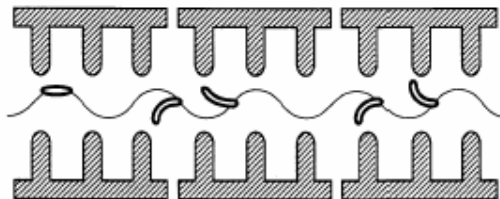


# INSTABILITIES IN LINACS

Massimo Ferrario

INFN-LNF

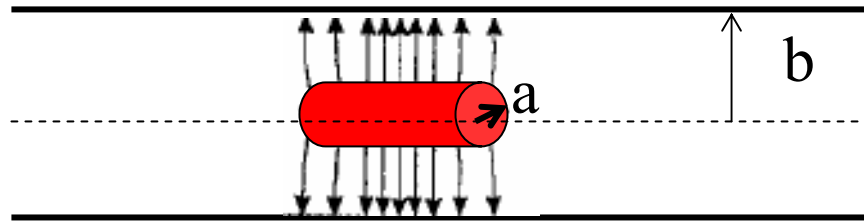


- **Introduction to Wake Fields**
- **Instability in Linacs: Beam Break Up**

# Circular Perfectly Conducting Pipe

- Beam at Center- Static Approximation  $\gamma \rightarrow \infty$

$$\rho = \frac{I}{\pi a^2 v}$$



$$\varphi(b) = 0$$

$$2\pi r l \epsilon_0 E_r = \rho \pi r^2 l$$

$$E_r = \frac{I}{2\pi \epsilon_0 a^2 v} r \quad \text{for } r \leq a$$

$$2\pi r l \epsilon_0 E_r = \rho \pi a^2 l$$

$$E_r = \frac{I}{2\pi \epsilon_0 v} \frac{1}{r} \quad \text{for } r > a$$

$$B_g = \frac{\beta}{c} E_r$$

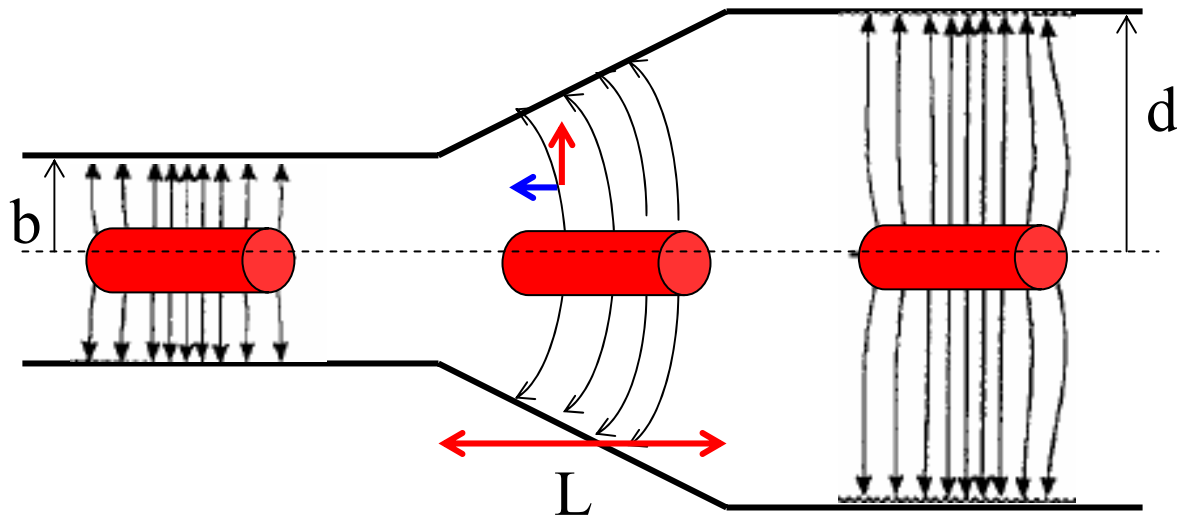
$$\varphi(r) = \int_r^b E_r(r') dr'$$

$$= \frac{I}{4\pi \epsilon_0 v} \left( 1 + 2 \ln \frac{b}{a} - \frac{r^2}{a^2} \right) \quad \text{for } r \leq a$$

$$= \frac{I}{2\pi \epsilon_0 v} \ln \frac{b}{r} \quad \text{for } a \leq r \leq b$$

$$\varphi(b) = 0$$

# Circular Perfectly Conducting Pipe with Transition



There is a longitudinal  $E_z(r,z)$  field in the transition and a test particle experience a voltage given by:

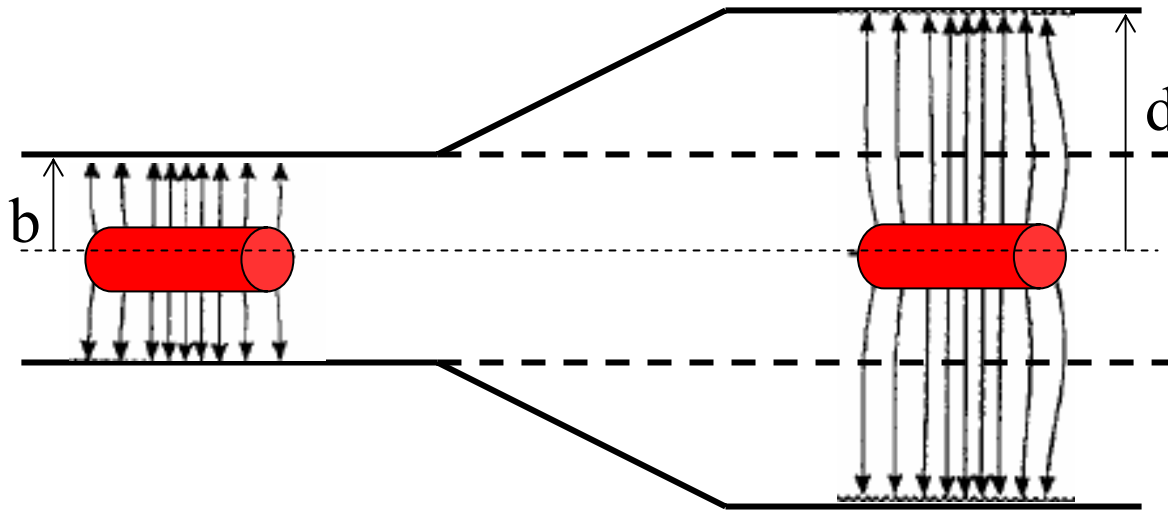
$$V = - \int_0^L E_z(r,z) dz = -(\varphi(r,L) - \varphi(r,0)) = -\frac{I}{2\pi\epsilon_0 v} \ln \frac{d}{b}$$

decelerating if  $d > b$

$$P_b = VI = \frac{I^2}{2\pi\epsilon_0 v} \ln \frac{d}{b}$$

Power lost by the beam

**For  $d > b$  the power is deposited to the energy of the fields:**  
**moving from left to right the beam induces the fields in the**  
**additional space available**



The additional **power passing through the right part of the beam pipe** is obtained by

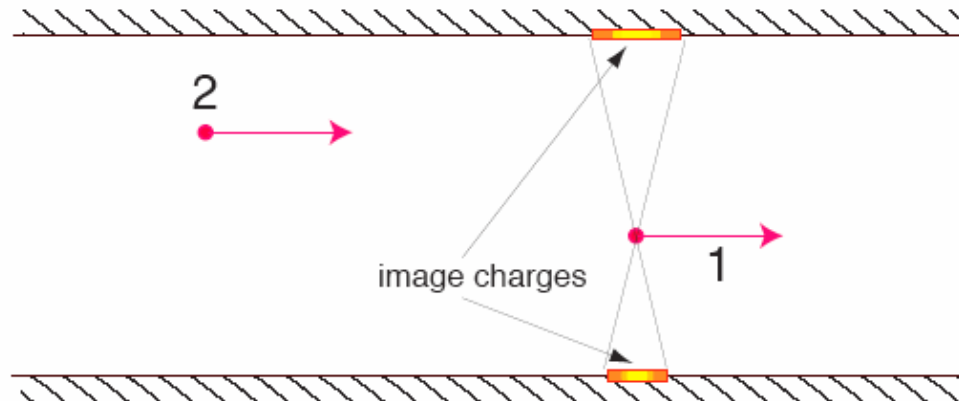
integrating the Poynting vector through the surface  $\Delta S = \pi(d^2 - b^2)$

$$P_{em} = \int_{\Delta S} \left( \frac{1}{\mu} \vec{E} \times \vec{B} \right) \cdot \hat{n} dS = \int_b^d \frac{E_r B_\theta}{\mu} 2\pi r dr = \frac{I^2}{2\pi\epsilon_0 v} \ln \frac{d}{b}$$

Notice that if  **$d < b$  the beam gains energy**. If  **$d \rightarrow \infty$  the power goes to infinity**, such an unphysical result is nevertheless consistent with the original assumption of an infinite energy beam ( **$\gamma \rightarrow \infty$** ).

## Scattered Field

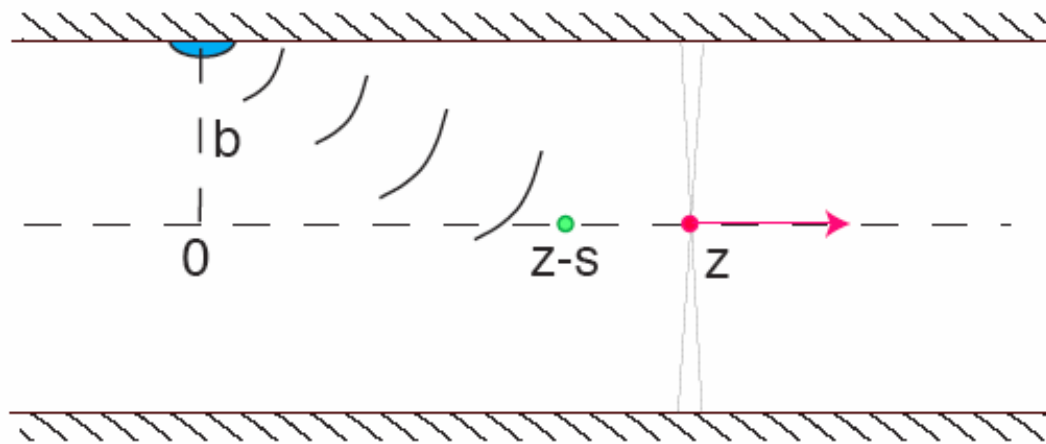
The image charges travel with the same particle velocity  $v$ .  
Since both the particles and the image charges move on parallel paths, in the limit  $\mathbf{v} = \mathbf{c}$  they do not interact with each other, no matter how close to the wall the particles are.



**FIGURE 2.** Particles traveling inside a perfectly conducting pipe of arbitrary cross section. Shown are the image charges on the wall generated by the leading charge.

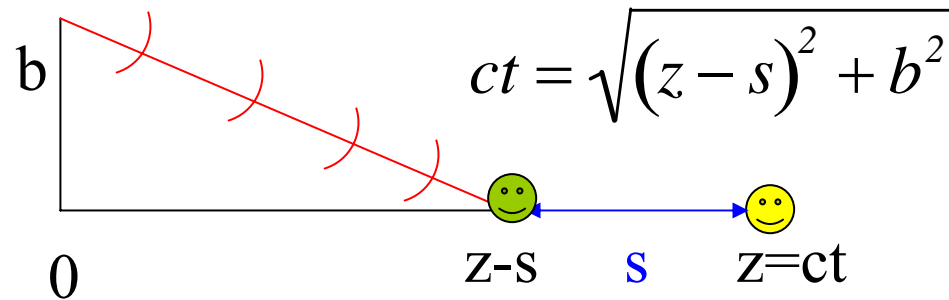
If a particle moves along a straight line with the speed of light, the electromagnetic field of this particle scattered off the boundary discontinuities will not overtake it and, furthermore, will not affect the charges that travel ahead of it.

The field can interact only with the trailing charges in the beam that move behind it.  
This constitutes the **principle of causality** in the theory of wake fields



**FIGURE 3.** A wall discontinuity located at  $z = 0$  scatters the electromagnetic field of an ultrarelativistic particle. When the particle moves to location  $z$ , the scattered field arrives to point  $z - s$ .

We can estimate the distance at which the electromagnetic field produced by a leading charge reaches a trailing particles traveling at a distance  $s$  behind.

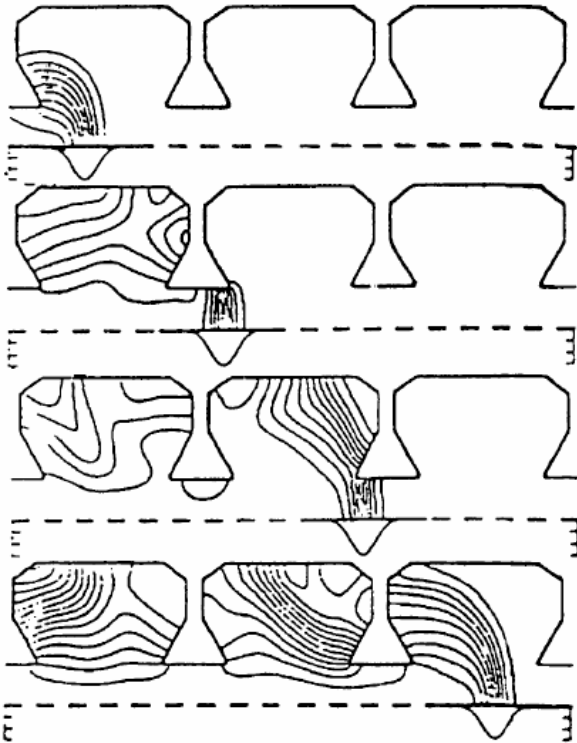


$$z^2 = (z-s)^2 + b^2 \quad \implies \quad z_{\text{catch-up}} \approx \frac{b^2}{2s} \quad \text{for } s \ll b$$

Only after the leading charge has traveled  $z_{\text{catch-up}}$  away from the discontinuity, can a particle at point  **$s$  behind it** feel the field generated by the discontinuity.



## Numerical Analysis

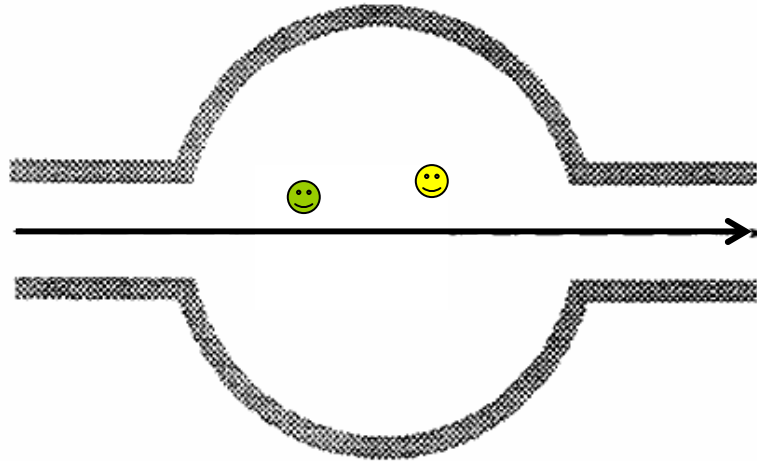


The study of the fields requires to solve the Maxwell equations in a given structure taking the beam current as source of fields. This is a quite complicated task for which it has been necessary to develop dedicated computer codes, which solve the e.m. problem in the frequency or in the time domain. There are several useful codes for the design of accelerator devices: **MAFIA**, **ABCI**, **URMEL**, etc...

## Theoretical Analysis

The parasitic fields depends on the particular charge distribution of the beam. It is therefore desirable to know what is the effect of a single charge, i.e. **find the Green function  $w$** , in order to reconstruct the fields produced by any charge distribution.

## Wake Potentials



$$\mathbf{F} = q \left[ E_z \hat{z} + (E_x - vB_y) \hat{x} + (E_y + vB_x) \hat{y} \right] \equiv \mathbf{F}_{\parallel} + \mathbf{F}_{\perp}$$

there can be two effects on the **test charge** :

- 1) a **longitudinal force** which **changes its energy**,
- 2) a **transverse force** which **deflects its trajectory**.

If we consider a device of length L:

the Energy Gain is:

$$U = \int_0^L F_z ds$$

the Transverse Deflecting Kick is:

$$M = \int_0^L \mathbf{F}_\perp ds$$

These quantities, normalised to the charges, are called *wake-potentials* (volt/coulomb) and are both function of the distance z.

Note that the integration is performed over a given path of the trajectory.

***Longitudinal wake potential***  
(Volt/Coulomb)

$$w_{\parallel} = -\frac{U}{q_o q}$$

**Energy Loss**

***Transverse wake potential***  
(Volt/Coulomb meter)

$$w_{\perp} = \frac{l}{r_o} \frac{M}{q_o q}$$

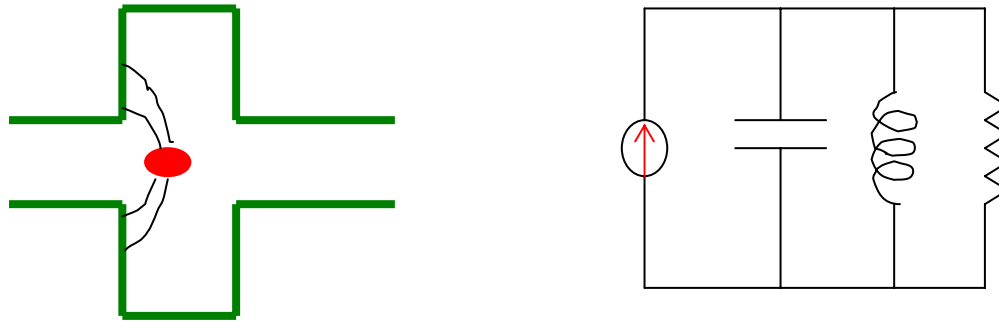
**Transverse Kick**

The sign minus in the longitudinal wake-potential means that the test charge loses energy when the wake is positive.

Positive transverse wake means that the transverse force is defocusing.

# Longitudinal wake potential of a resonant HOM

When a charge crosses a resonant structure, it excites the fundamental mode and high order modes (HOM). Each mode can be treated as an electric RLC circuit loaded by an impulsive current.



Just after the charge passage, the capacitor is charged with a voltage  $V_o = Cq_o$  and the electric field is  $E_{so} = V_o/l_o$ .

The time evolution of the electric field is governed by the same differential equation of the voltage

$$\ddot{V} + \frac{1}{RC} \dot{V} + \frac{1}{LC} V = \frac{1}{C} I$$

The passage of the impulsive current charges only the capacitor, which changes its potential by an amount  $V_c(0)$ .

This potential will oscillate and decay producing a current flow in the resistor and inductance.

For  $t > 0$  the potential satisfy the following equation and initial conditions:

$$\ddot{V} + \frac{1}{RC} \dot{V} + \frac{1}{LC} V = 0$$

$$V(t = 0^+) = \frac{q}{C} \equiv V_0$$

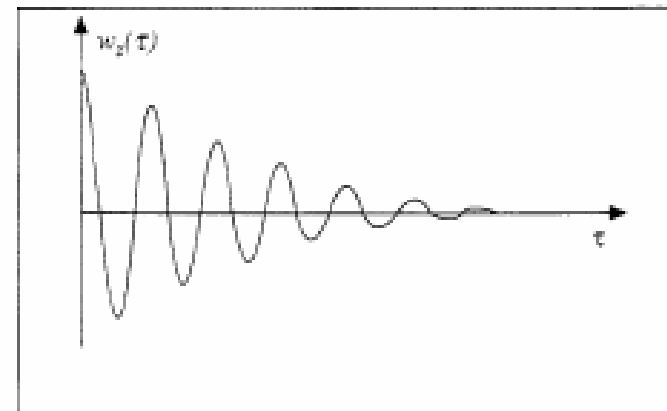
$$\dot{V}(t = 0^+) = \frac{\dot{q}}{C} = \frac{I(0^+)}{C} = \frac{V_0}{RC}$$

$$V(t) = V_0 e^{-\gamma t} \left[ \cos(\bar{\omega} z / c) - \frac{\gamma}{\bar{\omega}} \sin(\bar{\omega} t) \right]$$

$$\bar{\omega}^2 = \omega_r^2 - \gamma^2$$

putting  $z = -ct$  ( $z$  is negative behind the charge),

$$w_{//}(z) = \frac{-V(z)}{q_0} = w_0 e^{\gamma z / c} \left[ \cos(\bar{\omega} z / c) + \frac{\gamma}{\bar{\omega}} \sin(\bar{\omega} z / c) \right]$$



It is also useful to define the **loss factor** as the normalised energy lost by the **source charge  $q_0$**

$$k = - \frac{U(z=0)}{q_0^2}$$

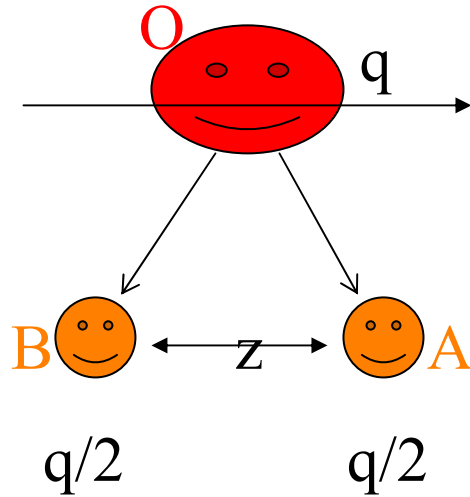
Although in general the loss factor is given by the longitudinal wake at  $z=0$ , for charges travelling with the light velocity the longitudinal wake potential is discontinuous at  $z=0$

The exact relationship between  $k$  and  $w(z \rightarrow 0)$  is given by the **beam loading theorem**:

$$k = \frac{w_{//}(z \rightarrow 0)}{2}$$

Causality requires that the longitudinal wake potential of a charge travelling with the velocity of light is discontinuous at the origin.

$$U_o = -q^2 k$$



$$U_A = -q_A^2 k = -\frac{q^2}{4} k$$

$$U_B = -q_B^2 k - q_A q_B W_{//}(z) \\ = -\frac{q^2}{4} k - \frac{q^2}{4} W_{//}(z)$$

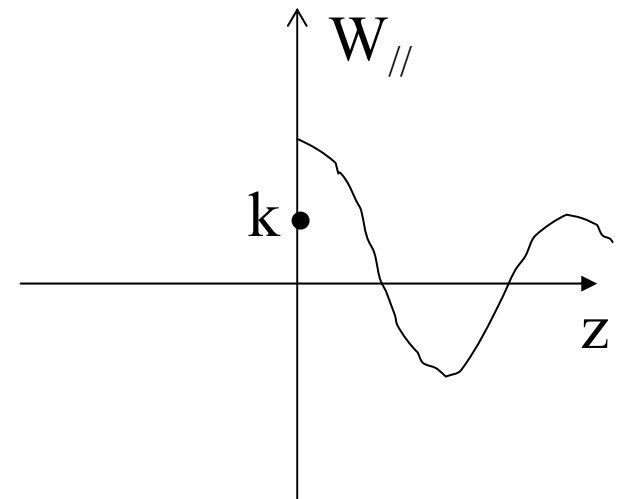
$$U_A + U_B = -\frac{q^2}{2} k - \frac{q^2}{4} W_{//}(z)$$

$$z \rightarrow 0 \quad U_o = -q^2 k = U_A + U_B$$

$$q^2 k = \frac{q^2}{2} k + \frac{q^2}{4} W_{//}(0)$$

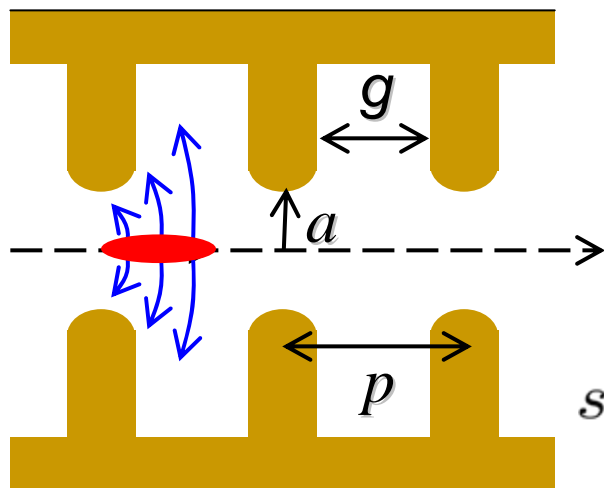
$$\frac{W_{//}(0)}{4} = \frac{k}{2}$$

$$k = \frac{W_{//}(0)}{2}$$





# Longitudinal Wakefields of RF Structures



**SLAC S-band:**

$$a \approx 11.6 \text{ mm}$$

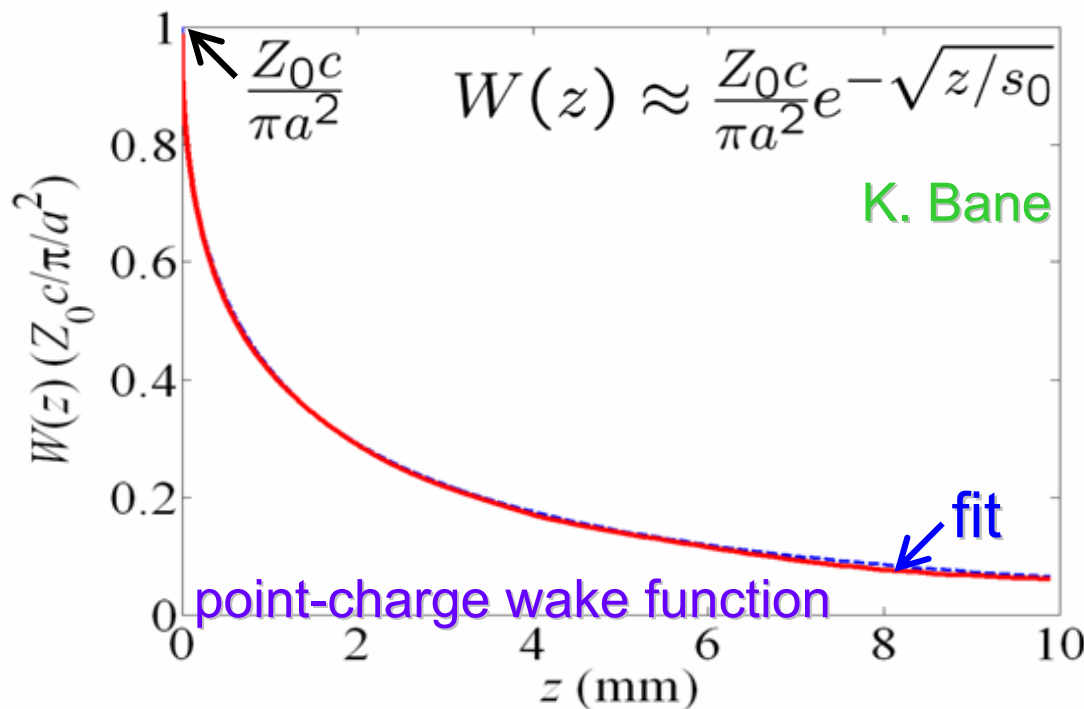
$$g \approx 29.2 \text{ mm}$$

$$p \approx 35.0 \text{ mm}$$

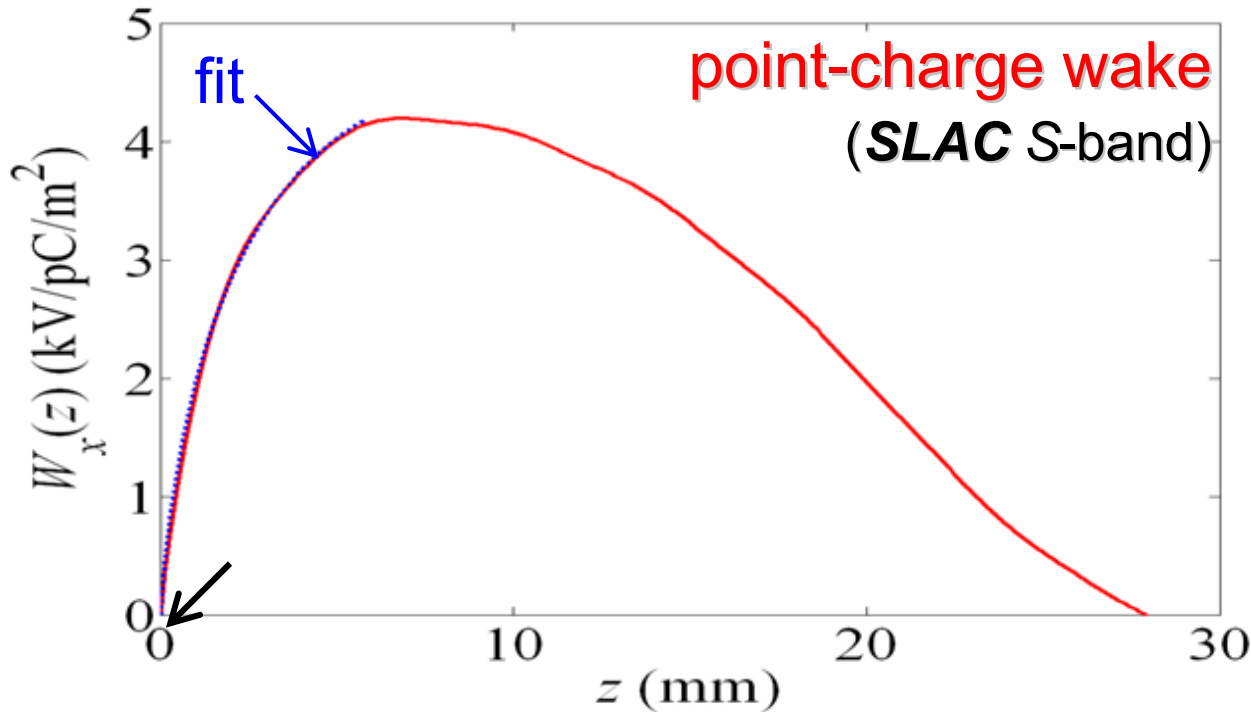
$$s_0 \approx 0.41 \frac{a^{1.8} g^{1.6}}{p^{2.4}}$$



$$W_{//} \propto \omega^2$$



# Transverse Wakefields



**SLAC S-band**  
 $s_1 \approx 0.56$  mm  
 $a \approx 11.6$  mm  
 $A \approx 1.13$   
 $z < \sim 6$  mm

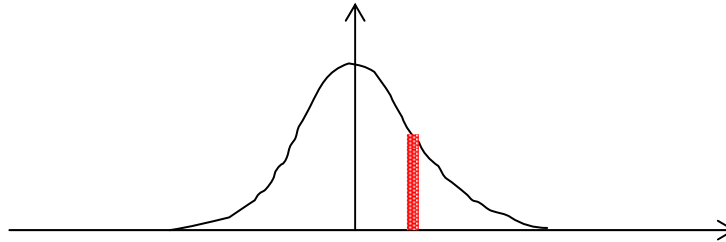
$$W_{\perp} \propto \omega^3$$

transverse point-charge wakefield function and short-range fit:

$$W_x(z) \approx A \frac{4Z_0cs_1}{\pi a^4} \left[ 1 - (1 + \sqrt{z/s_1})e^{-\sqrt{z/s_1}} \right], \quad z < 6 \text{ mm}$$

# Wake potentials and energy loss of a bunched distribution

When we have a bunch with density  $\lambda(z)$ , we may wonder what is the amount of energy lost or gained by a single charge  $e$  in the beam



To this end we calculate the effect on the charge from the whole bunch by means of the **convolution integral**:

$$U(z) = -e \int_{-\infty}^{\infty} w_{//}(z-z') \lambda(z') dz'$$

Which permits to define the **wake potential of a distribution**

$$W_{//}(z) = -\frac{U(z)}{qq_0}$$

The total energy lost by the bunch is computed summing up the loss of all particles:

$$U_{bunch} = -\frac{I}{e} \int_{-\infty}^{\infty} U(z') \lambda(z') dz'$$

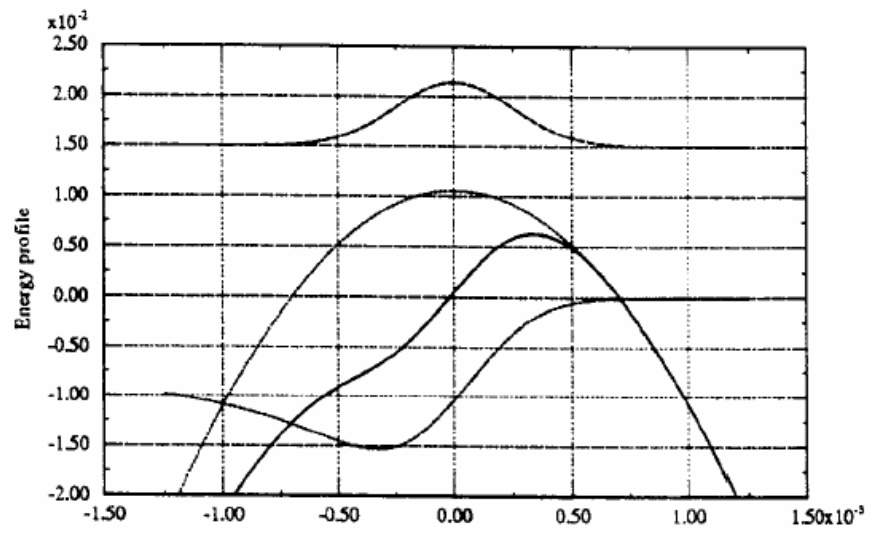


Fig. 5 Energy profile within the bunch sitting on the crest of the rf wave

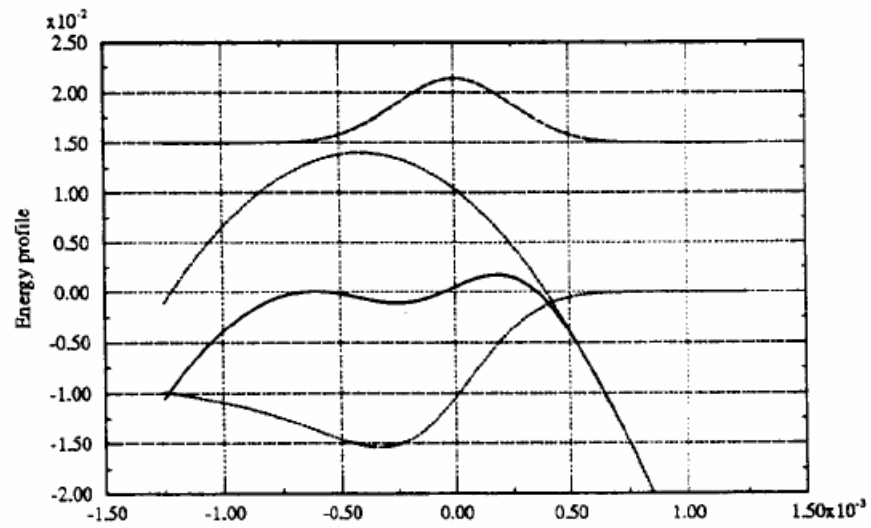


Fig. 6 Energy profile within the bunch after optimization of the rf phase

# **Wake Fields Effects In Linear Accelerators**

## Instabilities : driven oscillators

Consider an harmonic oscillator with natural frequency  $\omega$ , with an external excitation at frequency  $\Omega$

$$\ddot{x} + \omega^2 x = A \cos(\Omega t)$$

General solution:



$$x(t) = x^{free}(t) + x^{driven}(t)$$

$$\cos(\Omega t) \Rightarrow e^{i\Omega t}$$

$$x^{free}(t) = \tilde{x}_m^f e^{i\omega t}$$

$$x^{driven}(t) = \tilde{x}_m^d e^{i\Omega t}$$

substitution in the diff. equation:

$$(\omega^2 - \Omega^2) \tilde{x}_m^d e^{i\Omega t} = A e^{i\Omega t}$$

$$x^{driven}(t) = \frac{A}{(\omega^2 - \Omega^2)} e^{i\Omega t}$$

The general solution has to satisfy the initial condition at  $t=0$ . In our case we assume that the oscillator is at rest for  $t=0$ :

$$x^{free}(t=0) = -x^{driven}(t=0)$$

$$\tilde{x}_m^f = -\frac{A}{\omega^2 - \Omega^2}$$

thus we get:

$$x(t) = \frac{A}{\omega^2 - \Omega^2} [e^{i\Omega t} - e^{i\omega t}]$$

taking only the real part:

$$x(t) = \frac{A}{\omega^2 - \Omega^2} [\cos(\Omega t) - \cos(\omega t)]$$

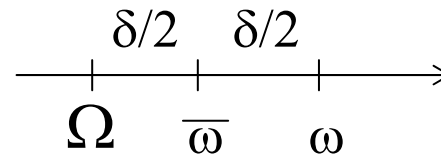
This expression is suitable for deriving the response of the oscillator driven at resonance or at frequency very close:

$$\omega - \Omega = \delta$$

$$\bar{\omega} = (\omega + \Omega) / 2$$

$$\omega = \bar{\omega} + \delta / 2$$

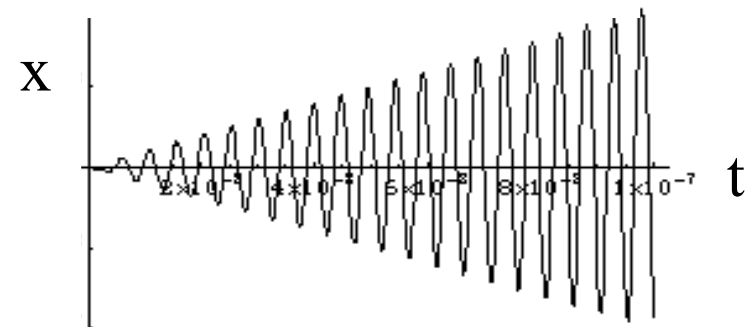
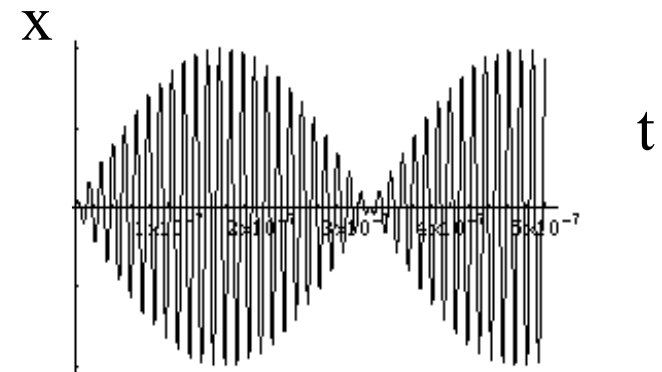
$$\Omega = \bar{\omega} - \delta / 2$$



$$x(t) = \frac{A}{2\bar{\omega}\delta} \left\{ [\cos(\bar{\omega}t)\cos(\delta t/2) + \sin(\bar{\omega}t)\sin(\delta t/2)] - [\cos(\bar{\omega}t)\cos(\delta t/2) + \sin(\bar{\omega}t)\sin(\delta t/2)] \right\}$$

$$x(t) = \frac{A}{\bar{\omega}\delta} \sin(\bar{\omega}t) \sin\left(\frac{\delta t}{2}\right) \equiv \frac{At}{2\bar{\omega}} \sin(\bar{\omega}t) \frac{\sin\left(\frac{\delta t}{2}\right)}{\frac{\delta t}{2}}$$

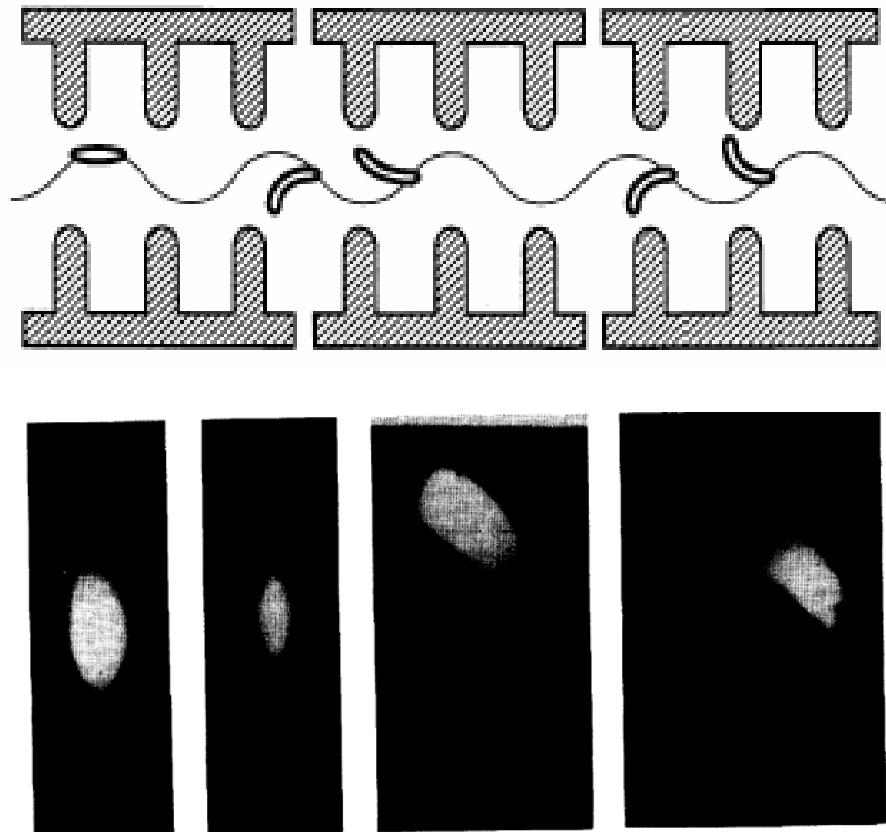
$$\lim_{\delta \rightarrow 0} x(t) = \frac{At}{2\bar{\omega}} \sin(\bar{\omega}t)$$





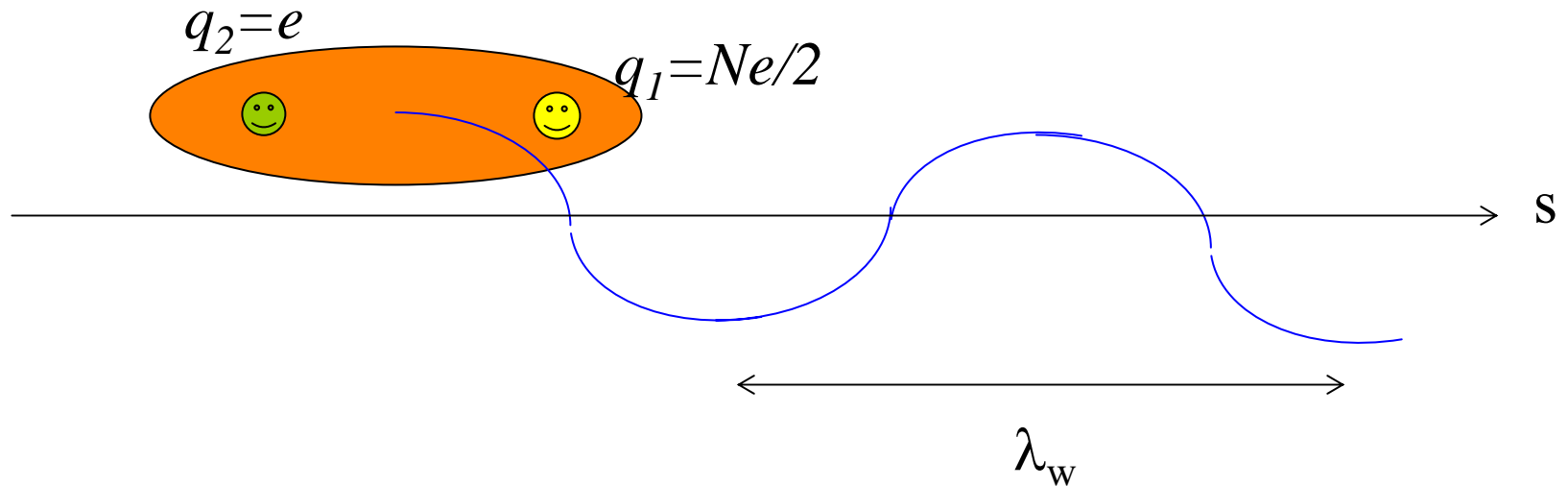
# Wake Fields Effects In Linear Accelerators: Beam Break Up

A beam injected off-center in a LINAC, because of the focusing quadrupoles, execute betatron oscillations. The displacement produces a transverse wake field in all the devices crossed during the flight, which deflects the trailing charges.



**Figure 3.4.** Four transverse beam profiles observed at the end of the SLAC linac are shown when the beam was carefully injected and injected with 0.2, 0.5, and 1 mm offsets. The beam sizes  $\sigma_x$  and  $\sigma_y$  are about 120  $\mu\text{m}$ . (Courtesy John Seeman, 1991.)

In order to understand the effect, we consider a simple model with only two charges  $q_1=Ne/2$  (leading = half bunch) and  $q_2=e$  (trailing = single charge).



the leading charge executes free betatron oscillations:

$$y_1(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right); \quad \frac{\omega_y}{c} = \frac{2\pi}{\lambda_w}$$

the trailing charge, at a distance  $z$  behind, over a length  $L_w$  experiences a deflecting force proportional to the displacement  $y_1$ , and dependent on the distance  $z$ :

$$\langle F_y^{self}(z, y_1) \rangle = \frac{Ne^2}{2L_w} w_{\perp}(z) y_1(s)$$

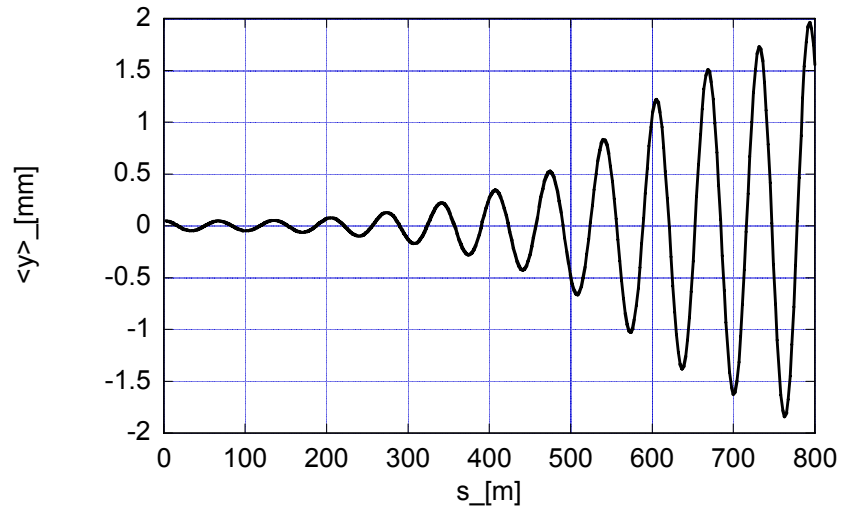
This force drives the motion of the trailing charge:

$$y_2'' + \left( \frac{\omega_y}{c} \right)^2 y_2 = \frac{Ne^2 w_{\perp}(z)}{2E_o L_w} \hat{y}_1 \cos\left( \frac{\omega_y}{c} s \right)$$

This is the typical equation of a resonator driven at the resonant frequency. The solution is given by the superposition of the “free” oscillation and a “forced” oscillation which, being driven at the resonant frequency, grows linearly with  $s$ .

$$y_2(s) = \hat{y}_2 \cos\left(\frac{\omega_y}{c} s\right) + y_2^{forced}$$

$$y_2^{forced} = \frac{cNe^2 w_{\perp}(z)s}{4\omega_y E_o L_w} \hat{y}_1 \sin\left(\frac{\omega_y}{c} s\right)$$

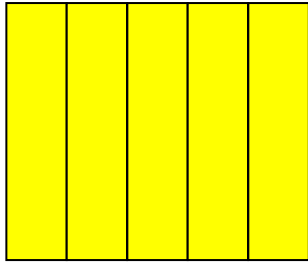


At the end of the LINAC of length  $L_L$ , the oscillation amplitude is grown by :

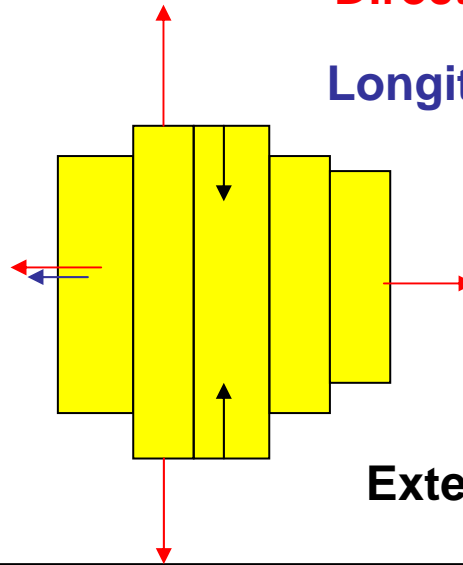
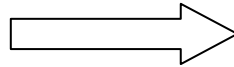
$$\left(\frac{\Delta \hat{y}_2}{\hat{y}_1}\right)_{max} = \frac{cNe w_{\perp}(z)L_L}{4\omega_y (E_o / e)L_w}$$

# The HOMDYN Model

On Axis



$\Delta t$

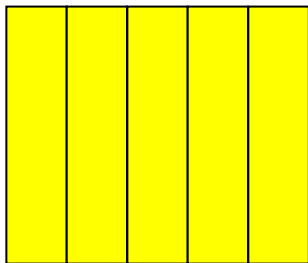


Direct Space Charge

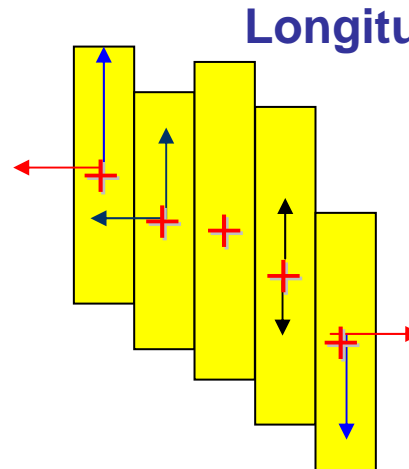
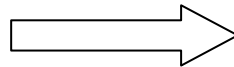
Longitudinal Wake Field

External Fields

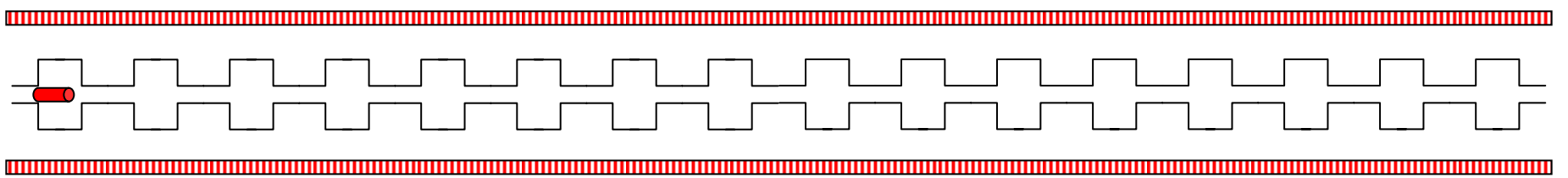
Off Axis



$\Delta t$



Longitudinal and Transverse  
Wake Field



QuickTime™ and a  
Animation decompressor  
are needed to see this picture.

## Balakin-Novokhatsky-Smirnov Damping

The BBU instability is quite harmful and hard to take under control even at high energy with a strong focusing, and after a careful injection and steering.

A simple method to cure it has been proposed observing that the strong oscillation amplitude of the bunch tail is mainly due to the **“resonant” driving**.

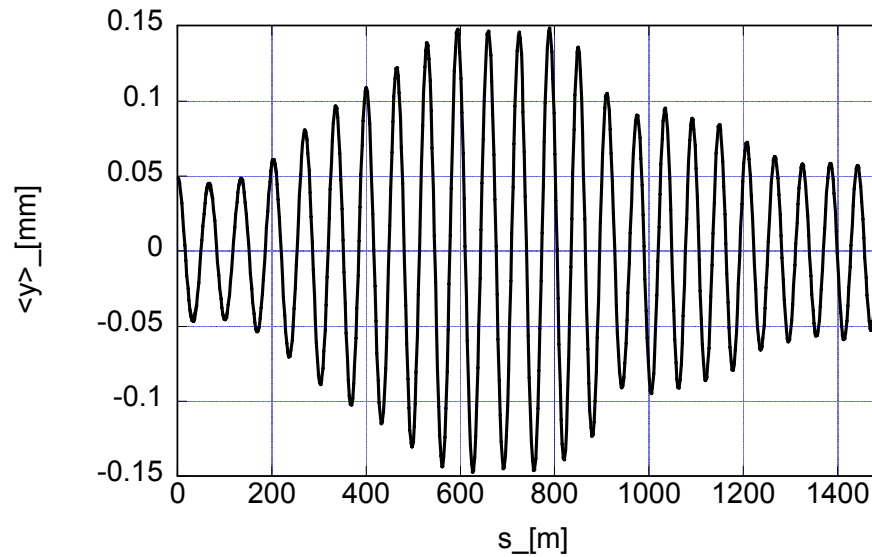
**If the tail and the head move with a different frequency, this effect can be significantly removed.**

Let us assume that the tail oscillate with a frequency  $\omega_y + \Delta\omega_y$ , the equation of motion reads:

$$y_2'' + \left( \frac{\omega_y + \Delta\omega_y}{c} \right)^2 y_2 = \frac{Ne^2 w_{\perp}(z)}{2E_o L_w} \hat{y}_1 \cos\left( \frac{\omega_y}{c} s \right)$$

the solution of which is:

$$y_2(s) = \frac{c^2 Ne^2 w_{\perp}(z)}{4\omega_y \Delta\omega_y E_o L_w} \hat{y}_I \left[ \cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right) - \cos\left(\frac{\omega_y}{c} s\right) \right]$$





by a suitable choice of  $\Delta\omega_y$ , it is possible to fully depress the oscillations of the tail.

$$\hat{y}_2 \cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right) + y_2(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right)$$

$$\Delta\omega_y = \frac{c^2 N e^2 w_{\perp}(z)}{4\omega_y E_o L_w}$$

The extra focusing at the tail can be obtained by:

- Using a RFQ, where head and tail see a different focusing strength,
- Exploit the **energy spread** across the bunch which, because of the chromaticity, induces a spread in the betatron frequency. An energy spread correlated with the position is attainable with the external accelerating voltage, or with the wake fields.

# **BNS HOMDYN Movie**

QuickTime™ and a  
Animation decompressor  
are needed to see this picture.

**THE END**

## References

**A. W. Chao - *Physics of collective beam instabilities in high energy accelerators* - Wiley, NY 1993**

**A. Mosnier - *Instabilities il Linacs* - CAS (Advanced) - 1994**

**L. Palumbo, V. Vaccaro, M. Zobov- *Wakes fields and Impedance* - CAS (Advanced) - 1994**

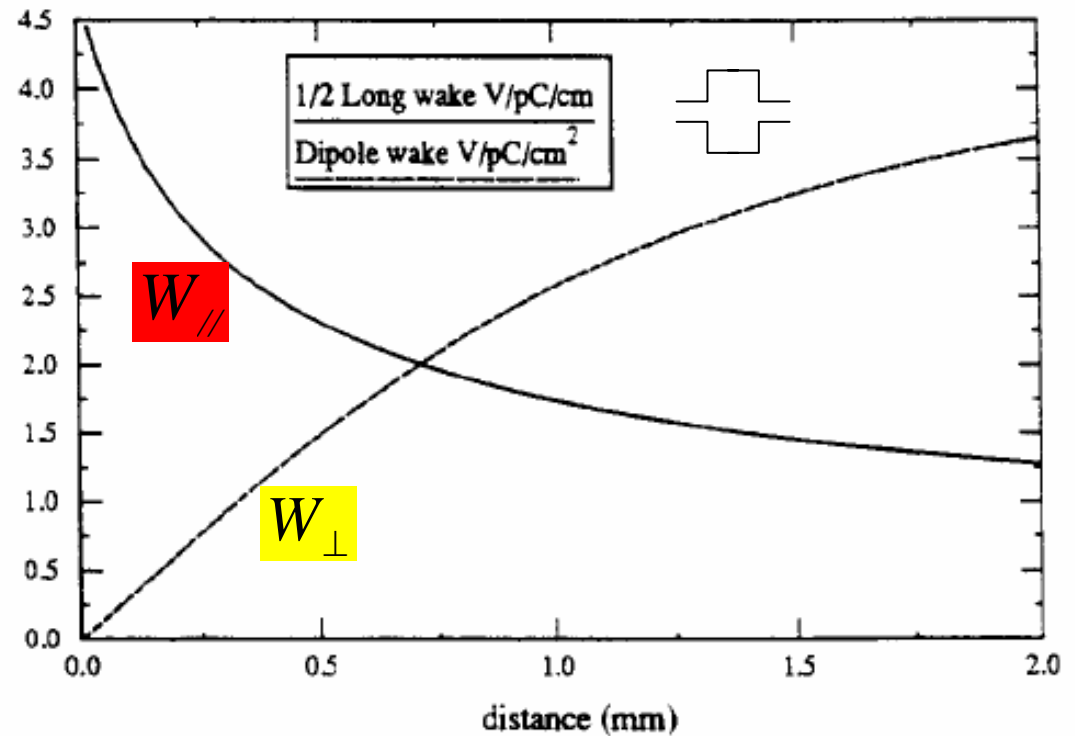
**G. V. Stupakov - *Wake and Impedance* - SLAC-PUB-8683**

**K.L.F. Bane, A. Mosnier, A. Novokhatsky, K. Yokoya-*Calculations of the Short Range Longitudinal Wakefields in the NLC Linac* - EPAC'98**

Relationship between transverse and longitudinal forces :  
**“Panofsky-Wenzel theorem”**.

$$\nabla_{\perp} F_{\parallel} = \frac{\partial}{\partial z} \mathbf{F}_{\perp}$$

$$\nabla_{\perp} w_{\parallel} = \frac{\partial}{\partial z} \mathbf{w}_{\perp}$$



## Coupling Impedance

The wake potentials are used for to study the beam dynamics in the time domain ( $s=vt$ ). If we take the equation of motion in the frequency domain, we need the Fourier transform of the wake potentials. Since these quantities have Ohms units are called *coupling impedances*:

*Longitudinal impedance ( $\Omega$ )*

$$Z_{//}(\omega) = \frac{1}{v} \int_{-\infty}^{\infty} w_{//}(z) e^{-i\frac{\omega z}{v}} dz$$

*Transverse impedance ( $\Omega/m$ )*

$$Z_{\perp}(\omega) = \frac{i}{v} \int_{-\infty}^{\infty} w_{\perp}(z) e^{-i\frac{\omega z}{v}} dz$$

## Coupling impedances of a HOM resonant mode

*Longitudinal Impedance:*

$$Z_{//}(\omega) = \frac{R_s}{1 - iQ_r \left( \frac{\omega_r}{\omega} - \frac{\omega}{\omega_r} \right)}$$

shunt impedance,  $R_s = \frac{w_0}{2\gamma}$

from the computer codes.

the quality factor,  $Q_r = \frac{\omega_r}{2\gamma}$

*Transverse Impedance:*

$$Z_{\perp}(\omega) = \frac{c}{\omega} \frac{R_{s\perp}}{1 - iQ_r \left( \frac{\omega_r}{\omega} - \frac{\omega}{\omega_r} \right)}$$