf{Y}$$

G is lower diagonal: $$\mathbf{G} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ g_{21} & 0 & 0 & 0 & \cdots \\ g_{31} & g_{32} & 0 & 0 & \cdots \\ g_{41} & g_{42} & g_{43} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Dispersive Emittance Growth

Consider effects of finite energy spread in beam δ_{RMS}

chromatic response matrix: $$\mathbf{Q}(\delta) = \mathbf{G}(\delta) \cdot \mathbf{diag}\left(\frac{\mathbf{K}}{1+\delta}\right) + \mathbf{I}$$

lattice chromaticity

dispersive kicks

$$\mathbf{G}(\delta) = \mathbf{G}(0) + \left.\frac{\partial \mathbf{G}}{\partial \delta}\right|_{\delta=0}$$

$$R_{34}(\delta) = R_{34}(0) + T_{346}\delta$$

dispersive orbit: $$\Delta\mathbf{y}(\delta) = -\left[\mathbf{Q}(\delta) - \mathbf{Q}(0)\right] \cdot \mathbf{Y}$$

What do we measure?

BPM readings contain additional errors:

$\mathbf{b}_{\text{offset}}$ static offsets of monitors wrt quad centres

$\mathbf{b}_{\text{noise}}$ one-shot measurement noise (resolution σ_{RES})

$$\mathbf{y}_{\text{BPM}} = -\underbrace{\mathbf{Q}\cdot\mathbf{Y}+\mathbf{b}_{\text{offset}}}_{\text{fixed from shot to shot}} + \underbrace{\mathbf{b}_{\text{noise}}+\mathbf{R}\cdot\mathbf{y}_0}_{\text{random (can be averaged to zero)}} \qquad \mathbf{y}_0 = \underset{\substack{\uparrow \\ \text{launch condition}}}{\begin{pmatrix} y_0 \\ y_0' \end{pmatrix}}$$

In principle: all BBA algorithms deal with $\mathbf{b}_{\text{offset}}$

BBA using Dispersion Free Steering (DFS)

– Find a set of steerer settings which minimise the dispersive orbit
– in practise, find solution that minimises difference orbit when ‘energy’ is changed
– Energy change:
  - true energy change (adjust linac phase)
  - scale quadrupole strengths

DFS

Problem:

$$\Delta\mathbf{y} = -\left[\mathbf{Q}(\frac{\Delta E}{E}) - \mathbf{Q}(0)\right]\left(\frac{\Delta E}{E}\right)\cdot\mathbf{Y}$$

$$\equiv \mathbf{M}\left(\frac{\Delta E}{E}\right)\cdot\mathbf{Y}$$

Note: taking difference orbit $\Delta\mathbf{y}$ removes $\mathbf{b}_{\mathrm{offset}}$

Solution (trivial): $\mathbf{Y} = \mathbf{M}^{-1}\cdot\Delta\mathbf{y}$

Unfortunately, not that easy because of noise sources:

$$\Delta\mathbf{y} = \mathbf{M}\cdot\mathbf{Y} + \mathbf{b}_{\mathrm{noise}} + \mathbf{R}\cdot\mathbf{y}_0$$

DFS example

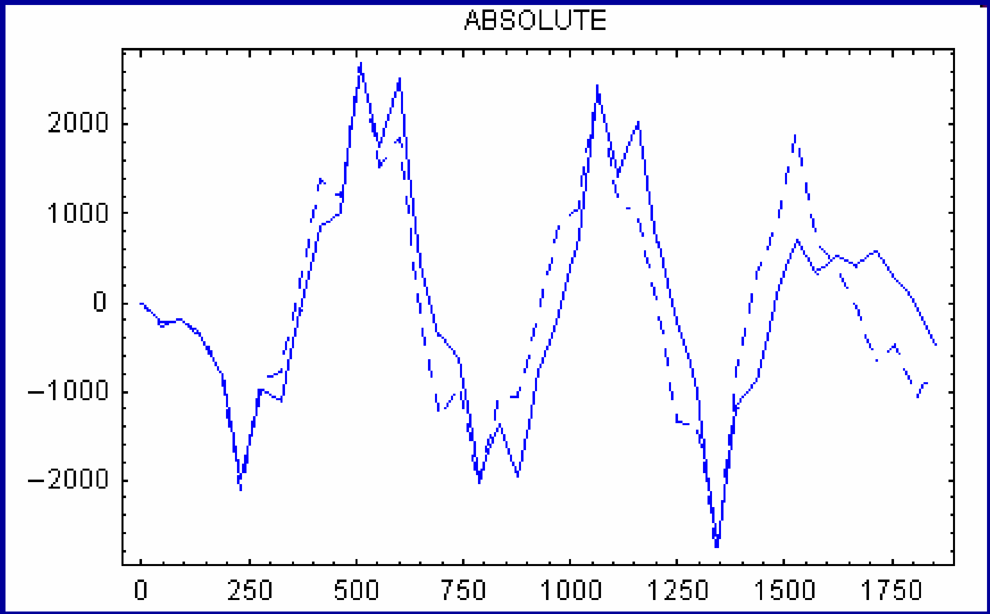
300μm random quadrupole errors

20% $\Delta E/E$

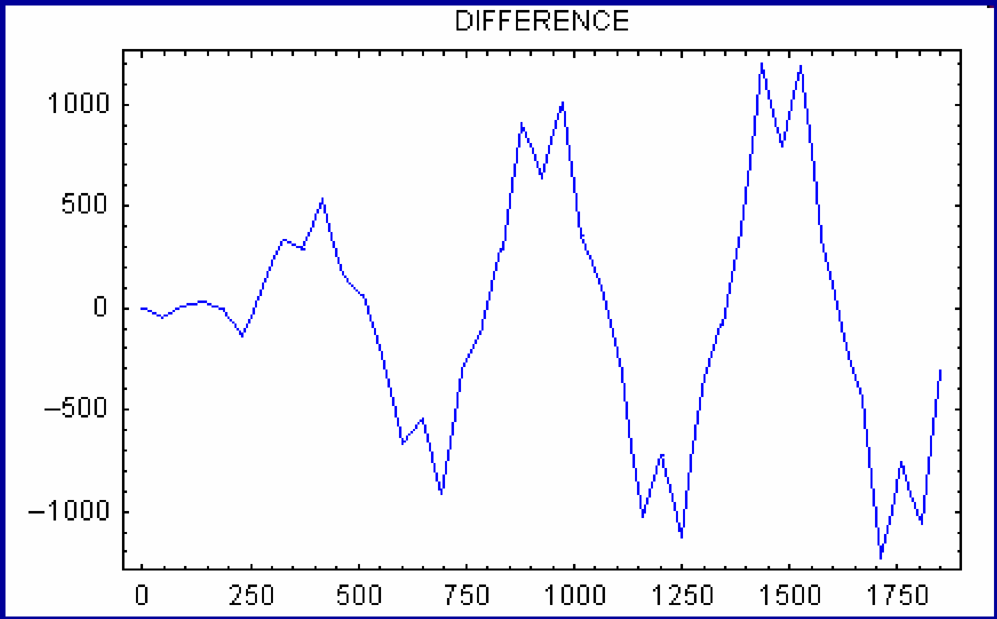
No BPM noise

No beam jitter

μm



μm

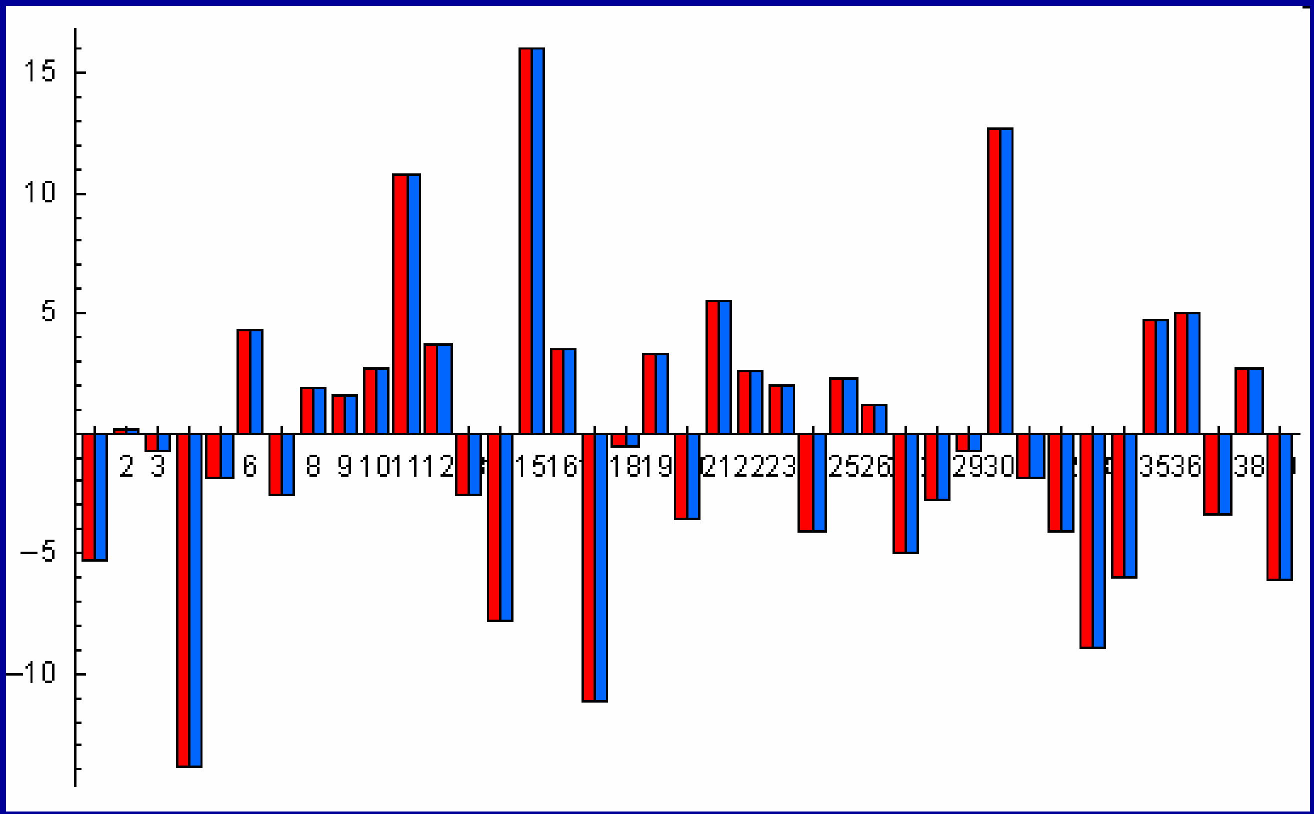


DFS example

Simple solve

$$\mathbf{Y} = \mathbf{M}^{-1} \cdot \Delta\mathbf{y}$$

In the absence of errors, works exactly



Resulting orbit is flat

⇒Dispersion Free

(perfect BBA)

original quad errors

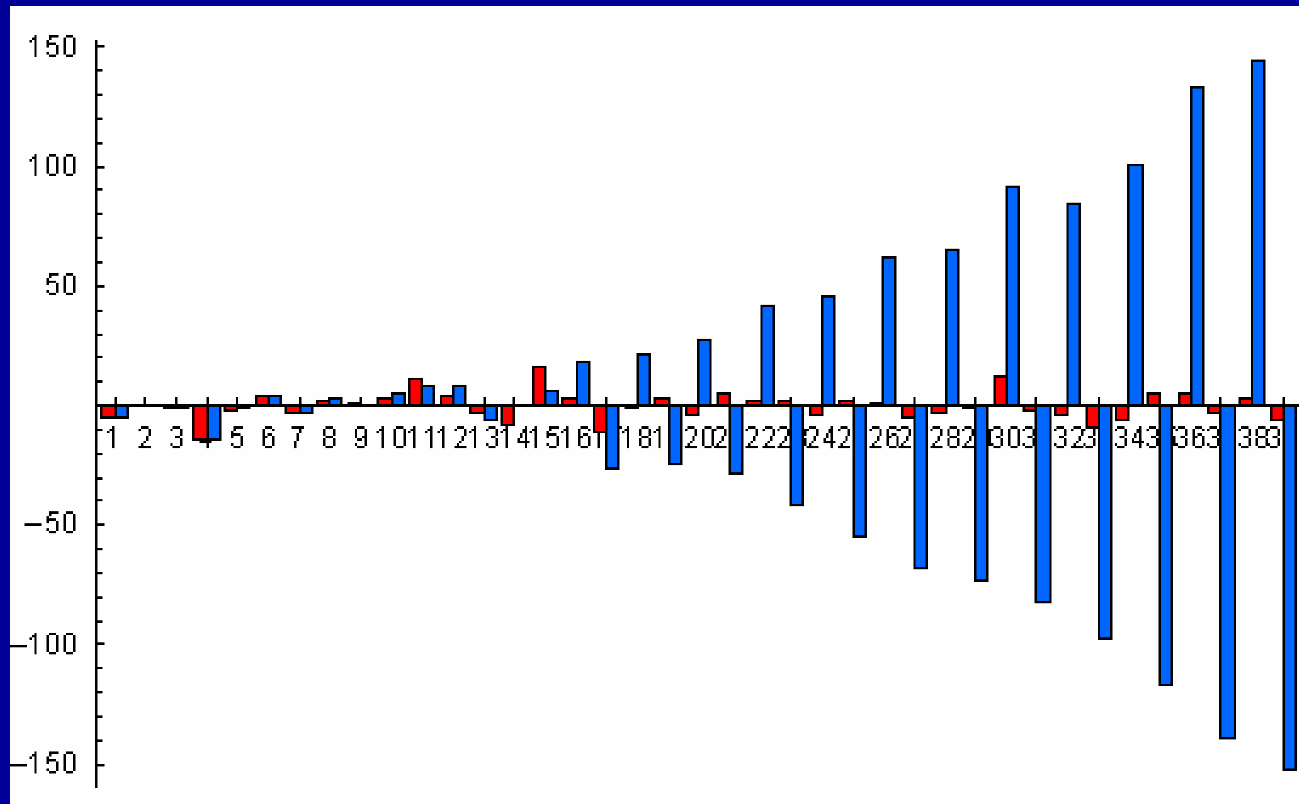
fitter quad errors

Now add 1μm random BPM noise to measured difference orbit

DFS example

Simple solve

$$\mathbf{Y} = \mathbf{M}^{-1} \cdot \Delta \mathbf{y}$$



Fit is ill-conditioned!

original quad errors

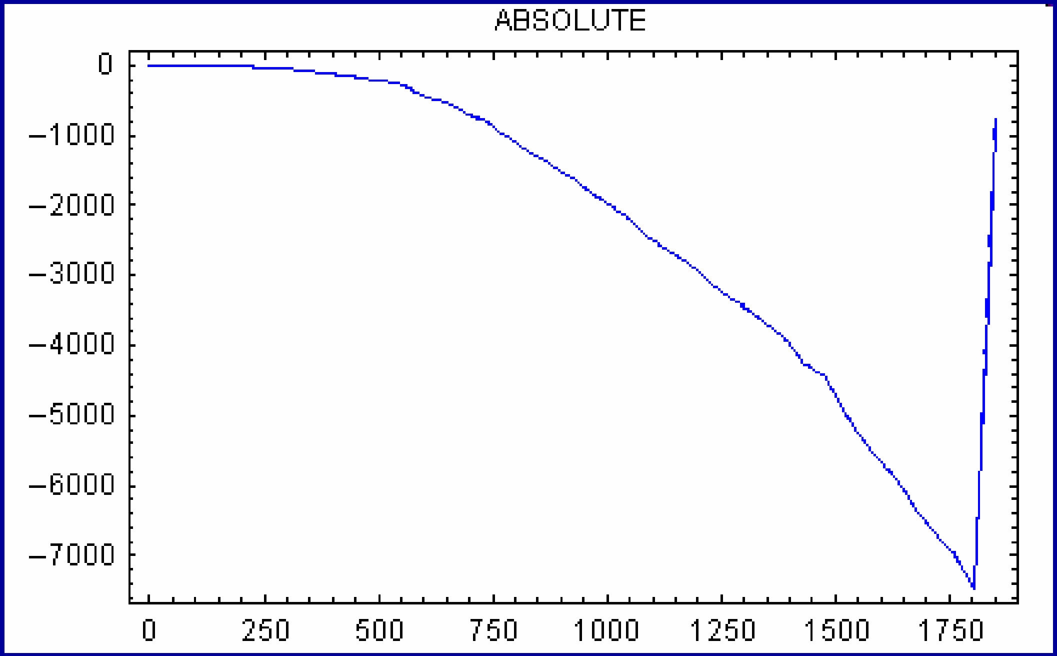
fitter quad errors

DFS example

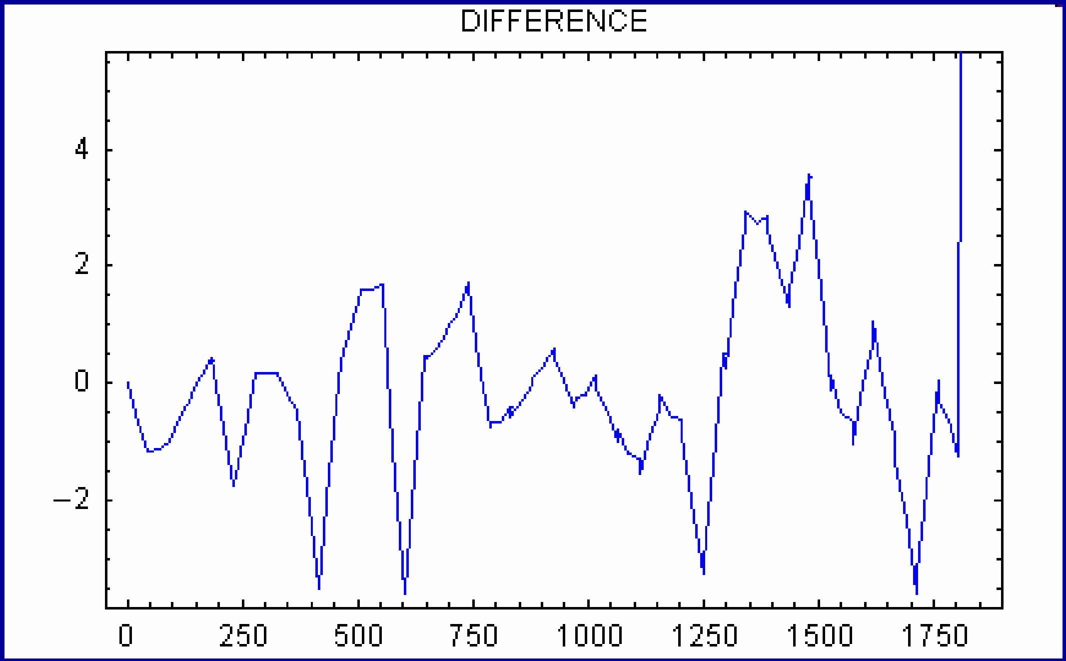
Solution is still Dispersion Free

but several mm off axis!

μm



μm



DFS: Problems

- Fit is ill-conditioned
  - with BPM noise DF orbits have very large unrealistic amplitudes.
  - Need to constrain the absolute orbit

minimise $$\frac{\Delta\mathbf{y}\cdot\Delta\mathbf{y}^{\mathrm{T}}}{2\sigma_{\mathrm{res}}^{2}}+\frac{\mathbf{y}\cdot\mathbf{y}^{\mathrm{T}}}{\sigma_{\mathrm{res}}^{2}+\sigma_{\mathrm{offset}}^{2}}$$

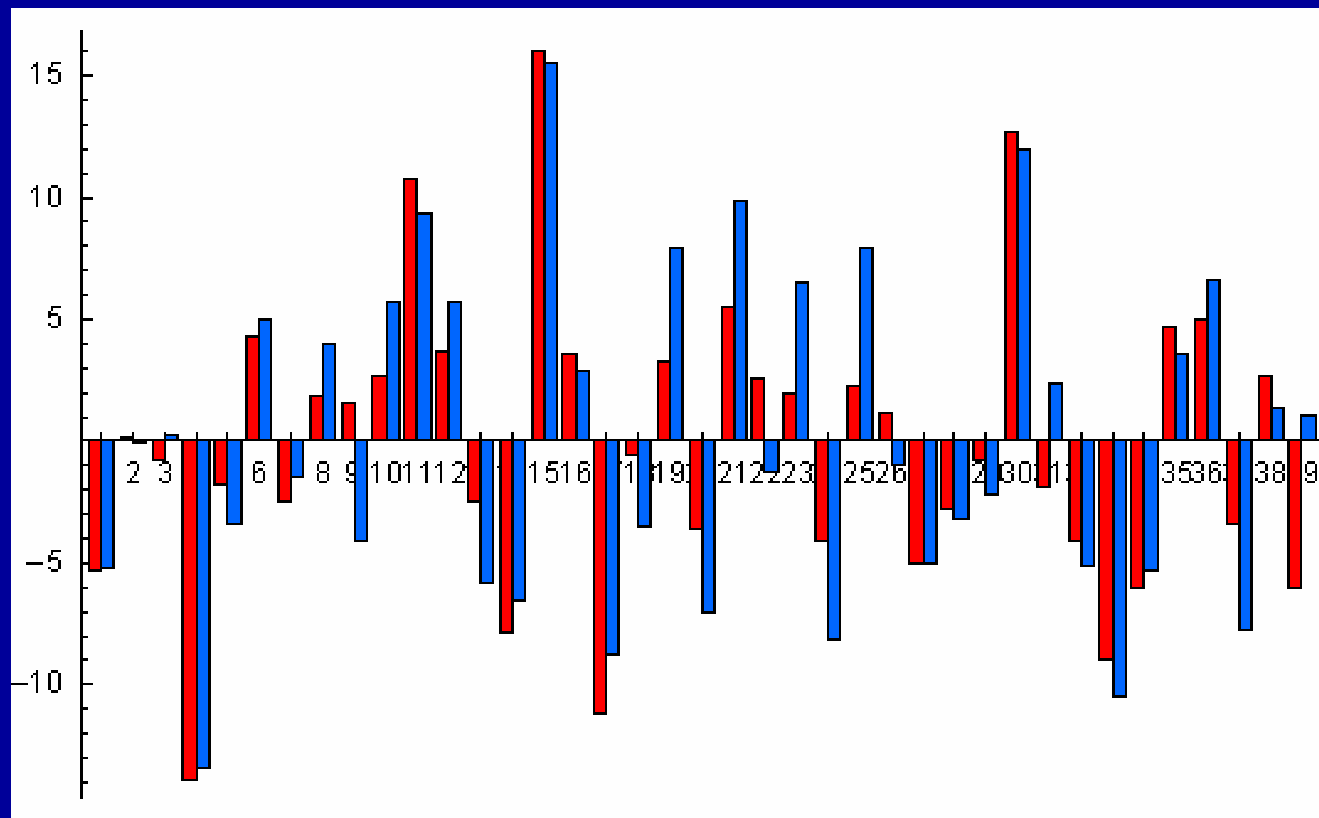
- Sensitive to initial launch conditions $\mathbf{R}\cdot\mathbf{y}_0$ (steering, beam jitter)
  - need to be fitted out or averaged away

DFS example

Minimise

$$\frac{\mathbf{\Delta y}\cdot\mathbf{\Delta y}^{\mathrm{T}}}{2\sigma_{\mathrm{res}}^{2}}+\frac{\mathbf{y}\cdot\mathbf{y}^{\mathrm{T}}}{\sigma_{\mathrm{res}}^{2}+\sigma_{\mathrm{offset}}^{2}}$$

absolute orbit now constrained



original quad errors

fitter quad errors

remember

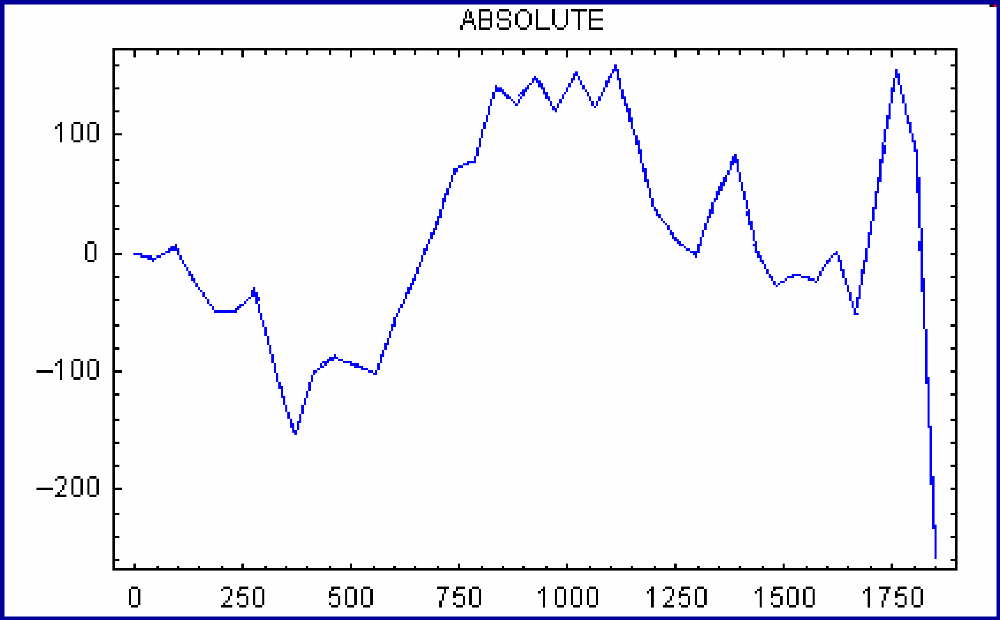
$\sigma_{\mathrm{res}} = 1\mu\mathrm{m}$

$\sigma_{\mathrm{offset}} = 300\mu\mathrm{m}$

DFS example

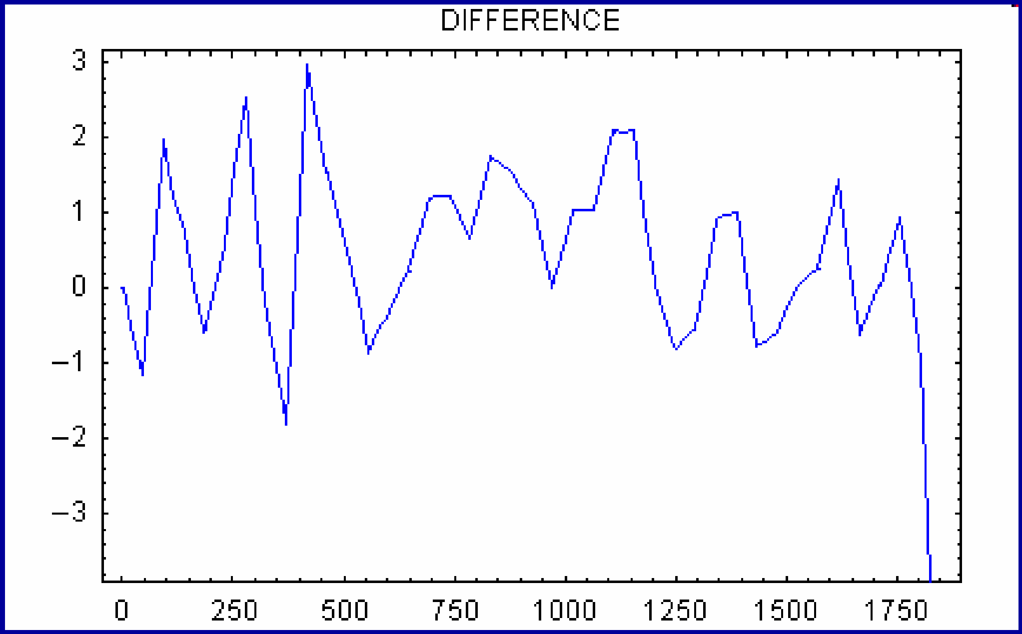
Solutions much better behaved!

μm



Orbit *not quite* Dispersion Free, but very close

μm



DFS practicalities

- Need to align linac in sections (bins), generally overlapping.
- Changing energy by 20%
  - quad scaling: only measures dispersive kicks from quads. Other sources ignored (not measured)
  - Changing energy upstream of section using RF better, but beware of RF steering (see initial launch)
  - dealing with energy mismatched beam may cause problems in practise (apertures)
- Initial launch conditions still a problem
  - coherent β-oscillation looks like dispersion to algorithm.
  - can be random jitter, or RF steering when energy is changed.
  - need good resolution BPMs to fit out the initial conditions.
- Sensitive to model errors (**M**)

Orbit Bumps

- Localised closed orbit bumps can be used to correct
  - Disperive
  - Wakefields
- “Global” correction (eg. end of linac) can only correct non-filamented part
  - Remaining linear correlation
- Need ‘emittance diagnostic’
  - Beam profile monitors
  - Other signal (eg. luminosity in the ILC)

I’ll Stop Here

Emittance Growth: Chromaticity

Chromatic kick from a thin-lens quadrupole:

$$\Delta x' = K\delta x; \quad \delta \equiv \Delta p / p$$

2nd-order moments:

$$\langle x^2 \rangle \rightarrow \langle x^2 \rangle$$

$$\langle xx' \rangle \rightarrow \langle xx' \rangle$$

$$\langle x'^2 \rangle \rightarrow \langle x'^2 \rangle + K^2 \langle \delta^2 \rangle \langle x^2 \rangle$$

Emittance Growth: Chromaticity

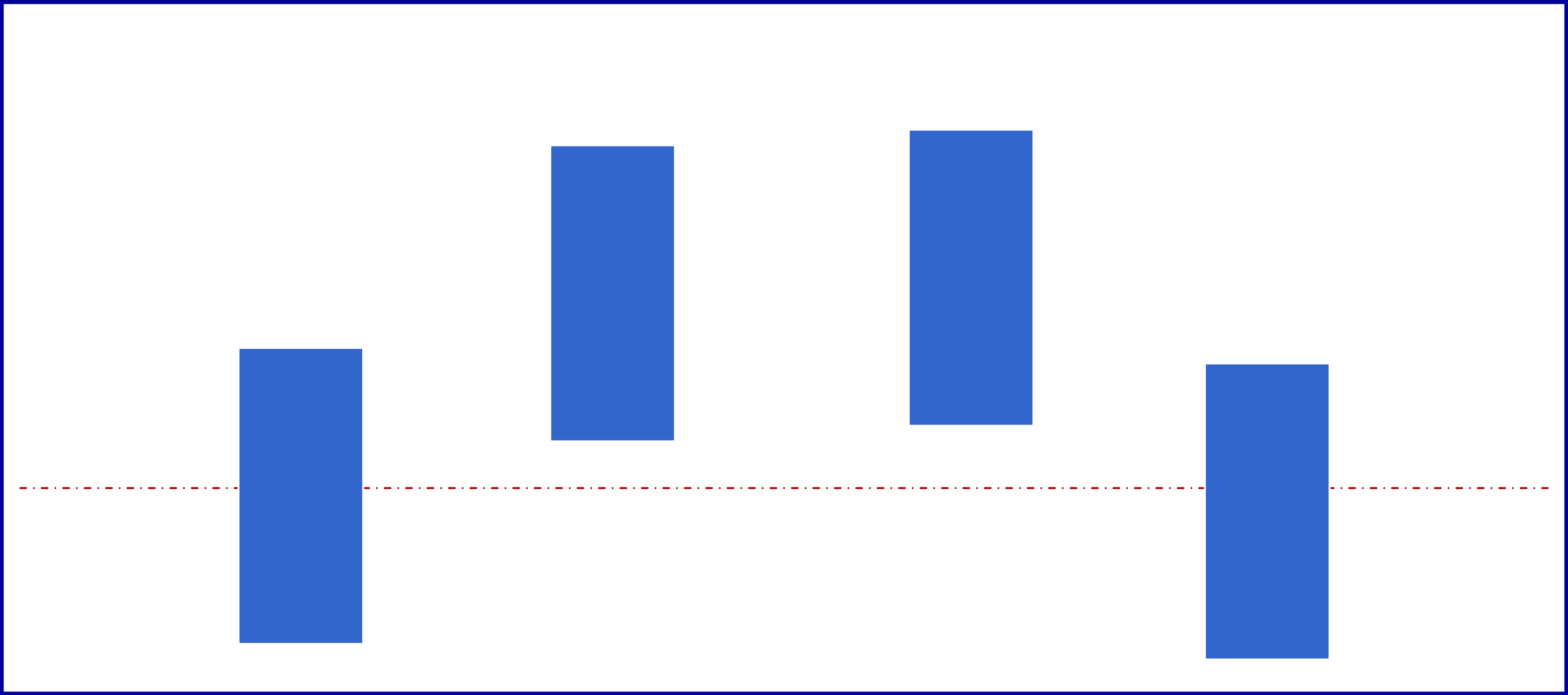
Chromatic kick from a thin-lens quadrupole:

$$\Delta x' = K\delta x; \quad \delta \equiv \Delta p / p$$

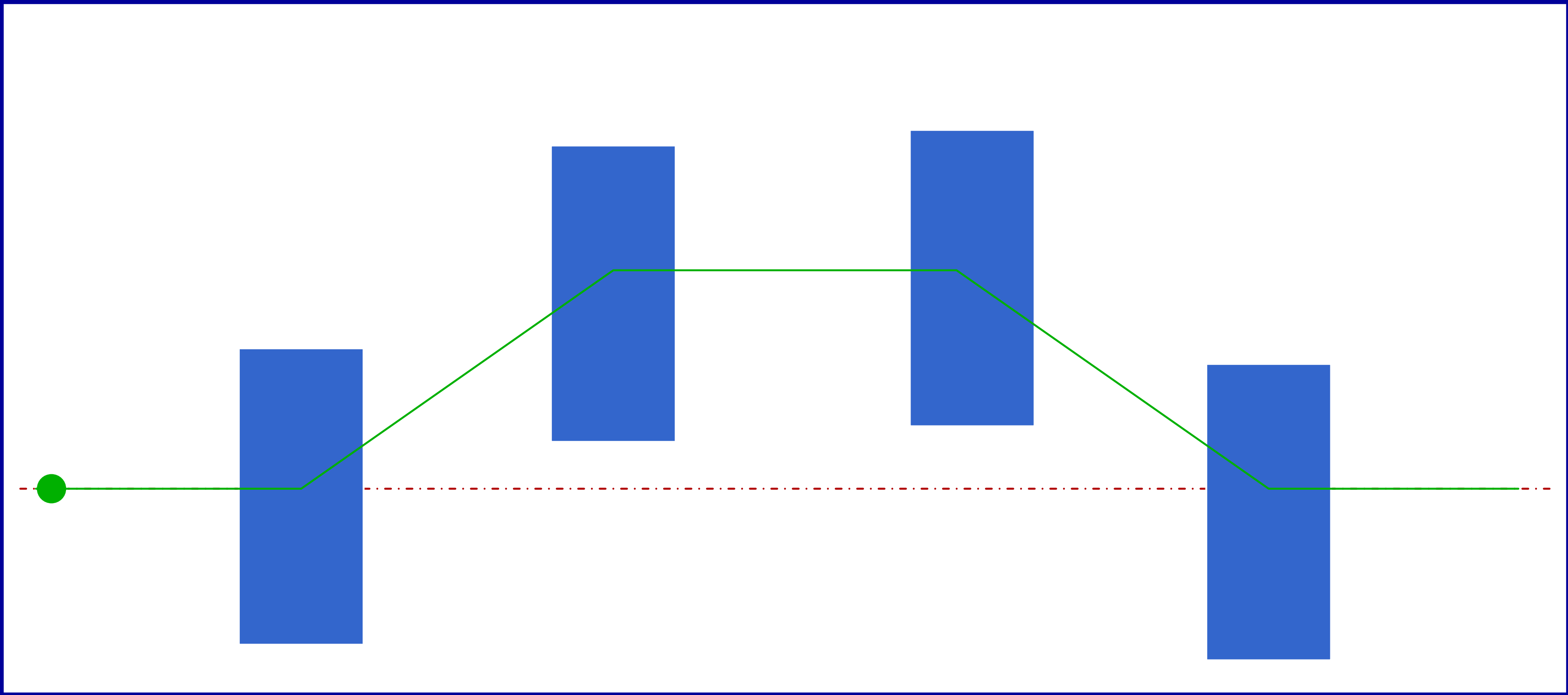
RMS emittance:

$$\left.\begin{aligned}
\varepsilon_x^2 &= \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle \\
&= \varepsilon_{x,0}^2 + K^2 \langle \delta^2 \rangle \langle x^2 \rangle^2 \\
&= \varepsilon_{x,0}^2 + K^2 \delta_{RMS}^2 \beta_x^4 \varepsilon_{x,0}^4
\end{aligned}\right\} \quad \boxed{\frac{\Delta\varepsilon_x}{\varepsilon_{x,0}} \approx \frac{1}{2} K^2 \delta_{RMS}^2 \beta_x^4 \varepsilon_{x,0}^2}$$

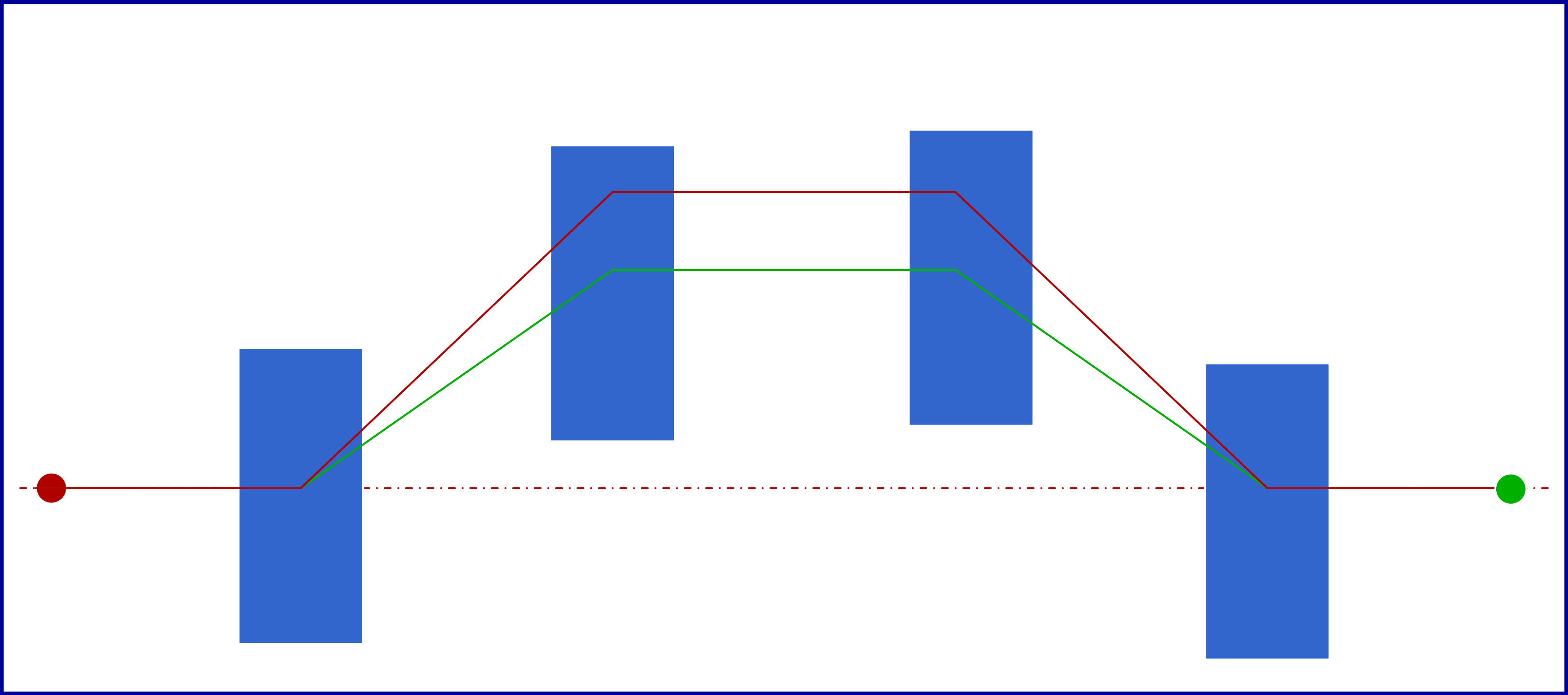
Synchrotron Radiation



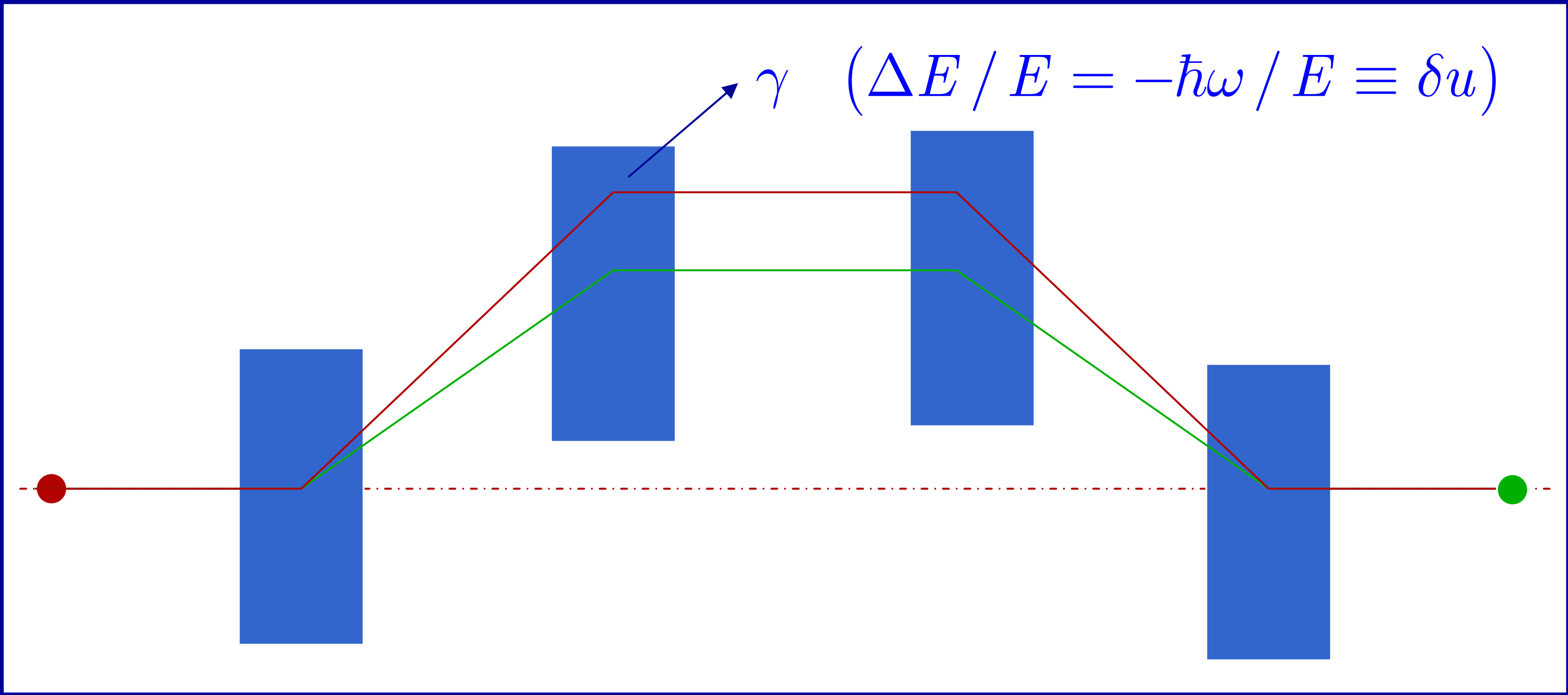
Synchrotron Radiation



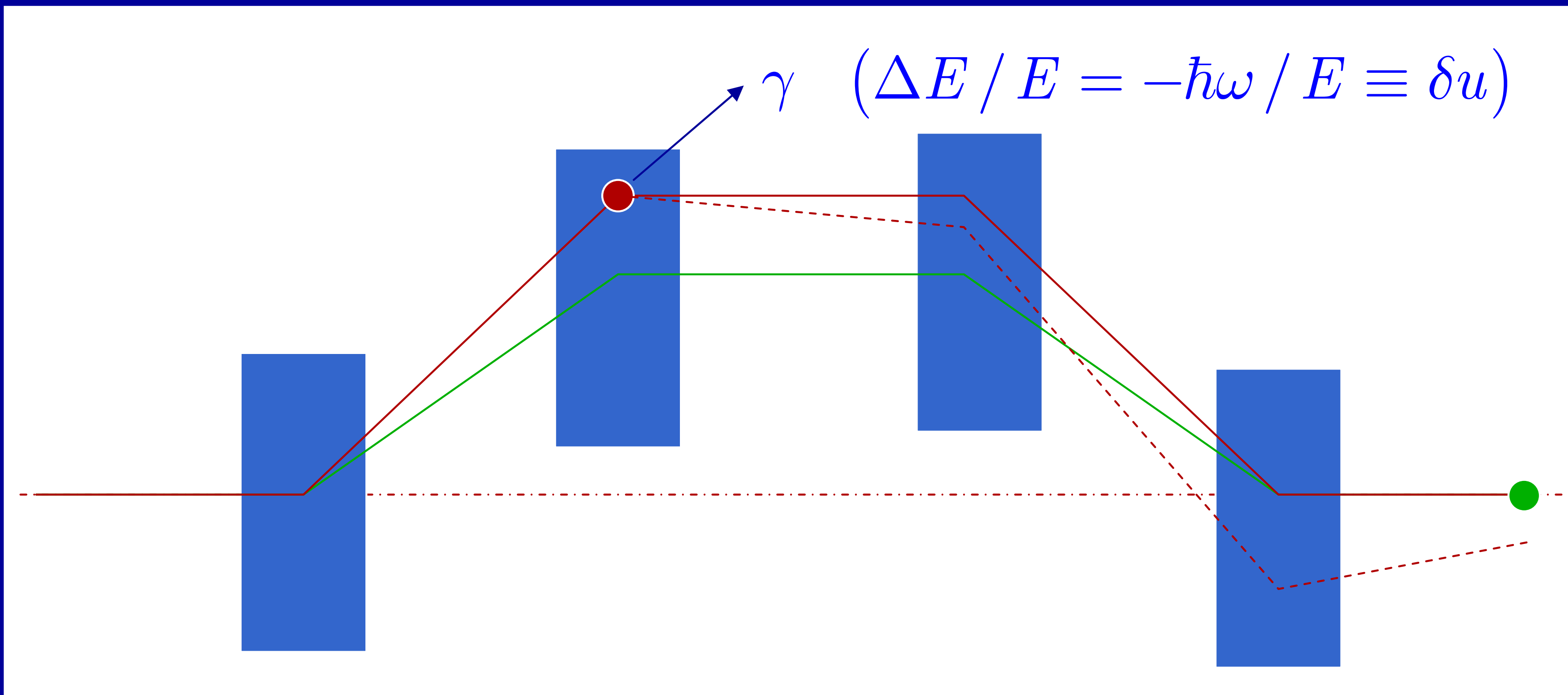
Synchrotron Radiation



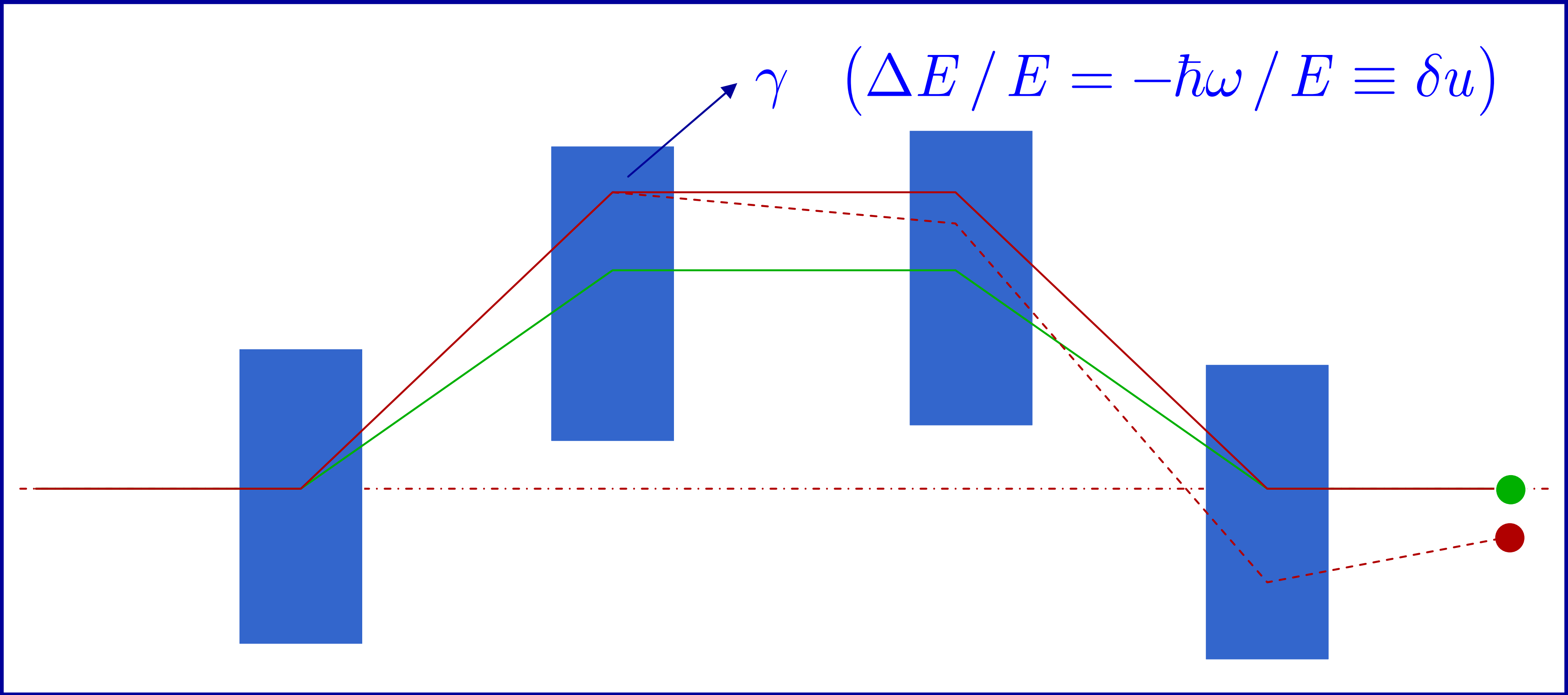
Synchrotron Radiation



Synchrotron Radiation



Synchrotron Radiation



$R_{16}(s)$ $\quad s = L$

$$\Delta x_i(L) = R_{16}(s)\delta u_i$$

$$\left\langle \Delta x_i^2 \right\rangle_\gamma = R_{16}^2(s)\left\langle \delta u_i^2 \right\rangle$$

$$\left\langle \delta u^2 \right\rangle \approx 4.13 \times 10^{-11} \frac{E^5[GeV]}{\rho^3[m]} \Delta s[m]$$

Synchrotron Radiation

$$\left\langle \Delta x^2 \right\rangle = C_\gamma E^5 \int_{s=0}^{L} \frac{R_{16}^2(s)}{\rho^3(s)} ds$$

$$C_\gamma \approx 4.13 \times 10^{-11} m^2 [GeV]^{-5}$$

$$\left\langle \Delta x'^2 \right\rangle = C_\gamma E^5 \int_{s=0}^{L} \frac{R_{26}^2(s)}{\rho^3(s)} ds$$

$$\left\langle \Delta x \Delta x' \right\rangle = C_\gamma E^5 \int_{s=0}^{L} \frac{R_{16}(s) R_{26}(s)}{\rho^3(s)} ds$$

phase space due to quantum excitation

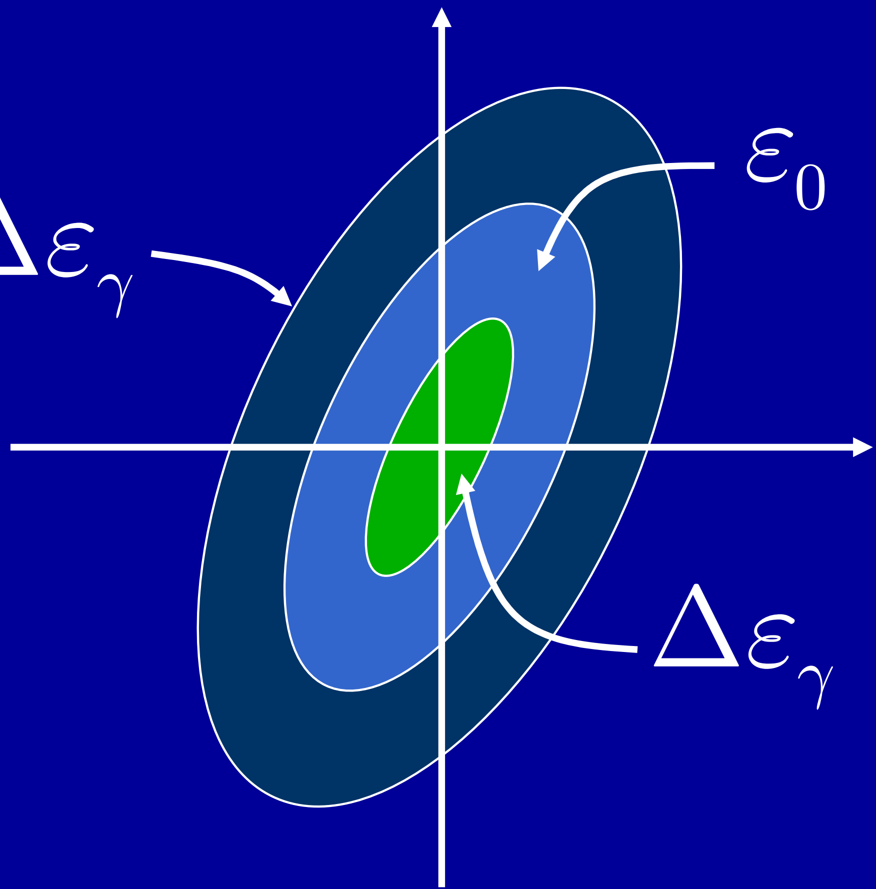
$$\boxed{\Delta\varepsilon_\gamma = C_\gamma E^5 \left[\int_0^L \frac{R_{16}^2(s)}{\rho^3(s)} ds \int_0^L \frac{R_{26}^2(s)}{\rho^3(s)} ds - \left(\int_0^L \frac{R_{16}(s) R_{26}(s)}{\rho^3(s)} ds \right)^2 \right]^{\frac{1}{2}}}$$

We have ignored the mean energy loss
(assumed to be small, or we have taken some suitable average)

Synchrotron Radiation

What is the additional emittance when our initial beam has a finite emittance?

$$\varepsilon = \varepsilon_0 + \Delta\varepsilon_\gamma$$



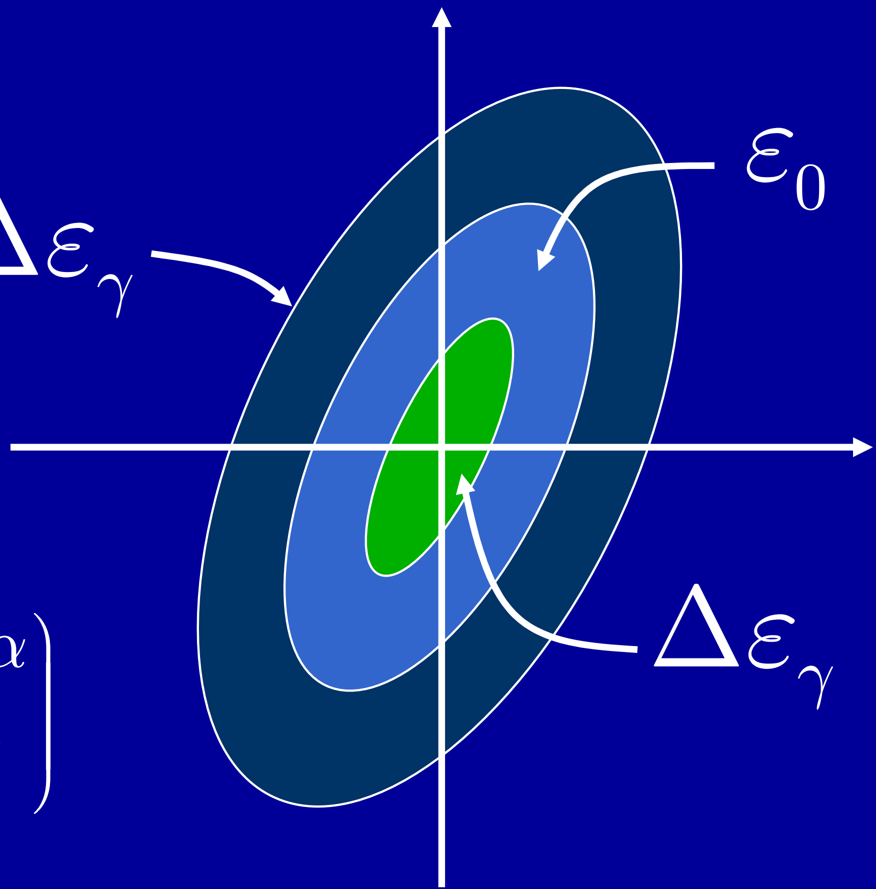
Quantum emission is uncorrelated, so we can add 2nd-order moments

When quantum induced phase space and original beam phase space are ‘geometrically similar’, just add emittances.

Synchrotron Radiation

What is the additional emittance when our initial beam has a finite emittance?

$$\varepsilon = \varepsilon_0 + \Delta\varepsilon_\gamma$$



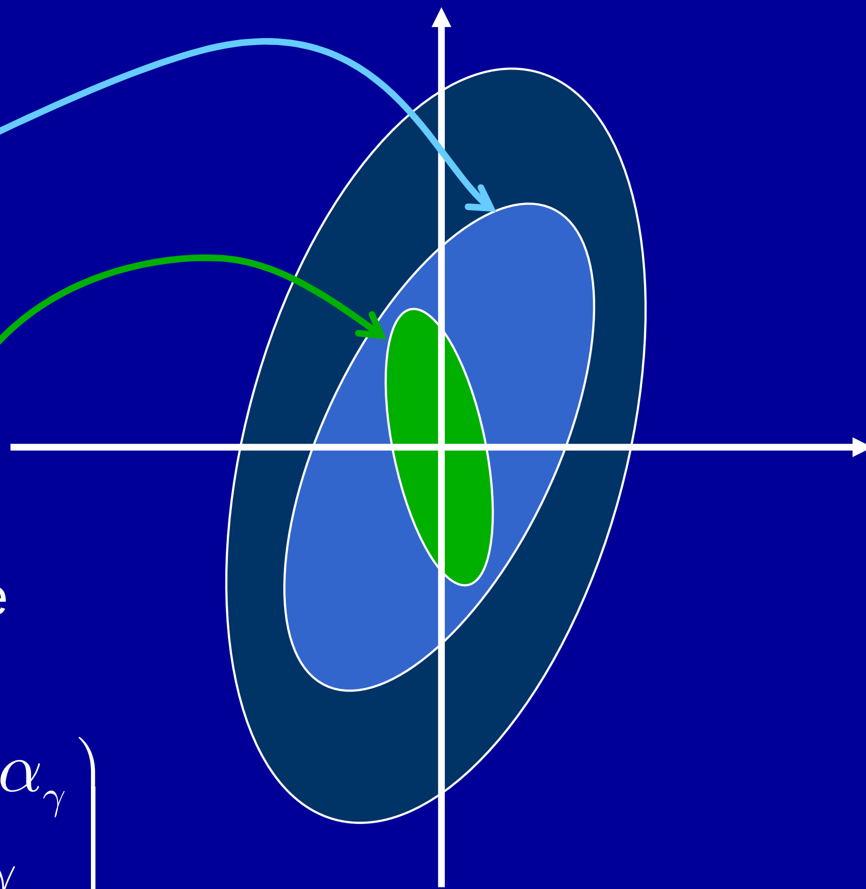
ellipse shape

$$\sigma = \varepsilon_0 \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} + \Delta\varepsilon_\gamma \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

$$= \left(\varepsilon_0 + \Delta\varepsilon_\gamma\right) \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

Synchrotron Radiation

When quantum induced phase space and initial beam phase space are *dissimilar*, an additional (cross) term must be included.



Initial beam ellipse

Q.E. beam ellipse

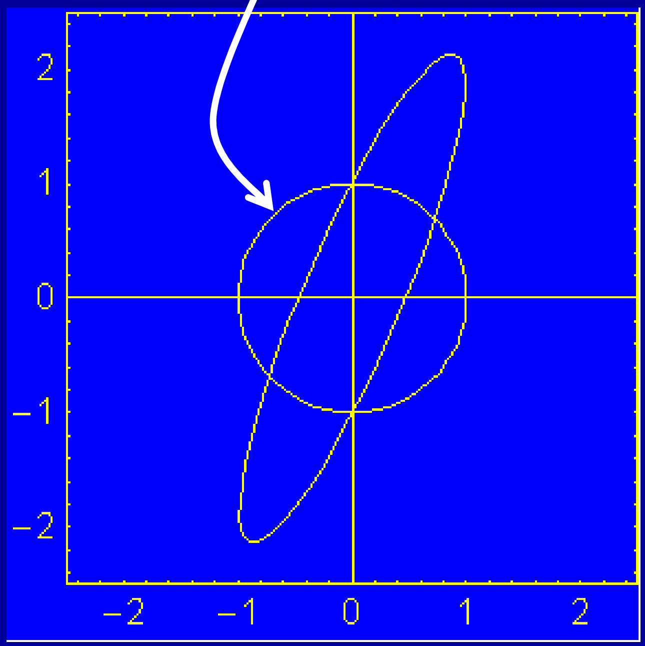
$$\sigma = \varepsilon_0 \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} + \Delta\varepsilon_\gamma \begin{pmatrix} \beta_\gamma & -\alpha_\gamma \\ -\alpha_\gamma & \gamma_\gamma \end{pmatrix}$$

$$\varepsilon^2 = |\sigma| = \varepsilon_0^2 + \Delta\varepsilon_\gamma^2 + 2\varepsilon_0 \Delta\varepsilon_\gamma \left[\frac{1}{2}\left(\gamma\beta_\gamma - 2\alpha\alpha_\gamma + \gamma_\gamma\beta\right)\right]$$

Beam Mismatch (Filamentation)

$$v \equiv \frac{\alpha x + \beta x'}{\sqrt{\beta}}$$

Matched beam (normalised to unit circle)



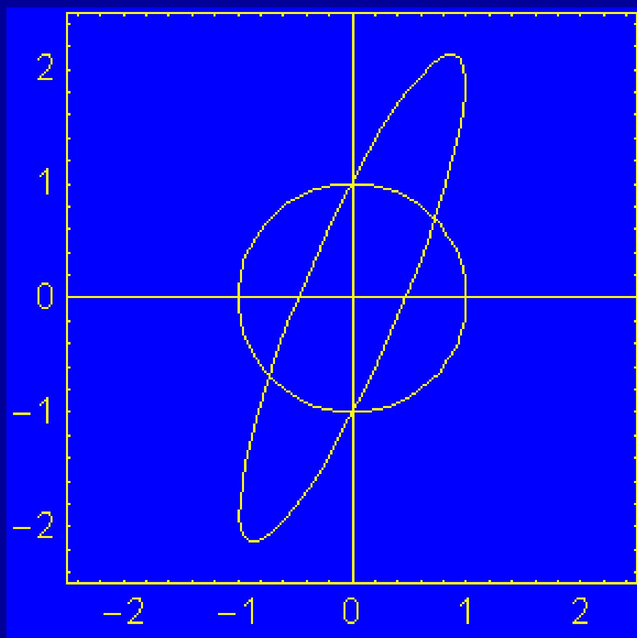
$$u \equiv \frac{x}{\sqrt{\beta}}$$

Mismatched beam (β-mismatch) rotates with nominal phase advance along beamline

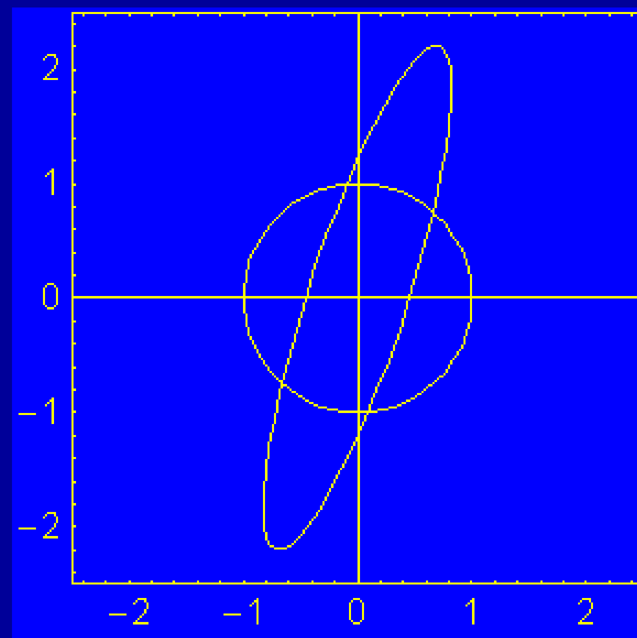
⇒ β-beat along machine (but emittance remains constant)

Beam Mismatch (Filamentation)

$$v \equiv \frac{\alpha x + \beta x'}{\sqrt{\beta}}$$



$$u \equiv \frac{x}{\sqrt{\beta}}$$



Mismatched beam (β-mismatch) rotates with nominal phase advance along beamline

⇒ β-beat along machine (but emittance remains constant)

Finite energy spread in beam + lattice chromaticity causes mismatch to "filament"

⇒ Emittance growth

Beam Mismatch (Filamentation)

$$\mathbf{M} = \frac{1}{\sqrt{\beta_0}} \begin{pmatrix} 1 & 0 \\ \alpha_0 & \beta_0 \end{pmatrix}$$

Normalisation matrix (matched beam)

$$\sigma = \varepsilon_0 \begin{pmatrix} \beta & -\alpha \\ -\alpha & \dfrac{(1+\alpha^2)}{\beta} \end{pmatrix}$$

Mismatched beam

$$R(\varphi) = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix}$$

Phase space rotation

$$\sigma_{fil} = \frac{1}{\pi} \int_0^{\pi} \mathbf{R}(\varphi) \cdot \mathbf{M} \cdot \sigma \cdot \mathbf{M}^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}}(\varphi) d\varphi$$

Fully filamented beam

$$\sqrt{\left|\sigma_{fil}\right|} = \frac{1}{2}\left[\gamma_0 \beta + \gamma \beta_0 - 2\alpha\alpha_0\right]\varepsilon_0$$

$$\boxed{\beta = 2\beta_0; \alpha = \alpha_0 = 0; \varepsilon_{fil} = 1.2\varepsilon_0}$$

Longitudinal Wake

Consider the TESLA wake potential $W_{\|}(z=ct)$

$$W_{\|}(z) \approx -38.1\left[\frac{\mathrm{V}}{\mathrm{pC}\cdot\mathrm{m}}\right]\left[1.165\exp\left(-\sqrt{\frac{s}{3.65\times10^{-3}[\mathrm{m}]}}\right)-0.165\right]$$

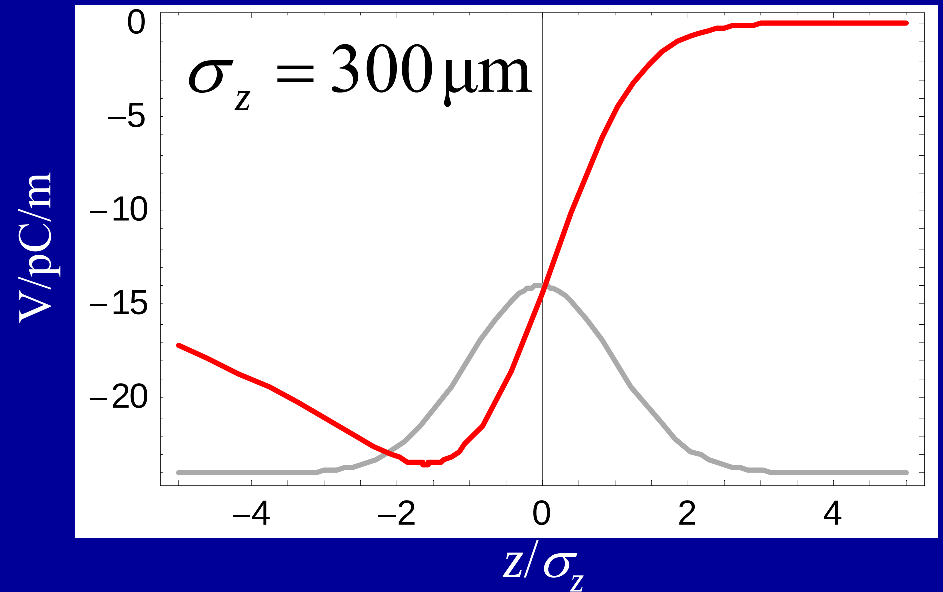
wake over bunch given by convolution: ($\rho(z)$ = long. charge dist.)

$$W_{\|,bunch}(z) = \int_{z'=z}^{\infty} W_{\|}(z'-z)\rho(z')dz'$$

average energy loss:

$$\langle \Delta E \rangle = q_b \int_{-\infty}^{\infty} W_{\|,bunch}(z)\rho(z)dz$$

For TESLA LC: $\langle \Delta E \rangle \approx -46\,\mathrm{kV/m}$



RMS Energy Spread

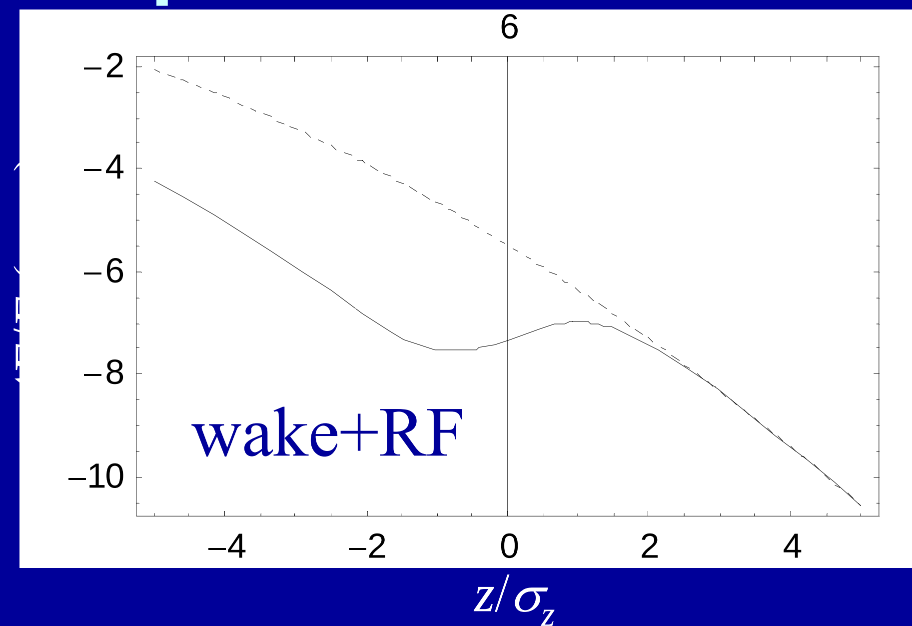
accelerating field along bunch:

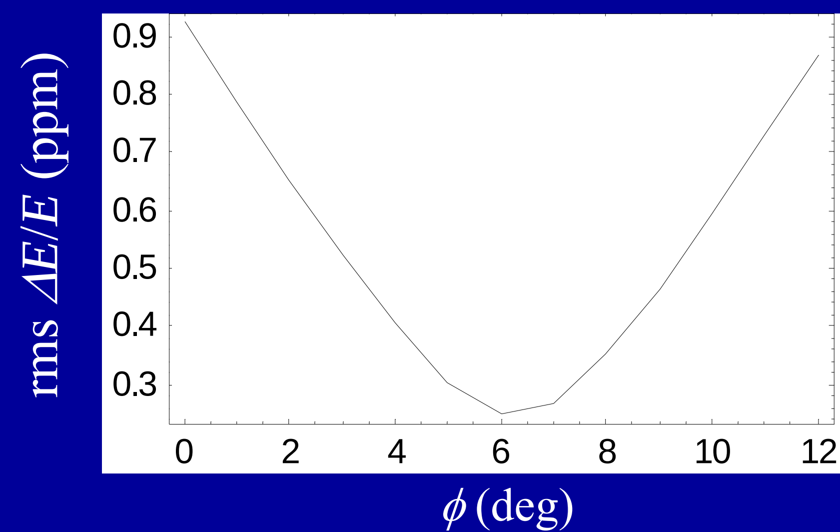
$$E(z) = q_b W_{\parallel,bunch}(z) + E_0 \cos(2\pi z/\lambda_{RF} + \phi)$$

Minimum energy spread along bunch achieved when bunch rides ahead of crest on RF.

Negative slope of RF compensates wakefield.

For TESLA LC, minimum at about $\phi \sim +6^{\circ}$





RMS Energy Spread