Lectures on Partial Differential Equations

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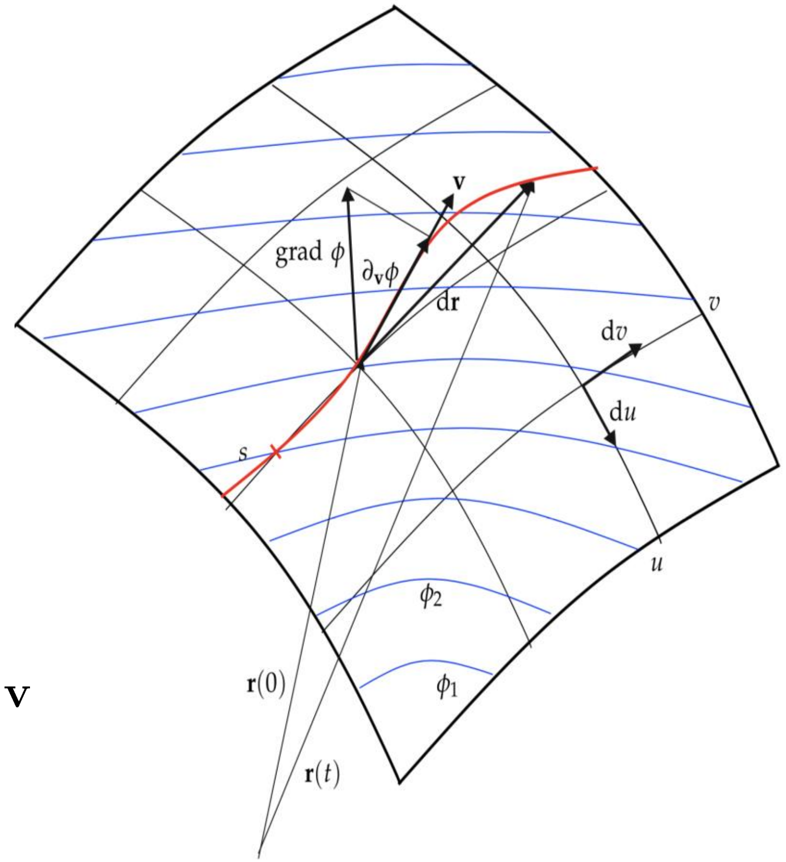
Foundations of Vector Analysis

Directional Derivative and the Total Differential

Space curve with $\mathbf{r}(t) = (x(t), y(t), z(t))$ parametrized such that $\mathbf{r}(0) = P$.

1-smooth scalar field $\phi : E_3 \to R : \mathbf{r} \mapsto \phi(\mathbf{r})$ expressed as $\phi(x,y,z)$, then $\phi(\mathbf{r}(t))$ at parameter (time) t.



$$\partial_{\mathbf{v}}\phi = \frac{\partial \phi}{\partial v} = \frac{\mathrm{d}}{\mathrm{d}t}[\phi(\mathbf{r}+t\mathbf{v})]_{t=0} = \lim_{t\to 0}\frac{\phi(\mathbf{r}+t\mathbf{v})-\phi(\mathbf{r})}{t}$$

$$\partial_{\mathbf{v}}\phi = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\mathbf{r}(t)) = \frac{\partial \phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{grad}\,\phi \cdot \mathbf{v}$$

$$\mathrm{grad}\phi = \frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

Best linear approximation of ϕ over displacement distance d***r***

$$\mathbf{dr} = \mathbf{v}\mathrm{d}t = \frac{\mathbf{v}}{v}v\mathrm{d}t = \mathbf{T}\,\mathrm{d}s \qquad \mathrm{d}\mathbf{a} = \mathbf{n}\,\mathrm{d}a = \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\mathrm{d}u\mathrm{d}v \qquad \mathrm{d}\mathbf{f} = \frac{\partial \mathbf{f}}{\partial x}\mathrm{d}x + \frac{\partial \mathbf{f}}{\partial y}\mathrm{d}y + \frac{\partial \mathbf{f}}{\partial z}\mathrm{d}z$$

Ideal Pole Shape of Conventional Magnets

Remember the Cauchy Schwarz inequality $|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \| \mathbf{a} \| \, \| \mathbf{b} \|,$

Thus for the directional derivative $|\partial_{\mathbf{v}} \phi| \leq |\operatorname{grad} \phi| \, |\mathbf{v}|$

This implies that the directional derivative takes its maximum when **v** points in the direction of the gradient. Therefore gradient points in the direction of the steepest ascent of *Φ* and is thus normal to the surface of equipotential.

The flux density **B** exits a highly permeable surface in normal direction. Therefore the pole shape of normal conducting magnets can be seen as an equipotential of the magnetic scalar potential.

Grad, Curl and Div in Cartesian Coordinates

$$\operatorname{grad}\phi := \frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

$$\operatorname{curl}\mathbf{g} = \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right)\mathbf{e}_z .$$

$$\operatorname{div}\mathbf{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} .$$

The First Lemma of Poincare

$$
\begin{aligned}
\operatorname{curl}\operatorname{grad}\phi = &\operatorname{curl}\left[\frac{1}{h_1}\frac{\partial\phi}{\partial u^1}\mathbf{e}_{u^1} + \frac{1}{h_2}\frac{\partial\phi}{\partial u^2}\mathbf{e}_{u^2} + \frac{1}{h_3}\frac{\partial\phi}{\partial u^3}\mathbf{e}_{u^3}\right] \\
&= \frac{1}{h_2 h_3}\left(\frac{\partial^2\phi}{\partial u^2\partial u^3} - \frac{\partial^2\phi}{\partial u^3\partial u^2}\right)\mathbf{e}_{u^1} \\
&\quad + \frac{1}{h_3 h_1}\left(\frac{\partial^2\phi}{\partial u^3\partial u^1} - \frac{\partial^2\phi}{\partial u^1\partial u^3}\right)\mathbf{e}_{u^2} \\
&\quad + \frac{1}{h_1 h_2}\left(\frac{\partial^2\phi}{\partial u^1\partial u^2} - \frac{\partial^2\phi}{\partial u^2\partial u^1}\right)\mathbf{e}_{u^3} = 0\,,
\end{aligned}
$$

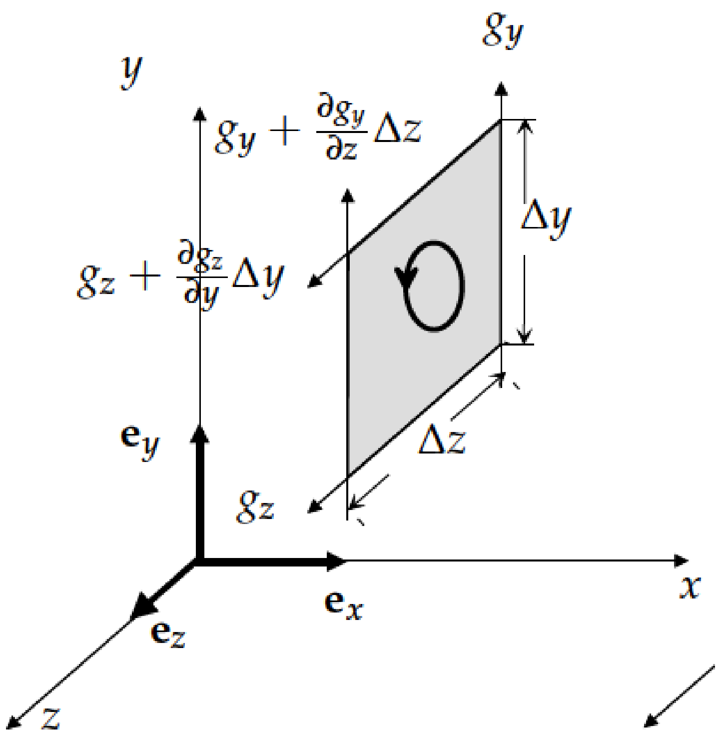
Ugly and not even a universal proof (orthogonality assumed)

Coordinate Free Definition of Grad, Curl, and Div

$$\int_{\mathscr{P}_1}^{\mathscr{P}_2} \mathbf{a} \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{P}_1}^{\mathscr{P}_2} \mathrm{grad}\,\phi \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{P}_1}^{\mathscr{P}_2} \mathrm{d}\phi = \phi(\mathscr{P}_2) - \phi(\mathscr{P}_1),$$

$$\mathbf{n} \cdot \mathrm{curl}\,\mathbf{g} = \lim_{a \to 0} \frac{\int_{\partial\mathscr{A}} \mathbf{g} \cdot \mathrm{d}\mathbf{r}}{a},$$

$$\mathrm{div}\,\mathbf{g} = \lim_{V \to 0} \frac{\int_{\partial\mathscr{V}} \mathbf{g} \cdot \mathrm{d}\mathbf{a}}{V},$$



The Boundary Operator

$$\partial(\partial\mathscr{V}) = \emptyset, \qquad \partial(\partial\mathscr{A}) = \emptyset,$$

$$\int_{\mathscr{V}} \operatorname{div}\,\operatorname{curl}\mathbf{g}\,\mathrm{d}V = \int_{\partial\mathscr{V}} \operatorname{curl}\mathbf{g}\cdot\mathrm{d}\mathbf{a} = \int_{\partial(\partial\mathscr{V})} \mathbf{g}\cdot\mathrm{d}\mathbf{r} = 0,$$

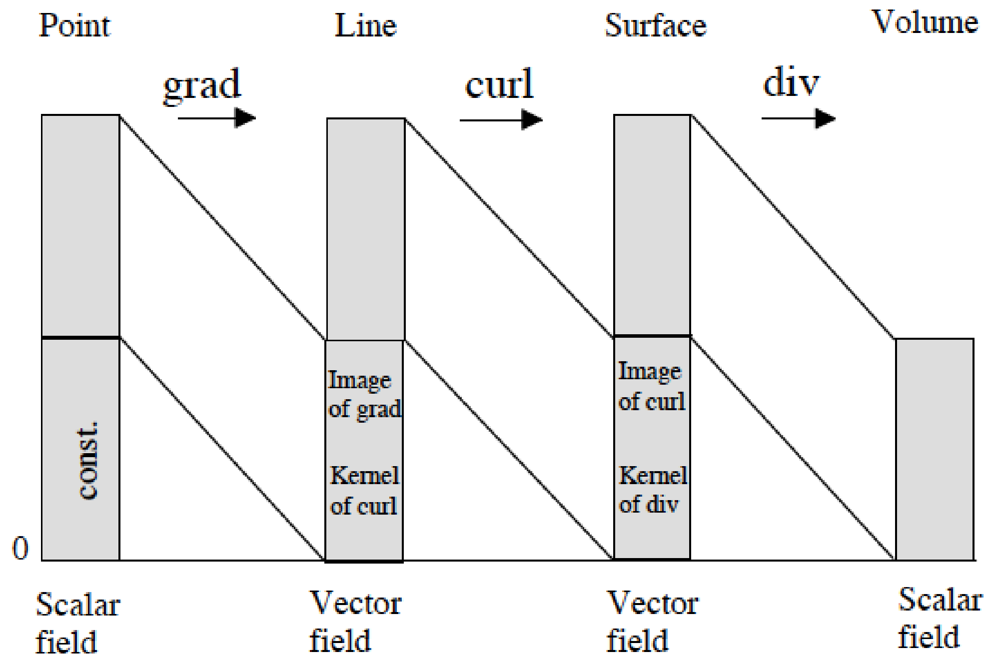
$$\int_{\mathscr{A}} \operatorname{curl}\,\operatorname{grad}\phi\cdot\mathrm{d}\mathbf{a} = \int_{\partial\mathscr{A}} \operatorname{grad}\phi\cdot\mathrm{d}\mathbf{r} = \phi|_{\partial(\partial\mathscr{A})} = 0,$$

Reversal of arguments yields two important statements (next slides):
Much nicer than writing it in coordinates

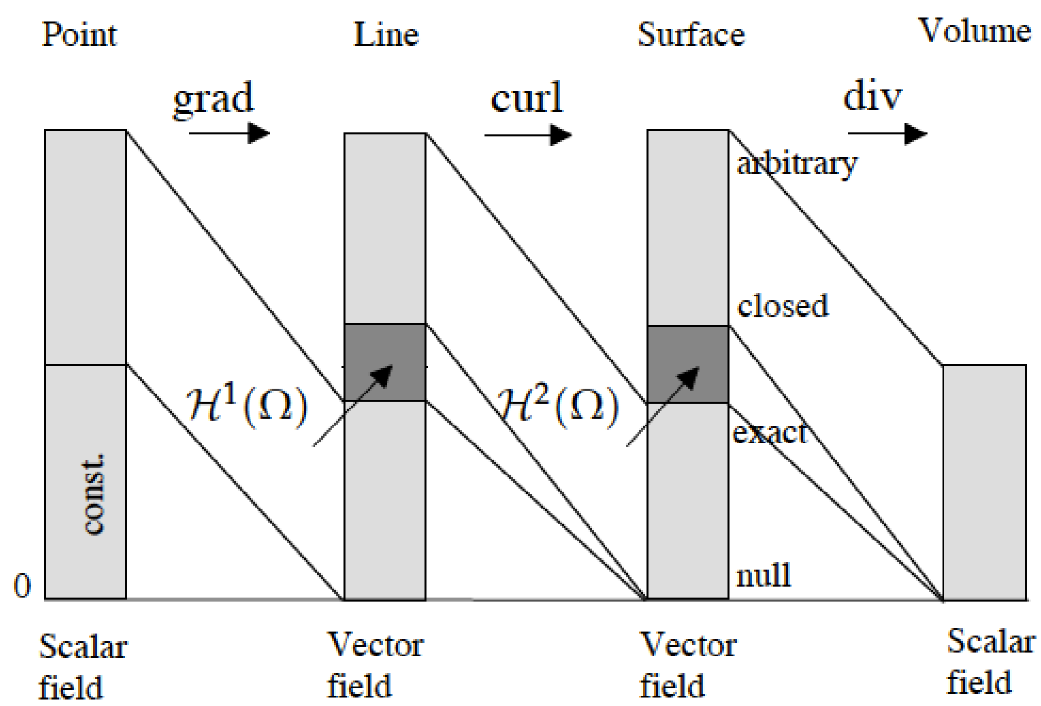
The second Lemma of Poincare (Contractible Domains)

$$\operatorname{div} \mathbf{b} = 0 \quad \rightarrow \quad \mathbf{b} = \operatorname{curl} \mathbf{a}.$$

$$\operatorname{curl} \mathbf{h} = 0 \quad \rightarrow \quad \mathbf{h} = \operatorname{grad} \phi.$$



Lemmata of Poincare (Non-Contractible Domains)



$$\mathcal{H}^1(\Omega) := \frac{\ker(\,\mathrm{curl}\,)}{\mathrm{im}(\,\mathrm{grad}\,)}$$

$$\mathcal{H}^2(\Omega) := \frac{\ker(\mathrm{div})}{\mathrm{im}(\,\mathrm{curl}\,)}$$

Toroidal domain Ω in a cylindrical coordinate system (r, φ, z):

$$H_\varphi = \frac{I}{2\pi r}$$

$$\mathrm{curl}\,\mathbf{H} = \frac{1}{r}\frac{\partial}{\partial r}(r H_\varphi) = 0$$

But $\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{s} = I$ and Ω, with $\oint_C \mathrm{grad}\,\phi \cdot \mathrm{d}\mathbf{s} = 0$

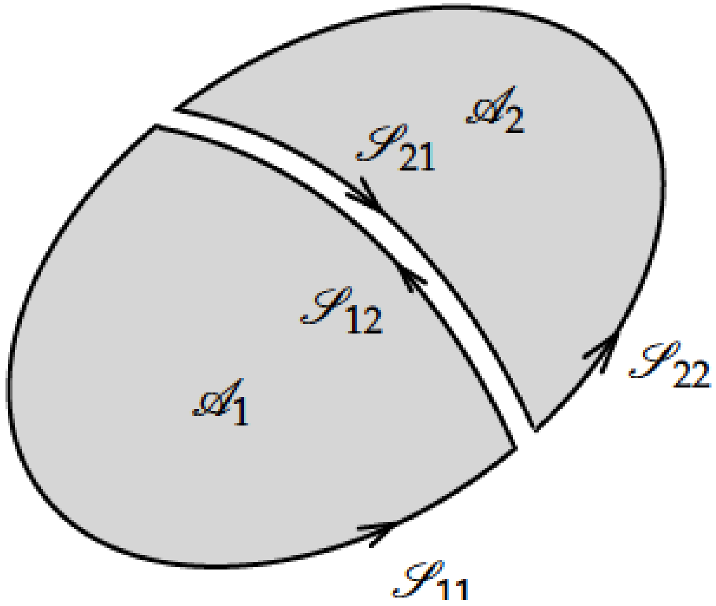
Domain Ω between two nested spheres centered at the origin.

$$D_R = \frac{Q}{4\pi R^2}\mathbf{e}_R$$

$$\mathrm{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial R}(R^2 D_R) = 0$$

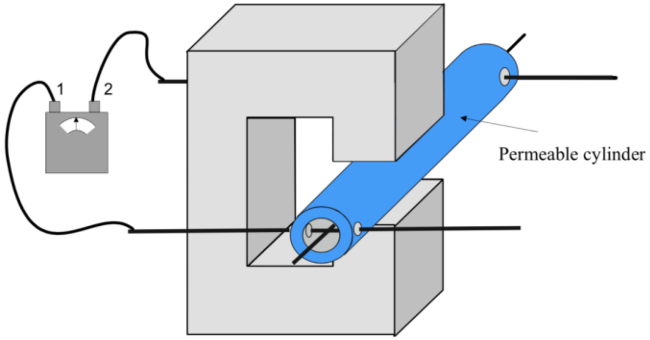
But $\oint_a \mathbf{D} \cdot \mathrm{d}\mathbf{a} = Q$ and $\oint_a \mathrm{curl}\,\mathbf{A} \cdot \mathrm{d}\mathbf{a} = 0$

Kelvin-Stokes Theorem



Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

No jump discontinuities (for example, co-moving shielding devices)

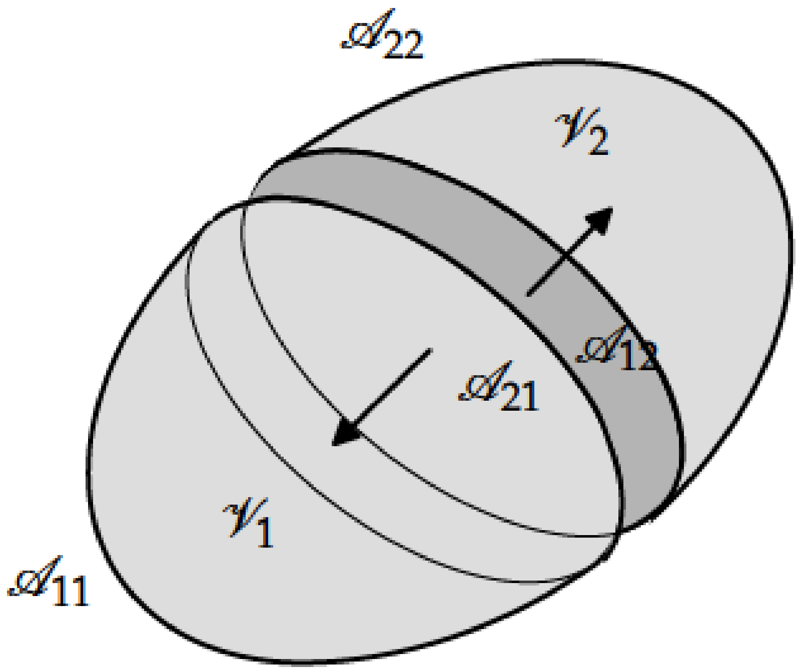


$$\int_{\partial\mathscr{A}} \mathbf{g}\cdot d\mathbf{r} = \int_{\mathscr{S}_1} \mathbf{g}\cdot d\mathbf{r} + \int_{\mathscr{S}_2} \mathbf{g}\cdot d\mathbf{r} = \int_{\mathscr{S}_{11}} \mathbf{g}\cdot d\mathbf{r} + \int_{\mathscr{S}_{22}} \mathbf{g}\cdot d\mathbf{r},$$

$$\int_{\partial\mathscr{A}} \mathbf{g}\cdot d\mathbf{r} = \lim_{I\to\infty}\sum_{i=1}^{I}\int_{\partial\mathscr{A}_i} \mathbf{g}\cdot d\mathbf{r} = \lim_{I\to\infty}\sum_{i=1}^{I}\Delta a_i \frac{1}{\Delta a_i}\int_{\partial\mathscr{A}_i} \mathbf{g}\cdot d\mathbf{r}$$

$$= \lim_{I\to\infty}\sum_{i=1}^{I}(\operatorname{curl}\mathbf{g})_i\cdot\mathbf{n}\,\Delta a_i = \int_{\mathscr{A}} \operatorname{curl}\mathbf{g}\cdot d\mathbf{a}\,.$$

Gauss' Theorem



Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

$$\int_{\partial\mathscr{V}} \mathbf{g}\cdot\mathrm{d}\mathbf{a} = \lim_{I\to\infty}\sum_{i=1}^{I}\int_{\partial\mathscr{V}_i}\mathbf{g}\cdot\mathrm{d}\mathbf{a} = \lim_{I\to\infty}\sum_{i=1}^{I}\Delta V_i\frac{1}{\Delta V_i}\int_{\partial\mathscr{V}_i}\mathbf{g}\cdot\mathrm{d}\mathbf{a}$$

$$= \lim_{I\to\infty}\sum_{i=1}^{I}(\operatorname{div}\mathbf{g})_i\,\Delta V_i = \int_{\mathscr{V}}\operatorname{div}\mathbf{g}\,\mathrm{d}V\,.$$

More Integral Theorems

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = \Big[g(x)f(x)\Big]_a^b - \int_a^b g(x)f'(x)\,\mathrm{d}x$$

Green's First

$$\int_{\mathcal{V}} \left(\operatorname{grad}\phi \cdot \operatorname{grad}\psi + \phi\nabla^2\psi\right)\,\mathrm{d}V = \int_{\partial\mathcal{V}} \phi\,\partial_{\mathbf{n}}\psi\,\mathrm{d}a$$

Green's Second

$$\int_{\Omega} \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\,\mathrm{d}V = \int_{\Gamma} (\phi\partial_{\mathbf{n}}\psi - \psi\partial_{\mathbf{n}}\phi)\,\mathrm{d}a$$

Vector Form of Green's Second

$$\int_{\mathcal{V}} \mathbf{a}\cdot\operatorname{curl}\mathbf{b}\,\mathrm{d}V = \int_{\mathcal{V}} \mathbf{b}\cdot\operatorname{curl}\mathbf{a}\,\mathrm{d}V - \int_{\partial\mathcal{V}} \mathbf{a}\cdot(\mathbf{b}\times\mathbf{n})\,\mathrm{d}a.$$

Generalization of the Integration by Parts Rule

$$-\int_{\mathcal{V}} \mathbf{a}\cdot\operatorname{grad}\phi\,\mathrm{d}V = \int_{\mathcal{V}} \phi\operatorname{div}\mathbf{a}\,\mathrm{d}V - \int_{\partial\mathcal{V}} \phi(\mathbf{a}\cdot\mathbf{n})\,\mathrm{d}a.$$

Stratton #1 and #2

$$\int_{\mathcal{V}} \operatorname{div}(\mathbf{a}\times\operatorname{curl}\mathbf{b})\mathrm{d}V = \int_{\partial\mathcal{V}} (\mathbf{a}\times\operatorname{curl}\mathbf{b})\cdot\mathbf{n}\,\mathrm{d}a$$

$$\int_{\mathcal{V}} (\mathbf{a}\operatorname{curl}\operatorname{curl}\mathbf{b} - \mathbf{b}\operatorname{curl}\operatorname{curl}\mathbf{a})\,\mathrm{d}V = \int_{\partial\mathcal{V}} (\mathbf{b}\times\operatorname{curl}\mathbf{a} - \mathbf{a}\times\operatorname{curl}\mathbf{b})\cdot\mathbf{n}\,\mathrm{d}a\,.$$

Maxwell's Equations in Different Avatars

Maxwell Equations

Integral Form

Local Form

Global Form

Laplace's Equation

Curl-Curl Equation

Harmonic Fields

Green's Functions

Weak-Forms

Kichhoff's Theorem

Lumped circuit calc. of NC magnets

Field quality in Accelerator magnets

The field of line-currents Coil-dominated magnets

FEM

BEM

DEM

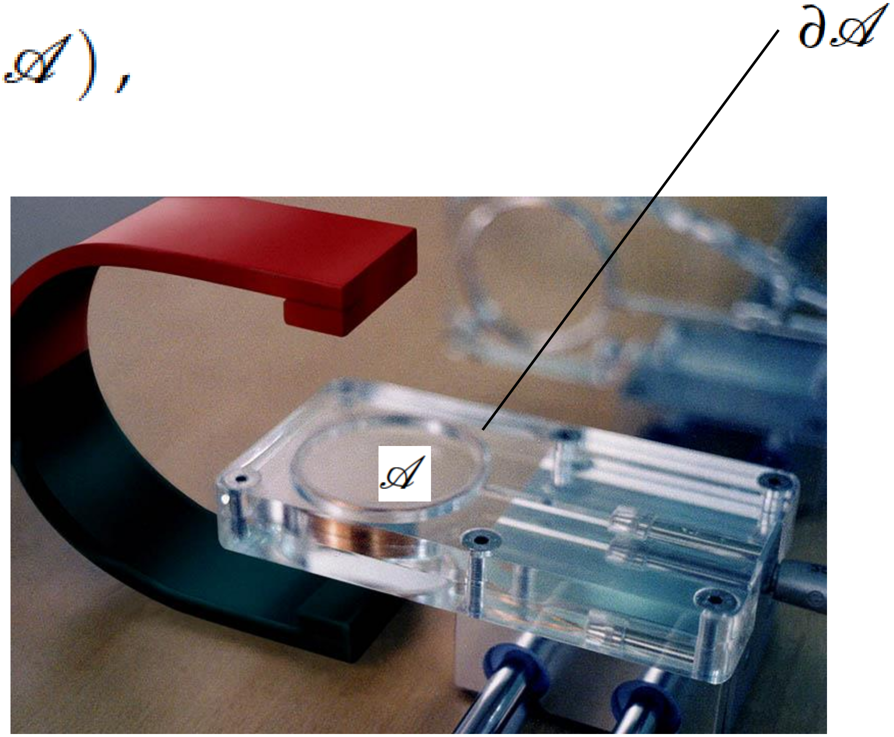
Maxwell Equations I: Global Form

Ampere + Maxwell extension
$$V_{\mathrm{m}}(\partial\mathscr{A}) = I(\mathscr{A}) + \frac{\mathrm{d}}{\mathrm{d}t}\Psi(\mathscr{A}),$$

Faraday
$$U(\partial\mathscr{A}) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathscr{A}),$$

Flux conservation
$$\Phi(\partial\mathscr{V}) = 0,$$

Gauss
$$\Psi(\partial\mathscr{V}) = Q(\mathscr{V}).$$



Required: Orientable manifolds

No switches, no Moebius strips

Maxwell Equations II: Integral Form

Global quantity	SI unit	Relation	SI unit	Field
MMF	$1\,\mathrm{A}$	$V_\mathrm{m}(\mathscr{S}) = \int_{\mathscr{S}} \mathbf{H}\cdot\mathrm{d}\mathbf{r}$	$1\,\mathrm{A\,m^{-1}}$	Magnetic field
Electric voltage	$1\,\mathrm{V}$	$U(\mathscr{S}) = \int_{\mathscr{S}} \mathbf{E}\cdot\mathrm{d}\mathbf{r}$	$1\,\mathrm{V\,m^{-1}}$	Electric field
Magnetic flux	$1\,\mathrm{V\,s}$	$\Phi(\mathscr{A}) = \int_{\mathscr{A}} \mathbf{B}\cdot\mathrm{d}\mathbf{a}$	$1\,\mathrm{V\,s\,m^{-2}}$	Magnetic flux density
Electric flux	$1\,\mathrm{A\,s}$	$\Psi(\mathscr{A}) = \int_{\mathscr{A}} \mathbf{D}\cdot\mathrm{d}\mathbf{a}$	$1\,\mathrm{A\,s\,m^{-2}}$	Electric flux density
Electric current	$1\,\mathrm{A}$	$I(\mathscr{A}) = \int_{\mathscr{A}} \mathbf{J}\cdot\mathrm{d}\mathbf{a}$	$1\,\mathrm{A\,m^{-2}}$	Electric current density
Electric charge	$1\,\mathrm{A\,s}$	$Q(\mathscr{V}) = \int_{\mathscr{V}} \rho\cdot\mathrm{d}V$	$1\,\mathrm{A\,s\,m^{-3}}$	Electric charge density

$$\int_{\partial\mathscr{A}} \mathbf{H}\cdot\mathrm{d}\mathbf{r} = \int_{\mathscr{A}} \mathbf{J}\cdot\mathrm{d}\mathbf{a} + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{A}} \mathbf{D}\cdot\mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{A}} \mathbf{E}\cdot\mathrm{d}\mathbf{r} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{A}} \mathbf{B}\cdot\mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{V}} \mathbf{B}\cdot\mathrm{d}\mathbf{a} = 0,$$

$$\int_{\partial\mathscr{V}} \mathbf{D}\cdot\mathrm{d}\mathbf{a} = \int_{\mathscr{V}} \rho\,\mathrm{d}V.$$

Required: Orientable manifolds, orientation, frame, metric, continuity

No switches, no Moebius strips

Maxwell's Equations in Local Form

$$\int_{\mathscr{A}} \operatorname{curl} \mathbf{g} \cdot d\mathbf{a} = \int_{\partial \mathscr{A}} \mathbf{g} \cdot d\mathbf{r},$$

$$\int_{\mathscr{V}} \operatorname{div} \mathbf{g} \, dV = \int_{\partial \mathscr{V}} \mathbf{g} \cdot d\mathbf{a},$$

$$\int_{\partial \mathscr{A}} \mathbf{H} \cdot d\mathbf{r} = \int_{\mathscr{A}} \mathbf{J} \cdot d\mathbf{a} + \frac{d}{dt} \int_{\mathscr{A}} \mathbf{D} \cdot d\mathbf{a},$$

$$\int_{\partial \mathscr{A}} \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \int_{\mathscr{A}} \mathbf{B} \cdot d\mathbf{a},$$

$$\int_{\partial \mathscr{V}} \mathbf{B} \cdot d\mathbf{a} = 0,$$

$$\int_{\partial \mathscr{V}} \mathbf{D} \cdot d\mathbf{a} = \int_{\mathscr{V}} \rho \, dV.$$

$$\int_{\mathscr{A}} \operatorname{curl} \mathbf{H} \cdot d\mathbf{a} = \int_{\mathscr{A}} \left(\mathbf{J} + \frac{\partial}{\partial t} \mathbf{D} \right) \cdot d\mathbf{a},$$

$$\int_{\mathscr{A}} \operatorname{curl} \mathbf{E} \cdot d\mathbf{a} = -\int_{\mathscr{A}} \frac{\partial}{\partial t} \mathbf{B} \cdot d\mathbf{a},$$

$$\int_{\mathscr{V}} \operatorname{div} \mathbf{B} \, dV = 0,$$

$$\int_{\mathscr{V}} \operatorname{div} \mathbf{D} \, dV = \int_{\mathscr{V}} \rho \, dV.$$

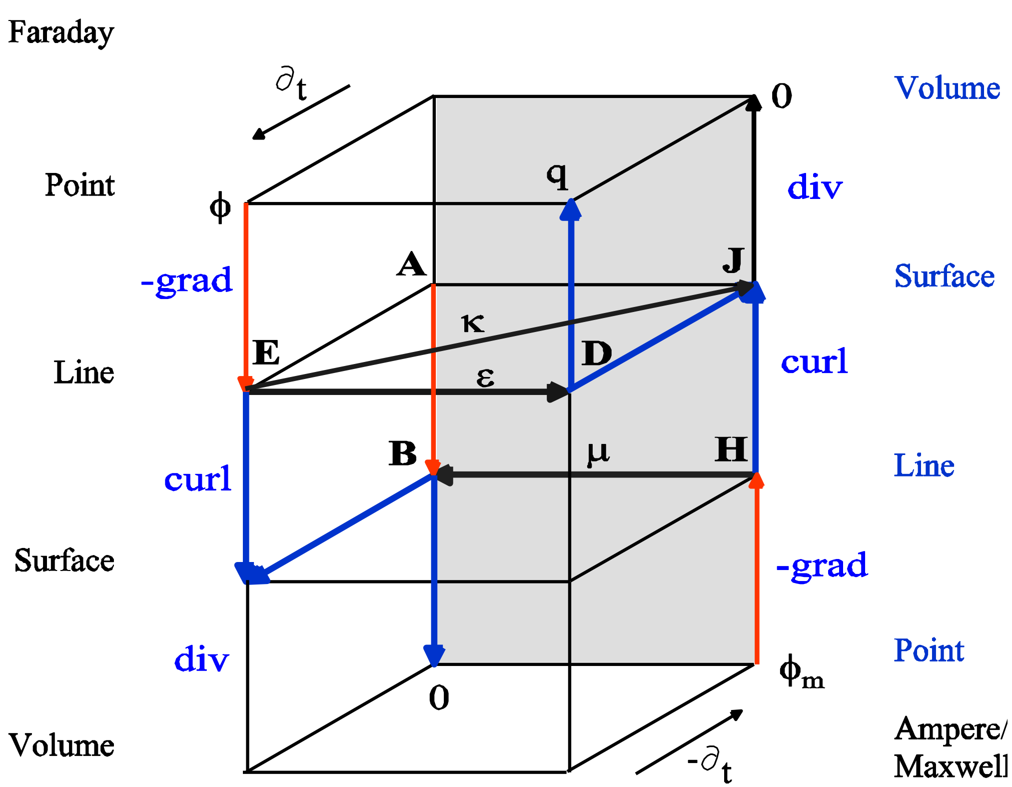
$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D},$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B},$$

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{div} \mathbf{D} = \rho.$$

Maxwell's House (Local Form)



$$\operatorname{curl}\mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D},$$

$$\operatorname{curl}\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B},$$

$$\operatorname{div}\,\mathbf{B} = 0,$$

$$\operatorname{div}\,\mathbf{D} = \rho.$$

Required: Orientable manifolds, orientation, frame, metric, continuity, contractible domains

No switches, no Moebius strips, no holes in surfaces, no bubbles in volumes, no internal boundaries

This simple form of constitutive equations are only true for linear (field-independent), homogeneous (position-independent), isotropic (direction-independent), lossless, and stationary media

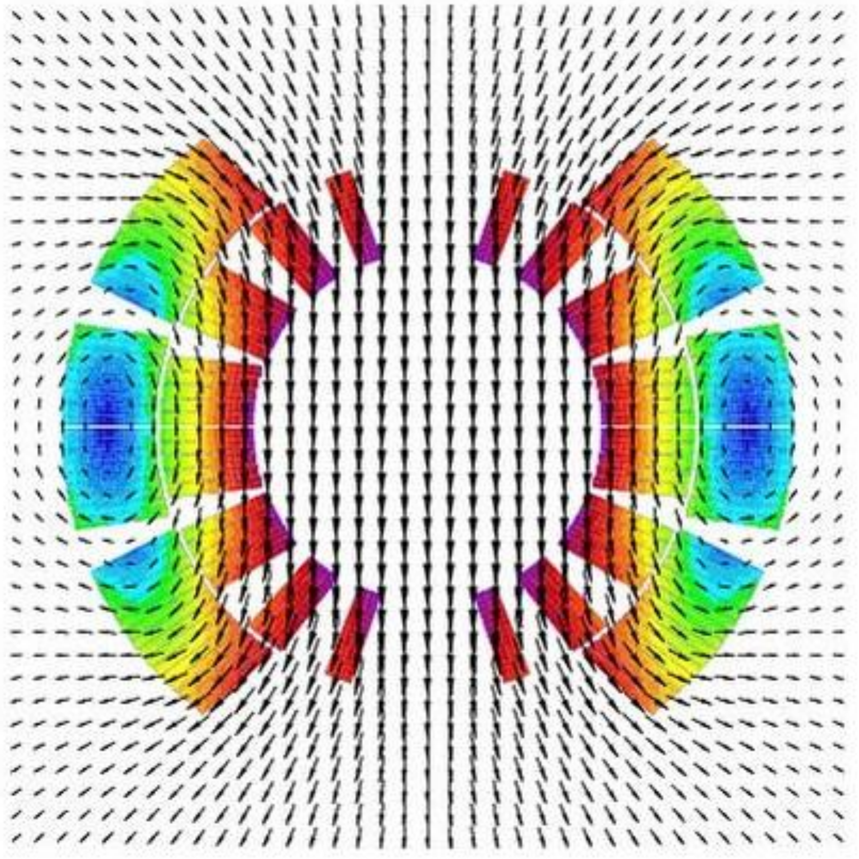
$$\mathbf{H} = -\operatorname{grad}\phi_{\mathrm{m}}^{\mathrm{red}} + \mathbf{T}$$

$$\mathbf{B} = \operatorname{curl}\mathbf{A}$$

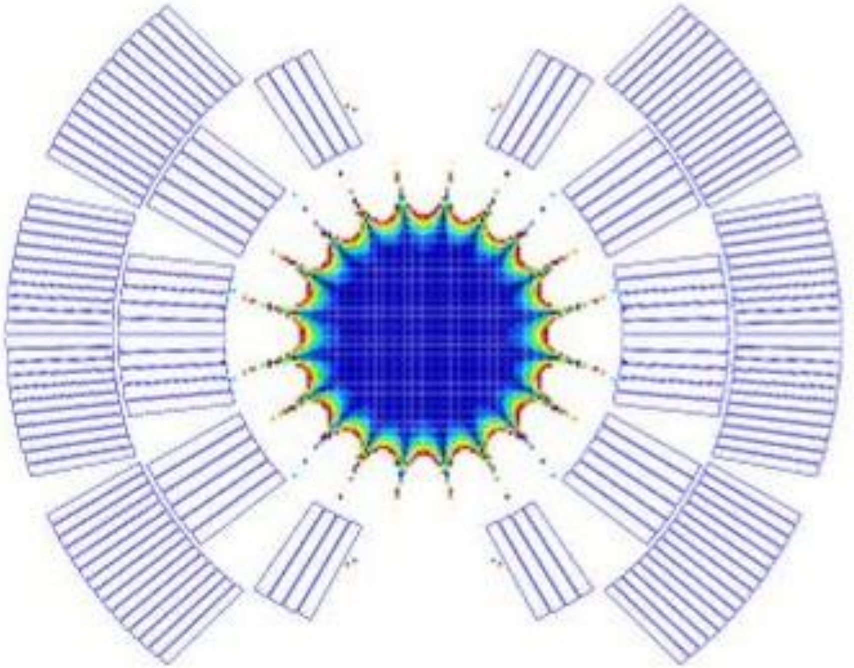
$$\mathbf{E} = -\operatorname{grad}\phi - \frac{\partial}{\partial t}\mathbf{A}$$

Harmonic Fields

Field Quality

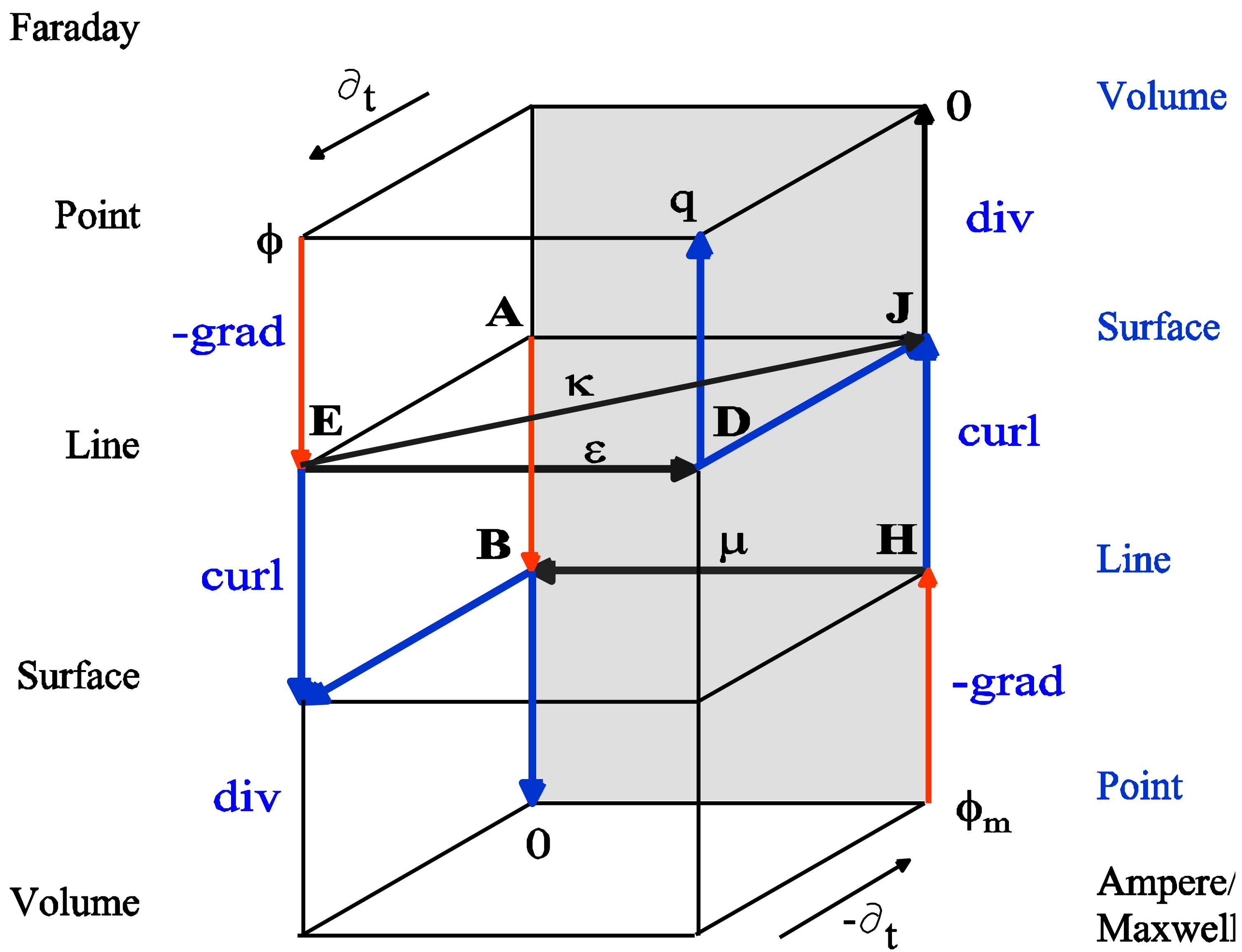


Field map



Good field region

Maxwell’s House



Maxwell's Facade

$$\operatorname{curl}\frac{1}{\mu}\operatorname{curl}\mathbf{A} = \mathbf{J}$$

$$\frac{1}{\mu_0}\operatorname{curl}\operatorname{curl}\mathbf{A} = \mathbf{J}$$

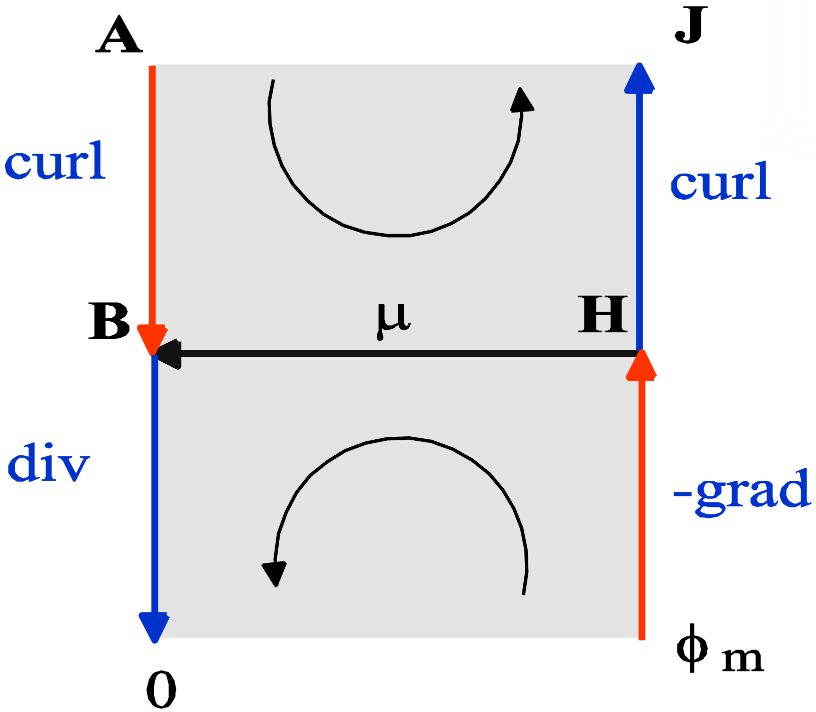
$$\nabla^2\mathbf{A} - \operatorname{grad}\operatorname{div}\mathbf{A} = 0$$

$$\nabla^2 A_z = 0$$

$$\operatorname{div}\mu\operatorname{grad}\phi_\mathrm{m} = 0$$

$$\mu_0\operatorname{div}\operatorname{grad}\phi_\mathrm{m} = 0$$

$$\nabla^2\phi_\mathrm{m} = 0$$



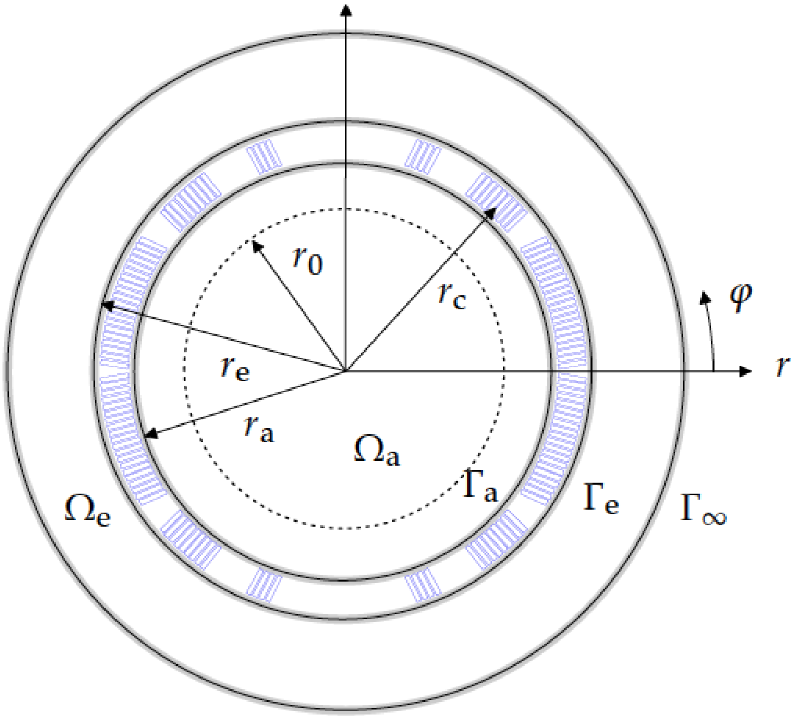
Solving of Boundary Value Problems

1. Governing equation in the air domain

$$\nabla^2 A_z = 0,$$

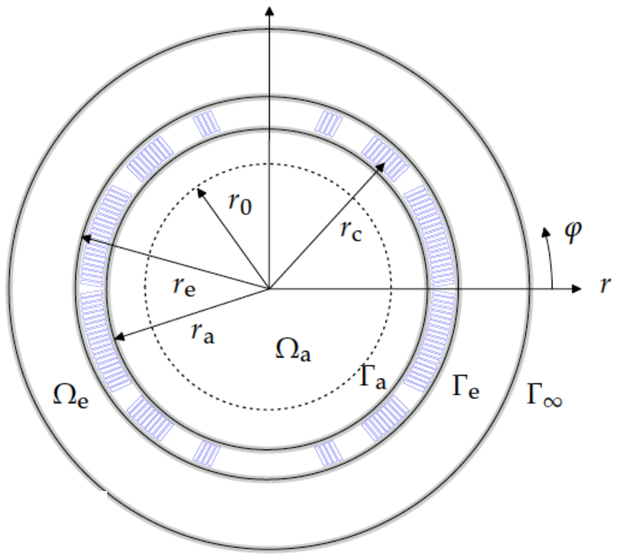
2. Chose a suitable coordinate system

$$r^2 \frac{\partial^2 A_z}{\partial r^2} + r \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0,$$



3. Find eigenfunctions. Coefficients are not know yet

$$A_z(r,\varphi) = \sum_{n=1}^{\infty} (\mathcal{E}_n r^n + \mathcal{F}_n r^{-n})(\mathcal{G}_n \sin n\varphi + \mathcal{H}_n \cos n\varphi).$$



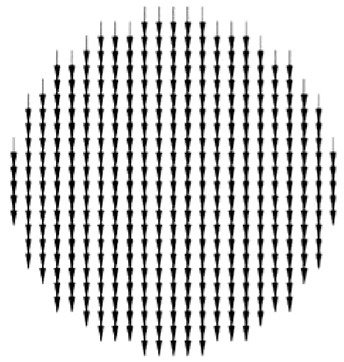
4. Incorporate a bit of knowledge and rename

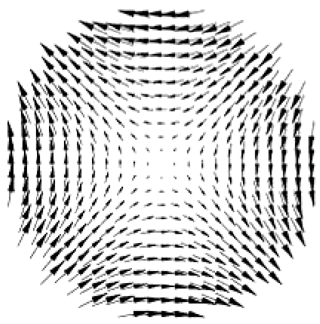
$$A_z(r,\varphi) = \sum_{n=1}^{\infty} r^n (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi).$$

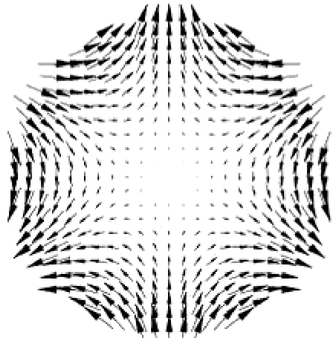
5. Calculate a field component

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

$$B_\varphi(r,\varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi),$$





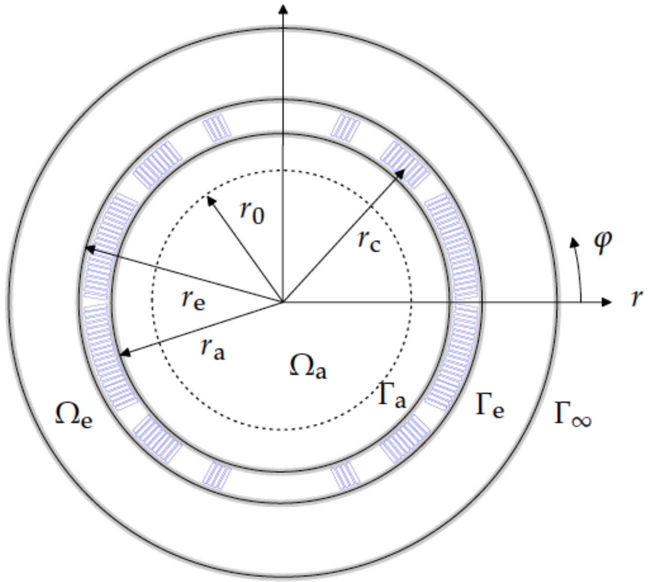


Solving of Boundary Value Problems

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1}(\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

6. Measure or calculate the field on a reference radius and perform Fourier analysis (develop into the eigenfunctions). Coefficients known here.

$$B_r(r_0,\varphi) = \sum_{n=1}^{\infty} (B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi),$$



7: Compare the known and unknown coefficients

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

$$B_r(r_0,\varphi) = \sum_{n=1}^{\infty} (B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi),$$

$$\mathcal{A}_n = \frac{1}{n\, r_0^{n-1}} A_n(r_0), \qquad \mathcal{B}_n = \frac{-1}{n\, r_0^{n-1}} B_n(r_0).$$

8. Put this into the original solution for the entire air domain

$$A_z(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r_0}{n}\left(\frac{r}{r_0}\right)^n (B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi).$$

9: Calculate fields and potential in the entire air domain

$$A_z(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r_0}{n}\left(\frac{r}{r_0}\right)^n (B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi).$$

$$B_r(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi)$$

$$B_\varphi(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi)$$

$$B_x(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\sin(n-1)\varphi + A_n(r_0)\cos(n-1)\varphi)$$

$$B_y(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\cos(n-1)\varphi - A_n(r_0)\sin(n-1)\varphi)$$

Multipoles and Scaling Laws

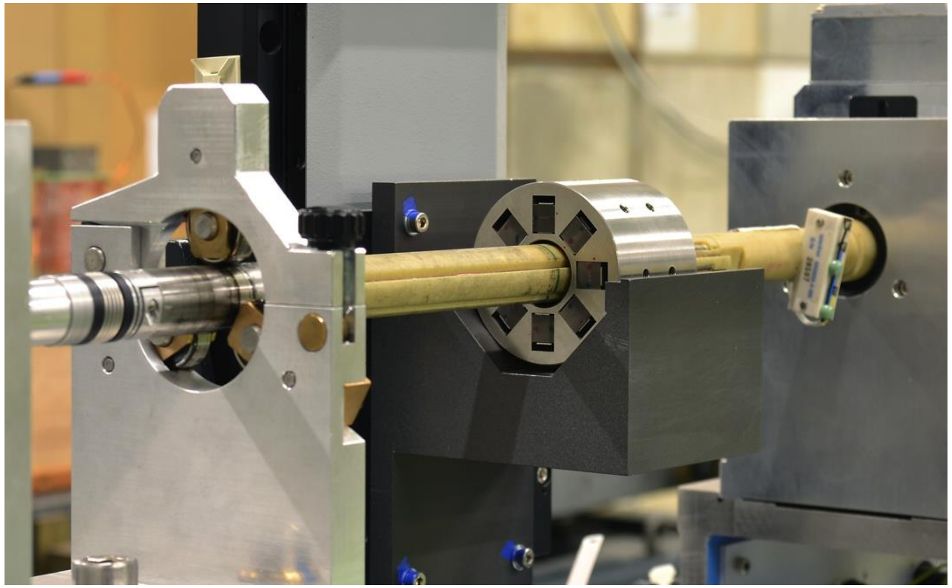
$$B_r(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi)$$

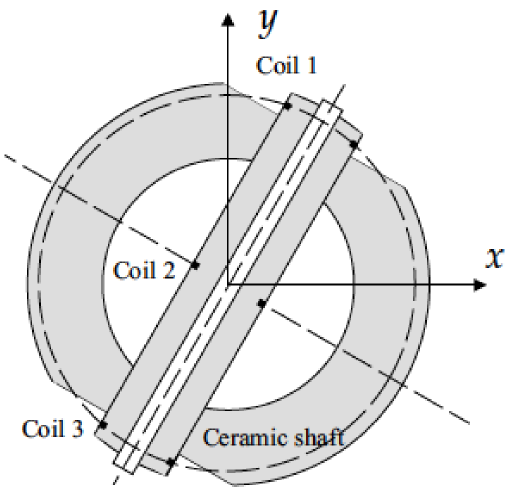
$$B_r(r,\varphi) = B_N \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-N} (b_n(r_0)\sin n\varphi + a_n(r_0)\cos n\varphi)\,.$$

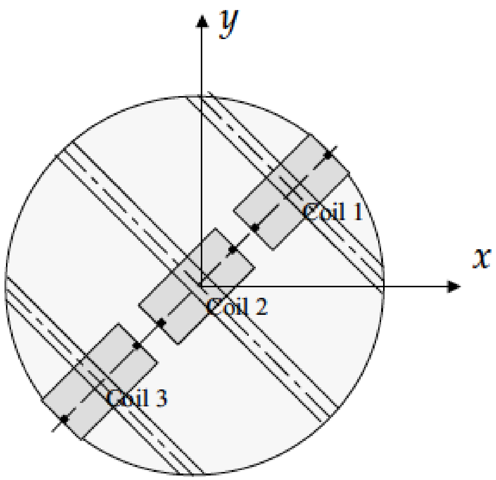
$$A_n(r_1) = \left(\frac{r_1}{r_0}\right)^{n-1} A_n(r_0), \qquad B_n(r_1) = \left(\frac{r_1}{r_0}\right)^{n-1} B_n(r_0),$$

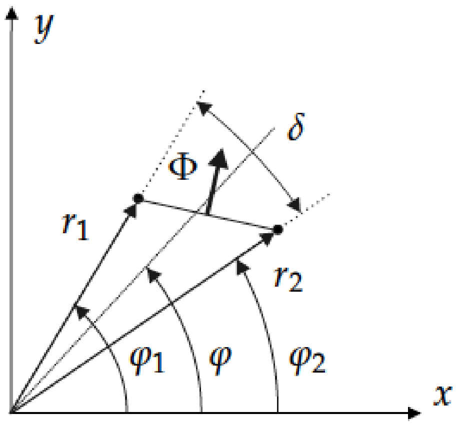
$$b_n(r_1) = \frac{B_n(r_1)}{B_N(r_1)} = \frac{\left(\frac{r_1}{r_0}\right)^{n-1} B_n(r_0)}{\left(\frac{r_1}{r_0}\right)^{N-1} B_N(r_0)} = \left(\frac{r_1}{r_0}\right)^{n-N} b_n(r_0),$$

Rotating Coil Measurements









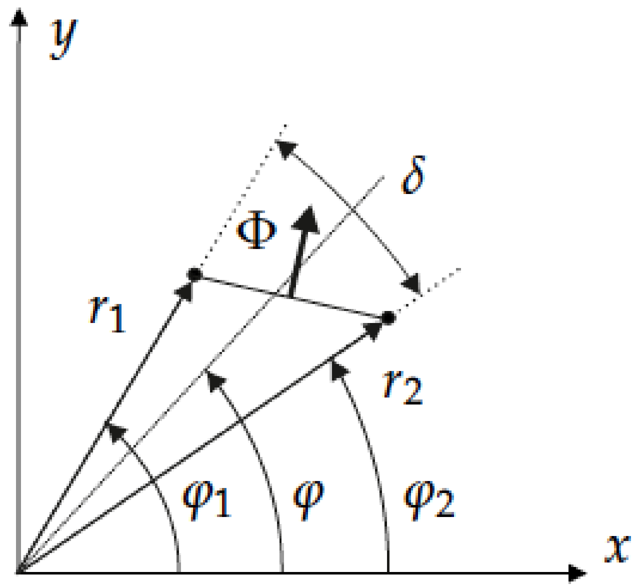
Tangential coil
Radial flux

Radial coil
Tangential flux

Rotating Coil Measurements

$$\Phi(\varphi) = N\int_{\mathscr{A}} \mathbf{B}\cdot \mathrm{d}\mathbf{a} = N\int_{\mathscr{A}} \operatorname{curl}\mathbf{A}\cdot \mathrm{d}\mathbf{a} = N\int_{\partial\mathscr{A}} \mathbf{A}\cdot \mathrm{d}\mathbf{r}$$
$$= N\ell\,[A_z(\mathscr{P}_1) - A_z(\mathscr{P}_2)]\,,$$

$$\Phi(\varphi) = N\ell\left[\sum_{n=1}^{\infty}\frac{r_0}{n}\left(\frac{r_2}{r_0}\right)^n (B_n(r_0)\cos n\varphi_2 - A_n(r_0)\sin n\varphi_2) - \sum_{n=1}^{\infty}\frac{r_0}{n}\left(\frac{r_1}{r_0}\right)^n (B_n(r_0)\cos n\varphi_1 - A_n(r_0)\sin n\varphi_1)\right],$$



$$\Phi(\varphi) = \sum_{n=1}^{\infty} S_n^{\mathrm{rad}}\,(B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi) + S_n^{\mathrm{tan}}\,(B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi)$$

$$S_n^{\mathrm{rad}} = \frac{N\ell}{n r_0^{n-1}}\left[r_2^n\,\cos\,n(\varphi_2-\varphi) - r_1^n\,\cos\,n(\varphi_1-\varphi)\right],$$
$$S_n^{\mathrm{tan}} = -\frac{N\ell}{n r_0^{n-1}}\left[r_2^n\,\sin\,n(\varphi_2-\varphi) - r_1^n\,\sin\,n(\varphi_1-\varphi)\right],$$

Cartesian Coordinates (Eigensolutions for the Ideal Dipole)

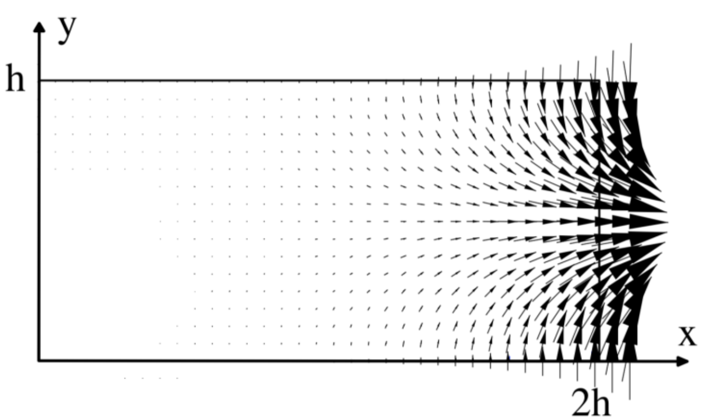
$$\phi_{\mathrm{m}} = X(x)Y(y)$$

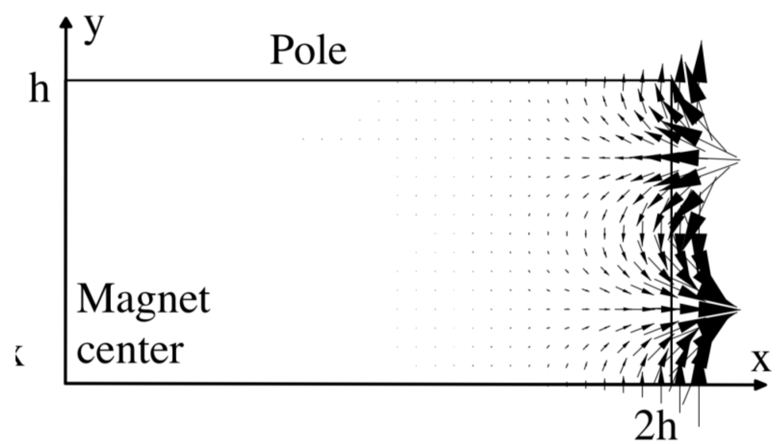
$$\underbrace{\frac{1}{X(x)}\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2}}_{p^2} + \underbrace{\frac{1}{Y(y)}\frac{\mathrm{d}^2 Y(y)}{\mathrm{d}y^2}}_{-p^2} = 0$$

$$X_p(x) = \mathcal{C}_p \cos px + \mathcal{D}_p \sin px\,,$$
$$Y_p(y) = \mathcal{E}_p \cosh py + \mathcal{F}_p \sinh py\,,$$

$$p = n\frac{2\pi}{\lambda} =: nk_0\,.$$

$$B_x(x,y) = \mu_0 \sum_{n=1}^{\infty} \mathcal{A}_n \sinh\left(\frac{n\pi}{h}x\right) \sin\left(\frac{n\pi}{h}y\right)\,,$$
$$B_y(x,y) = B_0 + \mu_0 \sum_{n=1}^{\infty} \mathcal{A}_n \cosh\left(\frac{n\pi}{h}x\right) \cos\left(\frac{n\pi}{h}y\right)\,.$$

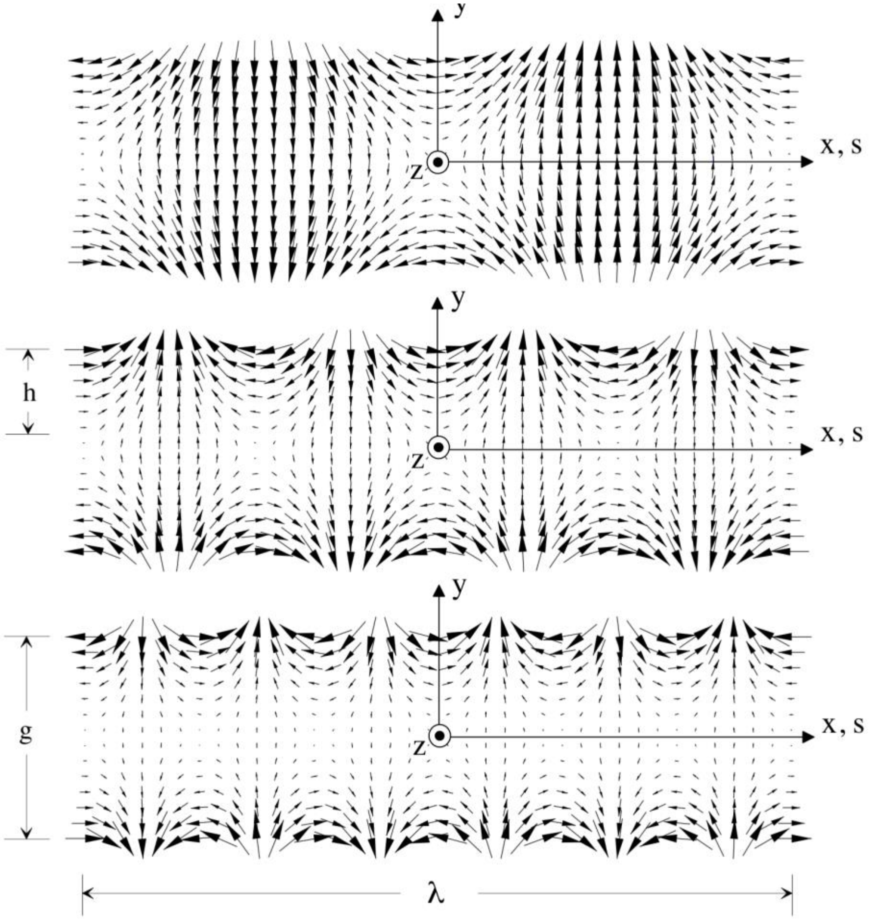




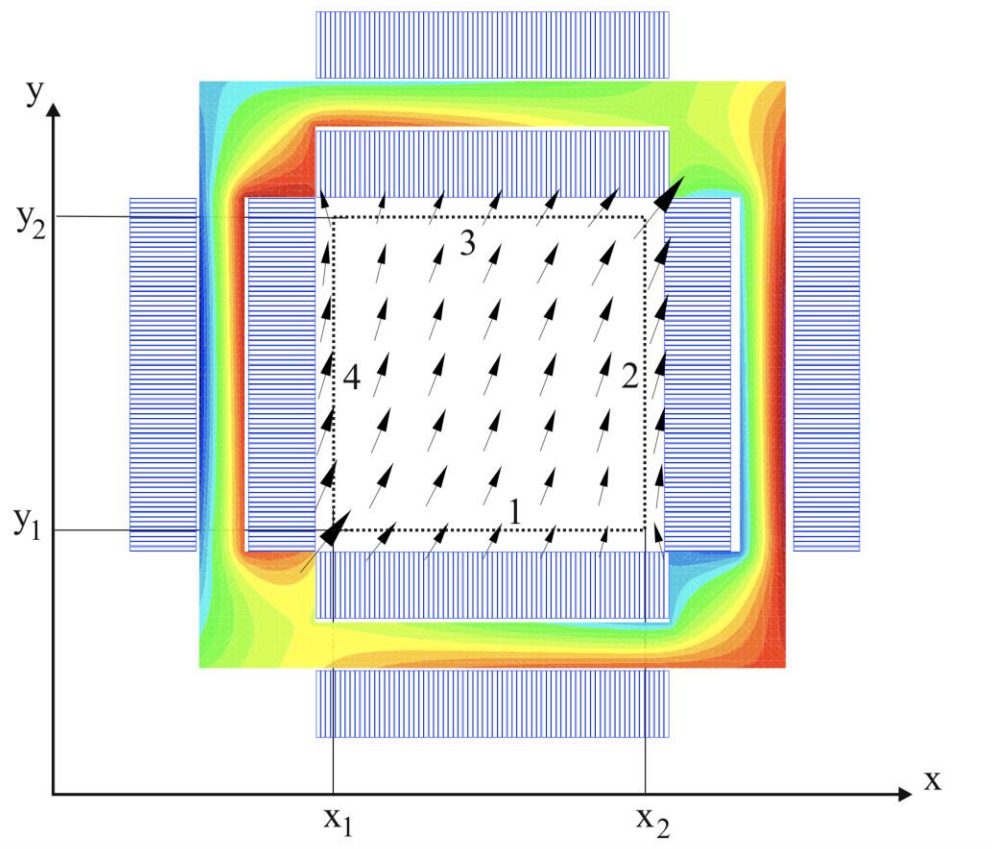
Cartesian Coordinates (Eigensolutions for the Wiggler)

$$B_x(x,y) = \mu_0 \sum_{n=1}^{\infty} \left(-\mathcal{A}_n \sin(nk_0x) + \mathcal{B}_n \cos(nk_0x)\right) \sinh(nk_0y) \ ,$$

$$B_y(x,y) = \mu_0 \sum_{n=1}^{\infty} \left(\mathcal{A}_n \cos(nk_0x) + \mathcal{B}_n \sin(nk_0x)\right) \cosh(nk_0y) \ .$$

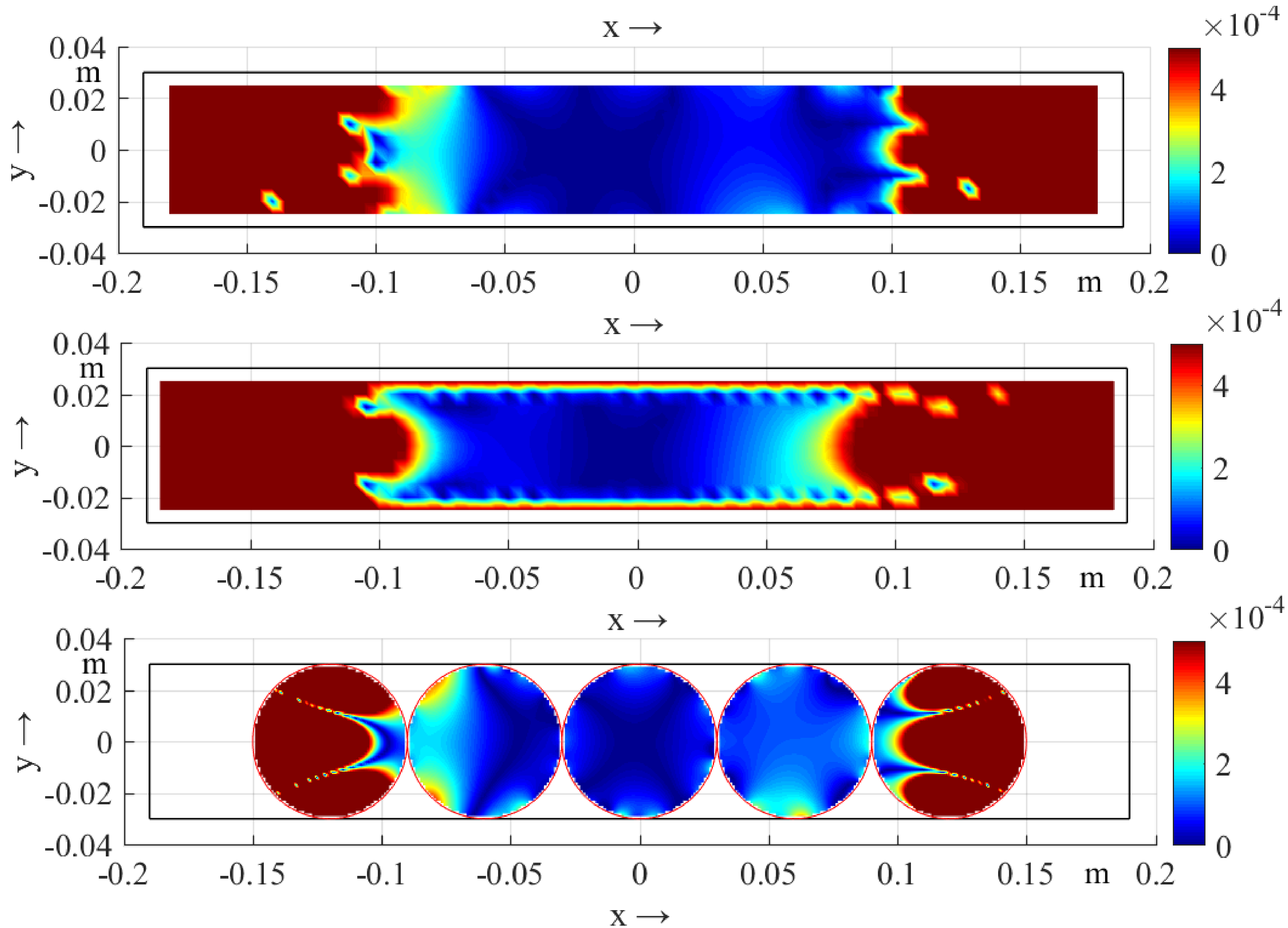


Determining the Coefficients



$$A_z^{(1)}(x,y) = \sum_n A_n^{(1)} \frac{\sinh\left(n\pi \frac{y_2-y}{x_2-x_1}\right)}{\sinh\left(n\pi \frac{y_2-y_1}{x_2-x_1}\right)} \sin\left(n\pi \frac{x_2-x}{x_2-x_1}\right)$$

Field Reconstruction from Boundary Data



Theorems on Harmonic Fields

Theorem 5.1 *If ϕ is harmonic in the closed contractible volume $\mathcal{V} \subset \Omega$ bounded by the surface $\partial\mathcal{V}$, the surface integral of the normal derivative of ϕ vanishes.*

Flux in = flux out

Theorem 5.2 *If ϕ is harmonic in the closed, contractible volume $\mathcal{V} \subset \Omega$, bounded by the surface $\partial\mathcal{V}$, with the same magnitude at all points on that surface, then ϕ is constant throughout $\mathcal{V}$ and equal to its value ϕ_0 on the boundary.*

Faraday cage

Theorem 5.3 *If ϕ is harmonic in the closed contractible volume $\mathcal{V} \subset \Omega$ bounded by $\partial\mathcal{V}$ and its value is specified at each point of that boundary, then ϕ is uniquely determined at all points inside the volume.*

Determine fields by Fourier analysis on boundary

Theorem 5.5 (Liouville) *If ϕ is a harmonic scalar field in E_n with an upper (or lower) bound, ϕ is constant.*

Watch out for singularities (sources of the field), maximum field at the boundary

Complex Potentials

$$\mathbf{H} = -\operatorname{grad}\phi = -\frac{\partial\phi}{\partial x}\mathbf{e}_x - \frac{\partial\phi}{\partial y}\mathbf{e}_y,$$

$$\mathbf{B} = \operatorname{curl}(\mathbf{e}_z A_z) = \frac{\partial A_z}{\partial y}\mathbf{e}_x - \frac{\partial A_z}{\partial x}\mathbf{e}_y\,.$$

This implies

$$\frac{\partial A_z}{\partial y} = -\mu_0\frac{\partial\phi}{\partial x} \quad \text{and} \quad \frac{\partial A_z}{\partial x} = \mu_0\frac{\partial\phi}{\partial y},$$

Which are the Cauchy Riemann equations of

$$w(z) := u(x,y) + iv(x,y) = A_z(x,y) + i\mu_0\phi(x,y)\,.$$

$$-\frac{\mathrm{d}w}{\mathrm{d}z} = -\frac{\partial A_z}{\partial x} - i\mu_0\frac{\partial\phi}{\partial x} = i\frac{\partial A_z}{\partial y} - \mu_0\frac{\partial\phi}{\partial y} = B_y(x,y) + iB_x(x,y) =: B(z).$$

Theorem 9.2 *Real and imaginary parts of a holomorphic function are harmonic functions.*

Proof. If $f(z) = f(x,y) = u(x,y) + iv(x,y)$ is holomorphic, the Cauchy–Riemann equations yield

$$\nabla^2 u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0,$$

$$\nabla^2 v = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 0.$$

Complex Representation of the Field in Accelerator Magnets

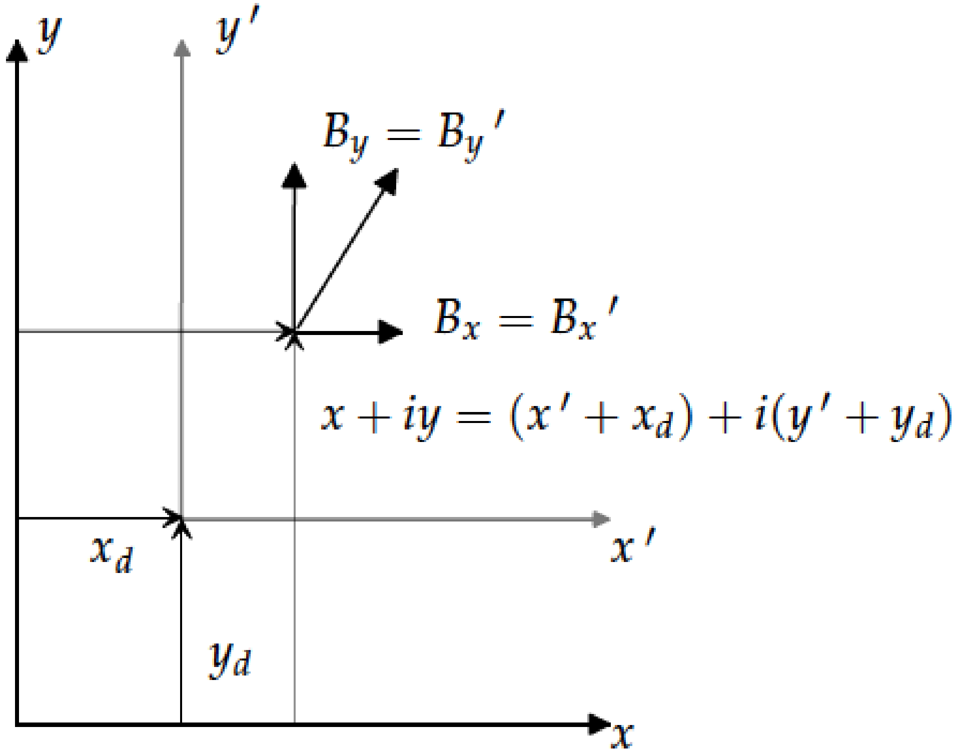
$$B_x = B_r \cos\varphi - B_\varphi \sin\varphi, \qquad B_y = B_r \sin\varphi + B_\varphi \cos\varphi,$$

$$B_y + iB_x = (B_\varphi + iB_r)e^{-i\varphi}.$$

$$\begin{aligned}
B_y + iB_x &= \sum_{n=1}^{\infty} (B_n(r_0) + i\,A_n(r_0)) \left(\frac{r}{r_0}\right)^{n-1} e^{i(n-1)\varphi} \\
&= \sum_{n=1}^{\infty} (B_n(r_0) + i\,A_n(r_0)) \left(\frac{z}{r_0}\right)^{n-1} \\
&= B_N \sum_{n=1}^{\infty} (b_n(r_0) + i\,a_n(r_0)) \left(\frac{z}{r_0}\right)^{n-1},
\end{aligned}$$

$$b_n = \frac{r^{n-1}}{B_N} \frac{1}{(n-1)!} \left.\frac{\mathrm{d}^{n-1} B_y}{\mathrm{d}x^{n-1}}\right|_{x=y=0}$$

Feed-down (Holomorphic Continuation)



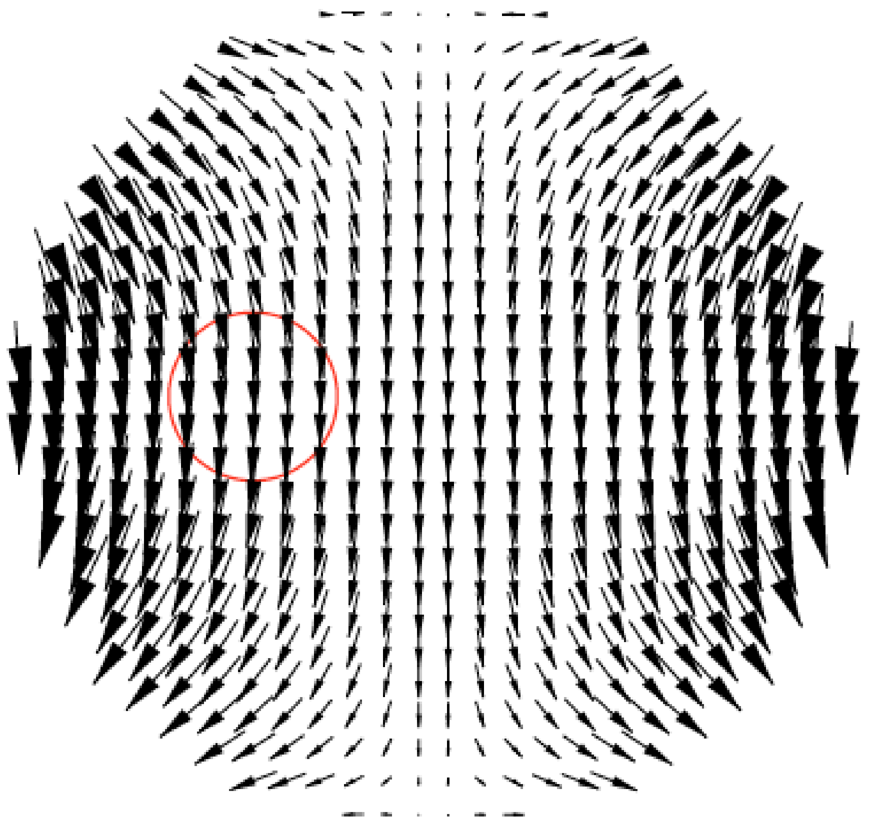
$$\sum_{n=1}^{\infty} C_n \left(\frac{z}{r_0}\right)^{n-1} \stackrel{!}{=} \sum_{n=1}^{\infty} C'_n \left(\frac{z'}{r_0}\right)^{n-1},$$

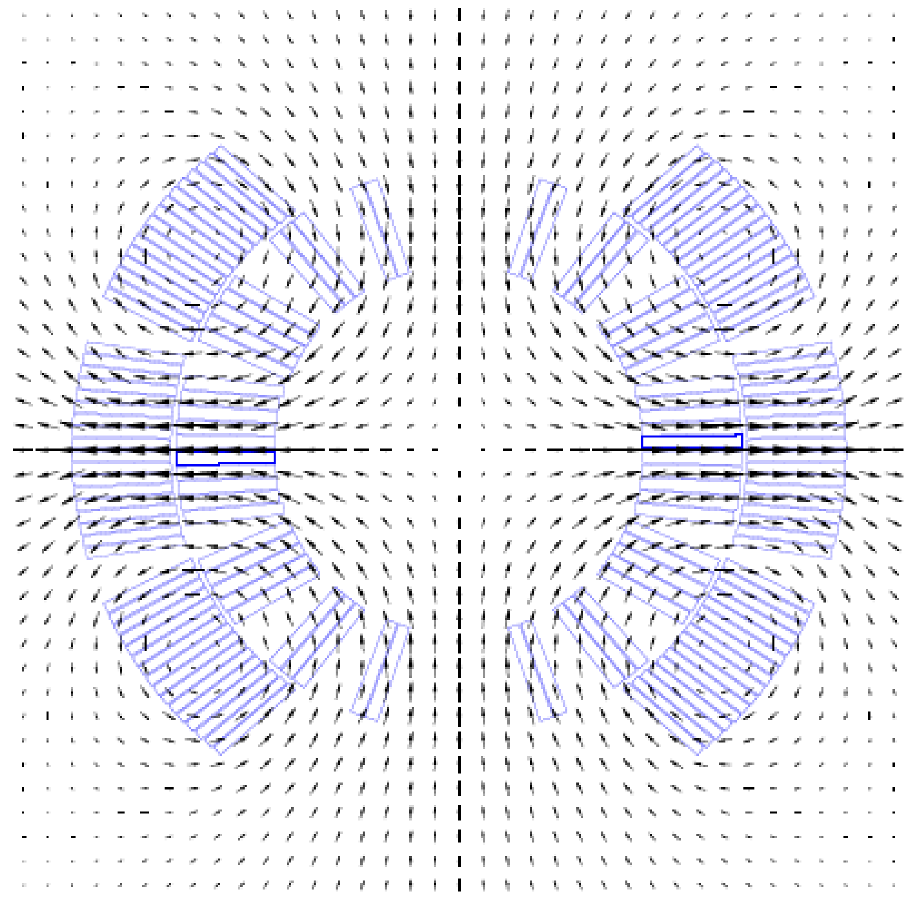
$$\binom{n}{p} = \frac{n!}{p!(n-p)!} \text{ for } 0 \le p \le n$$

$$C'_n = \sum_{k=n}^{\infty} C_k \binom{k-1}{n-1} \left(\frac{z_\mathrm{d}}{r_0}\right)^{k-n}.$$

$$C'_2 = C_2 + 2\,C_3 \left(\frac{z_\mathrm{d}}{r_0}\right) + 3\,C_4 \left(\frac{z_\mathrm{d}}{r_0}\right)^2 + \cdots,$$

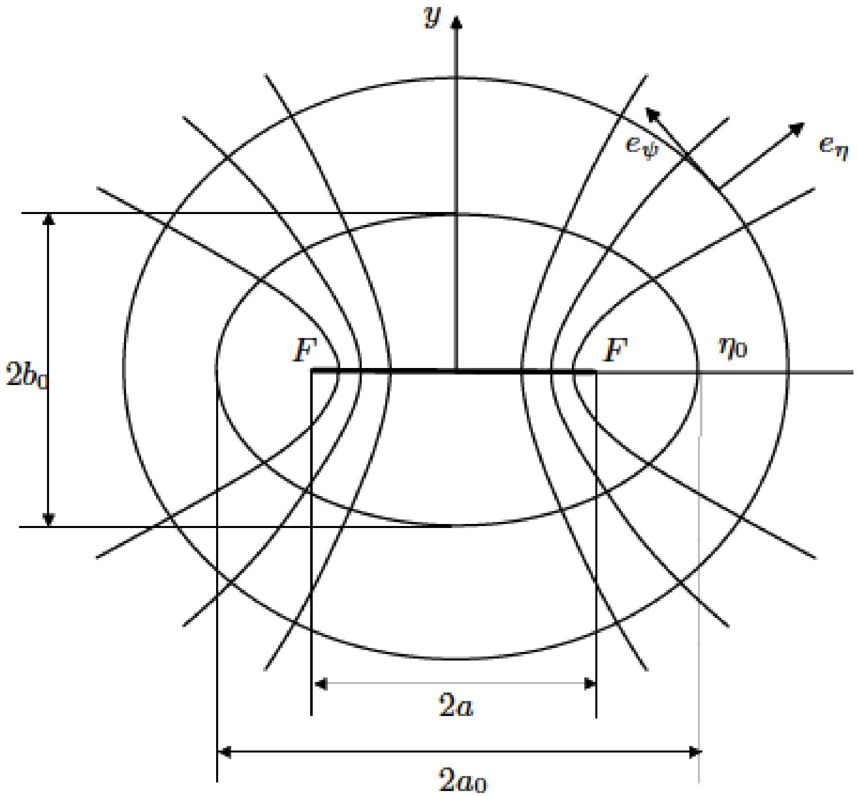
Feed-down: Enemy and Friend





Elliptical Harmonics

$$B_\eta(\eta,\psi) = \frac{1}{h_2}\sum_{n=1}^{\infty}\left(n\,\mathcal{A}_n \sinh n\eta \cos n\psi - n\,\mathcal{B}_n \cosh n\eta \sin n\psi\right) .$$



$$h_1 = h_2 = a\sqrt{\cosh^2\eta - \cos^2\psi}.$$

$$B_\eta = \frac{1}{h_1}\left(a\ \sinh\eta\cos\psi\, B_x + a\ \cosh\eta\sin\psi\, B_y\right) .$$

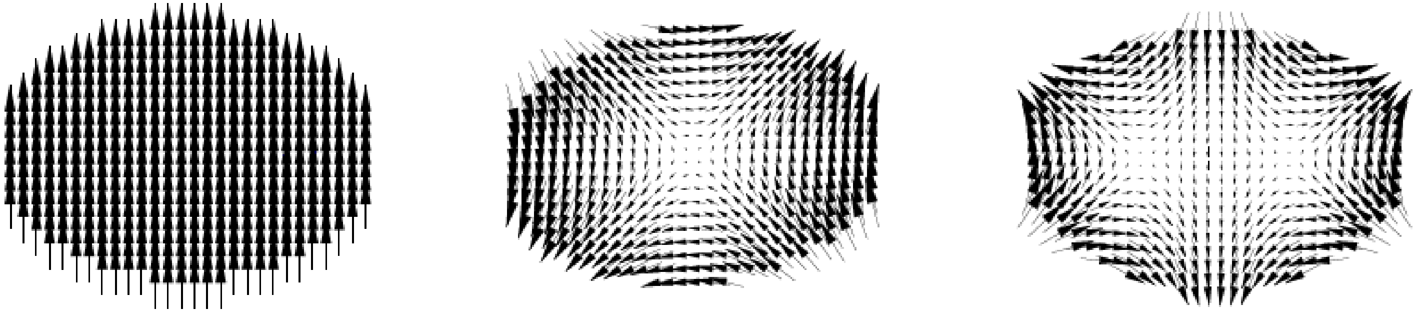
$$B_\eta(\eta_0,\psi) = \sum_{n=1}^{\infty}\left(B_n(\eta_0)\sin n\psi + A_n(\eta_0)\cos n\psi\right),$$

Solution: Use covariant derivative, i.e, differential forms (Auchmann, Kurz, Petrone, Russenschuck 2015)

Metric-Free Elliptic Multipoles

$$\tilde{B}_\eta = \frac{\partial A_z}{\partial \psi} \qquad \tilde{B}_\psi = \frac{\partial A_z}{\partial \eta}\,.$$

$$\tilde{B}_\eta(\eta,\psi) = \sum_{n=1}^{\infty} \left(n\mathcal{A}_n \sinh n\eta \cos n\psi - n\mathcal{B}_n \cosh n\eta \sin n\psi \right)\,.$$



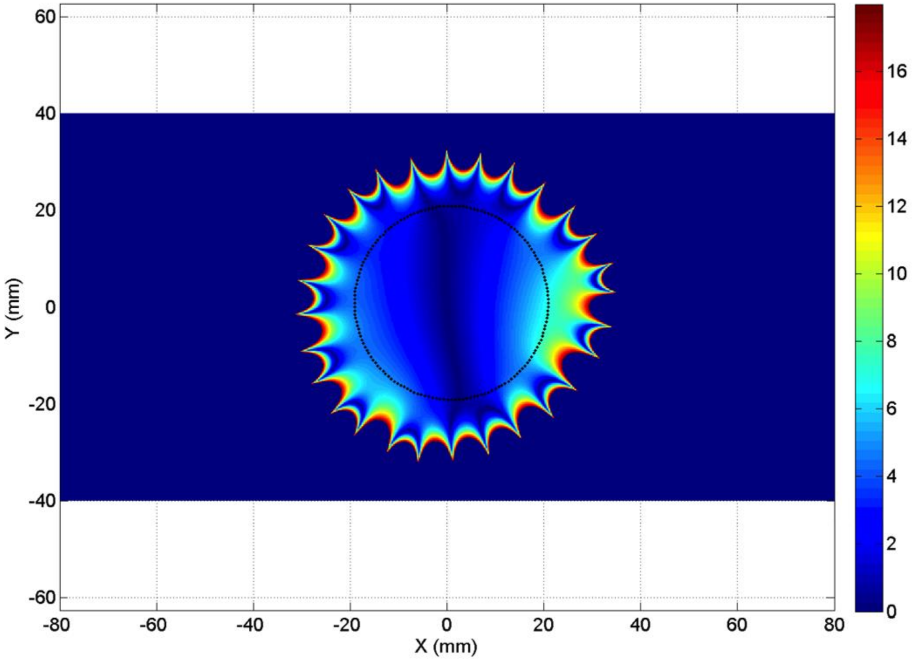
$$\tilde{B}_\eta(\eta_0,\psi) = \sum_{n=1}^{\infty} \left(\tilde{B}_n(\eta_0) \sin n\psi + \tilde{A}_n(\eta_0) \cos n\psi \right)\,,$$

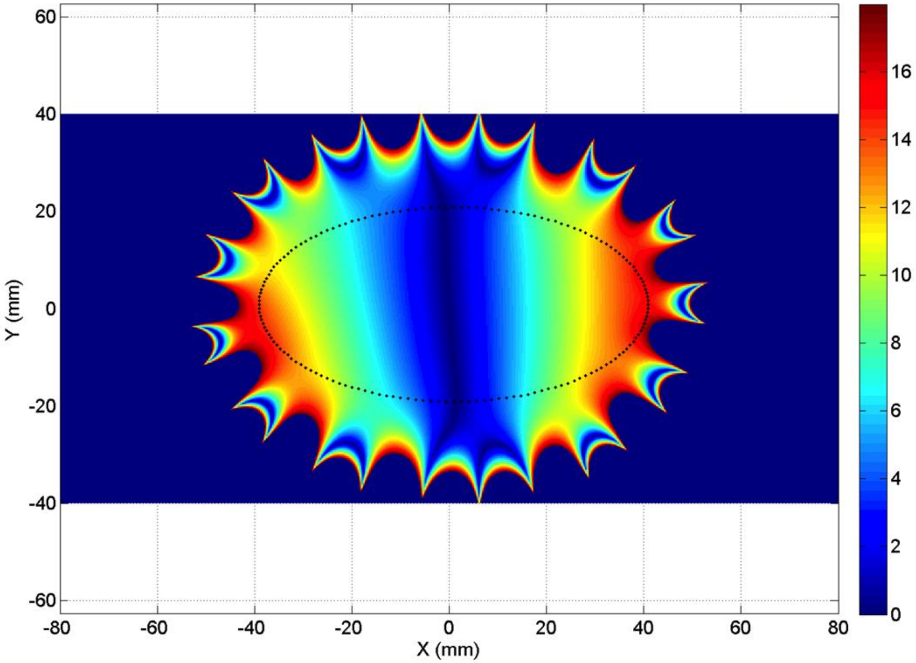
$$\mathcal{A}_n = \frac{1}{n \sinh n\eta_0} \tilde{A}_n(\eta_0), \qquad \mathcal{B}_n = -\frac{1}{n \cosh n\eta_0} \tilde{B}_n(\eta_0)\,,$$

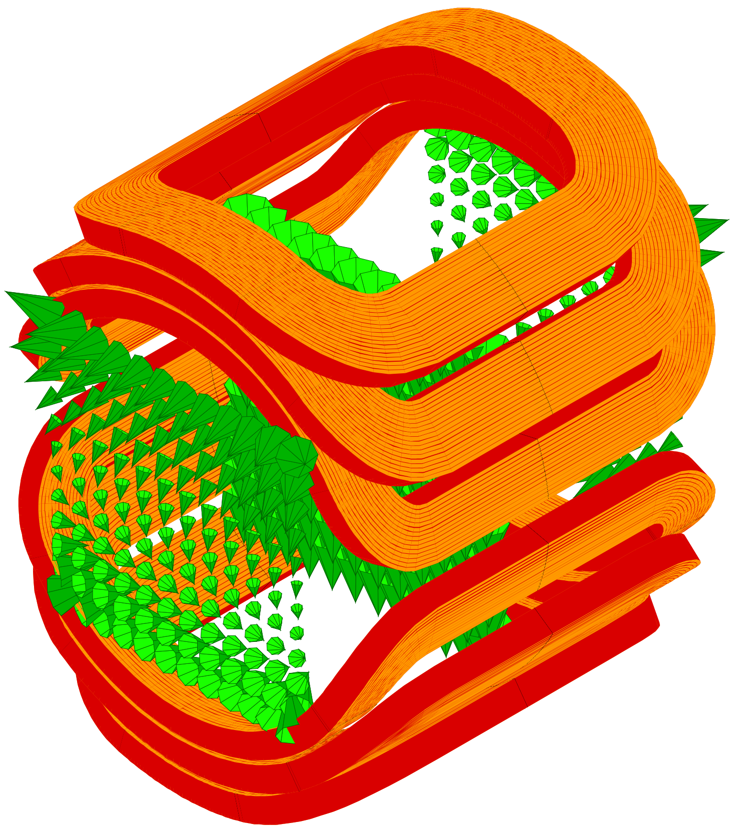
Results for the MM-Section's Calibration Magnets (ISR dipole)

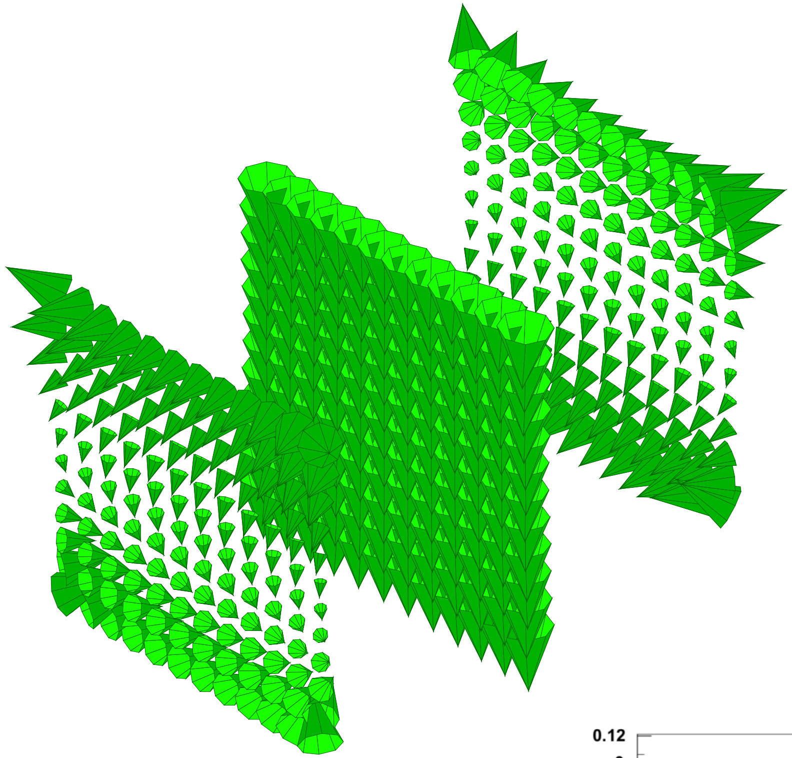
$$B_\eta(\eta,\psi) = \frac{1}{h_2}\sum_{n=1}^{\infty}\left(\tilde{B}_n(\eta_0)\frac{\cosh n\eta}{\cosh n\eta_0}\sin n\psi + \tilde{A}_n(\eta_0)\frac{\sinh n\eta}{\sinh n\eta_0}\cos n\psi\right),$$

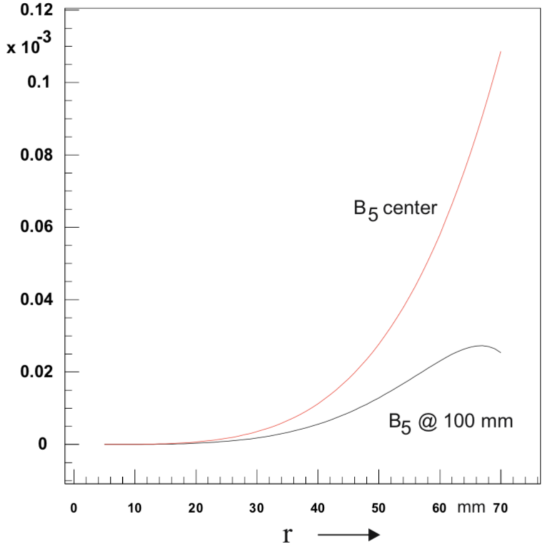
$$B_\psi(\eta,\psi) = \frac{1}{h_1}\sum_{n=1}^{\infty}\left(\tilde{B}_n(\eta_0)\frac{\sinh n\eta}{\cosh n\eta_0}\cos n\psi - \tilde{A}_n(\eta_0)\frac{\cosh n\eta}{\sinh n\eta_0}\sin n\psi\right).$$





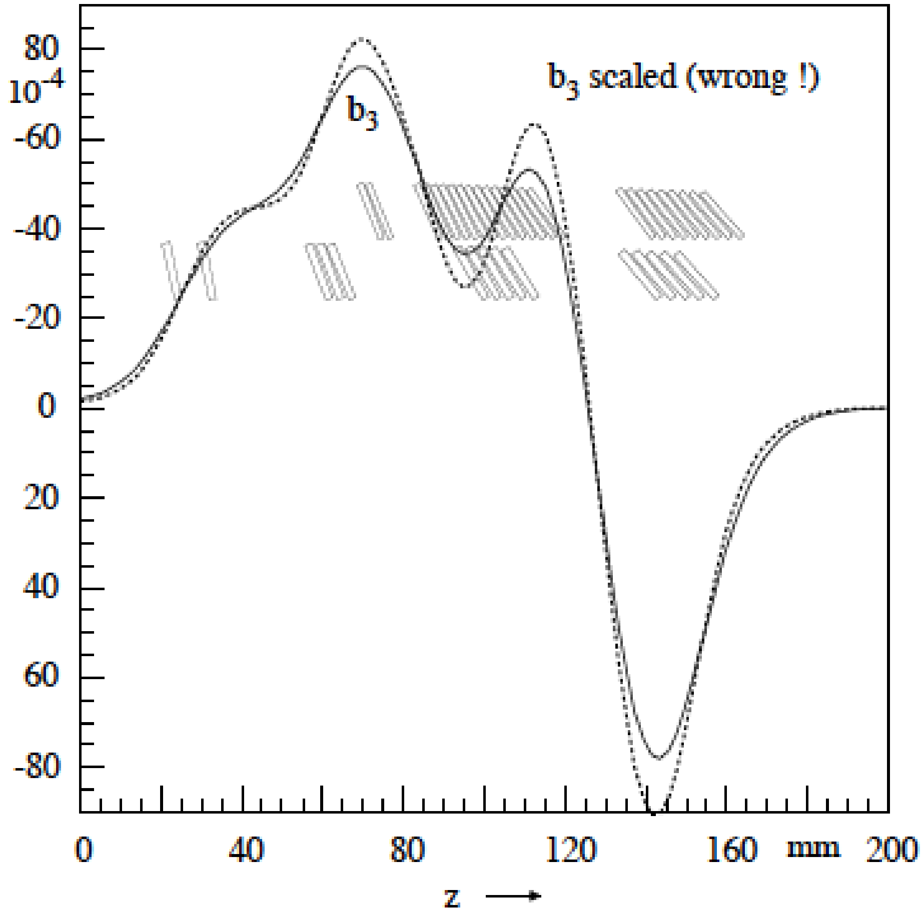






0.12
x 10-3
0.1
0.08
0.06
0.04
0.02
0
B5 center
B5 @ 100 mm
0 10 20 30 40 50 60 mm 70
r

Integrated Harmonics



Local transverse harmonics calculated at different reference radii and scaled with the 2D laws

$$b_n(r_1) = \left(\frac{r_1}{r_0}\right)^{n-N} b_n(r_0),$$

wrong

Integrated Harmonics

$$\nabla^2\phi_{\mathrm{m}}(x,y,z) = \frac{\partial^2\phi_{\mathrm{m}}(x,y,z)}{\partial x^2} + \frac{\partial^2\phi_{\mathrm{m}}(x,y,z)}{\partial y^2} + \frac{\partial^2\phi_{\mathrm{m}}(x,y,z)}{\partial z^2} = 0\,.$$

$$\overline{\phi}_{\mathrm{m}}(x,y) := \int_{-z_0}^{z_0} \phi_{\mathrm{m}}(x,y,z)\mathrm{d}z.$$

$$\begin{aligned}
\frac{\partial^2\overline{\phi}_{\mathrm{m}}(x,y)}{\partial x^2} + \frac{\partial^2\overline{\phi}_{\mathrm{m}}(x,y)}{\partial y^2} &= \int_{-z_0}^{z_0} \left(\frac{\partial^2\phi_{\mathrm{m}}}{\partial x^2} + \frac{\partial^2\phi_{\mathrm{m}}}{\partial y^2}\right)\mathrm{d}z \\
&= \int_{-z_0}^{z_0} \left(-\frac{\partial^2\phi_{\mathrm{m}}}{\partial z^2}\right)\mathrm{d}z = -\left.\frac{\partial\phi_{\mathrm{m}}}{\partial z}\right|_{-z_0}^{z_0} \\
&= H_z(-z_0) - H_z(z_0) \stackrel{!}{=} 0\,.
\end{aligned}$$

The 2D scaling laws hold for the integrated harmonics

Pseudo-Multipoles (Fourier Bessel Series)

$$\phi_{\mathrm{m}}(r,\varphi,z) = \left\{ \begin{array}{c} \cos n\varphi \\ \sin n\varphi \end{array} \right\} I_n(pr) \left\{ \begin{array}{c} \cos pz \\ \sin pz \end{array} \right\}$$

$$I_n(pr) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+n+1)} \left(\frac{pr}{2}\right)^{n+2k}$$

$$\phi_{\mathrm{m}} = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} r^{n+2k} (\mathcal{C}_{n+2k,n}(z) \sin n\varphi + \mathcal{D}_{n+2k,n}(z) \cos n\varphi)$$

$$
\frac{1}{r}\frac{\partial}{\partial r}\Big\{\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}(n+2k)r^{n+2k}\left(\mathcal{C}_{n+2k,n}(z)\sin n\varphi+\mathcal{D}_{n+2k,n}(z)\cos n\varphi\right)\Big\}
$$

$$
-\frac{1}{r^2}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}n^2\, r^{n+2k}\left(\mathcal{C}_{n+2k,n}(z)\sin n\varphi+\mathcal{D}_{n+2k,n}(z)\cos n\varphi\right)
$$

$$
+\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}r^{n+2k}\left(\mathcal{C}^{(2)}_{n+2k,n}(z)\sin n\varphi+\mathcal{D}^{(2)}_{n+2k,n}(z)\cos n\varphi\right)
$$

$$
=\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}(n+2k)^2\, r^{n+2k-2}\left(\mathcal{C}_{n+2k,n}(z)\sin n\varphi+\mathcal{D}_{n+2k,n}(z)\cos n\varphi\right)
$$

$$
-\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}n^2\, r^{n+2k-2}\left(\mathcal{C}_{n+2k,n}(z)\sin n\varphi+\mathcal{D}_{n+2k,n}(z)\cos n\varphi\right)
$$

$$
+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}r^{n+2k-2}\left(\mathcal{C}^{(2)}_{n+2k-2,n}(z)\sin n\varphi+\mathcal{D}^{(2)}_{n+2k-2,n}(z)\cos n\varphi\right)
$$

$$
=0\,, \qquad (
$$

Recursion for Cn,n

$$\mathcal{C}_{n+2k,n}(z)\left((n+2k)^2-n^2\right)+\mathcal{C}^{(2)}_{n+2k-2,n}(z)=0\,,$$

$$\mathcal{D}_{n+2k,n}(z)\left((n+2k)^2-n^2\right)+\mathcal{D}^{(2)}_{n+2k-2,n}(z)=0\,.$$

$$\mathcal{C}_{n+2k,n}(z)=\frac{1}{\prod_{m=1}^{k}(n^2-(n+2m)^2)}\,\mathcal{C}^{(2k)}_{n,n}(z)\,,$$

$$\phi_\mathrm{m} = \sum_{n=1}^{\infty} \left\{ \sum_{k=0}^{\infty} \frac{1}{\prod_{m=1}^{k} (n^2 - (n+2m)^2)} \mathcal{C}_{n,n}^{(2k)}(z) \right\} r^n \sin n\varphi + \sum_{n=1}^{\infty} \left\{ \sum_{k=0}^{\infty} \frac{1}{\prod_{m=1}^{k} (n^2 - (n+2m)^2)} \mathcal{D}_{n,n}^{(2k)}(z) \right\} r^n \cos n\varphi \,,$$

$$\phi_\mathrm{m} = \sum_{n=1}^{\infty} \left\{ \mathcal{C}_{n,n}(z) - \frac{\mathcal{C}_{n,n}^{(2)}(z)}{4(n+1)} r^2 + \frac{\mathcal{C}_{n,n}^{(4)}(z)}{32(n+1)(n+2)} r^4 - \frac{\mathcal{C}_{n,n}^{(6)}(z)}{384(n+1)(n+2)(n+3)} r^6 + \dots \right\} r^n \sin n\varphi + \sum_{n=1}^{\infty} \left\{ \mathcal{D}_{n,n}(z) - \frac{\mathcal{D}_{n,n}^{(2)}(z)}{4(n+1)} r^2 + \frac{\mathcal{D}_{n,n}^{(4)}(z)}{32(n+1)(n+2)} r^4 - \frac{\mathcal{D}_{n,n}^{(6)}(z)}{384(n+1)(n+2)(n+3)} r^6 + \dots \right\} r^n \cos n\varphi \,,$$

Mid-Plane Field

$$\frac{-1}{\mu_0} B_y(x, y=0, z) \approx$$

$$\mathcal{C}_{1,1}(z) \quad - \frac{\mathcal{C}_{1,1}^{(2)}(z)}{8} x^2 \quad + \frac{\mathcal{C}_{1,1}^{(4)}(z)}{192} x^4 \quad - \frac{\mathcal{C}_{1,1}^{(6)}(z)}{9216} x^6$$

$$+ 3\,\mathcal{C}_{3,3}(z)\, x^2 - \frac{3\,\mathcal{C}_{3,3}^{(2)}(z)}{16} x^4 + \frac{3\,\mathcal{C}_{3,3}^{(4)}(z)}{640} x^6$$

$$+ 5\,\mathcal{C}_{5,5}(z)\, x^4 - \frac{5\,\mathcal{C}_{5,5}^{(2)}(z)}{24} x^6$$

$$+ 7\,\mathcal{C}_{7,7}(z)\, x^6$$

Field Components from Pseudo-Multipoles

$$\phi_{\mathrm{m}}(r,\varphi) = \sum_{n=1}^{\infty} r^n (\widetilde{\mathcal{C}}_n(r,z)\sin n\varphi + \widetilde{\mathcal{D}}_n(z)\cos n\varphi)\,.$$

$$B_r(r,\varphi,z) = -\mu_0 \sum_{n=1}^{\infty} r^{n-1} (\overline{\mathcal{C}}_n(r,z)\sin n\varphi + \overline{\mathcal{D}}_n(r,z)\cos n\varphi)\,,$$

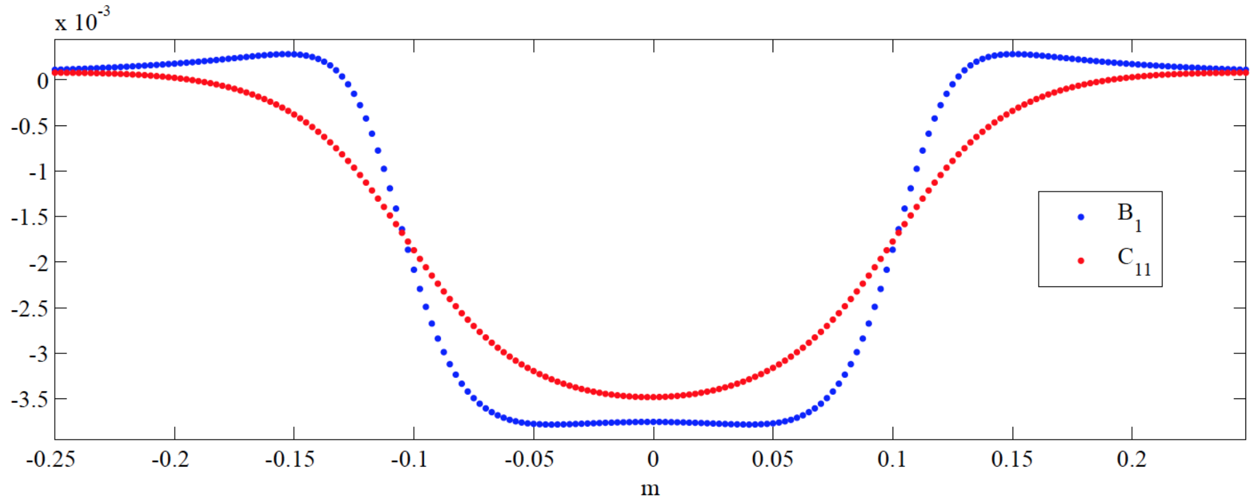
$$B_\varphi(r,\varphi,z) = -\mu_0 \sum_{n=1}^{\infty} n\, r^{n-1} (\widetilde{\mathcal{C}}_n(r,z)\cos n\varphi - \widetilde{\mathcal{D}}_n(r,z)\sin n\varphi)\,,$$

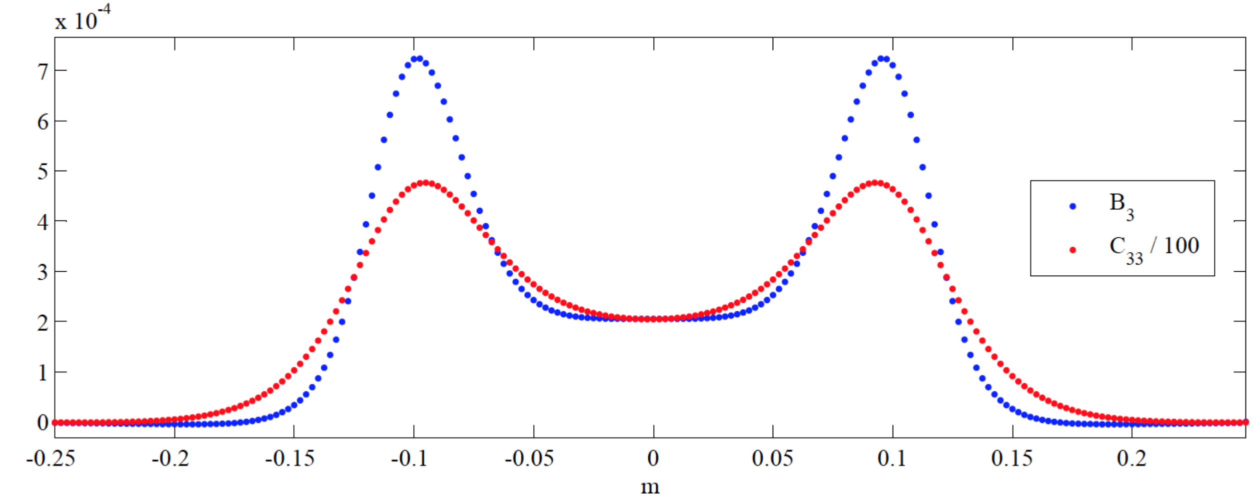
$$B_z(r,\varphi,z) = -\mu_0 \sum_{n=1}^{\infty} r^n \left(\frac{\partial \widetilde{\mathcal{C}}_n(r,z)}{\partial z} \sin n\varphi + \frac{\partial \widetilde{\mathcal{D}}_n(r,z)}{\partial z} \cos n\varphi \right)\,,$$

$$\overline{\mathcal{C}}_n(r,z) = n\,\mathcal{C}_{n,n}(z) - \frac{(n+2)\mathcal{C}_{n,n}^{(2)}(z)}{4(n+1)} r^2 + \frac{(n+4)\mathcal{C}_{n,n}^{(4)}(z)}{32(n+1)(n+2)} r^4 - \ldots$$

$$\widetilde{\mathcal{C}}_n(r,z) := \mathcal{C}_{n,n}(z) - \frac{\mathcal{C}_{n,n}^{(2)}(z)}{4(n+1)} r^2 + \frac{\mathcal{C}_{n,n}^{(4)}(z)}{32(n+1)(n+2)} r^4 - \ldots\,,$$

The Leading Term is NOT the Measured One





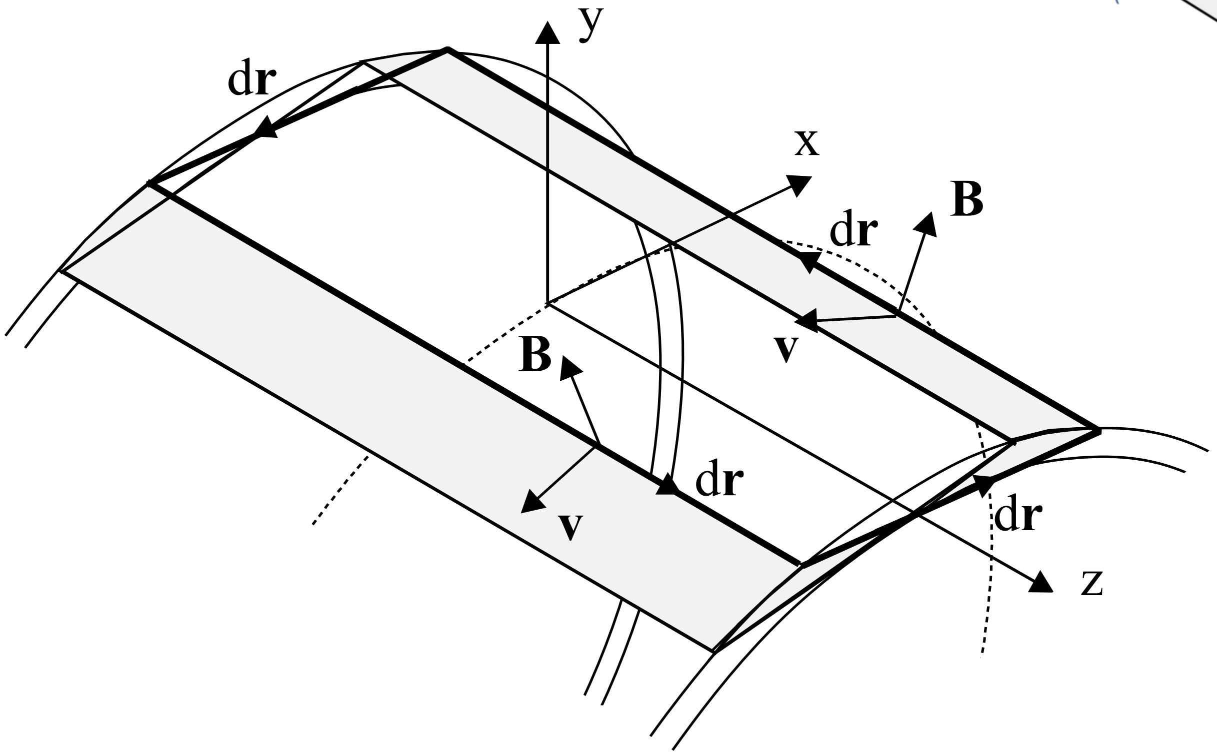
Fourier Transform for the Extractions of Cn,n

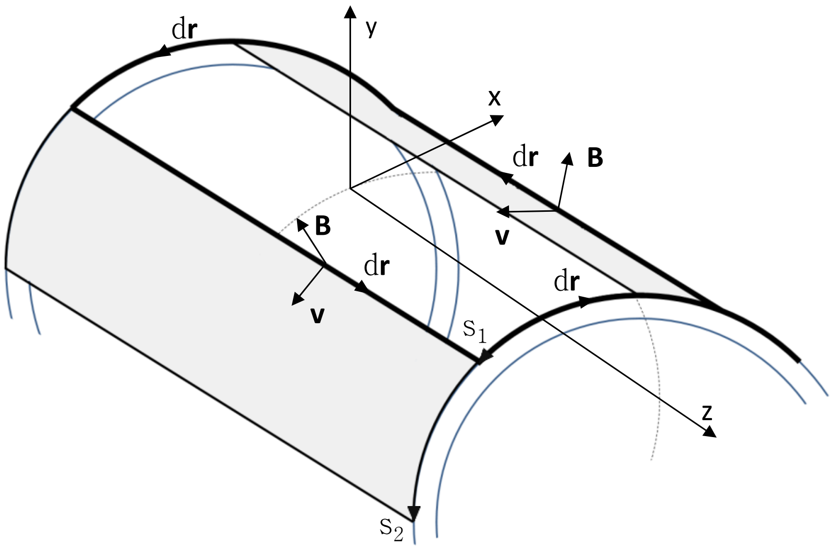
$$B_n(r_0,z) = -\mu_0\, r_0^{n-1}\overline{\mathcal{C}}_n(r_0,z) =$$
$$-\mu_0\, r_0^{n-1}\left(n\,\mathcal{C}_{n,n}(z) - \frac{(n+2)\mathcal{C}_{n,n}^{(2)}(z)}{4(n+1)}r_0^2 + \frac{(n+4)\mathcal{C}_{n,n}^{(4)}(z)}{32(n+1)(n+2)}r_0^4 - \dots\right).$$

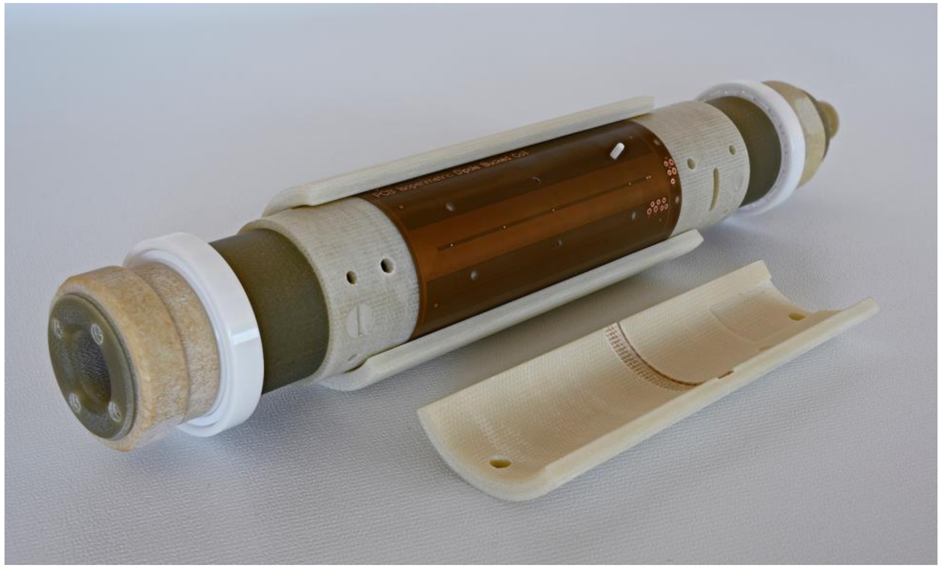
$$\mathcal{F}\{\mathcal{C}_{n,n}(z)\} = \frac{-\mathcal{F}\{B_n(r_0,z)\}}{\mu_0\, r_0^{n-1}\left(n - \frac{(n+2)(i\omega)^2}{4(n+1)}r_0^2 + \frac{(n+4)(i\omega)^4}{32(n+1)(n+2)}r_0^4 - \dots\right)}$$

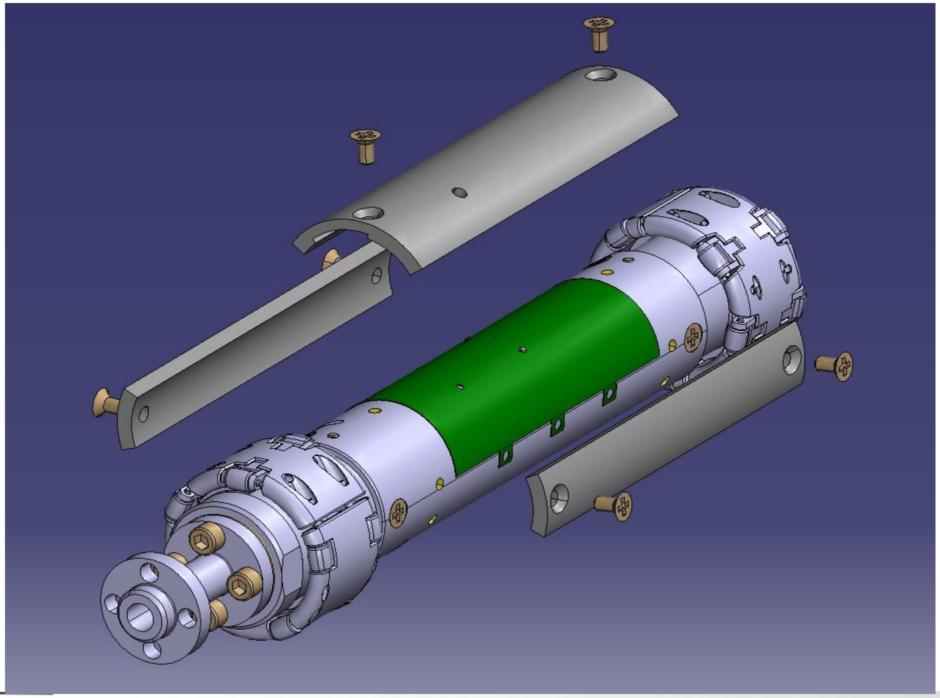
$$\mathcal{F}\{\mathcal{C}_{n,n}(z)\} = \frac{\mathcal{F}\{\tilde{B}_n(r_0,z)\}}{\mathcal{F}\{K_n(r_0,z)\}}\,\frac{-1}{\mu_0\, r_0^{n-1}\left(n - \frac{(n+2)(i\omega)^2}{4(n+1)}r_0^2 + \frac{(n+4)(i\omega)^4}{32(n+1)(n+2)}r_0^4 - \dots\right)}$$

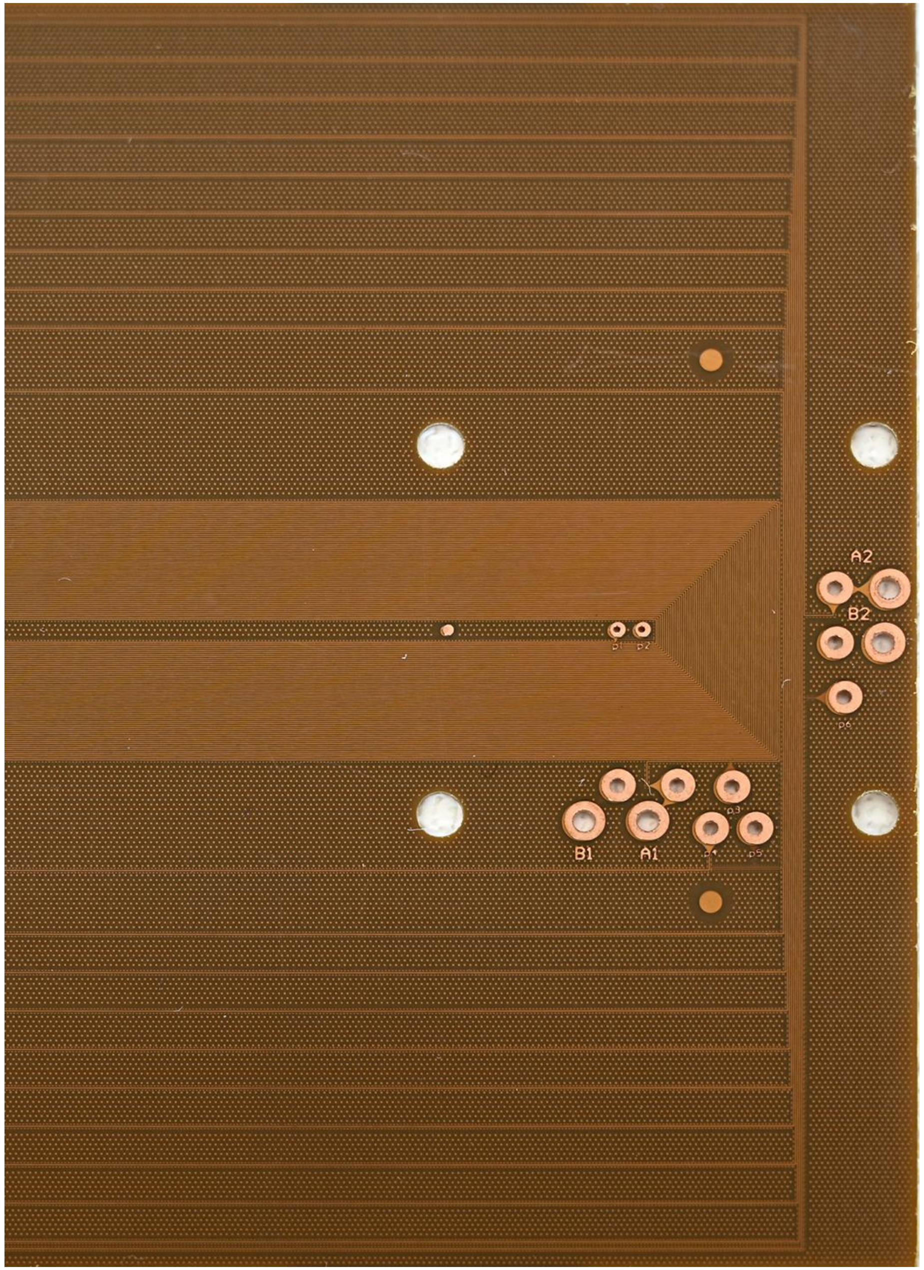
Classical Induction Coils Intercept the Bz Component

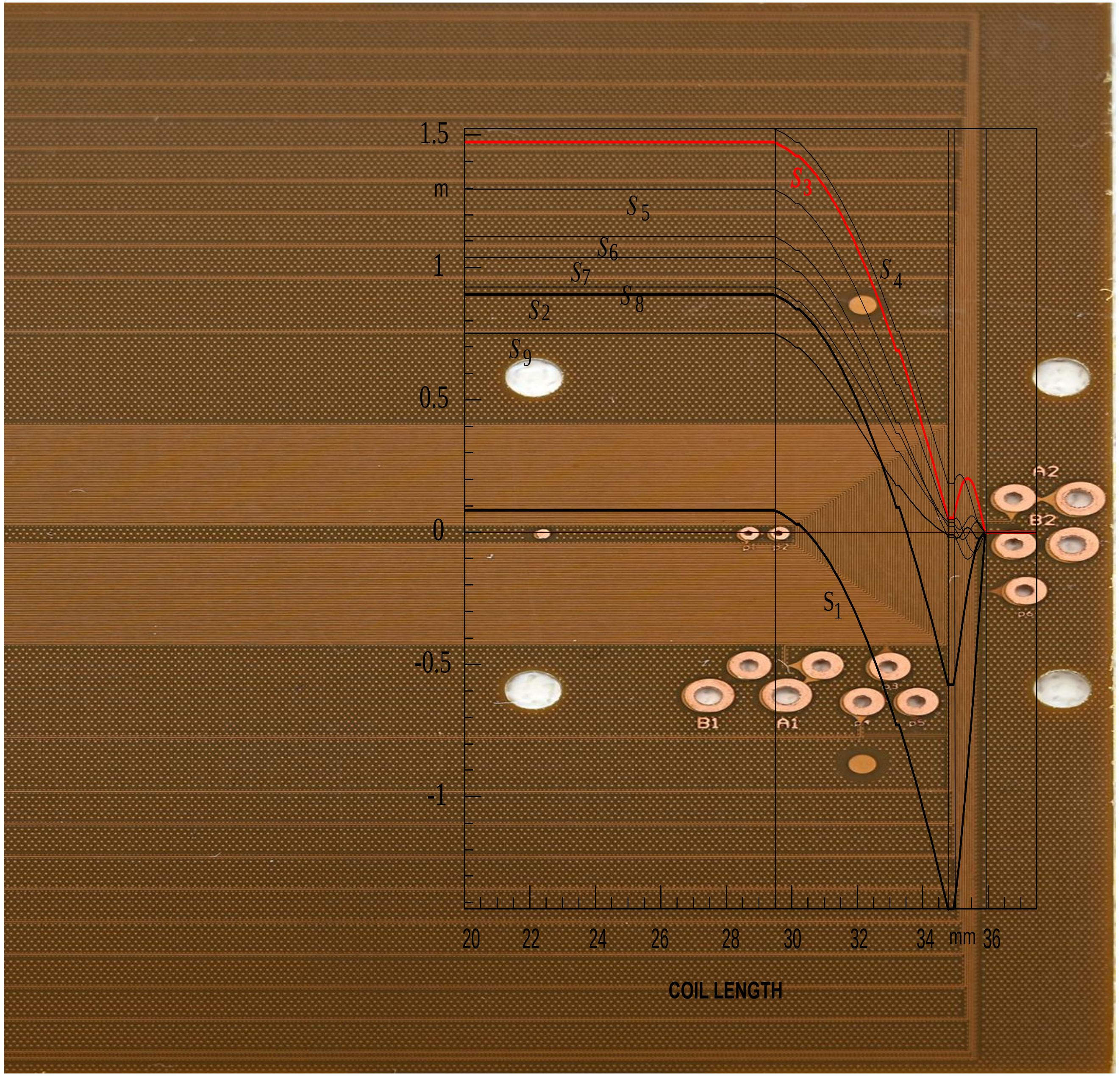




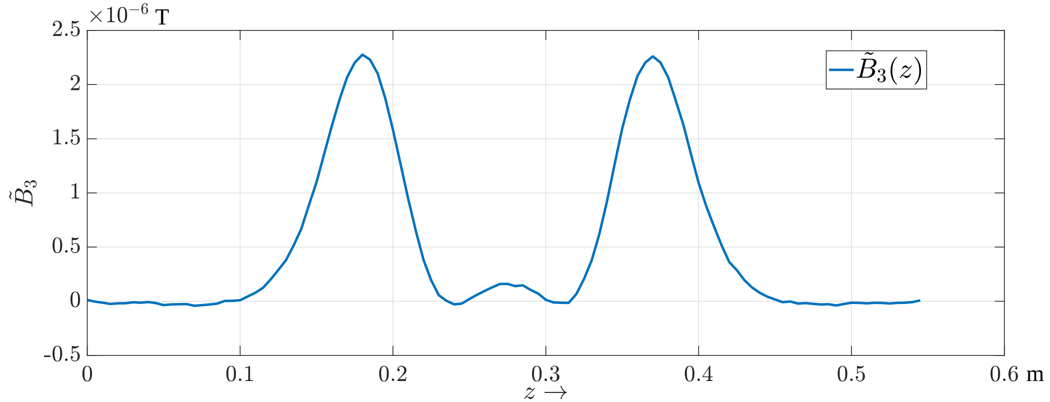


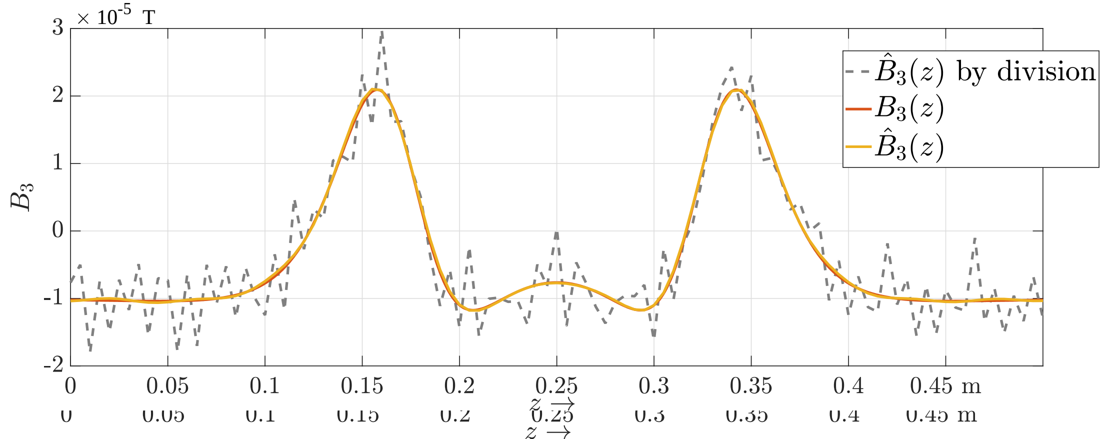


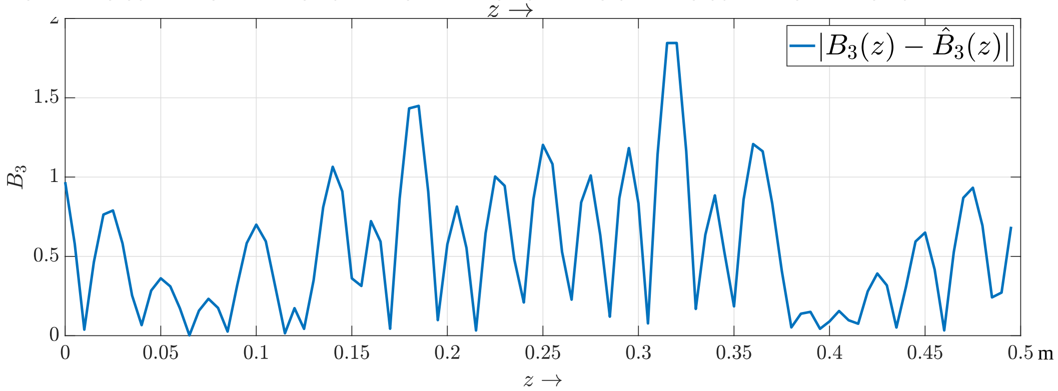

A2
B2
B1
A1

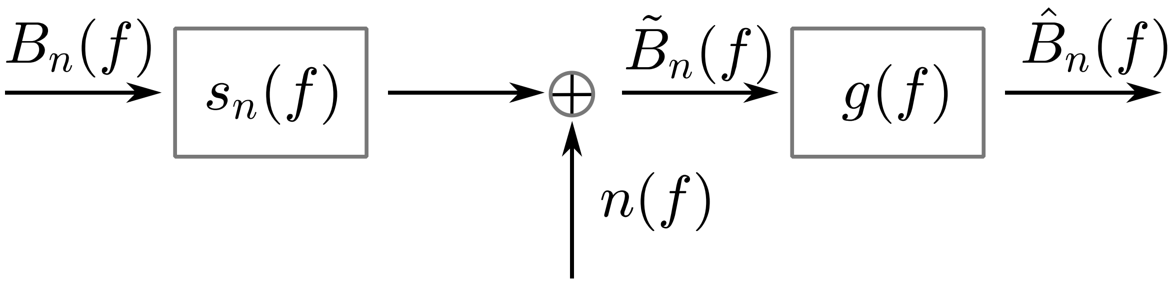


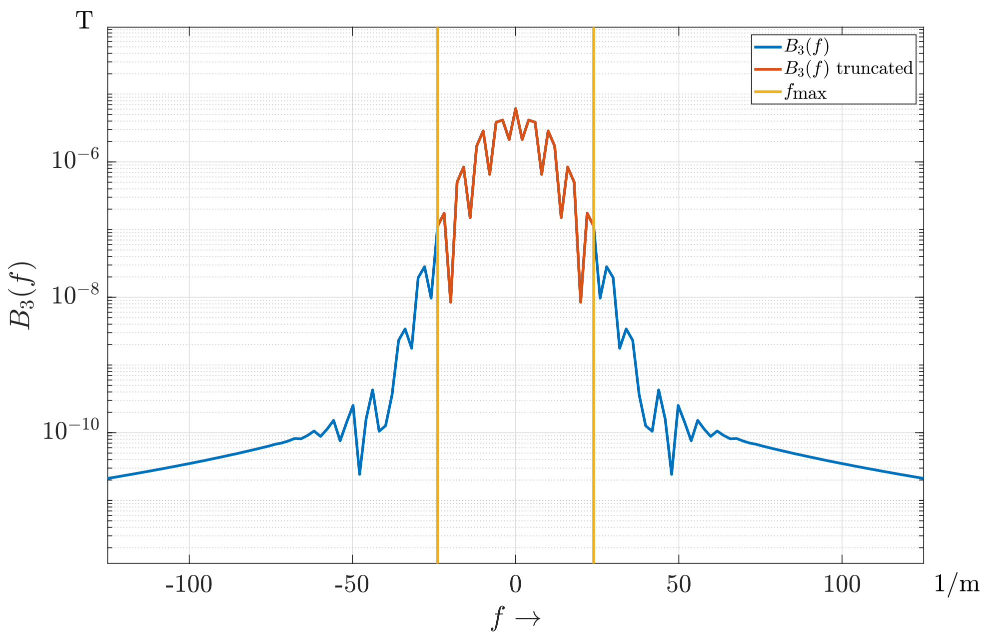
1.5
m
1
0.5
0
-0.5
-1
S_3
S_5
S_6
S_7
S_8
S_2
S_9
S_4
S_1
A2
B2
B1
A1
20 22 24 26 28 30 32 34 mm 36
COIL LENGTH

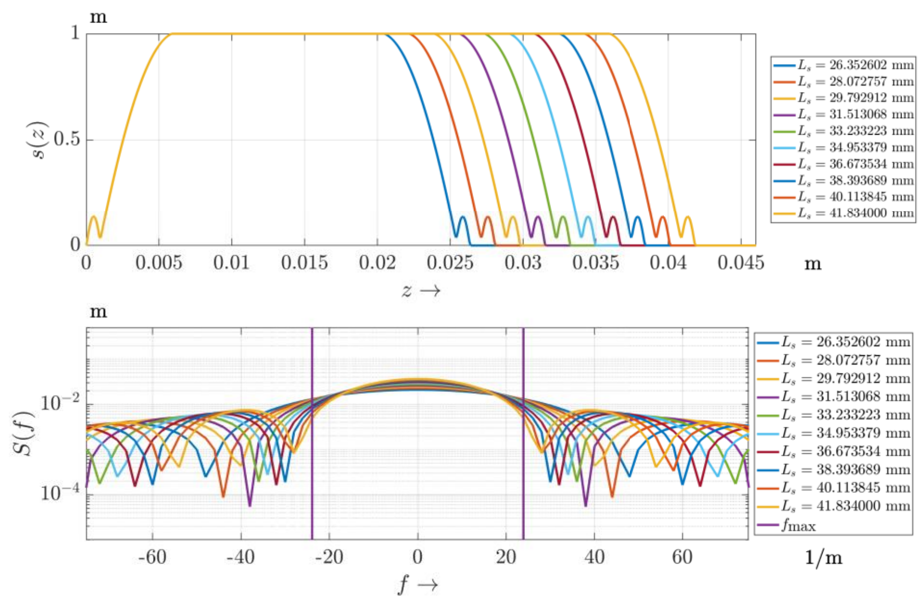

×10^−6 T
$\tilde{B}_3(z)$
$\tilde{B}_3$
$z \rightarrow$
0.6 m


× 10^-5 T
$\hat{B}_3(z)$ by division
$B_3(z)$
$\hat{B}_3(z)$
B_3
$z \rightarrow$
0.45 m


$|B_3(z) - \hat{B}_3(z)|$
B_3
$z \rightarrow$
0.5 m

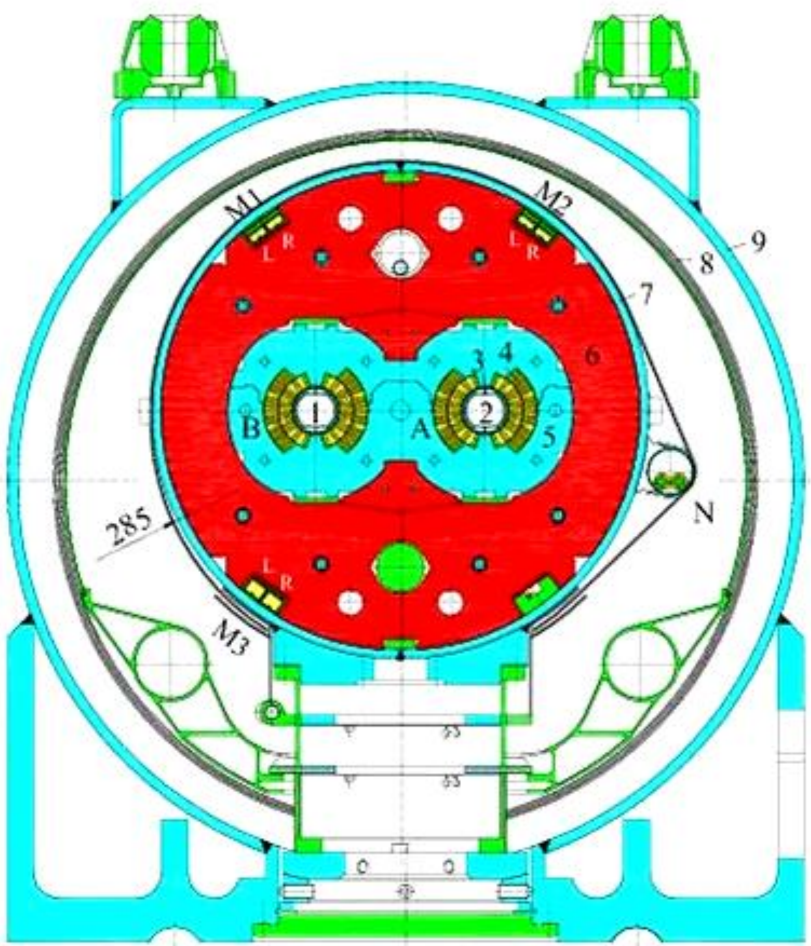

$B_n(f)$
$s_n(f)$
$\tilde{B}_n(f)$
$g(f)$
$\hat{B}_n(f)$
$n(f)$

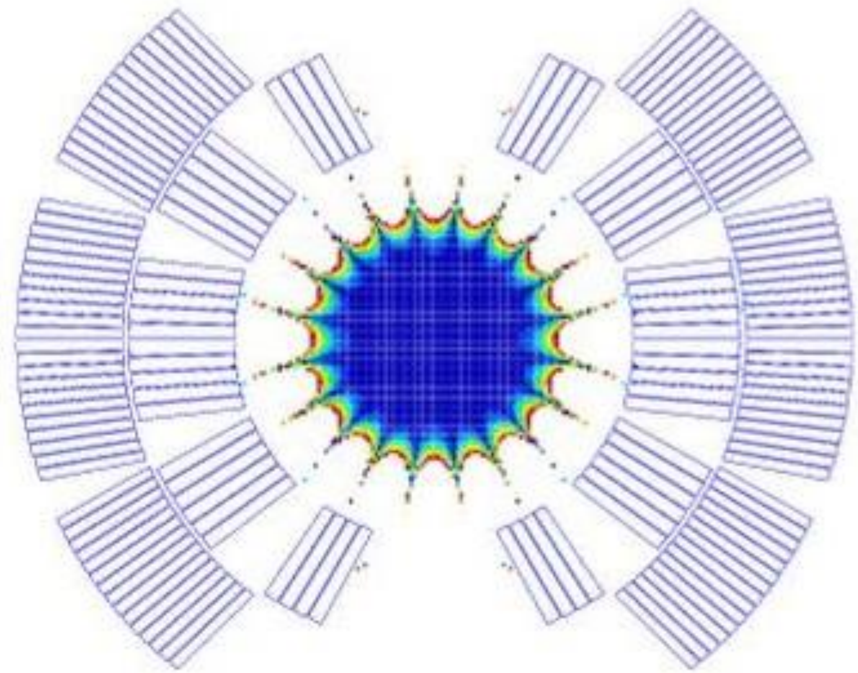

T
$B_3(f)$
$B_3(f)$ truncated
fmax
10^{-6}
10^{-8}
10^{-10}
-100
-50
0
50
100
1/m
$f \rightarrow$


m
1
0.5
0
$s(z)$
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
m
$z \rightarrow$
L_s = 26.352602 mm
L_s = 28.072757 mm
L_s = 29.792912 mm
L_s = 31.513068 mm
L_s = 33.233223 mm
L_s = 34.953379 mm
L_s = 36.673534 mm
L_s = 38.393689 mm
L_s = 40.113845 mm
L_s = 41.834000 mm
m
$S(f)$
10^{-2}
10^{-4}
-60
-40
-20
0
20
40
60
1/m
$f \rightarrow$
fmax

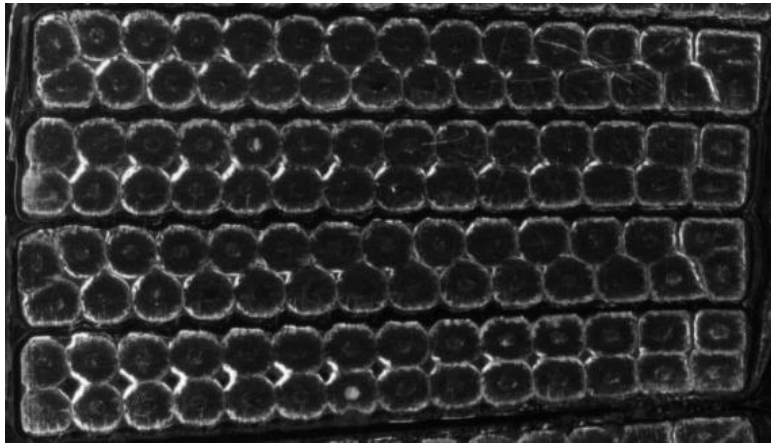
Field Singularities - The Green's Functions

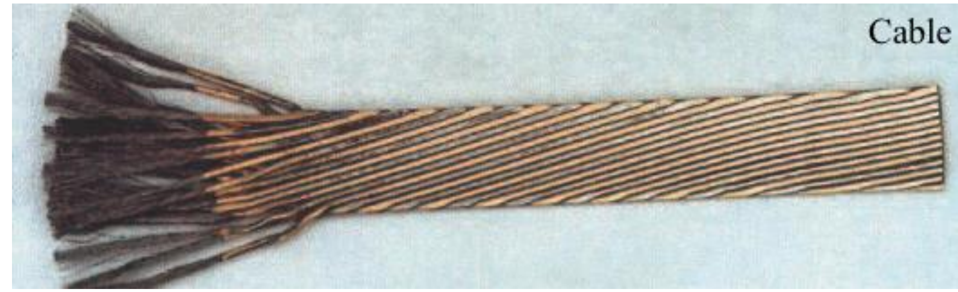
Cross-section of Cryodipole

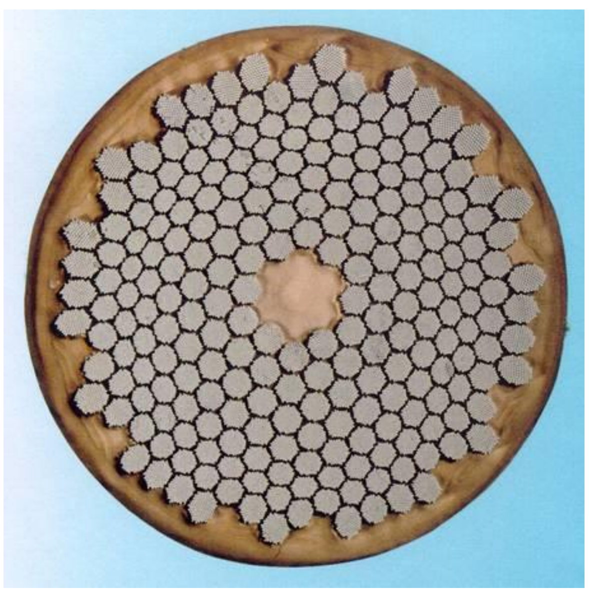




Rutherford (Roebel) Kabel, Strand, Nb-Ti Filament



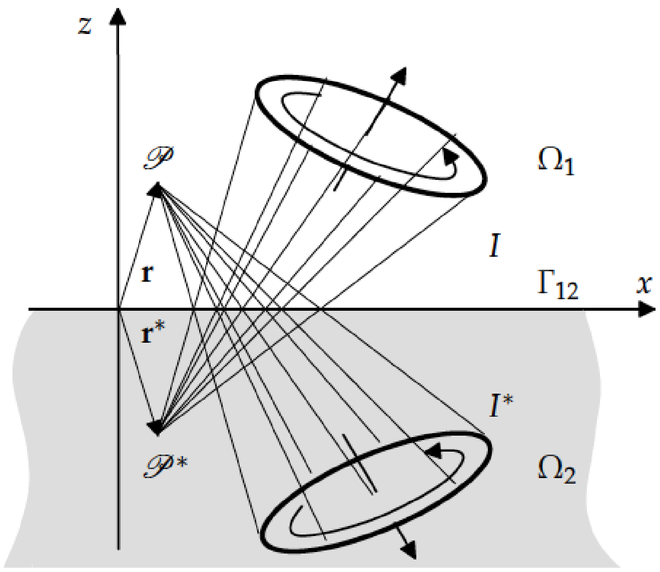


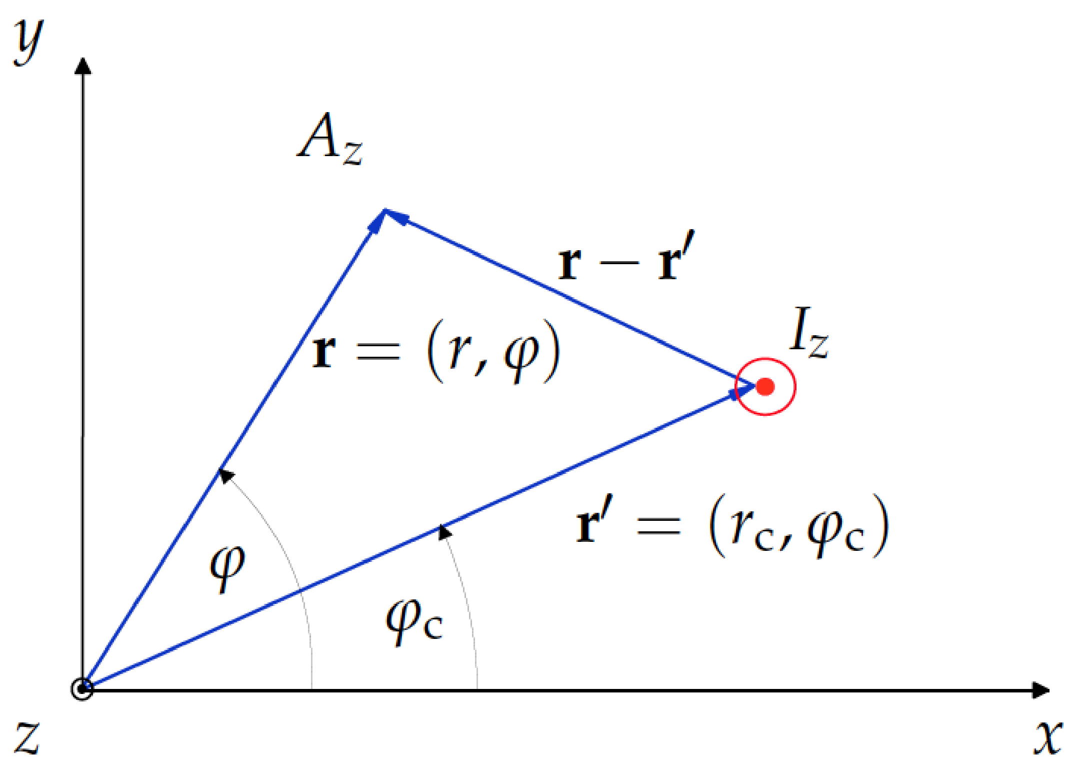


The Field of Line Currents

$$\mathbf{r} \mapsto \phi(|\mathbf{r} - \mathbf{r}'|)$$
$$\mathbf{r}' \mapsto \phi(|\mathbf{r} - \mathbf{r}'|)$$

$$\operatorname{grad} \phi(|\mathbf{r} - \mathbf{r}'|) = -\operatorname{grad}_{\mathbf{r}'} \phi(|\mathbf{r} - \mathbf{r}'|),$$
$$\operatorname{div} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|) = -\operatorname{div}_{\mathbf{r}'} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|),$$
$$\operatorname{curl} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|) = -\operatorname{curl}_{\mathbf{r}'} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|),$$
$$\nabla^2 \phi(|\mathbf{r} - \mathbf{r}'|) = \nabla^2_{\mathbf{r}'} \phi(|\mathbf{r} - \mathbf{r}'|).$$





Why bother? Reciprocity; except for sign it does not matter if we exchange the source and field points

Greens Functions of Free Space

$$\mathcal{L}_{\mathbf{r}'}\phi(\mathbf{r}') = -f(\mathbf{r}')$$

$$\mathcal{L}_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'),$$

$$\int_{\mathcal{V}} \mathcal{L}_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}')\, f(\mathbf{r})\mathrm{d}V = -\int_{\mathcal{V}} \delta(\mathbf{r}-\mathbf{r}')f(\mathbf{r})\mathrm{d}V = -f(\mathbf{r}').$$

$$\mathcal{L}_{\mathbf{r}'}\phi(\mathbf{r}') = \int_{\mathcal{V}} \mathcal{L}_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}')f(\mathbf{r})\mathrm{d}V = \mathcal{L}_{\mathbf{r}'}\int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')f(\mathbf{r})\mathrm{d}V,$$

$$\phi(\mathbf{r}') = \int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')f(\mathbf{r})\mathrm{d}V.$$

$$G_2(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{r_{\mathrm{ref}}}\right), \qquad G_3(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Green's Functions of Free Space

$$\phi(\mathbf{r}') = \int_{\mathscr{V}} G(\mathbf{r},\mathbf{r}') f(\mathbf{r}) \mathrm{d}V \,.$$

$$\phi(\mathbf{r}) = \int_{\mathscr{V}} G(\mathbf{r},\mathbf{r}') f(\mathbf{r}') \mathrm{d}V' \,.$$

$$\int_{\Omega} \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) \, \mathrm{d}V = \int_{\Gamma} \left(\phi \partial_{\mathbf{n}} \psi - \psi \partial_{\mathbf{n}} \phi\right) \, \mathrm{d}a$$

But what if boundaries are present?
Use Green's second identity (integration by parts)

$$\phi(\mathbf{r}) = \int_{\mathscr{V}} G(\mathbf{r},\mathbf{r}') f(\mathbf{r}') \mathrm{d}V' + \int_{\partial\mathscr{V}} \Big(-\phi(\mathbf{r}')\, \partial_{\mathbf{n}'} G(\mathbf{r},\mathbf{r}') + G(\mathbf{r},\mathbf{r}')\, \partial_{\mathbf{n}'} \phi(\mathbf{r}')\Big) \mathrm{d}a' \,.$$

Surface current

Surface density of dipole moments

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}\,, \qquad G_3(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V'\,,$$

$$\mathbf{A}(\mathbf{r}) = A_x \mathbf{e_x} + A_y \mathbf{e_y} + A_z \mathbf{e_z} = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V'.$$

This works only in Cartesian Coordinates

$$\mathbf{B}(\mathbf{r}) = \operatorname{curl} \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \operatorname{curl} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) \mathrm{d}V'$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \sum_{k=1}^{3} J_k(\mathbf{r}') (\mathbf{e}_i(\mathbf{r}) \cdot \mathbf{e}_k(\mathbf{r}'))\, \mathrm{d}V'. \quad \mathrm{d}V'$$

$$= \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \mathrm{d}V'.$$

But wait a minute: Are we finished? Are we sure that the divergence of the vector potential is zero as it was required for the Laplace equation?

$$\begin{aligned}
\operatorname{div} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \operatorname{div} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right) \mathrm{d}V' \\
&= \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \left(\mathbf{J}(\mathbf{r}') \cdot \operatorname{grad} \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) + \frac{1}{|\mathbf{r}-\mathbf{r}'|} \operatorname{div} \mathbf{J}(\mathbf{r}') \right) \mathrm{d}V' \\
&= \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \mathbf{J}(\mathbf{r}') \cdot \operatorname{grad} \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi} \int_{\mathscr{V}} \mathbf{J}(\mathbf{r}') \cdot \operatorname{grad}_{\mathbf{r}'} \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi} \int_{\mathscr{V}} \left(\operatorname{div}_{\mathbf{r}'} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|} \operatorname{div}_{\mathbf{r}'} \mathbf{J}(\mathbf{r}') \right) \mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi} \int_{\mathscr{V}} \operatorname{div}_{\mathbf{r}'} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right) \mathrm{d}V' = -\frac{\mu_0}{4\pi} \int_{\partial\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \cdot \mathrm{d}\mathbf{a}' .
\end{aligned}$$

Current loops must always be closed and must not leave the problem domain

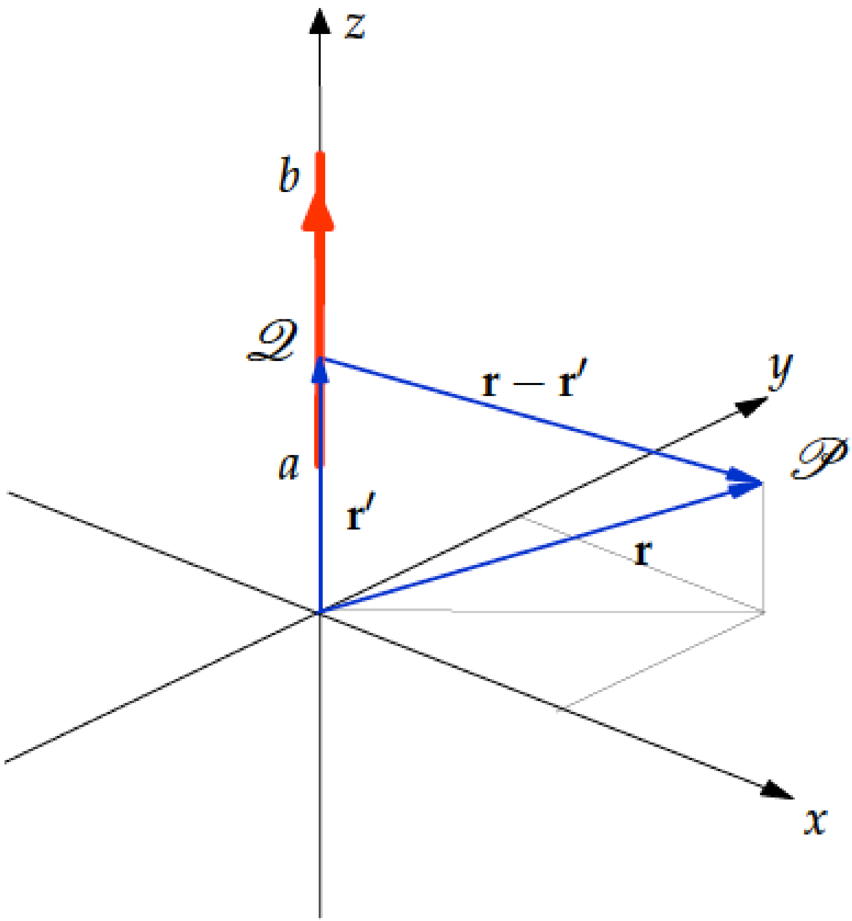
Biot-Savart's Law for Line Currents

$$\mathbf{A}(\mathbf{r}) = A_x \mathbf{e_x} + A_y \mathbf{e_y} + A_z \mathbf{e_z} = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V'.$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\mathscr{C}} \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\mathscr{C}} \frac{\mathrm{d}\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3},$$

Vector Potential of a Line Current



$$A_z(x,y,z) = \frac{\mu_0 I}{4\pi}\int_a^b \frac{\mathrm{d}z_c}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 I}{4\pi}\int_a^b \frac{\mathrm{d}z_c}{\sqrt{x^2+y^2+(z-z_c)^2}}$$

$$\frac{-\mu_0 I}{4\pi}\ln\left((z-z_c)+\sqrt{x^2+y^2+(z-z_c)^2}\right)\Bigg|_a^b$$

$$\frac{\mu_0 I}{4\pi}\ln\frac{z-a+\sqrt{x^2+y^2+(z-a)^2}}{z-b+\sqrt{x^2+y^2+(z-b)^2}}\,.$$

Field of a Line Current (Infinitely Long)

$$\lim_{a,b\to\pm\infty} \ln \frac{z-a+\sqrt{x^2+y^2+(z-a)^2}}{z-b+\sqrt{x^2+y^2+(z-b)^2}} = \lim_{a,b\to\pm\infty} \ln \frac{-a+|a|\sqrt{1+\frac{x^2+y^2}{a^2}}}{-b+|b|\sqrt{1+\frac{x^2+y^2}{b^2}}}$$

$$= \lim_{a,b\to\pm\infty} \ln \frac{-a-a(1+\frac{x^2+y^2}{2a^2}+\cdots)}{-b+b(1+\frac{x^2+y^2}{2b^2}+\cdots)} = \lim_{a,b\to\pm\infty} \ln \frac{-2a}{-b+b+\frac{x^2+y^2}{2b}}$$

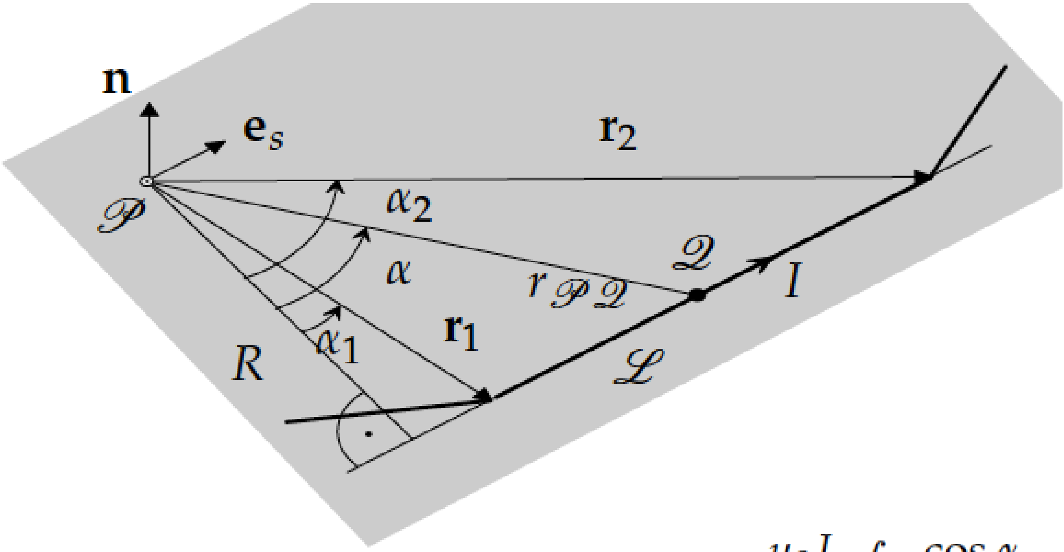
$$= \lim_{a,b\to\pm\infty} \ln \frac{-4ab}{x^2+y^2}\,.$$

$$A_z(x,y) = \lim_{a,b\to\pm\infty} \frac{\mu_0 I}{4\pi} \ln\left(\frac{-4ab}{x_0^2+y_0^2}\right) - \frac{\mu_0 I}{4\pi} \ln\left(\frac{x^2+y^2}{x_0^2+y_0^2}\right)\,.$$

Arbitrarily large but constant

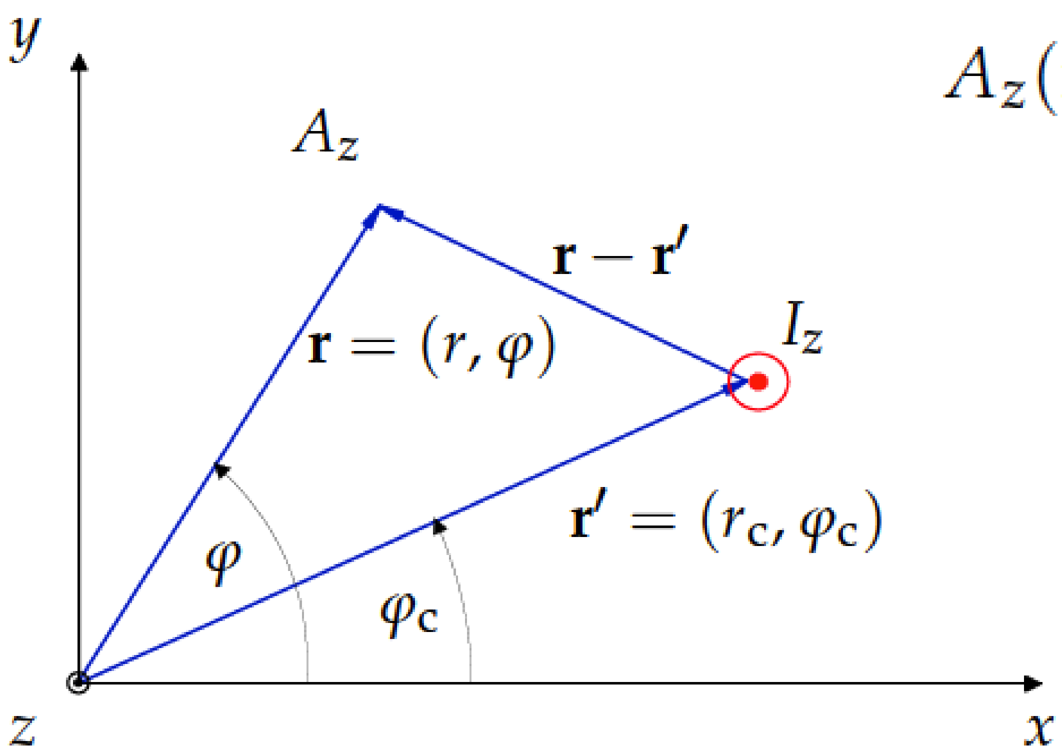
$$\mathbf{A}(x,y) = -\frac{\mu_0 I}{4\pi} \ln\left(\frac{x^2+y^2}{x_0^2+y_0^2}\right) \mathbf{e}_z = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{r}{r_{\mathrm{ref}}}\right) \mathbf{e}_z\,,$$

Field of a Line Current Segment



$$
\begin{aligned}
\mathbf{B}(\mathscr{P}) &= \frac{\mu_0 I}{4\pi}\int_{\mathscr{L}} \frac{\cos\alpha}{r^2_{\mathscr{P}\mathscr{Q}}}\mathrm{d}r' = \frac{\mu_0 I}{4\pi R}\mathbf{n}\int_{\alpha_1}^{\alpha_2}\cos\alpha\,\mathrm{d}\alpha = \frac{\mu_0 I}{4\pi R}\left(\sin\alpha_2 - \sin\alpha_1\right)\mathbf{n} \\
&= \frac{\mu_0 I}{4\pi}\,\frac{\cos\alpha_2 + \cos\alpha_1}{R}\,\frac{\sin\alpha_2 - \sin\alpha_1}{\cos\alpha_2 + \cos\alpha_1}\,\mathbf{n} \\
&= \frac{\mu_0 I}{4\pi}\left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|}\right)\frac{\sin(\alpha_2 - \alpha_1)}{1 + \cos(\alpha_2 - \alpha_1)}\,\mathbf{n} \\
&= \frac{\mu_0 I}{4\pi}\left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|}\right)\frac{\sin(\alpha_2 - \alpha_1)}{1 + \frac{\mathbf{r}_1\cdot\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}}\,\frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|\sin(\alpha_2 - \alpha_1)} \\
&= \frac{\mu_0 I}{4\pi}\,\frac{|\mathbf{r}_1| + |\mathbf{r}_2|}{|\mathbf{r}_1|\,|\mathbf{r}_2| + \mathbf{r}_1\cdot\mathbf{r}_2}\,\frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}\,,
\end{aligned}
$$

Expanding the Green's Function

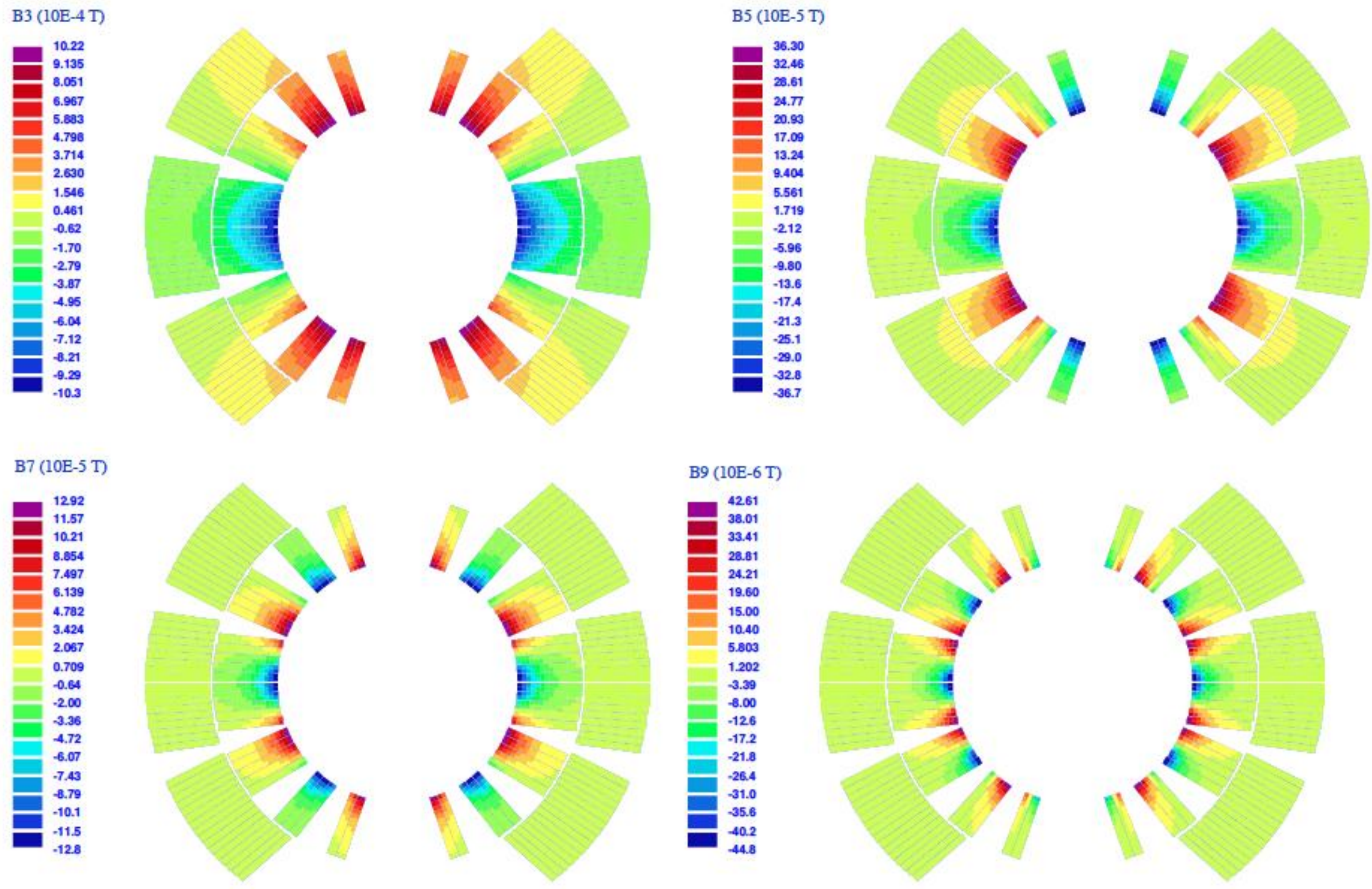


$$A_z(\mathbf{r}) = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{r_{\mathrm{ref}}}\right)$$

$$A_z(r,\varphi) = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{r_c}{r_{\mathrm{ref}}}\right) + \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{r_c}\right)^n \cos n(\varphi - \varphi_c)$$

$$B_n(r_0) = -\frac{\mu_0 I}{2\pi r_c} \left(\frac{r_0}{r_c}\right)^{n-1} \cos n\varphi_c, \qquad A_n(r_0) = \frac{\mu_0 I}{2\pi r_c} \left(\frac{r_0}{r_c}\right)^{n-1} \sin n\varphi_c.$$

Expanding the Green's Function II

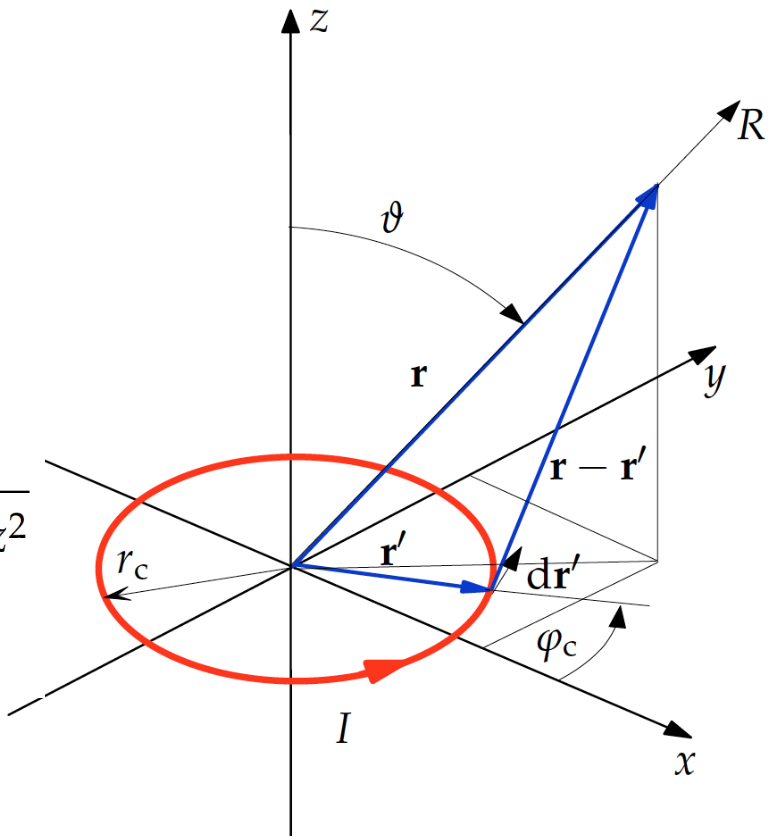


Field of a Ring Current

$$\mathbf{r}' = \cos\varphi_c r_c\,\mathbf{e}_x + \sin\varphi_c r_c\,\mathbf{e}_y$$

$$\mathrm{d}\mathbf{r}' = -\sin\varphi_c r_c\,\mathrm{d}\varphi_c\mathbf{e}_x + \cos\varphi_c r_c\,\mathrm{d}\varphi_c\mathbf{e}_y$$

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'| &= \sqrt{(x-x_c)^2+(y-y_c)^2+z^2}\\ &= \sqrt{(r\cos\varphi - r_c\cos\varphi_c)^2 + (r\sin\varphi - r_c\sin\varphi_c)^2 + z^2}\\ &= \sqrt{r^2+r_c^2+z^2-2rr_c\cos\varphi_c}\,,\end{aligned}$$

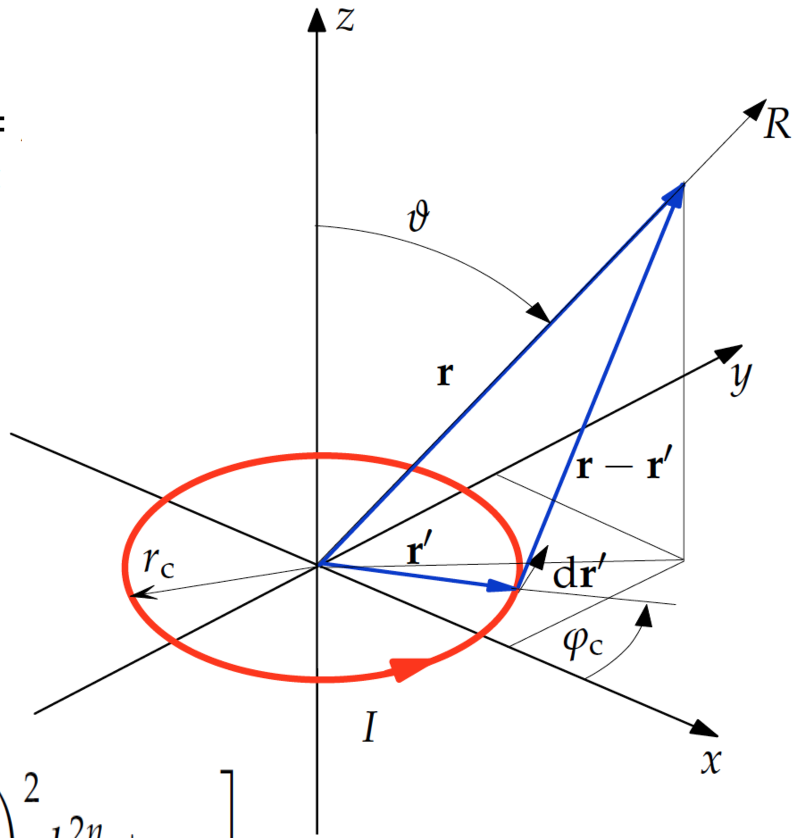


Field of a Ring Current

$$A_y(r,z) = \frac{\mu_0 I r_c}{2\pi} \int_0^{\pi} \frac{\cos\varphi_c \, d\varphi_c}{\sqrt{r^2 + r_c^2 + z^2 - 2 r r_c \cos\varphi_c}}$$

$$\psi := (\pi + \varphi_c)/2, \qquad k^2 := \frac{4 r r_c}{(r + r_c)^2 + z^2}$$

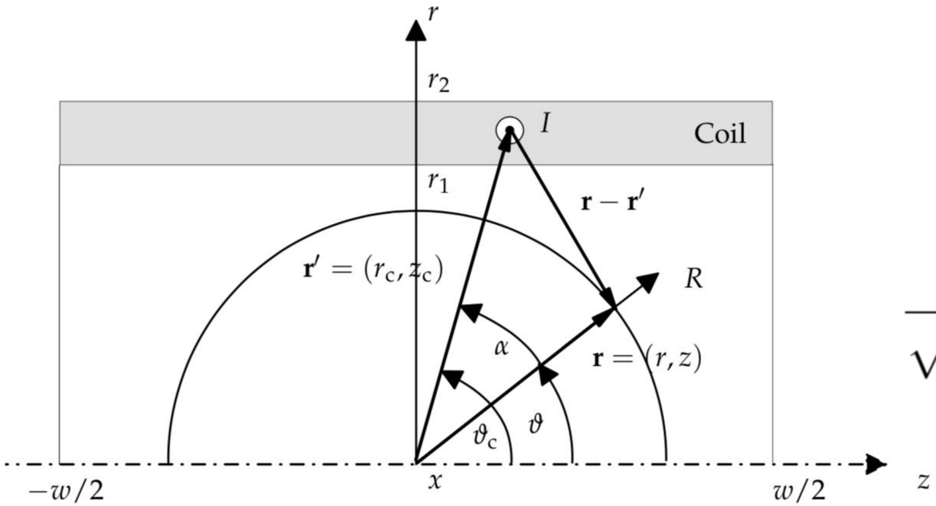
$$A_\varphi(r,z) = \frac{\mu_0 I r_c}{\pi\sqrt{(r + r_c)^2 + z^2}} \int_0^{\pi/2} \frac{2\sin^2\psi - 1}{\sqrt{1 - k^2 \sin^2\psi}} \, d\psi$$



$$K\left(\frac{\pi}{2},k\right) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots + \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 k^{2n} + \cdots\right]$$

$$A_\varphi(r,z) = \frac{\mu_0 I}{2\pi r}\sqrt{(r + r_c)^2 + z^2}\left[\left(1 - \frac{k^2}{2}\right) K\left(\frac{\pi}{2},k\right) - E\left(\frac{\pi}{2},k\right)\right]$$

Expanding the Green's Function



$$A_\varphi(R,\vartheta) = \sum_{n=1}^{\infty} \mu_0 \mathcal{A}_n R^n P_n^1(\cos\vartheta)\,,$$

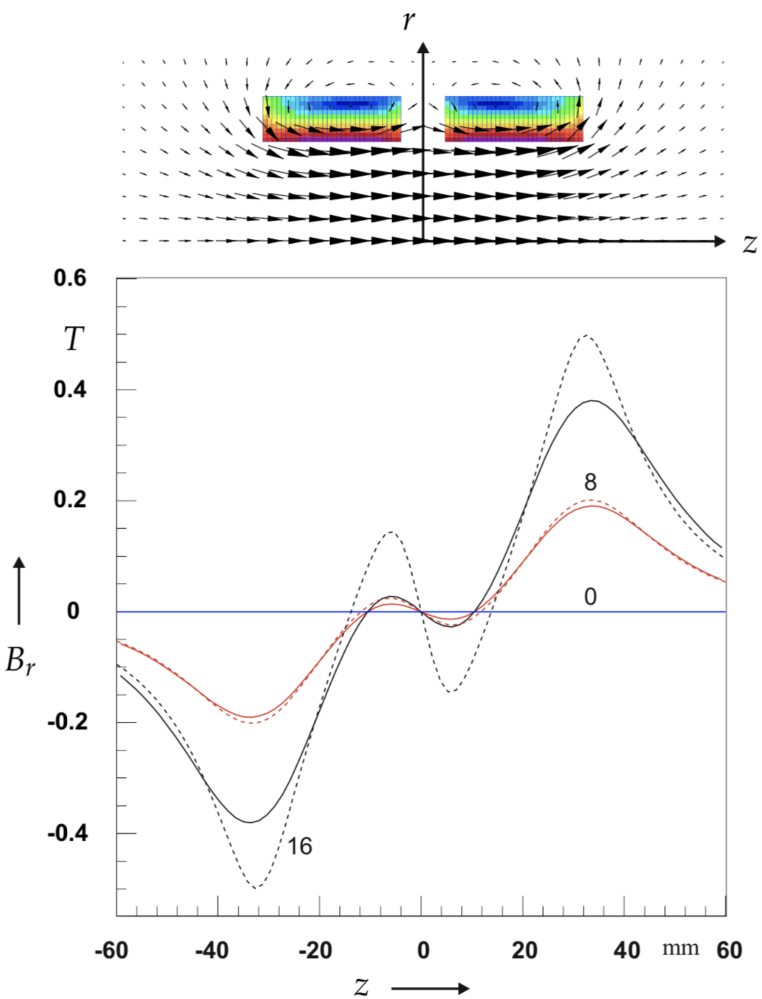
$$\frac{1}{\sqrt{|\mathbf{r}|^2 + |\mathbf{r}'|^2 - 2|\mathbf{r}||\mathbf{r}'|\cos\alpha}} = \frac{1}{|\mathbf{r}'|}\sum_{n=0}^{\infty}\left(\frac{|\mathbf{r}|}{|\mathbf{r}'|}\right)^n P_n(\cos\alpha)$$

$$\begin{aligned} A_\varphi &= \frac{\mu_0 I r_c}{2\pi}\int_0^\pi \frac{\cos\varphi_c \mathrm{d}\varphi_c}{\sqrt{r^2 + r_c^2 + (z-z_c)^2 - 2rr_c\cos\varphi_c}} \\ &= \frac{\mu_0 I r_c}{2\pi}\int_0^\pi \frac{\cos\varphi_c \mathrm{d}\varphi_c}{\sqrt{|\mathbf{r}|^2 + |\mathbf{r}'|^2 - 2|\mathbf{r}||\mathbf{r}'|(\cos\vartheta\cos\vartheta_c + \sin\vartheta\sin\vartheta_c\cos\varphi_c)}} \\ &= \frac{\mu_0 I r_c}{2}\frac{1}{|\mathbf{r}'|}\sum_{n=1}^{\infty}\left(\frac{|\mathbf{r}|}{|\mathbf{r}'|}\right)^n \frac{(n-1)!}{(n+1)!} P_n^1(\cos\vartheta) P_n^1(\cos\vartheta_c)\,. \end{aligned}$$

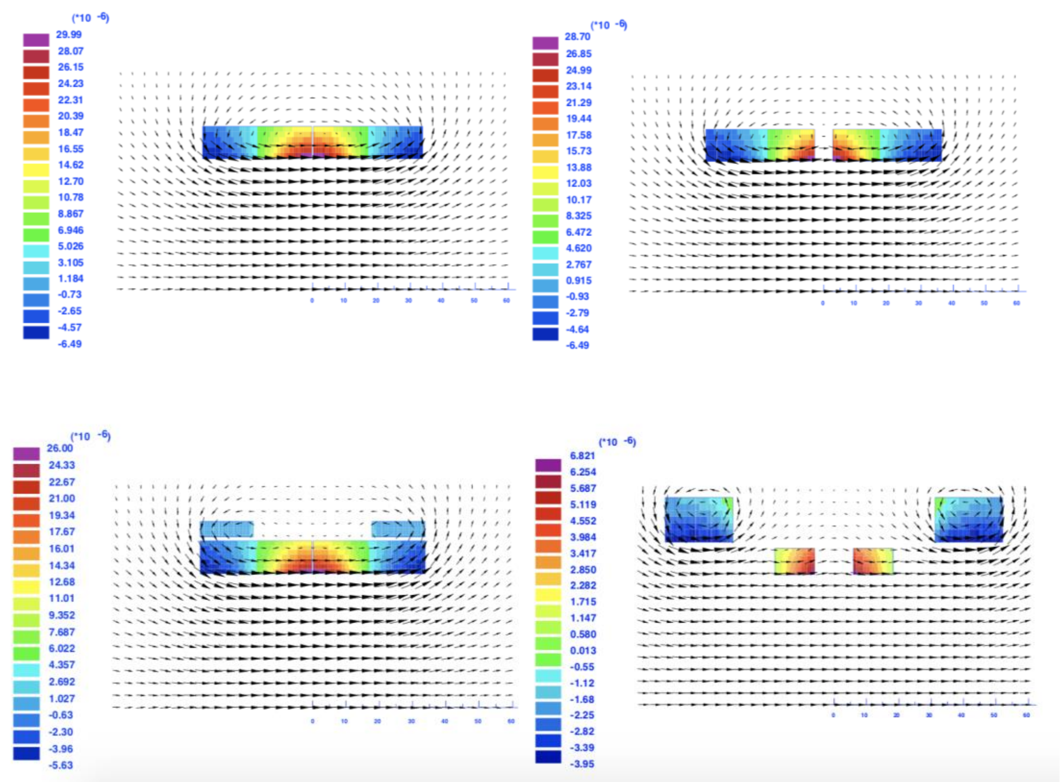
$$\mathcal{A}_n = \frac{I r_c}{2}\frac{1}{R_c^{n+1}}\frac{1}{n(n+1)}P_n^1(\cos\vartheta_c)$$

Split-Coil Solenoids

Field approximation up to first order (at different radii)



Optimization of the field homogeneity (suppressing the 3rd zonal harmonic)



Magnetic Dipole Moment

Far field approximation

$$A_\varphi(R,\vartheta) \approx \frac{\mu_0 I r_c^2 \pi}{4\pi}\frac{\sin\vartheta}{R^2} = \frac{\mu_0 m}{4\pi}\frac{\sin\vartheta}{R^2},$$

$$R = \sqrt{r^2+z^2} \text{ and } \sin\vartheta = r/R,$$

$$[m] = 1\,\mathrm{A\,m^2}.$$

Definition

$$m := I r_c^2 \pi$$

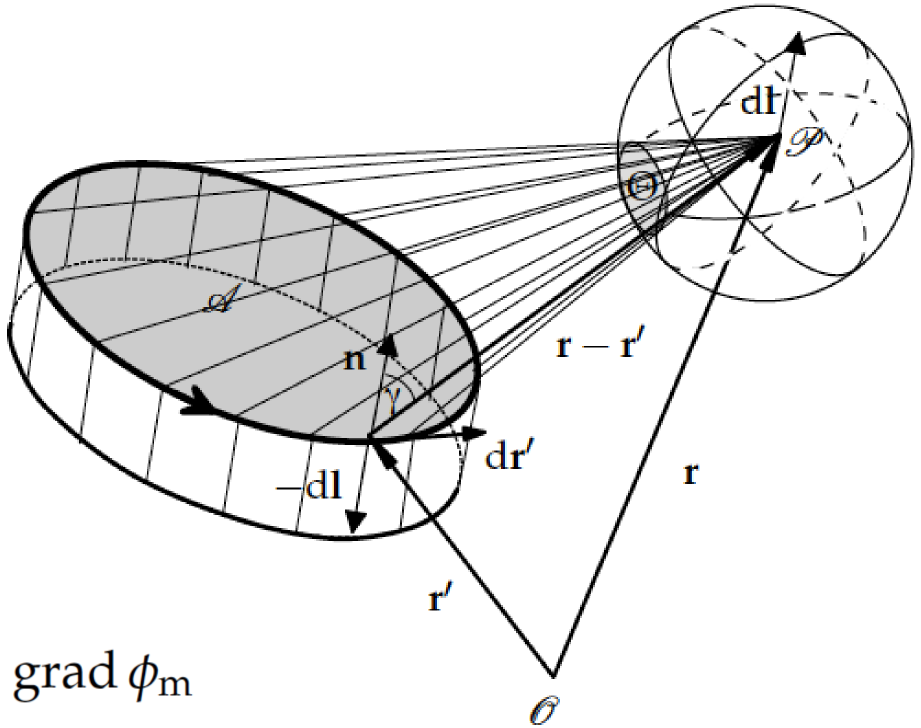
$$\mathbf{m} = I\mathbf{a},$$

$$\mathbf{m} = \frac{I}{2}\int_{\mathscr{C}} \mathbf{r}\times \mathrm{d}\mathbf{r},$$

$$\mathbf{M}(\mathbf{r}) := \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}V} = \frac{1}{2}\mathbf{r}\times\mathbf{J}(\mathbf{r}),$$

Solid Angle and Magnetic Scalar Potential

$$\mathrm{d}\Theta = -\int_{\partial\mathscr{A}} \frac{1}{|\mathbf{r}-\mathbf{r}'|^2}(\mathrm{d}\mathbf{l}\times\mathrm{d}\mathbf{r}')\cdot\mathbf{e}_R = -\int_{\partial\mathscr{A}} \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\cdot(\mathrm{d}\mathbf{l}\times\mathrm{d}\mathbf{r}')$$
$$= -\mathrm{d}\mathbf{l}\int_{\partial\mathscr{A}} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,.$$



Expressing $\mathrm{d}\Theta$ as $\operatorname{grad}\Theta\cdot\mathrm{d}\mathbf{l}$

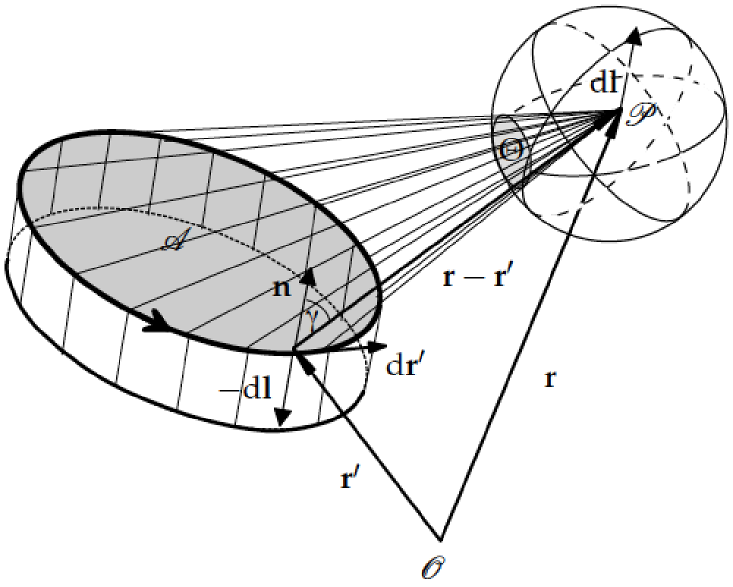
$$\operatorname{grad}\Theta = -\int_{\partial\mathscr{A}} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

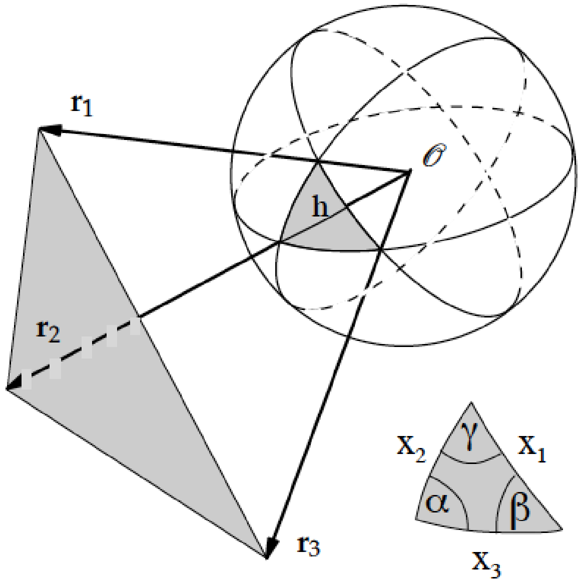
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_{\partial\mathscr{A}_c} \frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \mu_0\mathbf{H} = -\mu_0\operatorname{grad}\phi_\mathrm{m}$$

$$\phi_\mathrm{m}(\mathbf{r}) = \frac{I}{4\pi}\Theta$$

Solid angle (easy to compute) yields the magnetic scalar potential of a current loop

Solid Angle and Magnetic Scalar Potential

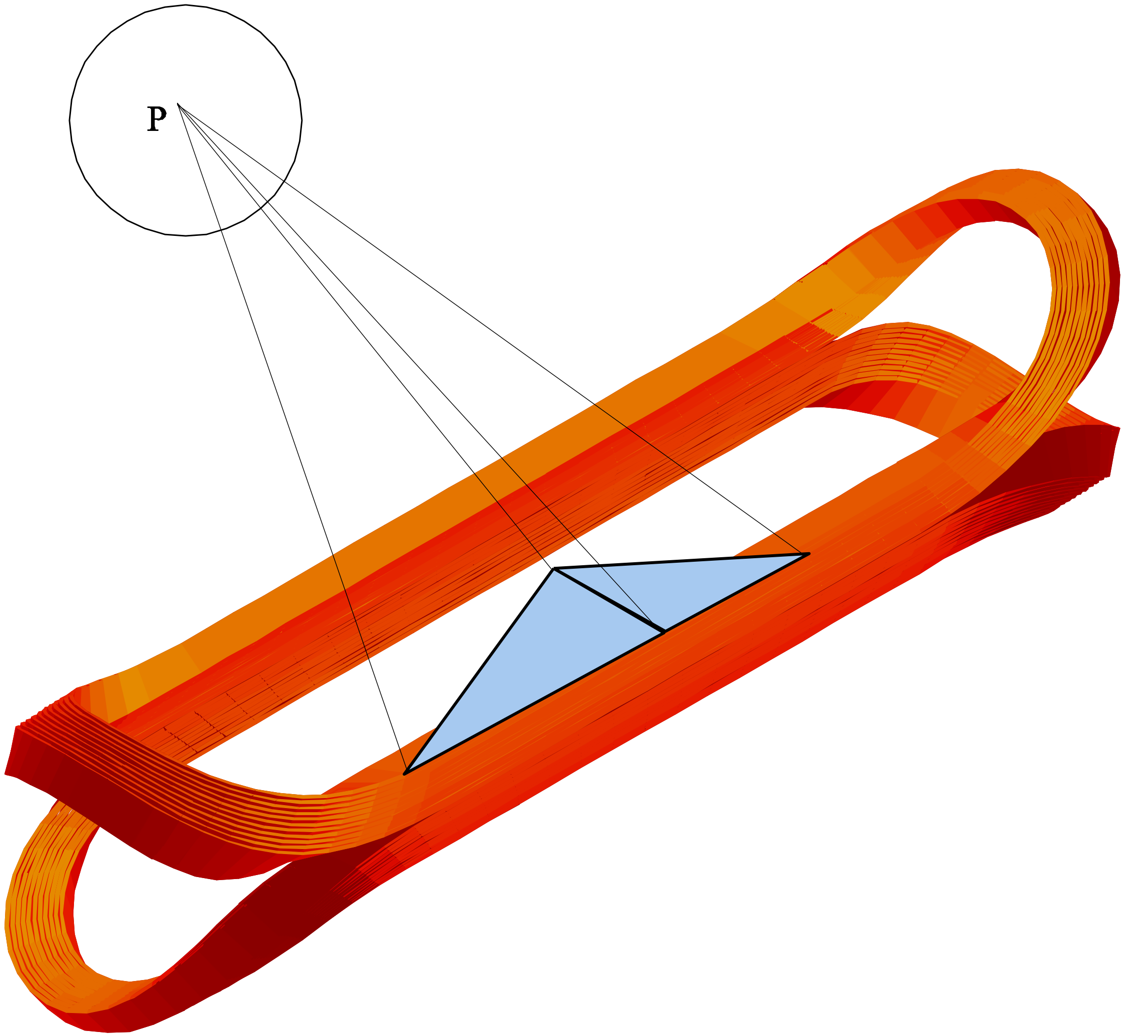




$$\Theta = \int_{\mathscr{A}} \frac{\cos\gamma}{R^2} \mathrm{d}a = \int_{\mathscr{A}} \frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|^3} \mathrm{d}a \,,$$

$$\tan\left(\frac{\Theta}{2}\right) = \frac{\mathbf{r}_1\cdot(\mathbf{r}_2\times\mathbf{r}_3)}{r_1 r_2 r_3 + (\mathbf{r}_1\cdot\mathbf{r}_2) r_3 + (\mathbf{r}_1\cdot\mathbf{r}_3) r_2 + (\mathbf{r}_2\cdot\mathbf{r}_3) r_1} \,.$$

Total Magnetic Scalar Potential



Finite-Element Shape Functions

The Model Problem (1-D)

or

Shape Functions

$$\Omega_j = [x_{n-1}, x_n]$$

$$\Omega = \bigcup_{j=1}^{J} \Omega_j$$

$$u_n = \alpha_{j1} + \alpha_{j2} x_n$$

Shape Functions

Cramer's rule

$$u_j(x) = \alpha_{j1} + \alpha_{j2}x = \frac{x_n - x}{x_n - x_{n-1}} u_{n-1} + \frac{-x_{n-1} + x}{x_n - x_{n-1}} u_n$$

Shape Functions

$$N_{j1}(x) = \frac{x_n - x}{x_n - x_{n-1}}$$

What have we won? We can express the field in the element as a function of the node potentials using known polynomials in the spatial coordinates

The Weighted Residual

What have we won? Removal of the second derivative, a way to incorporate Neumann boundary conditions

Galerkin's Method

$$\int_{\Omega_j} \frac{\mathrm{d}w_l(x)}{\mathrm{d}x} \sum_{k=1,2} \frac{\mathrm{d}N_{jk}(x)}{\mathrm{d}x} u^{(k)} \,\mathrm{d}\Omega_j = -\int_{\Omega_j} w_l(x) f(x) \,\mathrm{d}\Omega_j \,, \qquad l = 1, 2.$$

Linear equation system for the node potentials

Numerical Example

$$\{f_j\} = -\int_{x_{n-1}}^{x_n} \begin{pmatrix} N_{j1} \\ N_{j2} \end{pmatrix} C \mathrm{d}x = -C \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{x_n - x}{x_n - x_{n-1}} \\ \frac{-x_{n-1} + x}{x_n - x_{n-1}} \end{pmatrix} \mathrm{d}x$$

$$\begin{pmatrix} \frac{1}{L} & \frac{-1}{L} & 0 & 0 & 0 \\ \frac{1}{L} & \frac{2}{L} & \frac{-1}{L} & 0 & 0 \\ 0 & \frac{-1}{L} & \frac{2}{L} & \frac{-1}{L} & 0 \\ 0 & 0 & \frac{-1}{L} & \frac{2}{L} & \frac{-1}{L} \\ 0 & 0 & 0 & \frac{-1}{L} & \frac{1}{L} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = - \begin{pmatrix} 0.5\,CL \\ CL \\ CL \\ CL \\ 0.5\,CL \end{pmatrix}$$

Essential boundary conditions (Dirichlet)

$$\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} \frac{3L}{4} & \frac{L}{2} & \frac{L}{4} \\ \frac{L}{2} & L & \frac{L}{2} \\ \frac{L}{4} & \frac{L}{2} & \frac{2L}{4} \end{pmatrix} \begin{pmatrix} CL \\ CL \\ CL \end{pmatrix} = \begin{pmatrix} -0.375 \\ -0.5 \\ -0.375 \end{pmatrix}$$

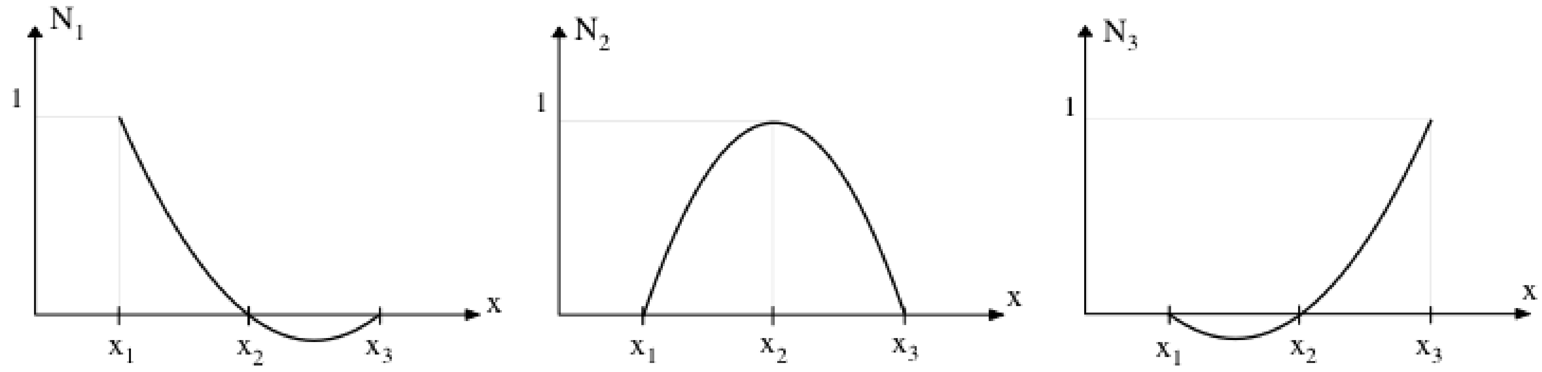
Higher order elements

$$u^{(1)} = \alpha_{j1} + \alpha_{j2}x_1 + \alpha_{j3}x_1^2$$
$$u^{(2)} = \alpha_{j1} + \alpha_{j2}x_2 + \alpha_{j3}x_2^2$$
$$u^{(3)} = \alpha_{j1} + \alpha_{j2}x_3 + \alpha_{j3}x_3^2$$

$$u_j(x) = \sum_{k=1}^{3} N_{jk}(x)u^{(k)}$$

$$N_{j1}(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}, \qquad N_{j2}(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$N_{j3}(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}.$$



Two Quadratic Elements

$$[k_j] = \int_{x_1}^{x_3} \begin{pmatrix} \frac{dN_{j1}}{dx}\frac{dN_{j1}}{dx} & \frac{dN_{j1}}{dx}\frac{dN_{j2}}{dx} & \frac{dN_{j1}}{dx}\frac{dN_{j3}}{dx} \\ \frac{dN_{j2}}{dx}\frac{dN_{j1}}{dx} & \frac{dN_{j2}}{dx}\frac{dN_{j2}}{dx} & \frac{dN_{j2}}{dx}\frac{dN_{j3}}{dx} \\ \frac{dN_{j3}}{dx}\frac{dN_{j1}}{dx} & \frac{dN_{j3}}{dx}\frac{dN_{j2}}{dx} & \frac{dN_{j3}}{dx}\frac{dN_{j3}}{dx} \end{pmatrix} dx$$

$$[k_j] = \begin{pmatrix} \frac{7}{6l} & \frac{-8}{6l} & \frac{1}{6l} \\ \frac{-8}{6l} & \frac{16}{6l} & \frac{-8}{6l} \\ \frac{1}{6l} & \frac{-8}{6l} & \frac{7}{6l} \end{pmatrix}$$

$$\{f_j\} = -\int_{x_1}^{x_3} \begin{pmatrix} N_{j1} \\ N_{j2} \\ N_{j3} \end{pmatrix} f(x)\, dx$$

$$\{f_j\} = -\frac{1}{3}c \begin{pmatrix} l \\ 4l \\ l \end{pmatrix}$$

$$\begin{pmatrix} \frac{2}{l} & \frac{-1}{l} & 0 \\ \frac{-1}{l} & \frac{2}{l} & \frac{-1}{l} \\ 0 & \frac{-1}{l} & \frac{2}{l} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = -\begin{pmatrix} cl \\ cl \\ cl \end{pmatrix}$$

$$\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = -\begin{pmatrix} \frac{3l}{4} & \frac{l}{2} & \frac{l}{4} \\ \frac{l}{2} & l & \frac{l}{2} \\ \frac{l}{4} & \frac{l}{2} & \frac{3l}{4} \end{pmatrix} \begin{pmatrix} cl \\ cl \\ cl \end{pmatrix} = \begin{pmatrix} -0.375 \\ -0.5 \\ -0.375 \end{pmatrix}$$

Shape Functions

$$A_j(\mathbf{x}) = A_{z_j}(x, y)$$

$$A^{(1)} = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$A^{(2)} = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$A^{(3)} = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

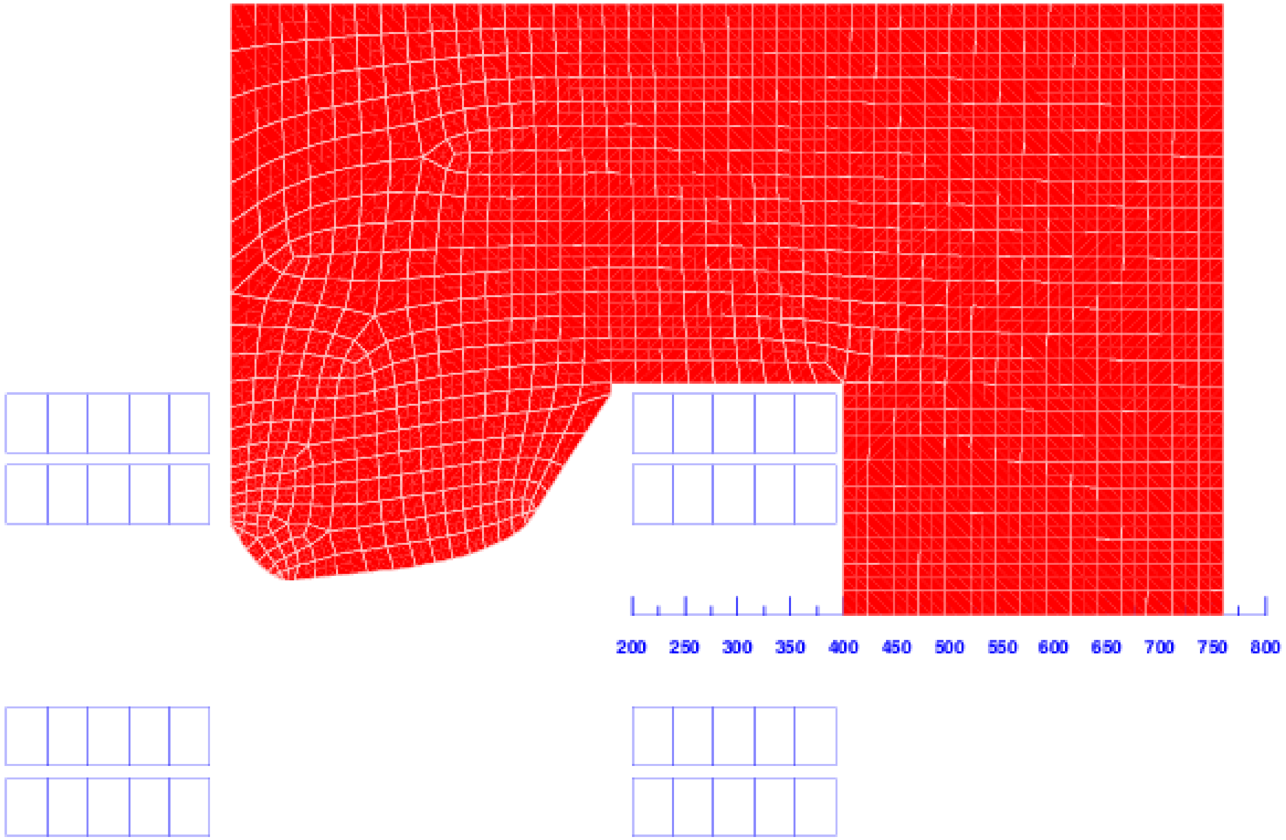
Mapped Elements

$$x = x(\xi, \eta, \zeta), \qquad y = y(\xi, \eta, \zeta), \qquad z = z(\xi, \eta, \zeta)$$

$$y_j(\boldsymbol{\xi}) = \sum_{k=1}^{K} N_k(\boldsymbol{\xi}) y^{(k)}$$

Use of the same shape functions for the transformation of the elements

Higher Order Elements



Higher accuracy of the field solution, but also better modeling of the iron contour

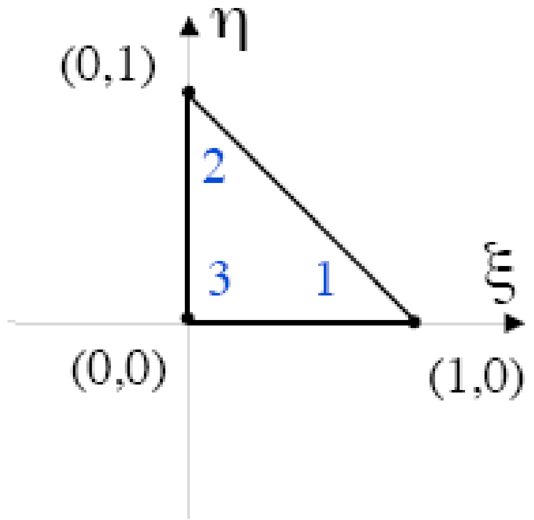
Mapped Elements

The picture can't be displayed.

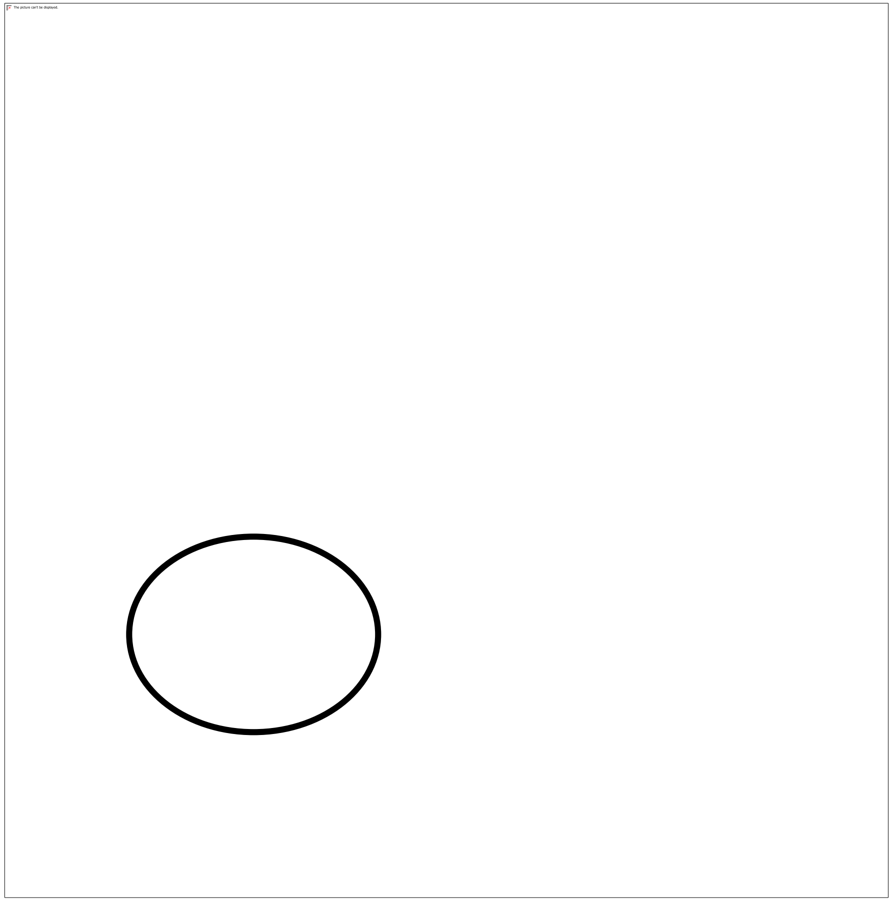
Transformation of Differential Operators

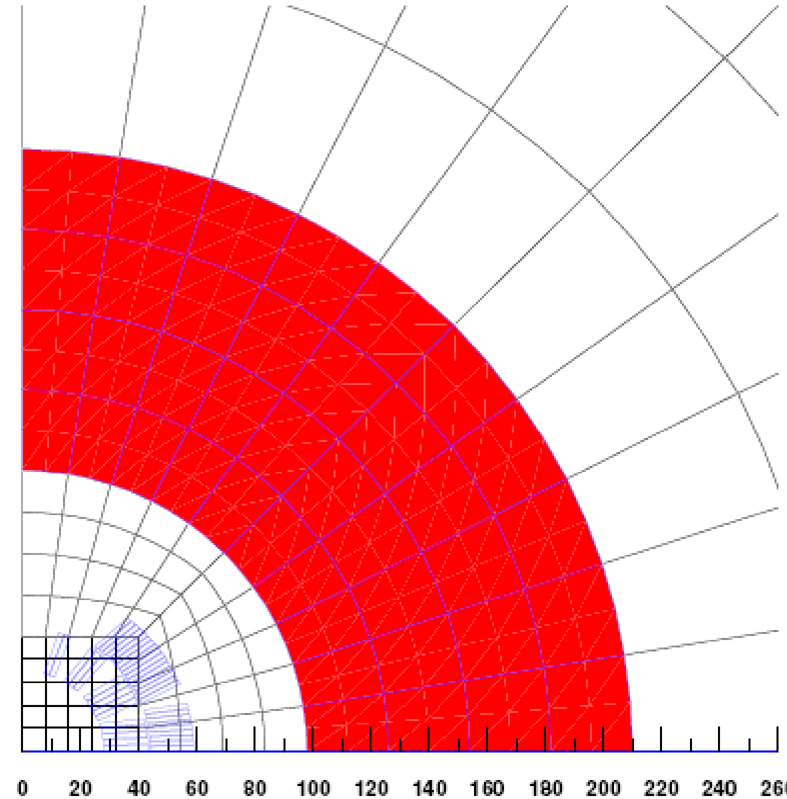
Complicated **Easy**

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} N_k = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} N_k = [J]_{T-1} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} N_k$$



Collinear Sides yield Singular Jacobi Matrices

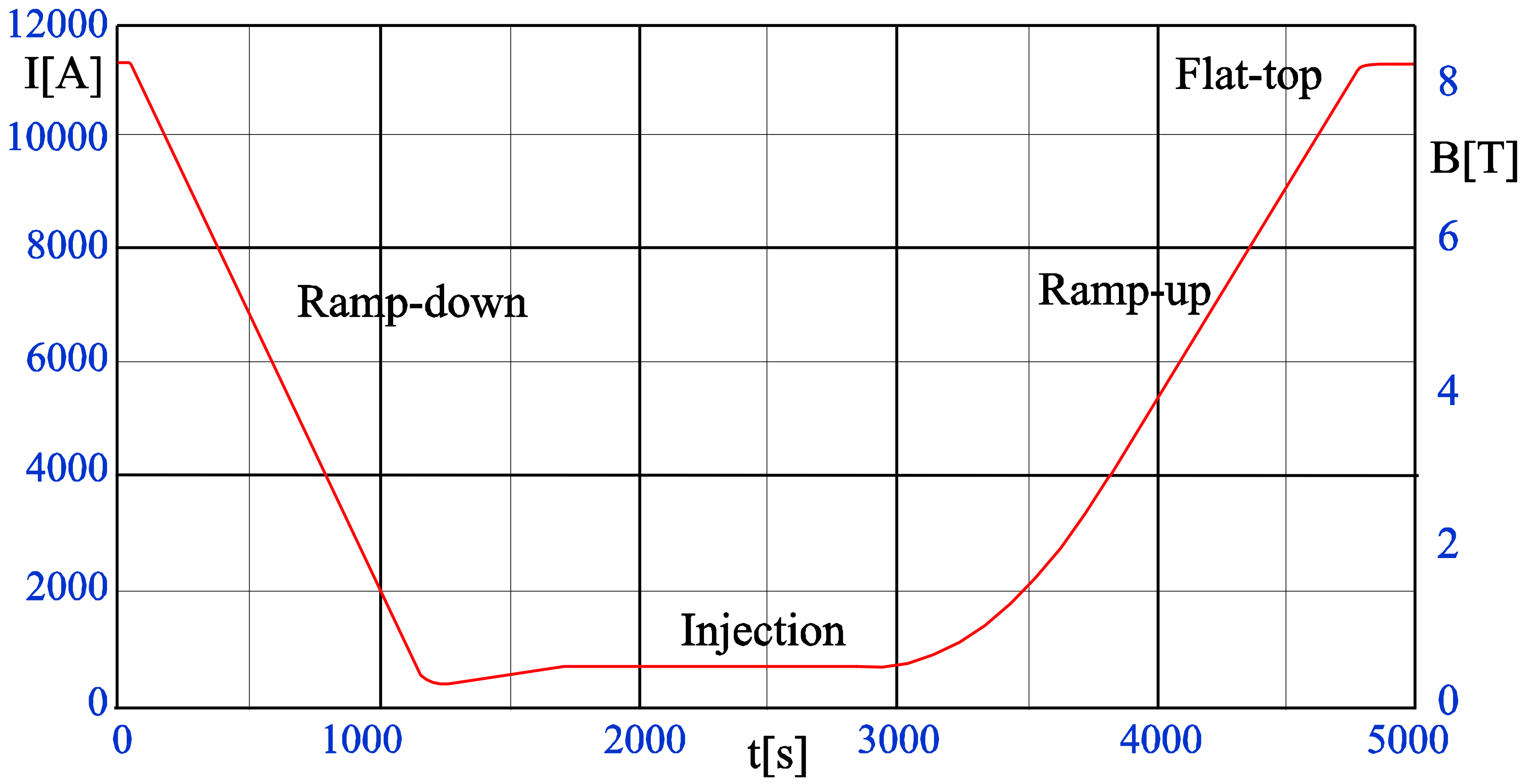


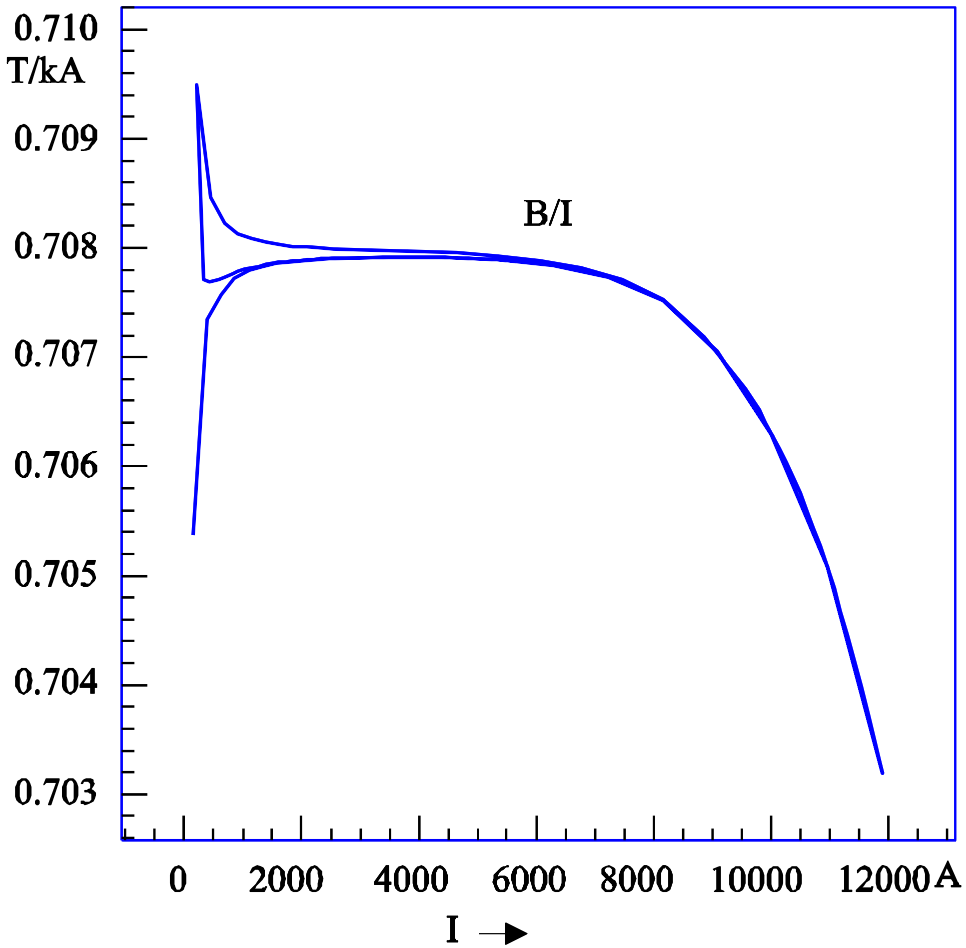


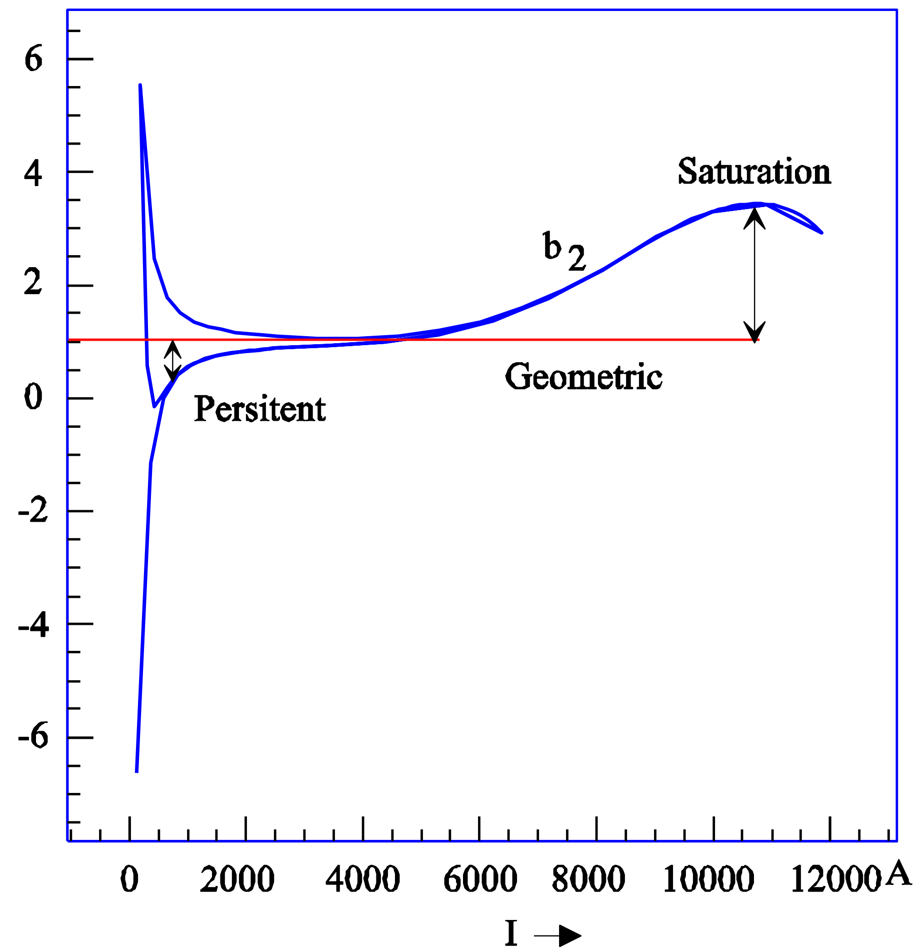
Note: Bad meshing is not a trivial offence

Numerical Methods for the Curl-Curl Equation

Excitation Cycle




0.710
T/kA
0.709
0.708
0.707
0.706
0.705
0.704
0.703
B/I
0
2000
4000
6000
8000
10000
12000
A
I


6
4
2
0
-2
-4
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Saturation
b 2
Geometric
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6000
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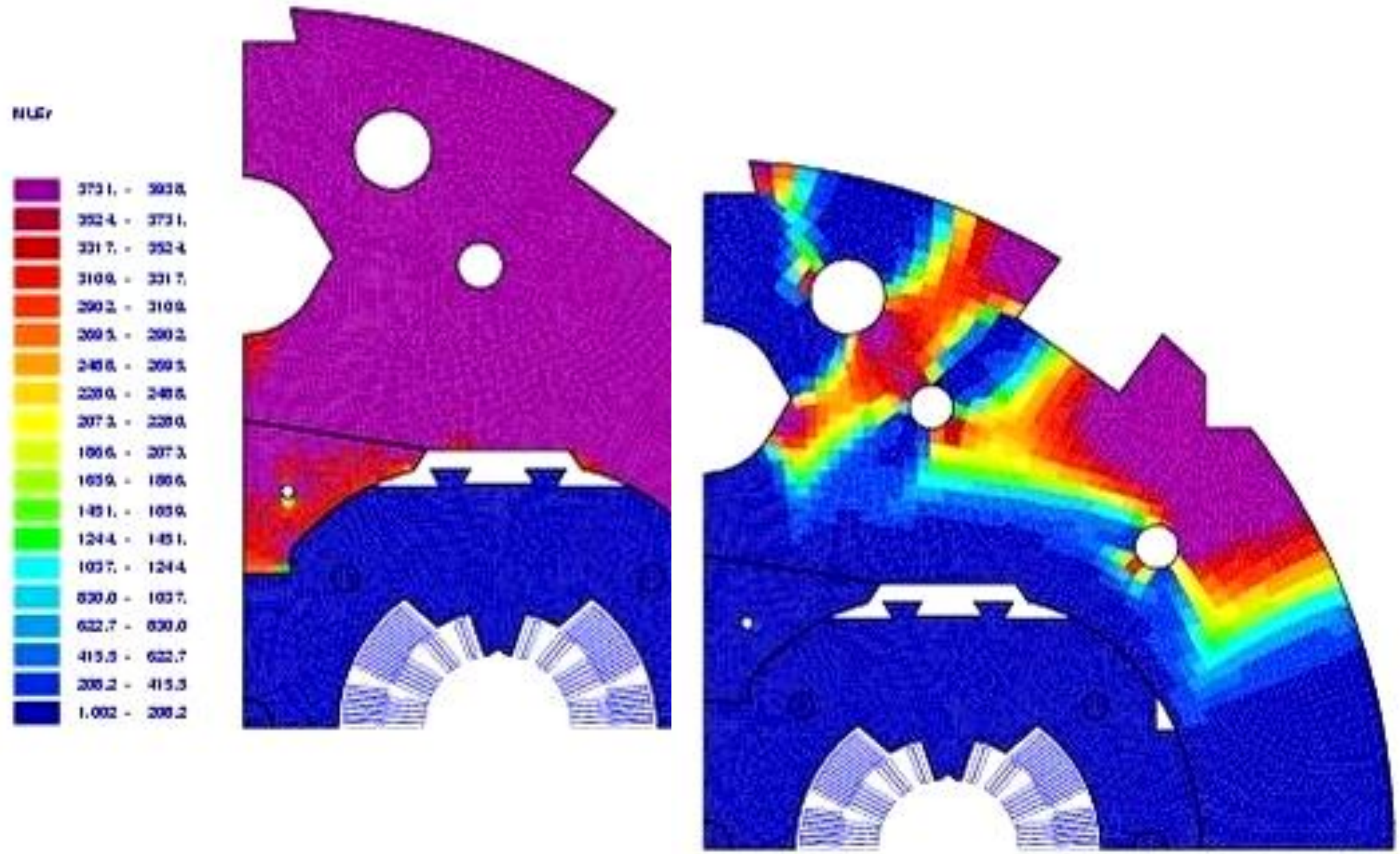
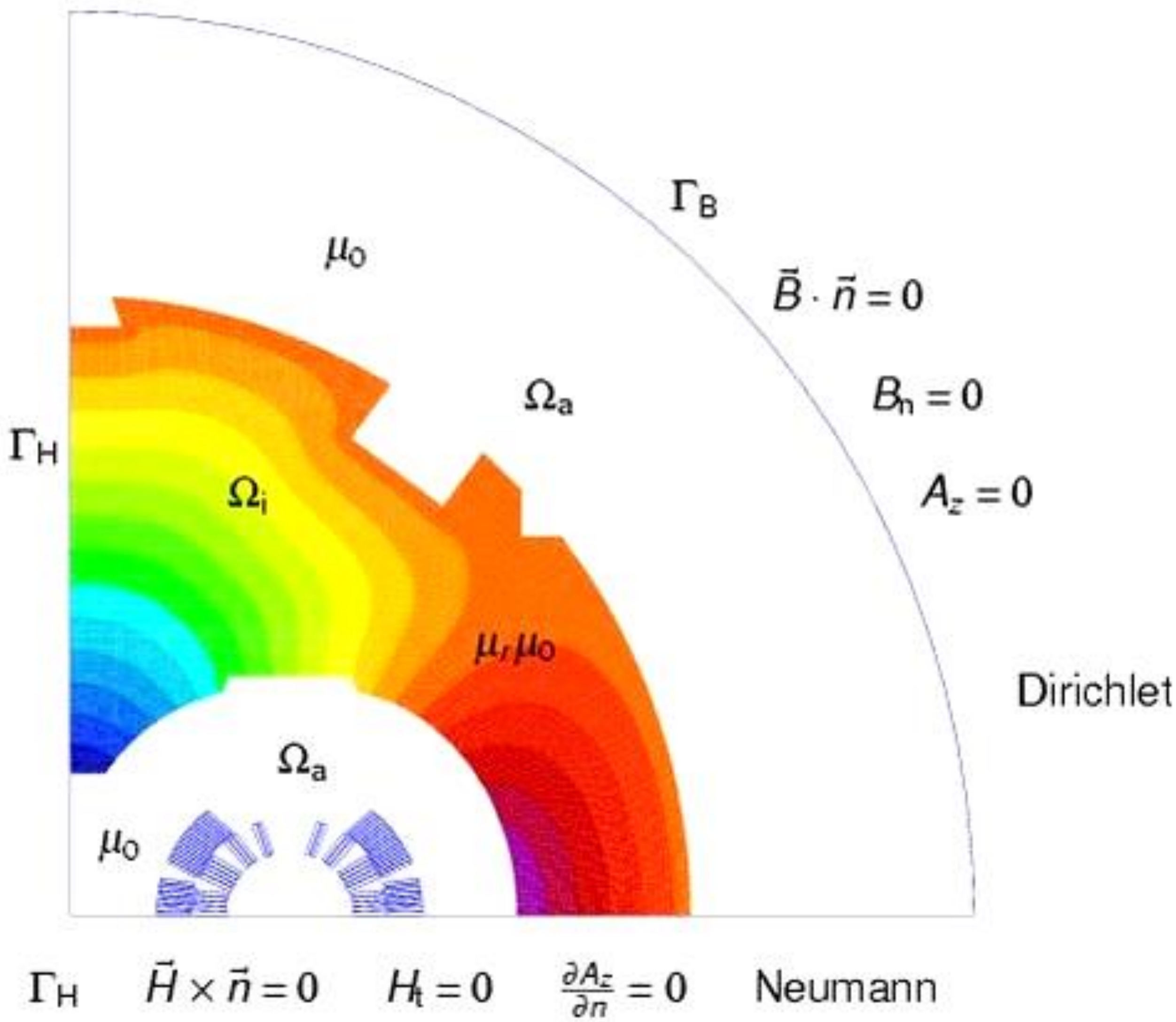
Saturation Effects in the Dipole Iron Yoke



The Problem Domain



Curl-Curl Equation

$$\mathbf{B} = \text{curl}\,\mathbf{A} \qquad \text{in}\,\Omega$$

$$\text{curl}\,\frac{1}{\mu}\,\text{curl}\,\mathbf{A} = \mathbf{J} \qquad \text{in}\,\Omega$$

$$\mathbf{H}_\text{t} = \mathbf{0} \quad \rightarrow \quad \frac{1}{\mu}(\text{curl}\,\mathbf{A}) \times \mathbf{n} \;=\; \mathbf{0} \;\; \text{on}\;\Gamma_H$$

$$B_\text{n} = 0 \quad \rightarrow \quad \mathbf{B} \cdot \mathbf{n} = \text{curl}\,\mathbf{A} \cdot \mathbf{n} \;=\; 0 \;\; \text{on}\;\Gamma_B$$

$$\left[\frac{1}{\mu}\,(\text{curl}\,\mathbf{A}) \times \mathbf{n}\right]_\text{ai} \;=\; \mathbf{0} \quad \text{on}\;\Gamma_\text{ai}$$

$$[\mathbf{A}]_\text{ai} \;=\; \mathbf{0} \quad \text{on}\;\Gamma_\text{ai}$$

Problem in 3-D: Gauging

$$\mathbf{A} \rightarrow \mathbf{A}' : \mathbf{A}' = \mathbf{A} + \text{grad}\,\psi$$

$$\text{div}\,\mathbf{A}' = q$$

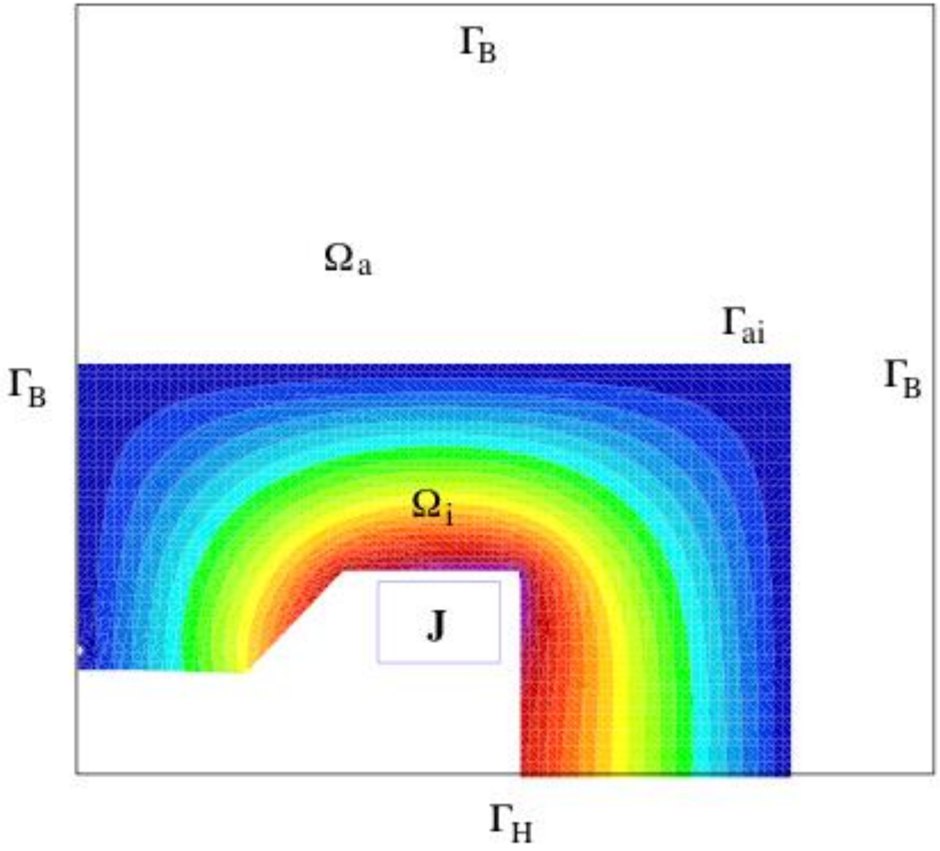
$$q = \text{div}\,\mathbf{A} + \nabla^2\psi$$

$$\frac{1}{\mu}\text{div}\,\mathbf{A} = 0 \quad \text{in}\;\Omega$$

$$\mathbf{A} \cdot \mathbf{n} = 0 \quad \text{on}\;\Gamma_H$$

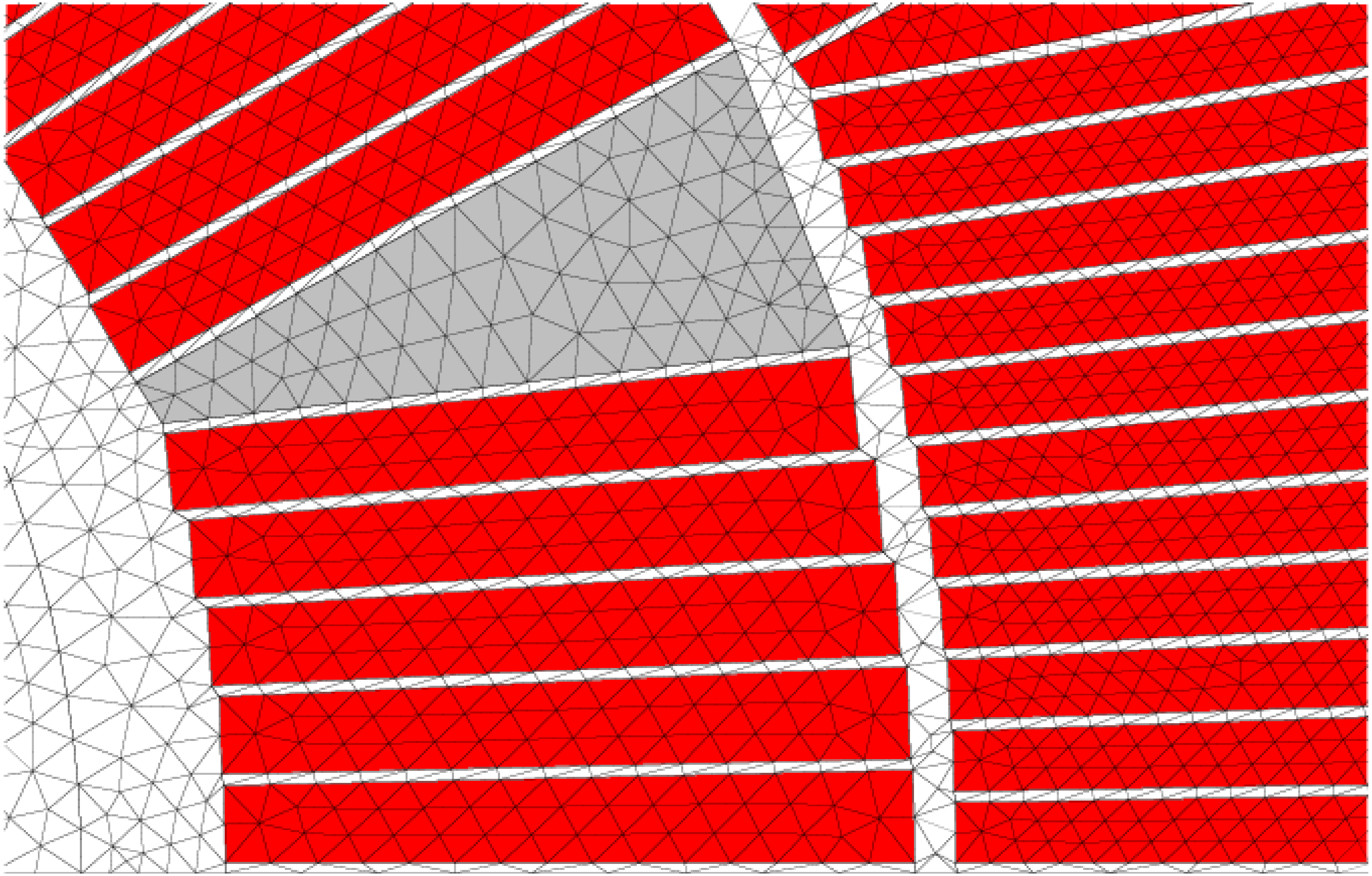
$$\text{curl}\,\frac{1}{\mu}\,\text{curl}\,\mathbf{A} - \;\text{grad}\;\frac{1}{\mu}\;\text{div}\;\mathbf{A} = \mathbf{J} \quad \text{in}\,\Omega$$

Weak Form in the FEM Problem

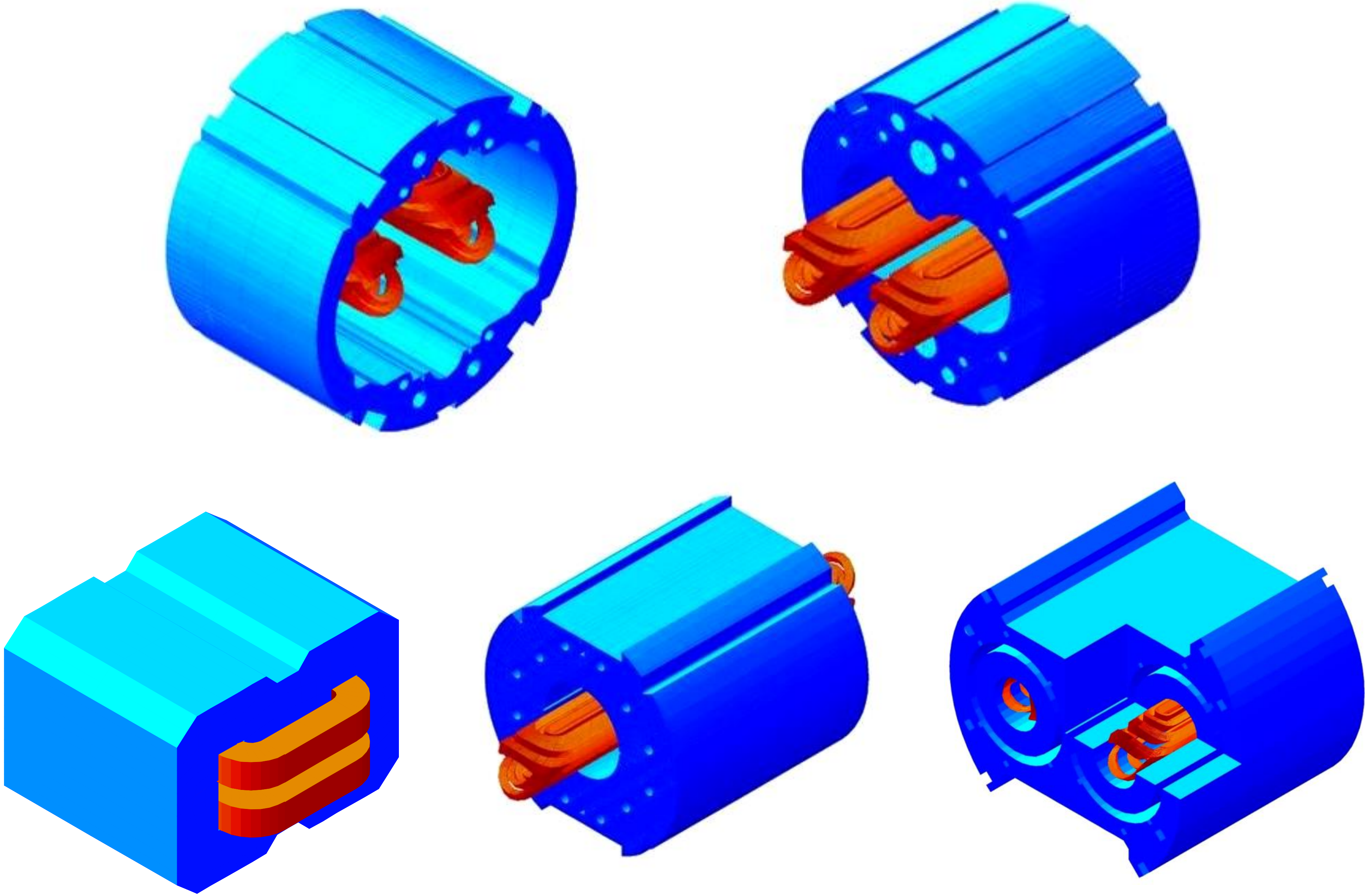


Weak Form in the FEM Problem

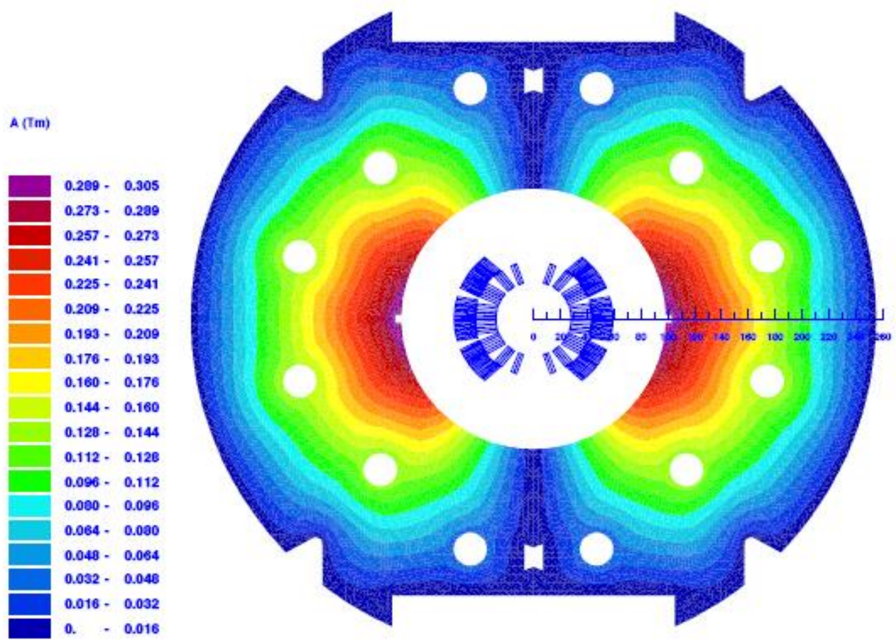
Meshing the Coil

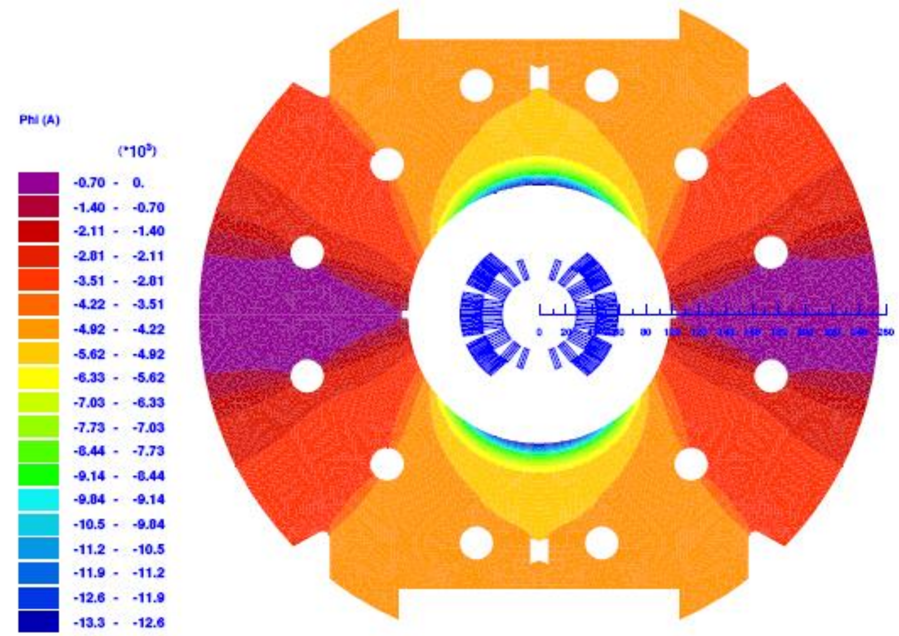


Magnet Extremities

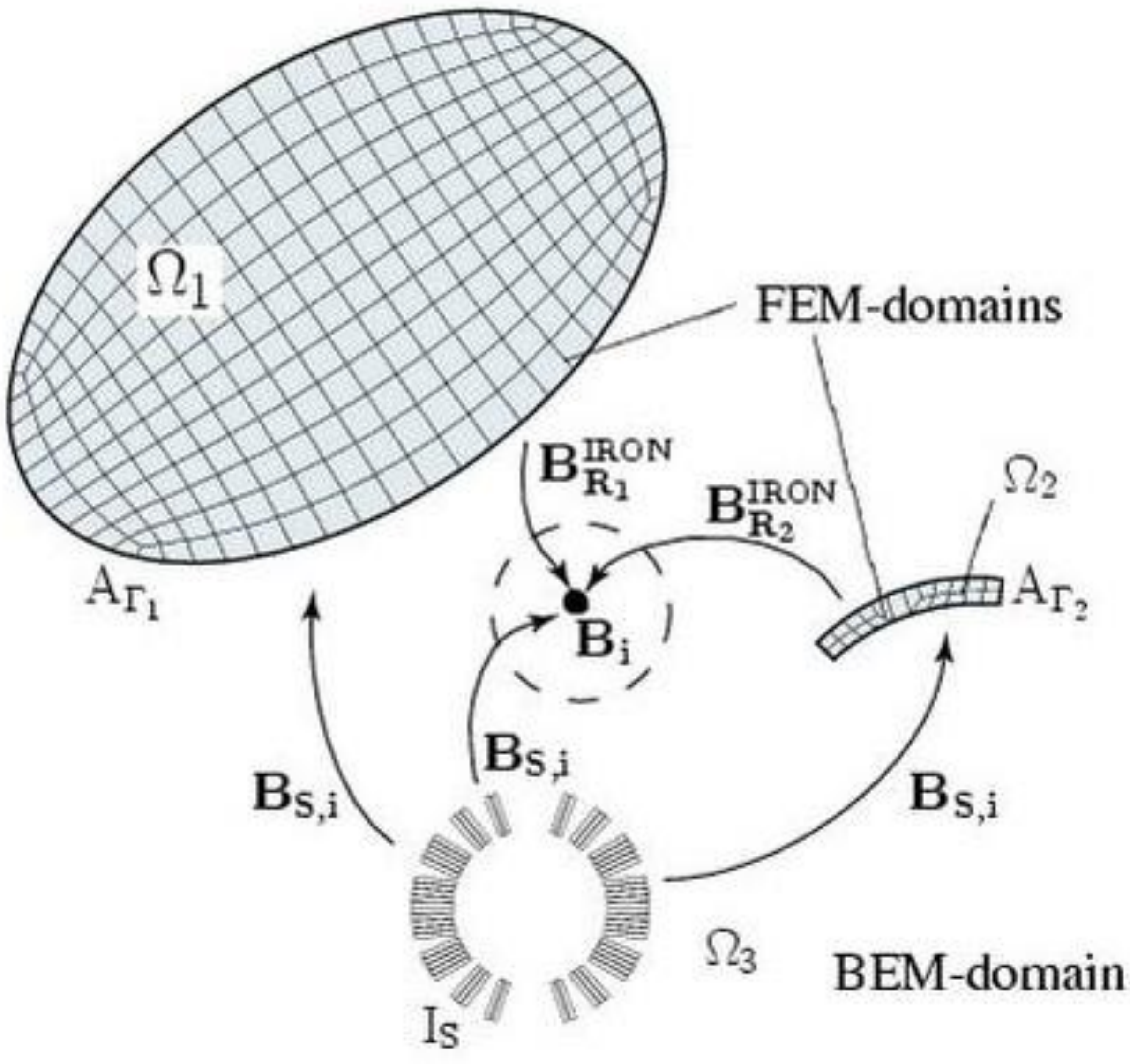


Vector Potential and Total Scalar Potential

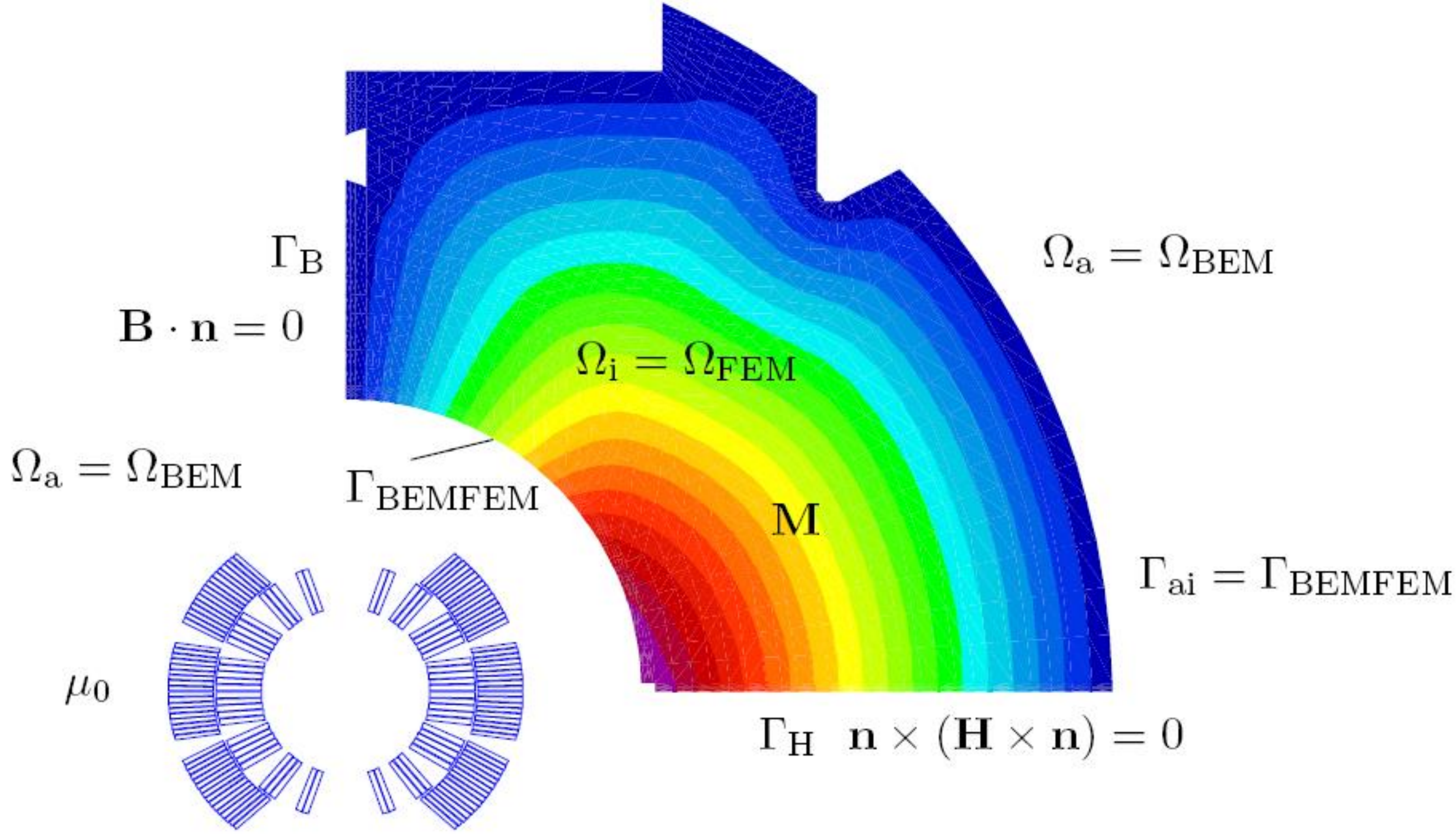




BEM-FEM Coupling (Elementary Model Problem)

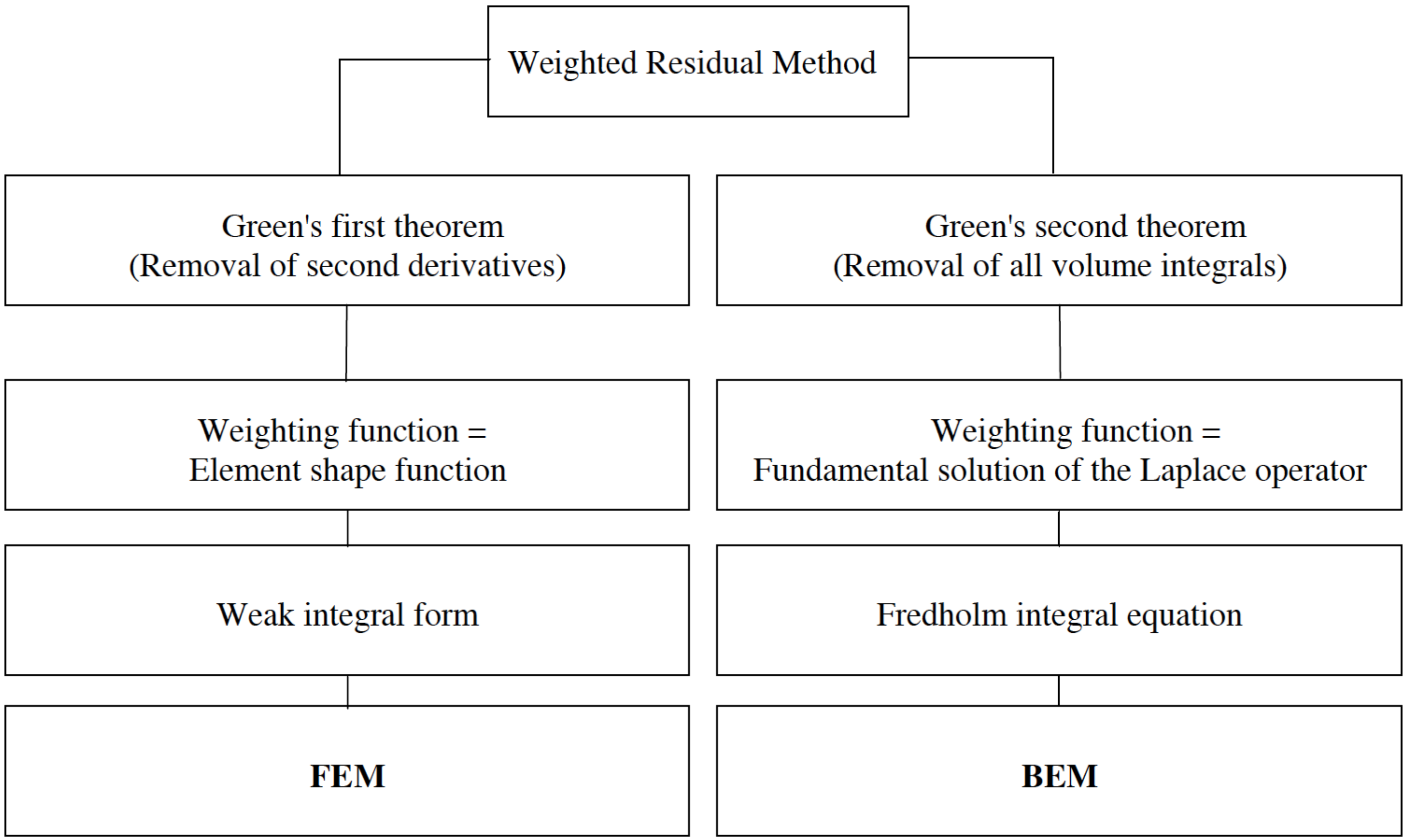


The Elementary Model Problem in Magnet Design



Green' s First and Second Identities in FEM and BEM

$$\int_\Omega (\operatorname{grad}\phi \cdot \operatorname{grad}\psi + \phi\nabla^2\psi)\,\mathrm{d}V = \int_\Gamma \phi \operatorname{grad}\psi \cdot \mathbf{n}\,\mathrm{d}a\,,$$



$$\int_\Omega \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\,\mathrm{d}V = \int_\Gamma (\phi\partial_{\mathbf{n}}\psi - \psi\partial_{\mathbf{n}}\phi)\,\mathrm{d}a\,,$$

The FEM Part (Vector Laplace Equation)

$$\boxed{-\frac{1}{\mu_0}\nabla^2\mathbf{A} = \mathbf{J} + \operatorname{curl}\mathbf{M}} \qquad \text{in } \Omega_i,$$

$$\mathbf{A}\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_H,$$

$$\frac{1}{\mu_0}\operatorname{div}\mathbf{A} = 0 \qquad \text{on } \Gamma_B,$$

$$\mathbf{n}\times(\mathbf{A}\times\mathbf{n}) = \mathbf{0} \qquad \text{on } \Gamma_B,$$

$$\frac{1}{\mu}(\operatorname{curl}\mathbf{A})\times\mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma_H,$$

$$\left[\frac{1}{\mu_0}\operatorname{div}\mathbf{A}_{\mathrm{a}}\right]_{\mathrm{ai}} = 0 \qquad \text{on } \Gamma_{\mathrm{ai}},$$

$$\boxed{\frac{1}{\mu_0}(\operatorname{curl}\mathbf{A}_{\mathrm{i}} - \mu_0\mathbf{M})\times\mathbf{n}_{\mathrm{i}} + \frac{1}{\mu_0}(\operatorname{curl}\mathbf{A}_{\mathrm{a}})\times\mathbf{n}_{\mathrm{a}} = \mathbf{0}} \qquad \text{on } \Gamma_{\mathrm{ai}},$$

$$[\mathbf{A}]_{\mathrm{ai}} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{ai}}.$$

FEM Part

$$\frac{1}{\mu_0}\int_{\Omega_\mathrm{i}} \operatorname{grad}(\mathbf{A}\cdot\mathbf{e}_a)\cdot\operatorname{grad} w_a \,\mathrm{d}\Omega_\mathrm{i} - \frac{1}{\mu_0}\oint_{\Gamma_\mathrm{ai}}\left(\frac{\partial\mathbf{A}}{\partial n_\mathrm{i}} - (\mu_0\mathbf{M}\times\mathbf{n}_\mathrm{i})\right)\cdot\mathbf{w}_a\,\mathrm{d}\Gamma_\mathrm{ai} = \int_{\Omega_\mathrm{i}}\mathbf{M}\cdot\operatorname{curl}\mathbf{w}_a\,\mathrm{d}\Omega_\mathrm{i}$$

$$[K]\{A\} - [T]\{Q\} = \{F(\mathbf{M})\}$$

BEM Part

Vector Laplace

Weighted Residual

From Green's second theorem:

$$\int_{\Omega_\mathrm{a}} A \nabla^2 w \, \mathrm{d}\Omega_\mathrm{a} = -\int_{\Omega_\mathrm{a}} \mu_0 J w \, \mathrm{d}\Omega_\mathrm{a} + \int_{\Gamma_\mathrm{ai}} A \frac{\partial w}{\partial n_a} \mathrm{d}\Gamma_\mathrm{ai} - \int_{\Gamma_\mathrm{ai}} \frac{\partial A}{\partial n_a} w \, \mathrm{d}\Gamma_\mathrm{ai}$$

Representation Formula (Fredholm Integral Equation)

$$\frac{\Theta}{4\pi}A(\mathbf{r}) = \int_\Gamma \partial_{\mathbf{n}_a} A(\mathbf{r}')\, u^*(\mathbf{r},\mathbf{r}')\, \mathrm{d}a' - \int_\Gamma A(\mathbf{r}')\, q^*(\mathbf{r},\mathbf{r}')\, \mathrm{d}a'$$

Single-layer potential

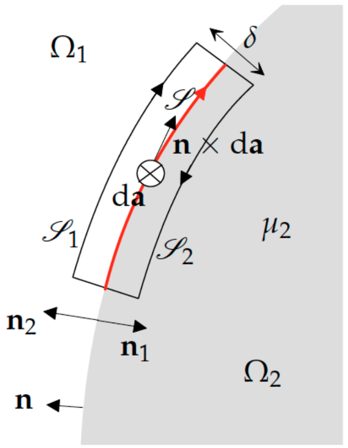
$$\boldsymbol{\alpha}(\mathbf{r}') := -\frac{1}{\mu}\partial_{\mathbf{n}_a} A(\mathbf{r}')$$

$$[\boldsymbol{\alpha}] = 1\ \mathrm{A\,m}^{-1}$$

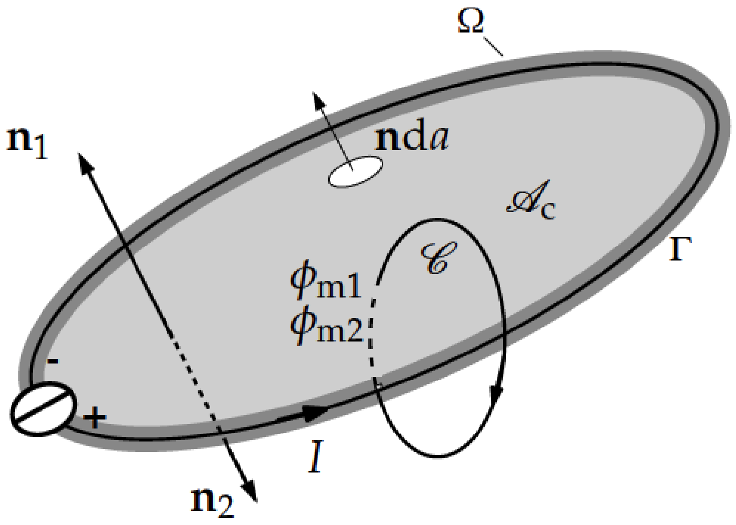
Double-layer potential

$$\boldsymbol{\tau}(\mathbf{r}') := \frac{1}{\mu}A(\mathbf{r}')$$

$$[\boldsymbol{\tau}] = 1\ \mathrm{A}$$



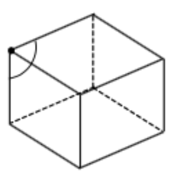
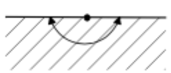
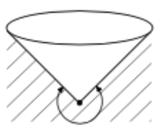
$$\boldsymbol{\alpha} = \mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2)$$



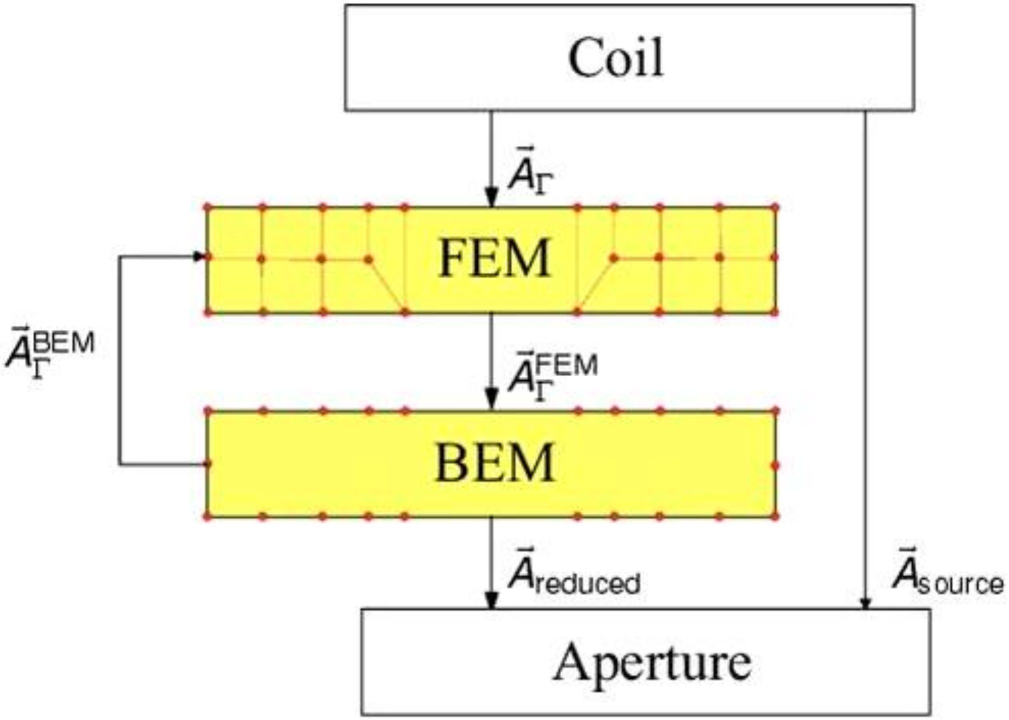
Point-Collocation (Compute One from the Other)

$$\frac{\Theta}{4\pi}A(\mathbf{r}) = \int_{\Gamma} \partial_{\mathbf{n}_\mathrm{a}} A(\mathbf{r}')\, u^*(\mathbf{r},\mathbf{r}')\, \mathrm{d}a' - \int_{\Gamma} A(\mathbf{r}')\, q^*(\mathbf{r},\mathbf{r}')\, \mathrm{d}a'$$

$$C(\mathbf{r}_p)A(\mathbf{r}_p) + \sum_{e=1}^{E} \int_{\Gamma_\mathrm{e}} -\partial_{\mathbf{n}_\mathrm{a}} A(\mathbf{r})\, u^*(\mathbf{r},\mathbf{r}_p)\mathrm{d}a + \sum_{e=1}^{E} \int_{\Gamma_\mathrm{e}} A(\mathbf{r})\, q^*(\mathbf{r},\mathbf{r}_p)\mathrm{d}a = 0$$

Ω_a	 90°Corner	 90° Cone inner	 Half-space	 90° Cone outer
Θ	$\frac{1}{2}\pi$	$(2-\sqrt{2})\pi$	2π	$(2+\sqrt{2})\,\pi$
$\frac{\Theta}{4\pi}$	$\frac{1}{8}$	$\frac{2-\sqrt{2}}{4}$	$\frac{1}{2}$	$\frac{2+\sqrt{2}}{4}$

BEM-FEM Coupling



BEM

$$\{Q\} = -[G]^{-1}[H]\{A\} + [G]^{-1}\{A_s\}$$

FEM

$$[K]\{A\} - [T]\{Q\} = \{F(\mathbf{M})\}$$

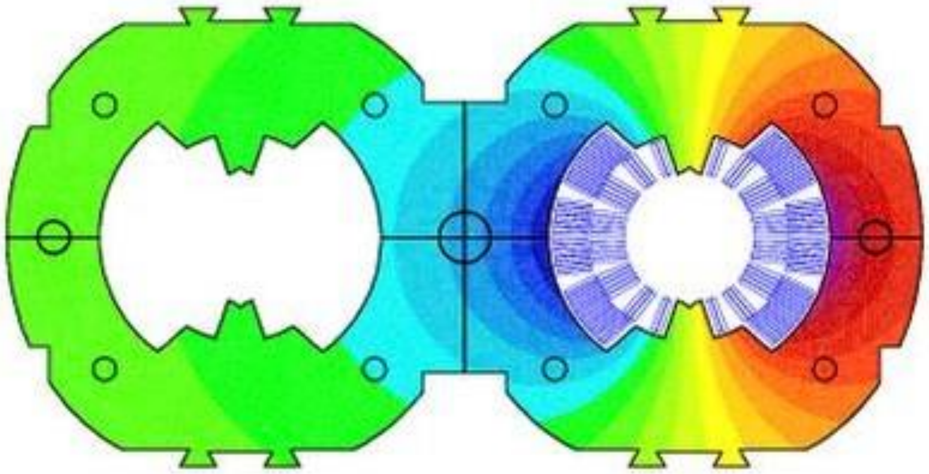
$$\left([K] + [T][G]^{-1}[H]\right)\{A\} = \{F(\mathbf{M})\} + [T][G]^{-1}\{A_s\}$$

$$[\overline{K}]\{A\} = \{\overline{F}(A_s, \mathbf{M})\}$$

Open Boundary Problems (1)

LHC Beam Screen

Open Boundary Problem (2)



Collared Coil
Field Problem

Collared Coil
Measurements in
Industry

Forces (N) in the Connection Ends of the LHC Main Dipole

I	Fx	Fy	Fz
1	-39.7	-44.0	-45.4
2	-6.5	3.7	-41.7
3	-6.1	88.3	-38.2
4	1.25	3.9	-28.5
5	48.1	-46.7	-48.5
Sum	-2.95	5.2	-202.3