Introduction to Transverse Beam Optics

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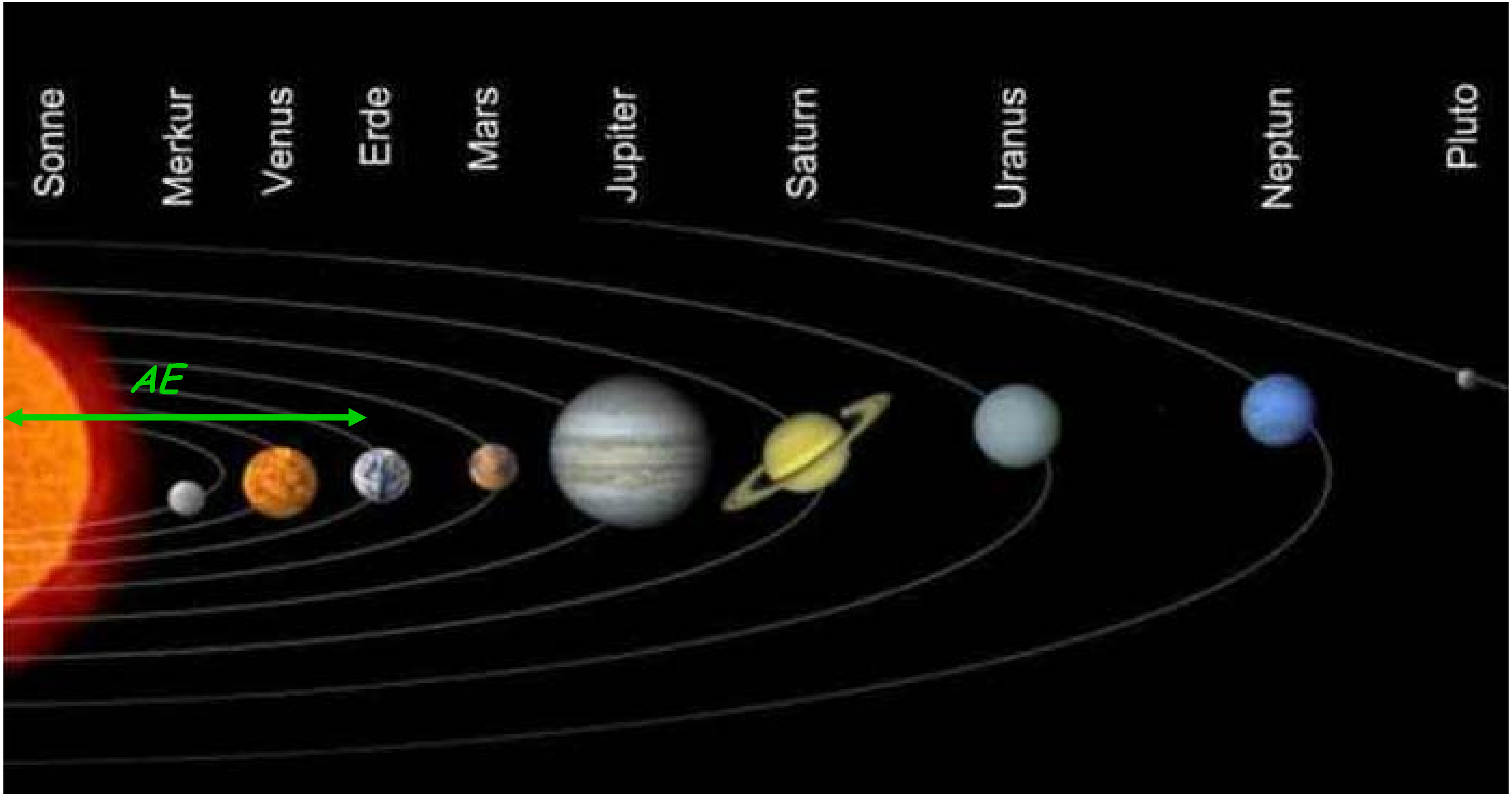
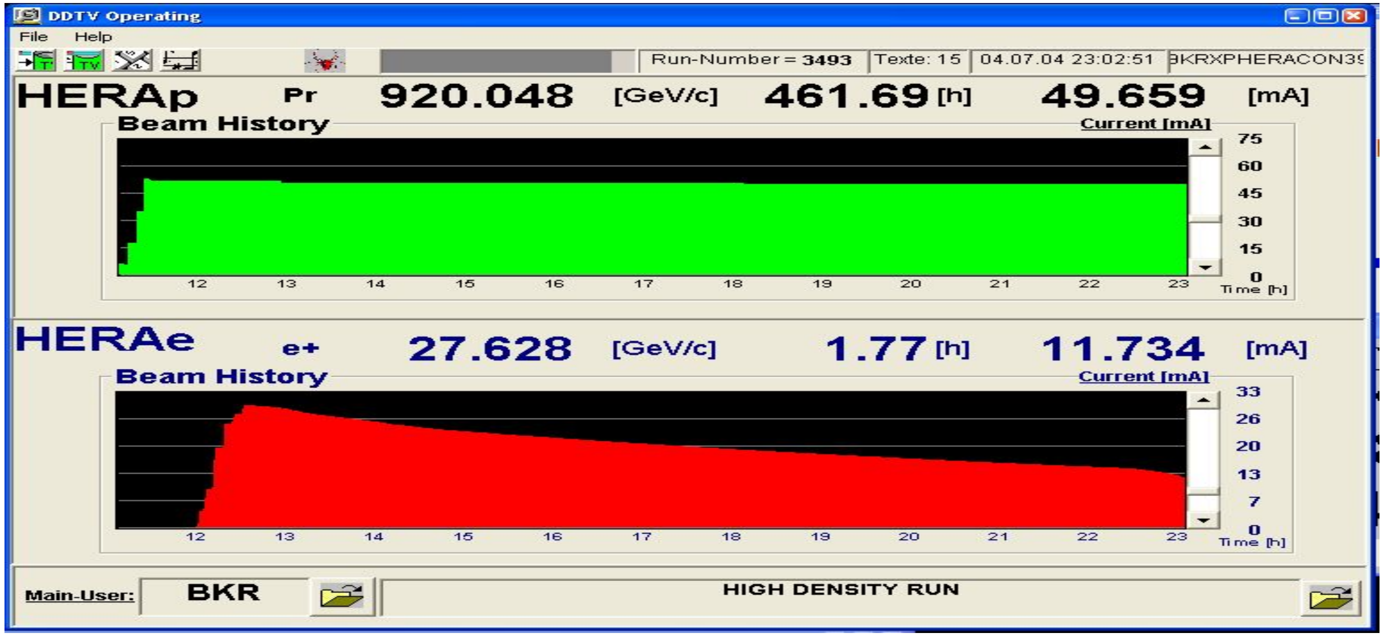
Largest storage ring: The Solar System

astronomical unit: average distance earth-sun
$1AE \approx 150 * 10^6$ km
Distance Pluto-Sun $\approx$ 40 AE



Luminosity Run of a typical storage ring:

HERA Storage Ring: Protons accelerated and stored for 12 hours
distance of particles travelling at about $v \approx c$
$L = 10^{10}$-10^{11} km
... several times Sun - Pluto and back



→ *guide the particles on a well defined orbit („design orbit")*
→ *focus the particles to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.*

Transverse Beam Dynamics:

0.) Introduction and Basic Ideas

„ ... in the end and after all it should be a kind of circular machine“

→ need transverse deflecting force

Lorentz force $$\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$$

typical velocity in high energy machines: $$v \approx c \approx 3*10^8 \, {}^{m}\!/_{s}$$

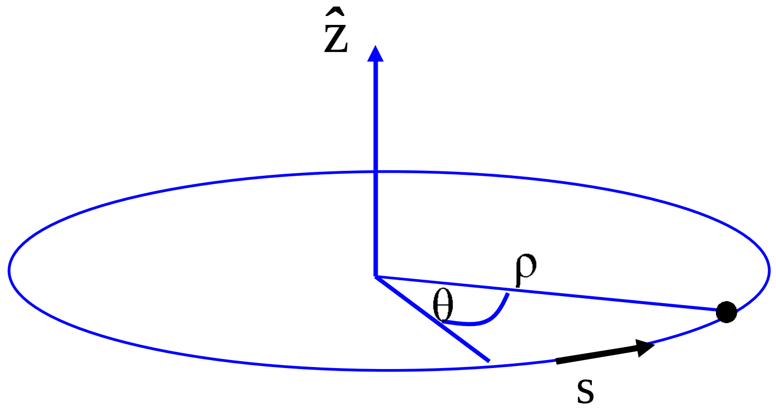
old greek dictum of wisdom:

if you are clever, you use magnetic fields in an accelerator wherever it is possible.

But remember: magn. fields act allways perpendicular to the velocity of the particle

→ only bending forces, → no „beam acceleration“

The ideal circular orbit



circular coordinate system

condition for circular orbit:

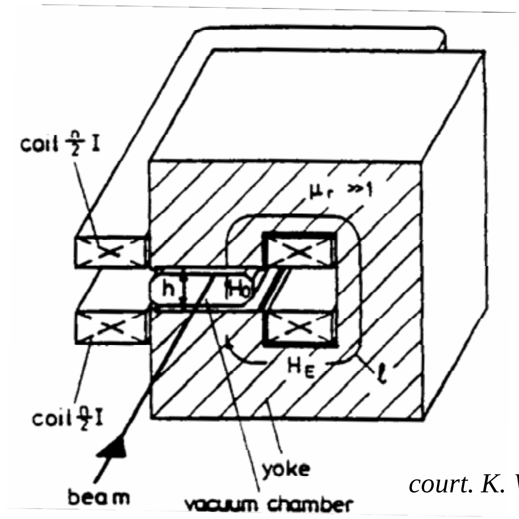
Lorentz force $\quad F_L = e * v * B$

centrifugal force $\quad F_{Zentr} = \frac{\gamma m_0 v^2}{\rho}$

$$\frac{\gamma m_0 \cancel{v}^2}{\rho} = e * \cancel{v} * B$$

$$\boxed{\frac{p}{e} = B * \rho}$$

I.) The Magnetic Guide Field



court. K. Wille

Dipole Magnets:

define the ideal orbit
homogeneous field created by two flat pole shoes

Field Calculation

3rd Maxwell equation for a static field: $$\vec{\nabla} \times \vec{H} = \vec{j}$$

according to Stokes theorem:

$$\int_S (\vec{\nabla} \times \vec{H})\vec{n}\, da = \oint \vec{H}\, d\vec{l} = \int_S \vec{j} \cdot \vec{n}\, da = N \cdot I$$

$$\oint \vec{H}\, d\vec{l} = H_0 * h + H_{Fe} * l_{Fe}$$

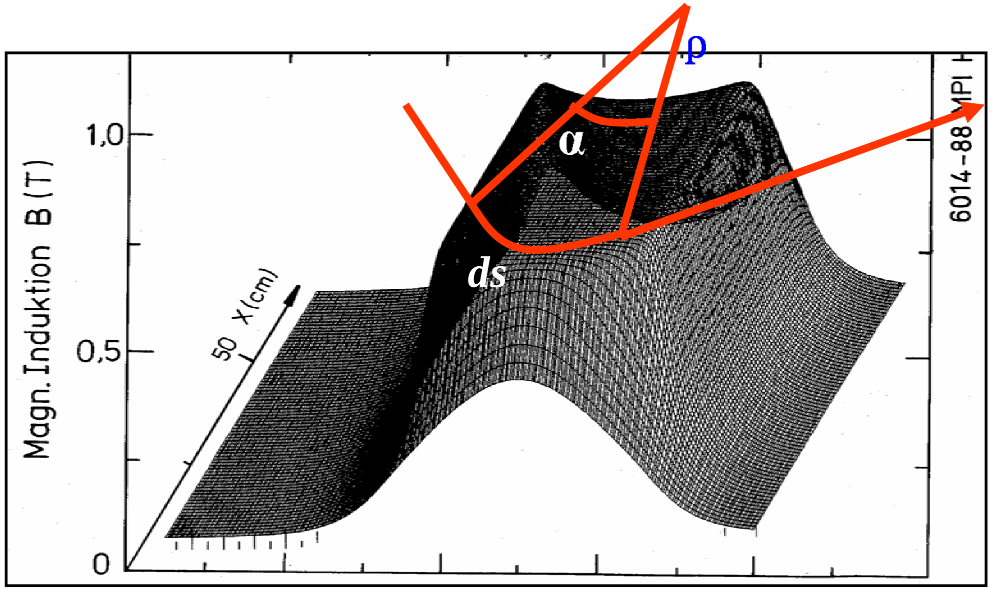
in matter we get with $\mu_r \approx 1000$

$$\oint \vec{H}\, d\vec{l} = H_0 * h + \cancel{\frac{H_0}{\mu_r} * l_{Fe}} \approx H_0 * h$$

Magnetic field of a dipole magnet:

$$H_0 = B_0 / \mu_0$$

$$B_0 = \frac{\mu_0 n I}{h}$$



field map of a storage ring dipole magnet

Normalise to momentum:

... remember $p/e = B^*\rho$

$$\frac{1}{\rho}\left[m^{-1}\right] = \frac{e \cdot B_0}{p} = 0.2998 \frac{B_0[T]}{p[GeV/c]}$$

„radius of curvature, bending strength"

Quadrupole Magnets:

required: ***focusing forces*** ***to keep trajectories in vicinity of the ideal orbit***

linear increasing Lorentz force

linear increasing magnetic field

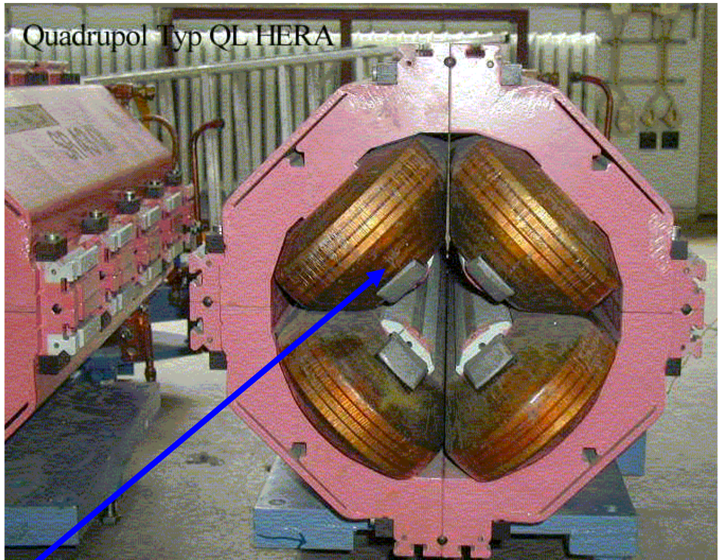
$$B_z = -g \cdot x \qquad B_x = -g \cdot z$$

at the location of the particle trajectory: no iron, no current

$$\vec{\nabla} \times \vec{B} = 0 \quad \rightarrow \quad \vec{B} = -\vec{\nabla} V$$

the magnetic field can be expressed as gradient of a scalar potential !

$$V(x,z) = g \cdot xz$$



equipotential lines (i.e. the surface of the iron contour) = hyperbolas

Calculation of the Quadrupole Field:

$$\oint \vec{H} d\vec{s} = N * I$$

$$B(r) = -g * r$$

gradient of a quadrupole field:

$$g = \frac{2\mu_0 nI}{r^2}$$

normalised quadrupole strength:

$$\boxed{k = \frac{g}{p/e}}$$

Separate Function Machines:

Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

II.) The equation of motion:

Linear approximation:

** ideal particle* → *design orbit*

** any other particle* → *coordinates x, z small quantities*
$x,z \ll \rho$

→ *magnetic guide field: only linear terms in x & z of B have to be taken into account*

Taylor Expansion of the B field:

$$B_z(x) = B_{z0} + \frac{dB_z}{dx}x + \cancel{\frac{1}{2!}\frac{d^2B_z}{dx^2}x^2} + \cancel{\frac{1}{3!}\frac{d^3B_z}{dx^3}x^3} + \ldots$$

… what about the vertical plane:

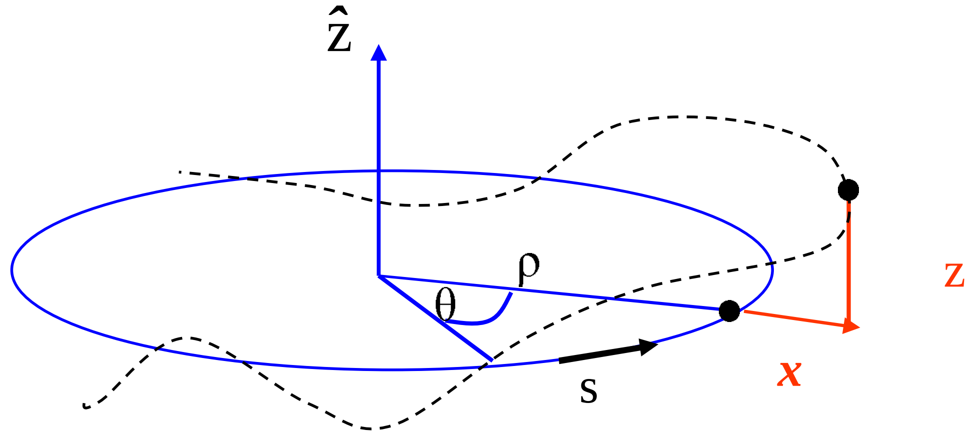
Maxwell:

$$\vec{\nabla} \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t} = 0$$

$$\Rightarrow \quad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x}$$

$$B_x(s,x,z) = \frac{\partial B_x}{\partial z} z$$

Equation of Motion:



Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

Force: $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

$$F = mv^2 / \rho$$

general trajectory:

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = eB_z v$$

develop for small x:

$$x \ll \rho$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1 - \frac{x}{\rho}) = eB_z v$$

guide field in linear approx.

$$B_z = B_0 + x\frac{\partial B_z}{\partial x}$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1 - \frac{x}{\rho}) = ev\left\{B_0 + x\frac{\partial B_z}{\partial x}\right\}$$

independent variable: $t \rightarrow s$

$$dx/_{dt} = dx/_{ds} * ds/_{dt}$$

$$x' = dx/_{ds}$$

$$x'' - \frac{1}{\rho}(1 - \frac{x}{\rho}) = \frac{eB_0}{mv} + \frac{exg}{mv}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = -\frac{1}{\rho} + kx$$

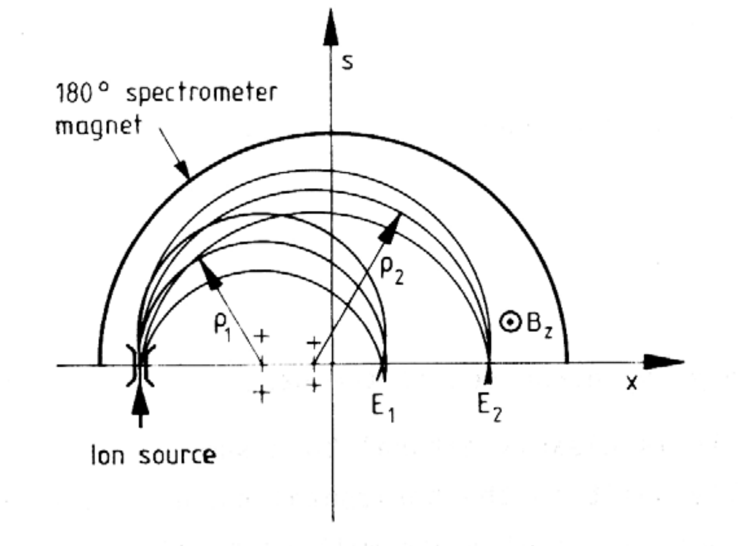
$$\boxed{x'' + x(\frac{1}{\rho^2} - k) = 0}$$

Remarks:

* $$x'' + \left(\frac{1}{\rho^2} - k\right) \cdot x = 0$$

… there seems to be a focusing even without a quadrupole gradient

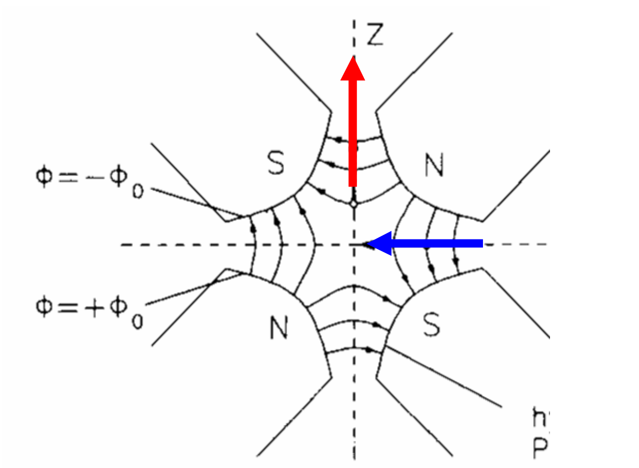
„weak focusing of dipole magnets“



Mass spectrometer: particles are separated according to their energy and focused due to the 1/ρ effect of the dipole

* ***Equation for the vertical motion:***

$$z'' + k \cdot z = 0$$

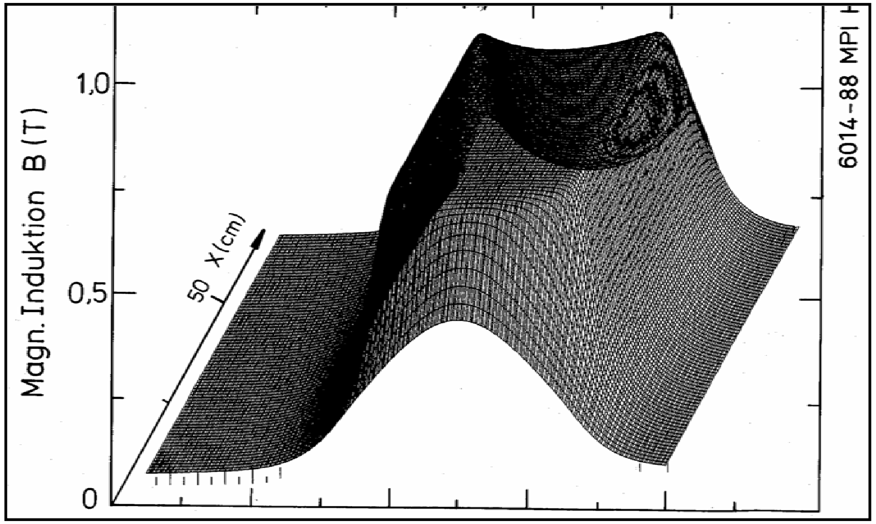


IV.) Solution of Trajectory Equations

Define … hor. plane: $K = 1/\rho^2 - k$

… vert. Plane: $K = k$

$$y'' + K * y = 0$$



K = const within a magnet

Differential Equation of harmonic oscillator … with spring constant K

Ansatz:

$$x(s) = a_1 \cdot \cos(\omega t) + a_2 \cdot \sin(\omega t)$$

general solution: linear combination of two independent solutions

Hor. Focusing Quadrupole K > 0:

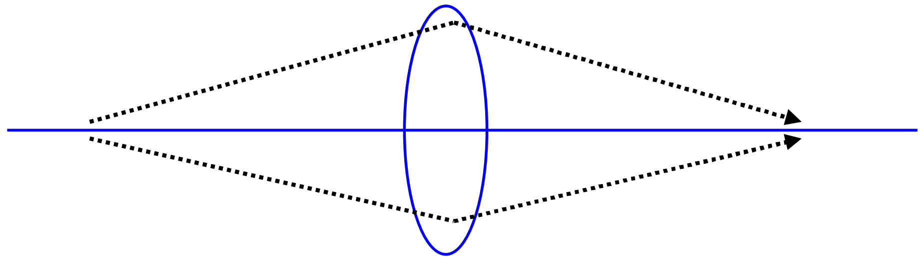
$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

$$\left.\begin{aligned} x(0) &= x_0 \\ x'(0) &= x'_0 \end{aligned}\right\} \quad \textit{starting conditions}$$

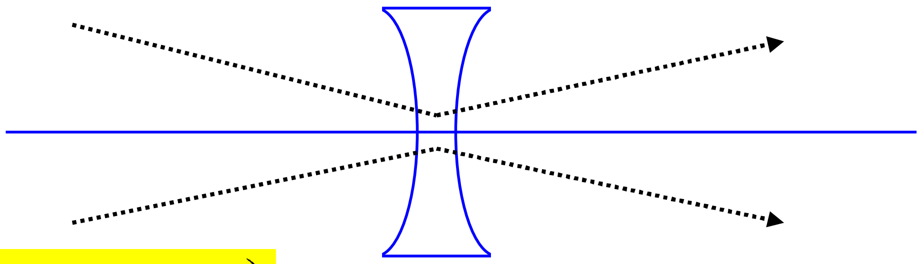
For convenience expressed in matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{foc} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_0$$



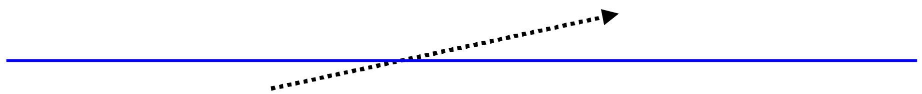
$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole: ***K < 0***



$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\,l \\ \sqrt{|K|}\sinh\sqrt{|K|}\,l & \cosh\sqrt{|K|}\,l \end{pmatrix}$$

drift space: ***K = 0***



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & z is uncoupled“

Thin Lens Approximation:

matrix of a quadrupole lens

$$M = \begin{pmatrix} \cos\sqrt{|k|}\,l & \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}\,l \\ -\sqrt{|k|}\sin\sqrt{|k|}\,l & \cos\sqrt{|k|}\,l \end{pmatrix}$$

in many practical cases we have the situation:

$$f = \frac{1}{k l_q} >> l_q$$

... focal length of the lens is much bigger than the length of the magnet

limes: $l \to 0$ ***while keeping*** $kl = const$

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \qquad M_z = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}$$

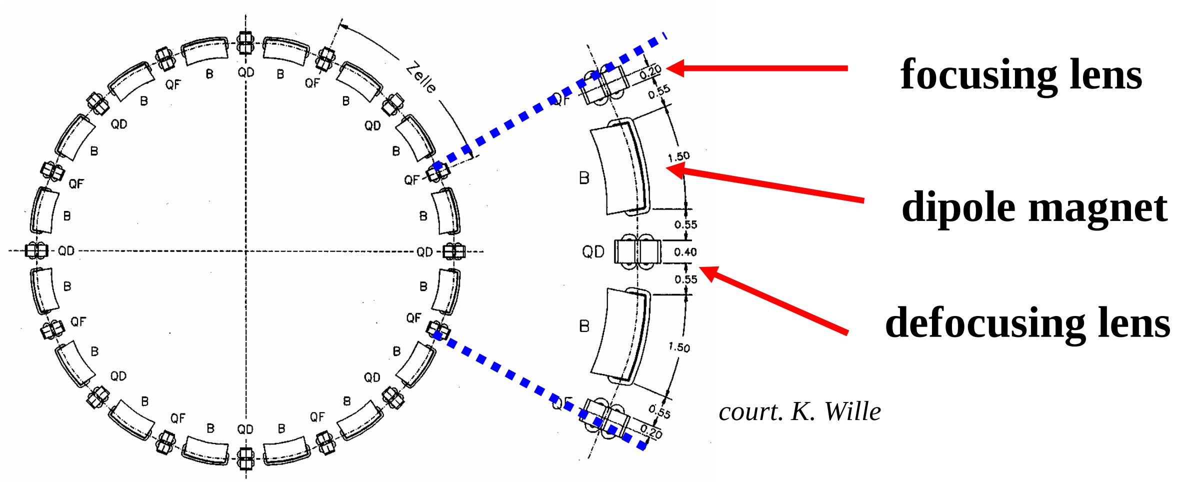
... usefull for fast (and in large machines still quite accurate) „back on the envelope calculations“ ... and for the guided studies !

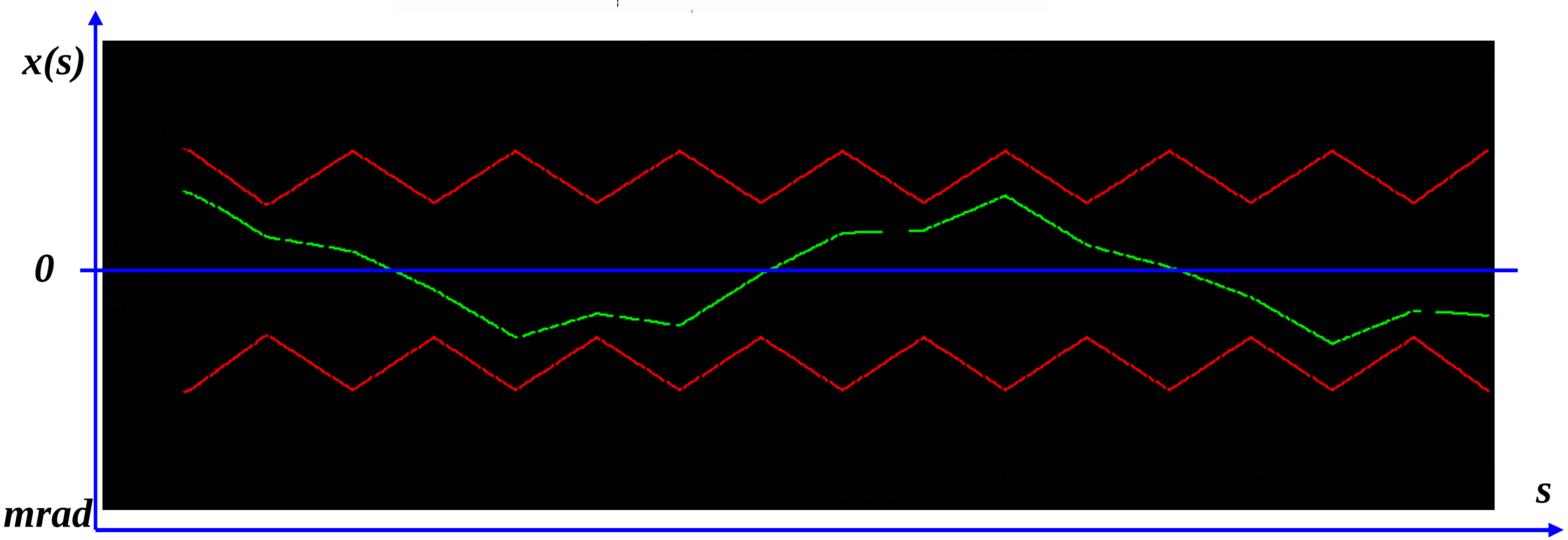
Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

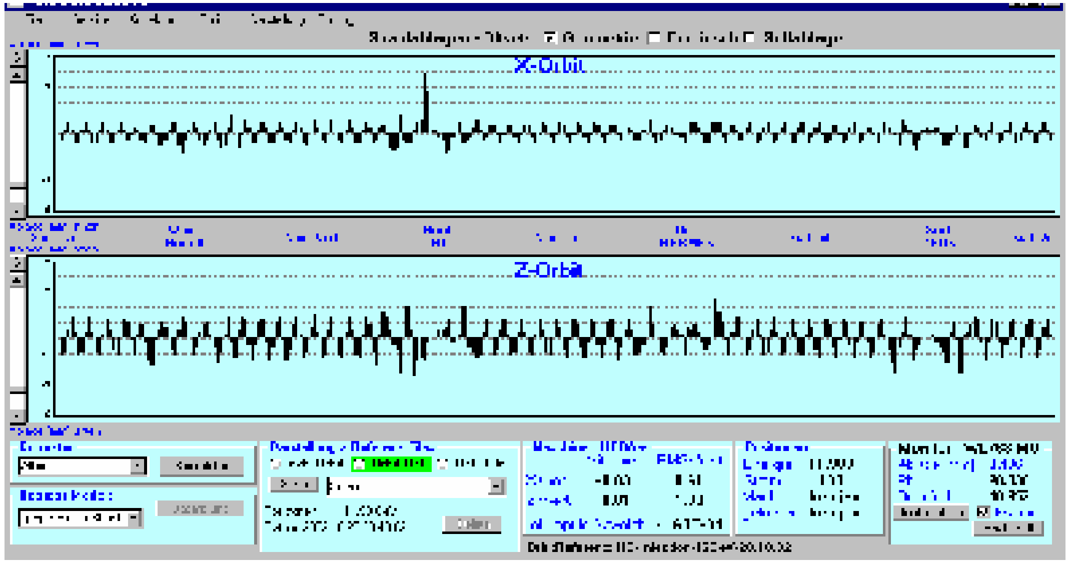
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(\text{s2,s1}) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$





typical values
in a strong
foc. machine:
x ≈ mm, x´ ≤ mrad

6.) Orbit & Tune:



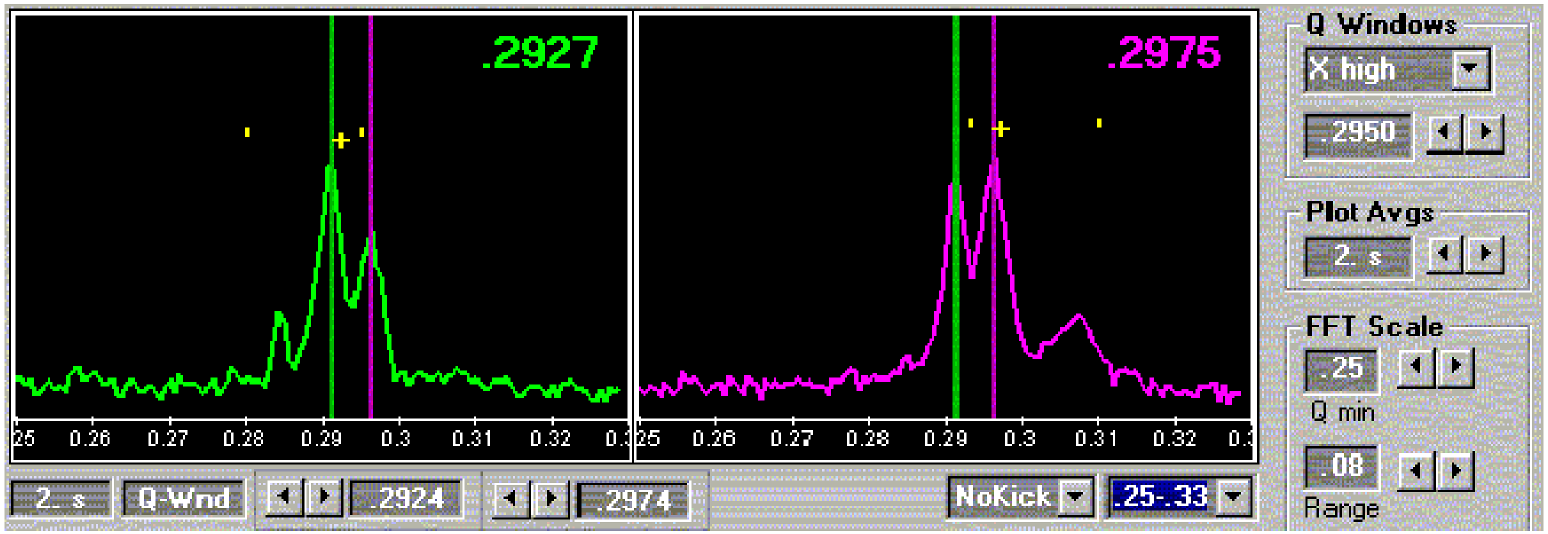
Tune: number of oscillations per turn

31.292
32.297

Relevant for beam stability:
non integer part

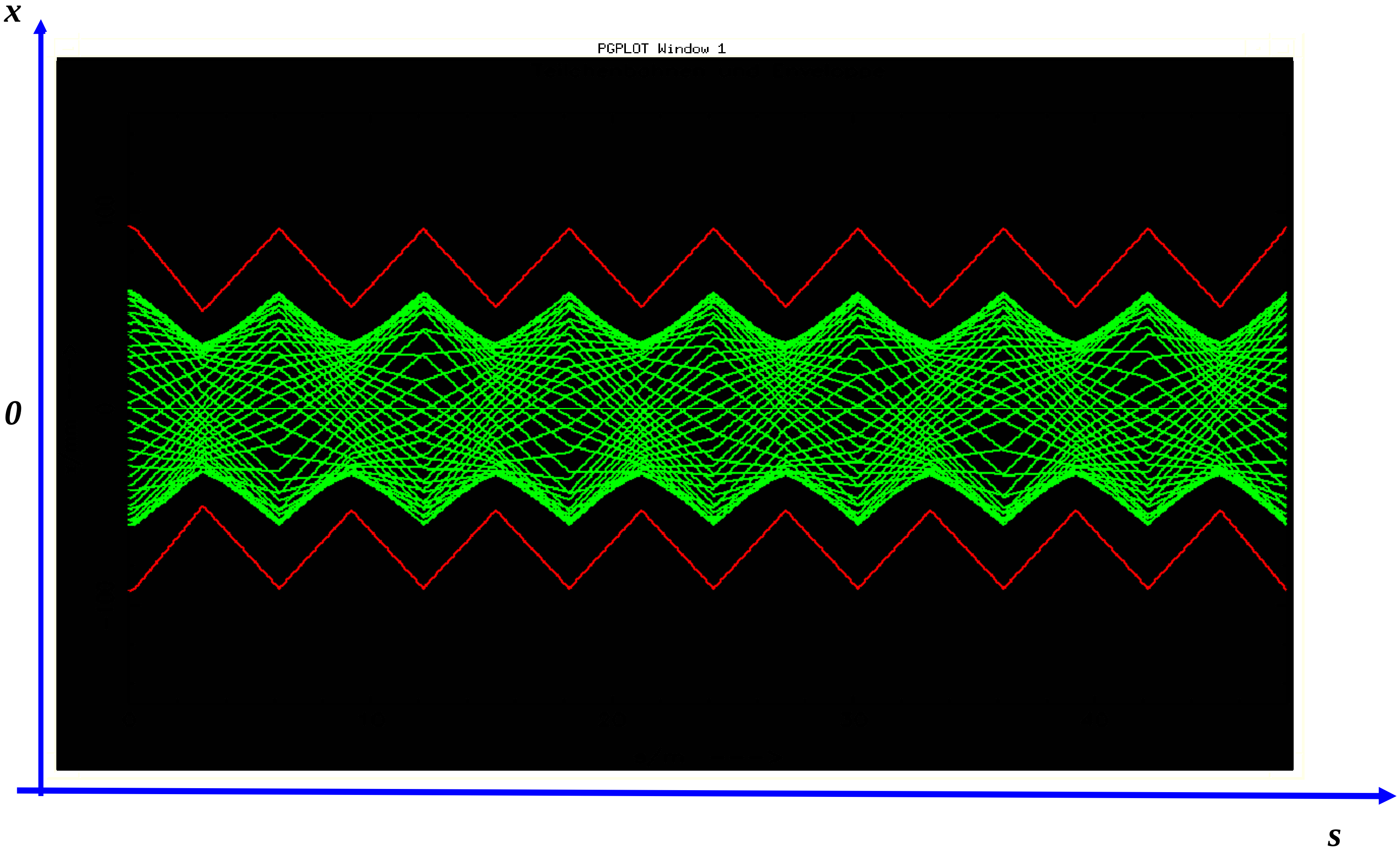
HERA revolution frequency: 47.3 kHz

$$0.292 * 47.3\ kHz = 13.81\ kHz$$



Question: ***what will happen, if the particle performs a second turn ?***

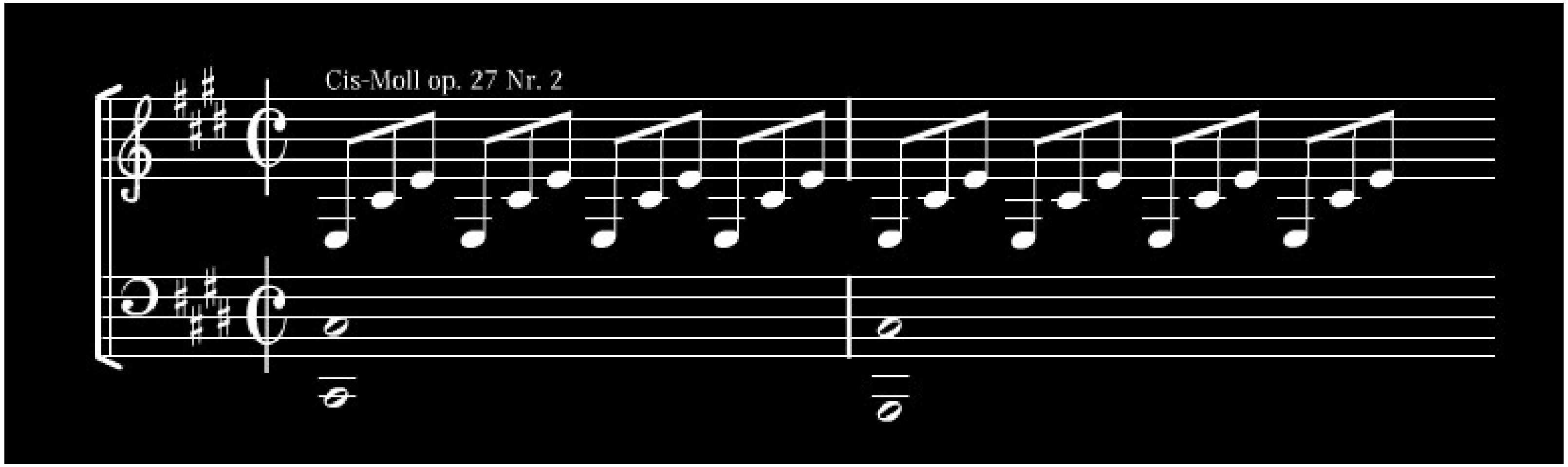
... or a third one or ... 10^{10} turns



19th century:

Ludwig van Beethoven: „Mondschein Sonate“

Sonate Nr. 14 in cis-Moll (op. 27/II, 1801)



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $$x''(s) - k(s)x(s) = 0$$

restoring force $\neq$ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

The Beta Function

General solution of Hill´s equation:

$$x(s)=\sqrt{\varepsilon}\sqrt{\beta(s)}\cdot\cos(\psi(s)+\phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L)=\beta(s)$$

Inserting (i) into the equation of motion …

$$\psi(s)=\int_0^s\frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.
For one complete revolution: number of oscillations per turn „Tune“

$$Q_y=\frac{1}{2\pi}\cdot\oint\frac{ds}{\beta(s)}$$

Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$(1) \quad x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \phi)$$

$$(2) \quad x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} * \{\alpha(s) * \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi)\}$$

from (1) we get

$$\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\varepsilon} * \sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2} \beta'(s)$$

$$\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}$$

Insert into (2) and solve for ε

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s) x'(s)^2$$

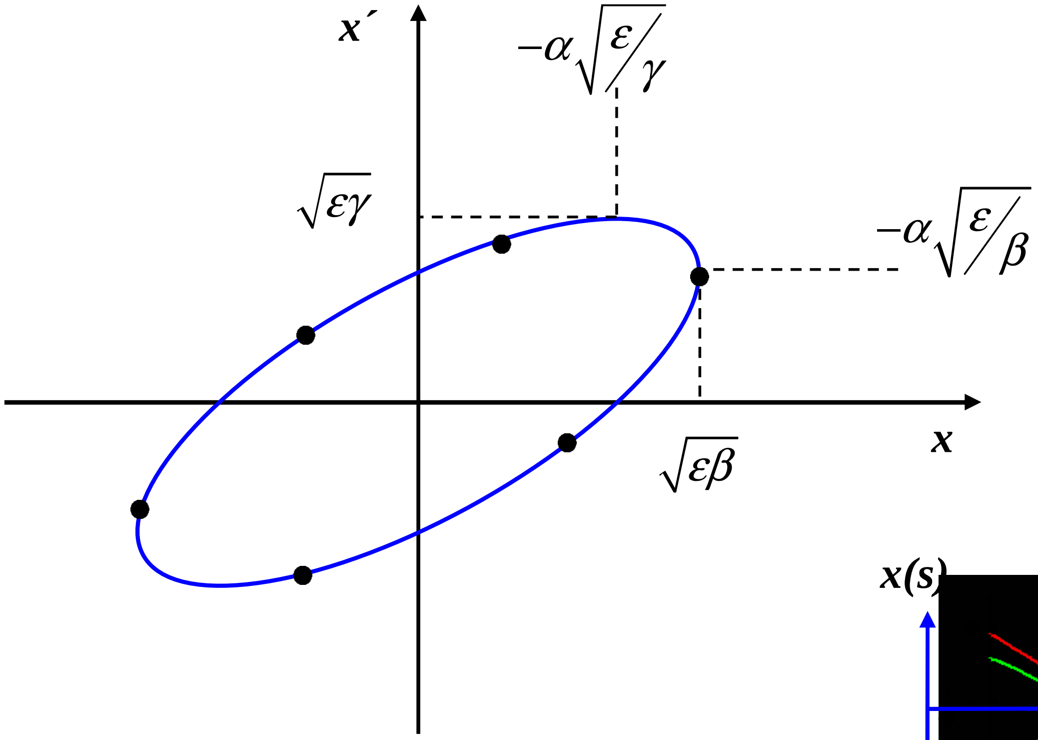
***** *ε* **is a** ***constant*** ***of the motion … it is independent of „s“***

*** parametric representation of an ellipse in the x x‘ space***

*** shape and orientation of ellipse are given by α, β, γ***

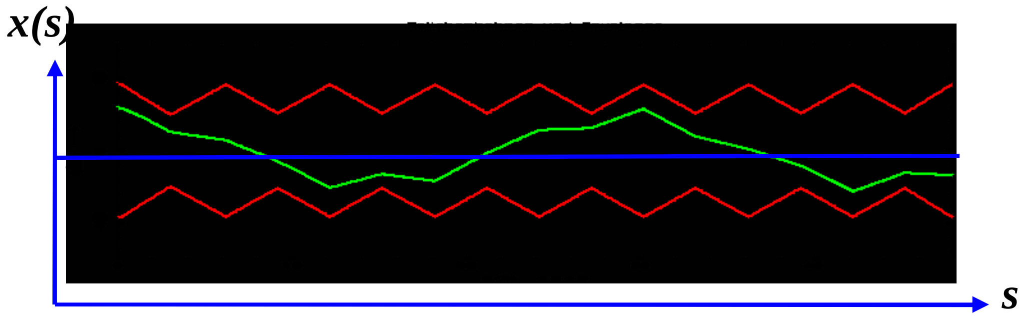
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

A = π*ε=const



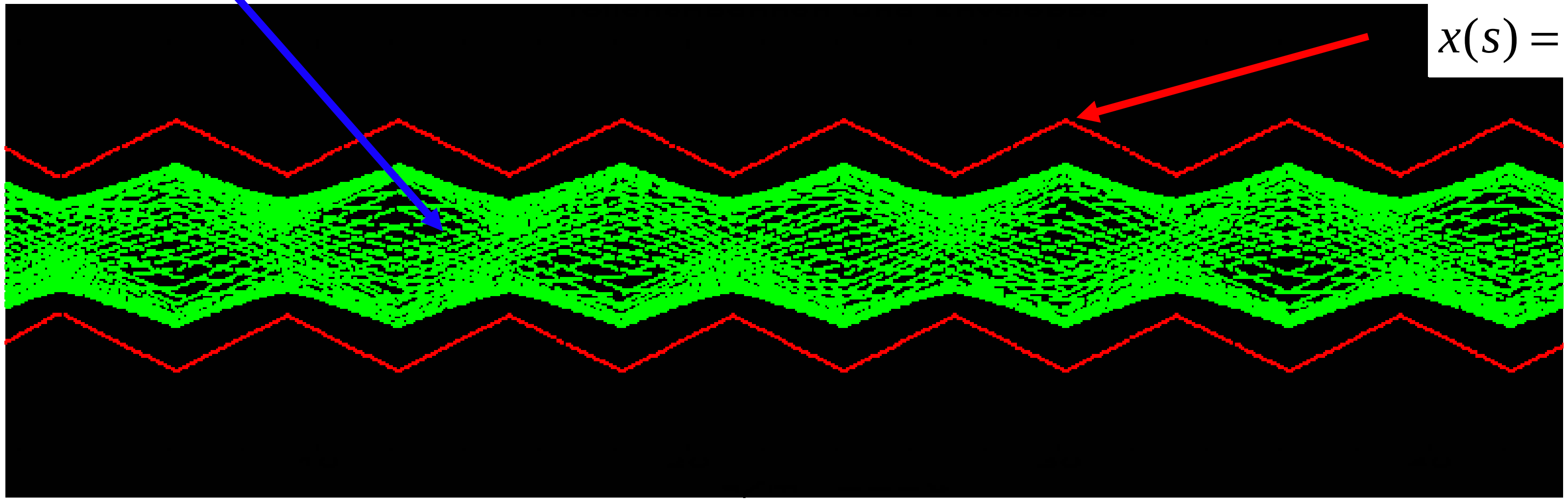
ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.
Scientifiquely spoken: area covered in transverse x, x´ phase space … and it is constant !!!

Ensemble of many (...all) possible particle trajectories

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \phi)$$

max. amplitude of all particle trajectories

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)}$$

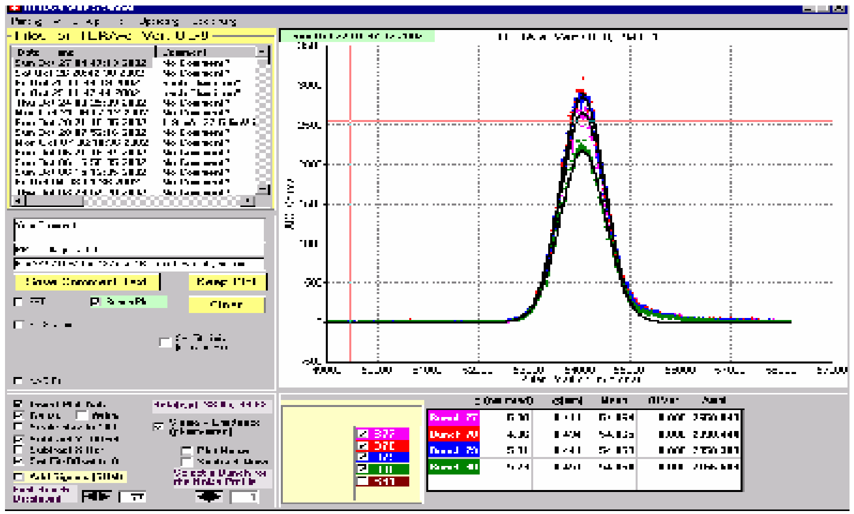


Beam Dimension:

determined by two parameters

$$\sigma = \sqrt{\varepsilon * \beta}$$

Example: transverse beam profile measured using a wirescanner



Transfer Matrix M ***… yes we had the topic already***

general solution of Hill´s equation

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right]$$

remember the trigonometrical gymnastics: sin(a+b)= … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point $s(0) = s_0$, where we put $\Psi(0) = 0$

$$\left.\begin{aligned} \cos\phi &= \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \\ \sin\phi &= -\frac{1}{\sqrt{\varepsilon}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{aligned}\right\}$$

inserting above …

$$x(s)=\sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s+\alpha_0\sin\psi_s\right\}x_0+\left\{\sqrt{\beta_s\beta_0}\sin\psi_s\right\}x'_0$$

$$x'(s)=\frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s\right\}x_0+\sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s-\alpha_s\sin\psi_s\right\}x'_0$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

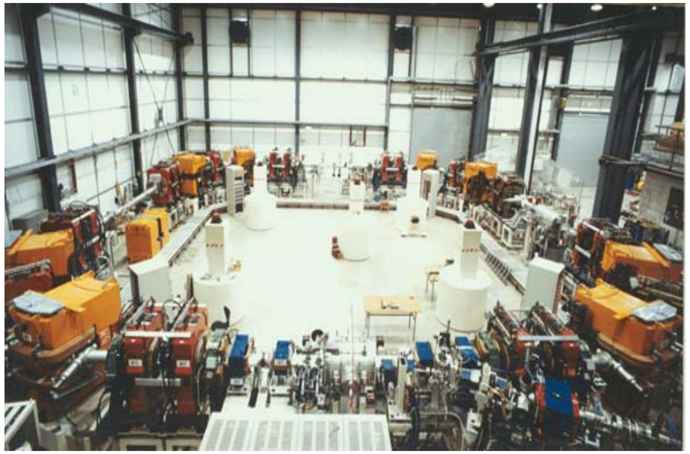
$$M=\begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s+\alpha_0\sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s-\alpha_s\sin\psi_s\right) \end{pmatrix}$$

*** we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***

*** and nothing but the α β γ at these positions.***

*** … !**

Stability Criterion:



Question: *what will happen, if we do not make too many mistakes and your particle performs one complete turn ?*

Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{1}} + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Matrix for N turns:

$$M^N = (1 \cdot \cos\psi + J \cdot \sin\psi)^N = 1 \cdot \cos N\psi + J \cdot \sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

$$\psi = real \quad \leftrightarrow \quad |\cos\psi| \leq 1 \quad \leftrightarrow \quad Tr(M) \leq 2$$

VIII.) Transformation of α, β, γ

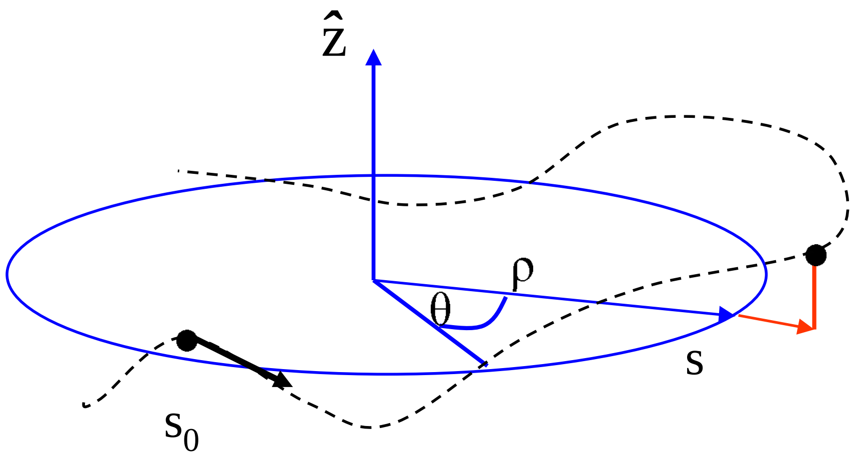
consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

since ε = const:

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0{}^2$$



express x_0, x'_0 as a function of x, x´.

... remember W = CS´-SC´ = 1

$$\left.\begin{aligned} \begin{pmatrix} x \\ x' \end{pmatrix}_0 &= M^{-1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_s \\ M^{-1} &= \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} x_0 &= S'x - Sx' \\ x_0' &= -C'x + Cx' \end{aligned}$$

inserting into ε

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

sort via x, x´and compare the coefficients to get ....

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)

V.) Résumé:

beam rigidity: $$B \cdot \rho = p/q$$

bending strength of a dipole: $$\frac{1}{\rho}\left[m^{-1}\right] = \frac{0.2998 \cdot B_0(T)}{p(GeV/c)}$$

focusing strength of a quadrupole: $$k\left[m^{-2}\right] = \frac{0.2998 \cdot g}{p(GeV/c)}$$

$$k\left[m^{-2}\right] = \frac{0.2998}{p(GeV/c)} \frac{2\mu_0 nI}{a_r^2}$$

focal length of a quadrupole: $$f = \frac{1}{k \cdot l_q}$$

equation of motion: $$x'' + Kx = \frac{1}{\rho}\frac{\Delta p}{p}$$

matrix of a foc. quadrupole: $$x_{s2} = M \cdot x_{s1}$$

$$M = \begin{pmatrix} \cos\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}l \\ -\sqrt{|K|}\sin\sqrt{|K|}l & \cos\sqrt{|K|}l \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$