Short Introduction to
(Classical) Electromagnetic Theory

(.. and applications to accelerators)

Werner Herr, CERN

(http://cern.ch/Werner.Herr/CAS2014/lectures/EM-Theory.pdf)

OUTLINE

- **Reminder of some mathematics, see also lecture R. Steerenberg**
- **Basic electromagnetic phenomena**
- **Maxwell's equations**
- **Lorentz force**
- **Motion of particles in electromagnetic fields**
- **Electromagnetic waves in vacuum**
- **Electromagnetic waves in conducting media**
  - **Waves in RF cavities**
  - **Waves in wave guides**

Reading Material

(1) **J.D. Jackson,** *Classical Electrodynamics* **(Wiley, 1998 ..)**

(2) **L. Landau, E. Lifschitz,** *Klassische Feldtheorie*, **Vol2. (Harri Deutsch, 1997)**

(3) **W. Greiner,** *Classical Electrodynamics*, **(Springer, February, 22nd, 2009)**

(4) **J. Slater, N. Frank,** *Electromagnetism*, **(McGraw-Hill, 1947, and Dover Books, 1970)**

(5) **R.P. Feynman,** *Feynman lectures on Physics*, **Vol2.**

For many details and derivations: (1) and (2)

Variables and units used in this lecture

Formulae use <u>SI units</u> throughout.

$\vec{E}$	=	**electric field [V/m]**
$\vec{H}$	=	**magnetic field [A/m]**
$\vec{D}$	=	**electric displacement [C/m^2]**
$\vec{B}$	=	**magnetic flux density [T]**
q	=	**electric charge [C]**
ρ	=	**electric charge density [C/m^3]**
$\vec{j}$	=	**current density [A/m^2]**
μ_0	=	**permeability of vacuum, 4** $\pi \cdot 10^{-7}$ **[H/m or N/A^2]**
ϵ_0	=	**permittivity of vacuum, 8.854** $\cdot 10^{-12}$ **[F/m]**
c	=	**speed of light, 2.99792458** $\cdot 10^{8}$ **[m/s]**

Scalar and vector fields

Electric phenomena: $\vec{E}$**,** $\vec{D}$ **and** Φ

Magnetic phenomena: $\vec{H}$**,** $\vec{B}$ **and** $\vec{A}$

→ **Need to know how to calculate with vectors (see lecture by R. Steerenberg)**

- **Scalar and vector products**
- **Vector calculus**

Vector calculus ...

We can define a special vector ∇ (sometimes written as $\vec{\nabla}$):

$$\nabla = (\frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ \frac{\partial}{\partial z})$$

It is called the "gradient" and invokes "partial derivatives".
It can operate on a scalar function $\phi(x, y, z)$:

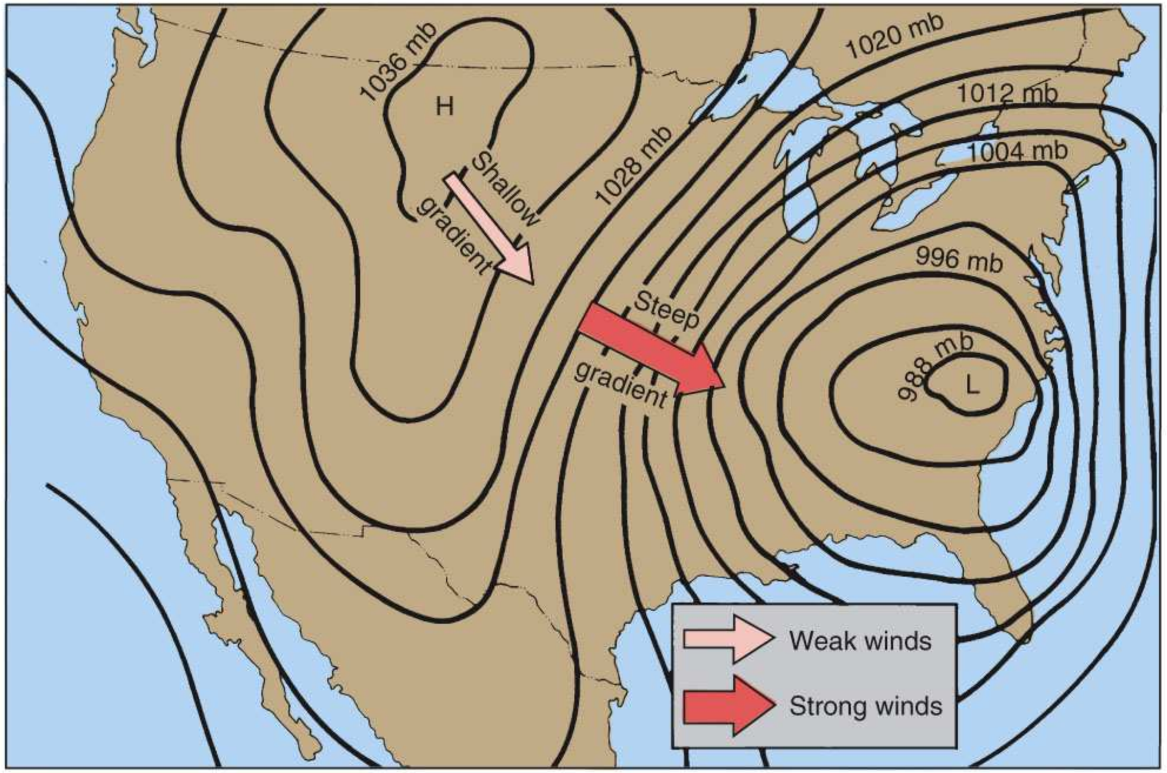
$$\nabla\phi = (\frac{\partial\phi}{\partial x}, \ \frac{\partial\phi}{\partial y}, \ \frac{\partial\phi}{\partial z}) \quad = \quad \vec{G} = (G_x, G_y, G_z)$$

and we get a vector $\vec{G}$. It is a kind of "slope" (steepness ..) in the 3 directions.

Example: $\phi(x, y, z) \ = \ C \cdot \mathbf{ln}(r^2)$ **with** $r \ = \ \sqrt{x^2 + y^2 + z^2}$

→ $\nabla\phi \ = \ (G_x, G_y, G_z) \ = \ (\frac{2C \cdot x}{r^2}, \ \frac{2C \cdot y}{r^2}, \ \frac{2C \cdot z}{r^2})$

Gradient (slope) of a scalar field



Lines of pressure (isobars)

Gradient is large (steep) where lines are close (fast change of pressure)

Vector calculus ...

The gradient ∇ can be used as scalar or vector product with a vector $\vec{F}$, sometimes written as $\vec{\nabla}$
Used as:

$$\nabla \cdot \vec{F} \qquad \text{or} \qquad \nabla \times \vec{F}$$

Same definition for products as before, ∇ treated like a "normal" vector, but results depends on how they are applied:
$\nabla \cdot \Phi$ is a vector
$\nabla \cdot \vec{F}$ is a scalar
$\nabla \times \vec{F}$ is a vector

Operations on vector fields ...

Two operations of ∇ have special names:

Divergence (scalar product of gradient with a vector):

$$\mathsf{div}(\vec{F}) \quad = \quad \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Physical significance: "amount of density", (see later)

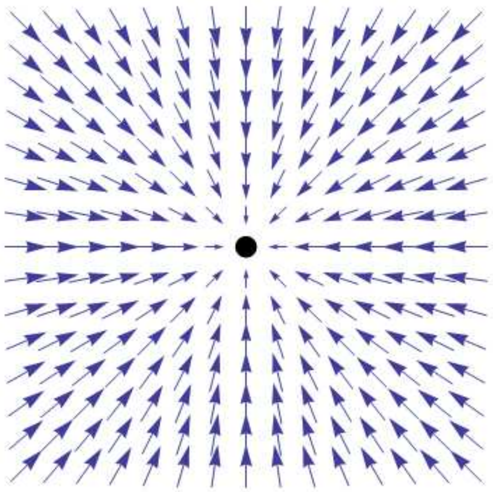
Curl (vector product of gradient with a vector):

$$\mathsf{curl}(\vec{F}) \quad = \quad \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Physical significance: "amount of rotation", (see later)

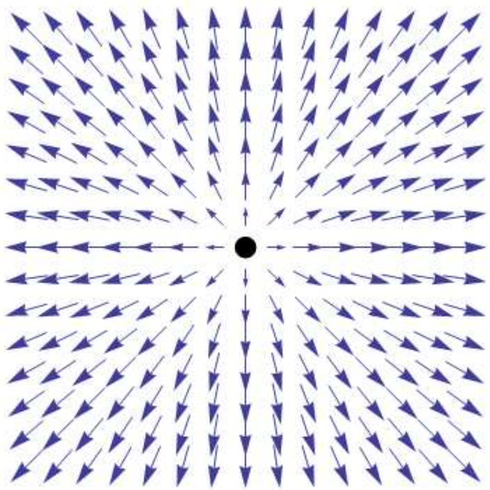
Meaning of Divergence of fields ...

Field lines of a vector field $\vec{F}$ seen from some origin:



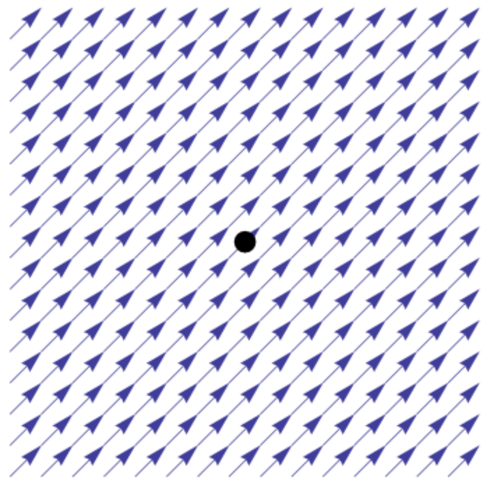
$\nabla \vec{F} < 0$

(sink)



$\nabla \vec{F} > 0$

(source)

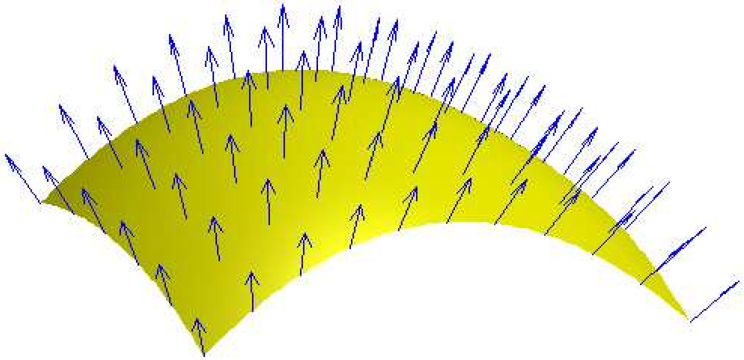


$\nabla \vec{F} = 0$

(fluid)

The divergence (scalar, a single number) characterizes what comes from (or goes to) the origin

How much comes out ?



Surface integrals: integrate field vectors passing (perpendicular) through a surface S (or area A), we obtain the Flux:

$$\rightarrow \quad \int\int_A \vec{F} \cdot d\vec{A}$$

Density of field lines through the surface

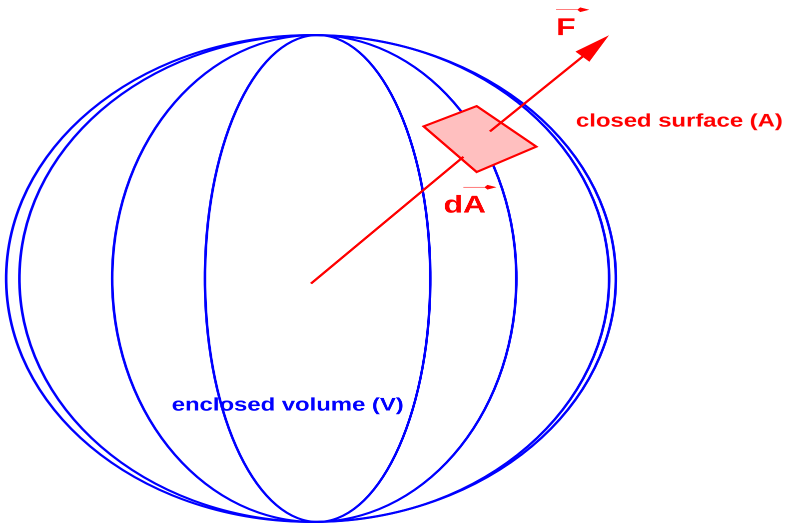
(e.g. amount of heat passing through a surface)

Surface integrals made easier ...

Gauss' Theorem:

<u>Integral through a closed surface (flux) is integral of divergence in the enclosed volume</u>

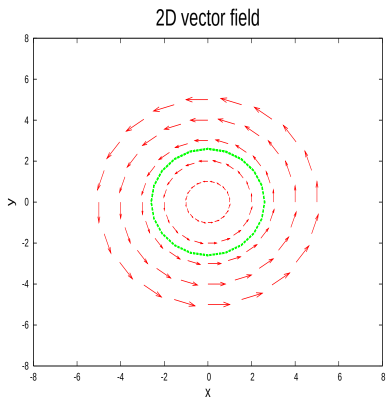
$$\int\int_A \vec{F} \cdot d\vec{A} = \int\int\int_V \nabla \cdot \vec{F} \cdot dV$$

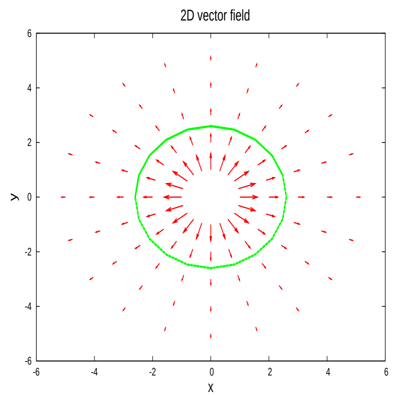


Relates surface integral to divergence

Meaning of curl of fields

The <u>curl</u> quantifies a rotation of vectors:





Line integrals: integrate field vectors along a line C:

$$\rightarrow \quad \oint_C \vec{F} \cdot d\vec{r}$$

"sum up" vectors (length) in <u>direction</u> of line C

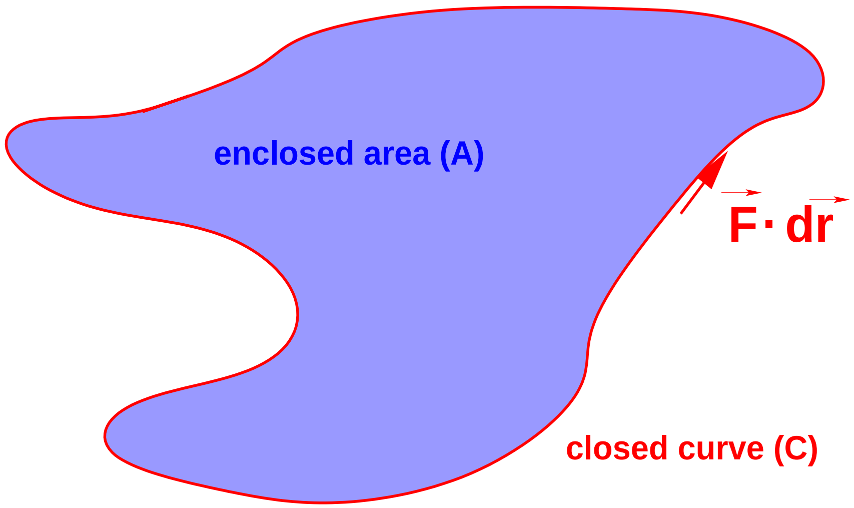
(e.g. work performed along a path ...)

Line integrals made easier ...

Stokes' Theorem:

Integral along a closed line is integral of curl in the enclosed area

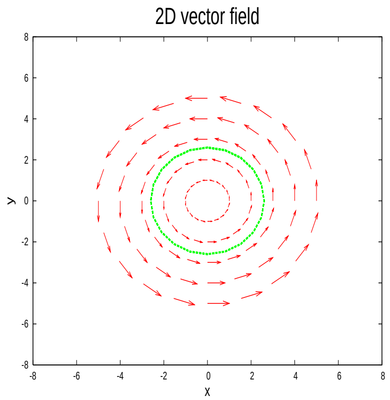
$$\oint_C \vec{F} \cdot d\vec{r} = \int\int_A \nabla \times \vec{F} \cdot d\vec{A}$$

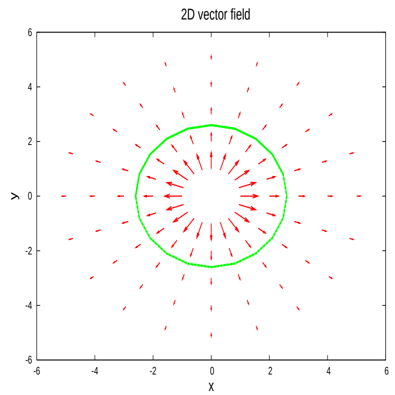


Relates line integral to curl

Integration of (vector-) fields

Two vector fields:





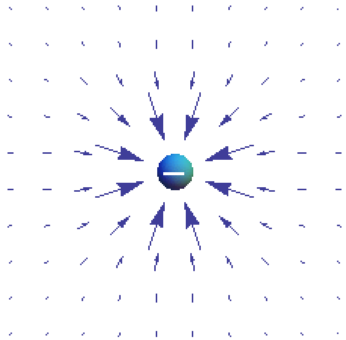
$\nabla \vec{F} = 0 \qquad \nabla \times \vec{F} \neq 0 \qquad\qquad \nabla \vec{F} \neq 0 \qquad \nabla \times \vec{F} = 0$

$$\oint_C \vec{F} \cdot d\vec{r} = \int\int_A \nabla \times \vec{F} \cdot d\vec{A}$$

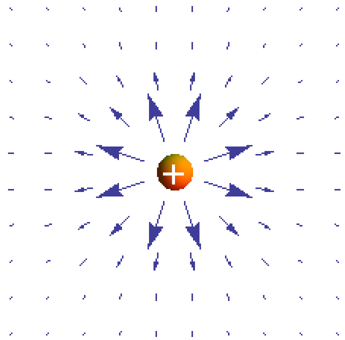
Line integral for second vector field vanishes ...

- APPLICATIONS to ELECTRODYNAMICS -

Electric fields from charges



(negative charges)



(positive charges)

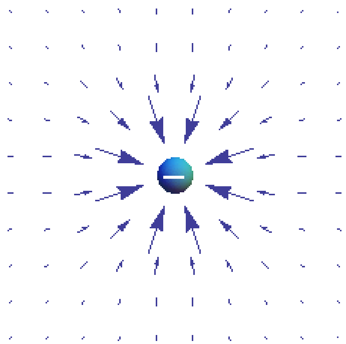
Assume fields from a positive or negative charge q

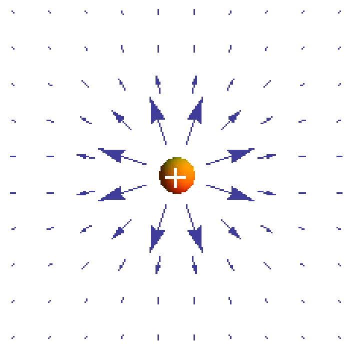
Electric field $\vec{E}$ is written as (Coulomb law):

$$\vec{E} \;=\; \frac{\pm q}{4\pi\epsilon_0} \cdot \frac{\vec{r}}{|r|^3}$$

with: $\vec{r} \;=\; (x, y, z),$ $\qquad |r| = \sqrt{x^2 + y^2 + z^2}$

Applying Divergence and charges ..





We can do the (non-trivial*)) computation of the divergence:

$$\mathrm{div}\vec{E} \;=\; \nabla\vec{E} \;=\; \frac{dE_x}{dx} + \frac{dE_y}{dy} + \frac{dE_z}{dz} \;=\; \frac{\rho}{\epsilon_0}$$

(negative charges) $\nabla \cdot \vec{E} < 0$

(positive charges) $\nabla \cdot \vec{E} > 0$

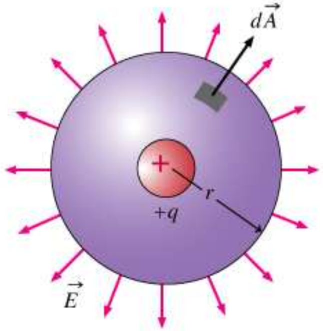
Divergence related to charge density ρ generating the field $\vec{E}$

*) **for a point charge for example ..**

More formal/general: Gauss's Theorem
(Maxwell's first equation ...)

$$\frac{1}{\epsilon_0} \int \int_A \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} \int \int \int_V \nabla \vec{E} \cdot dV = \frac{q}{\epsilon_0}$$

$$\nabla \vec{E} = \frac{\rho}{\epsilon_0}$$

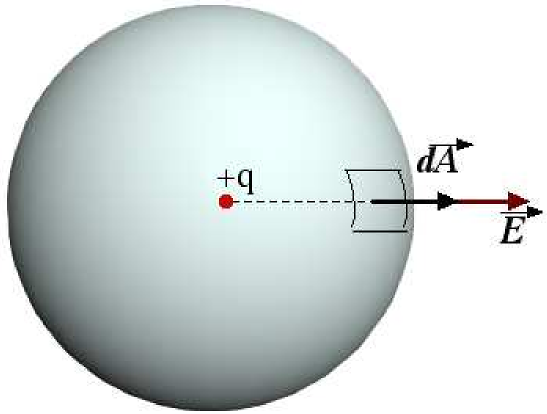


Flux of electric field $\vec{E}$ through any closed surface proportional to net electric charge q enclosed in the region (Gauss's Theorem).

Written with charge density ρ we get Maxwell's first equation:

$$\text{div}\vec{E} = \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Example: field from a charge q



A charge q generates a field $\vec{E}$ according to:

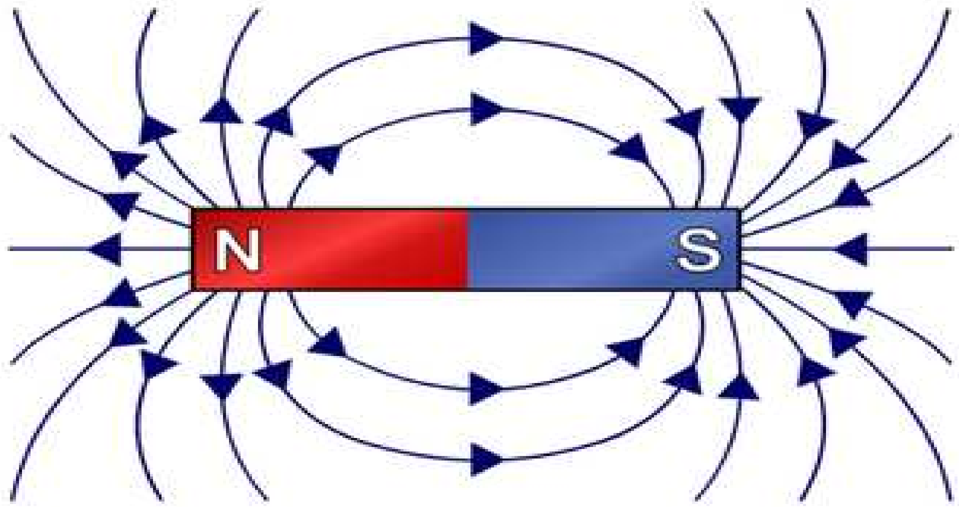
$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

Enclose it by a sphere: $\vec{E} = const.$ on a sphere (area is $4\pi \cdot r^2$):

$$\int\int_{sphere} \vec{E} \cdot d\vec{A} = \frac{q}{4\pi\epsilon_0}\int\int_{sphere}\frac{dA}{r^2} = \frac{q}{\epsilon_0}$$

Surface integral through sphere A is charge inside the sphere

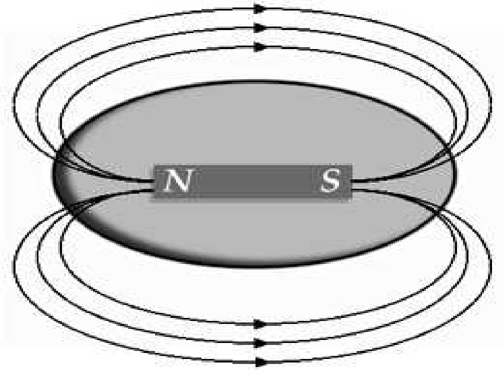
Divergence of magnetic fields



Definitions:

Magnetic field lines from North to South

Maxwell's second equation ...



$$\int\int_A \vec{B} \;\; d\vec{A} = \int\int\int_V \nabla\vec{B} \;\; dV \;\; = \;\; 0$$

$$\nabla\vec{B} = 0$$

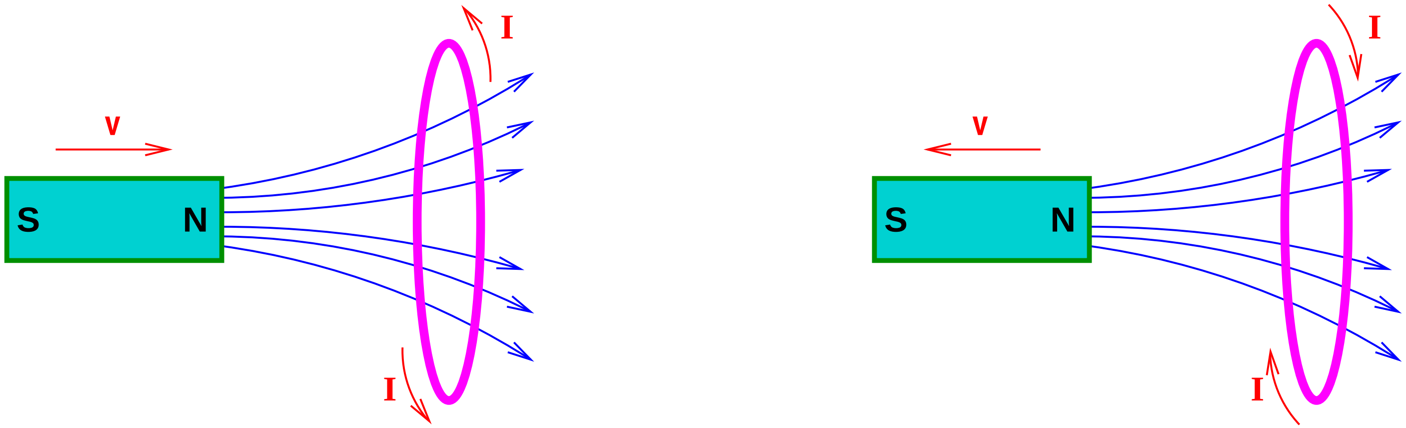
Closed field lines of magnetic flux density ($\vec{B}$): What goes out ANY closed surface also goes in, Maxwell's second equation:

$$\nabla\vec{B} = \mu_0\nabla\vec{H} = 0$$

➡ **Physical significance: no Magnetic Monopoles**

Maxwell's third equation ... (schematically)

Faradays law (electromagnetic induction):



- **Changing magnetic flux through area of a coil introduces electric current I**
- **Can be changed by moving <u>magnet</u> or <u>coil</u>**

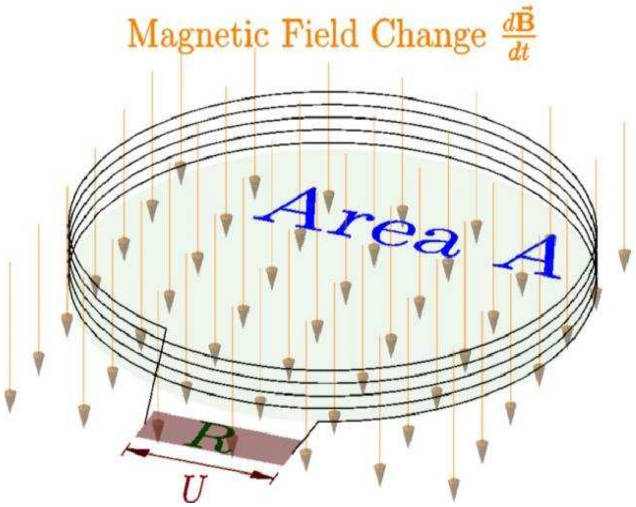
Maxwell's third equation ... (formally)

A changing flux Ω through an area A produces circulating electric field $\vec{E}$, i.e. a current I (Faraday)

$$-\frac{\partial \Omega}{\partial t} \;=\; \frac{\partial}{\partial t}\underbrace{\int_A \vec{B}\,d\vec{A}}_{flux\ \Omega} = \oint_C \vec{E}\cdot d\vec{r}$$

- **Flux can be changed by:**
  - **Change of magnetic field $\vec{B}$ with time t (e.g. transformers)**
  - **Change of area A with time t (e.g. dynamos)**

Formally: Maxwell's third equation ...



$$-\int_A \frac{\partial \vec{B}}{\partial t} d\vec{A} = \underbrace{\int_A \nabla \times \vec{E}\; d\vec{A} = \oint_C \vec{E} \cdot d\vec{r}}_{Stoke's formula}$$

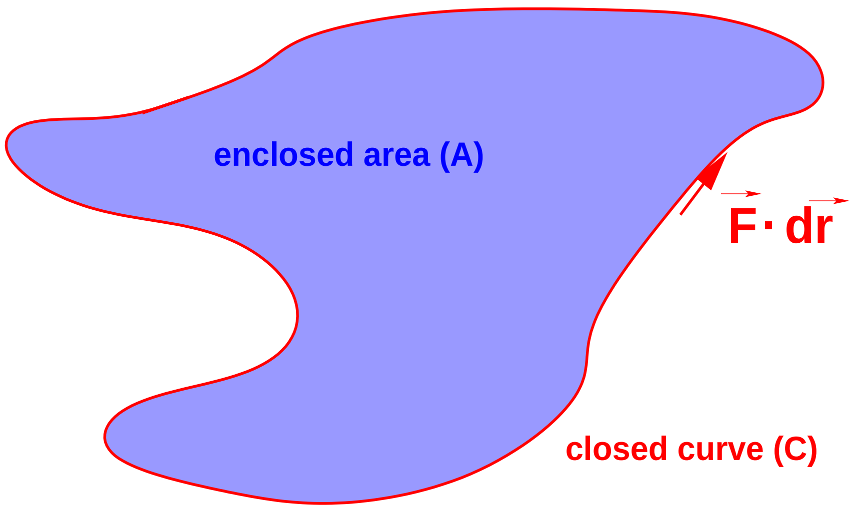
Changing magnetic field through an area induces circular electric field in coil around the area (Faraday)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Remember: large $curl$ **= strong circulating field**

More general:

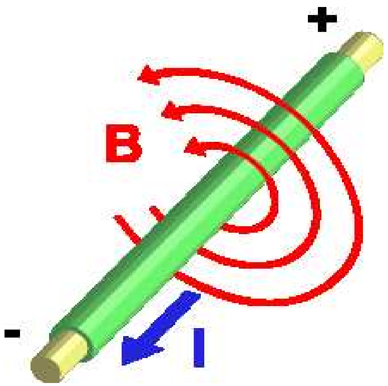
$$-\int_A \frac{\partial \vec{B}}{\partial t} d\vec{A} = \underbrace{\int_A \nabla \times \vec{E}\ d\vec{A} = \oint_C \vec{E} \cdot d\vec{r}}_{Stoke's formula}$$



Changing field through <u>any</u> area induces electric field in the (arbitrary) boundary

Maxwell's fourth equation (part 1) ...

From Ampere's law, for example current density $\vec{j}$:



<u>Static</u> electric current induces circulating magnetic field

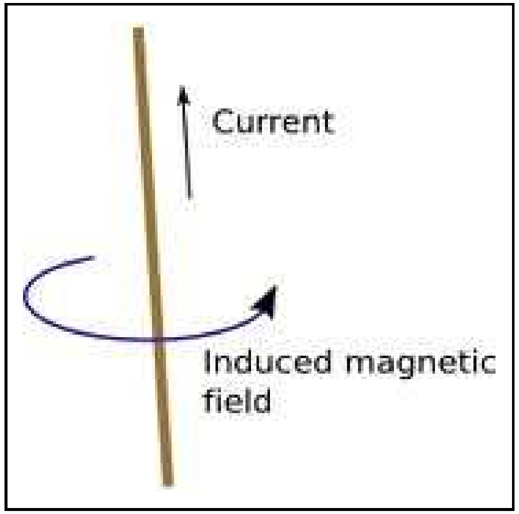
$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

or in integral form the currect density becomes the current I:

$$\int\int_A \nabla \times \vec{B} \;\; d\vec{A} \;\; = \;\; \int\int_A \mu_0 \vec{j} \;\; d\vec{A} \;\; = \;\; \mu_0 \vec{I}$$

Maxwell's fourth equation - application

For a static electric current I in a single wire we get Biot-Savart law (we have used Stoke's theorem and area of a circle $A = r^2 \cdot \pi$):



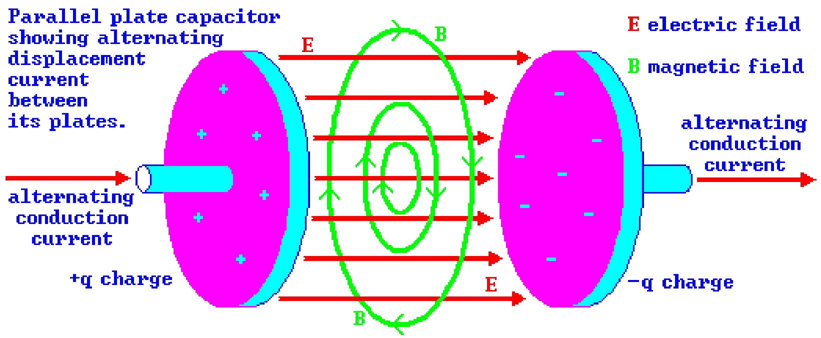
$$\vec{B} = \frac{\mu_0}{4\pi} \oint \vec{I} \cdot \frac{\vec{r} \cdot d\vec{r}}{r^3}$$

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{\vec{I}}{r}$$

For magnetic field calculations in electromagnets

Do we need an electric current ?

From displacement current, for example charging capacitor $\vec{j}_d$:



- **Defining a Displacement Current $\vec{I}_d$:**

  Not a current from <u>moving charges</u>

  But a current from <u>time varying electric fields</u>

Maxwell's fourth equation (part 2) ...

Displacement current I_d produces magnetic field, just like "actual currents" do ...

→ **Time varying electric field induce magnetic field (using the current density $\vec{j}_d$**

$$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Remember: strong *curl* = strong circulating field

Maxwell's complete fourth equation ...

Magnetic fields $\vec{B}$ can be generated by two ways:

$\nabla \times \vec{B} = \mu_0 \vec{j}$ **(electrical current)**

$\nabla \times \vec{B} = \mu_0 \vec{j_d} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$ **(changing electric field)**

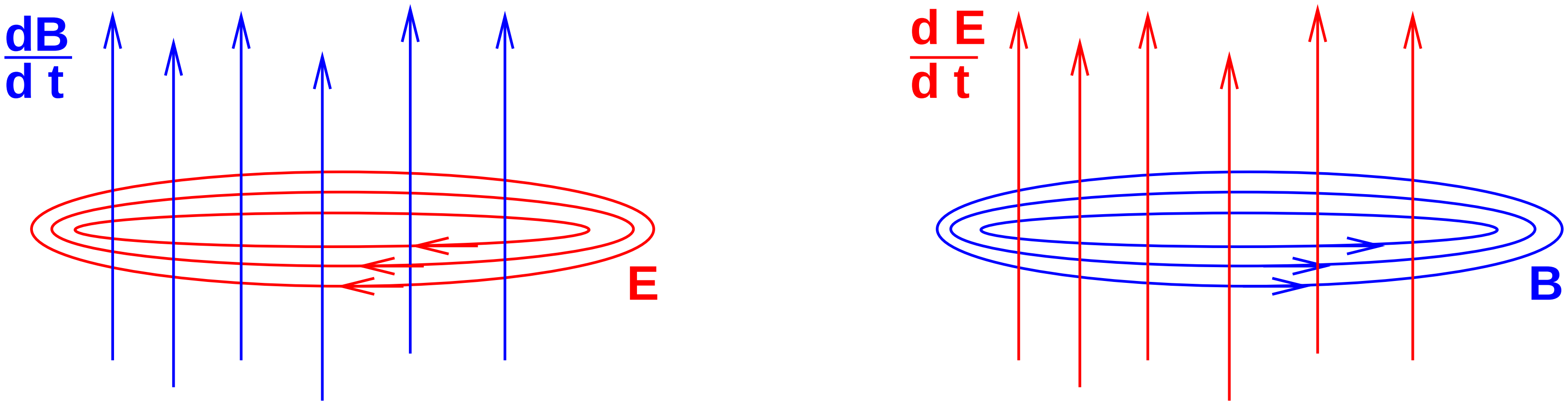
or putting them together:

$$\nabla \times \vec{B} = \mu_0 (\vec{j} + \vec{j_d}) = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

or in integral form (using Stoke's formula):

$$\underbrace{\oint_C \vec{B} \cdot d\vec{r} \ = \ \int_A \nabla \times \vec{B} \cdot d\vec{A}}_{Stoke's formula} = \int_A \left(\mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

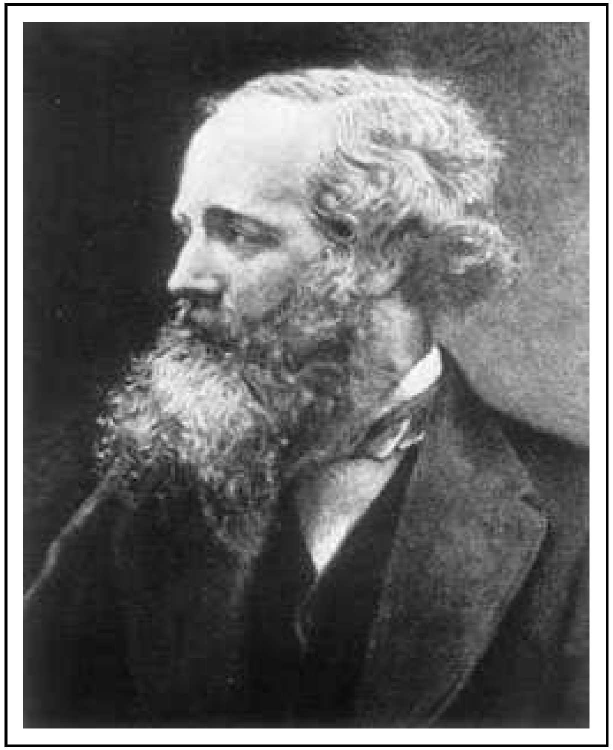
Summary: Static and Time Varying Fields



- **Time varying magnetic fields produce circulating electric field:** $\text{curl}(\vec{E}) = \nabla \times \vec{E} = -\frac{d\vec{B}}{\partial t}$
- **Time varying electric fields produce circulating magnetic field:** $\text{curl}(\vec{B}) = \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{d\vec{E}}{\partial t}$

because of the $\times$ **they are perpendicular:** $\vec{E} \perp \vec{B}$

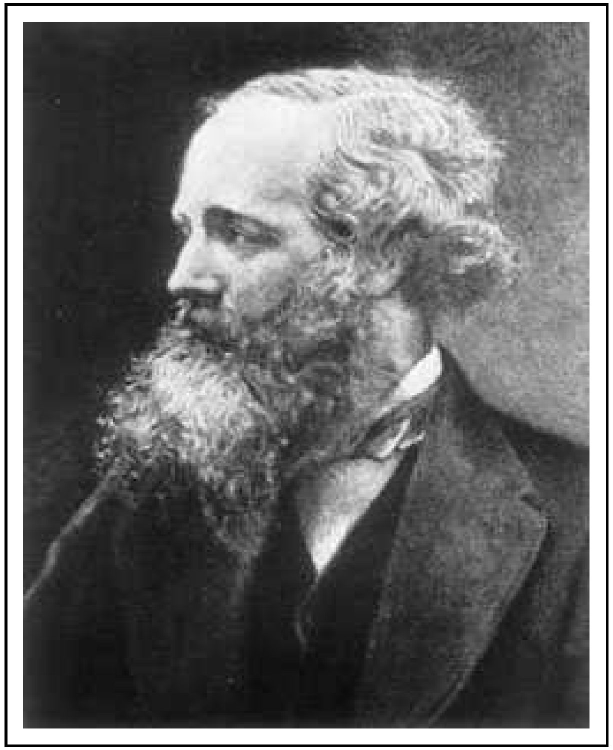
Put together: Maxwell's Equations



$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$
$$\int_A \vec{B} \cdot d\vec{A} = 0$$
$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt}\right) \cdot d\vec{A}$$
$$\oint_C \vec{B} \cdot d\vec{r} = \int_A \left(\mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}\right) \cdot d\vec{A}$$

Written in Integral form

Put together: Maxwell's Equations



$$\nabla \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

Written in Differential form

Maxwell in Physical terms

1. **Electric fields $\vec{E}$ are generated by charges and proportional to total charge**

2. **Magnetic monopoles do <u>not</u> exist**

3. **Changing magnetic flux generates circulating electric fields/currents**

4.1 **Changing electric flux generates circulating magnetic fields**

4.2 **Static electric current generates circulating magnetic fields**

Interlude and Warning !!

Maxwell's equation can be written in other forms.

Often used: cgs (Gaussian) units instead of SI units, example:

Starting from (SI):

$$\nabla \cdot \vec{E} \;=\; \frac{\rho}{\epsilon_0}$$

we would use:

$$\vec{E}_{cgs} \;=\; \tfrac{1}{c} \;\cdot \vec{E}_{SI} \qquad \text{and} \qquad \epsilon_0 \;=\; \tfrac{1}{4\pi \cdot c}$$

and arrive at (cgs):

$$\nabla \cdot \vec{E} \;=\; 4\pi \cdot \rho$$

Beware: there are more different units giving: $\nabla \cdot \vec{E} \;=\; \rho$

Electromagnetic fields in material

In vacuum:

$$\vec{D} = \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_0 \cdot \vec{H}$$

In a material:

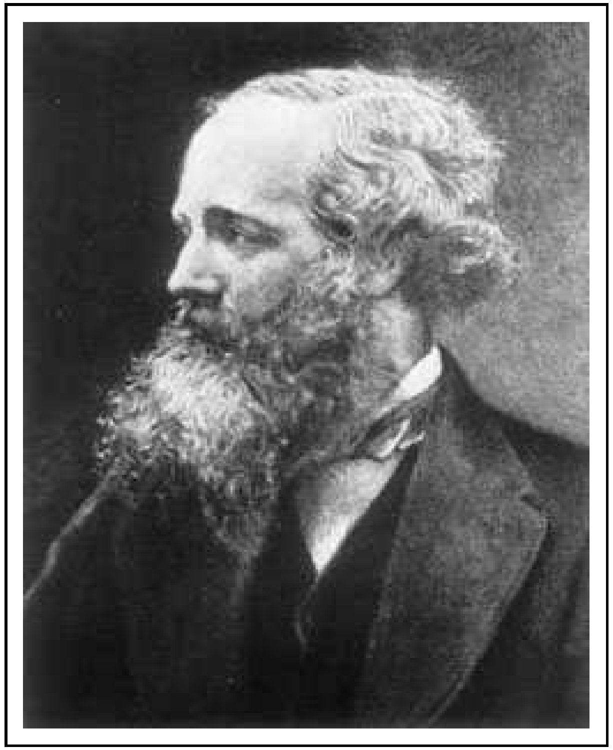
$$\vec{D} = \epsilon_r \cdot \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_r \cdot \mu_0 \cdot \vec{H}$$

ϵ_r **is <u>relative</u> permittivity** $\approx \quad [1 - 10^5]$

μ_r **is <u>relative</u> permeability** $\approx \quad [0(!) - 10^6]$

Origin: polarization and Magnetization

Once more: Maxwell's Equations



$$\nabla \vec{D} = \rho$$
$$\nabla \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$
$$\nabla \times \vec{H} = \vec{j} + \frac{d\vec{D}}{dt}$$

Re-factored in terms of the free current density $\vec{j}$ and free charge density ρ $(\mu_0 = 1, \epsilon_0 = 1)$:

Something on potentials:

Fields can be written as derivative of scalar and vector potentials $\Phi(x, y, z)$ and $\vec{A}(x, y, z)$:

	Electric fields	**Magnetic fields**
using:	$\vec{E} = -\nabla\Phi$	$\vec{B} = \nabla \times \vec{A}$
with:	$\nabla\vec{E} = \frac{\rho}{\epsilon_0}$	$\nabla \times \vec{B} = \mu_0\vec{j}$
➡	$\nabla^2\vec{\Phi} = -\frac{\rho}{\epsilon_0}$	$\nabla \times \nabla \times \vec{A} = \mu_0\vec{j}$
in short$^{*)}$:	$\Delta\vec{\Phi} = -\frac{\rho}{\epsilon_0}$	$\nabla^2 \times \vec{A} = \mu_0\vec{j}$

Potentials are linked to charge ρ and current $\vec{j}$

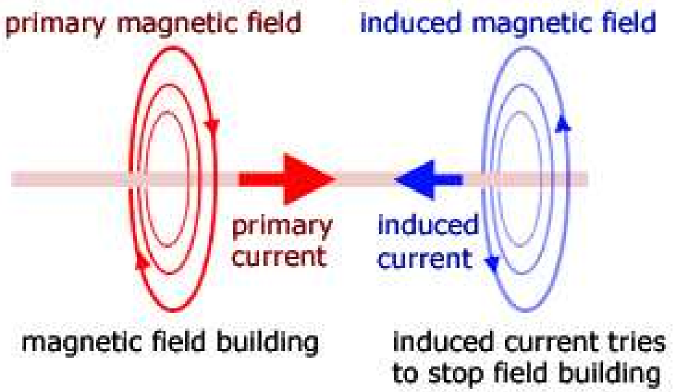
À bientôt ➡ Special Relativity ...

$^{*)}$**(with some vector analysis and definitions ...)**

Applications of Maxwell's Equations

- **Powering of magnets**
- **Lorentz force, motion in EM fields**
  - **Motion in electric fields**
  - **Motion in magnetic fields**
- **EM waves (in vacuum and in material)**
- **Boundary conditions**
- **EM waves in cavities and wave guides**

Powering and self-induction



- Induced magnetic flux $\vec{B}$ changes with changing current

➡ Induces a current and magnetic field $\vec{B}_i$ voltage in the conductor

➡ Induced current will oppose change of current (Lenz's law)

➡ We want to change a current to ramp a magnet ...

Powering and self-induction

- **Ramp rate defines required Voltage:**

$$U = -L\frac{\partial I}{\partial t}$$

Inductance L in Henry (H)

- **Example LHC:**
  - **Typical ramp rate: 10 A/s**
  - **With $L = $ 15.1 H per powering sector**

→ **Required Voltage is $\approx$ 150 V**

Lorentz force on charged particles

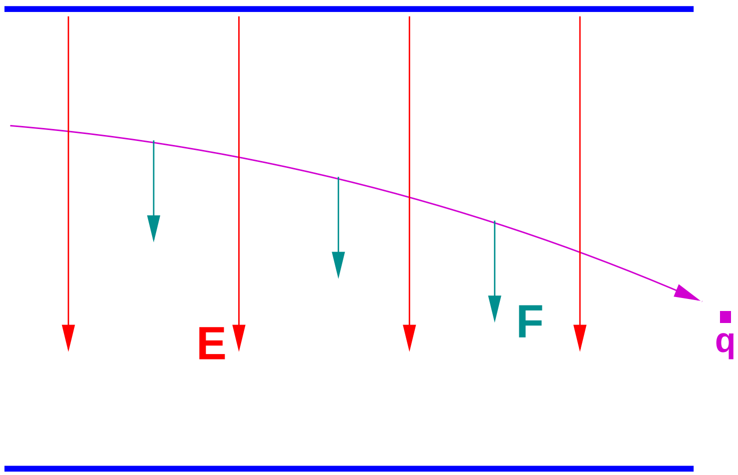
Moving ($\vec{v}$) charged (q) particles in electric ($\vec{E}$) and magnetic ($\vec{B}$) fields experience a force $\vec{f}$ like (Lorentz force):

$$\vec{f} \quad = \quad q \cdot (\vec{E} \; + \; \vec{v} \; \times \; \vec{B})$$

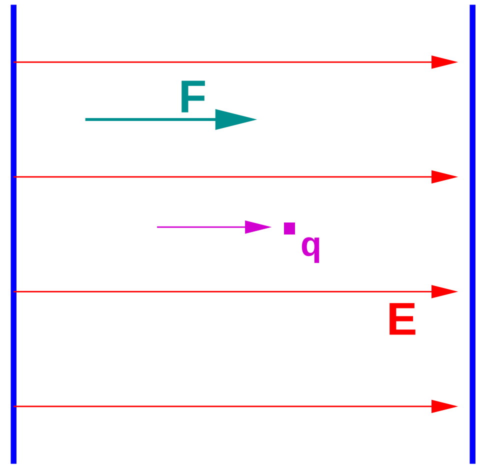
for the equation of motion we get (using Newton's law);

$$\frac{d}{dt}(m_0 \vec{v}) \quad = \quad \vec{f} \quad = \quad q \cdot (\vec{E} \; + \; \vec{v} \; \times \; \vec{B})$$

Motion in electric fields



$\vec{v} \perp \vec{E}$



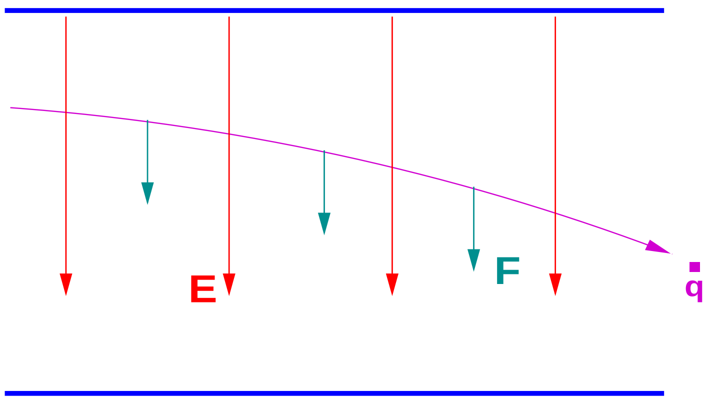
$\vec{v} \parallel \vec{E}$

Assume no magnetic field:

$$\frac{d}{dt}(m_0\vec{v}) = \vec{f} = q \cdot \vec{E}$$

Force always in direction of field $\vec{E}$, also for particles at rest.

Motion in electric fields



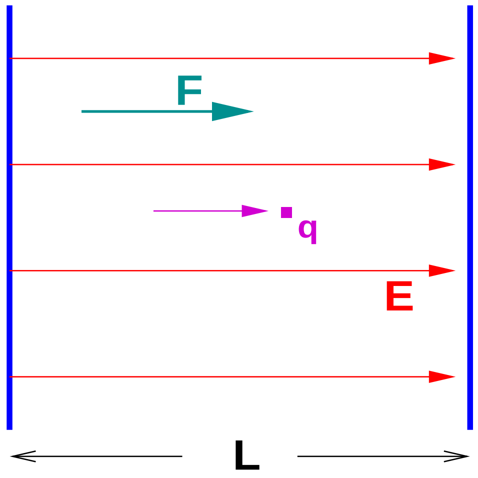
$$\frac{d}{dt}(m_0\vec{v}) = \vec{f} = q \cdot \vec{E}$$

The solution is:

$$\vec{v} = \frac{q \cdot \vec{E}}{m_0} \cdot t \quad \Rightarrow \quad \vec{x} = \frac{q \cdot \vec{E}}{2m_0} \cdot t^2 \qquad (\textbf{parabola})$$

Constant E-field deflects beams: TV, electrostatic separators (SPS,LEP)

Motion in electric fields



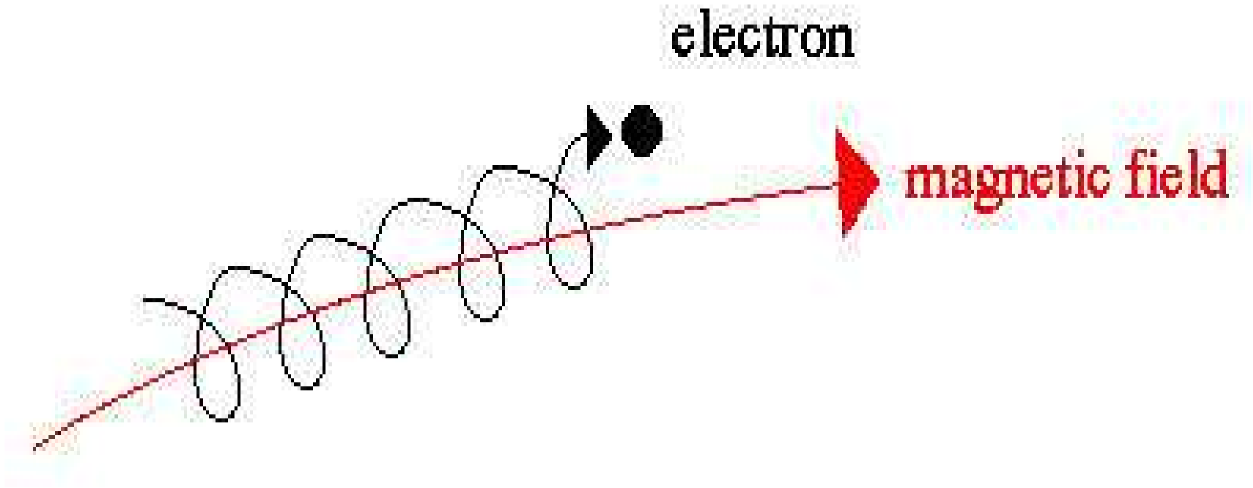
$$\frac{d}{dt}(m_0\vec{v}) = \vec{f} = q \cdot \vec{E}$$

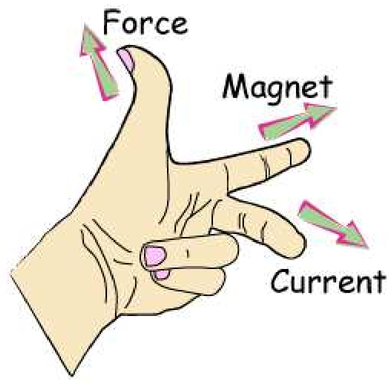
For constant field $\vec{E} = (E, 0, 0)$ in x-direction the kinetic energy gain is:

$$\Delta T = qE \cdot L$$

It is a line integral of the force along the path !
Constant E-field gives uniform acceleration over length L

Motion in magnetic fields





Assume first no electric field:

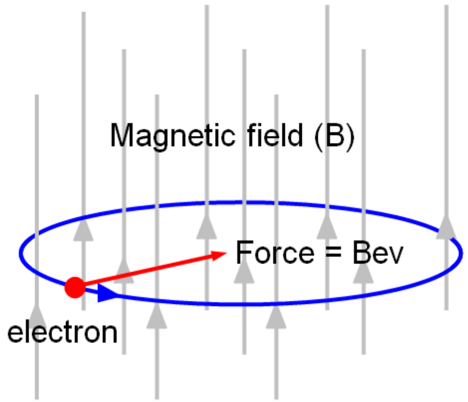
$$\frac{d}{dt}(m_0\vec{v}) \quad = \quad \vec{f} \quad = \quad q \cdot \vec{v} \;\times\; \vec{B}$$

Force is perpendicular to both, $\vec{v}$ **and** $\vec{B}$

No forces on particles at rest !

Particles will spiral around the magnetic field lines ...

Motion in magnetic fields



Assuming that $v_\perp$ is perpendicular to $\vec{B}$
We get a circular motion with radius ρ:

$$\rho = \frac{m_0 v_\perp}{q \cdot B}$$

defines the Magnetic Rigidity: $B \cdot \rho = \frac{m_0 v}{q} = \frac{p}{q}$

Magnetic fields deflect particles, but no acceleration (synchrotron, ..)

(but can accelerate in betatron !)

Motion in magnetic fields

Practical units:

$$B[T] \cdot \rho[m] \quad = \quad \frac{p[ev]}{c[m/s]}$$

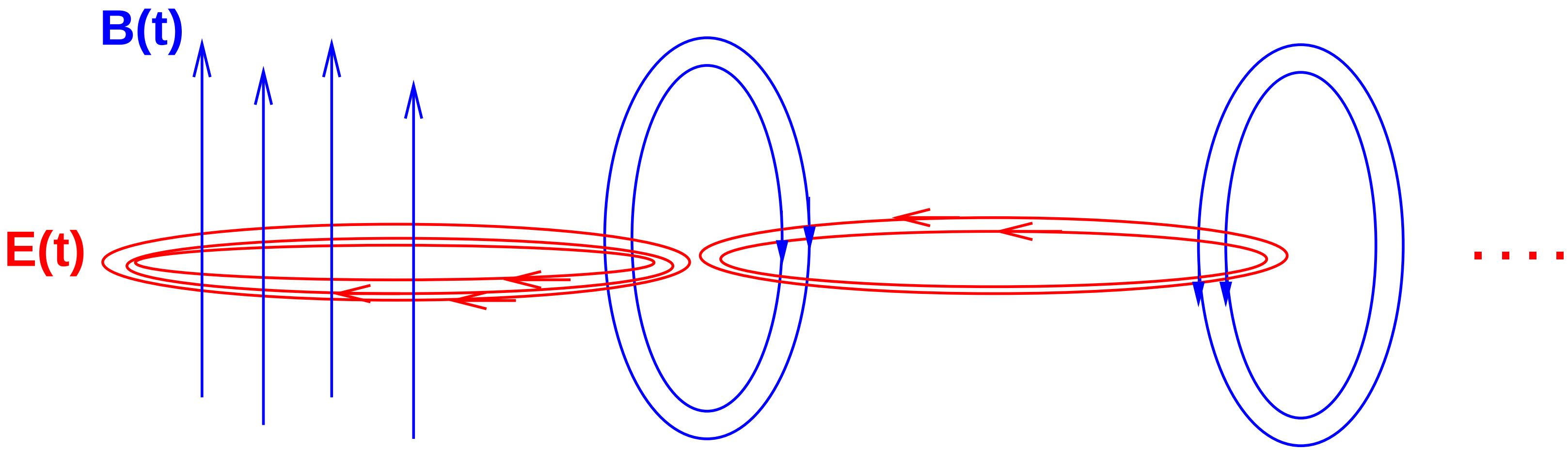
Example LHC:

B = 8.33 T, p = 7000 GeV/c ➡ ρ **= 2804 m**

Use of static fields (some examples, incomplete)

- **Magnetic fields**
  - **Bending magnets**
  - **Focusing magnets (quadrupoles)**
  - **Correction magnets (sextupoles, octupoles, orbit correctors, ..)**
- **Electric fields**
  - **Electrostatic separators (beam separation in particle-antiparticle colliders)**
  - **Very low energy machines**
- **What about non-static, time-varying fields ?**

Time Varying Fields (very schematic)



Time varying magnetic fields produce circulating electric fields

Time varying electric fields produce circulating magnetic fields

➡ **Can produce self-sustaining, propagating fields (i.e. waves)**

Electromagnetic waves in vacuum

Vacuum: only fields, no charges ($\rho = 0$), no current ($j = 0$) ...

From: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\Longrightarrow \quad \nabla \times (\nabla \times \vec{E}) = -\nabla \times (\frac{\partial \vec{B}}{\partial t})$$

$$\Longrightarrow \quad -(\nabla^2 \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\Longrightarrow \quad -(\nabla^2 \vec{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

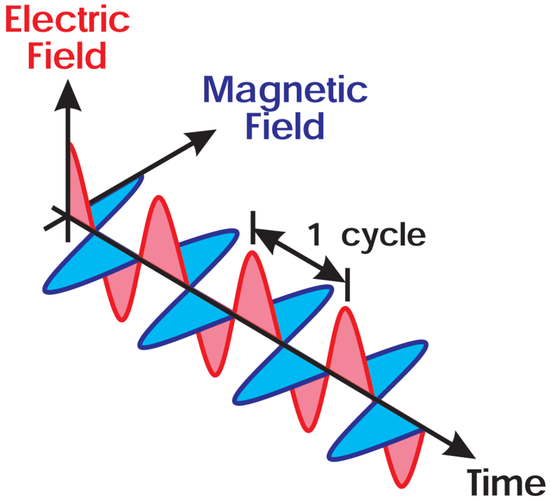
It happens to be: $\mu_0 \cdot \epsilon_0 = \frac{1}{c^2}$

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{E}}{\partial t^2}$$

Similar expression for the magnetic field:

$$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{B}}{\partial t^2}$$

Electromagnetic waves

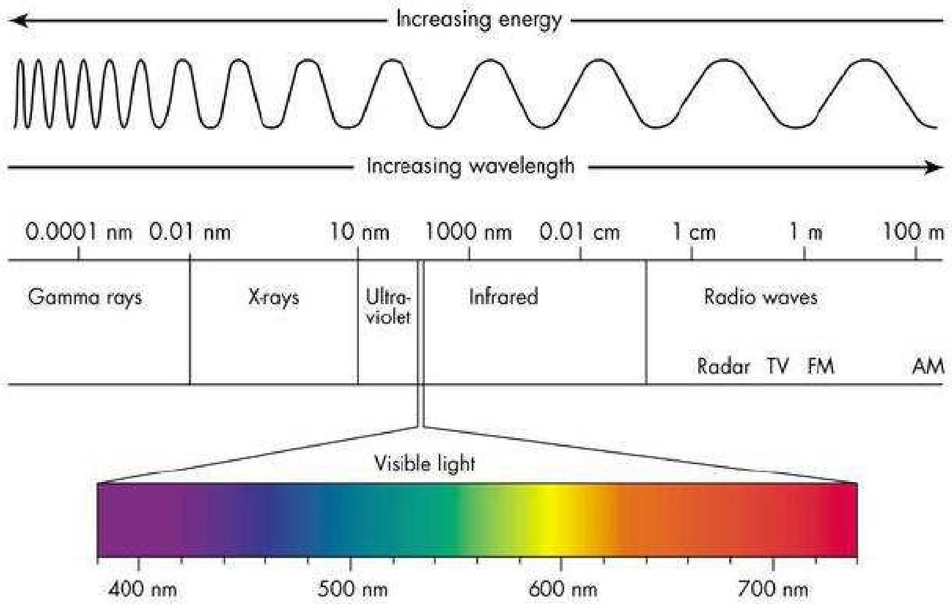


$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$
$$\vec{B} = \vec{B}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$
$$|\vec{k}| = \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad (\textbf{propagation vector})$$
$$\lambda = (\textbf{wave length, 1 cycle})$$
$$\omega = (\textbf{frequency} \cdot 2\pi)$$
$$c = \frac{\omega}{k} = (\textbf{wavevelocity})$$

Magnetic and electric fields are transverse to direction of propagation: $\vec{E} \perp \vec{B} \perp \vec{k}$

Velocity of wave in vacuum: 299792458.000 m/s

Spectrum of Electromagnetic waves



Example: yellow light ➡ $\approx\ 5 \cdot 10^{14}$ **Hz (i.e.** $\approx$ **2 eV !)**

gamma rays ➡ $\leq\ 3 \cdot 10^{21}$ **Hz (i.e.** $\leq$ **12 MeV !)**

LEP (SR) ➡ $\leq\ 2 \cdot 10^{20}$ **Hz (i.e.** $\approx$ **0.8 MeV !)**

Waves interacting with material

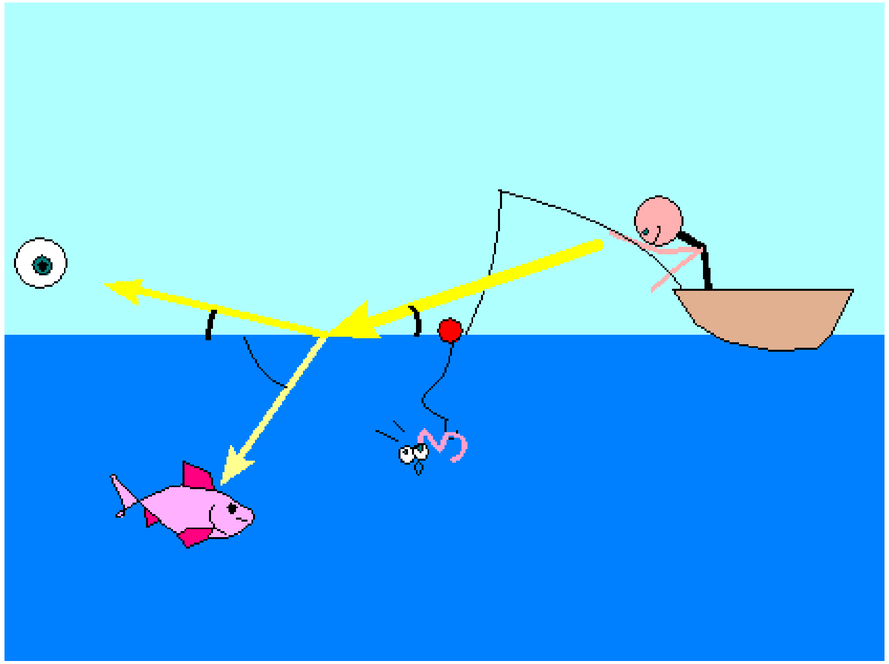
Need to look at the behaviour of electromagnetic fields at boundaries between different materials (air-glass, air-water, vacuum-metal, ...).

Have to consider two particular cases:

- **Ideal conductor (i.e. no resistance), apply to:**
  - **RF cavities**
  - **Wave guides**
- **Conductor with finite resistance, apply to:**
  - **Penetration and attenuation of fields in material (skin depth)**
  - **Impedance calculations**

Can be derived from Maxwell's equations, here only the results !

Observation: between air and water



- Some of the light is reflected
- Some of the light is transmitted and refracted

➡ Reason are <u>boundary conditions</u> for fields between two materials

Extreme case: ideal conductor

For an ideal conductor (i.e. no resistance) the tangential electric field must vanish, otherwise a surface current becomes infinite. Similar conditions for magnetic fields. We must have:

$$\vec{E}_{\|} = 0, \qquad \vec{B}_n = 0$$

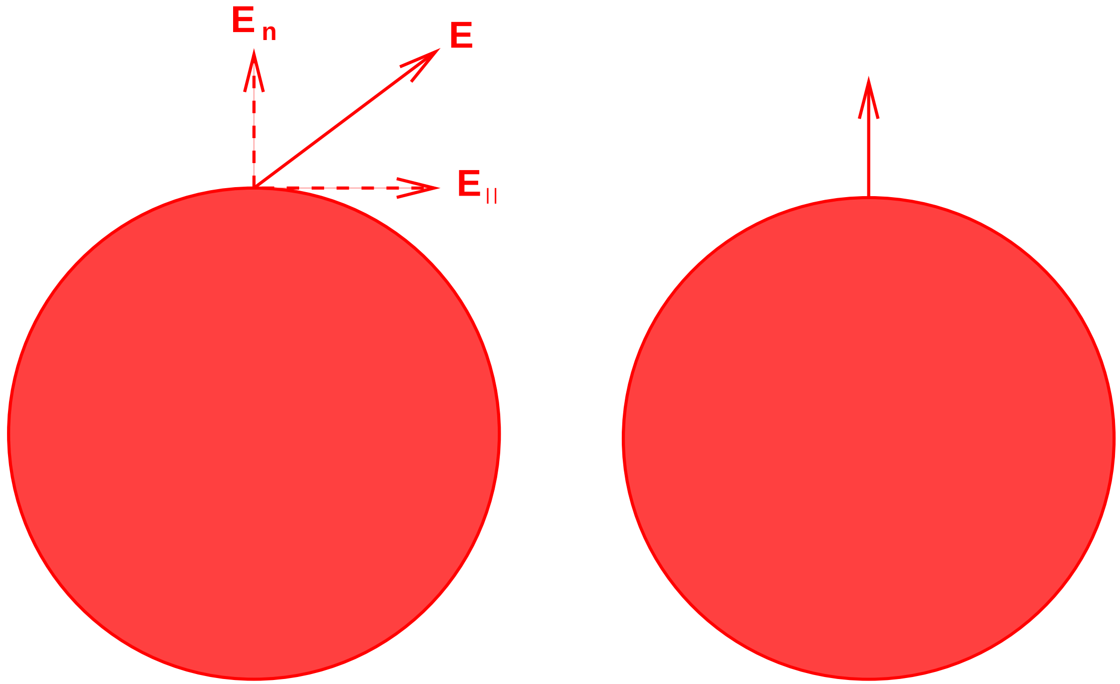
This implies:

- **All energy of an electromagnetic wave is reflected from the surface.**
- **Fields at any point in the conductor are zero.**
- **Only some fieldpatterns are allowed in waveguides and RF cavities**

A very nice lecture in R.P.Feynman, Vol. II

Boundary conditions: air and perfect conductor

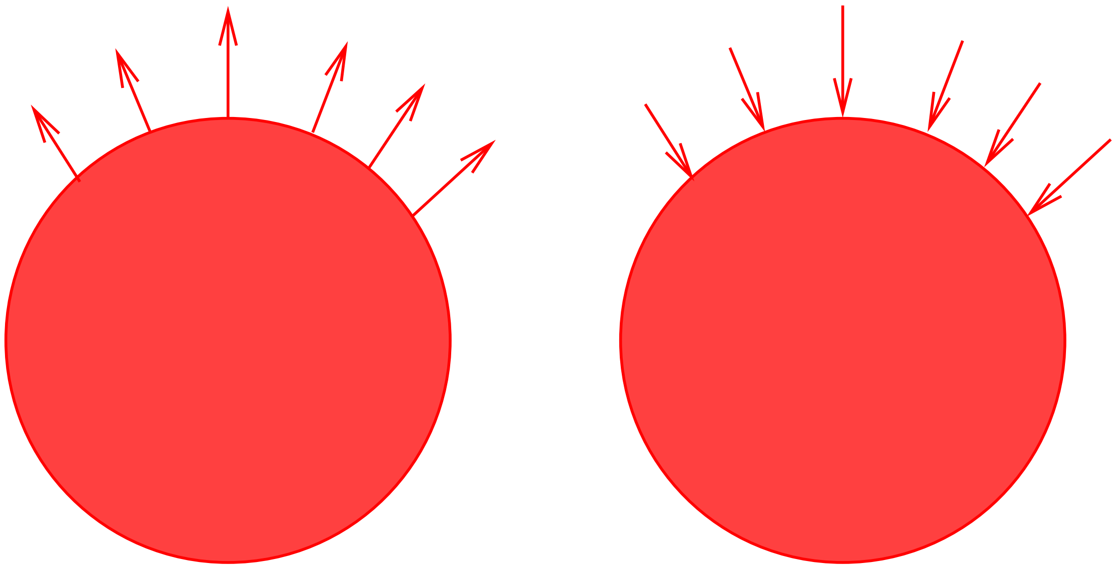
A simple case as demonstration ($\vec{E}$-fields on an ideally conducting sphere):



- **Field parallel to surface $E_{||}$ cannot exist (it would move charges and we get a surface current)**
- **Only field normal to surface E_n is possible**

Boundary conditions for fields

All electric field lines must be normal (perpendicular) to surface of a perfect conductor.



- **All conditions for $\vec{E}, \vec{D}, \vec{H}, \vec{B}$ can be derived from Maxwell's equations (see bibliography, e.g. R.P.Feynman or J.D.Jackson)**

General boundary conditions for fields*

Electromagnetic fields at boundaries between different materials with different permittivity and permeability ($\epsilon^a, \epsilon^b, \mu^a, \mu^b$).

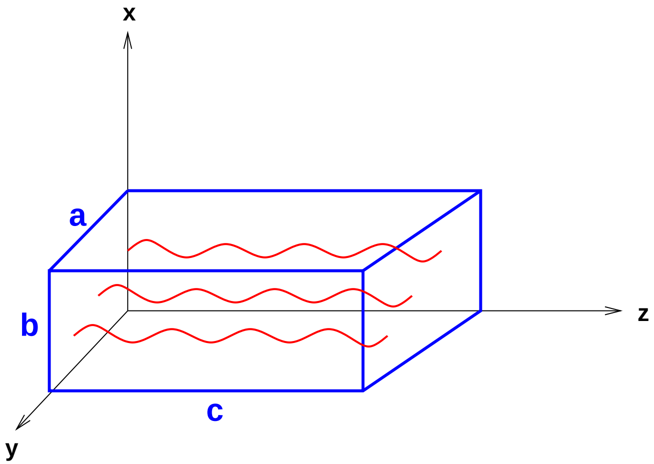
The requirements for the components are (summary of the results, not derived here !):

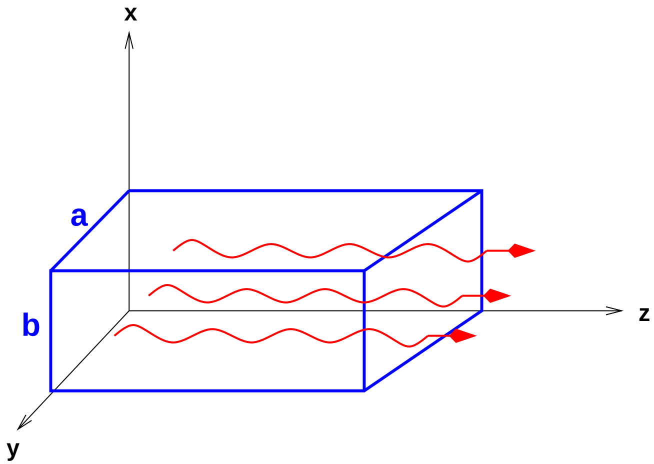
- $(E^a_{\parallel} \;=\; E^b_{\parallel}), (E^a_n \;\neq\; E^b_n)$
- $(D^a_{\parallel} \;\neq\; D^b_{\parallel}), (D^a_n \;=\; D^b_n)$
- $(H^a_{\parallel} \;=\; H^b_{\parallel}), (H^a_n \;\neq\; H^b_n)$
- $(B^a_{\parallel} \;\neq\; B^b_{\parallel}), (B^a_n \;=\; B^b_n)$

Conditions are used to compute reflection, refraction and refraction index n.

Examples: cavities and wave guides

Rectangular, conducting cavities and wave guides (schematic) with dimensions $a \times b \times c$ and $a \times b$:





- **RF cavity, fields can persist and be stored (reflection !)**
- **Plane waves can propagate along wave guides, here in z-direction**

Fields in RF cavities

Assume a rectangular RF cavity ($a,\ b,\ c$), ideal conductor.

Without derivations, the components of the fields are:

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

$$E_z = E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_x = \frac{i}{\omega}(E_{y0}k_z - E_{z0}k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_y = \frac{i}{\omega}(E_{z0}k_x - E_{x0}k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_z = \frac{i}{\omega}(E_{x0}k_y - E_{y0}k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

Consequences for RF cavities

Field must be zero at conductor boundary, only possible under the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

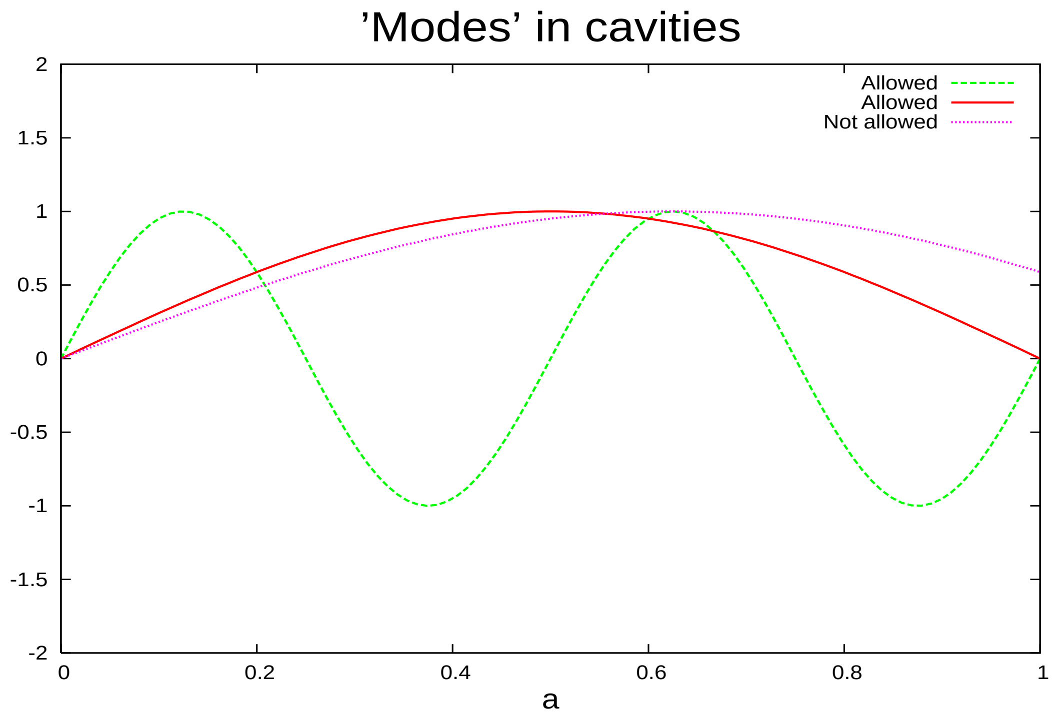
and for k_x, k_y, k_z we can write:

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \frac{m_z \pi}{c},$$

The integer numbers m_x, m_y, m_z are called mode numbers, important for shape of cavity !

It means that a half wave length $\lambda/2$ must always fit exactly the size of the cavity.

Allowed modes



- **Only modes which 'fit' into the cavity are allowed**
- $\frac{\lambda}{2} = \frac{a}{4}$, $\frac{\lambda}{2} = \frac{a}{1}$, $\frac{\lambda}{2} = \frac{a}{0.8}$
- **No electric field at boundaries, wave most have "nodes" at the boundaries**

Fields in wave guides

Similar considerations lead to (propagating) solutions in (rectangular) wave guides:

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

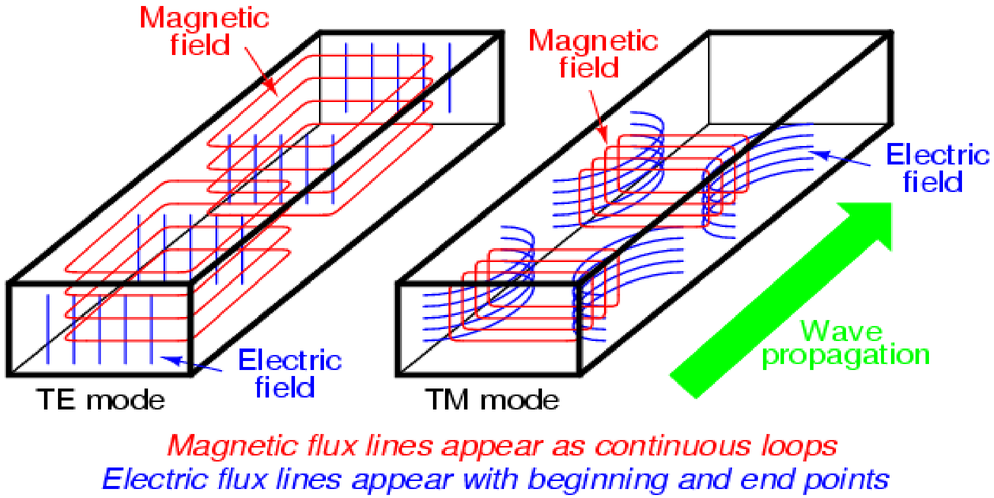
$$E_z = i \cdot E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$B_x = \frac{1}{\omega}(E_{y0} k_z - E_{z0} k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$B_y = \frac{1}{\omega}(E_{z0} k_x - E_{x0} k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$B_z = \frac{1}{i \cdot \omega}(E_{x0} k_y - E_{y0} k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

The fields in wave guides



- **Electric and magnetic fields through a wave guide**
- **Shapes are consequences of boundary conditions !**
- **Can be Transverse Electric (TE, no E-field in z-direction) or Transverse Magnetic (TM, no B-field in z-direction)**

Consequences for wave guides

Similar considerations as for cavities, no field at boundary.

We must satisfy again the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

This leads to modes like:

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b},$$

The numbers m_x, m_y are called mode numbers for planar waves in wave guides !

Consequences for wave guides

Re-writing the condition as:

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2$$

Propagation without losses requires k_z to be real, i.e.:

$$\frac{\omega^2}{c^2} > k_x^2 + k_y^2 = (\frac{m_x \pi}{a})^2 + (\frac{m_y \pi}{b})^2$$

which defines a cut-off frequency ω_c. For lowest order mode:

$$\omega_c \quad = \quad \frac{\pi \cdot c}{a}$$

- **Above cut-off frequency: propagation without loss**
- **Below cut-off frequency: attenuated wave (means it does not "really fit" and k is complex).**

Other case: finite conductivity

Starting from Maxwell equation:

$$\nabla \times \vec{H} = \vec{j} \ + \ \frac{d\vec{D}}{dt} \ = \ \underbrace{\overbrace{\sigma \cdot \vec{E}}^{\vec{j}}}_{Ohm's\ law} + \epsilon \frac{d\vec{E}}{dt}$$

Wave equations:

$$\vec{E} \ = \ \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}, \qquad \vec{H} \ = \ \vec{H}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$

We have:

$$\frac{d\vec{E}}{dt} = i\omega \cdot \vec{E}, \qquad \frac{d\vec{H}}{dt} = i\omega \cdot \vec{H}, \qquad \nabla \times \vec{E} = -i\vec{k} \times \vec{E}, \qquad \nabla \times \vec{H} = -i\vec{k} \times \vec{H}$$

Put together:

$$\vec{k} \times \vec{H} \ = \ i\sigma \cdot \vec{E} \ - \ \omega\epsilon \cdot \vec{E} \ = \ (i\sigma - \omega\epsilon) \cdot \vec{E}$$

Finite conductivity - Skin Depth

Starting from:

$$\vec{k} \times \vec{H} \;=\; i\sigma \cdot \vec{E} \;-\; \omega\epsilon \cdot \vec{E} \;=\; (i\sigma - \omega\epsilon) \cdot \vec{E}$$

With $\vec{B} \;=\; \mu \; \vec{H}$**:**

$$\nabla \times \vec{E} \;=\; -i\vec{k} \times \vec{E} \;=\; -\frac{\partial \vec{B}}{\partial t} \;=\; -\mu \frac{\partial \vec{H}}{\partial t} \;=\; -i\omega\mu\vec{H}$$

Multiplication with $\vec{k}$**:**

$$\vec{k} \times (\vec{k} \times \vec{E}) \;=\; \omega\mu(\vec{k} \times \vec{H}) \;=\; \omega\mu(i\sigma - \omega\epsilon) \cdot \vec{E}$$

After some calculus and $\vec{E} \;\perp\; \vec{H} \;\perp\; \vec{k}$**:**

$$\boxed{k^2 \;=\; \omega\mu(-i\sigma \;+\; \omega\epsilon)}$$

Finite conductivity - Skin Depth

With:

$$k^2 \; = \; \omega\mu(-i\sigma \; + \; \omega\epsilon)$$

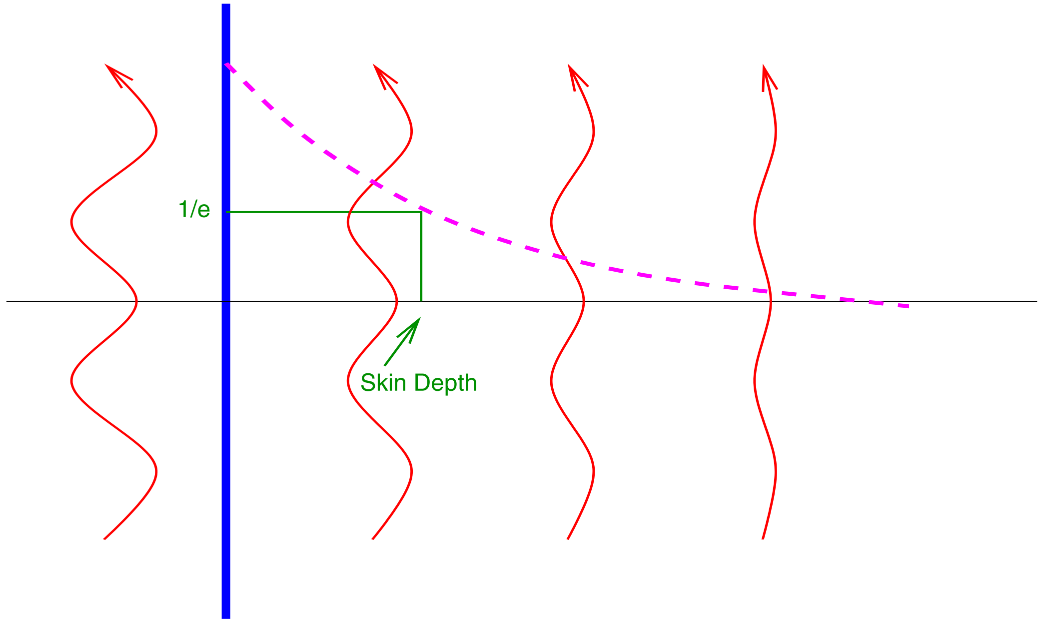
For a good conductor $\sigma \gg \omega\epsilon$**:**

$$k^2 \; \approx \; - i\omega\mu\sigma \quad \rightarrow \quad k \; \approx \; \sqrt{\frac{\omega\mu\sigma}{2}}(1-i) \; = \; \frac{1}{\delta}(1-i)$$

$$\delta \; = \; \sqrt{\frac{2}{\omega\mu\sigma}}$$

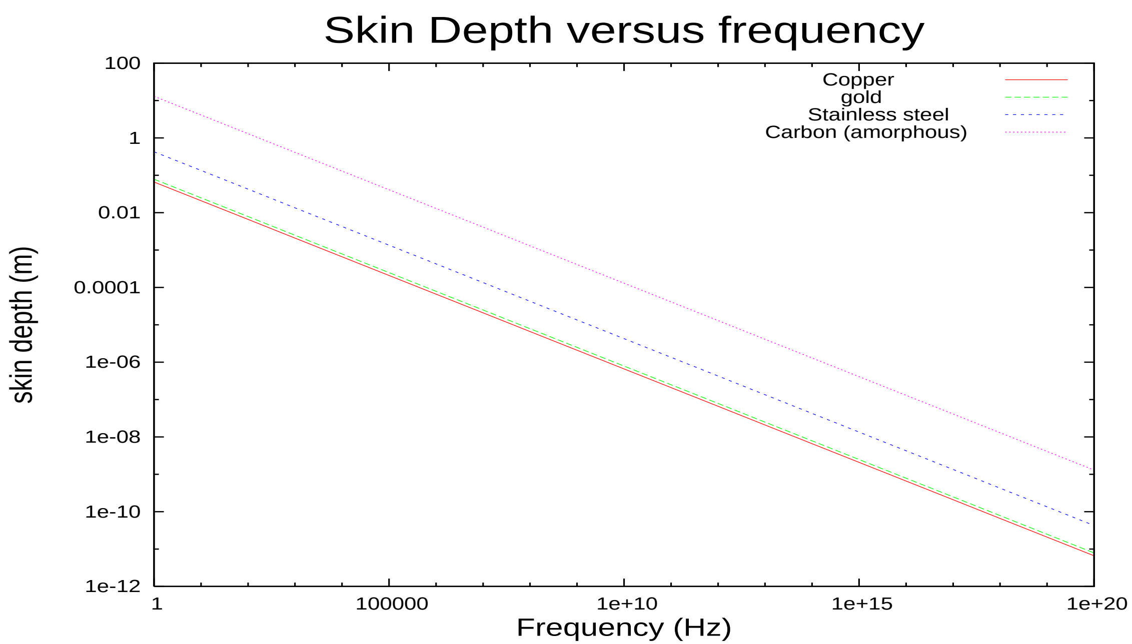
is the <u>Skin Depth</u>

Attenuated waves - skin depth



- **Waves in conducting material are attenuated**
- **Defines Skin depth (attenuation to 1/e)**
- **Wave form:** $e^{i(\omega t - kx)} = e^{i(\omega t - (1-i)x/\delta)} = e^{\frac{-x}{\delta}} \cdot e^{i(\omega t - \frac{x}{\delta})}$

Skin depth - examples



- **Copper: 1 GHz $\delta \approx$ 2.1 μm, 50 Hz $\delta \approx$ 10 mm**
- **Gold 50 Hz $\delta \approx$ 11 mm**
- **Q1: why do we use many cables for power lines ??**
- **Q2: why are SC cables very thin ?**

Skin Depth - beam dynamics

For metal walls thicker than δ we get Resistive Wall Impedances, see later on collective effects.

$$Z(\omega) \propto \delta \propto \omega^{-1/2}$$

- **Largest impedance at low frequencies**
- **Cause longitudinal and transverse instabilities (see later)**

Done ...

- **Review of basics and Maxwell's equations**
- **Lorentz force**
- **Motion of particles in electromagnetic fields**
- **Electromagnetic waves in vacuum**
- **Electromagnetic waves in conducting media**
  - **Waves in RF cavities**
  - **Waves in wave guides**
  - **Penetration of waves in material**

To make things easier:
say "good bye" to Maxwell and "hello" to Einstein ...

- BACKUP SLIDES -

Scalar products

Define a scalar product for (usual) vectors like: $\vec{a} \cdot \vec{b}$,

$$\vec{a} = (x_a, y_a, z_a) \qquad \vec{b} = (x_b, y_b, z_b)$$

$$\vec{a} \cdot \vec{b} = (x_a, y_a, z_a) \cdot (x_b, y_b, z_b) = (x_a \cdot x_b + y_a \cdot y_b + z_a \cdot z_b)$$

This product of two vectors is a <u>scalar</u> (number) not a vector.

(on that account: Scalar Product)

Example:

$$(-2, 2, 1) \cdot (2, 4, 3) = -2 \cdot 2 + 2 \cdot 4 + 1 \cdot 3 = 7$$

Vector products (sometimes cross product)

Define a vector product for (usual) vectors like: $\vec{a} \times \vec{b}$**,**

$$\vec{a} = (x_a, y_a, z_a) \qquad \vec{b} = (x_b, y_b, z_b)$$

$$\begin{aligned} \vec{a} \times \vec{b} &= (x_a, y_a, z_a) \times (x_b, y_b, z_b) \\ &= (\underbrace{y_a \cdot z_b - z_a \cdot y_b}_{x_{ab}}, \underbrace{z_a \cdot x_b - x_a \cdot z_b}_{y_{ab}}, \underbrace{x_a \cdot y_b - y_a \cdot x_b}_{z_{ab}}) \end{aligned}$$

This product of two vectors is a <u>vector</u>, not a scalar (number), (on that account: Vector Product)

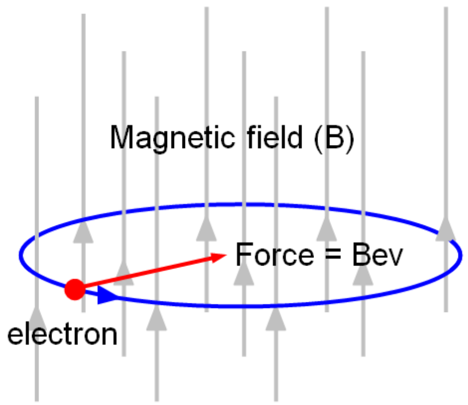
Example 1:

$(-2, 2, 1) \times (2, 4, 3) = (2, 8, -12)$

Example 2 (two components only in the $x - y$ **plane):**

$(-2, 2, 0) \times (2, 4, 0) = (0, 0, -12)$ **(see R. Steerenberg)**

Is that the full truth ?



If we have a circulating E-field along the circle of radius R ?

➡ **should get acceleration !**

Remember Maxwell's third equation:

$$\oint_C \vec{E} \cdot d\vec{r} \quad = \quad -\frac{d}{dt}\int_A \vec{B} \cdot d\vec{A}$$

$$\Rightarrow 2\pi R E_\theta \quad = \quad -\frac{d\Phi}{dt}$$

Motion in magnetic fields

- **This is the principle of a Betatron**
  - **Time varying magnetic field creates circular electric field !**
  - **Time varying magnetic field deflects the charge !**

For a constant radius we need:

$$-\frac{m \cdot v^2}{R} = e \cdot v \cdot B \quad \Rightarrow \quad B = -\frac{p}{e \cdot R}$$

$$\frac{\partial}{\partial t} B(r,t) = -\frac{1}{e \cdot R}\frac{dp}{dt}$$

$$\Rightarrow \quad B(r,t) = \frac{1}{2}\frac{1}{\pi R^2}\int\int B dS$$

B-field on orbit must be half the average over the circle

➡ **Betatron condition**

Some popular confusion ..

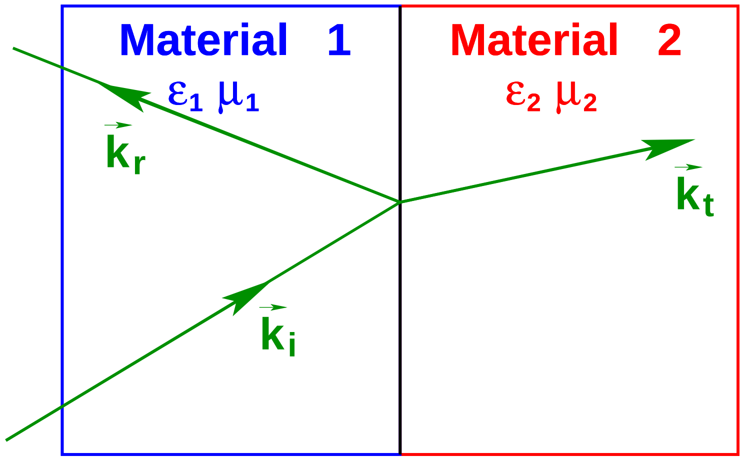
V.F.A.Q: why this strange mixture of $\vec{E}, \vec{D}, \vec{B}, \vec{H}$??

Materials respond to an applied electric E field and an applied magnetic B field by producing their own internal charge and current distributions, contributing to E and B. Therefore H and D fields are used to re-factor Maxwell's equations in terms of the free current density $\vec{j}$ and free charge density ρ:

$$
\begin{aligned}
\vec{H} &= \frac{\vec{B}}{\mu_0} - \vec{M} \\
\vec{D} &= \epsilon_0 \vec{E} + \vec{P}
\end{aligned}
$$

$\vec{M}$ **and** $\vec{P}$ **are** *Magnetization* **and** *Polarisation* **in material**

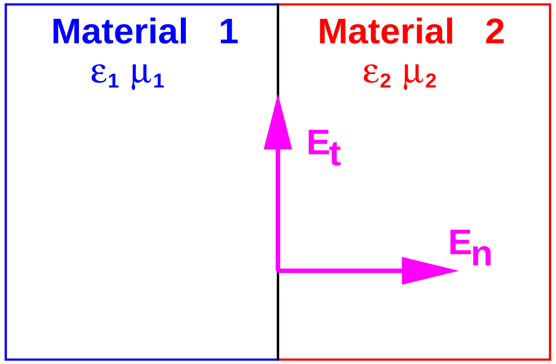
Boundary conditions for fields

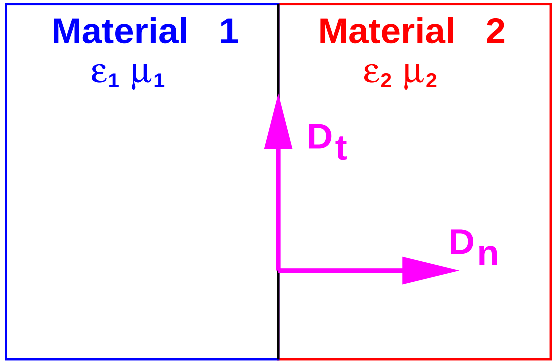


What happens when an incident wave $(\vec{K}_i)$ encounters a boundary between two different media ?

- **Part of the wave will be reflected $(\vec{K}_r)$, part is transmitted $(\vec{K}_t)$**
- **What happens to the electric and magnetic fields ?**

Boundary conditions for fields

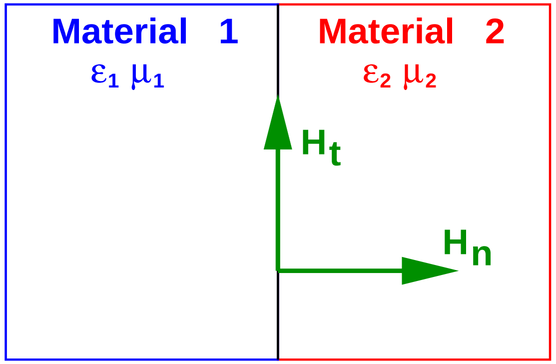


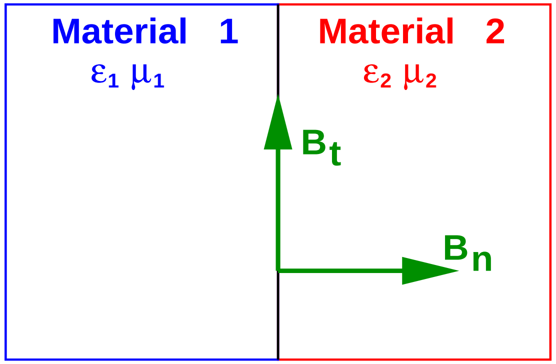


Assuming <u>no</u> surface charges:

- **<u>tangential $\vec{E}$-field</u> constant across boundary** $(E_{1t} = E_{2t})$
- **<u>normal $\vec{D}$-field</u> constant across boundary** $(D_{1n} = D_{2n})$

Boundary conditions for fields





Assuming no surface currents:

- tangential $\vec{H}$-field constant across boundary ($H_{1t} = H_{2t}$)
- normal $\vec{B}$-field constant across boundary ($B_{1n} = B_{2n}$)