Hamiltonian Dynamics (see also CAS Advanced Course)

The menu:

- **Lagrange and Hamiltonian formalism, including:**
  - Lie operators and Lie transformations
  - Invariants of motion
  - examples: beam-beam, sextupoles, octupoles, ..
- **Non-linear Normal Forms**
  - the concept and what can we get from it
  - examples: beam-beam, sextupoles, octupoles, ..

Frequently asked Question: Why not just Newton's law and Lorentz force ?

Newton requires rectangular coordinates and time, trajectories with e.g. "curvature" or "torsion"*) need to introduce "reaction forces".
For example: LHC has locally non-planar (cork-screw) "design" orbits !

For linear dynamics done by ad hoc introduction of new coordinate frame

With Hamiltonian it is free: The formalism is "coordinate invariant"

Automatically solves problems with curvature and torsion. Concepts such as Symplecticity, Liouville Theorem, etc. follow naturally from the Hamiltonian treatment, i.e. without hand-waving (or wrong !) arguments

For complicated systems (e.g. nonlinear, coupling, radiation, spin, etc.): makes our life a lot easier (and in many cases possible !)

*) **E.g. solenoids, helical wigglers, helical separation**

Describe the particle's motion by a function L (Lagrange function)

$$L(\ q_1(t), ...q_n(t),\ \dot{q}_1(t), ...\dot{q}_n(t), t\ ) \quad \text{short :} \quad L(q_i, \dot{q}_i, t)$$

$q_1(t), ...q_n(t)$ **... generalized coordinates** $^{*)}$

$\dot{q}_1(t), ...\dot{q}_n(t)$ **... generalized velocities**

The integral $S = \int L(\ q_i(t), \dot{q}_i(t), t\ )\ dt$ **defines the action** $^{**)}$

Without proof or derivation:

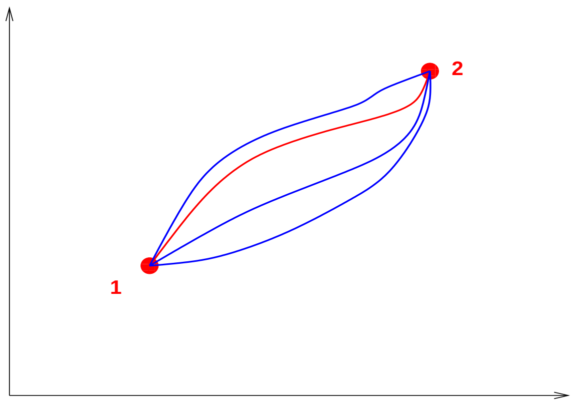
$L = T - V$ **= kinetic energy - potential energy**

$^{*)}$ q_i **can stand for any coordinate and any particle,** n **can be a very large number**

$^{**)}$ **Confusion alert: action J (a variable) and S (a functional) are different things**

Hamilton principle (stationary action)

$$S = \int_1^2 L(q_i, \dot{q}_i, t)dt = \text{stationary}$$



Hamiltonian principle: system moves from 1 to 2 such that the action S becomes stationary, i.e. $\delta S = 0$

Is fulfilled when: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$ **(Euler - Lagrange equation)**

For the Lagrangian of a (relativistic) free particle we must write

$$L_{free} \; = \; -\,mc^2 \sqrt{1 \; - \; \beta_r^2} \; = \; -\,mc^2 \sqrt{1 \; - \; (\frac{v}{c})^2} \; = \; - \; \frac{mc^2}{\gamma}$$

Using for the electromagnetic Lagrangian a form (without derivation, any textbook):

$$L_{em} \; = \; \frac{e}{c} \; v \cdot \vec{A} \; - \; e\phi$$

The complete Lagrangian is:

$$L \; = \; - \; \frac{mc^2}{\gamma} \; + \; \left(\frac{e}{c} \cdot \vec{v} \cdot \vec{A} \; - \; e \cdot \phi \right)$$

For our purpose it is an advantage to use Hamiltonian

From Lagrangian to Hamiltonian ..

- Generalized momenta instead of velocities
- q_i and p_i are independent and on equal footing, q_i and $\dot{q}_i$ are not

  Closely tied in with Symplecticity

  We use from now on: $q_i \quad \Rightarrow \quad x_i$

The generalized momenta p_i we derive from L as:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \quad \longrightarrow \quad \vec{p} - \frac{Q}{c}\vec{A}$$

Note: the canonical momenta are linked to the fields $\vec{A}$!!!!!

Once we know what the canonical momenta p_i are: the Hamiltonian is a (Legendre-)transformation of the Lagrangian:

$$H(x_i, p_i, t) = \sum_i \dot{x}_i p_i - L(x_i, \dot{x}_i, t)$$

Without proof, the Hamiltonian is: $H = T + V$ **= kinetic energy + potential energy**

From Hamilton's principle*) we obtain 2 first order equations of motion (Hamilton equations):

$$\frac{\partial H}{\partial x_i} = -\dot{p}_i = -\frac{dp_i}{dt}, \qquad \frac{\partial H}{\partial p_i} = \dot{x}_i = \frac{dx_i}{dt}$$

Canonical coordinates:

- **The Hamilton equations have always the same form (the Hamiltonian itself in general not)**
- **Form the basis for calculating conserved quantities (invariants ..)**
- **Basic requirement for Liouville's theorem**

Next step: Hamiltonian for electro-magnetic fields →

*) **Backup slides or any textbook on classical mechanics**

Hamiltonian for a (ultra relativistic, i.e. $\gamma \gg 1,\ \beta \approx\ 1$) particle in an electro-magnetic field is given by (any textbook on Electrodynamics):

$$H(\vec{x}, \vec{p}, t) = c\sqrt{(\vec{p} - e\vec{A}(\vec{x}, t))^2 + m_0^2 c^2} + e\Phi(\vec{x}, t) \qquad \text{(ugly...)}$$

where $\vec{A}(\vec{x}, t)$, $\Phi(\vec{x}, t)$ are the vector and scalar potentials (i.e. the V)

Using canonical variables (2D*)) and the design path length s as independent variable (bending field B_0 in y-plane) and no electric fields:

$$H \;=\; \overbrace{-(1+\frac{x}{\rho})}^{\text{due to t} \to \text{s}} \cdot \overbrace{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}}^{kinematic} + \overbrace{\frac{x}{\rho} + \frac{x^2}{2\rho^2}}^{\text{due to t} \to s} - \overbrace{\frac{A_s(x,y)}{B_0\rho}}^{normalized}$$

where $p \;=\; \sqrt{E^2/c^2 \;-\; m^2c^2}$ total momentum, $\delta = (p - p_0)/p_0$ is relative momentum deviation and $A_s(x, y)$ longitudinal component of the vector potential.

Find a formulation suited for beam dynamics in accelerators →

After square root expansion*) and sorting A_s contributions:

$$H = \overbrace{\frac{p_x^2 + p_y^2}{2(1+\delta)}}^{kinematic} - \overbrace{\underbrace{\frac{x\delta}{\rho}}_{bending} + \underbrace{\frac{x^2}{2\rho^2}}_{focusing}}^{dipole} + \overbrace{\frac{k_1}{2}(x^2 - y^2)}^{quadrupole} + \overbrace{\frac{k_2}{6}(x^3 - 3xy^2)}^{sextupole} + \ \ldots$$

using : $\quad k_n = k_n^{(n)} = \dfrac{1}{B\rho}\dfrac{\partial^n B_y}{\partial x^n} \qquad \left(k_n^{(s)} = \dfrac{1}{B\rho}\dfrac{\partial^n B_x}{\partial x^n} \right)$

- **Each element has a component in the Hamiltonian (but see in a few moments ..)**
- **Basis to extend the linear to a nonlinear formalism**

*) remember : $\sqrt{1+\alpha} = 1 + \dfrac{\alpha}{2} - \dfrac{\alpha^2}{8} + \dfrac{\alpha^3}{16} + \ldots.$

Hamiltonians of some machine elements (3D)

In general for multipole n:

$$H_n \;=\; \frac{1}{1+n}\mathcal{R}e\left[(k_n + \mathrm{i}k_n^{(s)})(x + \mathrm{i}y)^{n+1}\right] \;+\; \frac{p_x^2 \;+\; p_y^2}{2(1+\delta)}$$

We get for some important types (normal components k_n only):

drift space: $$H = -\sqrt{(1+\delta)^2 \;-\; p_x^2 \;-\; p_y^2}$$

dipole: $$H = -\frac{-x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

quadrupole: $$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

sextupole: $$H = \frac{1}{3}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

octupole: $$H = \frac{1}{4}k_3(x^4 - 6x^2y^2 + y^4) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

A first application - the simplest possible:

Keeping only the lower orders (focusing) and $\delta = 0$ we have:

$$H = \frac{p_x^2 + p_y^2}{2} - \underbrace{\frac{x^2}{2\rho^2(s)}}_{\text{dipole}} + \underbrace{\frac{k_1(s)}{2}(x^2 - y^2)}_{\text{quadrupole}}$$

Putting it into Hamilton's equations (for x, ditto for y):

$$\frac{\partial H}{\partial x} = -\frac{dp_x}{ds} = -x\left(\frac{1}{\rho^2(s)} - k_1(s)\right) \qquad \frac{\partial H}{\partial p_x} = \frac{dx}{ds} = p_x$$

it follows immediately:

$$\frac{d^2x}{ds^2} + \left(\frac{1}{\rho(s)^2} - k_1(s)\right) x = 0 \qquad \frac{d^2y}{ds^2} + k_1(s)\, y = 0$$

Hill's equations are a direct consequence of Hamiltonian treatment of EM fields to lower orders (without invoking the moon and hand-waving arguments !)

Introduce <u>Poisson brackets</u> for a differential operator (n = DOF):

$$[f,g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i} \right)$$

Here the variables x_i, p_i are canonical variables, f and g are (arbitrary) functions of x_i and p_i, (so far just a definition).

We can now write (using the Hamiltonian H for $g(x_i, p_i)$ in the above):

$$f = x_i \Rightarrow \quad [x_i, H] = \frac{\partial H}{\partial p_i} \quad \left(= \frac{dx_i}{dt} \right) \quad \rightarrow \quad [x_i, H] = \frac{dx_i}{dt}$$

$$f = p_i \Rightarrow \quad [p_i, H] = \frac{\partial H}{\partial x_i} \quad \left(= -\frac{dp_i}{dt} \right) \quad \rightarrow \quad [p_i, H] = -\frac{dp_i}{dt}$$

Poisson brackets encode time evolution of x and p

Having the principal equations:

$$[x_i, H] = \frac{dx_i}{dt} \qquad [p_i, H] = -\frac{dp_i}{dt}$$

They give the state of a system at a time $t + dt$ given the state at t (or $s + ds$), i.e. the time evolution of the dynamical system

we have a mapping from one place to another and a procedure for the numerical integration

The Poisson bracket of the Hamiltonian with a variable provides the evolution of <u>this</u> variable

The numerical studies of dynamical systems using Hamiltonian maps is the <u>only</u> sensible method in the era of fast computers !

It holds more generally for any function F(x, p) of canonical coordinates:

$$[F, H] = \frac{dF}{dt}$$

The Poisson bracket of the Hamiltonian with a function provides the evolution of this function

Not relevant for us, but for F(x, p, t) (to avoid complaints):

$$[F, H] = \frac{dF}{dt} - \frac{\partial F}{\partial t}$$

Question: if ρ is the particle distribution function, what is:

$$[\rho, H] = -\frac{d\rho}{dt}$$

We can define a symbolic operator: $:f:g \stackrel{def}{=} [f,g]$

where $:f:$ **is an operator acting on the function** g**:**

$$:f: \;=\; [f,\;\,] = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial x_i}\right) = \overbrace{\frac{\partial f}{\partial x}\frac{\partial}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial}{\partial x}}^{\text{for 1D}}$$

The operator $:f:$ **is (a special form of) a Lie Operator**

Lie operators are Poisson brackets "in waiting"

Look at special cases of the functions $g(x,p)$**:**

$$g = x \quad\rightarrow\quad [f,\;x] \;=\; :f:x \quad = \quad \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial x}{\partial x}\right)$$

$$g = p \quad\rightarrow\quad [f,\;p] \;=\; :f:p \quad = \quad \left(\frac{\partial f}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial p}{\partial x}\right)$$

In passing: Useful formulae for calculations (and examples)

Some common special (very useful) cases for f:

$$: x : \;=\; \frac{\partial}{\partial p} \qquad\qquad : p : \;=\; -\frac{\partial}{\partial x}$$

$$: x :^2 \;=\; \overbrace{: x :: x :}^{\text{applied twice}} \;=\; \frac{\partial^2}{\partial p^2} \qquad\qquad : p :^2 \;=\; \overbrace{: p :: p :}^{\text{applied twice}} \;=\; \frac{\partial^2}{\partial x^2}$$

$$: xp : \;=\; p\frac{\partial}{\partial p} - x\frac{\partial}{\partial x} \qquad\qquad : x :: p : \;=\; : p :: x : \;=\; -\frac{\partial^2}{\partial x \partial p}$$

$$: x^2 : \;=\; 2x\frac{\partial}{\partial p} \qquad\qquad : p^2 : \;=\; -2p\frac{\partial}{\partial x}$$

$$: x^n : \;=\; n \cdot x^{n-1}\frac{\partial}{\partial p} \qquad\qquad : p^n : \;=\; -n \cdot p^{n-1}\frac{\partial}{\partial x}$$

Applied to some simple (but most important) cases:

With x coordinate, p momentum (as generators):

$$:p:x \;=\; -1$$

$$:p^2:\; x \;=\; -2p\frac{\partial x}{\partial x} \;=\; -2p$$

$$:p:p \;=\; 0$$

$$:p^2:\; p \;=\; -2p\frac{\partial p}{\partial x} \;=\; 0$$

$$:p^2:\; xp^2 \;=\; -2p\frac{\partial xp^2}{\partial x} \;=\; -2p^3$$

$$(:p^2:)^2\, x \;=\; :p^2:(:p^2:x) \;=\; :p^2:(-2p) \;=\; 0$$

$$(:p^2:)^2\, p \;=\; :p^2:(:p^2:p) \;=\; :p^2:(0) \;=\; 0$$

Applied to some simple (but most important) cases:

With x coordinate (as generators), p momentum:

$$:x:\, x \;=\; 0 \qquad\qquad :x^2:\; x \;=\; -\,2x\frac{\partial x}{\partial p} \;=\; 0$$

$$\boxed{:x:\, p \;=\; 1} \qquad\qquad :x^2:\; p \;=\; -\,2x\frac{\partial p}{\partial p} \;=\; -\,2x$$

$$:x^2:\; xp^2 \;=\; -\,2x\frac{\partial xp^2}{\partial p} \;=\; -\,4px^2$$

$$(:x^2:)^2\; x \;=\; :x^2:(:x^2:x) \;=\; :x^2:(0) \;=\; 0$$

$$(:x^2:)^2\; p \;=\; :x^2:(:x^2:p) \;=\; :x^2:(-2x) \;=\; 0$$

How to use them for our purpose ?

$:H:\mathbf{g}$ **describes evolution of $\mathbf{g}$ over an <u>infinitesimal</u> distance dL**

$$\frac{dg}{ds} = [g, H] = (:-H:dL)\,\mathbf{g}$$

Accelerators typically larger than dL: we need to describe the evolution of $\mathbf{g}$ over a <u>finite</u> distance $L = dL \cdot n$

➡ **have to apply the map n times**

$$(:-H:L)\,\mathbf{g} = (:-H:\underbrace{dL \cdot n}_{L})\,\mathbf{g} = (:-H:dL)^n\,\mathbf{g}$$

We know how to compute powers and write them as:

$$(:f:)^2 g =: f:(:f:g) = [f,[f,g]]$$

$$(:f:)^3 g =: f:(:f:(:f:g)) = [f,[f,[f,g]]] \quad \text{etc.}$$

then we can construct/define an exponential operator:

like : $e^{x} = \sum_{i=0}^{\infty} \frac{1}{i!}(x)^i \quad \Rightarrow \quad e^{:f:} \stackrel{def}{=} \sum_{i=0}^{\infty} \frac{1}{i!}(:f:)^i$

$$e^{:f:} = 1 + :f: + \frac{1}{2!}(:f:)^2 + \frac{1}{3!}(:f:)^3 + \ldots$$

The operator $e^{:f:}$ is called a Lie Transformation

A special transformation:

We have from the Hamiltonian equations for the motion through an element with the Hamiltonian H for the element of length L (s as independent variable):

$$\frac{dg}{ds} = [g, H] =: -H : g \quad \Rightarrow \quad (: -H :)^k g = \frac{d^k g}{ds^k}$$

$$\Rightarrow \quad g(s) = \sum_{k=0}^{\infty} \frac{s^k}{k!} \left(\frac{d^k g}{ds^k} \right) = \sum_{k=0}^{\infty} \frac{s^k}{k!} (- : H :)^k g = e^{:-sH:} g$$

For the motion through an element of length L and a Hamiltonian H:

$$\Rightarrow \quad g(\mathrm{L}) = e^{:-LH:} \; g(0)$$

Acting on the phase space coordinates (shown for 1D here):

$$\begin{pmatrix} x \\ p \end{pmatrix}_2 = e^{:f:} \begin{pmatrix} x \\ p \end{pmatrix}_1$$

for the components: $x_2 = e^{:f:} x_1$ **and** $p_2 = e^{:f:} p_1$

- **Lie transforms describe how to go from one point $(x, p)_1$ to another $(x, p)_2$ → they are maps**
- **Crux of the matter: Not restricted to be matrices !!**
- **The generator f describes the element(s) between 1 and 2**

The miracle:

Lie transformations are always symplectic, no matter what is f

What is f ?

- The generator f is the Hamiltonian H of the element (or a sequence of many elements) !
- The Hamiltonian describes the exact motion from 1 to 2
- For an element of length L the generator f is: $f = L \cdot H$

For example a sextupole (remember the Hamiltonian components):

$$\begin{pmatrix} x \\ p \end{pmatrix}_2 = \exp\left(L \; : \underbrace{\frac{p_x^2 + p_y^2}{2(1+\delta)} + \frac{k}{6}(x^3 - 3xy^2)}_{H_{sextupole}} :\right) \begin{pmatrix} x \\ p \end{pmatrix}_1$$

Instead of multiplications, one performs a more general operation

(examples follow ..)

Another neat package with useful formulae:

With a constant, f, g, h arbitrary functions:

$$:a: \;=\; 0 \quad\longrightarrow\quad e^{:a:} \;=\; 1$$

$$:f:a \;=\; 0 \quad\longrightarrow\quad e^{:f:}a \;=\; a$$

$$e^{:f:}[g,h] = [e^{:f:}g,\; e^{:f:}h]$$

$$e^{:f:}(g \cdot h) = e^{:f:}g \;\cdot\; e^{:f:}h$$

and very important:

$$\mathcal{M}\, g(x) \;=\; e^{:f:}\, g(x) \;=\; g(e^{:f:}\, x)$$

e.g. $e^{:f:}\, x^2 \;=\; (e^{:f:}\, x)^2$

$$\mathcal{M}^{-1}\, g(x) \;=\; (e^{:f:})^{-1}\, g(x) \;=\; e^{-:f:}\, g(x)$$

$\left(\text{this is not} \quad \frac{1}{e^{:f:}} \; !\right)$

If we know the Hamiltonian H of a machine element then:

$$e^{:H:} x_1 = x_2 \qquad \text{and} \qquad e^{:H:} p_1 = p_2$$

It transforms the variables x and p, but that is not all:

This is true for <u>any function</u> of x and p

i.e. <u>any</u> property of a particle or the entire beam:

$$e^{:H:} f_1(x,p) = f_2(x,p)$$

e.g.: $x^2, \quad x \cdot p, \quad x^2 + p^2, ..$

- **H and f can be complicated, any nonlinear contraption**
- **Used for: spin, synchrotron radiation, ..**

Not possible with matrices ...

A (most) important feature - assume we have the map:

$$\mathcal{M} = e^{:f:}$$

we can write it in a different form, one transformation for each power (factorization):

$$e^{:f:} = e^{:f_2:} e^{:f_3:} e^{:f_4:} ...$$

Here f_k are power series of k-th order.

The miracle:

since all exponential maps are symplectic, one can truncate the factorized map at any order k ... and it remains symplectic !!

This was not possible with Power Series !

We can get closer to the best solution while remaining symplectic

Full Drift space

The exact Hamiltonian in two transverse dimensions and with a relative momentum deviation δ is (full Hamiltonian with $\vec{A}(\vec{x}, t)$ = 0):

$$H = -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \quad \Rightarrow \quad f_{drift} = L \cdot H$$

The <u>exact</u> map for a drift space is now (do <u>not</u> use x and x' !):

$$\begin{aligned}
x^{new} &= x + L \cdot \frac{p_x}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}} \\
p_x^{new} &= p_x \\
y^{new} &= y + L \cdot \frac{p_y}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}} \\
p_y^{new} &= p_y
\end{aligned}$$

In 2D and with $\delta \neq 0$ it is a complicated beast !!

In practice the map can (often) be simplified to the well known form.

Let's apply it to polynomials:

For example:

$$e^{:f:} x^3 \quad = \quad ??$$

Looking at the effect of a drift space on x^3:

$$e^{:f:} x^3 \quad = \quad e^{:-\frac{1}{2}Lp^2:} x^3$$

we would get:

$$\underbrace{e^{:-\frac{1}{2}Lp^2:} x^3 \quad = \quad (e^{:-\frac{1}{2}Lp^2:} x)^3}_{\text{with useful formula}} \quad = \quad x^3 + 3x^2Lp + 3xL^2p^2 + L^3p^3$$

Note:

$$\underbrace{e^{:-\frac{1}{2}Lp^2:} x^2 \quad = \quad (e^{:-\frac{1}{2}Lp^2:} x)^2}_{\text{with useful formula}} \quad = \quad x^2 + 2xLp + L^2p^2$$

(→ evolution of x^2 in a drift space)

Try a Lie transformation with $f = -L \cdot k \cdot x^2/2 \;=\; L \cdot H$:

$$e^{:-Lkx^2/2:}x \;=\; x - \frac{1}{2}L \underbrace{:kx^2:x}_{=0} + 0 + ..$$

$$= \boxed{\mathbf{x}}$$

$$e^{:-Lkx^2/2:}p \;=\; p - \frac{1}{2}L \underbrace{:kx^2:p}_{=kLx} + 0 + ...$$

$$= \boxed{\mathbf{p + kL \cdot x}}$$

Transformation of a thin quadrupole of length L and strength k !!

For:

$$f = -\frac{L}{2}kx^2 - \frac{L}{2}p^2 = -\frac{L}{2}(kx^2 + p^2)$$

we write for the transformation (map):

$$e^{:f:}x = e^{:-\frac{L}{2}(kx^2+p^2):}x$$
$$e^{:f:}p = e^{:-\frac{L}{2}(kx^2+p^2):}p$$

Remember:

$$e^{:f:}x = \sum_{n=0}^{\infty}\frac{:f:^n}{n!}x \qquad e^{:f:}p = \sum_{n=0}^{\infty}\frac{:f:^n}{n!}p$$

from the useful formulae (for the operators):

$$:\mathbf{f}:^{\mathbf{2n}} x = (-1)^n k^n L^{2n} \cdot x \qquad :\mathbf{f}:^{\mathbf{2n+1}} x = (-1)^{n+1} k^n L^{2n+1} \cdot p$$

we would get (rather straightforward with the above expressions, and some intelligent sorting):

$$e^{:-\frac{L}{2}(kx^2+p^2):} x = \sum_{n=0}^{\infty} \left(\overbrace{\frac{(-1)^n (\sqrt{k}L)^{2n}}{(2n)!}}^{:\mathbf{f}:^{\mathbf{2n}}} \right) \cdot x + \sum_{n=0}^{\infty} \left(\overbrace{\frac{(-1)^{n+1} (\sqrt{k}L)^{2n+1}}{(2n+1)!}}^{:\mathbf{f}:^{\mathbf{2n+1}}} \right) \cdot \frac{1}{\sqrt{k}} \cdot p$$

$$e^{:-\frac{L}{2}(kx^2+p^2):} p = \sum_{n=0}^{\infty} \left(\frac{(-1)^n (\sqrt{k}L)^{2n}}{(2n)!} \right) \cdot p - \sum_{n=0}^{\infty} \left(\frac{(-1)^{n+1} (\sqrt{k}L)^{2n+1}}{(2n+1)!} \right) \cdot \sqrt{k} \cdot x$$

Starting from:

$$f_{quad} = -\frac{L}{2}(kx^2 + p^2)$$

we finally have obtained:

$$e^{:f:}x = \cos(\sqrt{k}L)\cdot x + \frac{1}{\sqrt{k}}\sin(\sqrt{k}L)\cdot p$$

$$e^{:f:}p = -\sqrt{k}\sin(\sqrt{k}L)\cdot x + \cos(\sqrt{k}L)\cdot p$$

→ **Thick, focusing quadrupole, 1D !**

Comes directly from the Hamiltonian from first principles, no need to assume a solution of an equation of motion ...

A key point: the transformation/maps are done without the use of numerical integration !!

Extension: general monomials

Monomials in x and p of orders n and m $(x^n p^m)$

$$e^{:\,a x^n p^m\,:}$$

gives for the map (for $n \neq m$):

$$e^{:\,a x^n p^m\,:}x = x \cdot [1 + a(n-m)x^{n-1}p^{m-1}]^{m/(m-n)}$$
$$e^{:\,a x^n p^m\,:}p = p \cdot [1 + a(n-m)x^{n-1}p^{m-1}]^{n/(n-m)}$$

gives for the map (for $n = m$):

$$e^{:\,a x^n p^n\,:}x = x \cdot e^{-anx^{n-1}p^{n-1}}$$
$$e^{:\,a x^n p^n\,:}p = p \cdot e^{anx^{n-1}p^{n-1}}$$

A special case ... (a useful one)

If the matrix represents one complete turn, it has a simpler form

$$\begin{pmatrix} \cos\mu + \alpha\, sin(\mu) & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha\, sin(\mu) \end{pmatrix}$$

and f becomes the Courant-Snyder invariant (derivation in backup slides):

$$e^{:h:} = e^{:-\mu \cdot \frac{1}{2}(\gamma x^2 + 2\alpha xp + \beta p^2):} = e^{:-\mu \cdot J_x:}$$

The (linear) normal form transformation was:

$$\frac{1}{2}\underbrace{(\gamma x^2 + 2\alpha xp + \beta p^2)}_{ellipse} \Longrightarrow \frac{1}{2}\underbrace{(x^2 + p^2)}_{circle} = J_x$$

Written in our normal (simple) form, i.e. with the invariant J_x :

$$e^{:h:} = e^{:-\mu \cdot J_x:} \stackrel{defines}{\Longrightarrow} e^{:f_1:}$$ (the generator f_1 of the transformation)

Note: for a n-turn-matrix we have $e^{:-n \cdot \mu \cdot J_x:}$

Physical Meaning:

The invariant J_x is directly related to the effective Hamiltonian h.

A particularly important transformation:

$$\mathcal{M}\, J_x \;=\; e^{:-\mu J_x:}\, J_x \;=\; J_x$$

The constant area of the ellipse is conservation of energy

For a 3D linear system we have for f_3:

$$f_2 \;=\; -\frac{\mu_x}{2}(x^2+p_x^2)-\frac{\mu_y}{2}(y^2+p_y^2)-\overbrace{\frac{C}{2}}^{???}\alpha_c\delta^2 \;=\; -\mu_x J_x-\mu_y J_y-\frac{C}{2}\alpha_c\delta^2$$

Many machine elements

We want again a One-Turn-map for the ring (is now a Lie-transform, but with a single generator)

$$\mathcal{M}_{ring} = e^{:h_{eff}:}$$

- **We must combine N machine elements m_i by applying one transformation after the other*):**

$$e^{:h:} = e^{:m_1:} e^{:m_2:} \ldots e^{:m_N:} \qquad (\text{e.g. FODO cell}: \; = e^{:f_{QF}:} e^{:f_D:} e^{:f_{QD}:} \ldots e^{:f_D:})$$

- **Not restricted to matrices, i.e. linear elements ...**

→ **Need a procedure to combine Lie transforms**

*) **Apply left to right (matrices right to left)**

To combine/concatenate:

$$e^{:h:} = e^{:f:} e^{:g:}$$

We can use the formula (Baker-Campbell-Hausdorff (BCH)):

$$\begin{aligned} h = f \quad & + g + \tfrac{1}{2}[f,g] + \tfrac{1}{12}[f,[f,g]] + \tfrac{1}{12}[g,[g,f]] \\ & + \tfrac{1}{24}[f,[g,[g,f]]] - \tfrac{1}{720}[g,[g,[g,[g,f]]]] \\ & - \tfrac{1}{720}[f,[f,[f,[f,g]]]] + \tfrac{1}{360}[g,[f,[f,[f,g]]]] + \ldots \end{aligned}$$

or:

$$\begin{aligned} h = f \quad & + g + \tfrac{1}{2} :f: g + \tfrac{1}{12} :f:^2 g + \tfrac{1}{12} :g:^2 f \\ & + \tfrac{1}{24} :f::g:^2 f - \tfrac{1}{720} :g:^4 f \\ & - \tfrac{1}{720} :f:^4 g + \tfrac{1}{360} :g::f:^3 g + \ldots \end{aligned}$$

Stay calm: Software packages exist → LIEART, LIEMATH, LIEMAP, ...

Some simple tractable cases:

1. If f and g commute (i.e. $[f, g] = [g, f] = 0$) then concatenation is (exact):

$$h = f + g$$

2. If $[f, g] = [g, f] = scalar$ then concatenation is (exact):

$$h = f + g + \frac{1}{2}[f, g]$$

Other simple cases exist .. (in fact: many of the terms are zero !)

Example thin magnets, i.e. we neglect higher orders:

1. H_k **is the Hamiltonian of a thin multipole of order** k
2. H_D **is the Hamiltonian of a drift space (length of magnet)**

For the combination we can write (both are Hamiltonians):

$$H_{kD} = H_k + H_D \quad (= \quad H_D + H_k)$$

or alternatively:

$$H_{kD} = \frac{1}{2}H_D + H_k + \frac{1}{2}H_D$$

What does this correspond to ??

A frequently applicable case:

$$e^{:h:} = e^{:f:} e^{:g:}$$

if one of them (f or g) is small, can truncate the series and get a very useful formula.

Assume g is small compared to f:

$$e^{:f:} e^{:g:} \;=\; e^{:h:} \;=\; \exp\left[: f + \left(\frac{:f:}{1 - e^{-:f:}} \right) g : \right]$$

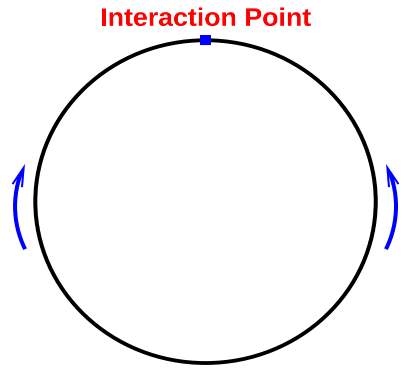
(How to use it: next example ...)

Some comments:

- Applied to simple (linear) cases, the formalism looks complicated and rather awkward !
  Seems we need more effort to get the same result.
  Doing concatenation by hand can drive you crazy !
- Its power lies in the application to nonlinear problems :
  - Lie transformations generate transfer maps
  - They are always symplectic
  - They can be applied when the equation of motion is not integrable !! (because they use only differentiation)
  - The formalism does not change when coupling or nonlinearities are added

The effort does NOT increase with the complexity of the problem !

A (challenging) real life example: beam-beam interaction



- Linear beam transport around the machine
- Beam-beam interaction localized and very nonlinear, cannot be treated as "spectator" (ideally requires self-consistent treatment)
- But essential to understand single-particle stability

We need to know:

- How do particles behave in phase space ?
- Do we have an invariant (stable beam) and how to compute it

We look for invariants - start with single IP

Here in 1D, same treatment for higher dimensions

Linear transfer around the machine $e^{:f_1:}$ **and beam-beam interaction** $e^{:B:}$

It is factorized into the two parts (see before):

$$e^{:f_1:}\, e^{:B:} = e^{:h:}$$

with (see before):

$$f_1 = -\frac{\mu}{2}(x^2 + p_x^2) = \mu \cdot J_x$$

with the usual transformation to action - angle variables

$$x = \sqrt{2J\beta}\,\cos\Psi, \qquad p = -\sqrt{\frac{2J}{\beta}}\sin\Psi$$

Beam-Beam part B(x):

For a Gaussian beam we have for for the kick/force $b(x)$ of the beam-beam interaction (derived from the fields, see e.g. [WH1]):

$$b(x) = \frac{N \cdot e^2}{4\pi\epsilon_0 mc^2\gamma} \cdot \frac{2}{x}(1 - e^{\frac{-x^2}{2\sigma^2}}) \quad \overbrace{\Longrightarrow}^{\text{for simplicity}} \quad b(x) = \frac{2}{x}(1 - e^{\frac{-x^2}{2\sigma^2}})$$

For the generator (potential of the beam-beam force = H) we get (extremely non-linear due to exponential !):

$$B(x) = \int_0^x dx' b(x')$$

and written as Fourier series (will soon be clear why):

$$B(x)^{*)} = \sum_{n=-\infty}^{\infty} c_n(J) e^{in\Psi} \qquad \text{with} \qquad c_n(J) = \frac{1}{2\pi}\int_0^{2\pi} d\Psi \, B(x) \, e^{-in\Psi}$$

*) **Note:** $x = \sqrt{2J\beta}\cos\Psi$

We evaluate the expression (because the beam-beam part is much smaller than the rest of the machine, typically 10^{-5}):

$$e^{:\mu J_x:}\, e^{:B:} \;=\; e^{:h:} \;=\; \exp\left[:\mu J_x + \left(\frac{:\mu J_x:}{1-e^{-:\mu J_x:}}\right) B :\right]$$

To do that we can now use (again) useful properties of Lie operators

For each n-th component of B (i.e. $\propto$ $\mathrm{e}^{in\Psi}$):

$$:\mu J_x: e^{in\Psi} = in\mu \cdot e^{in\Psi}, \qquad g(:\mu J_x:) e^{in\Psi} = g(in\mu) \cdot e^{in\Psi}$$

where we have used:

with $\quad g(:\mu J_x:) \;=\; \dfrac{1}{1-e^{-:\mu J_x:}} \qquad \Rightarrow \qquad g(in\mu) \;=\; \dfrac{1}{1-e^{-in\mu}}$

gives immediately for h:

$$h = \overbrace{-\,\mu J}^{\text{no beam-beam}} \quad + \left(\sum_n c_n(J) \; \cdot in\mu \; \cdot \; \frac{1}{1 - e^{-in\mu}} \; \cdot \; e^{in\Psi} \right)$$

or written differently:

$$h = -\mu J + \left(\sum_n c_n(J) \frac{n\mu}{2\sin(\frac{n\mu}{2})} \; e^{(in\Psi + i\frac{n\mu}{2})} \right)$$

Note: we can use the identical procedure for other "lenses"

Some inspection - analysis of h

$$h = \underbrace{-\mu J}_{linear} + \sum_n c_n(J) \frac{n\mu}{2\sin(\frac{n\mu}{2})} \; e^{(in\Psi + i\frac{n\mu}{2})}$$

On resonance:

$$Q = \frac{p}{n} = \frac{\mu}{2\pi}$$

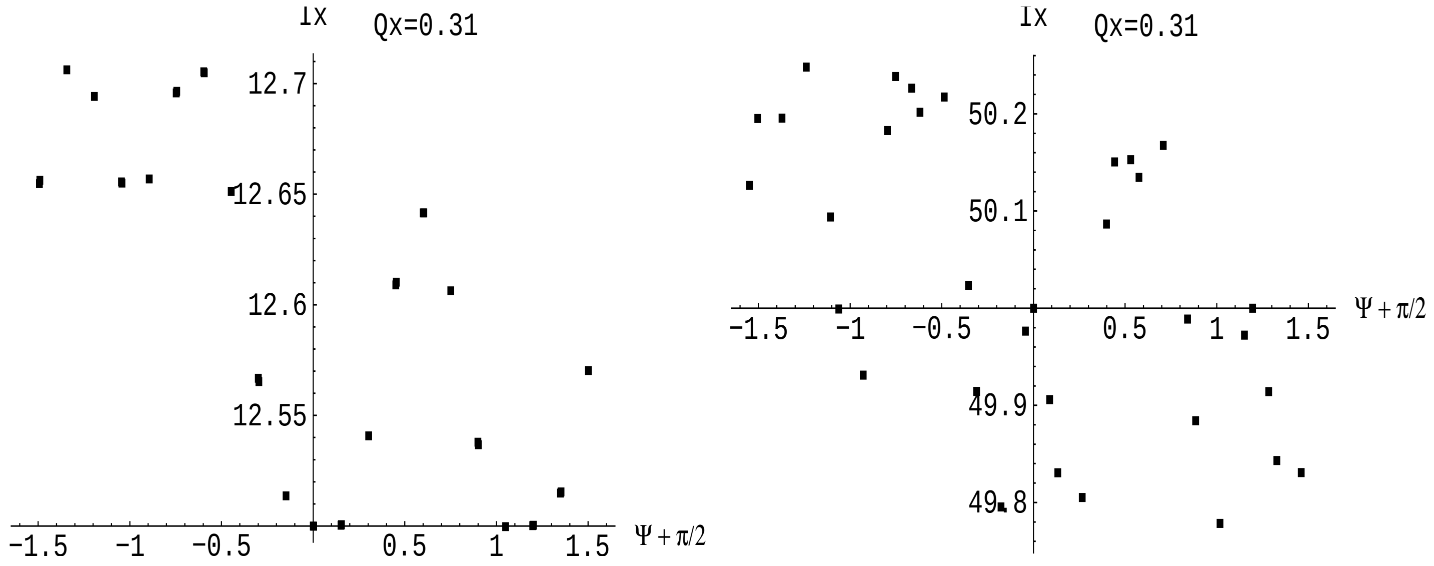
with $c_n \neq 0$**:**

$$\sin(\frac{n\pi p}{n}) = \sin(p\pi) \equiv 0 \quad \forall \; integer \quad p$$

and h **diverges, find automatically all resonance conditions**

Not a big deal, but can we also reproduce the distorted phase space (in action angle variables) ?

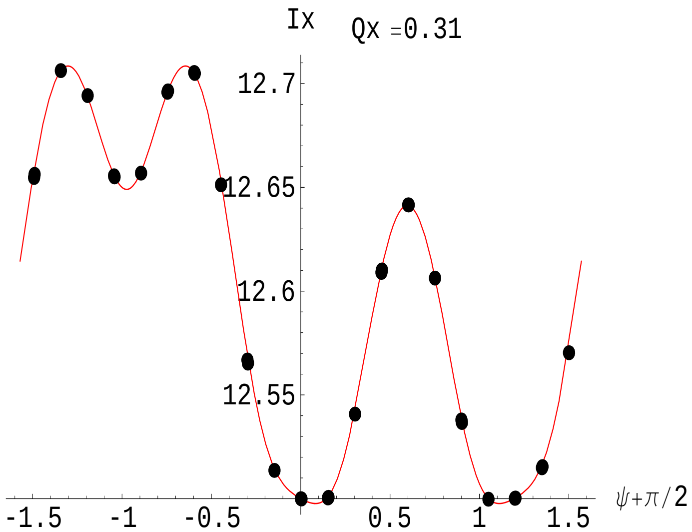
Invariant from tracking: Poincaré section of one IP

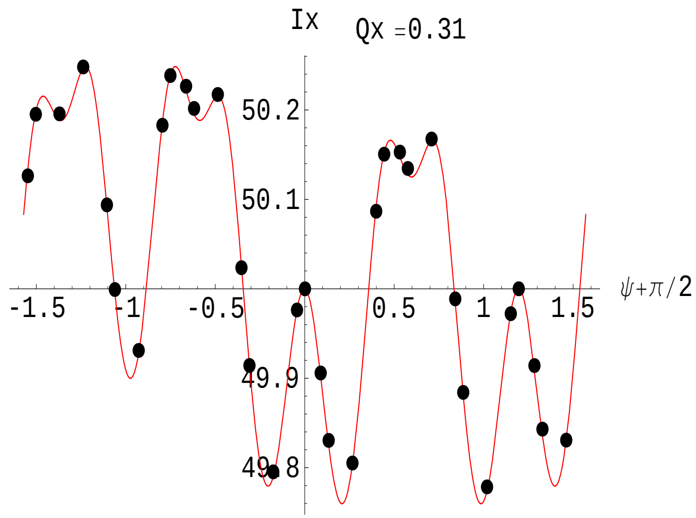


- **Phase space (action-angle) coordinates plotted each turn**
- **Shown for particle amplitudes of 5σ_x and 10σ_x**

Without beam-beam: a straight line

Invariant versus tracking: one IP





➡ **Shown for particle amplitudes of 5σ_x and 10σ_x**

one can reproduce and analyse the motion ...

works also for more than one interaction point (see backup slides), for LHC we treat up to 124 interactions per turn

First summary: Lie transforms and integrators

- **We have powerful tools to describe nonlinear (and obviously linear) elements**
- **They are always symplectic !**
- **Can be combined to form a ring (and therefore a nonlinear One-Turn-Map)**
- **Tools and programs are available for their manipulation and computation**
- **How do we analyse the maps ? → Normal Forms**

Normal forms nonlinear case

Normal form transformations can be generalized for nonlinear maps. If $\mathcal{M}$ is our usual one-turn-map, we try to find a transformation:

$$\mathcal{N} = \mathcal{A}\mathcal{M}\mathcal{A}^{-1} \qquad \text{as before, but now } \mathcal{M} \text{ is non-linear}$$

again $\mathcal{N}$ is a simple form (like the rotation we had before)

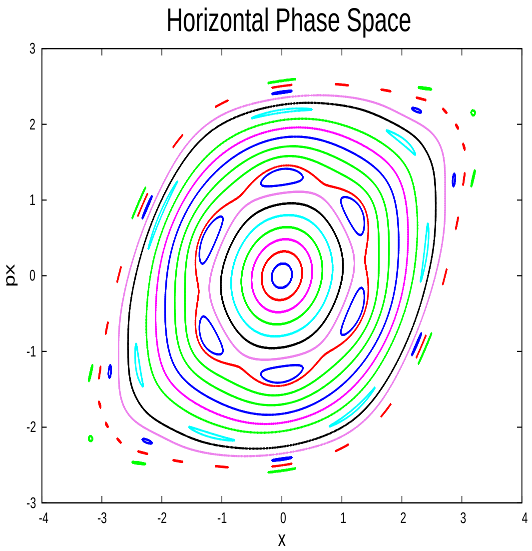
Of course we now do not have matrices, we use a Lie transform F to describe the transform $\mathcal{A}$:

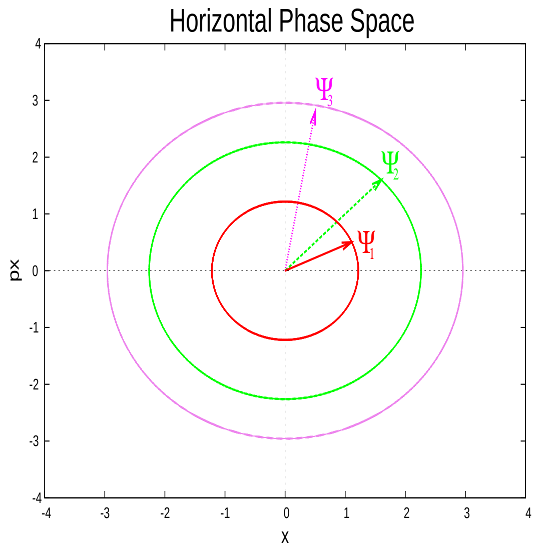
$$\mathcal{N} = \overbrace{e^{-:h:}}^{\text{simple form}} = \mathcal{A}\mathcal{M}\mathcal{A}^{-1} = e^{:F:}\mathcal{M}e^{-:F:}$$

The objects $\mathcal{A}$ and $\mathcal{A}^{-1}$ describe the transformation between the "ideal" and "real" motion.

Note: the inverse of $e^{:F:}$ is just $e^{-:F:}$!

Use beam-beam example:





- **Non-resonant contours can (maybe) transformed into a circle** *)
- **More complicated transformation F required**
- **Transform to coordinates where map is a rotation (as before)**

But: Rotation angle (i.e. phase advance) is amplitude dependent:

$\Psi \Rightarrow \Psi(J) \qquad \Psi_3 > \Psi_2 > \Psi_1$

*) **I have picked some of the amplitudes with closed contours**

The transformation $\mathcal{A} = e^{-:F:}$ should be the transformation to produce a simple form

"Simple" means: Remove the dependence on Ψ_x and Ψ_y

$$\mathcal{M} = e^{h(J_x, \not{\Psi_x}, J_y, \not{\Psi_y})} \Rightarrow e^{:F:}\mathcal{M}e^{-:F:} = e^{h_{eff}(J_x, J_y)} = \mathcal{N}$$

Once we know $h_{eff}(J_x, J_y)$ we can derive everything !

$\mathcal{A}$ analyses again the complexity of the motion, e.g. amplitude of the wiggles etc.

Formalism and software tools exist to find F (see e.g. Chao[1)] or E.Forest, M. Berz, J. Irwin, SSC-166)

[1)] A. Chao, Lecture Notes on Topics in Accelerator Physics, 2001

Normal forms - nonlinear case

Once we can write the map as (now example in 3D):

$$\mathcal{N} = e^{- : h_{eff}(J_x, J_y, \delta) :}$$

where h_{eff} depends only on J_x, J_x, and δ, then we have the tunes:

$$Q_x(J_x, J_y, \delta) = \frac{1}{2\pi} \frac{\partial h_{eff}}{\partial J_x}$$

$$Q_y(J_x, J_y, \delta) = \frac{1}{2\pi} \frac{\partial h_{eff}}{\partial J_y}$$

and the change of path length:

$$\Delta z = -\frac{\partial h_{eff}}{\partial \delta}$$

Particles with different J_x, J_y and δ have different tunes:

→ **Dependence on J is amplitude detuning, dependence on δ are the chromaticities !**

How does h_{eff} look like ?

The effective Hamiltonian can always be written (here to 3rd order) as:

$$
\begin{aligned}
h_{eff} = \quad & + \quad \mu_x J_x + \mu_y J_y + \frac{1}{2}\alpha_c \delta^2 \\
& + \quad c_{x1} J_x \delta + c_{y1} J_y \delta + c_3 \delta^3 \\
& + \quad c_{xx} J_x^2 + c_{xy} J_x J_y + c_{yy} J_y^2 + c_{x2} J_x \delta^2 + c_{y2} J_y \delta^2 + c_4 \delta^4
\end{aligned}
$$

and then tune depends on action J and momentum deviation δ:

$$
Q_x(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_x} = \frac{1}{2\pi}\left(\mu_x + \overbrace{2c_{xx}J_x + c_{xy}J_y}^{detuning} + \overbrace{c_{x1}\delta + c_{x2}\delta^2}^{chromaticity}\right)
$$

$$
Q_y(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_y} = \frac{1}{2\pi}\left(\mu_y + \overbrace{2c_{yy}J_y + c_{xy}J_x}^{detuning} + \overbrace{c_{y1}\delta + c_{y2}\delta^2}^{chromaticity}\right)
$$

What’s the meaning of it ?

- μ_x, μ_y: **linear phase advance or (2π)*tunes for rings**
- $\frac{1}{2}\alpha_c, c_3, c_4$: **linear and nonlinear "momentum compaction"**
- c_{x1}, c_{y1}: **first order chromaticities**
- c_{x2}, c_{y2}: **second order chromaticities**
- c_{xx}, c_{xy}, c_{yy}: **detuning with amplitude**

The coefficients are the various <u>aberrations</u> of the optics

A few examples (in brief - no derivation)

Example 1: sextupole

A linear map (3D !) followed by a single (weak) sextupole:

$$\mathcal{M} = e^{- : \mu J_x + \mu J_y + \frac{1}{2}\alpha_c \delta^2 :} e^{: k(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)} :}$$

we get for h_{eff} (see e.g. [AC1, EF]):

$$h_{eff} = \mu_x J_x + \mu_y J_y + \frac{1}{2}\alpha_c \delta^2 - kD^3\delta^3 - 3k\beta_x J_x D\delta + 3k\beta_y J_y D\delta$$

Then it follows:

$$Q_x(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_x} = \frac{1}{2\pi}(\mu_x - 3k\beta_x D\delta)$$

$$Q_y(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_y} = \frac{1}{2\pi}(\mu_y + 3k\beta_y D\delta)$$

No tune shift/spread in first order ...

Side note:

Before the Normal Form Transformation, the Hamiltonian h (1D) is:

$$h(J,\Psi) \;=\; -\mu{\cdot}J - \frac{3}{8}{\cdot}\mu{\cdot}k{\cdot}(2\beta J)^{3/2}{\cdot}\left[\frac{\sin(3\Psi+\frac{3\mu}{2})}{\sin(\frac{3\mu}{2})} \;-\; \frac{\sin(\Psi+\frac{\mu}{2})}{\sin(\frac{\mu}{2})}\right] \;=\; const.$$

Example 2: octupole (1D - to emphasize important part)

Starting with: $\mathcal{M} = e^{-:\mu J_x:} e^{:f_4:} = e^{-:\mu J_x:} e^{:k_3 \cdot \frac{x^4}{4}:}$

we get (without derivation, see[EF1, AW]):

$$M = e^{-:F:} \, e^{\overbrace{:-\mu J + \frac{3}{8} k_3 \cdot J^2:}^{\mathrm{h_{eff}} \; = \; \mathrm{R}}} \, e^{:F:}$$

Note: the normalized map (our most simple map):

$$R = \exp : -\mu J + \frac{3}{8} k_3 \cdot J^2 : \quad \rightarrow \quad Q = \frac{1}{2\pi}(\mu + \frac{3}{4} k_3 J)$$

is again a rotation in phase space, but the rotation angle (tune) now depends linearly on the amplitude J

Particles with different amplitudes have different tunes → <u>tune spread</u>

Example 3: once more beam-beam ...

We had:

$$h = -\mu J + \sum_n c_n(J) \frac{n\mu}{2\sin(\frac{n\mu}{2})} e^{(in\Psi + i\frac{n\mu}{2})}$$

a normal form transformation takes away the angular dependence (see before) and we have only:

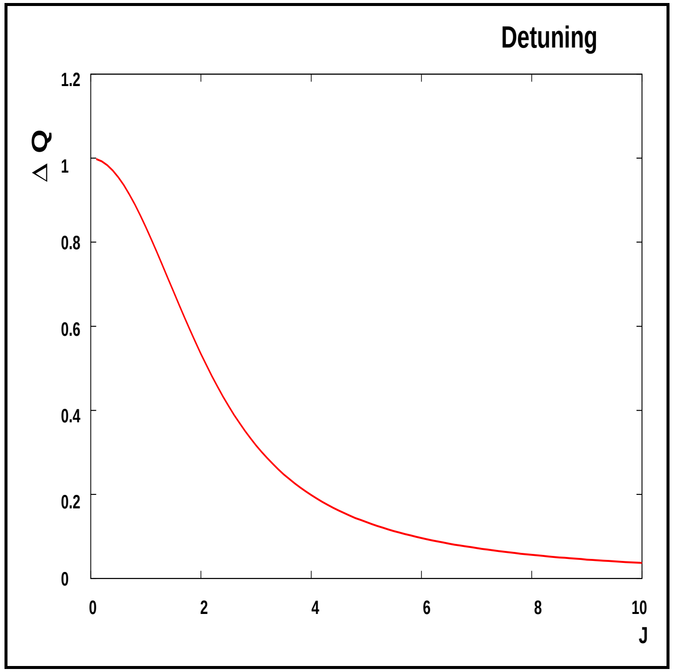
$$h_{eff} \;=\; -\mu J + c_0(J) \;= const. \qquad \text{(for } c_0(J) \text{ see e.g. [AC1])}$$

$$\Delta Q \;=\; \frac{\partial h_{eff}}{\partial J} \;=\; \frac{\partial c_0(J)}{\partial J} \;=\; \left(\frac{N \cdot e^2}{4\pi\epsilon_0 mc^2\gamma}\right) \cdot \frac{2}{J}\left[1 - I_0(\frac{J}{2}) \cdot e^{\frac{-J}{2}}\right]$$

I_0 is the modified Bessel function

Different amplitudes J imply different tunes → tune spread

Amplitude detuning



Detuning is amplitude dependent

Very nonlinear (unlike octupoles)

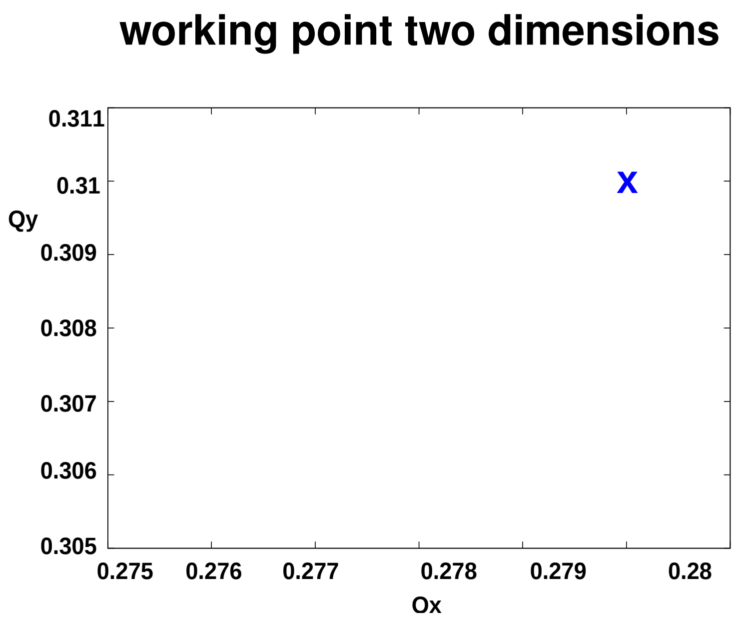
Largest effect for small amplitudes

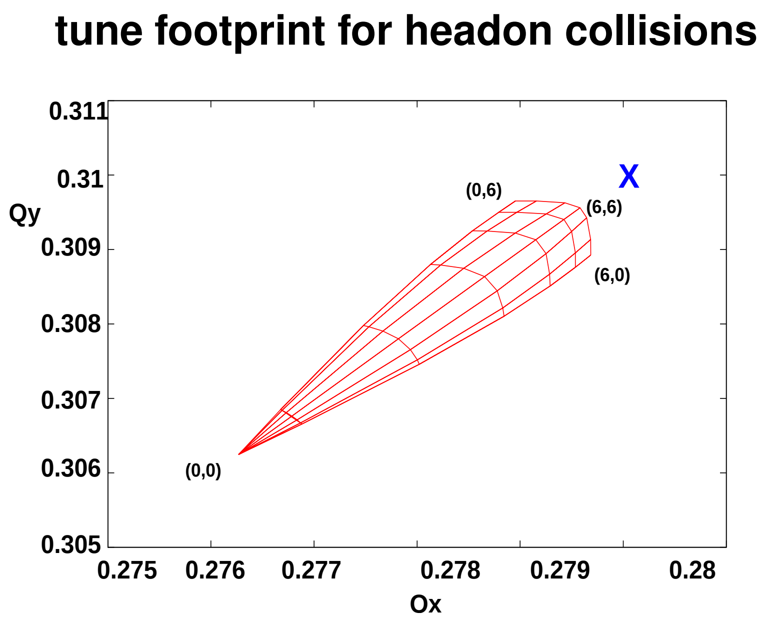
For calculations : see proceedings

Advanced CAS (Trondheim, 2013)

$$\Delta Q = \frac{\partial h_{eff}}{\partial J} = \frac{\partial c_0(J)}{\partial J} = \left(\frac{N \cdot e^2}{4\pi\epsilon_0 mc^2\gamma}\right) \cdot \frac{2}{J}\left[1 - I_0(\frac{J}{2}) \cdot e^{\frac{-J}{2}}\right]$$

Tunes in tune grid, now in 2D: with and without beam-beam



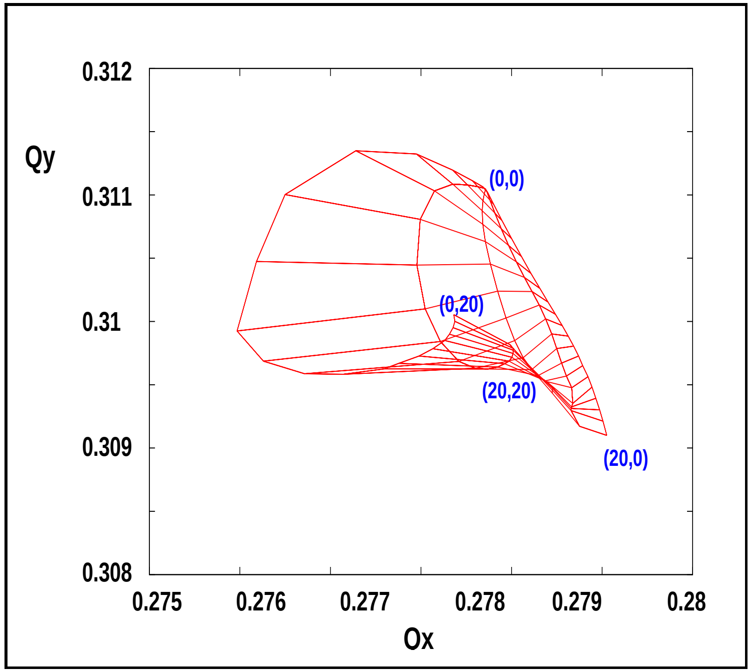


force for 2D ⇨ $$b_{x,y}(x,y) = \frac{x,y}{x^2 + y^2} \cdot \left(1 - \exp\left(\frac{-(x^2+y^2)}{2\sigma^2}\right)\right)$$

- **Without beam-beam: all particles at the same tune (at X)**
- **With beam-beam: all particles have a different tune !**

Here for a single collision, LHC has many ...

It can be worse:



Beam – beam with offset beams

(so – called "Long Range" interactions)

Very different behaviour

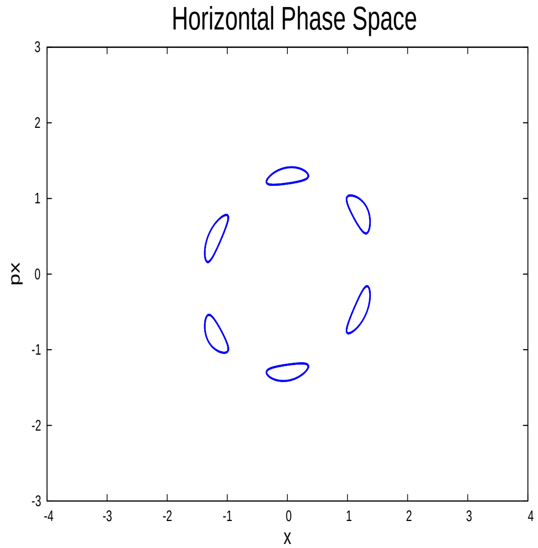
Here calculated for 1 interaction

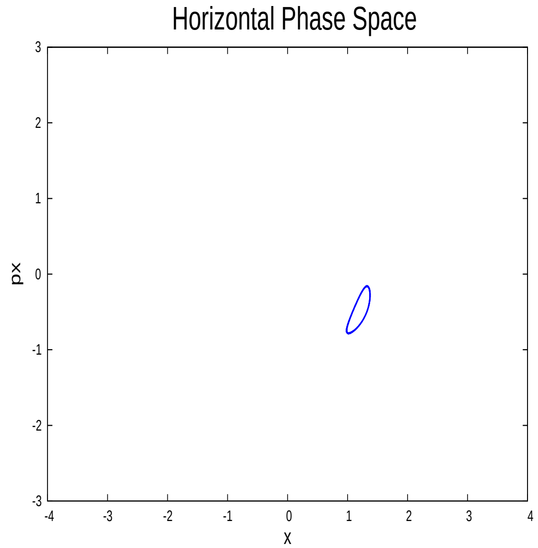
(LHC has 120(!) of them)

Analysis of the h_{eff} allows relevant predictions and optimization, e.g.

W.Herr, D. Kaltchev, "Analysis of long range studies in the LHC"**, in ICFA beam-beam workshop, CERN-2014-004**

What about particle on resonance (beam-beam again):





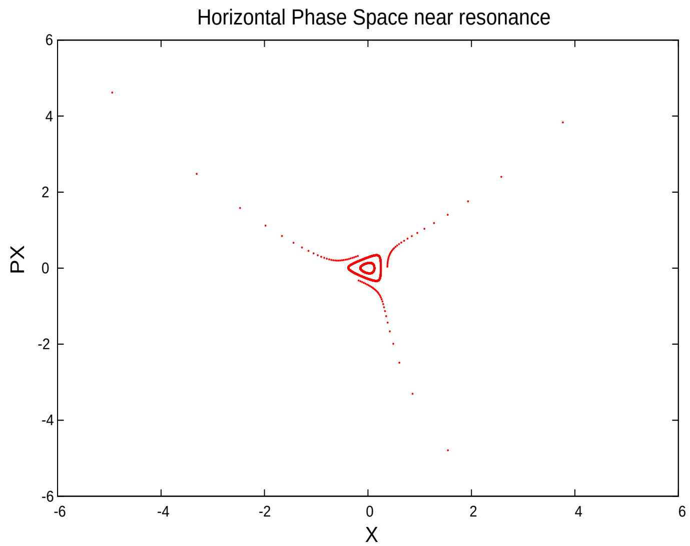
Particle "jumps" from one island to the next each turn, i.e. move fast, big jumps

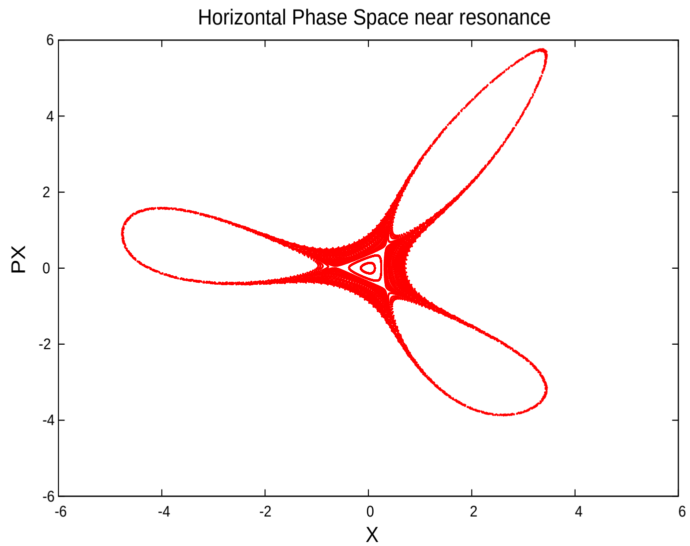
Stroboscopic analysis: use only every nth turn (6th in this example)

→ **Particle moves slowly around the (now lonely) island**

→ **Can be analysed (very involved, for a simple example see [AC1])**

Are nonlinear effects always bad ??





- **Left: close to 3rd order resonance with sextupole, particles are lost (or extracted)**
- **Right: close to 3rd order resonance with sextupole <u>and</u> octupole**
- **Octupole has stabilizing effect due to strong detuning**

Is it always bad ??

Landau Damping

- Octupole or space charge or beam-beam (!) introduce large tune spread
- Tune spread within beam suppresses coherent beam oscillations (Landau Damping)
- Tune spread from Normal Form analysis allows to compute the Stability Diagram
- Stability Diagram determines optimal operating conditions, maximum intensity, maximum allowed impedance
- Ion (e.g. proton) storage rings <u>cannot</u> work without Landau Damping (e.g. LHC relies on it)

Many nonlinear elements

Assume: $\mathcal{M} = e^{-:\mu J_x:} e^{:f_3:} e^{:f_4:} e^{:f_B:} e^{:f_x:}$

The map can be (most of the time) factorized

Since we get an analytical expression for h_{eff}, we can insert a "correction element" f_x

Examples:

- **Chromaticity correction with sextupoles**
- **Final focus linear collider**

Putting it together

Conventional tools and methods fail for nonlinear (i.e. realistic) systems

But we can provide a suitable framework for complex systems

- **The main steps needed:**
  - **Get the (linear or nonlinear) map from the Hamiltonian**
  - **Lie maps are the natural extension from linear to nonlinear dynamics**
  - **Always symplectic and allow analytical solutions**
  - **Normal Form analysis to obtain all relevant properties**
- **Recommendation:**
  - **Without Hamiltonians you can do linear dynamics, but completely fail for nonlinear effects**
  - **Right from the start use an approach which leads automatically into the application of advanced concepts and methods**