Analysis techniques

(applied to non-linear dynamics)

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ΑΡΙΣΤΟΤΕΛΕΙΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΘΕΣΣΑΛΟΝΙΚΗΣ

Summary

- Phase space dynamics – fixed point analysis
- Poincaré map
- Motion close to a resonance
- Onset of chaos
- Chaos detection methods

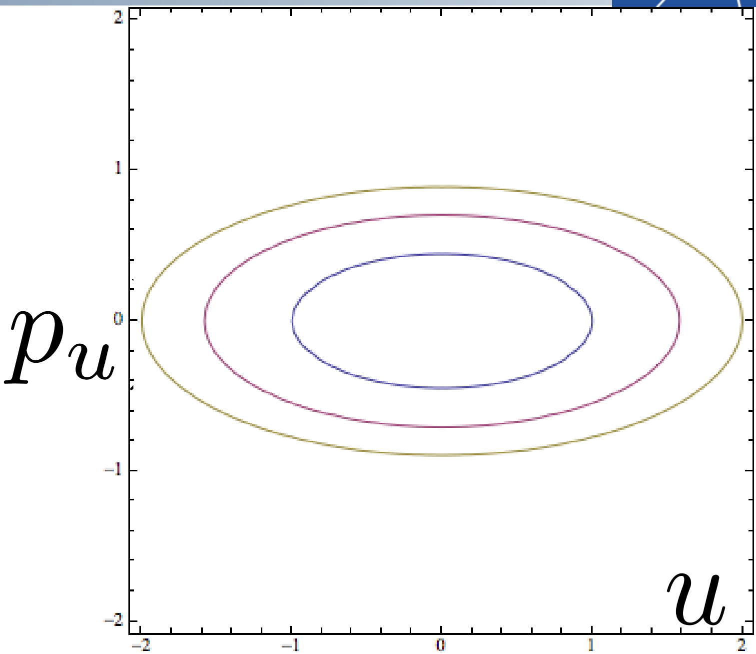
Phase space dynamics - Fixed point analysis

Phase space dynamics

- Valuable description when examining trajectories in **phase space** (u, p_u)
- Existence of integral of motion imposes geometrical constraints on phase flow
- For the simple **harmonic oscillator**

$$H = \frac{1}{2}\left(p_u^2 + \omega_0^2 u^2\right)$$

phase space curves are **ellipses** around the equilibrium point parameterized by the Hamiltonian (energy)



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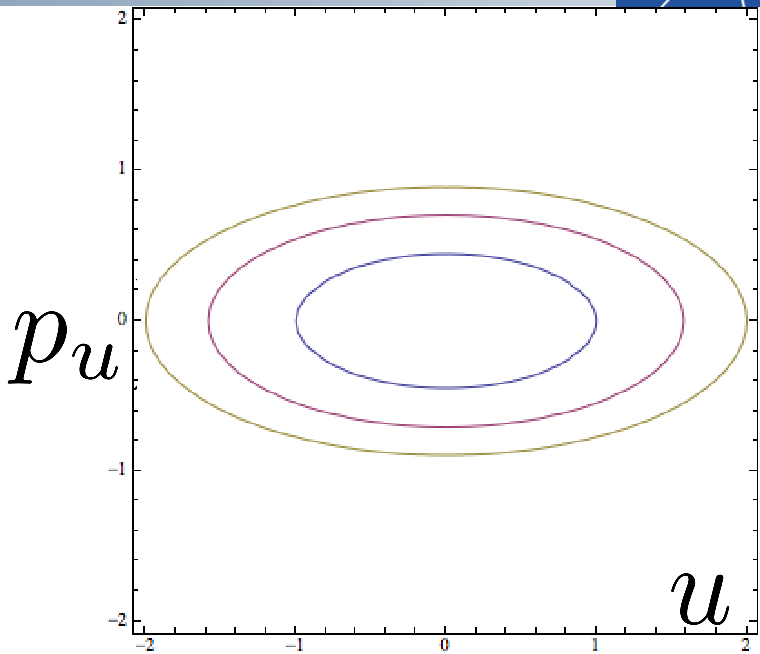
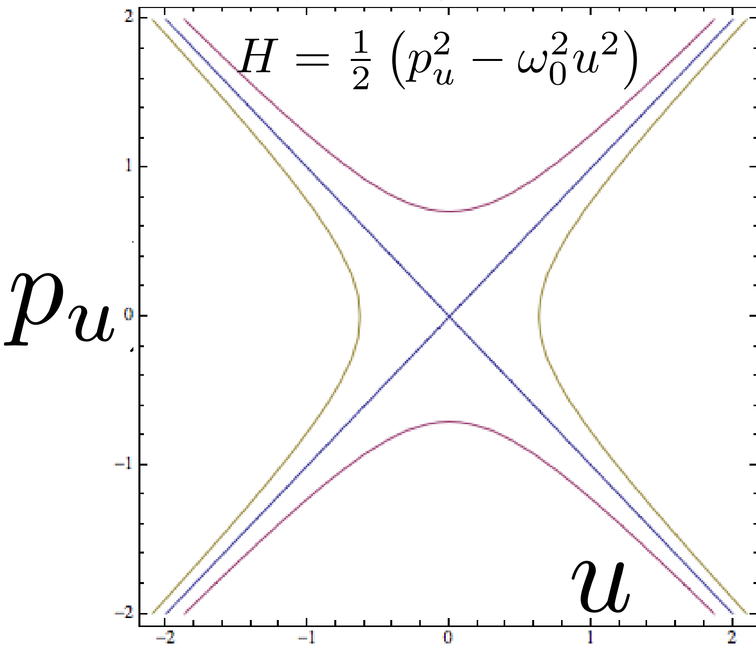
- By simply **changing** the **sign** of the potential in the harmonic oscillator, the phase trajectories become **hyperbolas**, symmetric around the equilibrium point where two straight lines cross, moving towards and away from it





Non-linear oscillators

- Conservative non-linear oscillators have Hamiltonian

$$H = E = \frac{1}{2}p_u^2 + V(u)$$

  with the potential being a general (polynomial) function of positions
- **Equilibrium points** are associated with extrema of the potential

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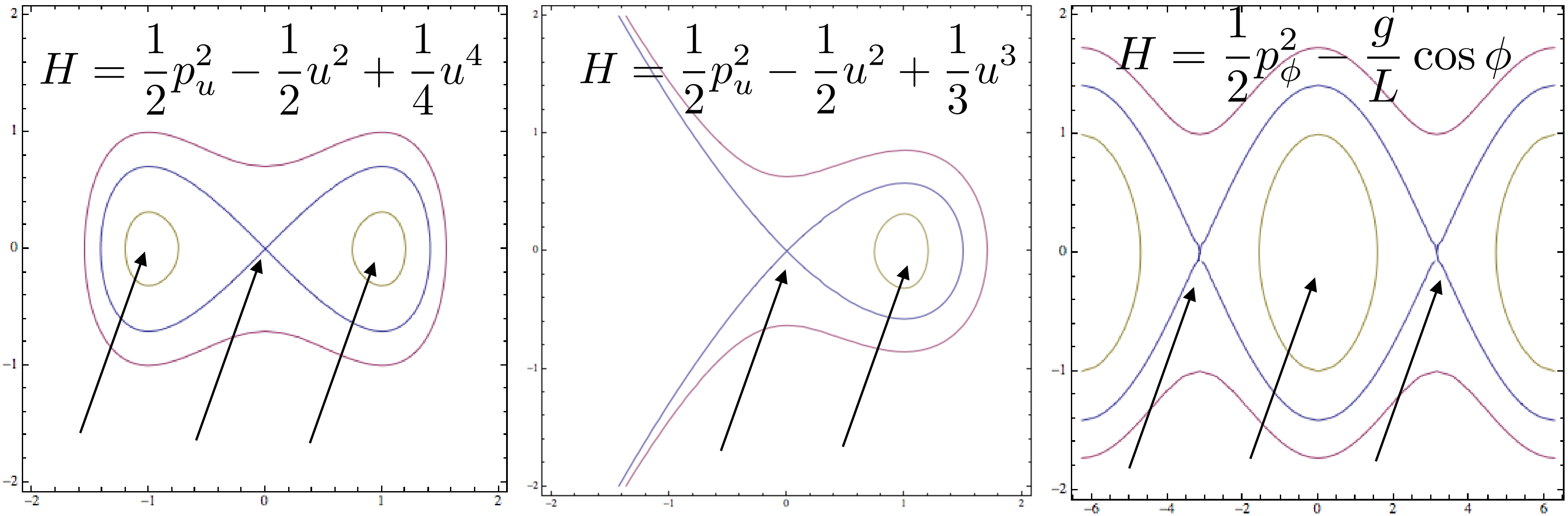

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- **Equilibrium points** are associated with extrema of the potential
- Considering three non-linear oscillators
  - **Quartic** potential (left): two minima and one maximum
  - **Cubic** potential (center): one minimum and one maximum
  - **Pendulum** (right): periodic minima and maxima

- Consider a general second order system
$$\begin{aligned}\frac{du}{dt} &= f_1(u,p_u)\\ \frac{dp_u}{dt} &= f_2(u,p_u)\end{aligned}$$
- Equilibrium or "**fixed**" points $f_1(u_0,p_{u0}) = f_2(u_0,p_{u0}) = 0$ are determinant for topology of trajectories at their vicinity

Fixed point analysis

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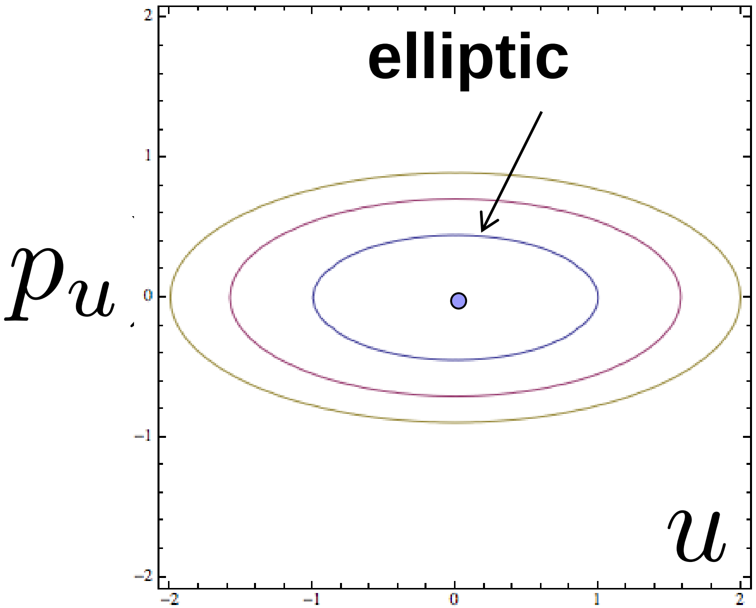
- Equilibrium or "**fixed**" points $f_1(u_0, p_{u0}) = f_2(u_0, p_{u0}) = 0$ are determinant for topology of trajectories at their vicinity
- The **linearized equations** of motion at their vicinity are

$$\frac{d}{dt}\begin{bmatrix}\delta u\\ \delta p_u\end{bmatrix} = \mathcal{M}_J\begin{bmatrix}\delta u\\ \delta p_u\end{bmatrix} = \underbrace{\begin{bmatrix}\frac{\partial f_1(u_0,p_{u0})}{\partial u} & \frac{\partial f_1(u_0,p_{u0})}{\partial p_u}\\ \frac{\partial f_2(u_0,p_{u0})}{\partial u} & \frac{\partial f_2(u_0,p_{u0})}{\partial p_u}\end{bmatrix}}_{\textbf{Jacobian}\text{ matrix}}\begin{bmatrix}\delta u\\ \delta p_u\end{bmatrix}$$

- Fixed point nature is revealed by **eigenvalues** of $\mathcal{M}_J$, i.e. solutions of the characteristic polynomial $\det|\mathcal{M}_J - \lambda\mathbf{I}| = 0$

Fixed point for conservative systems

- For **conservative systems** of 1 degree of freedom, the second order characteristic polynomial for any fixed point has two possible solutions:
  - Two **complex eigenvalues** with opposite sign, corresponding to **elliptic** fixed points. Phase space flow is described by **ellipses**, with particles evolving clockwise or anti-clockwise



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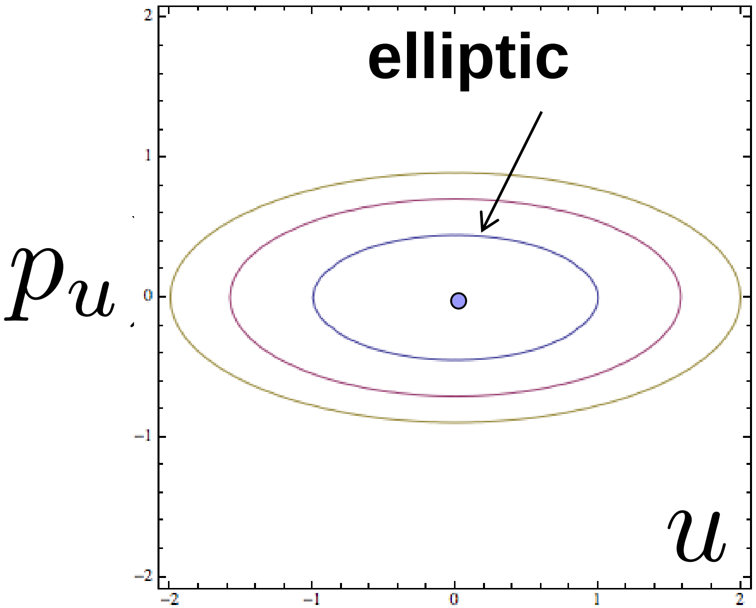
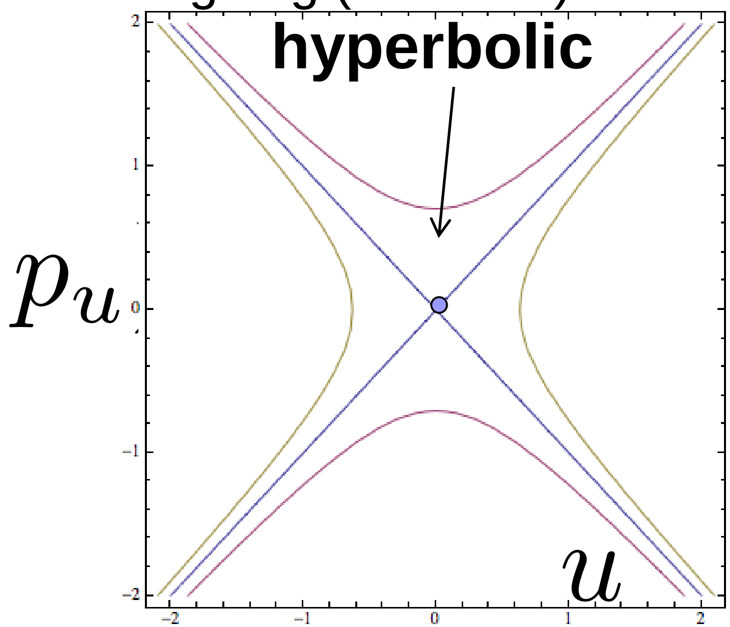
  - Two **real eigenvalues** with opposite sign, corresponding to **hyperbolic** (or saddle) fixed points. Flow described by two lines (or manifolds), incoming (stable) and outgoing (unstable)





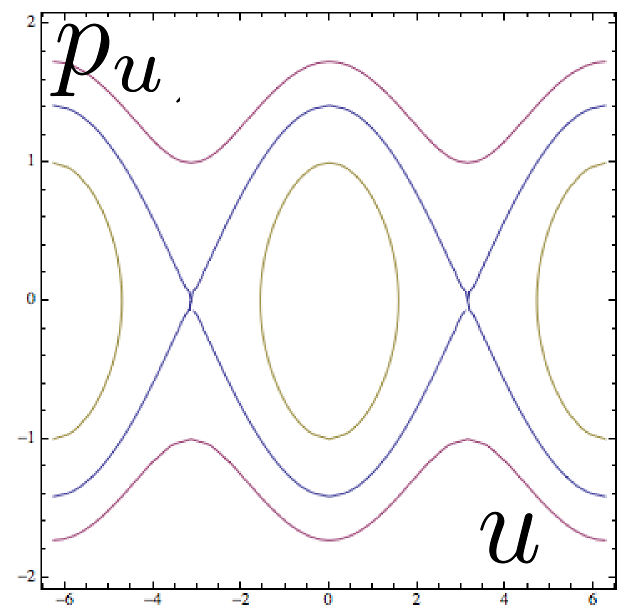
Pendulum fixed point analysis

- The "fixed" points for a pendulum can be found at

$$(\phi_n, p_\phi) = (\pm n\pi, 0)\ ,\ n = 0, 1, 2\ldots$$

- The Jacobian matrix is $\begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos\phi_n & 0 \end{bmatrix}$

- The eigenvalues are $\lambda_{1,2} = \pm i\sqrt{\frac{g}{L}\cos\phi_n}$



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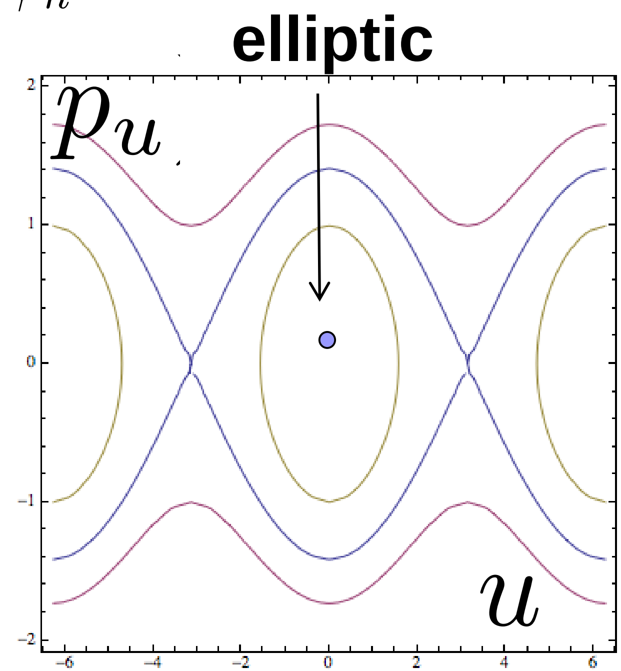

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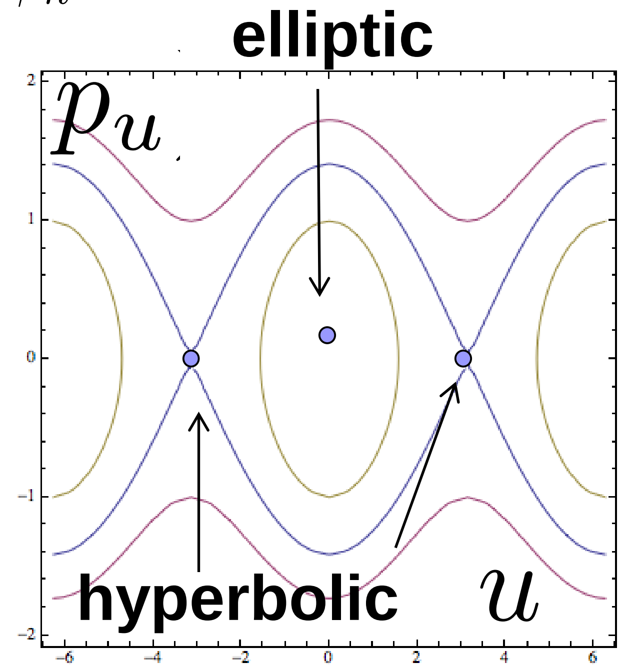
  - $\phi_n = (2n+1)\pi$, for which $\lambda_{1,2} = \pm\sqrt{\frac{g}{L}}$ corresponding to **hyperbolic** fixed points
  - The **separatrix** are the stable and unstable manifolds through the hyperbolic points, separating bounded **librations** and unbounded **rotations**



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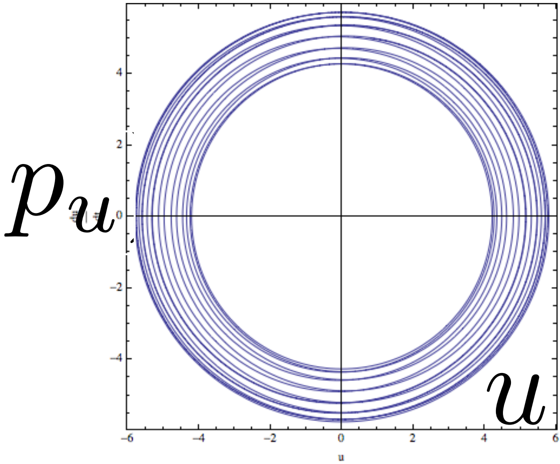
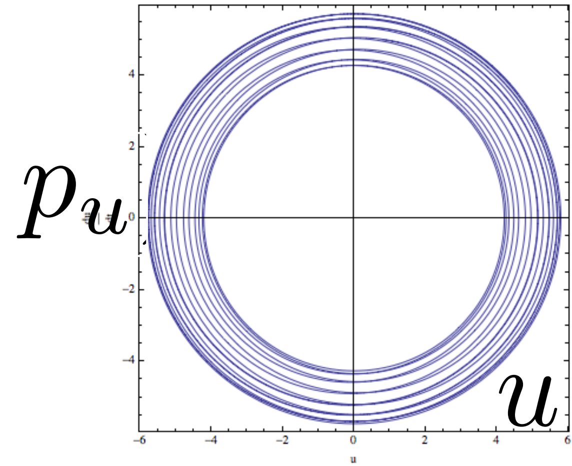


- Plotting the evolution in phase space, provides trajectories that **intersect** each other
- The phase space has **time** as **extra dimension**

Phase space for time-dependent systems



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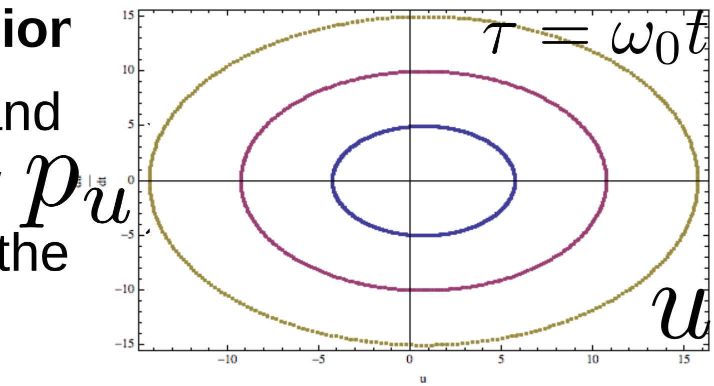

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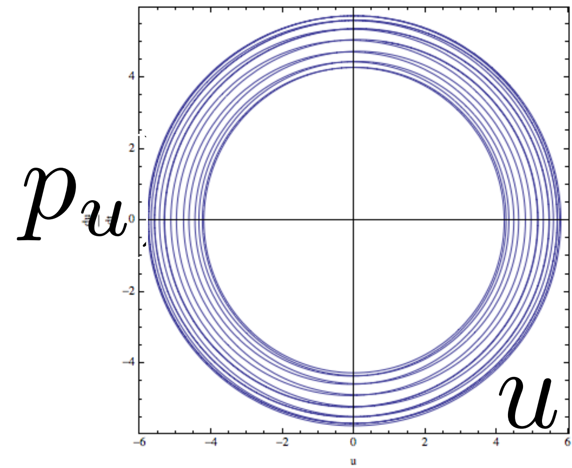
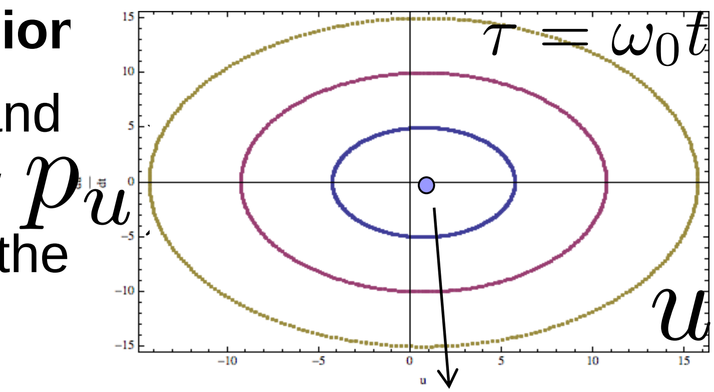
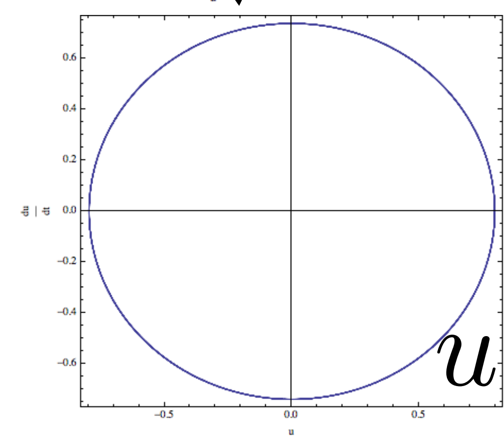
- This is the simplest version of a **Poincaré surface of section**, which is useful for studying geometrically phase space of multi-dimensional systems
- The **fixed point** in the surface of section is now a periodic orbit

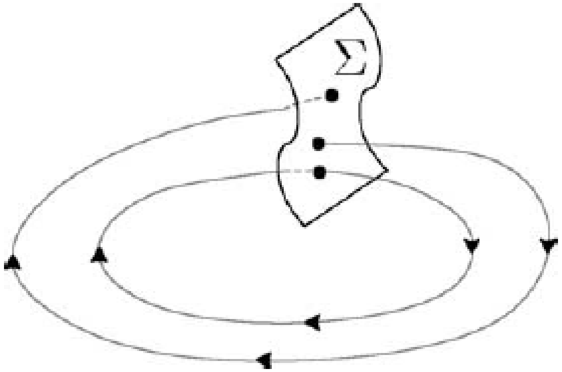






Poincaré map

- **First recurrence** or **Poincaré map** (or surface of section) is defined by the intersection of trajectories of a dynamical system, with a fixed surface in phase space



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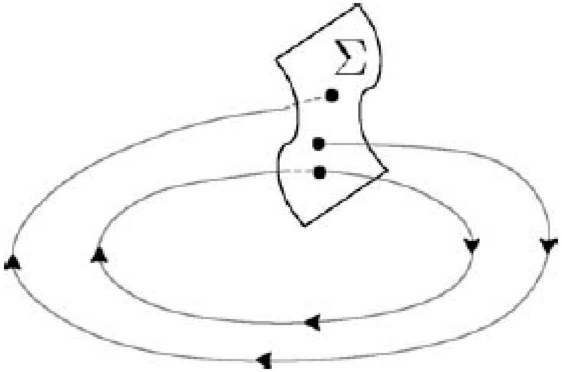

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- For an **autonomous** Hamiltonian system $H(\mathbf{q},\mathbf{p})$ (no **explicit** time dependence), it can be chosen to be any fixed surface in phase space, e.g. $q_i = 0$
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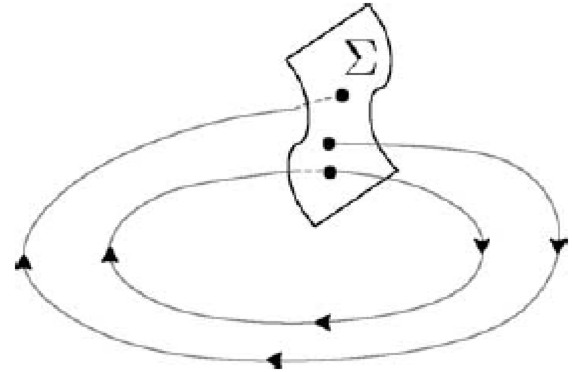

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- For a **non-autonomous** Hamiltonian system $H(\mathbf{q},\mathbf{p},t)$ (explicit time dependence), which is **periodic,** it can be chosen as the period $t = T_0$
- In a system with n degrees of freedom (or $n+1$ including time), the phase space has $2n$ (or $2n+2$) dimensions
- By fixing the value of the Hamiltonian to H_0, the motion on a Poincaré map is reduced to $2n-2$ (or $2n$)

Poincaré map

- Particularly useful for a system with **2 degrees** of **freedom**, or **1 degree** of **freedom + time**, as the motion on Poincaré map is described by 2-dimensional curves
- For continuous system, numerical techniques exist to produce the Poincaré map exactly (e.g. M.Henon Physica D 5, 1982)

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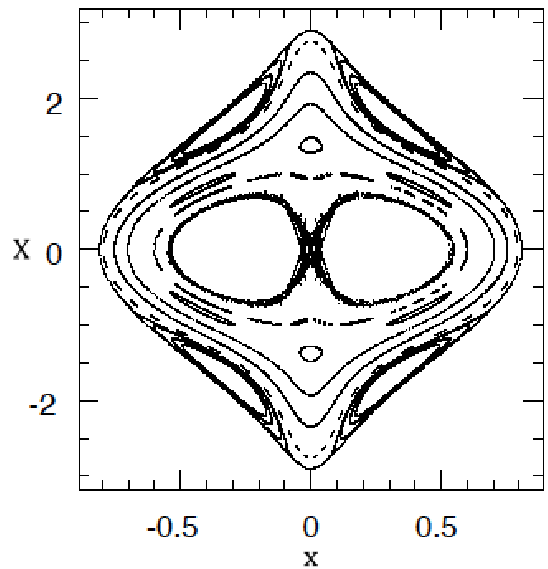
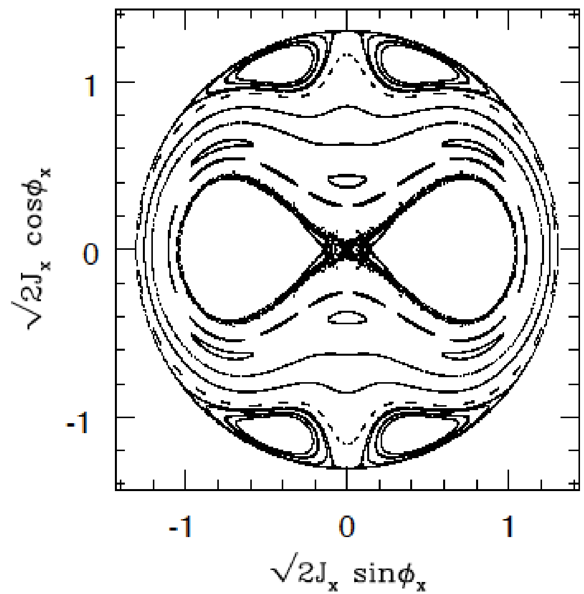
- Example from Astronomy: the logarithmic galactic potential

$$H_q(x,y,X,Y) = \frac{1}{2}(X^2+Y^2) + \ln\left(x^2 + \frac{y}{q^2} + R_c^2\right)$$

$$(x,y,X,Y) \mapsto (\phi_x,\phi_y,J_x,J_y)$$

$$y = 0$$



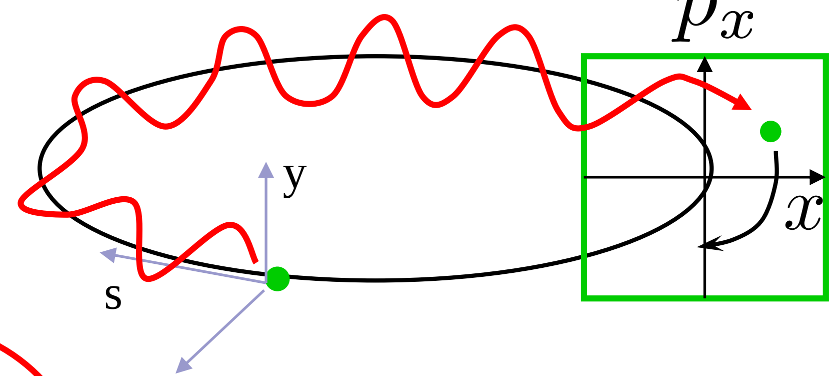


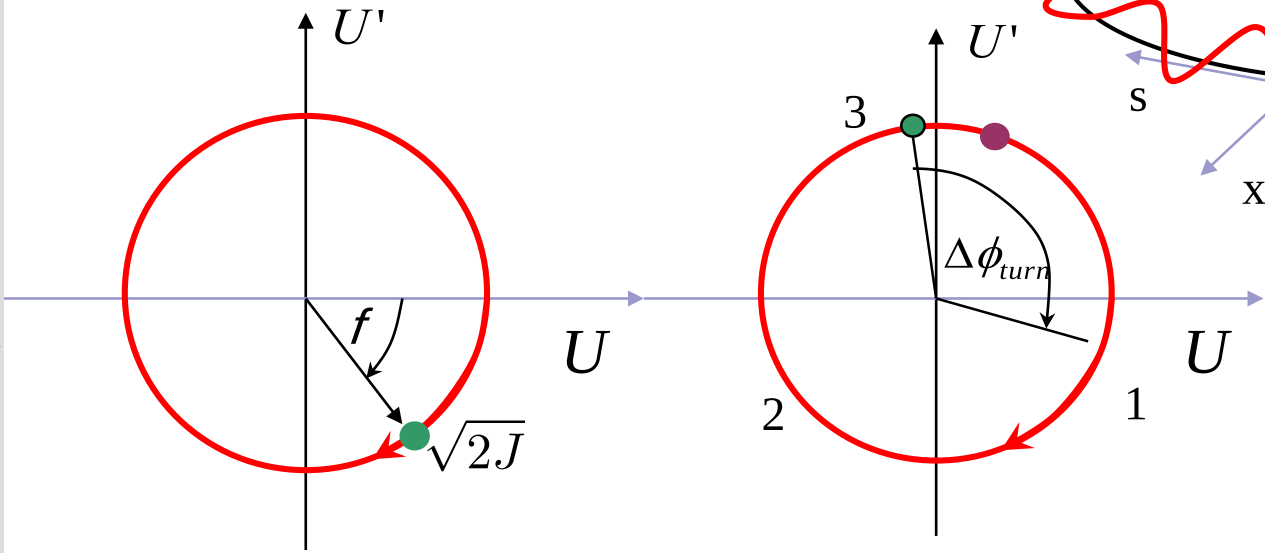
$$\phi_y = 0$$

Poincaré Section for a ring

- Record the particle coordinates at one location in a ring
- Unperturbed motion lies on a circle in normalized coordinates (simple rotation)

Poincaré Section:





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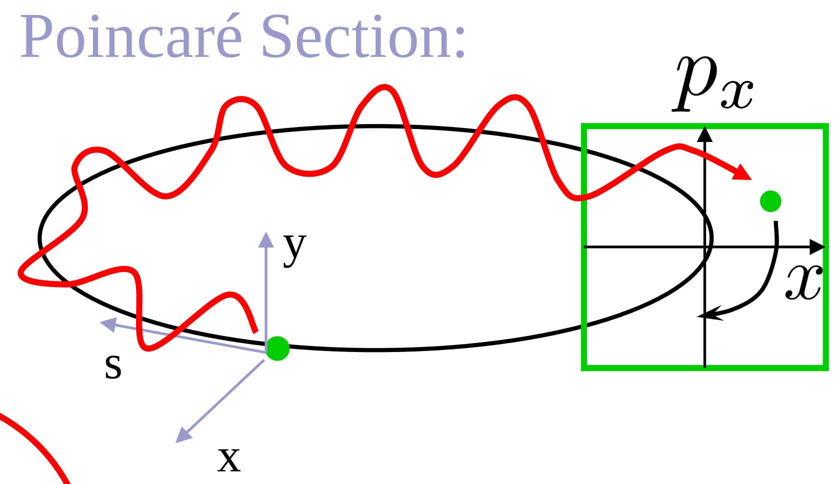
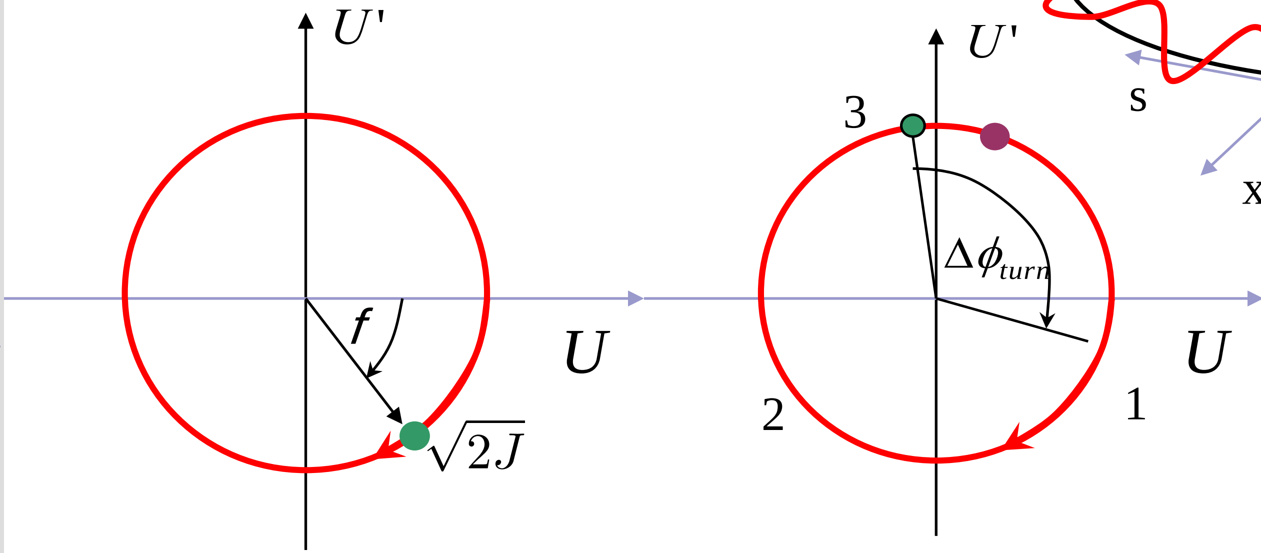
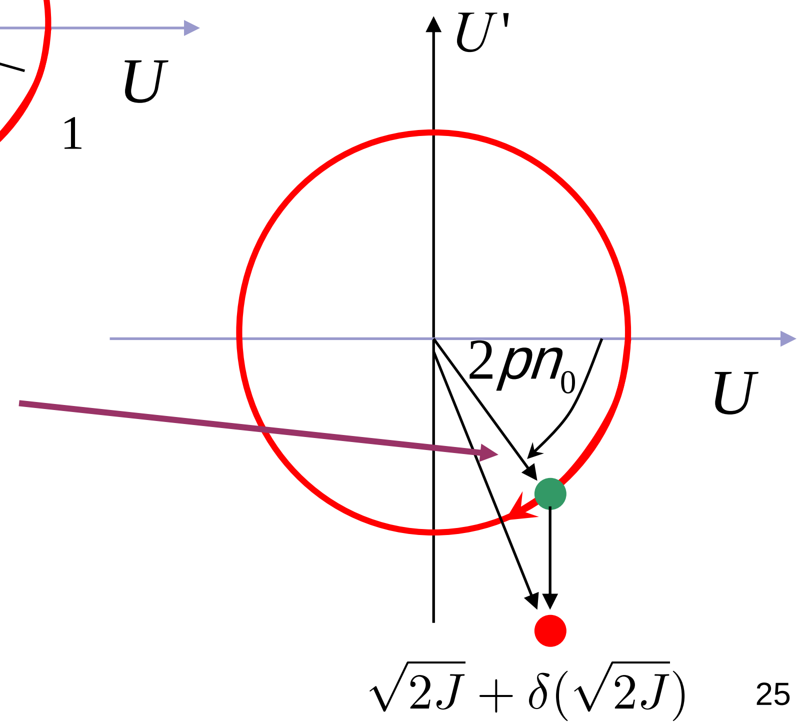




- Resonance condition corresponds to a periodic orbit or fixed points in phase space
- For a non-linear kick, the radius will change by $\delta(\sqrt{2J})$ and the particles stop lying on circles



Example: Single Octupole

- Simple **map** with single octupole kick with integrated strength k_3 + rotation with phase advances (μ_x, μ_y)

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  x1  = x
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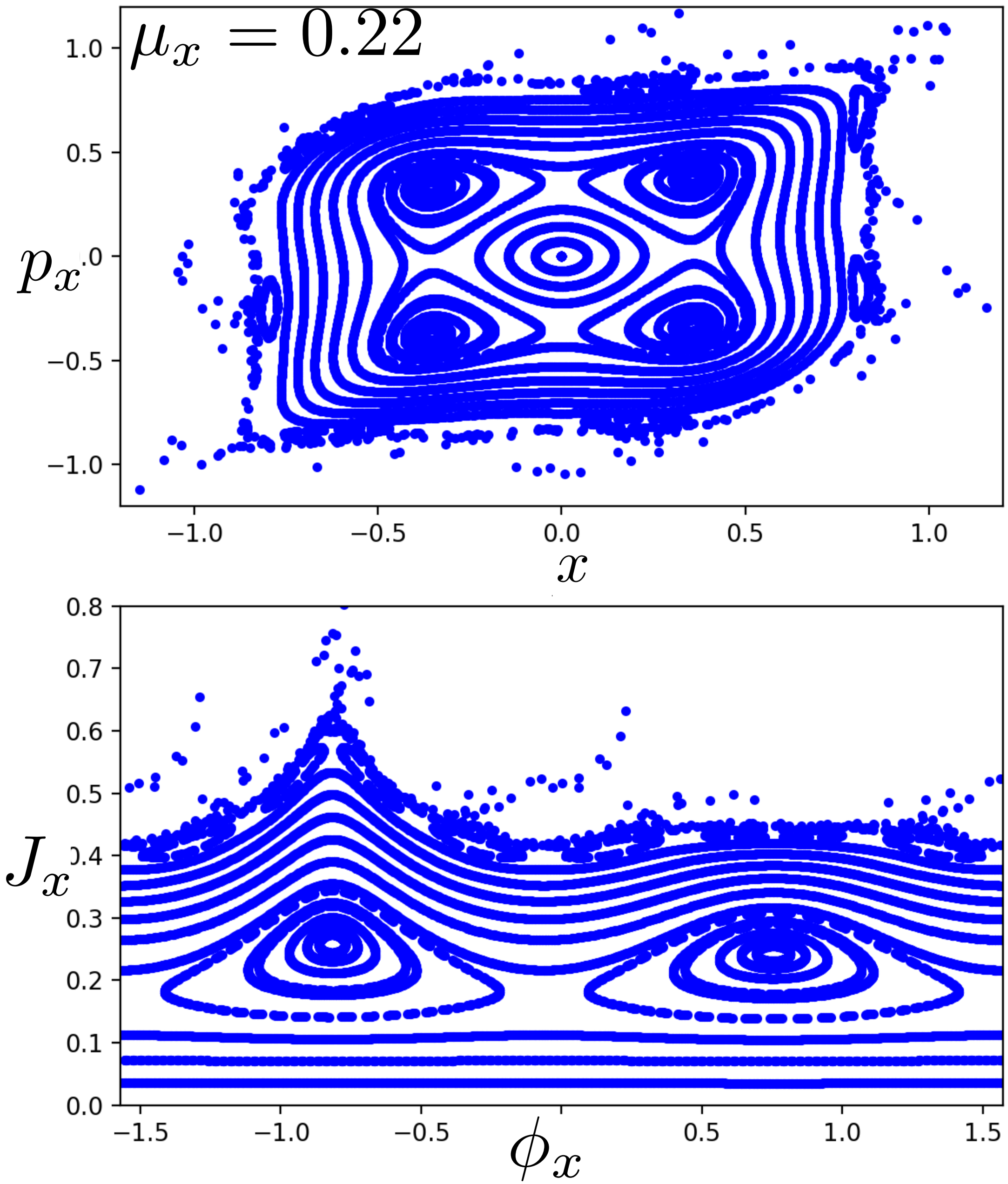

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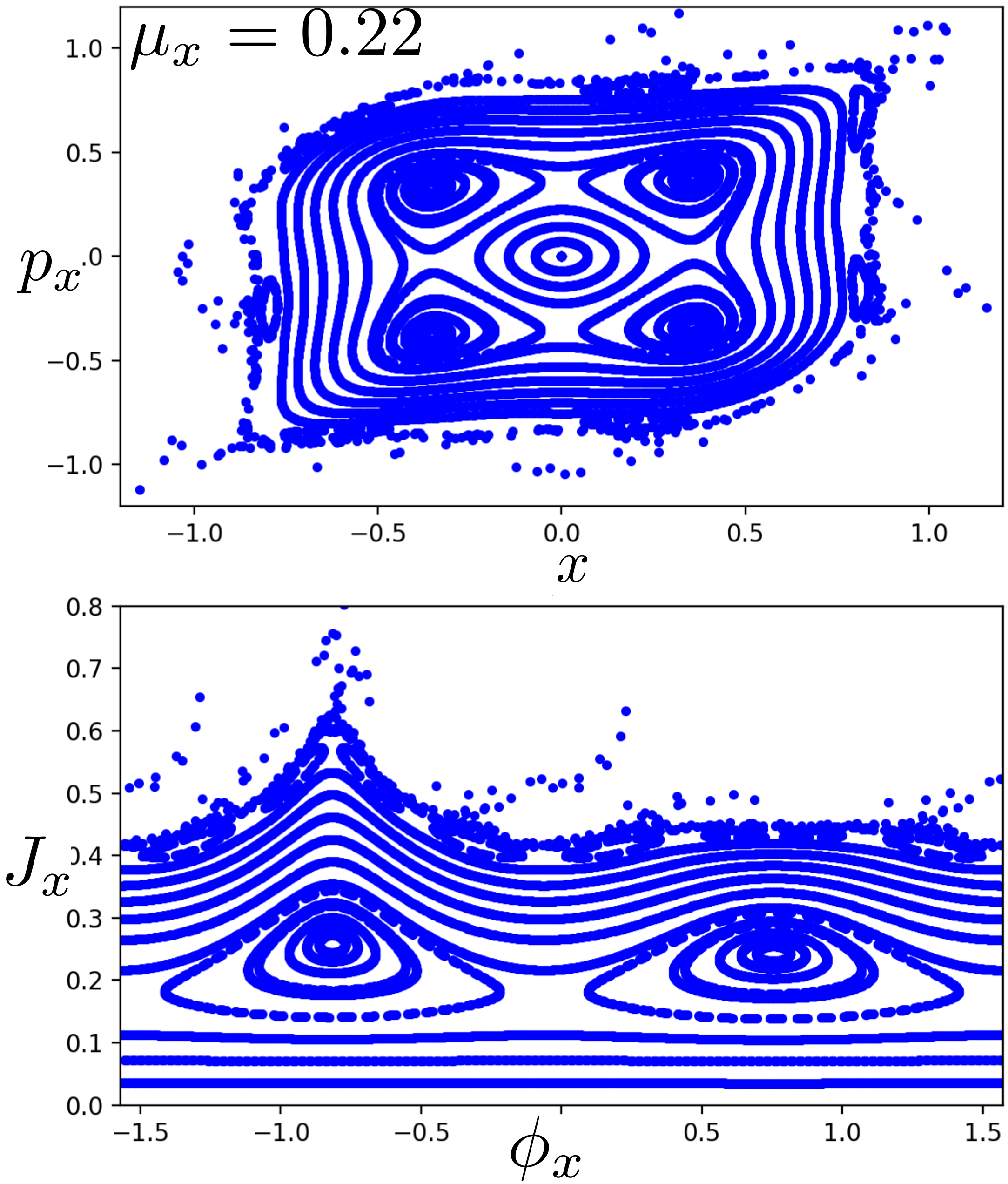
- **Appearance of invariant curves** ("distorted" circles), where "action" is an integral of motion
- **Resonant islands** with stable and separatrices with unstable fixed points
- **Chaotic motion**
- Electromagnetic fields coming from multi-pole expansions (polynomials) do not bound phase space and chaotic trajectories may eventually escape to infinity (**Dynamic Aperture**)
- For some fields like beam-beam and space-charge this is not true, i.e. chaotic motion leads to halo formation



Motion close to a resonance

Secular perturbation theory

- The vicinity of a resonance $n_1\omega_1 + n_2\omega_2 = 0$, can be studied through **secular perturbation theory** (see appendix)
- A canonical transformation is applied such that the new variables are in a frame remaining **on top** of the **resonance**
- If one frequency is slow, one can average the motion and remain only with a **1 degree of freedom Hamiltonian** which looks like the one of the **pendulum**
- Thereby, one can find the location and nature of the fixed points measure the width of the resonance

Fixed points for general multi-pole

- For **any polynomial perturbation** of the form x^k the "resonant" Hamiltonian is written as

$$\hat{H}_2 = \delta J_2 + \alpha(J_2) + J_2^{k/2} A_{kp} \cos(k\psi_2)$$

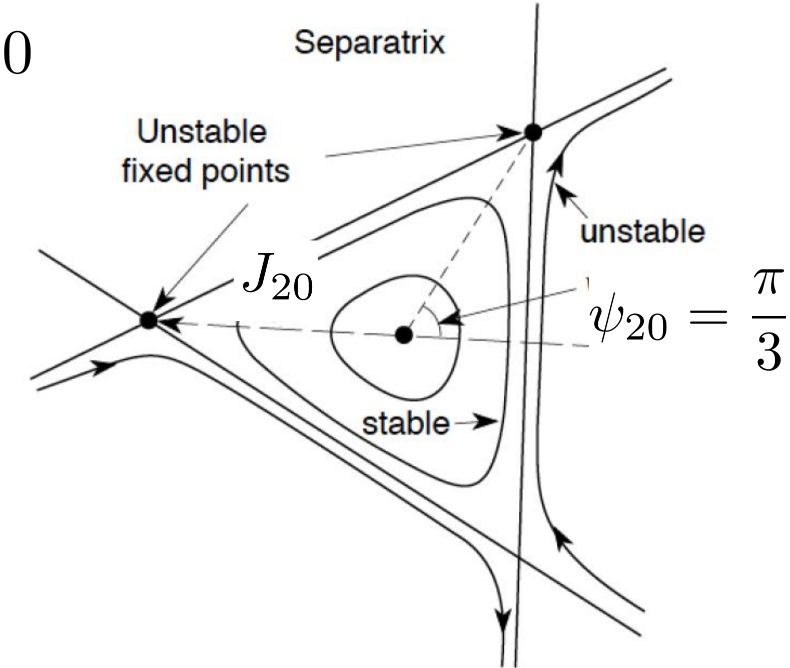
- With the **distance** to the resonance defined as $\nu = \frac{p}{3} + \delta \ , \quad \delta << 1$
- The non-linear shift of the tune is described by the term $\alpha(J_2)$
- The conditions for the fixed points are

$$\sin(k\psi_2) = 0 \ , \quad \delta + \frac{\partial\alpha(J_2)}{\partial J_2} + \frac{k}{2} J_2^{k/2-1} A_{kp} \cos(k\psi_2) = 0$$

- There are fixed points for which $\cos(k\psi_{20}) = -1$ and the fixed points are **stable** (elliptic). They are surrounded by ellipses
- There are also fixed points for which $\cos(k\psi_{20}) = 1$ and the fixed points are **unstable** (hyperbolic). The trajectories are hyperbolas

Fixed points for 3rd order resonance

- The Hamiltonian for a sextupole close to a third order resonance is $\hat{H}_2 = \delta J_2 + J_2^{3/2} A_{3p} \cos(3\psi_2)$
- Note the absence of the non-linear tune-shift term (in this 1st order approximation!)
- By setting the Hamilton's equations equal to zero, three fixed points can be found at $\psi_{20} = \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \quad J_{20} = \left(\frac{2\delta}{3A_{3p}}\right)^2$
- For $\frac{\delta}{A_{3p}} > 0$ all three points are unstable
- Close to the elliptic one at $\psi_{20} = 0$ the motion in phase space is described by circles that they get more and more distorted to end up in the "triangular" **separatrix** uniting the unstable fixed points
- The tune separation from the resonance is $\delta = \frac{3A_{3p}}{2} J_{20}^{1/2}$



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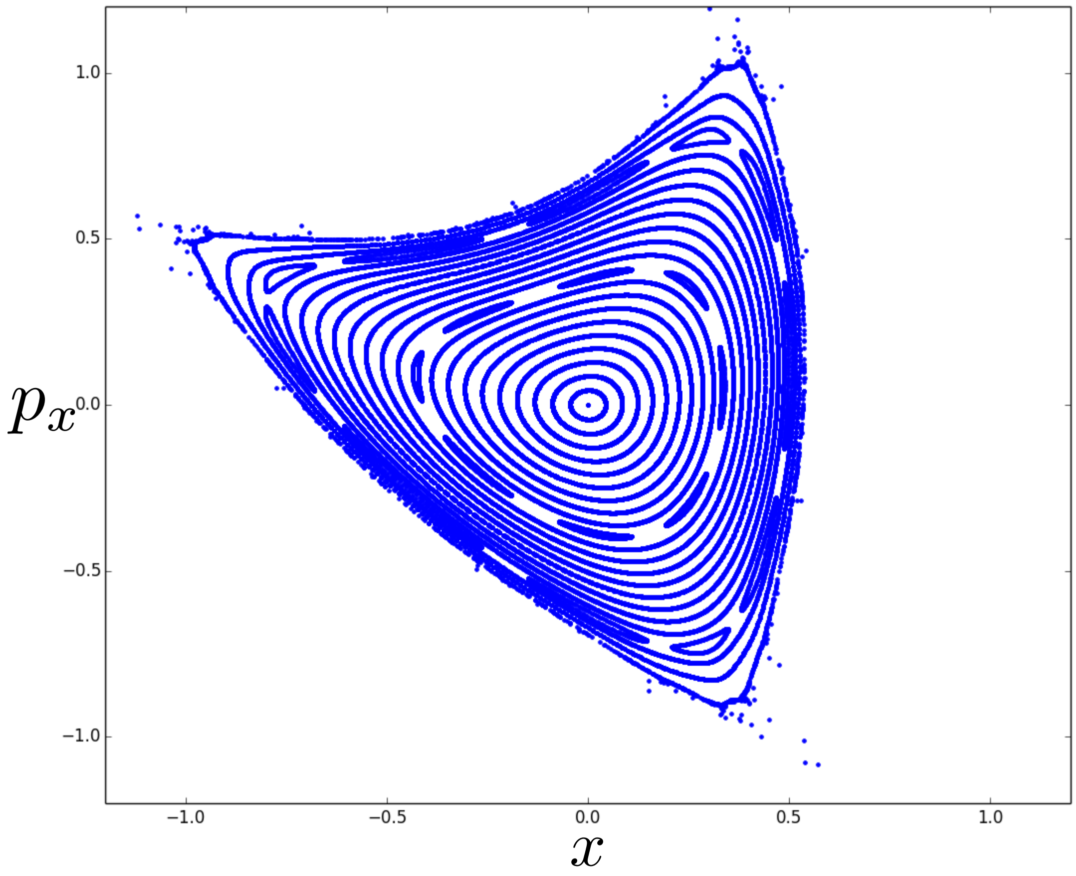
  return x1,px1,y1,py1
```

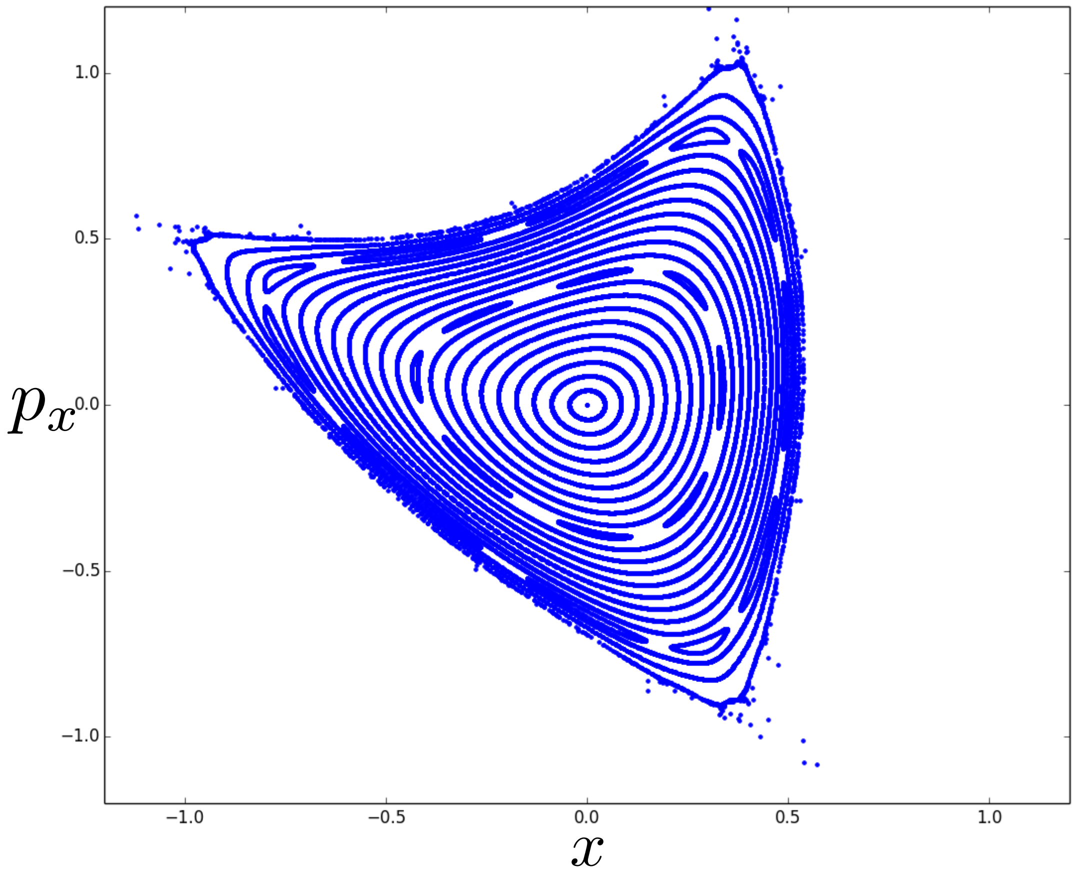
- Restrict motion in (x, p_x) plane i.e. $y_0 = p_{y0} = 0$
- Iterate for a number of "turns" (here 1000)

$$\mu_x = 0.38$$



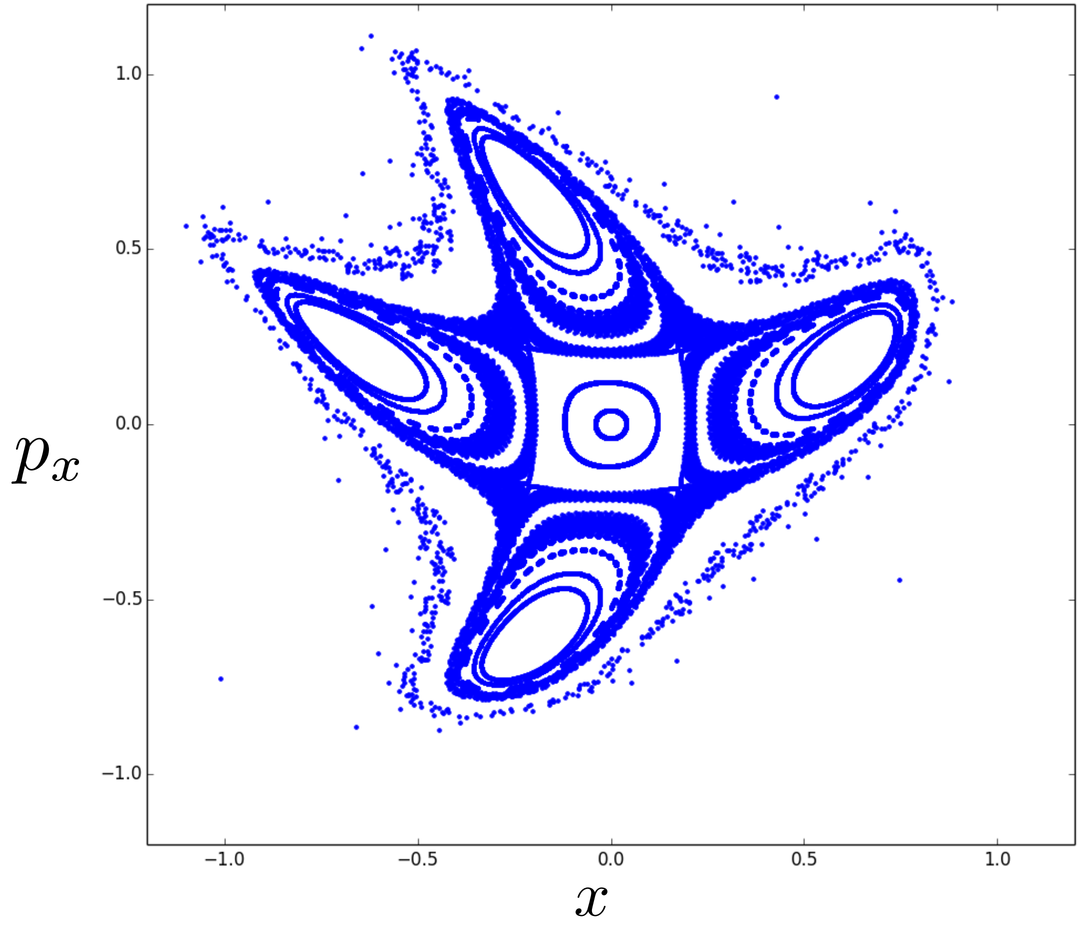
- Appearance of 3rd order resonance for certain phase advance
- … but also 4th order resonance

$$\mu_x = 0.38$$



- Appearance of 3rd order resonance for certain phase advance
- … but also 4th order resonance

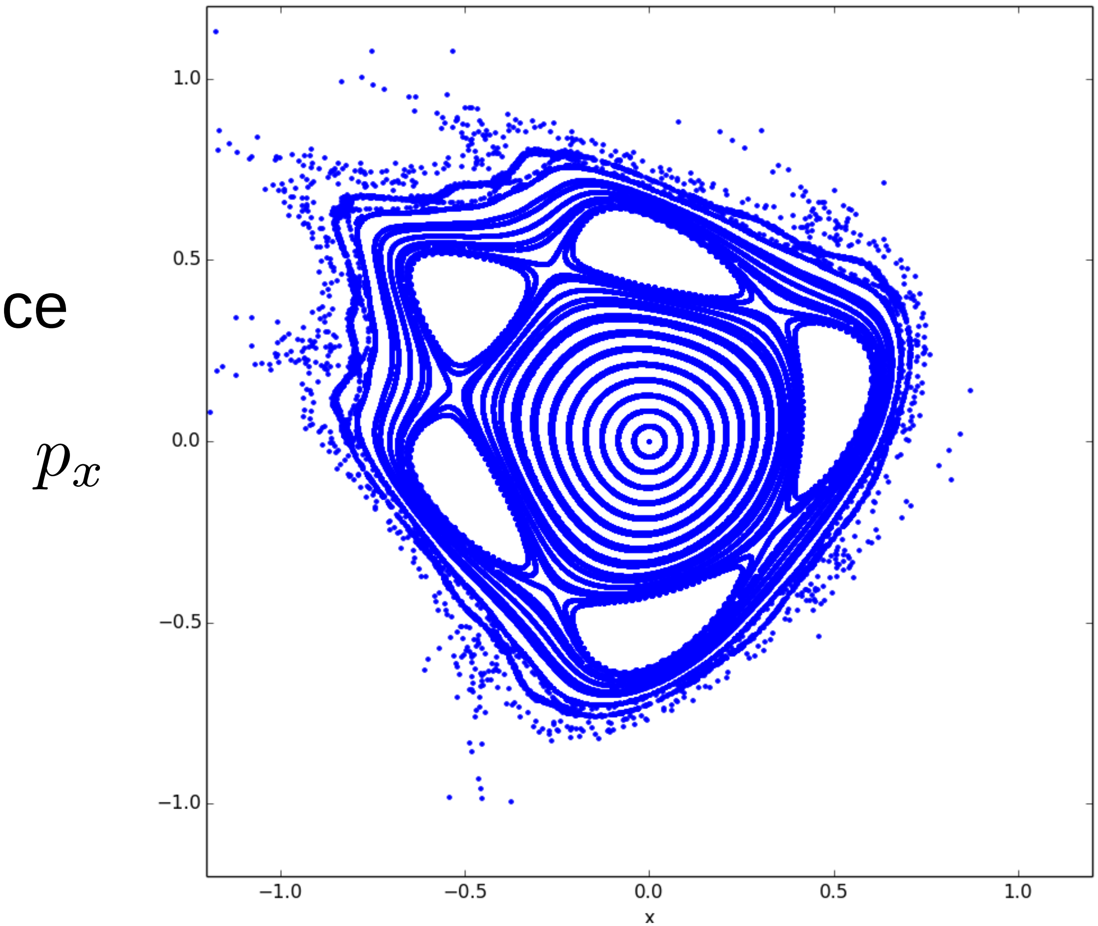
$$\mu_x = 0.253$$



Example: Single Sextupole

- Appearance of 3rd order resonance for certain phase advance
- … but also 4th order resonance
- … and 5th order resonance

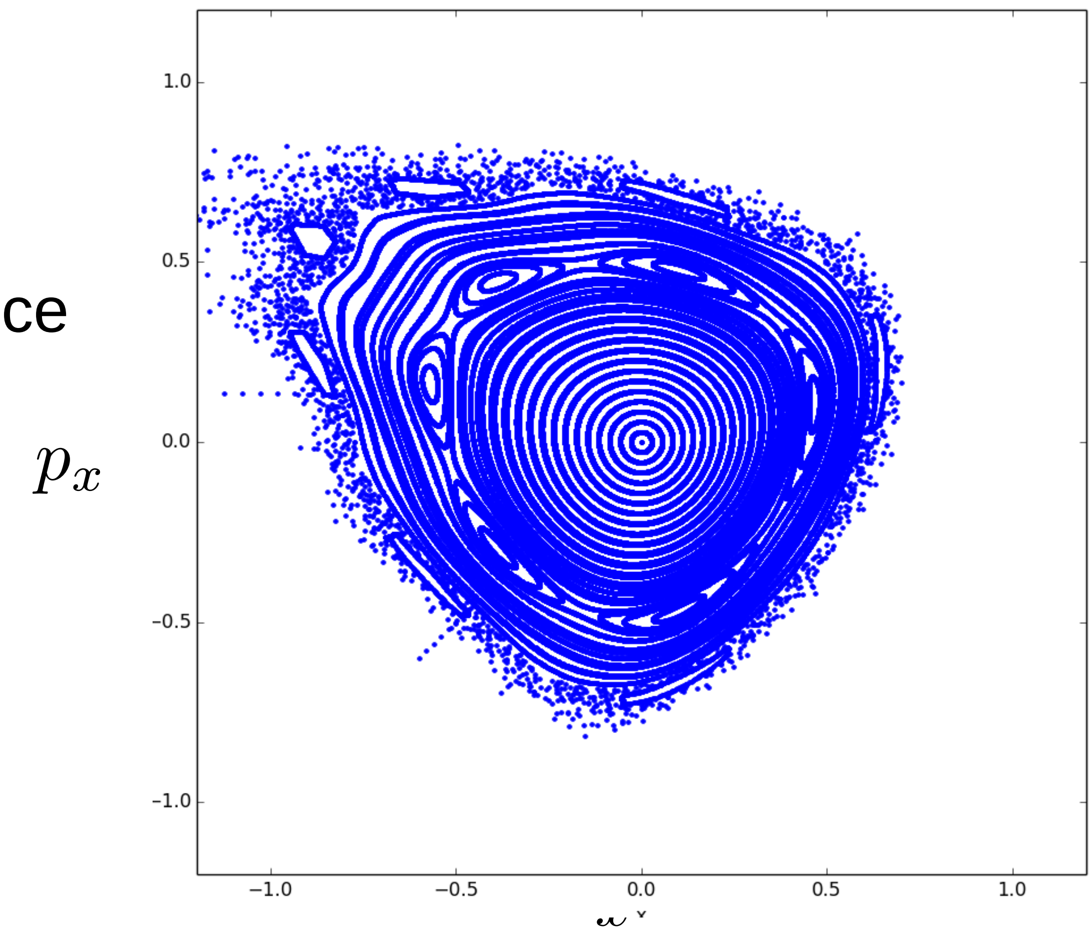
$$\mu_x = 0.21$$



Example: Single Sextupole

- Appearance of 3rd order resonance for certain phase advance
- … but also 4th order resonance
- … and 5th order resonance
- … and 6th order and 7th order and several higher orders…

$$\mu_x = 0.18$$



- The resonant Hamiltonian close to the **4th order resonance** is written as

$$\hat{H}_2 = \delta J_2 + cJ_2^2 + J_2^2 A_{4p} \cos(4\psi_2)$$

- The **fixed points** are found by taking the derivative over the two variables and setting them to zero, i.e.

$$\sin(4\psi_2) = 0 \ , \quad \delta + 2cJ_2 + 2J_2 A_{kp} \cos(4\psi_2) = 0$$

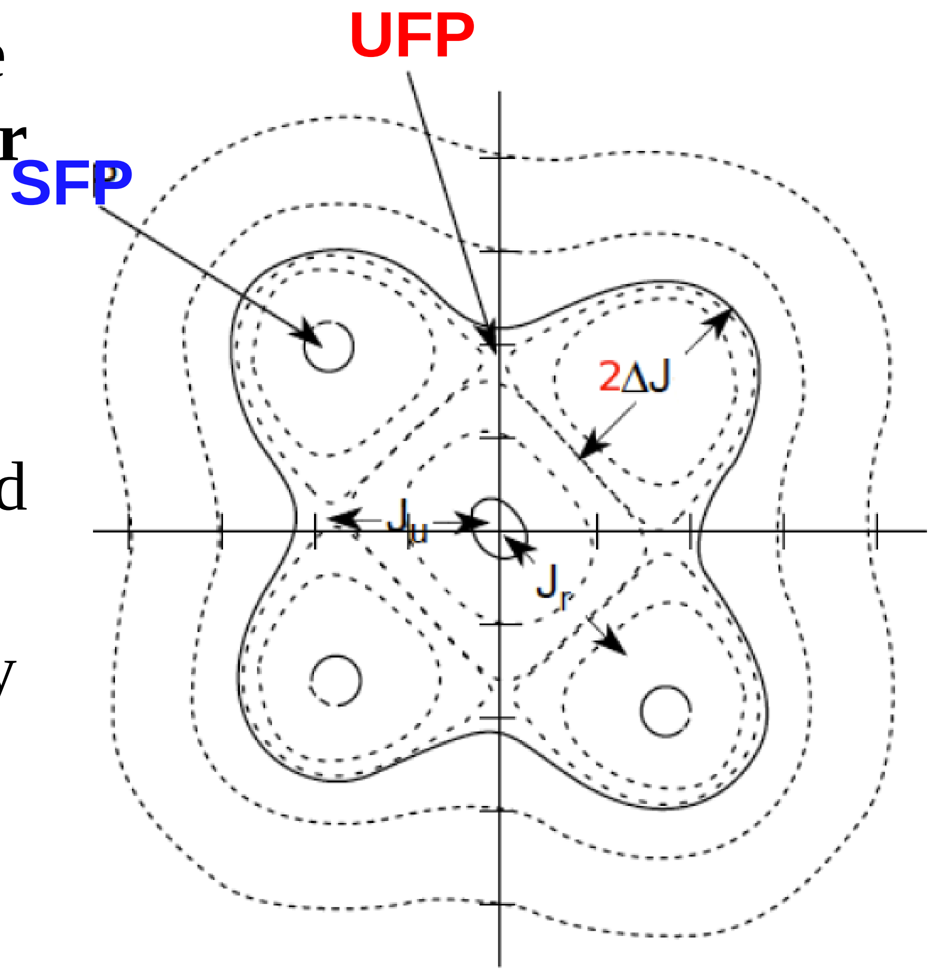
- The fixed points are at

$$\psi_{20} = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$$

- For **half** of them, there is a minimum in the potential as $\cos(4\psi_{20}) = -1$ and they are **elliptic** and **half** of them they are **hyperbolic** as $\cos(4\psi_{20}) = 1$

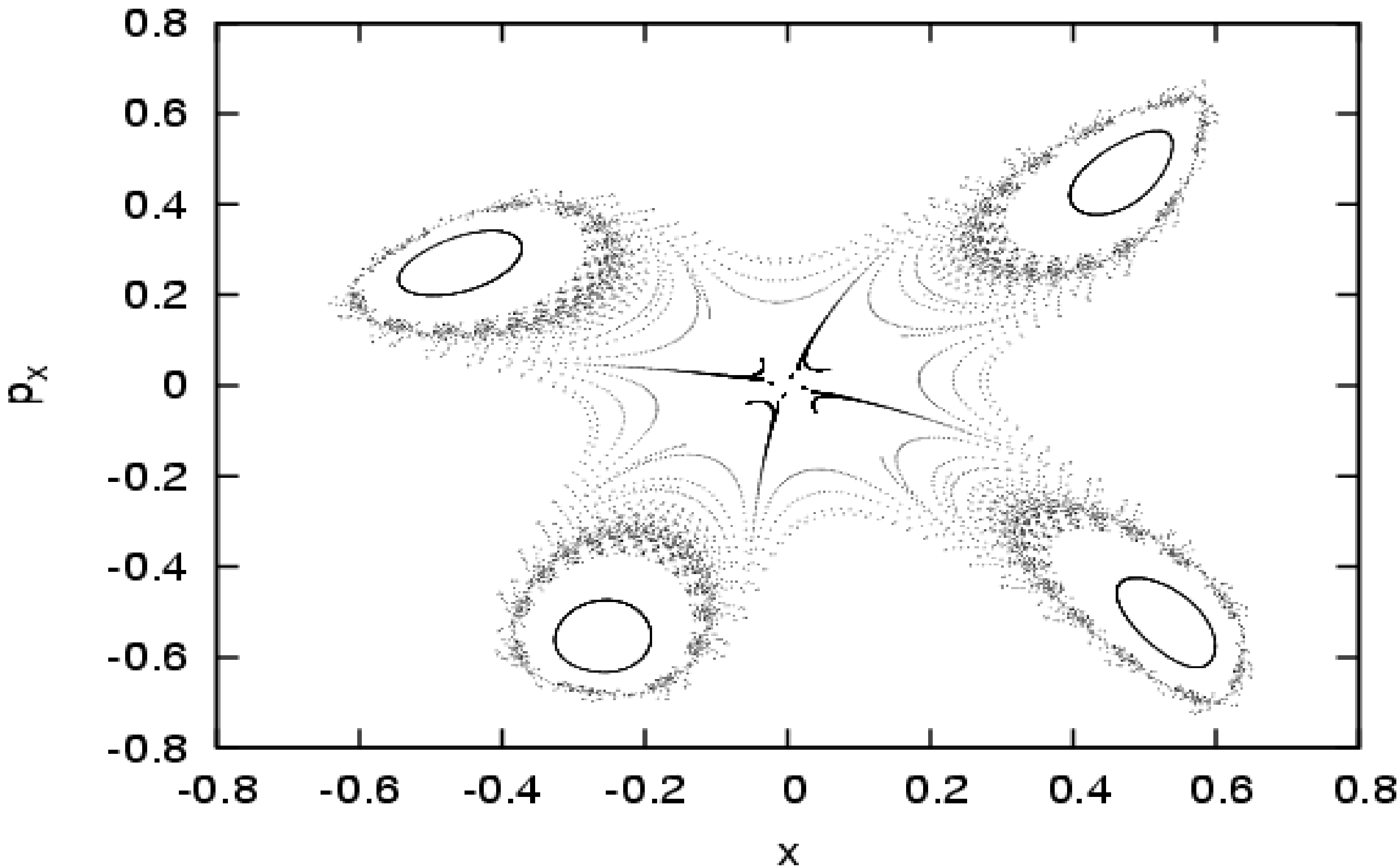
Topology of an octupole resonance

- **Regular motion** near the center, with curves getting more deformed towards a **rectangular shape**
- The **separatrix** passes through 4 unstable fixed points, but motion seems well contained
- **Four stable fixed points** exist and they are surrounded by stable motion (**islands of stability**)
- Question: Can the **central fixed point** become **hyperbolic** (answer in the appendix)



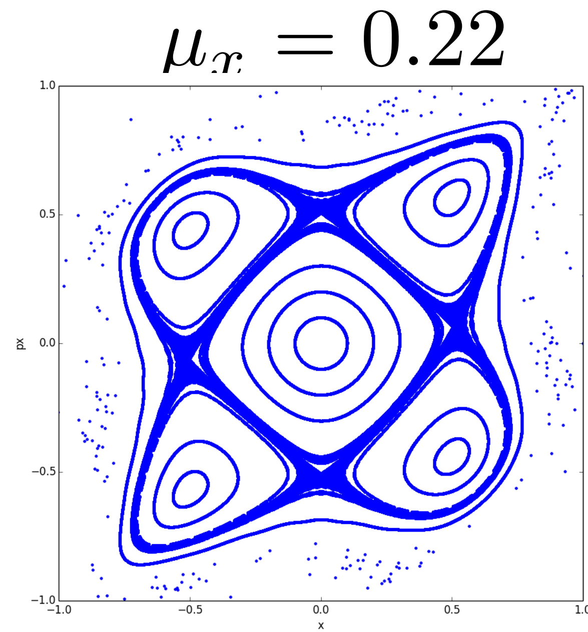
Octupole with hyperbolic central fixed point

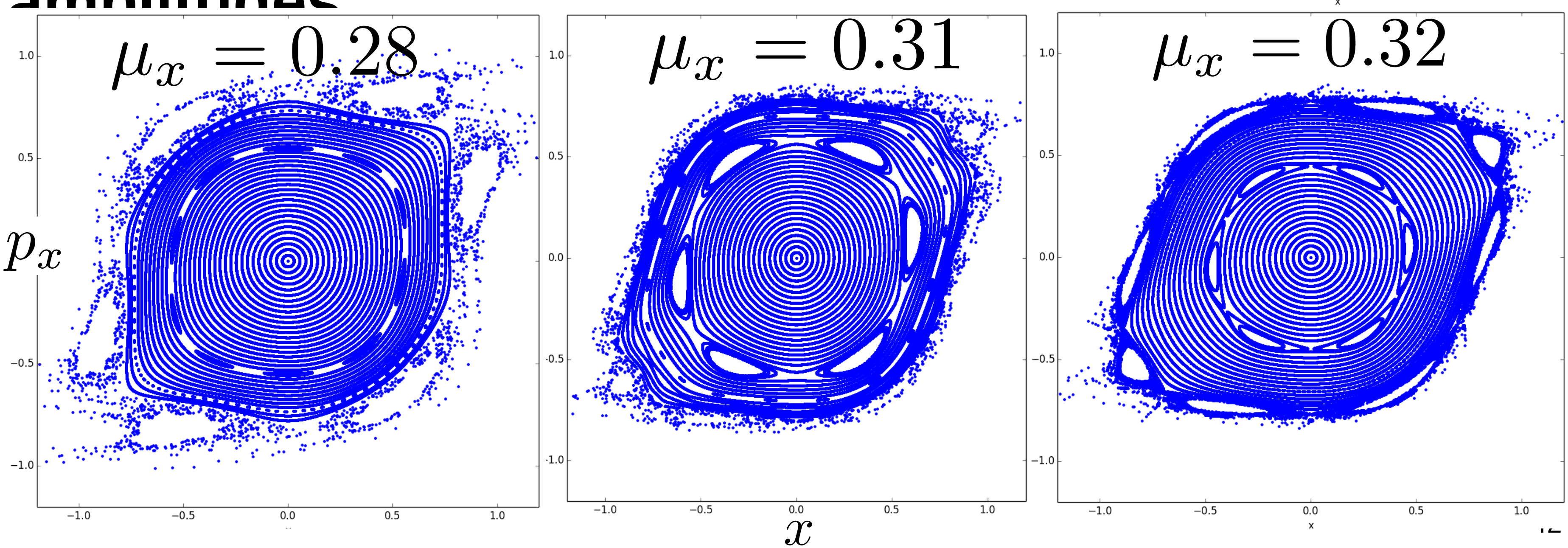
- Now, if $c = 0$ the solution for the action is $J_{20} = 0$
- So there is **no minima** in the potential, i.e. the central fixed point is **hyperbolic**



Single Octupole

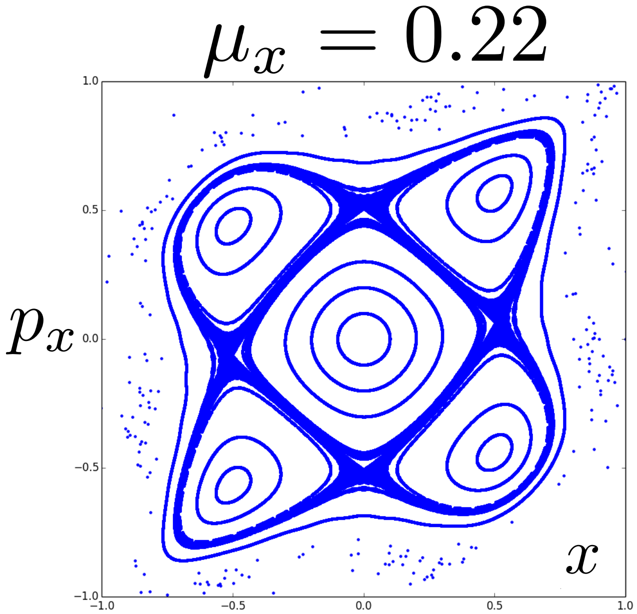
- As for the sextupole, the octupole can excite any resonance
- Multi-pole magnets can excite **any resonance** order
- It depends on the **tunes**, **strength** of the **magnet** and particle **amplitudes**

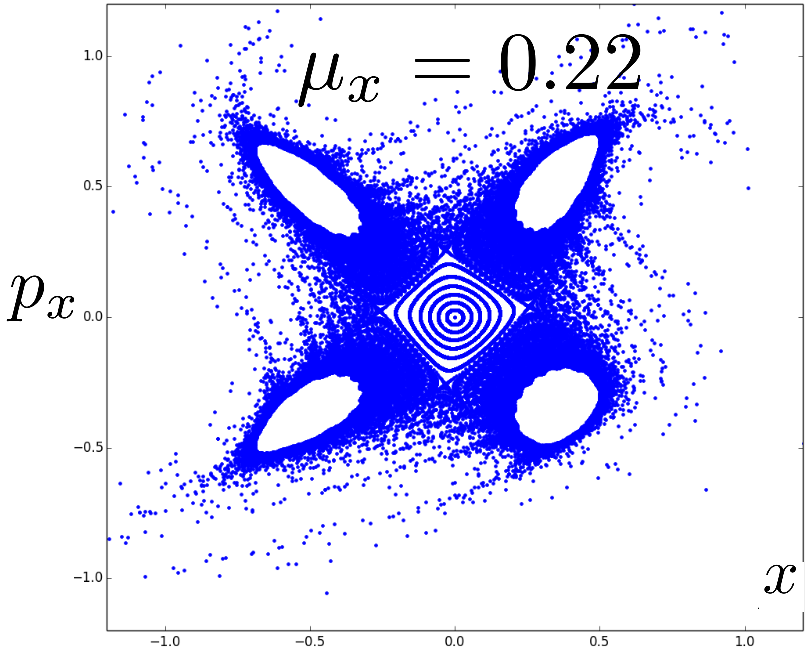




Single Octupole + Sextupole

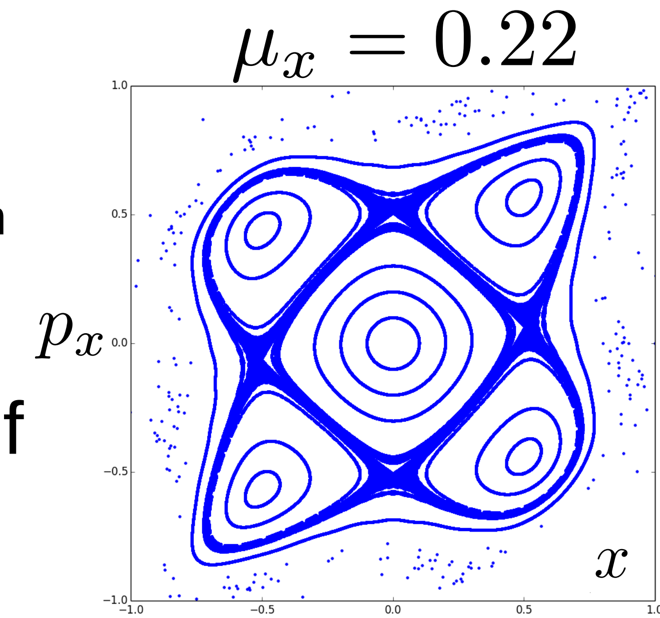
- Adding a sextupole and an octupole increases the chaotic motion region, when close to the 4th order resonance

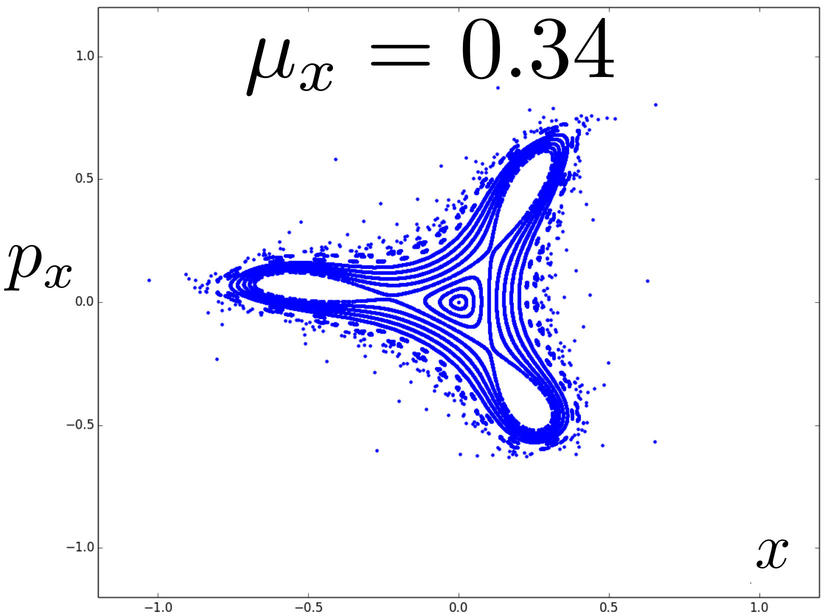


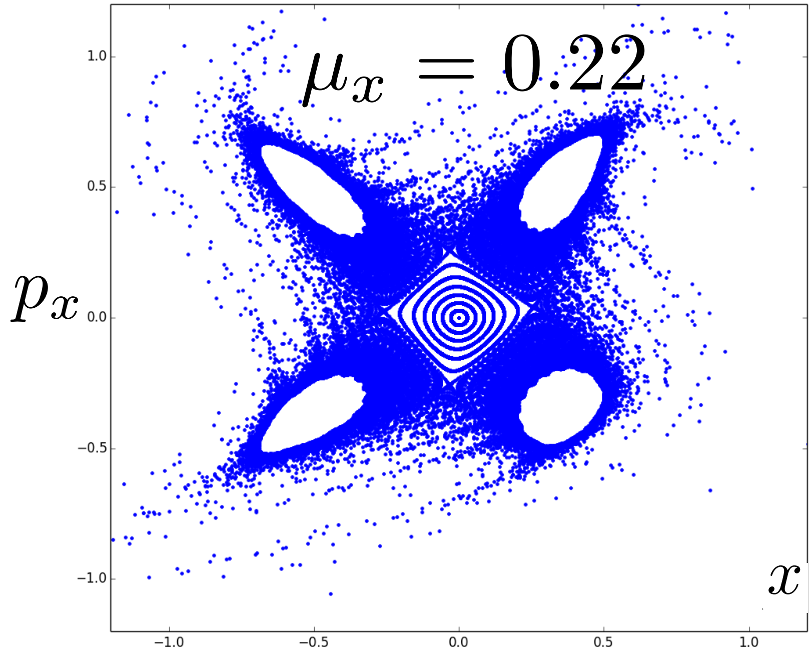


Single Octupole + Sextupole

- Adding a sextupole and an octupole increases the chaotic motion region, when close to the 4th order resonance
- But also allows the appearance of **3rd order resonance stable fixed points**



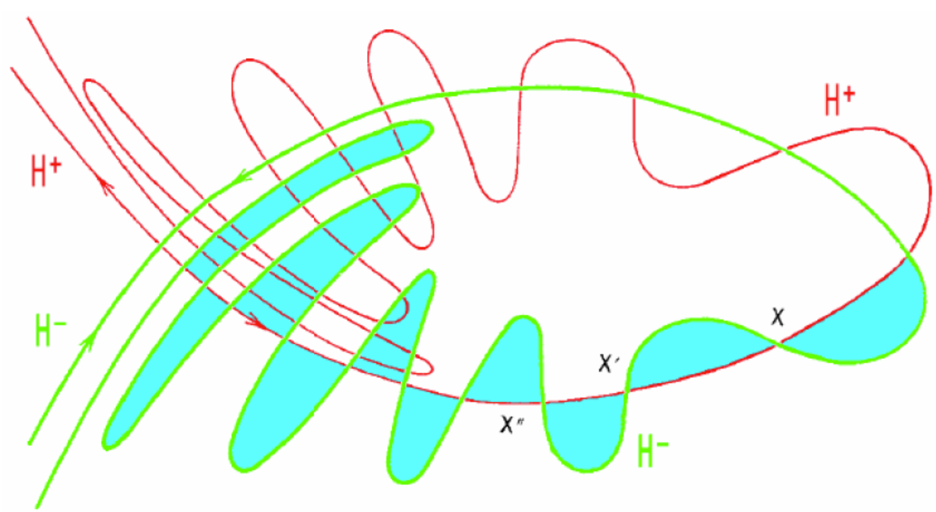


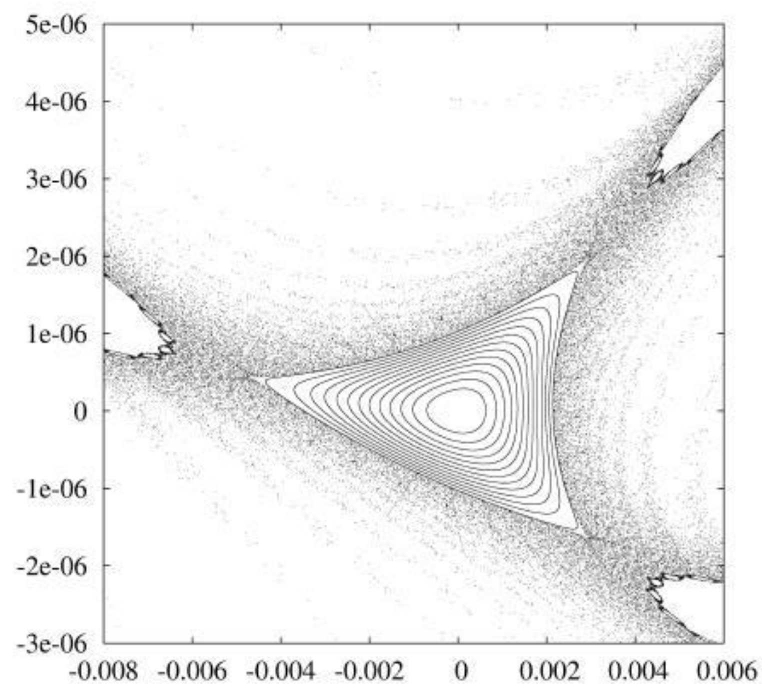


Onset of chaos

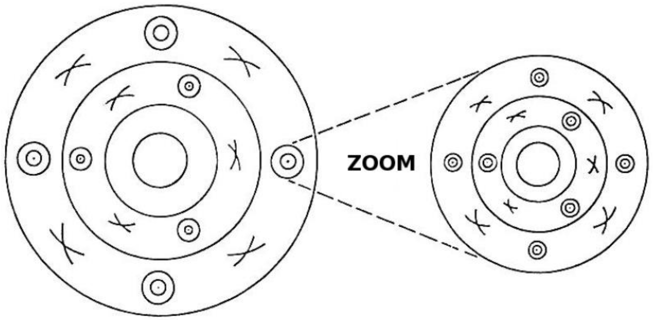
Path to chaos

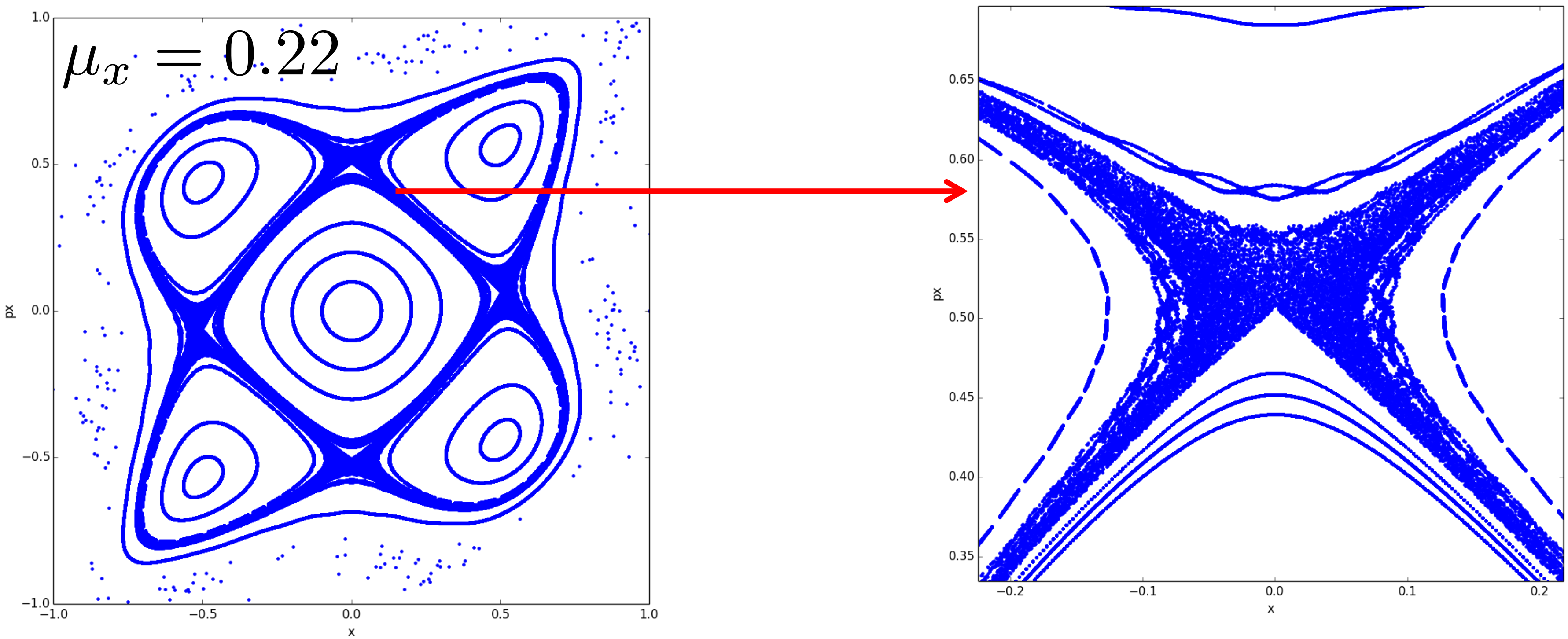
- When **perturbation** becomes **higher**, motion around the **separatrix** becomes **chaotic** (producing **tongues** or **splitting** of the separatrix)
- **Unstable** fixed points are indeed the **source of chaos** when a perturbation is added





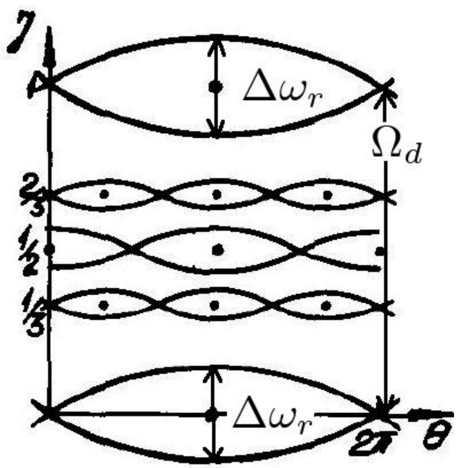
- **Poincare-Birkhoff** theorem states that under **perturbation** of a **resonance** only an **even number** of **fixed points** survives (**half stable** and the other **half unstable**)
- **Themselves** get **destroyed** when perturbation gets **higher**, etc. (**self-similar** fixed points)
- Resonance **islands grow** and **resonances** can **overlap** allowing diffusion of particles





Resonance overlap criterion

- When perturbation grows, the resonance island width grows
- **Chirikov** (1960, 1979) proposed a **criterion** for the **overlap** of two **neighboring resonances** and the onset of orbit diffusion
- The **distance** between two resonances is $\delta\hat{J}_{1\ n,n'} = \frac{2\left(\frac{1}{n_1+n_2} - \frac{1}{n_1'+n_2'}\right)}{\left|\left.\frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\ 2}}\right|_{\hat{J}_1=\hat{J}_{10}}\right|}$
- The **simple overlap criterion** is
  $\Delta\hat{J}_{n\ max} + \Delta\hat{J}_{n'\ max} \geq \delta\hat{J}_{n,n'}$



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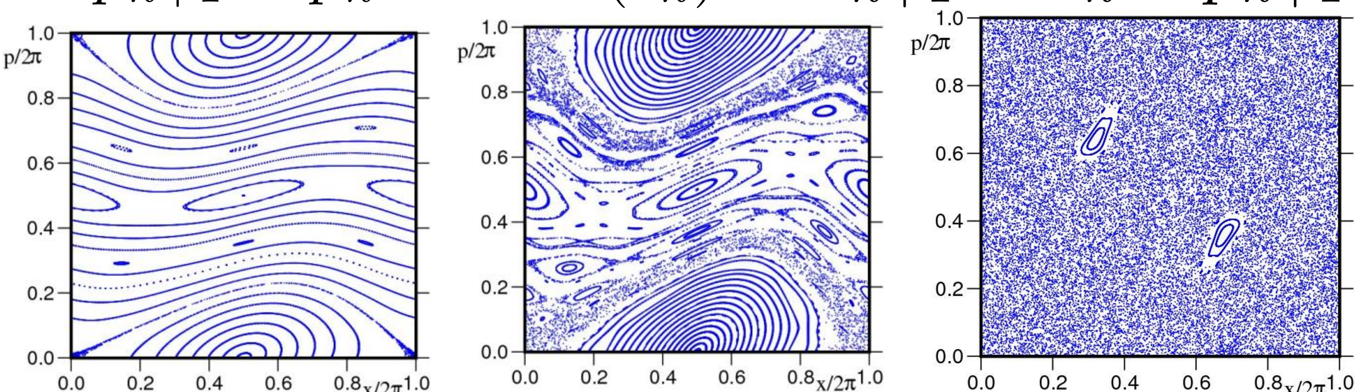
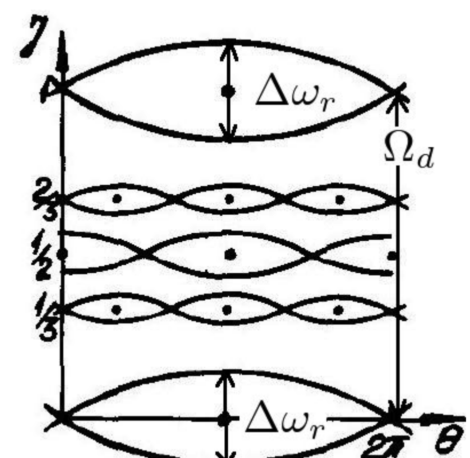
  $$\Delta\hat{J}_{n\ max} + \Delta\hat{J}_{n'\ max} \geq \delta\hat{J}_{n,n'}$$
- Considering the **width** of **chaotic layer** and **secondary islands**, the “two thirds” rule apply $\Delta\hat{J}_{n\ max} + \Delta\hat{J}_{n'\ max} \geq \frac{2}{3}\delta\hat{J}_{n,n'}$
- Example: **Chirikov’s standard map**

$$p_{n+1} = p_n + K\sin(\theta_n) \qquad \theta_{n+1} = \theta_n + p_{n+1}$$





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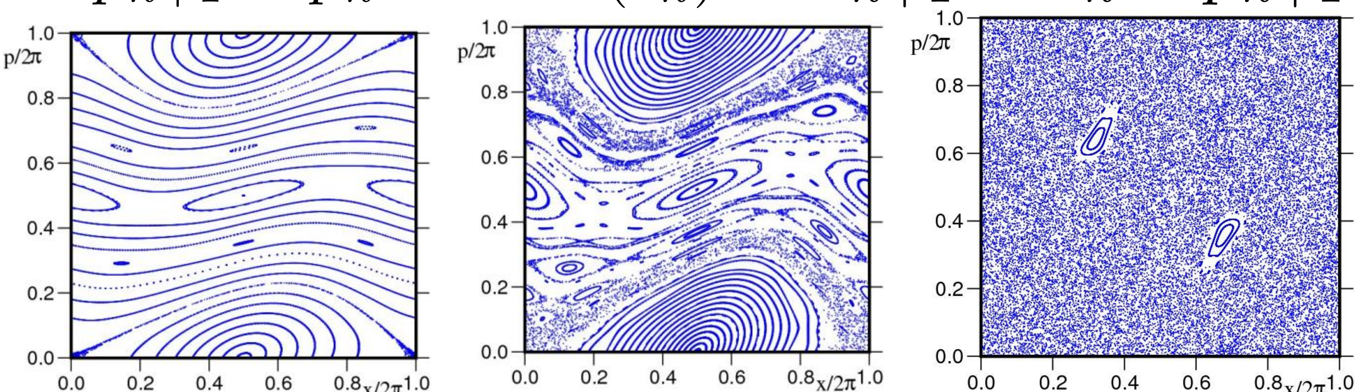
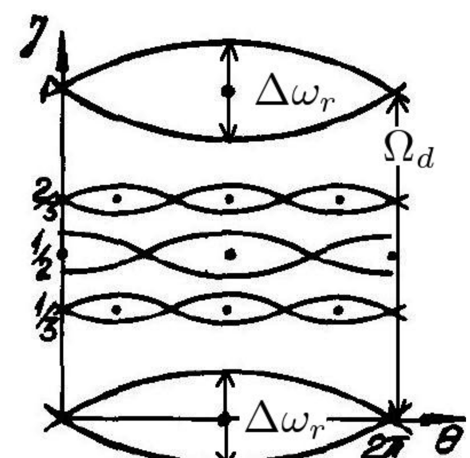
- The **simple overlap criterion** is

$$\Delta \hat{J}_{n\ max} + \Delta \hat{J}_{n'\ max} \geq \delta \hat{J}_{n,n'}$$

- Considering the **width** of **chaotic layer** and **secondary islands**, the "two thirds" rule apply $\Delta \hat{J}_{n\ max} + \Delta \hat{J}_{n'\ max} \geq \frac{2}{3} \delta \hat{J}_{n,n'}$
- The main limitation is the **geometrical nature** of the criterion (**difficulty** to be **extended** for **> 2 degrees of freedom**)

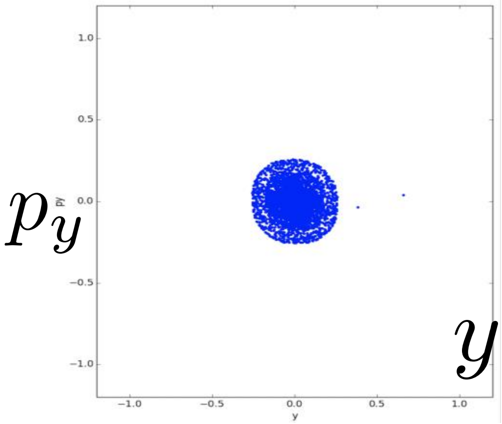
$$p_{n+1} = p_n + K \sin(\theta_n) \qquad \theta_{n+1} = \theta_n + p_{n+1}$$



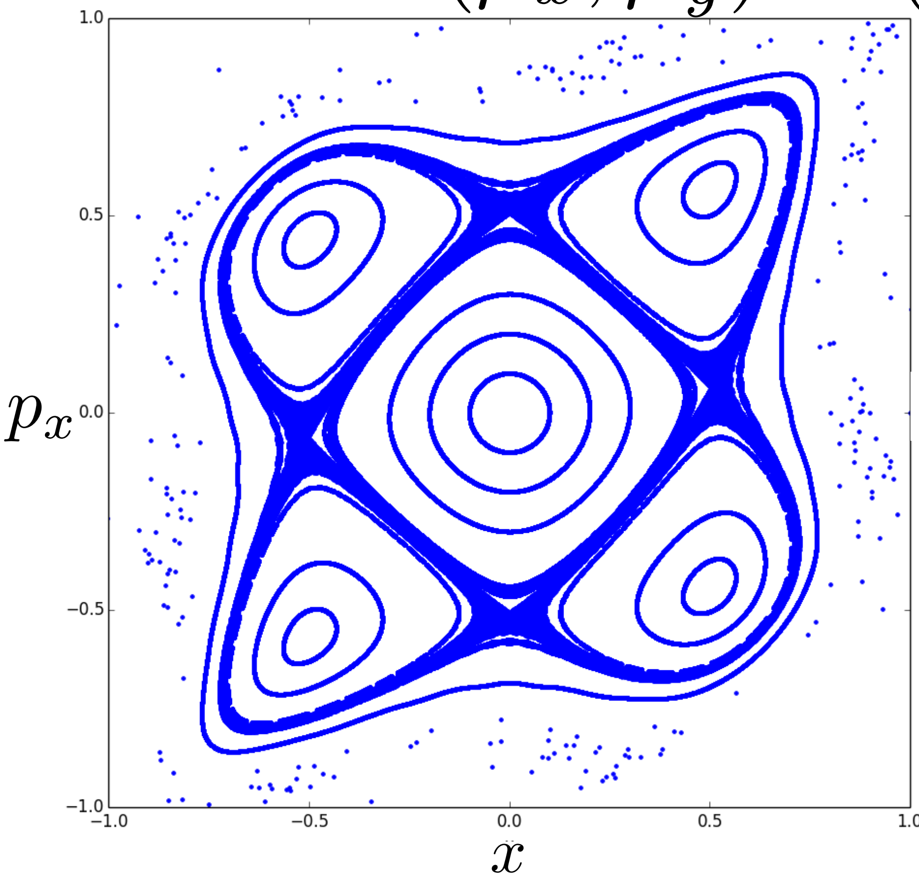


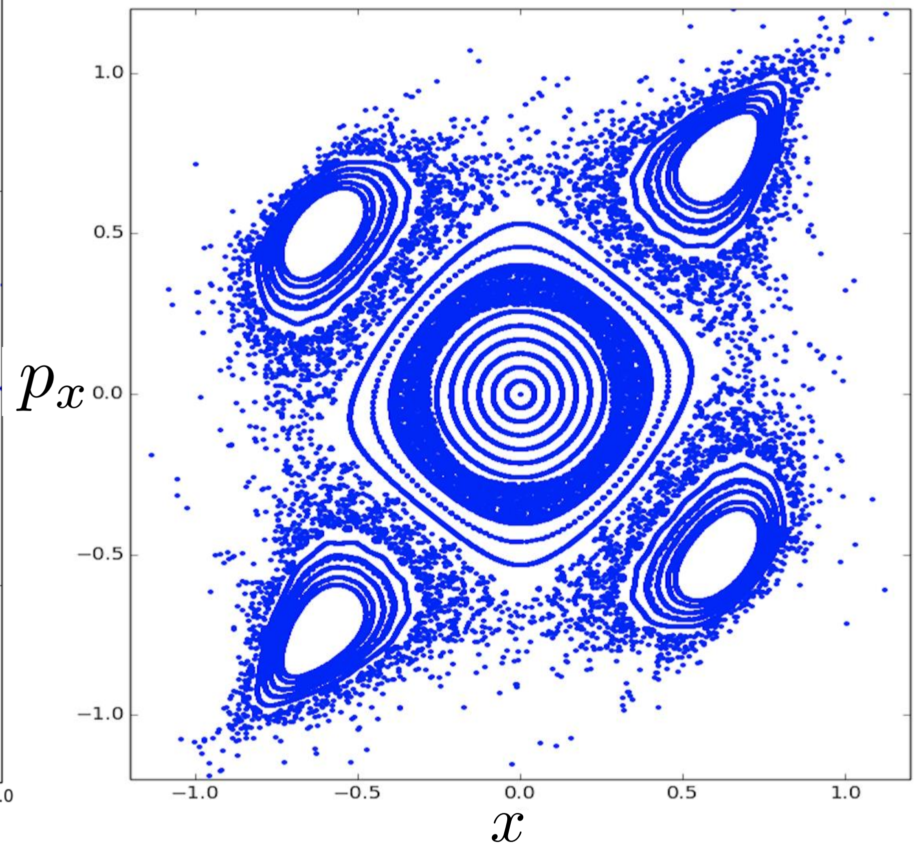
Increasing dimensions

- For $(y_0, p_{y0}) \neq (0, 0)$, i.e. by adding another degree of freedom **chaotic motion** is **enhanced**



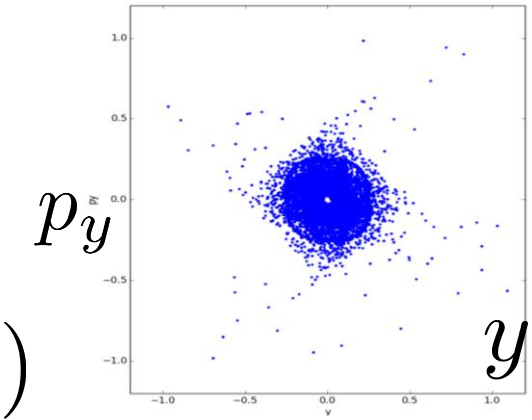
$$(\mu_x, \mu_y) = (0.22, 0.24)$$





Increasing dimensions

- For $(y_0, p_{y0}) \neq (0,0)$, i.e. by adding another degree of freedom **chaotic motion** is **enhanced**
- At the same time, **analysis** of phase space on **surface of section** becomes **difficult** to interpret, as these are projections of 4D objects on a 2D plane



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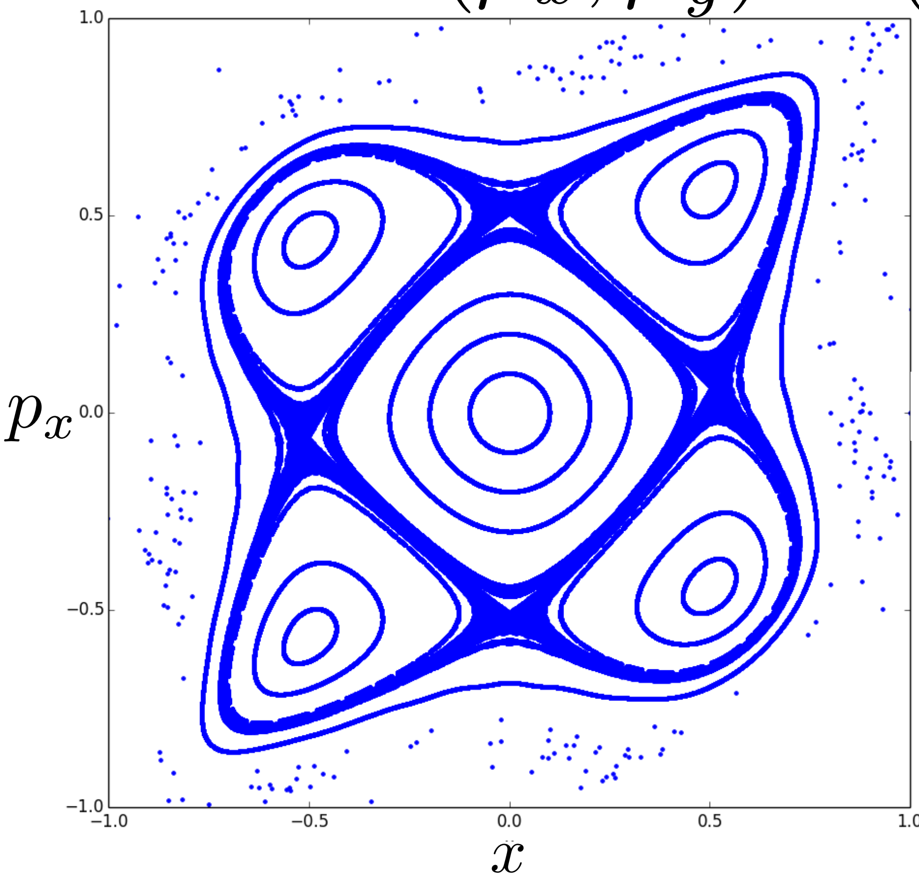

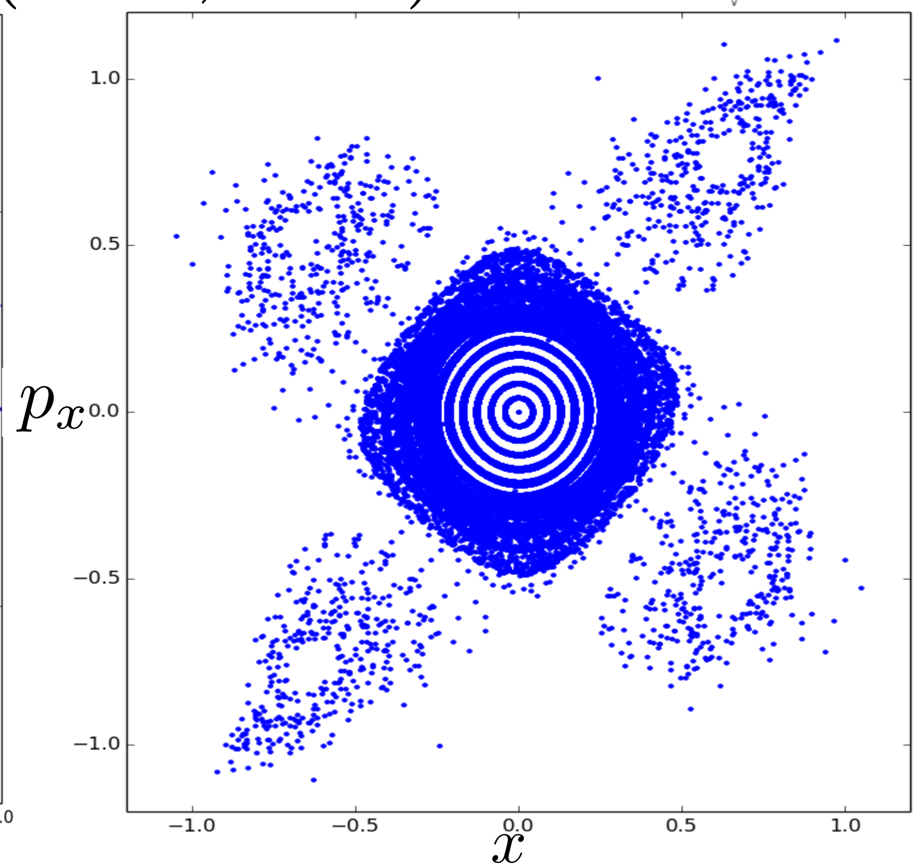

- Computing/measuring **dynamic aperture** (DA) or particle survival

  A. Chao et al., PRL 61, 24, 2752, 1988;
  F. Willeke, PAC95, 24, 109, 1989.

- Computation of Lyapunov exponents

  F. Schmidt, F. Willeke and F. Zimmermann, PA, 35, 249, 1991;
  M. Giovannozi, W. Scandale and E. Todesco, PA 56, 195, 1997

- Variance of unperturbed action (a la Chirikov)

  B. Chirikov, J. Ford and F. Vivaldi, AIP CP-57, 323, 1979
  J. Tennyson, SSC-155, 1988;
  J. Irwin, SSC-233, 1989

- Fokker-Planck diffusion coefficient in actions

  T. Sen and J.A. Elisson, PRL 77, 1051, 1996

- **Frequency map analysis**

Appendix

The pendulum

- An important non-linear equation which can be integrated is the one of the pendulum, for a string of length L and gravitational constant g

$$\frac{d^2\phi}{dt^2} + \frac{g}{L}\sin\phi = 0$$

- For small displacements it reduces to an harmonic oscillator with frequency $\omega_0 = \sqrt{\frac{g}{L}}$
- The integral of motion (scaled energy) is

$$\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - \frac{g}{L}\cos\phi = I_1 = E'$$

and the quadrature is written as $t = \int \frac{d\phi}{\sqrt{2(I_1 + \frac{g}{L}\cos\phi)}}$
assuming that for $t = 0\ ,\quad \phi = 0$

Solution for the pendulum

- Using the substitutions $\cos\phi = 1 - 2k^2\sin^2\theta$ with $k = \sqrt{1/2(1+I_1 L/g)}$, the integral is

$$t = \sqrt{\frac{L}{g}}\int_0^{\theta}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$ and can be solved using

Jacobi elliptic functions: $\phi(t) = 2\arcsin\left[k\ \mathrm{sn}\left(t\sqrt{\frac{g}{L}}, k\right)\right]$

- For recovering the period, the integration is performed between the two extrema, i.e. $\phi = 0$ and $\phi = \arccos(-I_1 L/g)$, corresponding to $\theta = 0$ and $\theta = \pi/2$ for which

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = 4\sqrt{\frac{L}{g}}\mathcal{F}(\pi/2, k)$$

i.e. the complete elliptic integral multiplied by four

Secular perturbation theory

- Consider a general two degrees of freedom Hamiltonian:

$$H(\mathbf{J},\boldsymbol{\varphi}) = H_0(\mathbf{J}) + \varepsilon H_1(\mathbf{J},\boldsymbol{\varphi})$$

  with the perturbed part periodic in angles:

$$H_1(\mathbf{J},\boldsymbol{\varphi}) = \sum_{k_1,k_1} H_{k_1,k_2}(J_1,J_2)\exp[i(k_1\varphi_1 + k_2\varphi_2)]$$

- The resonance $n_1\omega_1 + n_2\omega_2 = 0$ prevents the convergence of the series

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- The resonance $n_1\omega_1 + n_2\omega_2 = 0$ prevents the convergence of the series
- A **canonical transformation** can be applied for eliminating one action: $(\mathbf{J},\boldsymbol{\varphi}) \longmapsto (\hat{\mathbf{J}},\hat{\boldsymbol{\varphi}})$ using the generating function $F_r(\hat{\mathbf{J}},\boldsymbol{\varphi}) = (n_1\varphi_1 - n_2\varphi_2)\hat{J}_1 + \varphi_2\hat{J}_2$
- The relationships between new and old variables are

$$J_1 = n_1\hat{J}_1 \quad , \qquad J_2 = \hat{J}_2 - n_2\hat{J}_1$$

$$\hat{\varphi}_1 = n_1\varphi_1 - n_2\varphi_2 \quad , \qquad \hat{\varphi}_2 = \varphi_2$$

- This transformation put the system in a **rotating frame**, where the rate of change $\dot{\hat{\varphi}}_1 = n_1\dot{\varphi}_1 - n_2\dot{\varphi}_2$ measures the deviation from resonance

Secular perturbation theory

- The **transformed Hamiltonian** is $\hat{H}(\hat{\mathbf{J}},\hat{\varphi}) = \hat{H}_0(\hat{\mathbf{J}}) + \varepsilon \hat{H}_1(\hat{\mathbf{J}},\hat{\varphi})$ with the perturbation written as

$$\hat{H}_1(\hat{\mathbf{J}},\hat{\varphi}) = \sum_{k_1,k_2} H_{k_1,k_2}(\hat{\mathbf{J}}) \exp\left\{\frac{i}{n_1}\left[k_1\hat{\varphi}_1 + (k_1 n_2 + k_2 n_1)\hat{\varphi}_1\right]\right\}$$

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- This transformation assumes that $\dot{\varphi}_2$ is the **slow frequency** and the Hamiltonian can be **averaged** over the corresponding angle to obtain

$\bar{H}(\hat{\mathbf{J}},\hat{\varphi}) = \bar{H}_0(\hat{\mathbf{J}}) + \varepsilon \bar{H}_1(\hat{\mathbf{J}},\hat{\varphi}_1)$ with $\bar{H}_0(\hat{\mathbf{J}}) = \hat{H}_0(\hat{\mathbf{J}})$ and

$$\bar{H}_1(\hat{\mathbf{J}},\hat{\varphi}_1) = \langle \hat{H}_1(\hat{\mathbf{J}},\hat{\varphi}_1)\rangle_{\hat{\varphi}_2} = \sum_{p=-\infty}^{+\infty} H_{-pn_1,pn_2}(\hat{\mathbf{J}}) \exp(-ip\hat{\varphi}_1)$$

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- The averaging eliminated one angle and thus $\hat{J}_2 = J_2 + J_1 \frac{n_2}{n_1}$ is an **invariant** of motion
- This means that the Hamiltonian has effectively only **one degree of freedom** and it is **integrable**

Secular perturbation theory

- Assuming that the **dominant Fourier harmonics** for $p = 0, \pm 1$ the Hamiltonian is written as

$$\bar{H}(\hat{\boldsymbol{J}}, \hat{\boldsymbol{\phi}}_{\mathbf{1}}) = \bar{H}_0(\hat{\mathbf{J}}) + \varepsilon \bar{H}_{0,0}(\hat{\mathbf{J}}) + 2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}}) \cos \hat{\varphi}_1$$

- **Fixed points** $(\hat{J}_{10}, \hat{\phi}_{10})$ (i.e. periodic orbits) in phase space $(\hat{J}_1, \hat{\phi}_1)$ are defined by $\frac{\partial \bar{H}}{\partial \hat{J}_1} = 0, \quad \frac{\partial \bar{H}}{\partial \hat{\phi}_1} = 0$

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- Introduce **moving reference** on **fixed point** and expand $\bar{H}(\hat{\mathbf{J}})$ around it $\Delta \hat{J}_1 = \hat{J}_1 - \hat{J}_{10}$
- Hamiltonian describing motion near a resonance:
$$\bar{H}_r(\Delta \hat{J}_1, \hat{\phi}_1) = \frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}}\bigg|_{\hat{J}_1 = \hat{J}_{10}} \frac{(\Delta \hat{J}_1)^2}{2} + 2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}}) \cos \hat{\varphi}_1$$

Secular perturbation theory

- Assuming that the **dominant Fourier harmonics** for $p = 0, \pm 1$ the Hamiltonian is written as

$$\bar{H}(\hat{\boldsymbol{J}}, \hat{\boldsymbol{\phi}}_1) = \bar{H}_0(\hat{\mathbf{J}}) + \varepsilon \bar{H}_{0,0}(\hat{\mathbf{J}}) + 2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}}) \cos \hat{\varphi}_1$$

- **Fixed points** $(\hat{J}_{10}, \hat{\phi}_{10})$ (i.e. periodic orbits) in phase space $(\hat{J}_1, \hat{\phi}_1)$ are defined by $\frac{\partial \bar{H}}{\partial \hat{J}_1} = 0, \quad \frac{\partial \bar{H}}{\partial \hat{\phi}_1} = 0$
- Introduce **moving reference** on **fixed point** and expand $\bar{H}(\hat{\mathbf{J}})$ around it $\Delta \hat{J}_1 = \hat{J}_1 - \hat{J}_{10}$
- Hamiltonian describing motion near a resonance:

$$\bar{H}_r(\Delta \hat{J}_1, \hat{\phi}_1) = \frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}}\bigg|_{\hat{J}_1 = \hat{J}_{10}} \frac{(\Delta \hat{J}_1)^2}{2} + 2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}}) \cos \hat{\varphi}_1$$

- Motion near a typical resonance is like the one of the **pendulum**!!! The **libration frequency** and the resonance half width are

$$\hat{\omega}_1 = \left(2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}}) \frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}}\bigg|_{\hat{J}_1 = \hat{J}_{10}}\right)^{1/2} \qquad \Delta \hat{J}_{1\,max} = 2\left(\frac{2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}})}{\frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}}\Big|_{\hat{J}_1 = \hat{J}_{10}}}\right)^{1/2}$$

- The single resonance accelerator Hamiltonian (Hagedorn (1957), Schoch (1957), Guignard (1976, 1978))

$$H(J_x, J_y, \phi_x, \phi_y, s) = \frac{1}{R}(\nu_x J_x + \nu_y J_y) + g_{n_x,n_y} \frac{2}{R} J_x^{\frac{k_x}{2}} J_y^{\frac{k_y}{2}} \cos(n_x\phi_x + n_y\phi_y + \phi_0 - p\theta)$$

with $g_{n_x,n_y} e^{i\phi_0} = g_{j,k,l,m;p}$

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- From the **generating function**

$$F_r(\phi_x,\phi_y,\hat{J}_x,\hat{J}_y,s)=(n_x\phi_x+n_y\phi_y-p\theta)\hat{J}_x+\phi_y\hat{J}_y$$

the relationships between old and new variables are

$$\hat{\phi}_x=(n_x\phi_x+n_y\phi_y-p\theta)\;,\quad J_x=n_x\hat{J}_x$$

$$\hat{\phi}_y=\phi_y\;,\quad J_y=n_y\hat{J}_x+\hat{J}_y$$

- The following Hamiltonian is obtained

$$\hat{H}(\hat{J}_x,\hat{J}_y,\hat{\phi}_x)=\frac{(n_x\nu_x+n_y\nu_y-p)\hat{J}_x+\hat{J}_y}{R}+g_{n_x,n_y}\frac{2}{R}(n_x\hat{J}_x)^{\frac{k_x}{2}}(n_y\hat{J}_x+\hat{J}_y)^{\frac{k_y}{2}}\cos(\hat{\phi}_x+\phi_0)$$

Resonance widths

- There are two integrals of motion
  - The Hamiltonian, as it is **independent** on "time"
  - The **new action** $\hat{J}_y$ as the Hamiltonian is independent on $\hat{\phi}_y$
- The **two invariants** in the **old variables** are written as:

$$c_1 = \frac{J_x}{n_x} - \frac{J_y}{n_y}$$

$$c_2 = (\nu_x - \frac{p}{n_x + n_y})J_x + (\nu_y - \frac{p}{n_x + n_y})J_y + 2g_{n_x,n_y} J_x^{\frac{k_x}{2}} J_y^{\frac{k_y}{2}} \cos(n_x\phi_x + n_y\phi_y + \phi_0 - p\theta)$$

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- Two cases can be distinguished
  - n_x , n_y have **opposite** sign, i.e. **difference** resonance, the motion is the one of an ellipse, so bounded
  - n_x , n_y have the **same** sign, i.e. **sum** resonance, the motion is the one of an hyperbola, so **not** bounded
- These are **first order** perturbation theory considerations
- The **distance** from the resonance is obtained as

$$\Delta = \frac{g_{n_x,n_y}}{R} J_x^{\frac{k_x-2}{2}} J_y^{\frac{k_y-2}{2}} (k_x n_x J_x + k_y n_y J_y)$$