Lectures on Partial Differential Equations

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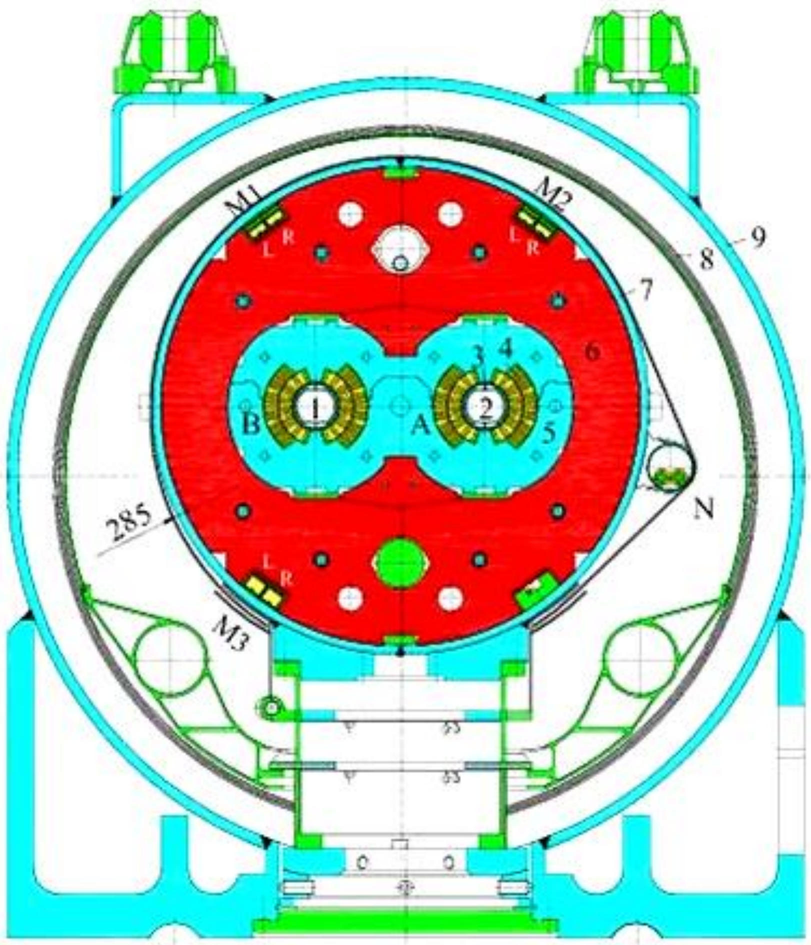
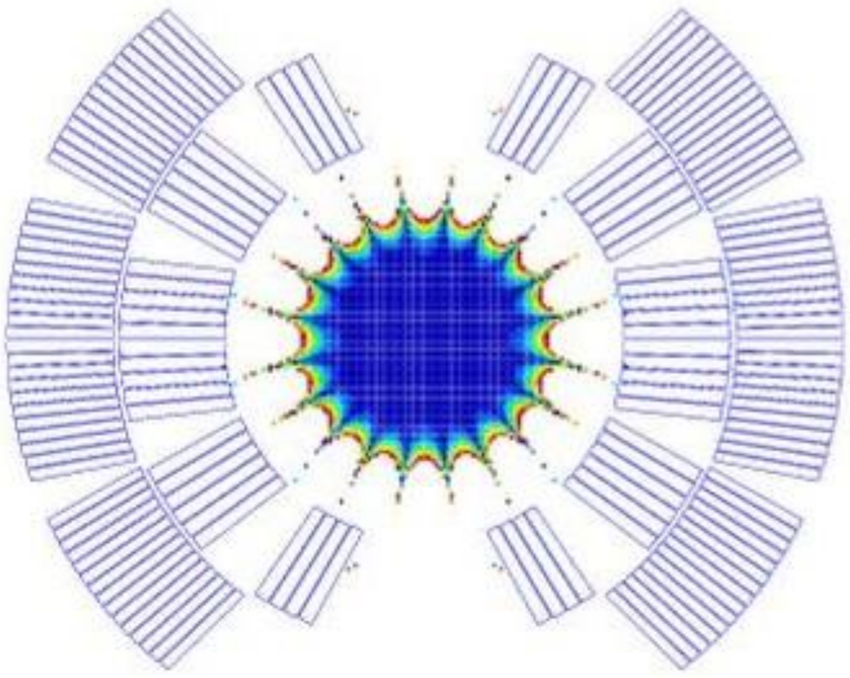
Field Singularities
FEM-BEM

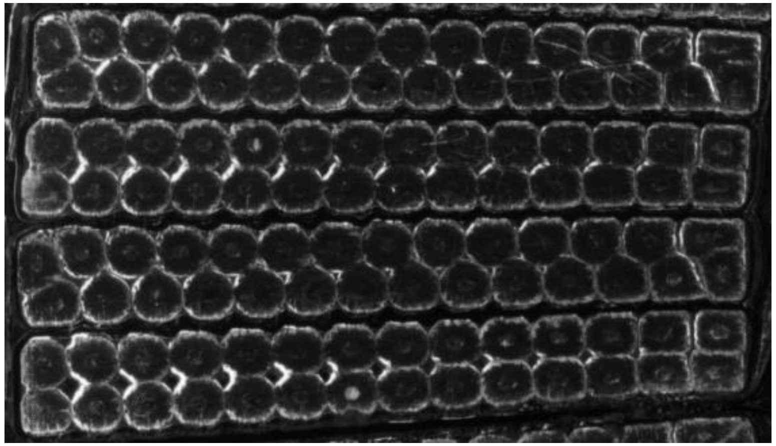
The Green's Functions
Green's Theorems

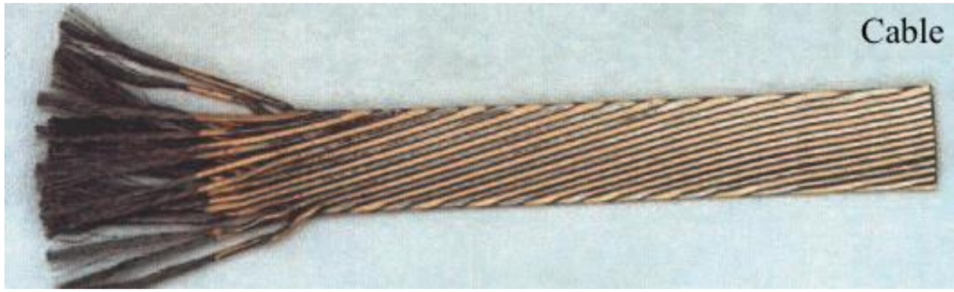
Cross-section of Cryodipole

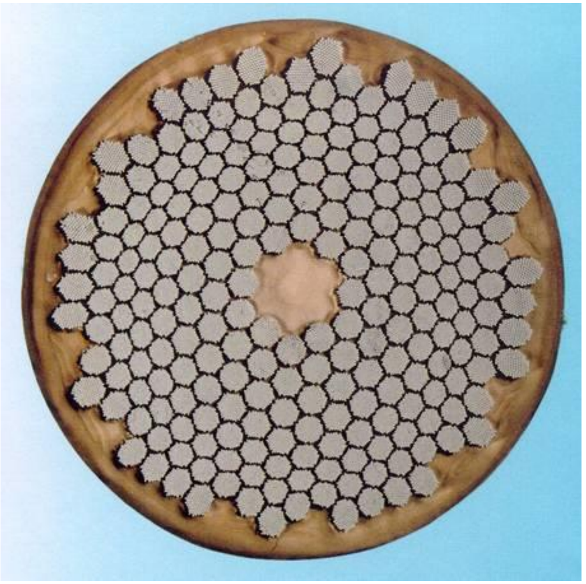




Rutherford (Roebel) Kabel, Strand, Nb-Ti Filament



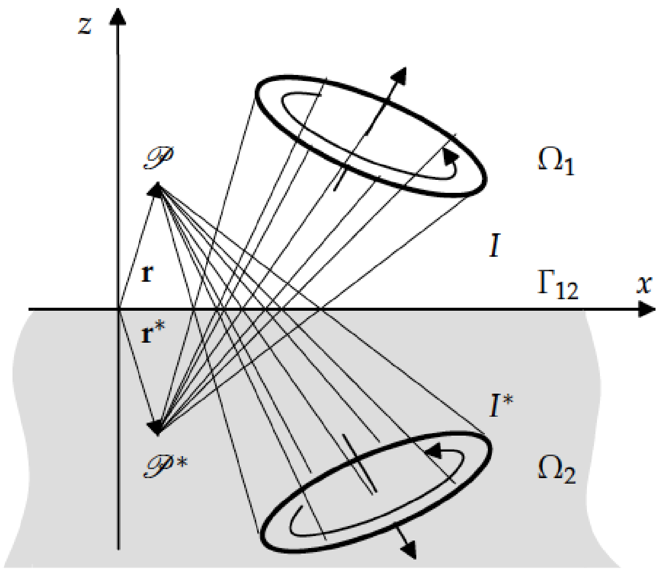




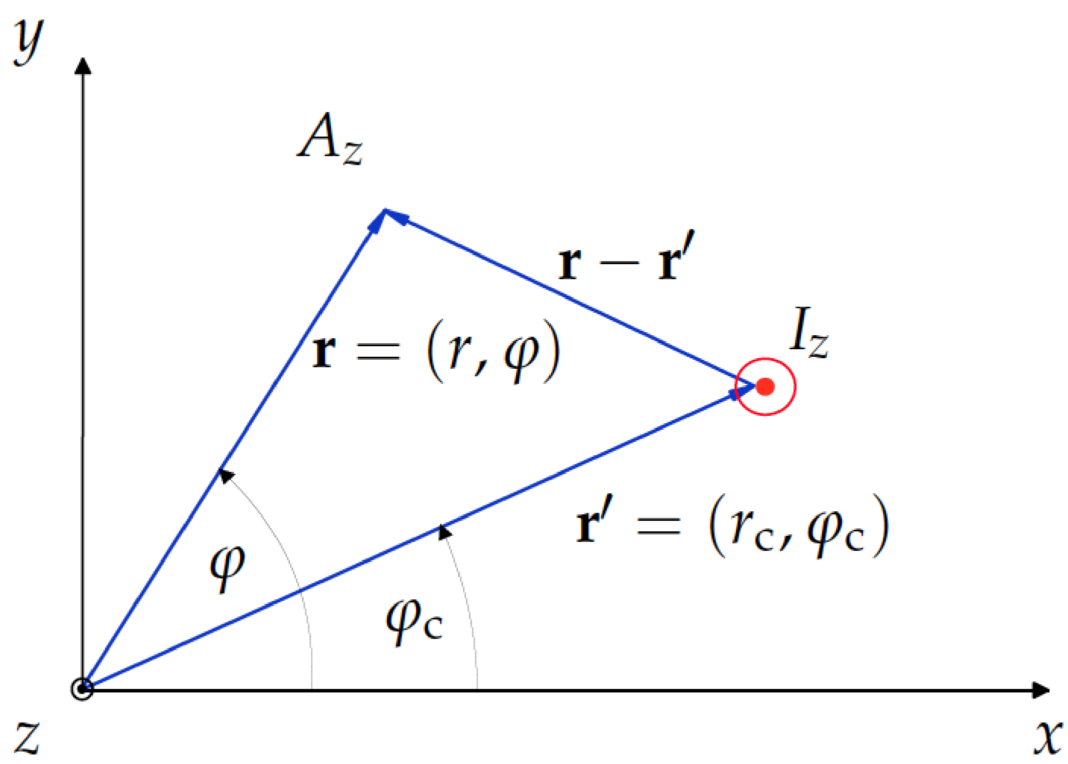
The Field of Line Currents

$$\mathbf{r} \mapsto \phi(|\mathbf{r} - \mathbf{r}'|)$$
$$\mathbf{r}' \mapsto \phi(|\mathbf{r} - \mathbf{r}'|)$$

$$\operatorname{grad} \phi(|\mathbf{r} - \mathbf{r}'|) = -\operatorname{grad}_{\mathbf{r}'} \phi(|\mathbf{r} - \mathbf{r}'|),$$
$$\operatorname{div} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|) = -\operatorname{div}_{\mathbf{r}'} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|),$$
$$\operatorname{curl} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|) = -\operatorname{curl}_{\mathbf{r}'} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|),$$
$$\nabla^2 \phi(|\mathbf{r} - \mathbf{r}'|) = \nabla^2_{\mathbf{r}'} \phi(|\mathbf{r} - \mathbf{r}'|).$$



Why bother? Reciprocity; except for sign it does not matter if we exchange the source and field points



Greens Functions of Free Space

$$\mathcal{L}_{\mathbf{r}'}\phi(\mathbf{r}') = -f(\mathbf{r}')$$

$$\mathcal{L}_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'),$$

$$\int_{\mathcal{V}} \mathcal{L}_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}')\, f(\mathbf{r})\mathrm{d}V = -\int_{\mathcal{V}} \delta(\mathbf{r}-\mathbf{r}')f(\mathbf{r})\mathrm{d}V = -f(\mathbf{r}').$$

$$\mathcal{L}_{\mathbf{r}'}\phi(\mathbf{r}') = \int_{\mathcal{V}} \mathcal{L}_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}')f(\mathbf{r})\mathrm{d}V = \mathcal{L}_{\mathbf{r}'}\int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')f(\mathbf{r})\mathrm{d}V,$$

$$\phi(\mathbf{r}') = \int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')f(\mathbf{r})\mathrm{d}V.$$

$$G_2(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{r_{\mathrm{ref}}}\right), \qquad G_3(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Green's Functions of Free Space

$$\phi(\mathbf{r}') = \int_{\mathscr{V}} G(\mathbf{r},\mathbf{r}') f(\mathbf{r}) \mathrm{d}V \,.$$

$$\phi(\mathbf{r}) = \int_{\mathscr{V}} G(\mathbf{r},\mathbf{r}') f(\mathbf{r}') \mathrm{d}V' \,.$$

$$\int_\Omega \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) \, \mathrm{d}V = \int_\Gamma \left(\phi \partial_{\mathbf{n}} \psi - \psi \partial_{\mathbf{n}} \phi\right) \, \mathrm{d}a$$

But what if boundaries are present?
Use Green's second identity (integration by parts)

$$\phi(\mathbf{r}) = \int_{\mathscr{V}} G(\mathbf{r},\mathbf{r}') f(\mathbf{r}') \mathrm{d}V' + \int_{\partial\mathscr{V}} \Big(-\phi(\mathbf{r}') \, \partial_{\mathbf{n}'} G(\mathbf{r},\mathbf{r}') + G(\mathbf{r},\mathbf{r}') \, \partial_{\mathbf{n}'} \phi(\mathbf{r}') \Big) \mathrm{d}a' \,.$$

Surface current

Surface density of dipole moments

Biot-Savart's Law

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \qquad G_3(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V',$$

$$\mathbf{A}(\mathbf{r}) = A_x\mathbf{e_x} + A_y\mathbf{e_y} + A_z\mathbf{e_z} = \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'.$$

This works only in Cartesian Coordinates

$$\mathbf{B}(\mathbf{r}) = \operatorname{curl}\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \operatorname{curl}\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V'$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \frac{1}{|\mathbf{r}-\mathbf{r}'|}\sum_{k=1}^{3} J_k(\mathbf{r}')(\mathbf{e}_i(\mathbf{r})\cdot\mathbf{e}_k(\mathbf{r}'))\,\mathrm{d}V'. \quad \mathrm{d}V'$$

$$= \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\mathrm{d}V'.$$

Biot Savart' s Law

But wait a minute: Are we finished? Are we sure that the divergence of the vector potential is zero as it was required for the Laplace equation?

$$
\begin{aligned}
\operatorname{div}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \operatorname{div}\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \left(\mathbf{J}(\mathbf{r}')\cdot\operatorname{grad}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) + \frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{div}\mathbf{J}(\mathbf{r}')\right)\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \mathbf{J}(\mathbf{r}')\cdot\operatorname{grad}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi}\int_{\mathscr{V}} \mathbf{J}(\mathbf{r}')\cdot\operatorname{grad}_{\mathbf{r}'}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi}\int_{\mathscr{V}} \left(\operatorname{div}_{\mathbf{r}'}\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{div}_{\mathbf{r}'}\mathbf{J}(\mathbf{r}')\right)\mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi}\int_{\mathscr{V}} \operatorname{div}_{\mathbf{r}'}\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' = -\frac{\mu_0}{4\pi}\int_{\partial\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\cdot\mathrm{d}a'\,.
\end{aligned}
$$

Current loops must always be closed and must not leave the problem domain

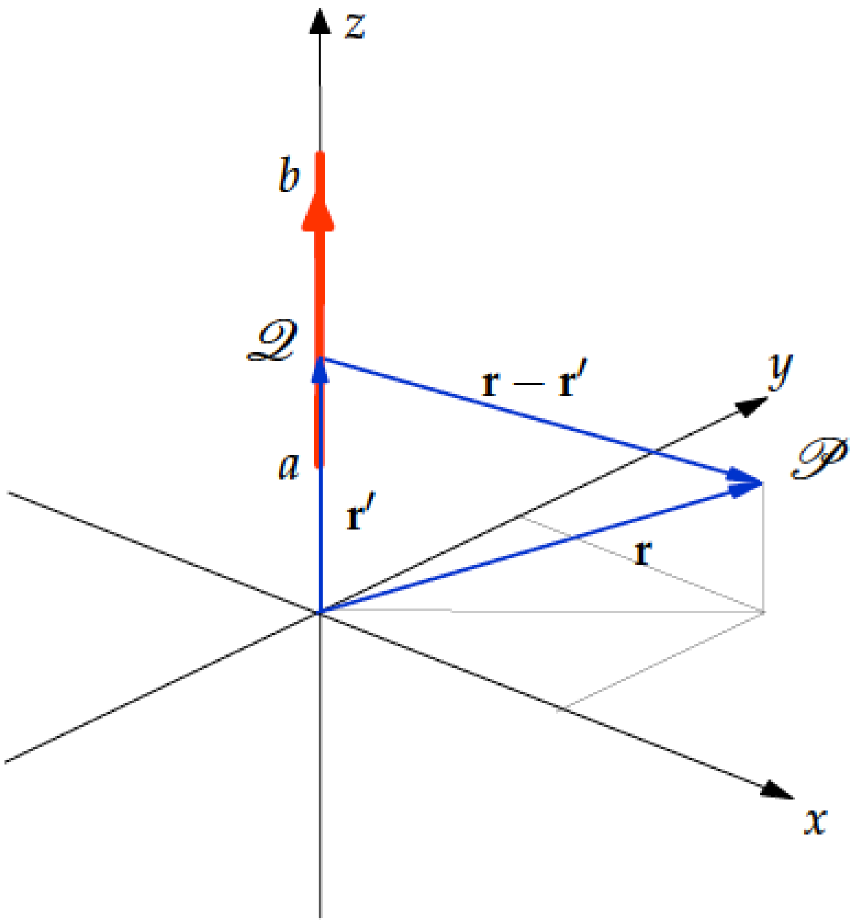
Biot-Savart's Law for Line Currents

$$\mathbf{A}(\mathbf{r}) = A_x\mathbf{e_x} + A_y\mathbf{e_y} + A_z\mathbf{e_z} = \frac{\mu_0}{4\pi}\int_{\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'.$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_{\mathscr{C}} \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_{\mathscr{C}} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3},$$

Vector Potential of a Line Current



$$A_z(x,y,z) = \frac{\mu_0 I}{4\pi}\int_a^b \frac{\mathrm{d}z_c}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 I}{4\pi}\int_a^b \frac{\mathrm{d}z_c}{\sqrt{x^2+y^2+(z-z_c)^2}}$$

$$\frac{-\mu_0 I}{4\pi}\ln\left((z-z_c)+\sqrt{x^2+y^2+(z-z_c)^2}\right)\Bigg|_a^b$$

$$\frac{\mu_0 I}{4\pi}\ln\frac{z-a+\sqrt{x^2+y^2+(z-a)^2}}{z-b+\sqrt{x^2+y^2+(z-b)^2}}\,.$$

Field of a Line Current (Infinitely Long)

$$\lim_{a,b\to\pm\infty} \ln \frac{z-a+\sqrt{x^2+y^2+(z-a)^2}}{z-b+\sqrt{x^2+y^2+(z-b)^2}} = \lim_{a,b\to\pm\infty} \ln \frac{-a+|a|\sqrt{1+\frac{x^2+y^2}{a^2}}}{-b+|b|\sqrt{1+\frac{x^2+y^2}{b^2}}}$$

$$= \lim_{a,b\to\pm\infty} \ln \frac{-a-a(1+\frac{x^2+y^2}{2a^2}+\cdots)}{-b+b(1+\frac{x^2+y^2}{2b^2}+\cdots)} = \lim_{a,b\to\pm\infty} \ln \frac{-2a}{-b+b+\frac{x^2+y^2}{2b}}$$

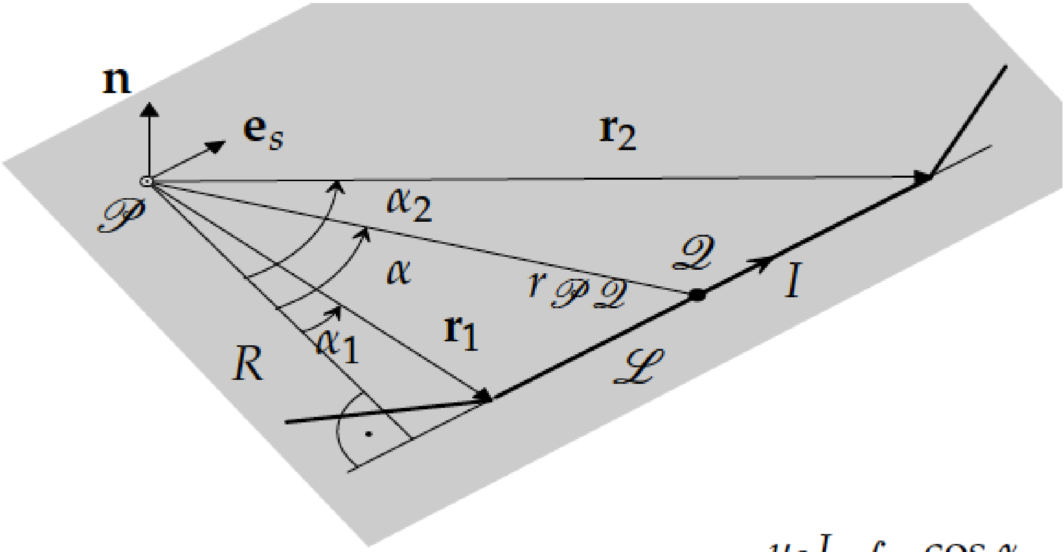
$$= \lim_{a,b\to\pm\infty} \ln \frac{-4ab}{x^2+y^2} .$$

$$A_z(x,y) = \lim_{a,b\to\pm\infty} \frac{\mu_0 I}{4\pi} \ln\left(\frac{-4ab}{x_0^2+y_0^2}\right) - \frac{\mu_0 I}{4\pi} \ln\left(\frac{x^2+y^2}{x_0^2+y_0^2}\right) .$$

Arbitrarily large but constant

$$\mathbf{A}(x,y) = -\frac{\mu_0 I}{4\pi} \ln\left(\frac{x^2+y^2}{x_0^2+y_0^2}\right) \mathbf{e}_z = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{r}{r_{\text{ref}}}\right) \mathbf{e}_z ,$$

Field of a Line Current Segment



$$\mathbf{B}(\mathscr{P}) = \frac{\mu_0 I}{4\pi}\int_{\mathscr{L}} \frac{\cos\alpha}{r^2_{\mathscr{P}\mathscr{Q}}}\mathrm{d}r' = \frac{\mu_0 I}{4\pi R}\mathbf{n}\int_{\alpha_1}^{\alpha_2}\cos\alpha\,\mathrm{d}\alpha = \frac{\mu_0 I}{4\pi R}(\sin\alpha_2 - \sin\alpha_1)\,\mathbf{n}$$

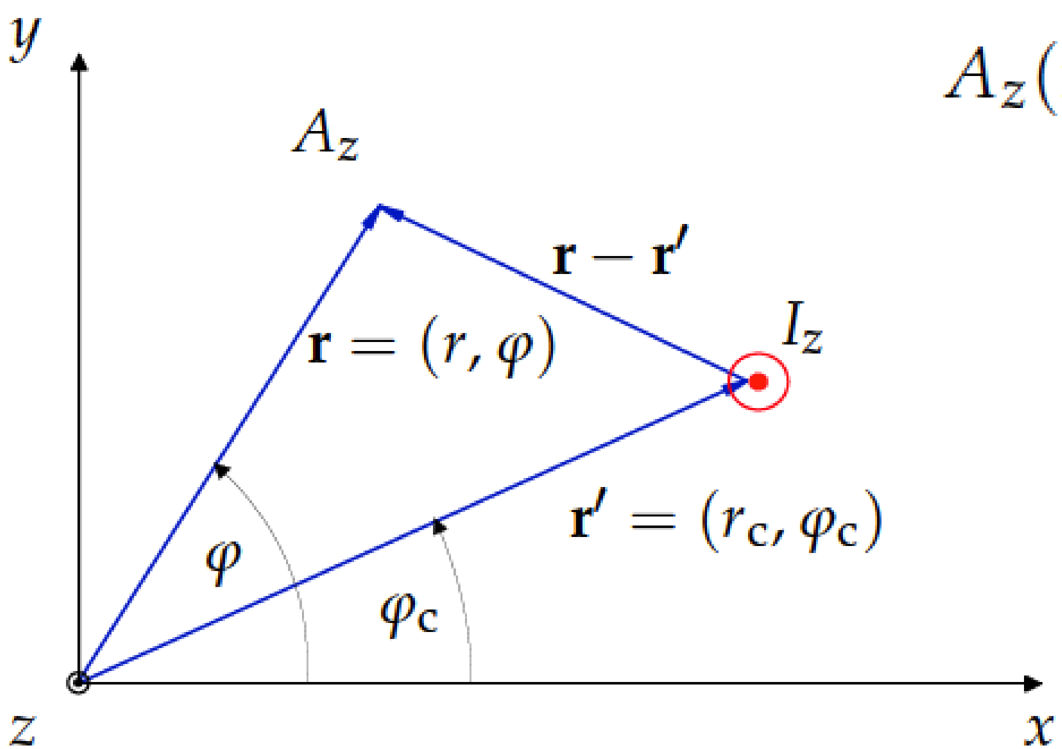
$$= \frac{\mu_0 I}{4\pi}\,\frac{\cos\alpha_2 + \cos\alpha_1}{R}\,\frac{\sin\alpha_2 - \sin\alpha_1}{\cos\alpha_2 + \cos\alpha_1}\,\mathbf{n}$$

$$= \frac{\mu_0 I}{4\pi}\left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|}\right)\frac{\sin(\alpha_2 - \alpha_1)}{1 + \cos(\alpha_2 - \alpha_1)}\,\mathbf{n}$$

$$= \frac{\mu_0 I}{4\pi}\left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|}\right)\frac{\sin(\alpha_2 - \alpha_1)}{1 + \frac{\mathbf{r}_1\cdot\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}}\,\frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|\sin(\alpha_2 - \alpha_1)}$$

$$= \frac{\mu_0 I}{4\pi}\,\frac{|\mathbf{r}_1| + |\mathbf{r}_2|}{|\mathbf{r}_1|\,|\mathbf{r}_2| + \mathbf{r}_1\cdot\mathbf{r}_2}\,\frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}\,,$$

Expanding the Green's Function

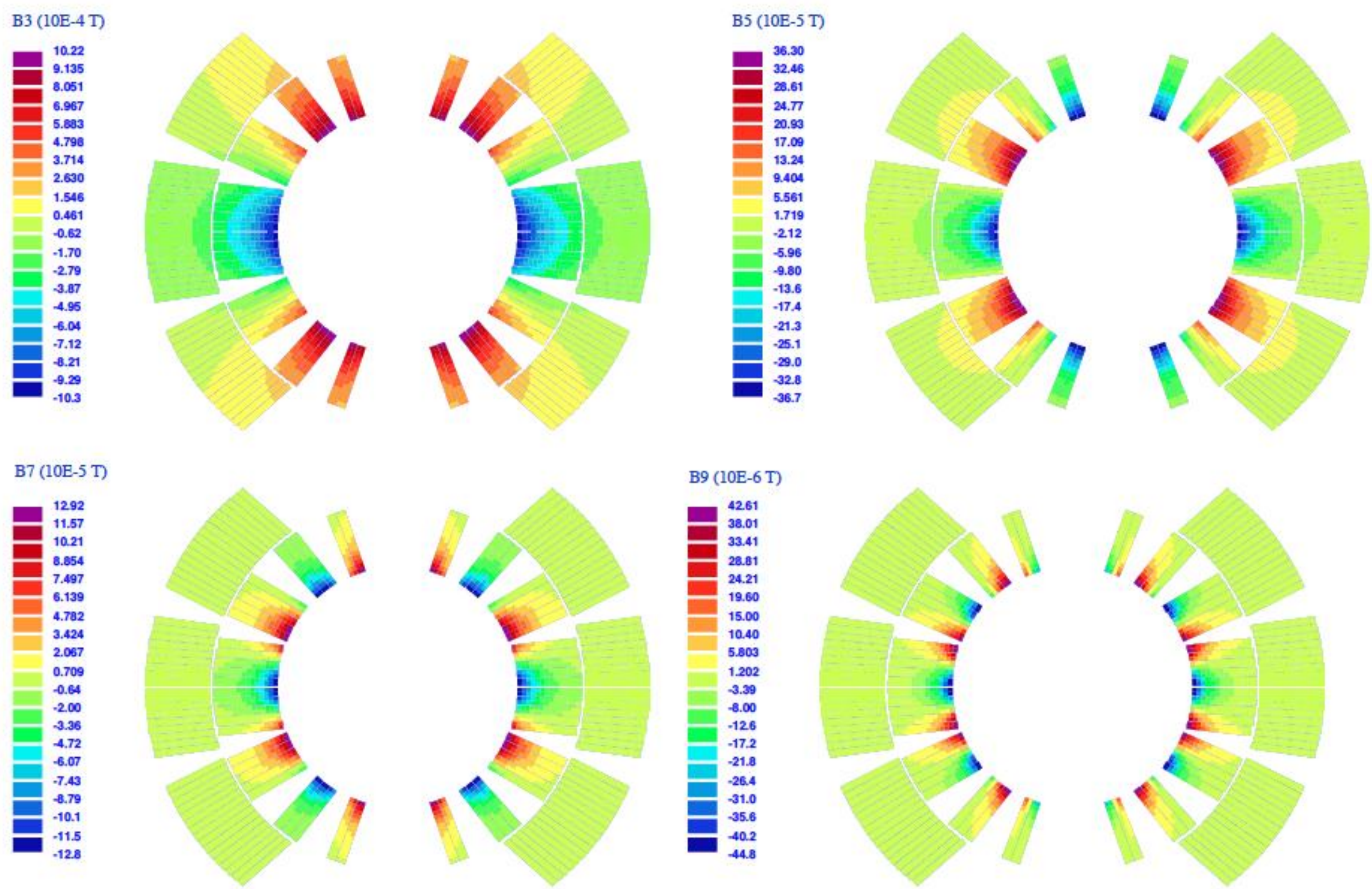


$$A_z(\mathbf{r}) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{r_{\mathrm{ref}}}\right)$$

$$A_z(r,\varphi) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{r_c}{r_{\mathrm{ref}}}\right) + \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{r_c}\right)^n \cos n(\varphi-\varphi_c)$$

$$B_n(r_0) = -\frac{\mu_0 I}{2\pi r_c}\left(\frac{r_0}{r_c}\right)^{n-1}\cos n\varphi_c, \qquad A_n(r_0) = \frac{\mu_0 I}{2\pi r_c}\left(\frac{r_0}{r_c}\right)^{n-1}\sin n\varphi_c.$$

Expanding the Green's Function II

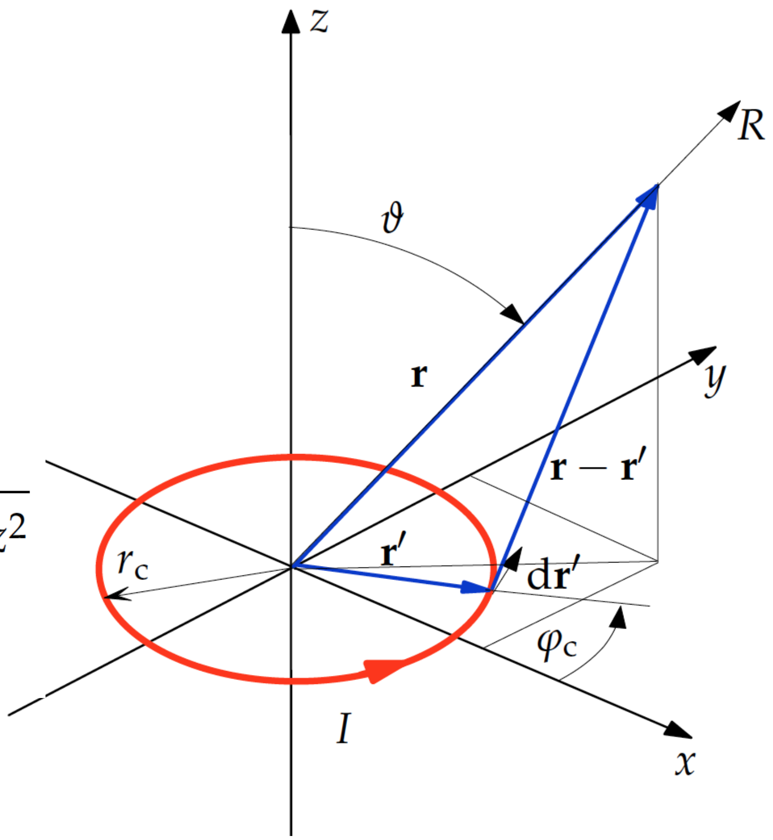


Field of a Ring Current

$$\mathbf{r}' = \cos\varphi_c r_c\,\mathbf{e}_x + \sin\varphi_c r_c\,\mathbf{e}_y$$

$$\mathrm{d}\mathbf{r}' = -\sin\varphi_c r_c\,\mathrm{d}\varphi_c \mathbf{e}_x + \cos\varphi_c r_c\,\mathrm{d}\varphi_c \mathbf{e}_y$$

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'| &= \sqrt{(x-x_c)^2+(y-y_c)^2+z^2}\\ &= \sqrt{(r\cos\varphi - r_c\cos\varphi_c)^2 + (r\sin\varphi - r_c\sin\varphi_c)^2 + z^2}\\ &= \sqrt{r^2+r_c^2+z^2-2rr_c\cos\varphi_c}\,,\end{aligned}$$

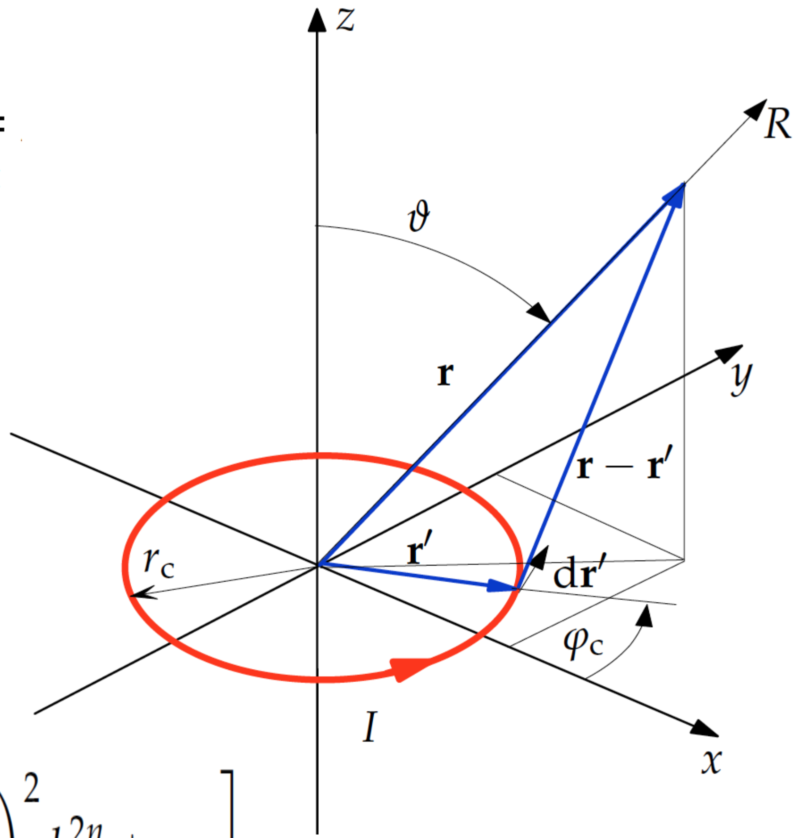


Field of a Ring Current

$$A_y(r,z) = \frac{\mu_0 I r_c}{2\pi}\int_0^{\pi} \frac{\cos\varphi_c \,\mathrm{d}\varphi_c}{\sqrt{r^2 + r_c^2 + z^2 - 2 r r_c \cos\varphi_c}}$$

$$\psi := (\pi + \varphi_c)/2, \qquad k^2 := \frac{4 r r_c}{(r + r_c)^2 + z^2}$$

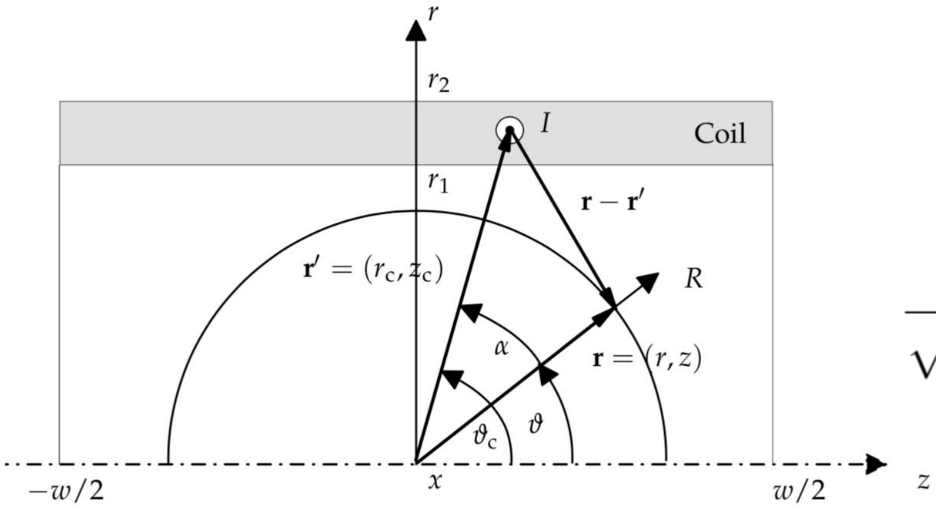
$$A_\varphi(r,z) = \frac{\mu_0 I r_c}{\pi\sqrt{(r+r_c)^2 + z^2}}\int_0^{\pi/2} \frac{2\sin^2\psi - 1}{\sqrt{1 - k^2 \sin^2\psi}}\,\mathrm{d}\psi$$



$$K\left(\frac{\pi}{2},k\right) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots + \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 k^{2n} + \cdots\right]$$

$$A_\varphi(r,z) = \frac{\mu_0 I}{2\pi r}\sqrt{(r+r_c)^2 + z^2}\left[\left(1 - \frac{k^2}{2}\right)K\left(\frac{\pi}{2},k\right) - E\left(\frac{\pi}{2},k\right)\right]$$

Expanding the Green's Function



$$A_\varphi(R,\vartheta) = \sum_{n=1}^{\infty} \mu_0 \mathcal{A}_n R^n P_n^1(\cos\vartheta)\,,$$

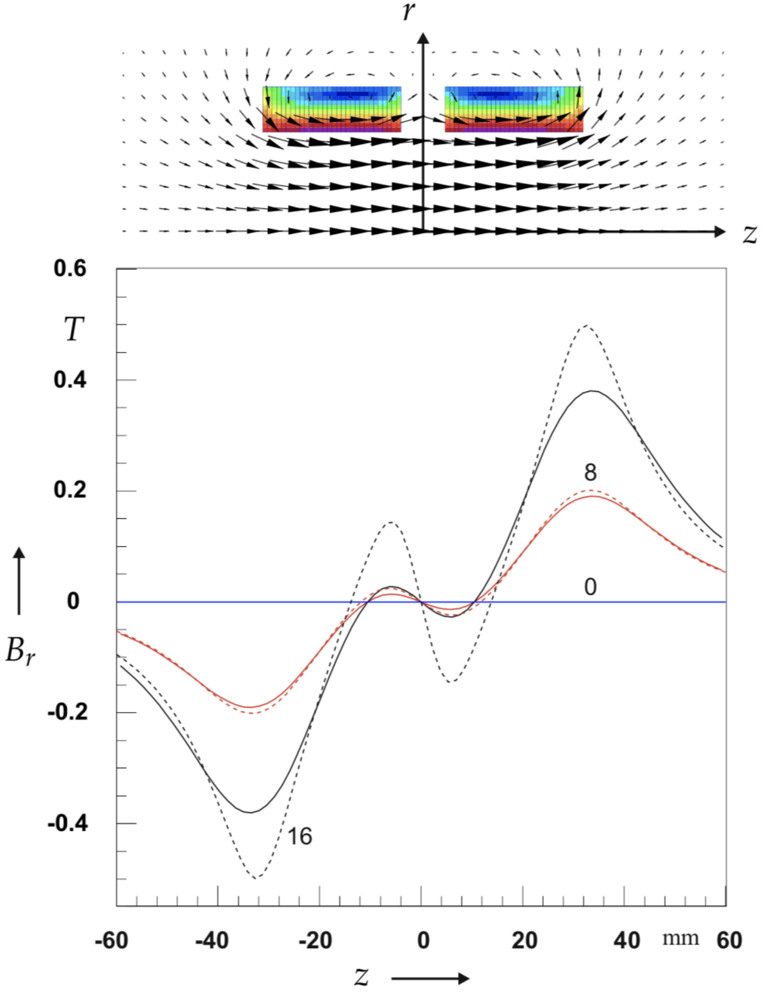
$$\frac{1}{\sqrt{|\mathbf{r}|^2 + |\mathbf{r}'|^2 - 2|\mathbf{r}||\mathbf{r}'|\cos\alpha}} = \frac{1}{|\mathbf{r}'|}\sum_{n=0}^{\infty}\left(\frac{|\mathbf{r}|}{|\mathbf{r}'|}\right)^n P_n(\cos\alpha)$$

$$\begin{aligned} A_\varphi &= \frac{\mu_0 I r_c}{2\pi}\int_0^\pi \frac{\cos\varphi_c \mathrm{d}\varphi_c}{\sqrt{r^2 + r_c^2 + (z-z_c)^2 - 2 r r_c \cos\varphi_c}} \\ &= \frac{\mu_0 I r_c}{2\pi}\int_0^\pi \frac{\cos\varphi_c \mathrm{d}\varphi_c}{\sqrt{|\mathbf{r}|^2 + |\mathbf{r}'|^2 - 2|\mathbf{r}||\mathbf{r}'|(\cos\vartheta\cos\vartheta_c + \sin\vartheta\sin\vartheta_c\cos\varphi_c)}} \\ &= \frac{\mu_0 I r_c}{2}\frac{1}{|\mathbf{r}'|}\sum_{n=1}^{\infty}\left(\frac{|\mathbf{r}|}{|\mathbf{r}'|}\right)^n \frac{(n-1)!}{(n+1)!} P_n^1(\cos\vartheta) P_n^1(\cos\vartheta_c)\,. \end{aligned}$$

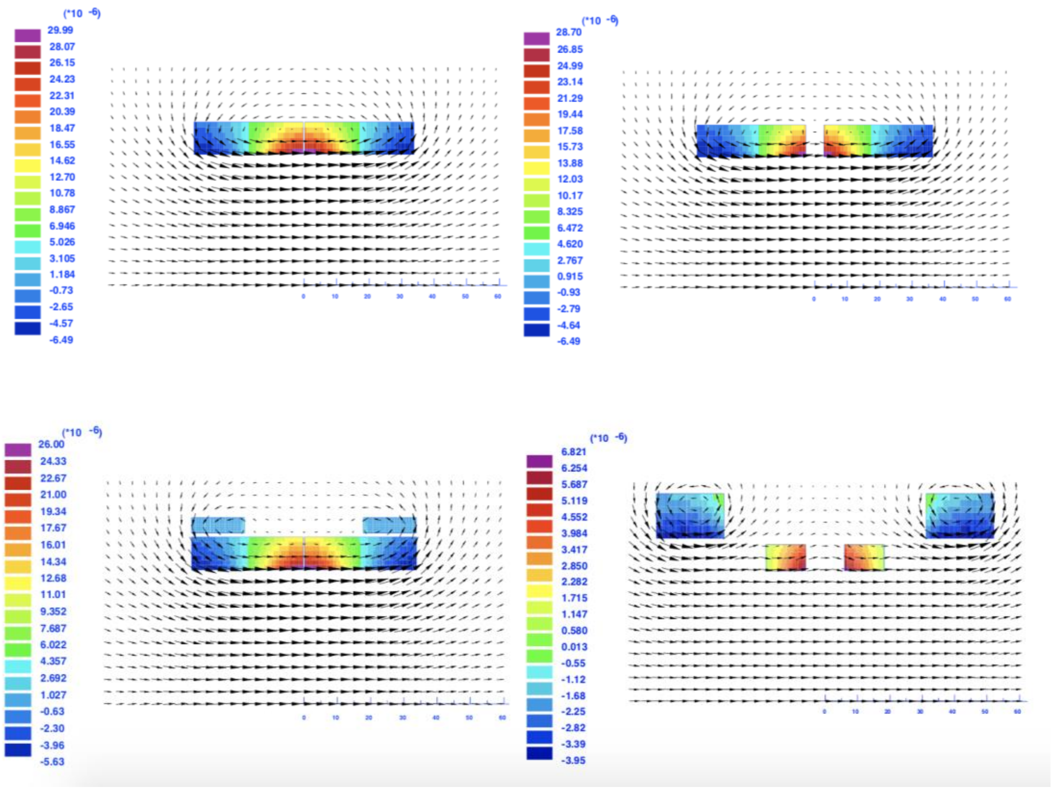
$$\mathcal{A}_n = \frac{I r_c}{2}\frac{1}{R_c^{n+1}}\frac{1}{n(n+1)} P_n^1(\cos\vartheta_c)$$

Split-Coil Solenoids

Field approximation up to first order (at different radii)



Optimization of the field homogeneity (suppressing the 3rd zonal harmonic)



Magnetic Dipole Moment

Far field approximation

$$A_\varphi(R,\vartheta) \approx \frac{\mu_0 I r_c^2 \pi}{4\pi} \frac{\sin\vartheta}{R^2} = \frac{\mu_0 m}{4\pi} \frac{\sin\vartheta}{R^2},$$

$$R = \sqrt{r^2 + z^2} \text{ and } \sin\vartheta = r/R,$$

$$[m] = 1\,\mathrm{A\,m^2}.$$

Definition

$$m := I r_c^2 \pi$$

$$\mathbf{m} = I\mathbf{a},$$

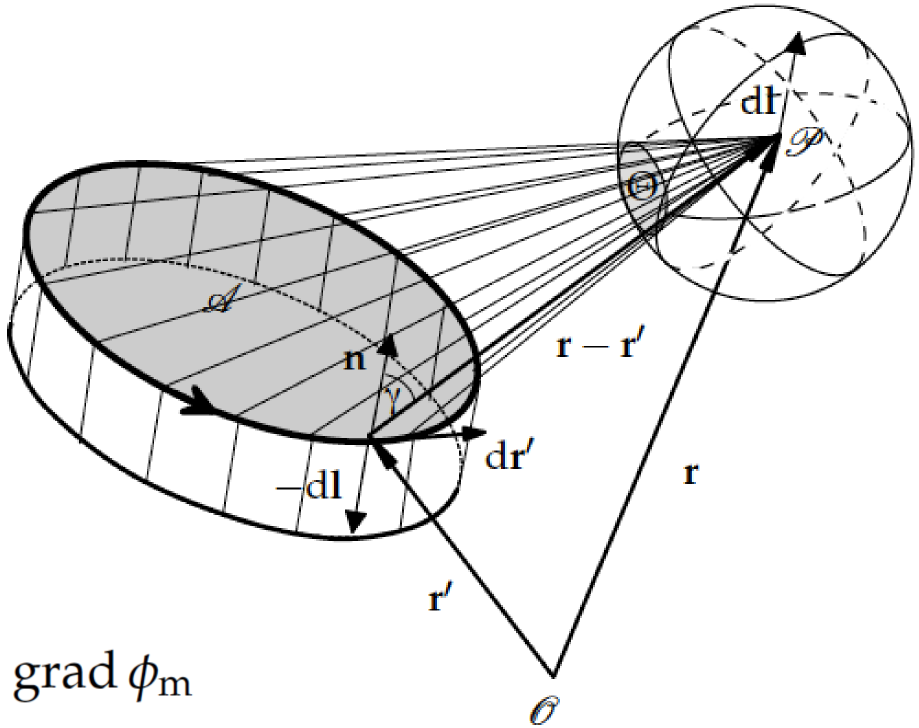
$$\mathbf{m} = \frac{I}{2}\int_{\mathscr{C}} \mathbf{r} \times \mathrm{d}\mathbf{r},$$

$$\mathbf{M}(\mathbf{r}) := \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}V} = \frac{1}{2}\mathbf{r} \times \mathbf{J}(\mathbf{r}),$$

Solid Angle and Magnetic Scalar Potential

$$\mathrm{d}\Theta = -\int_{\partial\mathscr{A}} \frac{1}{|\mathbf{r}-\mathbf{r}'|^2}(\mathrm{d}\mathbf{l}\times\mathrm{d}\mathbf{r}')\cdot\mathbf{e}_R = -\int_{\partial\mathscr{A}} \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\cdot(\mathrm{d}\mathbf{l}\times\mathrm{d}\mathbf{r}')$$

$$= -\mathrm{d}\mathbf{l}\int_{\partial\mathscr{A}} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,.$$



Expressing $\mathrm{d}\Theta$ as $\operatorname{grad}\Theta\cdot\mathrm{d}\mathbf{l}$

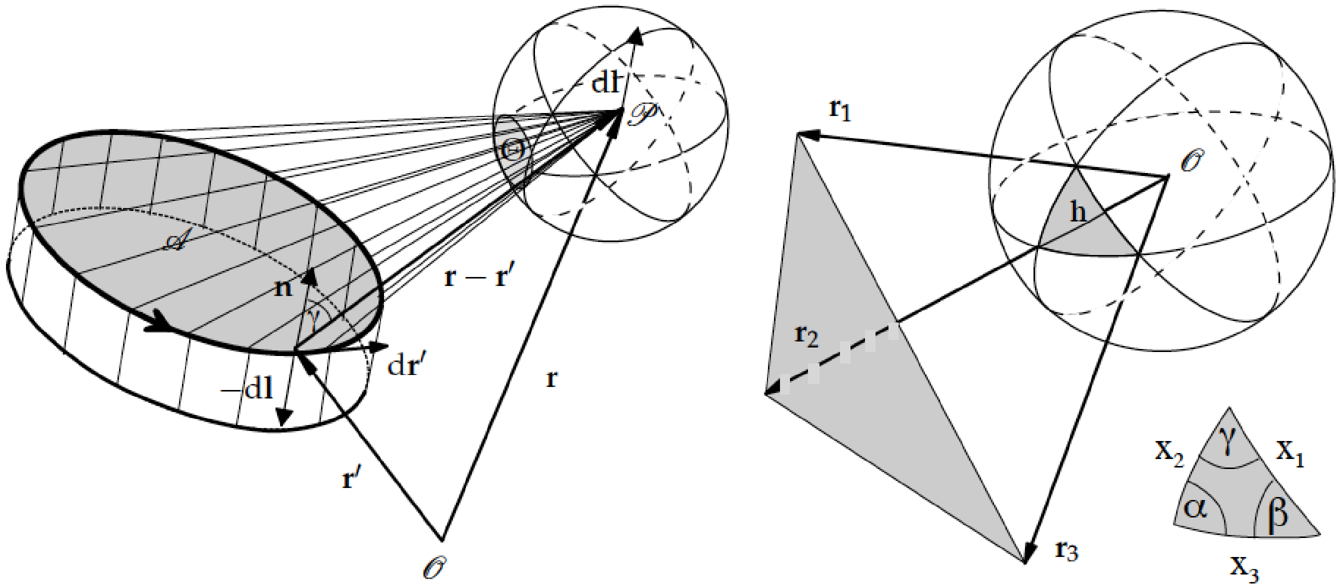
$$\operatorname{grad}\Theta = -\int_{\partial\mathscr{A}} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_{\partial\mathscr{A}_c} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \mu_0\mathbf{H} = -\mu_0\operatorname{grad}\phi_m$$

$$\phi_m(\mathbf{r}) = \frac{I}{4\pi}\Theta$$

Solid angle (easy to compute) yields the magnetic scalar potential of a current loop

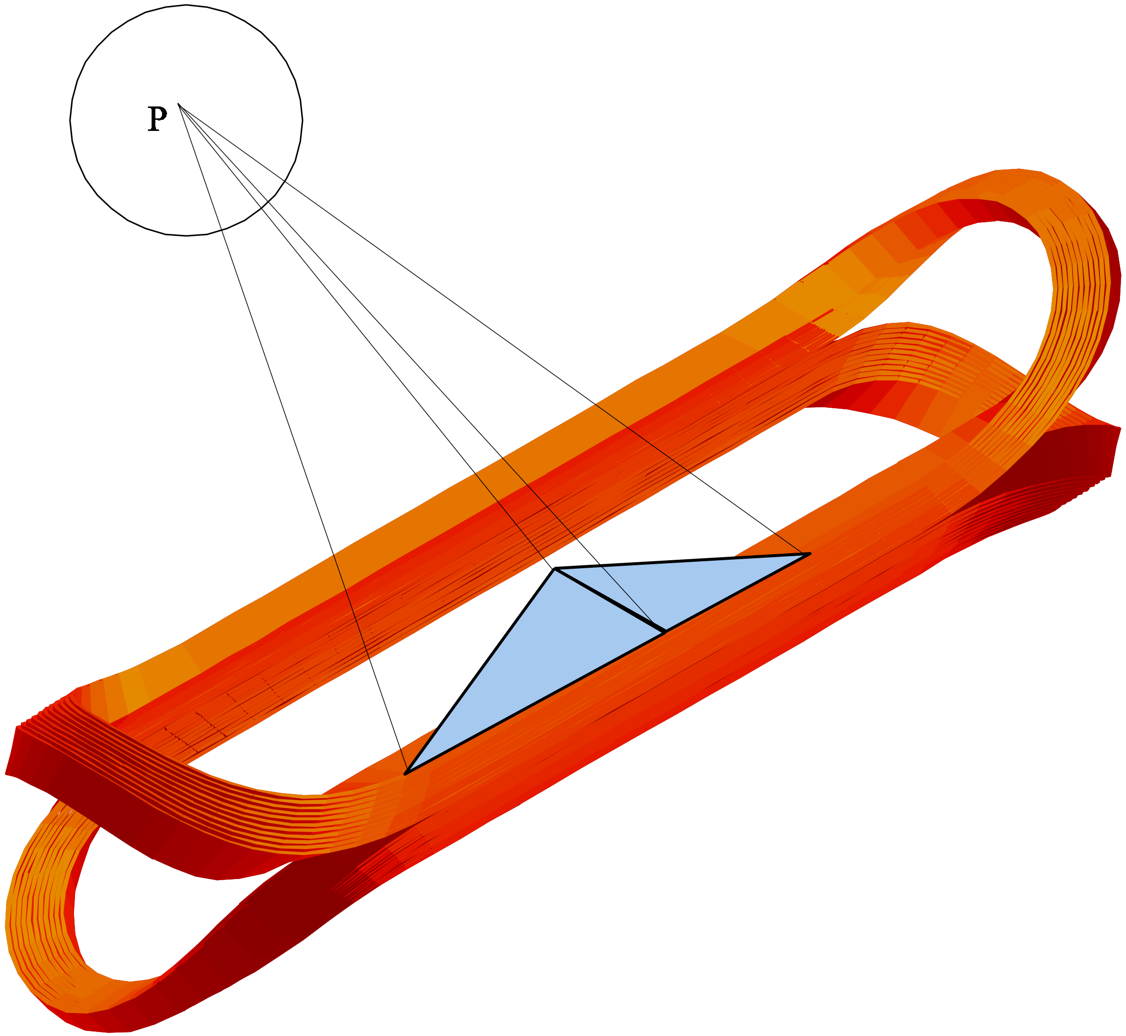
Solid Angle and Magnetic Scalar Potential



$$\Theta = \int_{\mathscr{A}} \frac{\cos\gamma}{R^2}\mathrm{d}a = \int_{\mathscr{A}} \frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|^3}\mathrm{d}a\,,$$

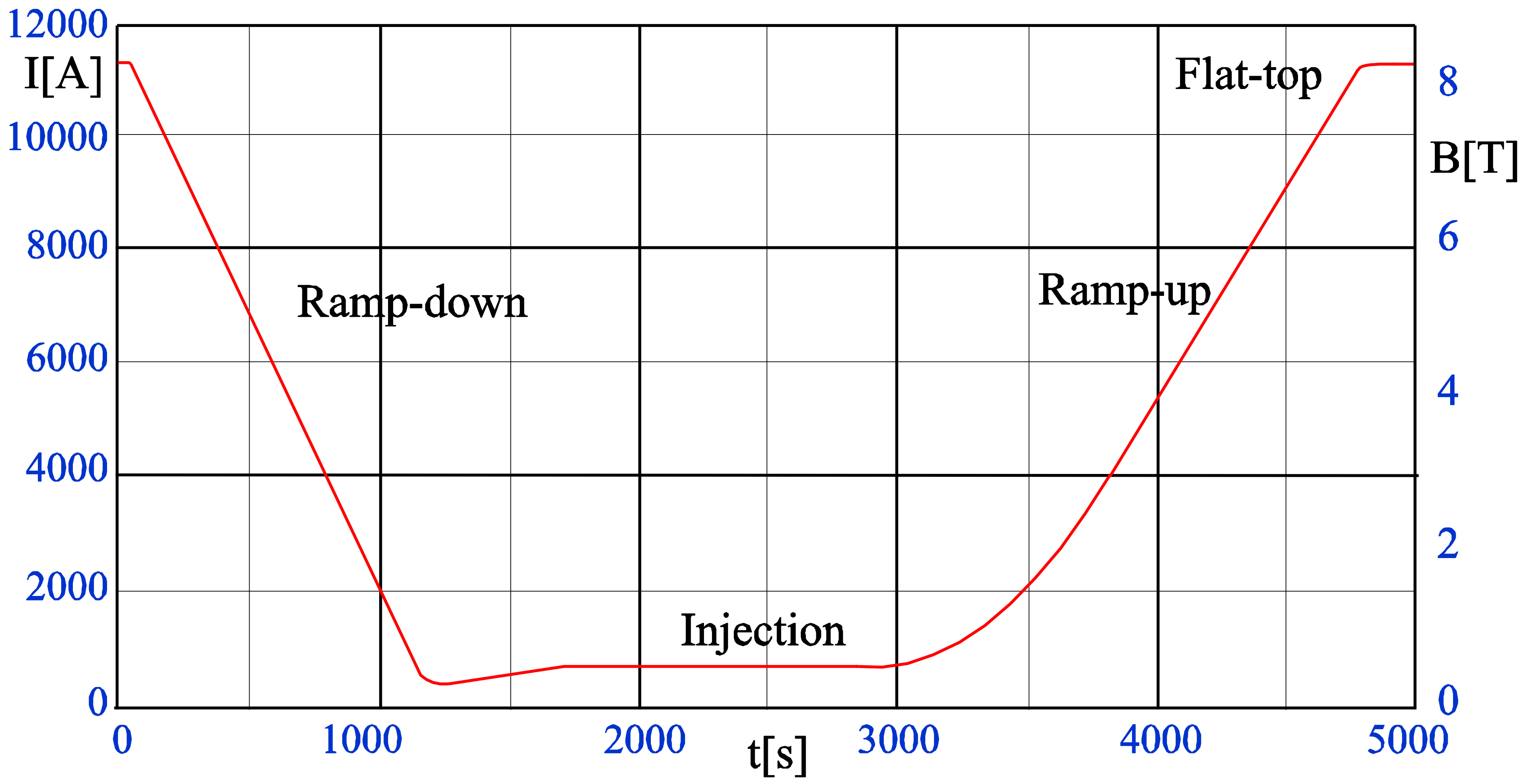
$$\tan\left(\frac{\Theta}{2}\right) = \frac{\mathbf{r}_1\cdot(\mathbf{r}_2\times\mathbf{r}_3)}{r_1 r_2 r_3 + (\mathbf{r}_1\cdot\mathbf{r}_2)r_3 + (\mathbf{r}_1\cdot\mathbf{r}_3)r_2 + (\mathbf{r}_2\cdot\mathbf{r}_3)r_1}\,.$$

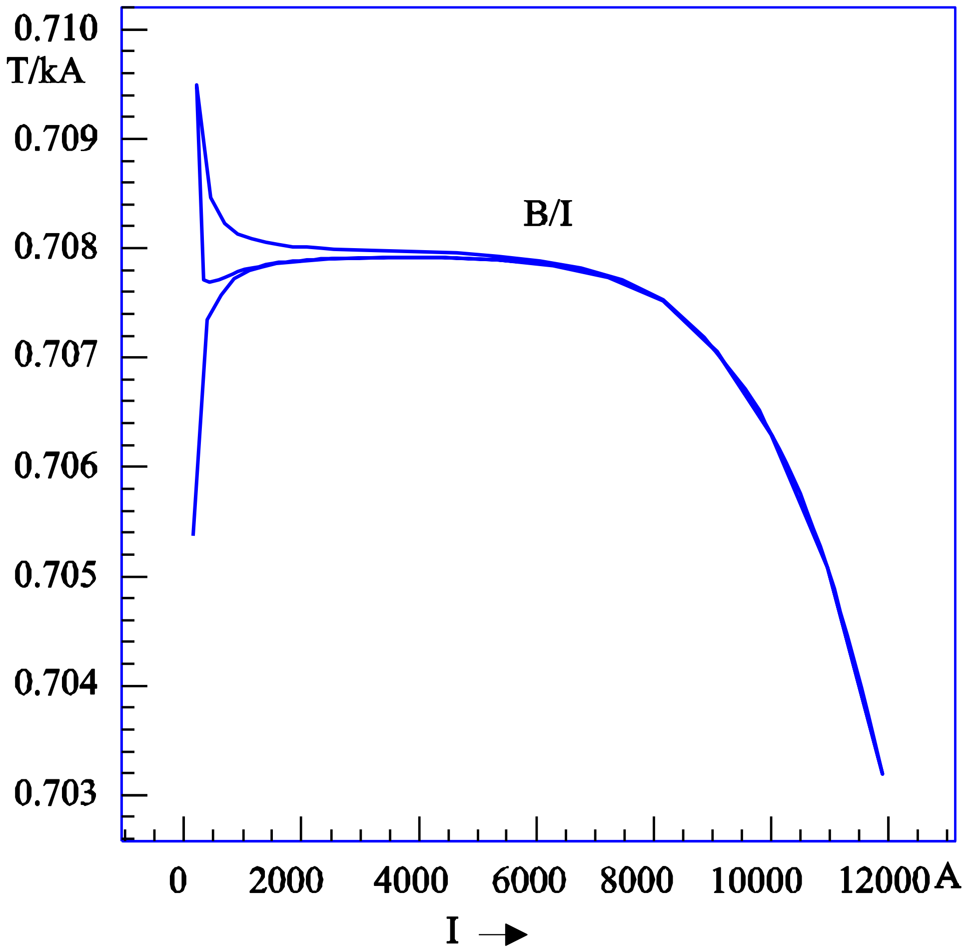
Total Magnetic Scalar Potential



Numerical Methods for the Curl-Curl and Vector-Laplace Equations

Excitation Cycle




0.710
T/kA
0.709
0.708
0.707
0.706
0.705
0.704
0.703
B/I
0
2000
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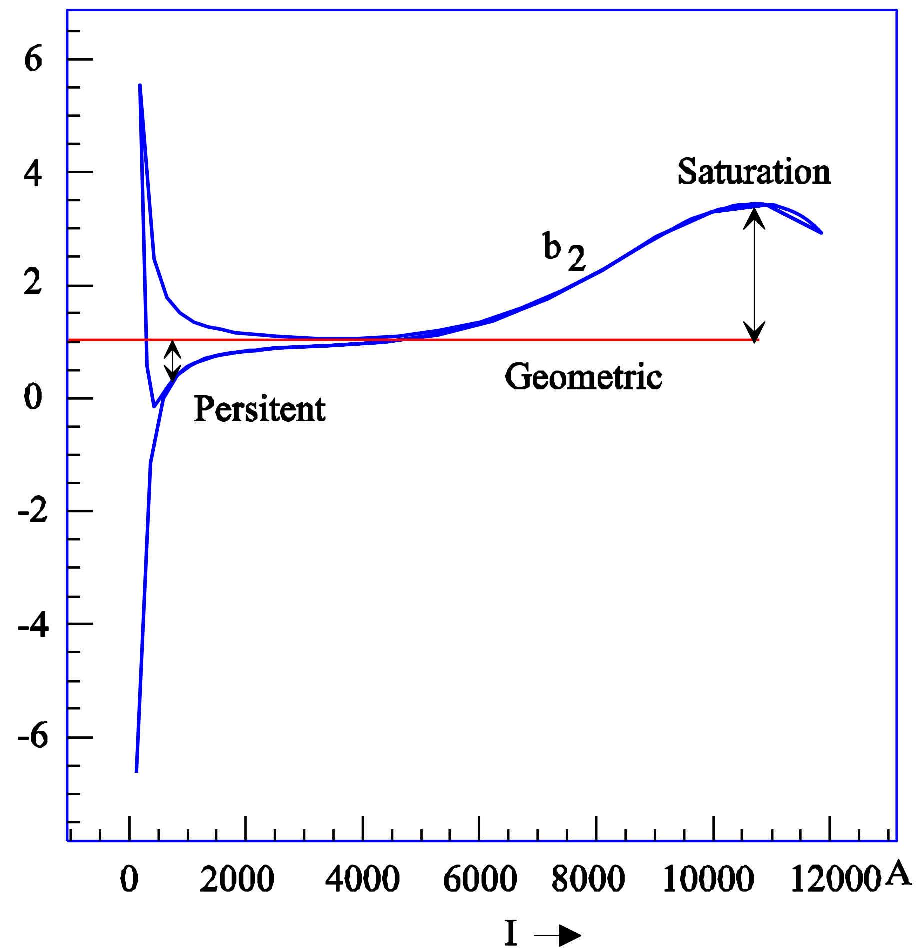
8000
10000
12000
A
I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4
2
0
-2
-4
-6
Saturation
b_2
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0
2000
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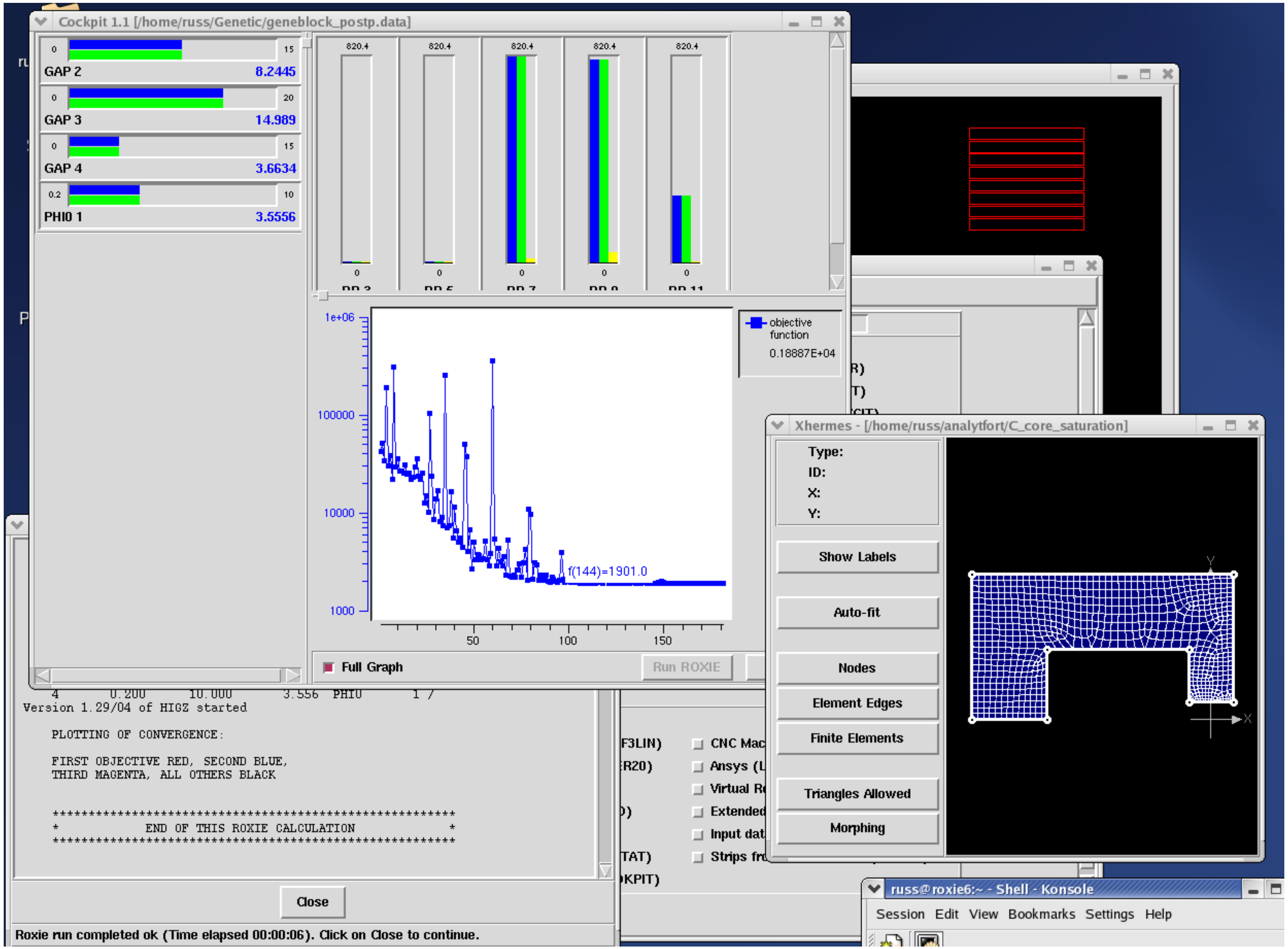
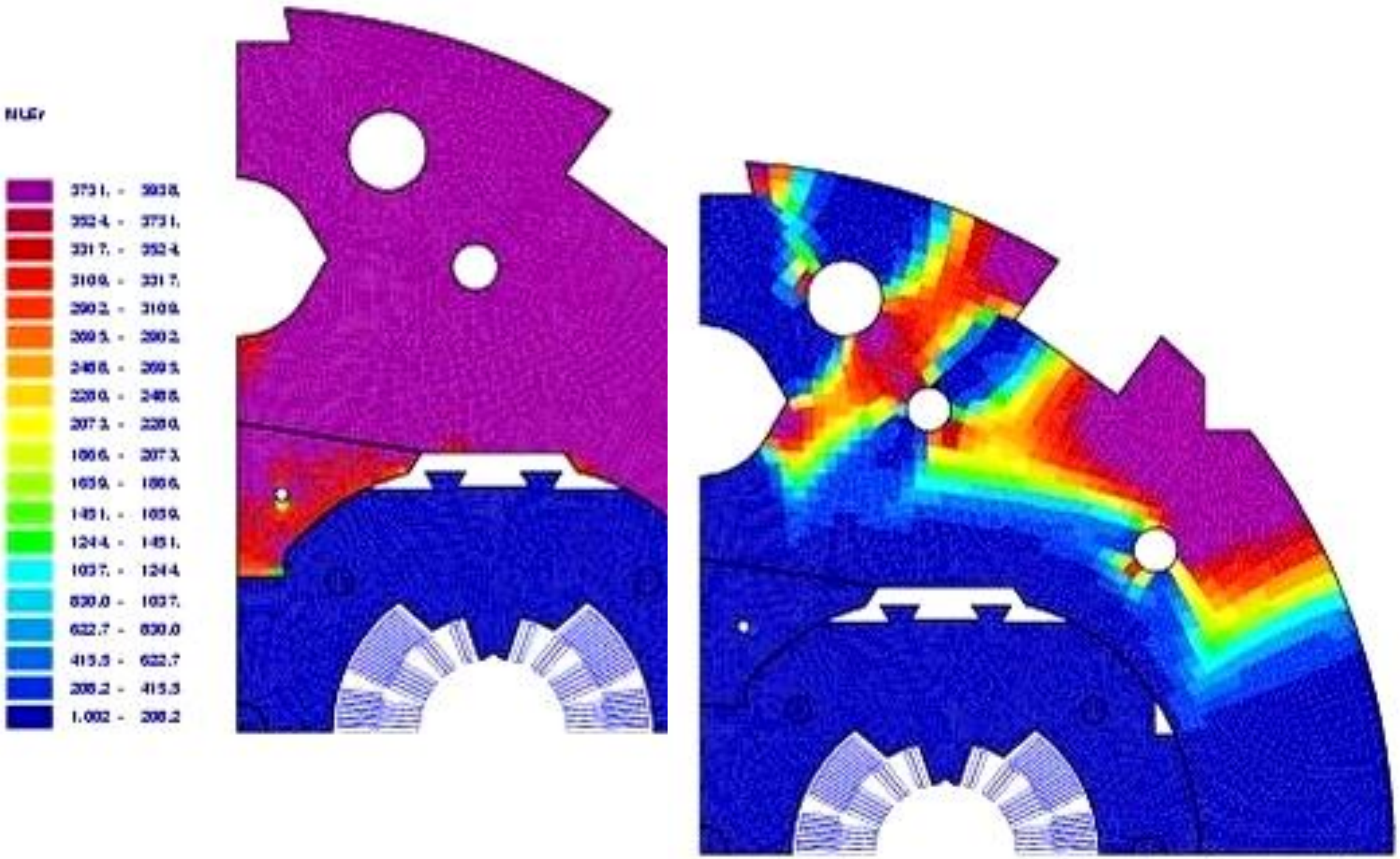
I

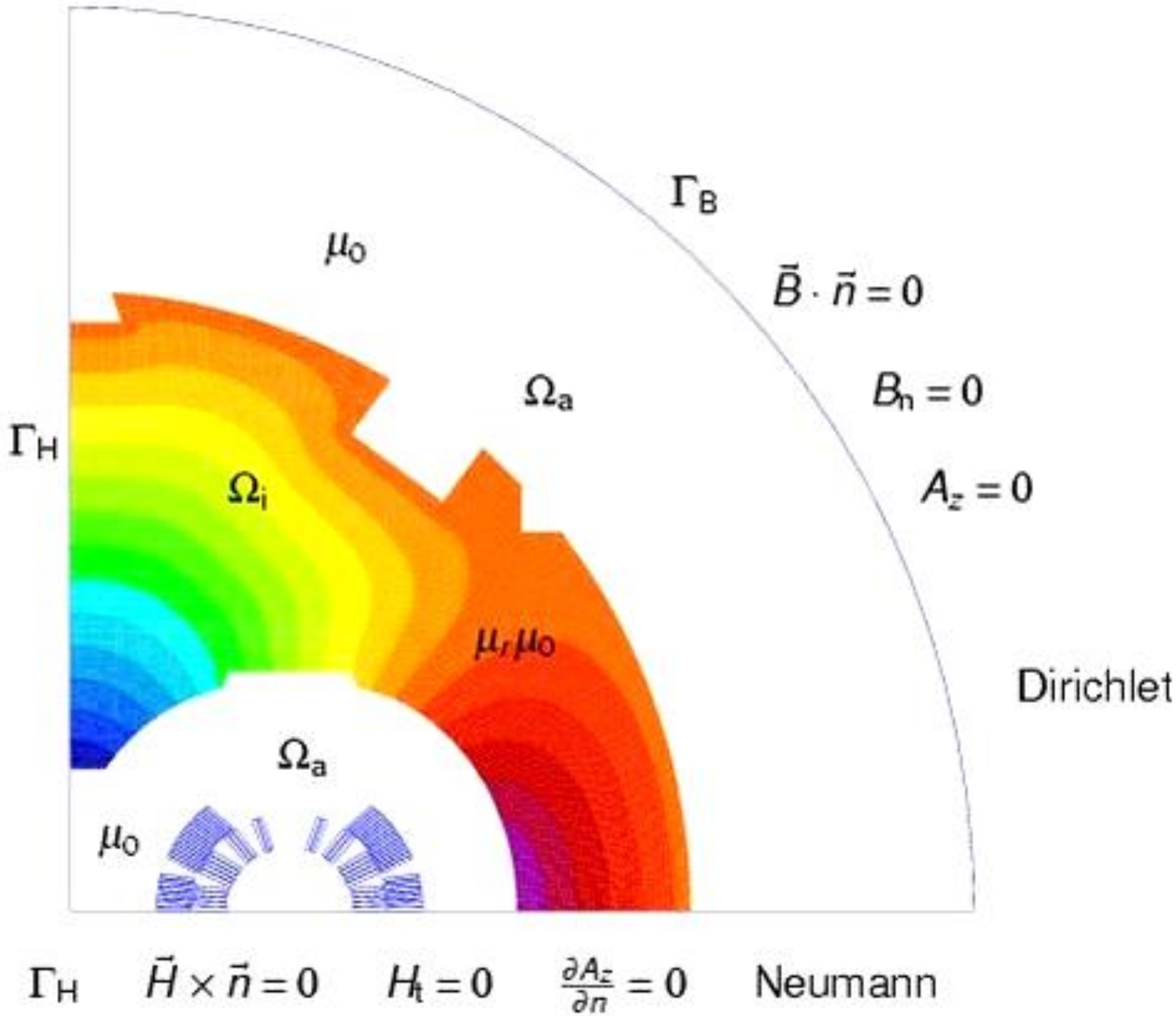
The CERN Field Computation Program ROXIE



Saturation Effects in the Dipole Iron Yoke



The Problem Domain



Curl-Curl Equation

$$\mathbf{B} = \mathrm{curl}\,\mathbf{A} \quad \text{in}\,\Omega$$

$$\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A} = \mathbf{J} \quad \text{in}\,\Omega$$

$$\mathbf{H}_\mathrm{t} = \mathbf{0} \quad \rightarrow \quad \frac{1}{\mu}(\mathrm{curl}\,\mathbf{A}) \times \mathbf{n} \; = \; \mathbf{0} \;\; \text{on}\; \Gamma_H$$

$$B_\mathrm{n} = 0 \quad \rightarrow \quad \mathbf{B}\cdot\mathbf{n} = \mathrm{curl}\,\mathbf{A}\cdot\mathbf{n} \; = \; 0 \;\; \text{on}\; \Gamma_B$$

$$\left[\frac{1}{\mu}\,(\mathrm{curl}\,\mathbf{A})\times\mathbf{n}\right]_\mathrm{ai} \; = \; \mathbf{0} \;\; \text{on}\; \Gamma_\mathrm{ai}$$

$$[\mathbf{A}]_\mathrm{ai} \; = \; \mathbf{0} \;\; \text{on}\; \Gamma_\mathrm{ai}$$

Problem in 3-D: Gauging

$$\mathbf{A} \rightarrow \mathbf{A}' : \mathbf{A}' = \mathbf{A} + \mathrm{grad}\,\psi$$

$$\mathrm{div}\,\mathbf{A}' = q$$

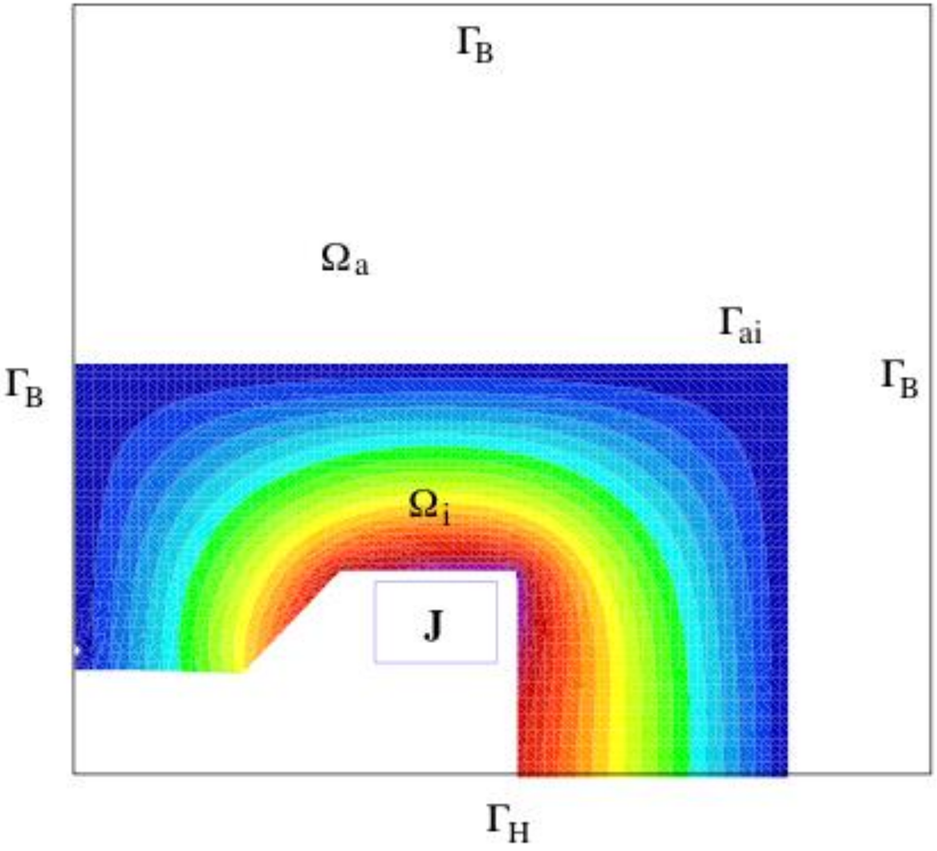
$$q = \mathrm{div}\,\mathbf{A} + \nabla^2\psi$$

$$\frac{1}{\mu}\mathrm{div}\,\mathbf{A} = 0 \;\; \text{in}\; \Omega$$

$$\mathbf{A}\cdot\mathbf{n} = 0 \;\; \text{on}\; \Gamma_H$$

$$\boxed{\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A} - \;\mathrm{grad}\,\frac{1}{\mu}\,\mathrm{div}\,\mathbf{A} = \mathbf{J} \;\; \text{in}\,\Omega}$$

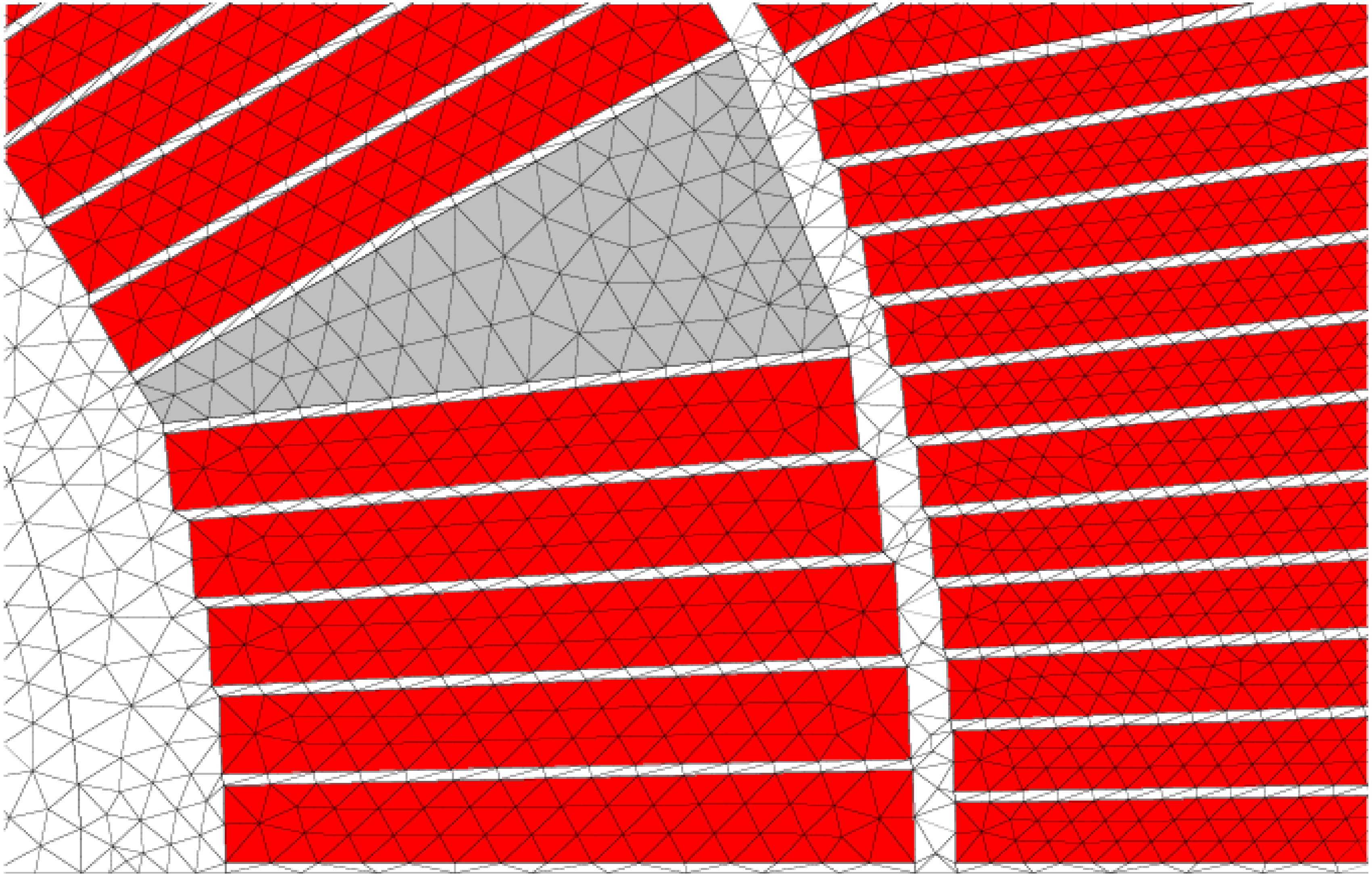
Weak Form in the FEM Problem



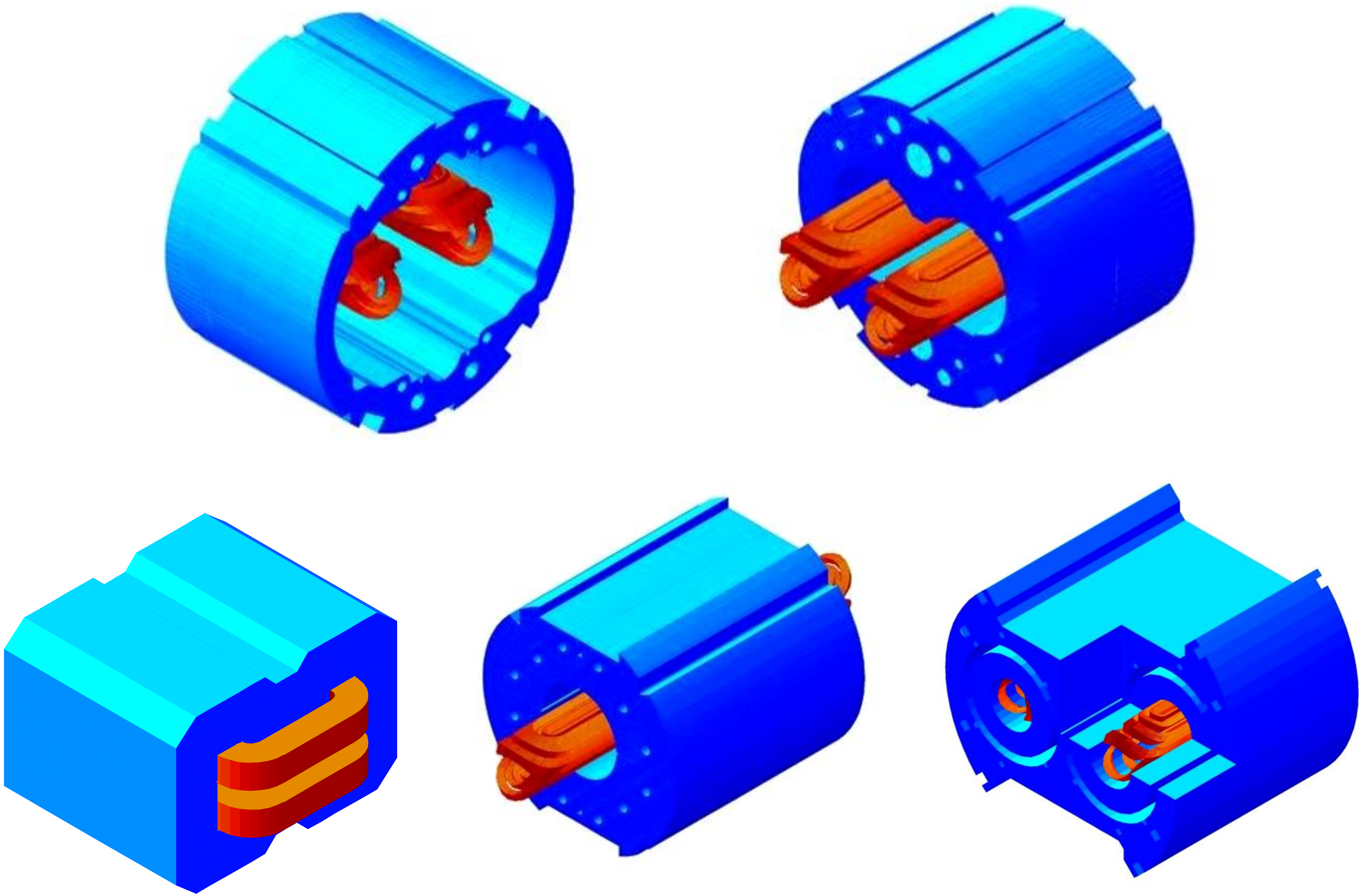
Weak Form in the FEM Problem

Galerkin: Element shape functions (polynomial approximations) = weighting function

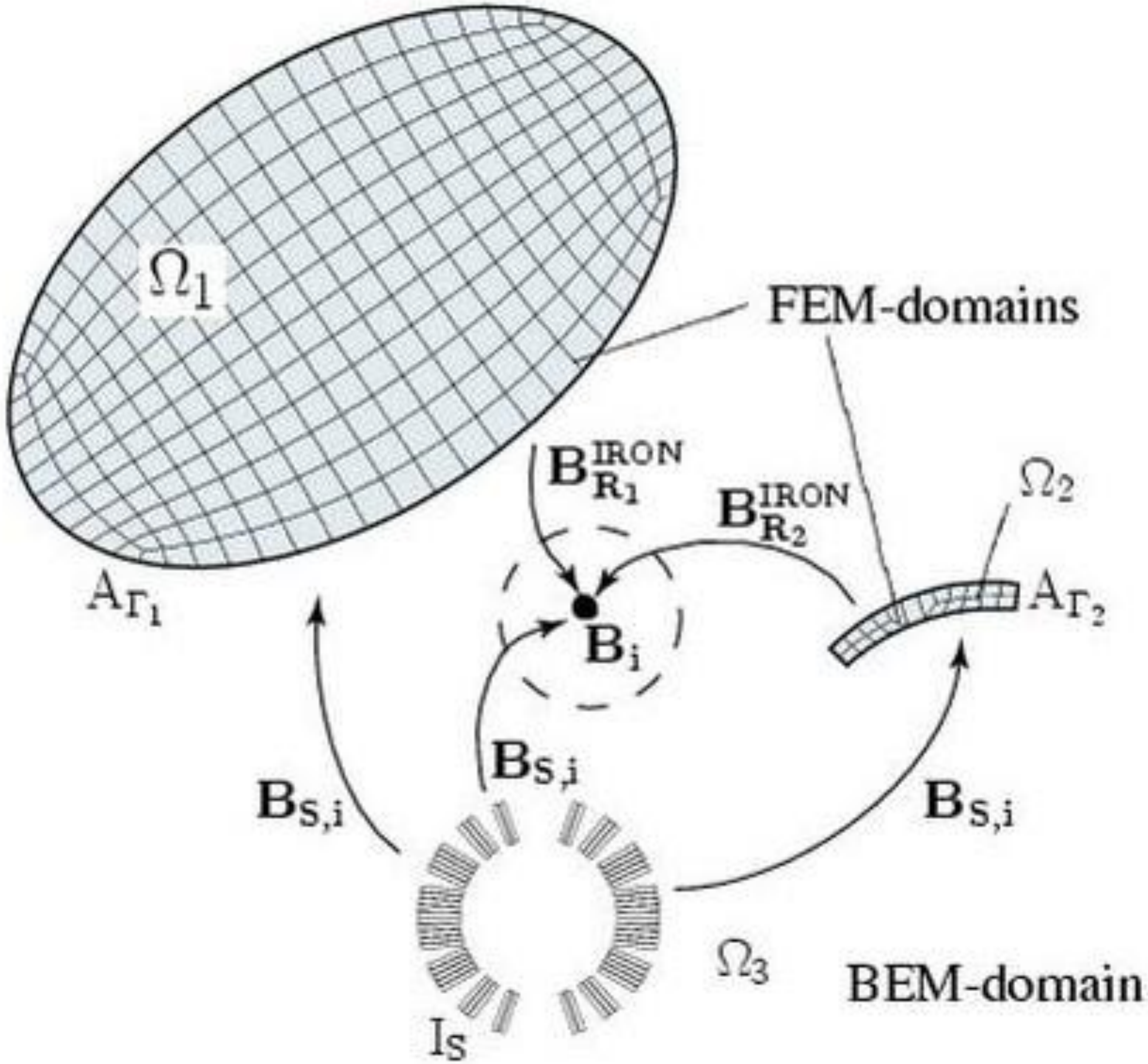
Meshing the Coil



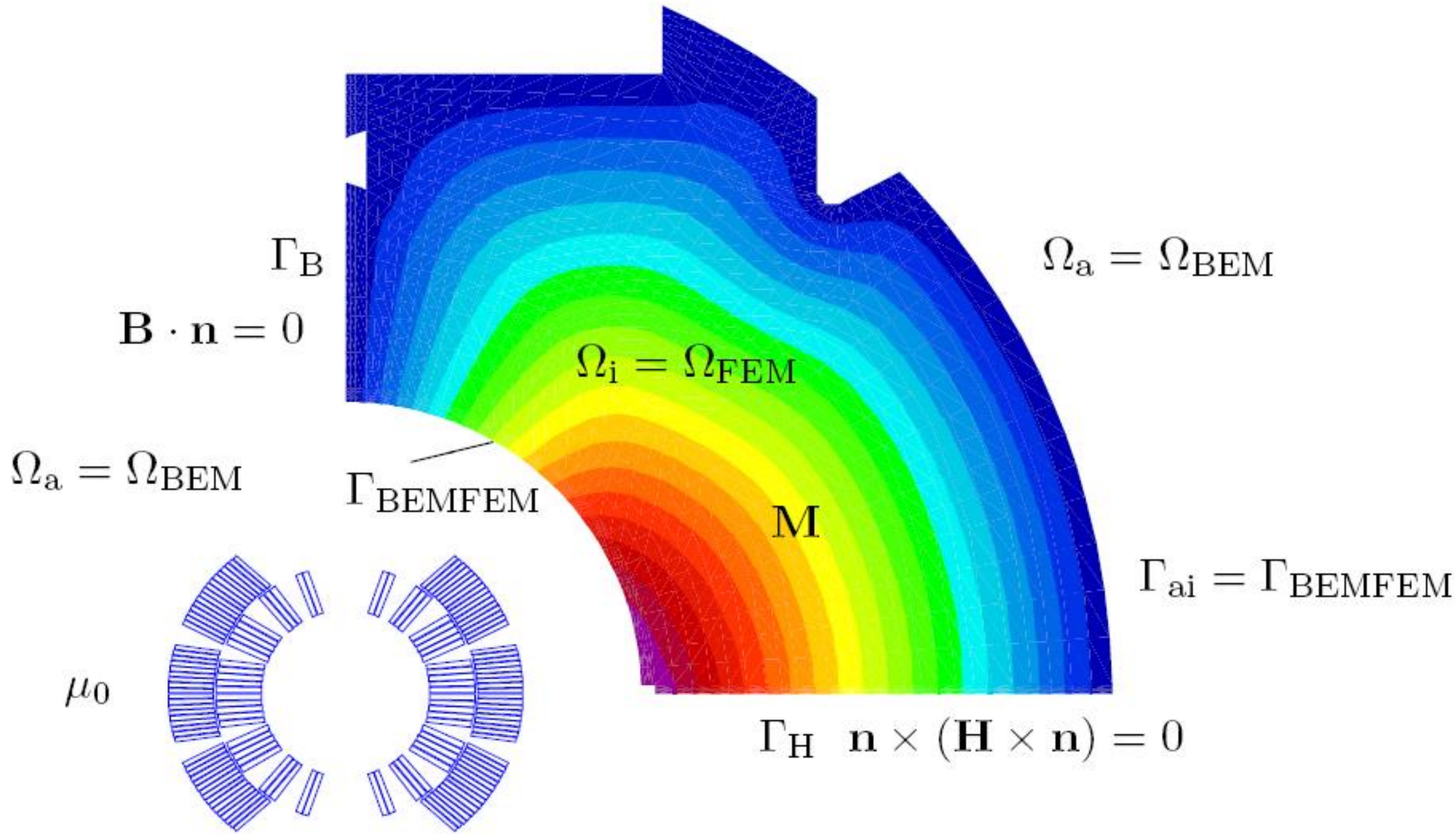
Magnet Extremities



BEM-FEM Coupling (Elementary Model Problem)

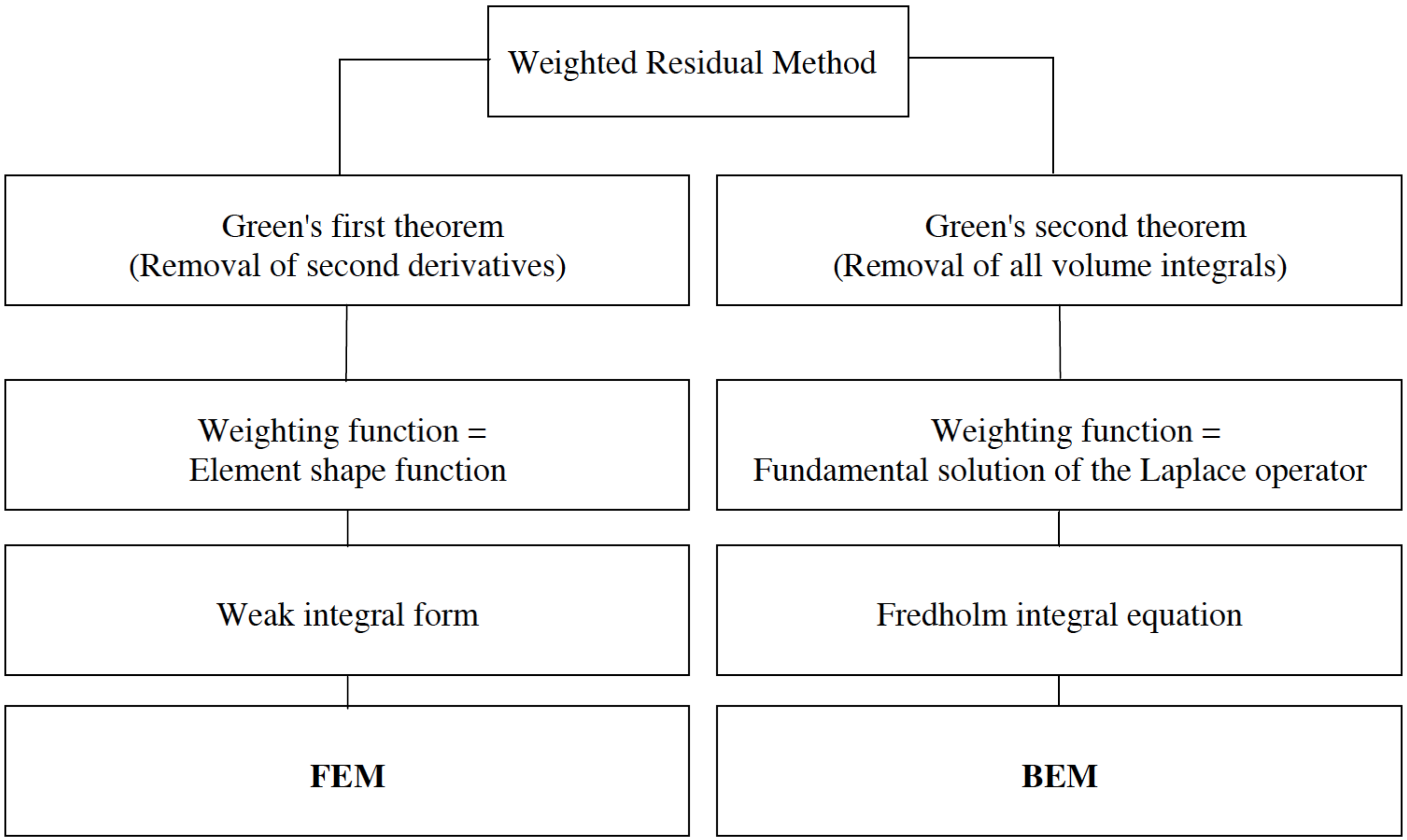


The Elementary Model Problem in Magnet Design



Green's First and Second Identities in FEM and BEM

$$\int_\Omega \left(\operatorname{grad}\phi \cdot \operatorname{grad}\psi + \phi\nabla^2\psi \right) \mathrm{d}V = \int_\Gamma \phi \operatorname{grad}\psi \cdot \mathbf{n} \, \mathrm{d}a \,,$$



$$\int_\Omega \left(\phi\nabla^2\psi - \psi\nabla^2\phi \right) \mathrm{d}V = \int_\Gamma \left(\phi\partial_{\mathbf{n}}\psi - \psi\partial_{\mathbf{n}}\phi \right) \mathrm{d}a \,,$$

$$\mathbf{B} = \mu_0\mathbf{H} + \mathbf{P}_{\mathrm{m}}(\mathbf{H}) = \mu_0(\mathbf{H} + \mathbf{M}(\mathbf{H}))$$

Vector-Laplace

$$\boxed{-\frac{1}{\mu_0}\nabla^2\mathbf{A} = \mathbf{J} + \operatorname{curl}\mathbf{M}} \qquad \text{in } \Omega_i,$$

$$\mathbf{A}\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_H,$$

$$\frac{1}{\mu_0}\operatorname{div}\mathbf{A} = 0 \qquad \text{on } \Gamma_B,$$

$$\mathbf{n}\times(\mathbf{A}\times\mathbf{n}) = \mathbf{0} \qquad \text{on } \Gamma_B,$$

$$\frac{1}{\mu}(\operatorname{curl}\mathbf{A})\times\mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma_H,$$

$$\left[\frac{1}{\mu_0}\operatorname{div}\mathbf{A}_{\mathrm{a}}\right]_{\mathrm{ai}} = 0 \qquad \text{on } \Gamma_{\mathrm{ai}},$$

$$\boxed{\frac{1}{\mu_0}(\operatorname{curl}\mathbf{A}_{\mathrm{i}} - \mu_0\mathbf{M})\times\mathbf{n}_{\mathrm{i}} + \frac{1}{\mu_0}(\operatorname{curl}\mathbf{A}_{\mathrm{a}})\times\mathbf{n}_{\mathrm{a}} = \mathbf{0}} \qquad \text{on } \Gamma_{\mathrm{ai}},$$

$$[\mathbf{A}]_{\mathrm{ai}} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{ai}}.$$

FEM Part (Weak Formulation and Matrix Representation)

$$\frac{1}{\mu_0}\int_{\Omega_\mathrm{i}} \mathrm{grad}\,(\mathbf{A}\cdot\mathbf{e}_a)\cdot\mathrm{grad}\,w_a\,\mathrm{d}\Omega_\mathrm{i} - \frac{1}{\mu_0}\oint_{\Gamma_\mathrm{ai}}\left(\frac{\partial\mathbf{A}}{\partial n_\mathrm{i}} - (\mu_0\mathbf{M}\times\mathbf{n}_\mathrm{i})\right)\cdot\mathbf{w}_a\,\mathrm{d}\Gamma_\mathrm{ai} = \int_{\Omega_\mathrm{i}}\mathbf{M}\cdot\mathrm{curl}\,\mathbf{w}_a\,\mathrm{d}\Omega_\mathrm{i}$$

$$[K]\{A\} - [T]\{Q\} = \{F(\mathbf{M})\}$$

BEM Part (Air-Domain – No Iron, but Current Sources)

Vector Laplace

Weighted Residual

Apply Green's second theorem:

$$\int_{\Omega_\mathrm{a}} A \nabla^2 w \mathrm{d}\Omega_\mathrm{a} = -\int_{\Omega_\mathrm{a}} \mu_0 J w \, \mathrm{d}\Omega_\mathrm{a} + \int_{\Gamma_\mathrm{ai}} A \frac{\partial w}{\partial n_a} \mathrm{d}\Gamma_\mathrm{ai} - \int_{\Gamma_\mathrm{ai}} \frac{\partial A}{\partial n_a} w \mathrm{d}\Gamma_\mathrm{ai}$$

Biot-Savart

Representation Formula (Fredholm Integral Equation)

$$\frac{\Theta}{4\pi}A(\mathbf{r}) + \int_{\Gamma} Q\,u^*(\mathbf{r},\mathbf{r}')\,\mathrm{d}a' + \int_{\Gamma} A\,q^*(\mathbf{r},\mathbf{r}')\;\mathrm{d}a' = \int_{\Omega_\mathrm{a}} \mu_0 J u^*(\mathbf{r},\mathbf{r}')\,\mathrm{d}V',$$

Single-layer potential

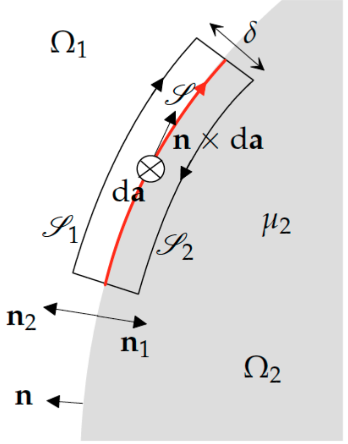
$$\boldsymbol{\alpha}(\mathbf{r}') := -\frac{1}{\mu}\partial_{\mathbf{n}_\mathrm{a}} A(\mathbf{r}')$$

$$[\boldsymbol{\alpha}] = 1\ \mathrm{A\,m}^{-1}$$

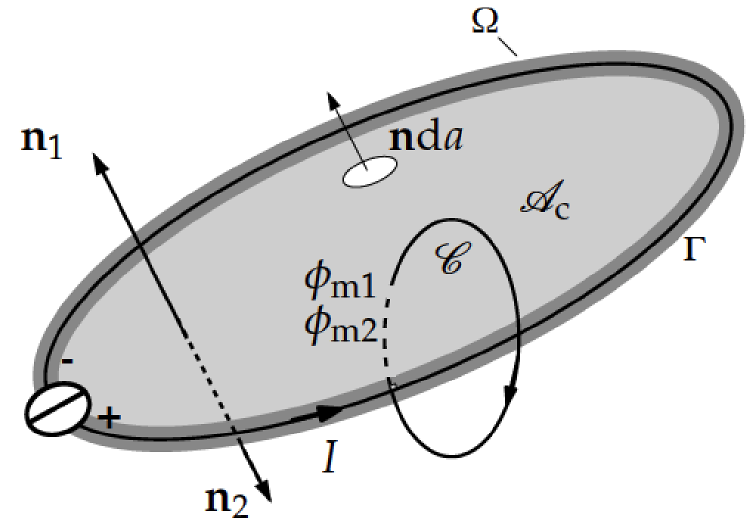
Double-layer potential

$$\boldsymbol{\tau}(\mathbf{r}') := \frac{1}{\mu}A(\mathbf{r}')$$

$$[\boldsymbol{\tau}] = 1\ \mathrm{A}$$



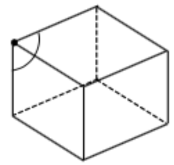
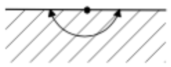
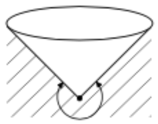
$$\boldsymbol{\alpha} = \mathbf{n}\times(\mathbf{H}_1 - \mathbf{H}_2)$$



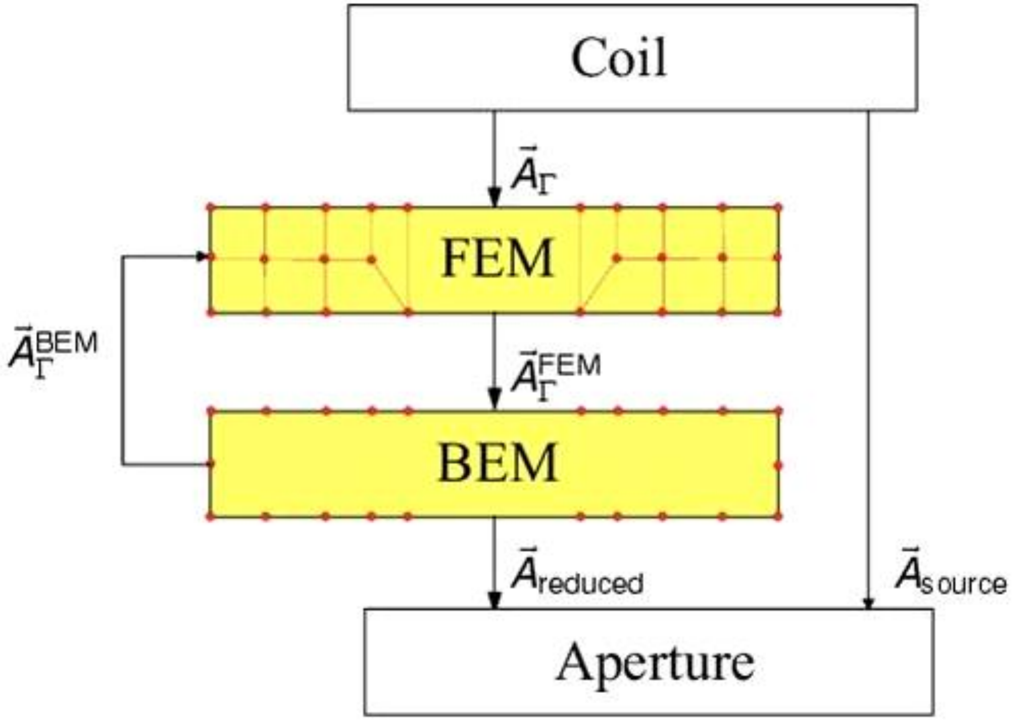
Node-Collocation (Compute One from the Other)

$$\frac{\Theta}{4\pi} A(\mathbf{r}) + \int_{\Gamma} Q\, u^*(\mathbf{r},\mathbf{r}')\,\mathrm{d}a' + \int_{\Gamma} A\, q^*(\mathbf{r},\mathbf{r}')\;\mathrm{d}a' = \int_{\Omega_a} \mu_0 J u^*(\mathbf{r},\mathbf{r}')\,\mathrm{d}V',$$

$$[G]\{Q\} + [H]\{A\} = \{A_s\}$$

Ω_a	 90°Corner	 90° Cone inner	 Half-space	 90° Cone outer
Θ	$\frac{1}{2}\pi$	$(2-\sqrt{2})\pi$	2π	$(2+\sqrt{2})\,\pi$
$\frac{\Theta}{4\pi}$	$\frac{1}{8}$	$\frac{2-\sqrt{2}}{4}$	$\frac{1}{2}$	$\frac{2+\sqrt{2}}{4}$

BEM-FEM Coupling



BEM

$$[G]\{Q\} + [H]\{A\} = \{A_s\}$$

FEM

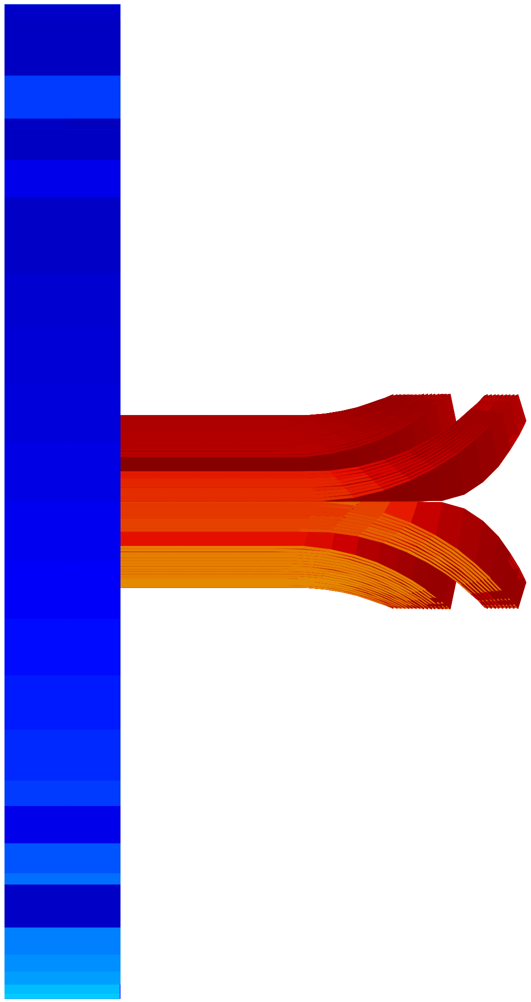
$$[K]\{A\} - [T]\{Q\} = \{F(\mathbf{M})\}$$

$$\left([K] + [T][G]^{-1}[H]\right)\{A\} = \{F(\mathbf{M})\} + [T][G]^{-1}\{A_s\}$$

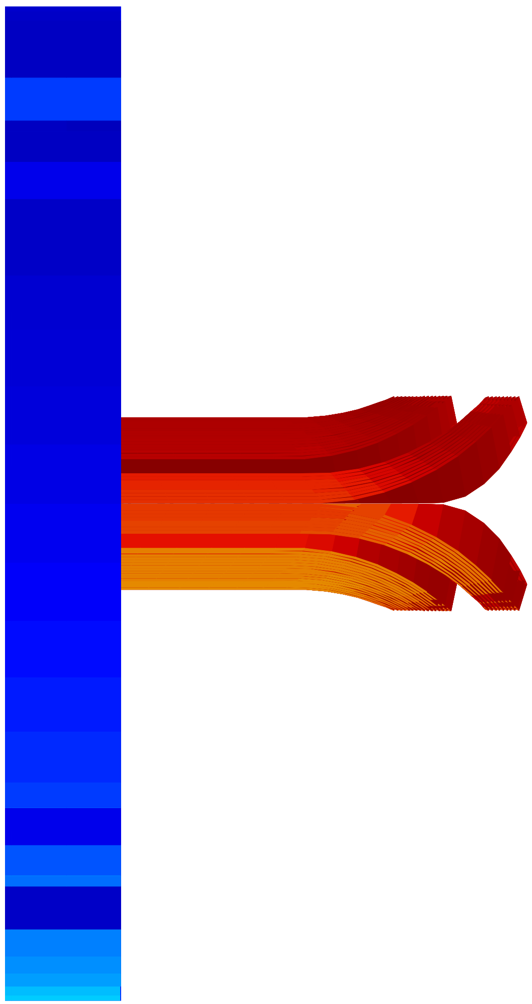
Open Boundary Problems (1)

LHC Beam Screen

Source Field



Reduced Field



Total Field