Coherence in Beams

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Contents

This lecture will address some **basic issues**

-- details and solutions will be given in later lectures

--- or (even better) in your own studies

- Why coherence ?
- Basics on wave propagation
- Interference
- Transverse coherence
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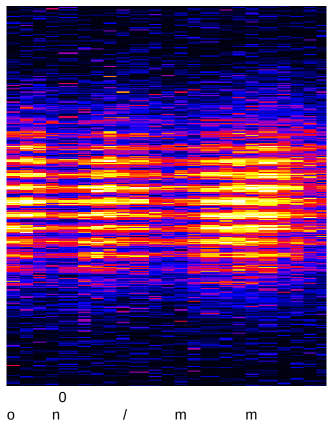
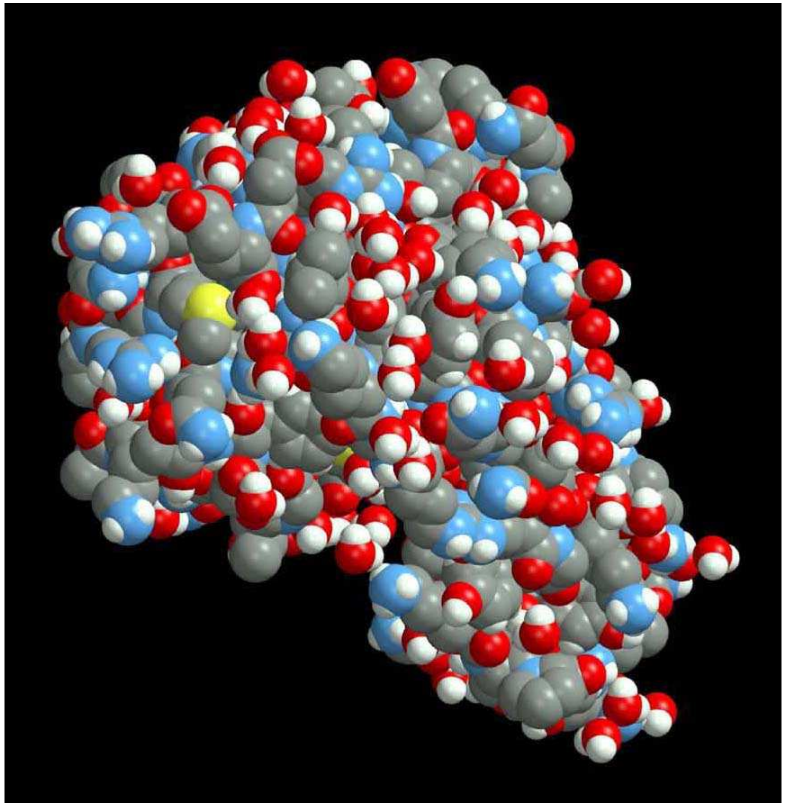
- Correlation functions
- Coherent matter waves



Why coherence?

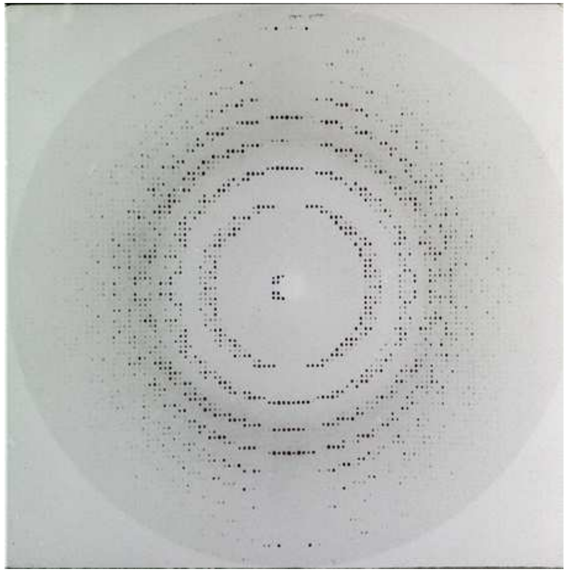
Electro-magnetic radiation comes in waves: amplitudes and phases
-- and we have to cope with it !



LYSOZYME , MW=19,806

Structure of biological macromolecule

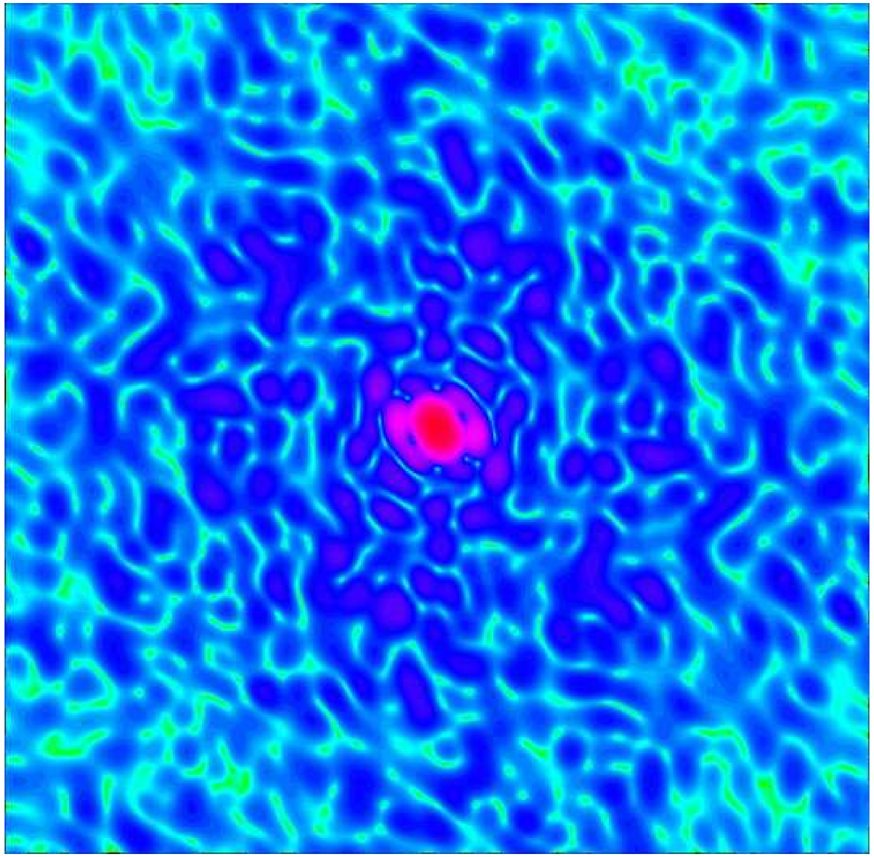
reconstructed from diffraction pattern of protein <u>crystal</u>:

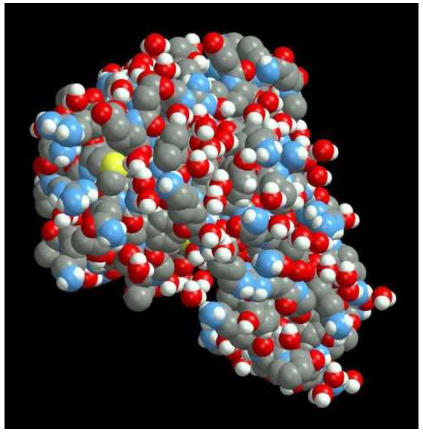


Images courtesy Janos Hajdu

Coherence of a single photon pulse from an FEL

courtesy Janos Hajdu





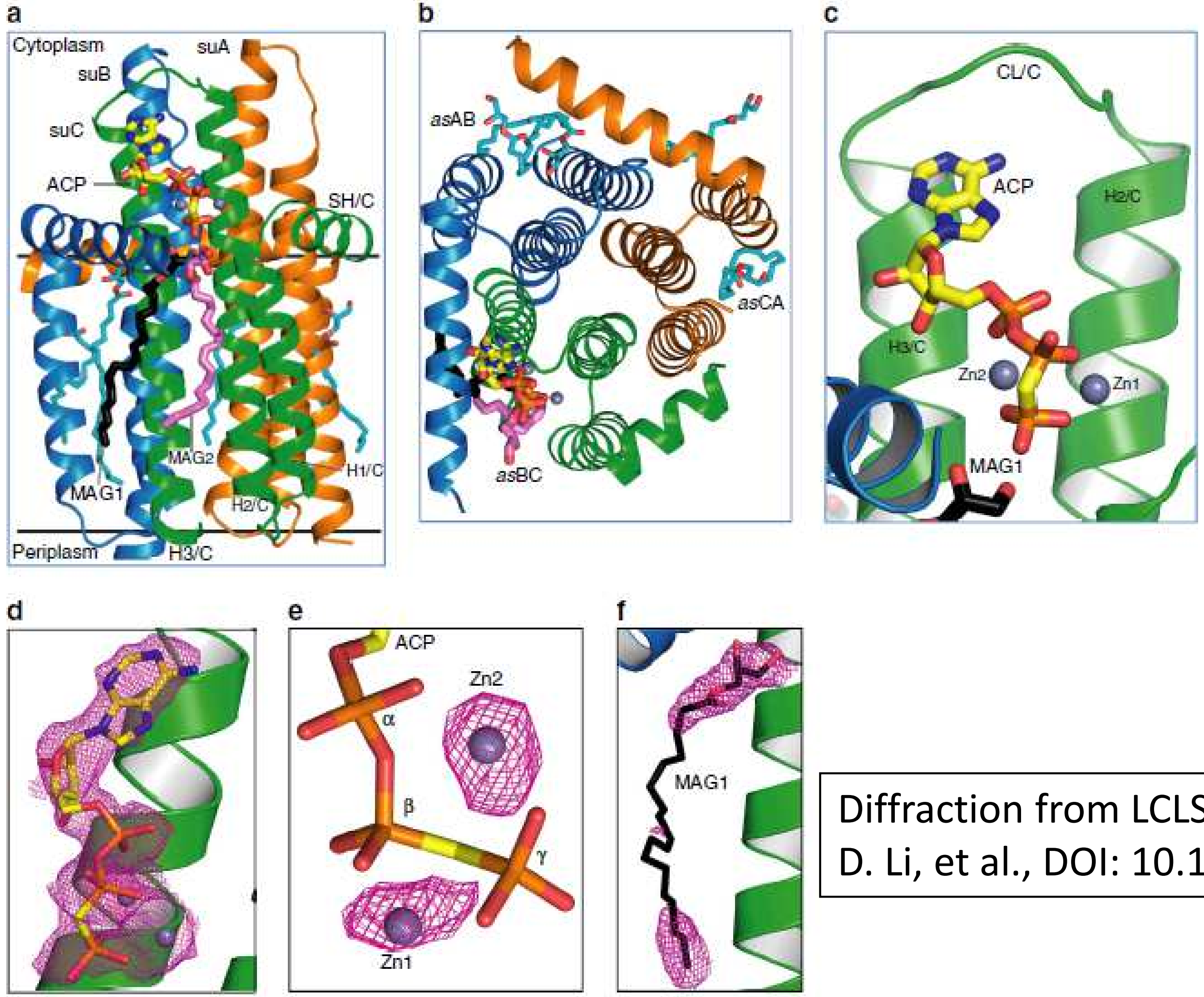
SINGLE MACROMOLECULE

simulated image

Needs very high radiation power @ $\lambda \approx 1$Å

CAS on FELs and ERLs, Hamburg 2016 Jörg Rossbach, Univ HH 4

Why Coherence ?

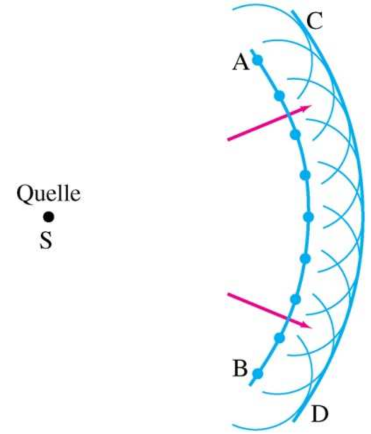


Diffraction from LCLS X-ray FEL/Stanford:
D. Li, et al., DOI: 10.1038/ncomms10140 (2016)

Basics on wave propagation

1. At some distance from the source, a wave front propagates independently → **Huygens Principle:**
   Each point on the wave front can be considered the origin of a spherical wave, the new wave front being the envelope of these wavelets
   → applicable for e.m., water, acoustic, matter waves

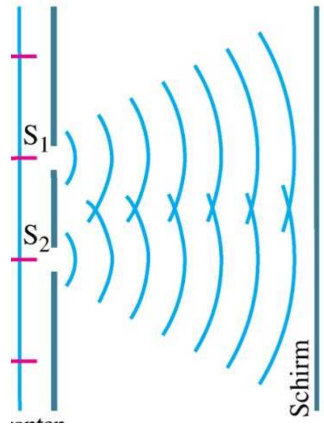
   → Basics for diffraction



2. The e.m. wave equation is **linear**

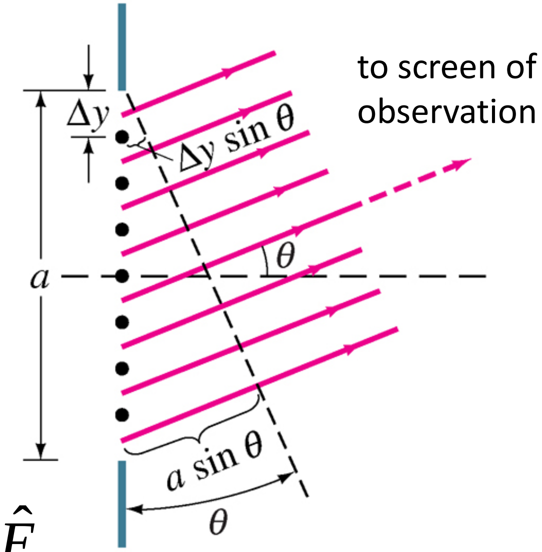
→ If two waves $\vec{E}_1(\vec{r},t)$, $\vec{E}_2(\vec{r},t)$ are solutions, then $\vec{E}_{sum}(\vec{r},t) = \vec{E}_1(\vec{r},t) + \vec{E}_2(\vec{r},t)$ will be a solution as well !

→ Basics for interference



Diffraction from a slit

Consider a plane wave (complex amplitude $\hat{E}_0$) arriving from the left at slit of width a



We use Huygens principle and consider the radiation field consisting of wavelets from tiny areas of size Δy . We observe under an angle θ from a "very far distance".

We call the complex electric wave vector of the first wavelet $\hat{E}_1$.

The next wavelet is: $\hat{E}_1 \cdot e^{i\Delta\delta}$ with $\Delta\delta = 2\pi \dfrac{\Delta y \sin\theta}{\lambda}$

And the last wavelet: $\hat{E}_1 \cdot e^{i\delta_N}$ with $\delta_N = 2\pi \dfrac{a \cdot \sin\theta}{\lambda}$

The intensity at the screen is $I \propto \left|\hat{E}_{sum}\right|^2 = \left|\hat{E}_1 + \ldots + \hat{E}_1 e^{in\Delta\delta} + \ldots + \hat{E}_1 e^{i\delta_N}\right|^2$

The amplitude of each wavelet is $\hat{E}_1 = \dfrac{\Delta y}{a}\hat{E}_0 = \hat{E}_0 \dfrac{\lambda}{2\pi a \sin\theta}\Delta\delta$ $\rightarrow$

Diffraction from a slit

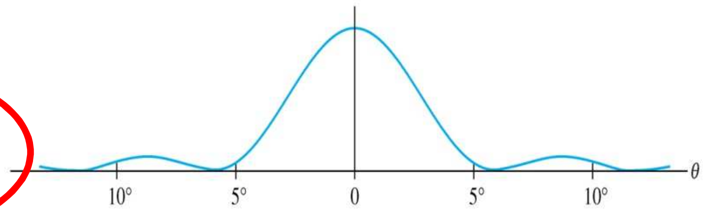
Intensity at the screen:

$$I \propto \left|\hat{E}_{sum}\right|^2 = \left(\frac{\lambda}{2\pi a \sin\theta}\right)^2 \left|\hat{E}_0 \Delta\delta\left(1+\ldots+e^{in\Delta\delta}+\ldots+e^{i\delta_N}\right)\right|^2$$

$$\Rightarrow \left(\frac{\lambda}{2\pi a\sin\theta}\right)^2 \left|\hat{E}_0 \int_{\delta=0}^{\delta=\delta_N} e^{i\delta}\, d\delta\right|^2 = \left(\frac{\lambda}{2\pi a\sin\theta}\right)^2 \left|-\hat{E}_0 i\left(e^{i\delta_N}-1\right)\right|^2$$

$$I \propto \left(\frac{\lambda}{2\pi a\sin\theta}\right)^2 \hat{E}_0\left(e^{i\delta_N}-1\right)\hat{E}_0^*\left(e^{-i\delta_N}-1\right) = \left(\frac{\lambda}{2\pi a\sin\theta}\right)^2 2E_0^2\left(1-\cos\delta_N\right) =$$

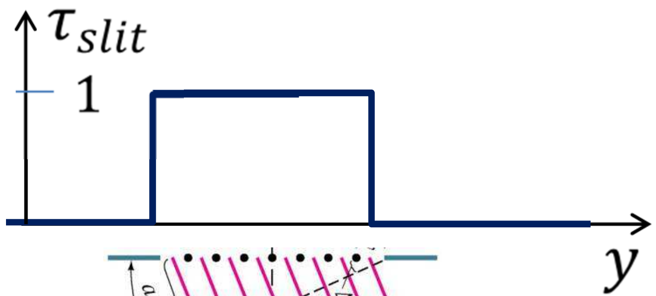
$$= 4\left(\frac{\lambda}{2\pi a\sin\theta}\right)^2 E_0^2 \sin^2\frac{\delta_N}{2} = E_0^2 \frac{\sin^2\left(\frac{\pi a\sin\theta}{\lambda}\right)}{\left(\frac{\pi a \sin\theta}{\lambda}\right)^2} \propto sinc^2\left(\frac{\pi a\sin\theta}{\lambda}\right)$$

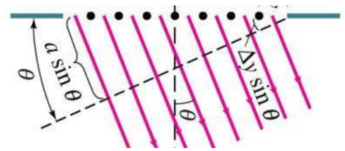


(a) Beugungsfaktor, $(\sin^2\beta/2)/(\beta/2)^2$ vs. θ

Can be interpreted as a Fourier transform of the transmission function τ_{slit} of the slit (note $\delta \approx 2\pi y\theta/\lambda$ is a linear function of y):

Note: I consider small angles → $\sin\theta \approx \theta$



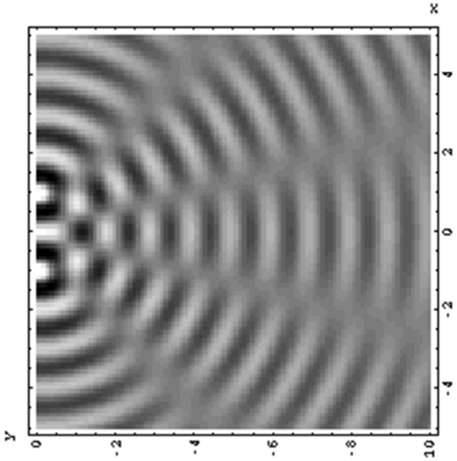


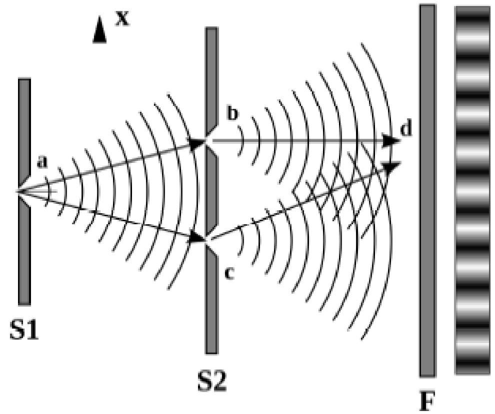
$$\int_{\delta=0}^{\delta=\delta_N} \frac{e^{i\delta}}{\theta} d\delta \propto \int_{y=y_1}^{y=y_2} e^{i\frac{2\pi\theta}{\lambda}y} dy = \int_{-\infty}^{+\infty} \tau_{slit}(y) e^{i\frac{2\pi\theta}{\lambda}y} dy = FT\left(\tau(y)\right)$$

Diffraction from two narrow slits

Makes it simple to calculate more complicated slit systems:

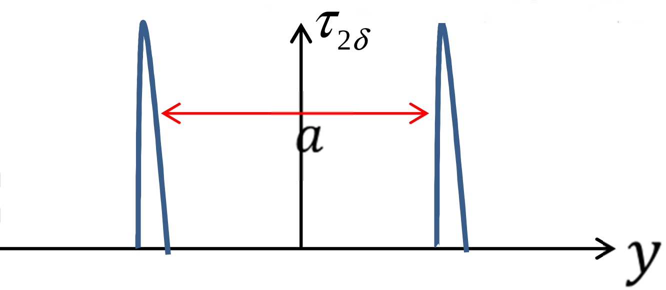
e.g. Young's double slit experiment (1801)





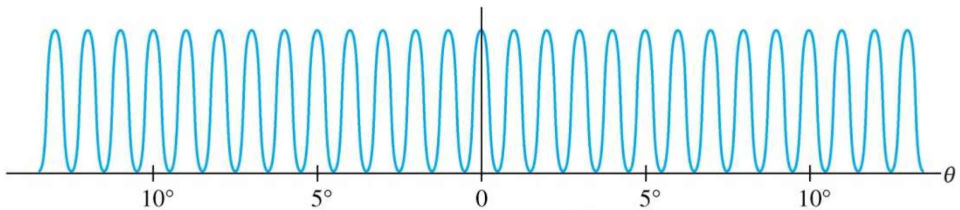
Fourier transform of two delta functions at ±a/2:

$$\int_{-\infty}^{+\infty}\left(\underbrace{\delta\left(y+\frac{a}{2}\right)+\delta\left(y-\frac{a}{2}\right)}_{\tau_{2\delta}}\right)e^{i\frac{2\pi\theta}{\lambda}y}dy = e^{i\pi a\theta/\lambda}+e^{-i\pi a\theta/\lambda}=2\cos\left(\frac{\pi a\theta}{\lambda}\right)$$



$$I(\theta)\propto\left|FT\left(\tau_{2\delta}(y)\right)\right|^2$$

$$\propto\left(\cos\left(\frac{\pi a\theta}{\lambda}\right)\right)^2=\tfrac{1}{2}\left(1+\cos\left(\frac{2\pi a\theta}{\lambda}\right)\right):$$



(b) Interferenzfaktor, $\cos^2\frac{\delta}{2}$ vs. θ

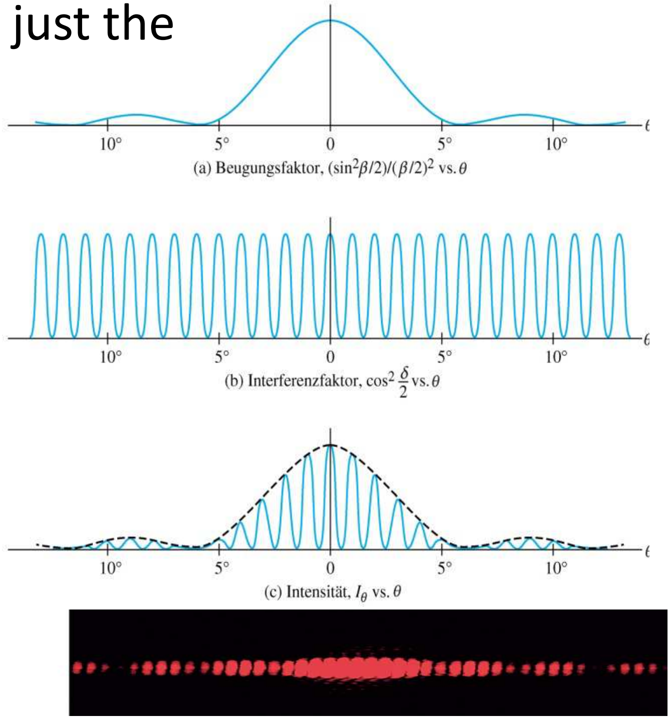
Diffraction from two real slits

We can even take into account the finite width of the slits:

$$\tau_{realistic}(y) = \int \tau_{2\delta}(y' - y) \cdot \tau_{slit}(y') dy'$$

The Fourier transform of such a convolution integral is just the product of the Fourier transforms:

$$I(\theta) \propto \left| FT\left(\tau_{realistic}(y)\right) \right|^2 = \left| FT(\tau_{2\delta}) \cdot FT(\tau_{slit}) \right|^2 :$$



Note 1:
"diffraction from two slits" (or of a finite number of separated waves) is mostly called "**interference**". Interference from a continuum of waves is called "**diffraction**".

There is no fundamental difference !

Note 2:
Due to interference of two waves, there are suddenly locations of permanently ZERO intensity (energy density) where each individual wave did generate intensity.
→ Energy density is re-distributed in space due to interference !

Far field condition

We have assumed a plane wave arriving at the slits.
→ Typically realized by a point source in very far distance L.

How far is “far”?

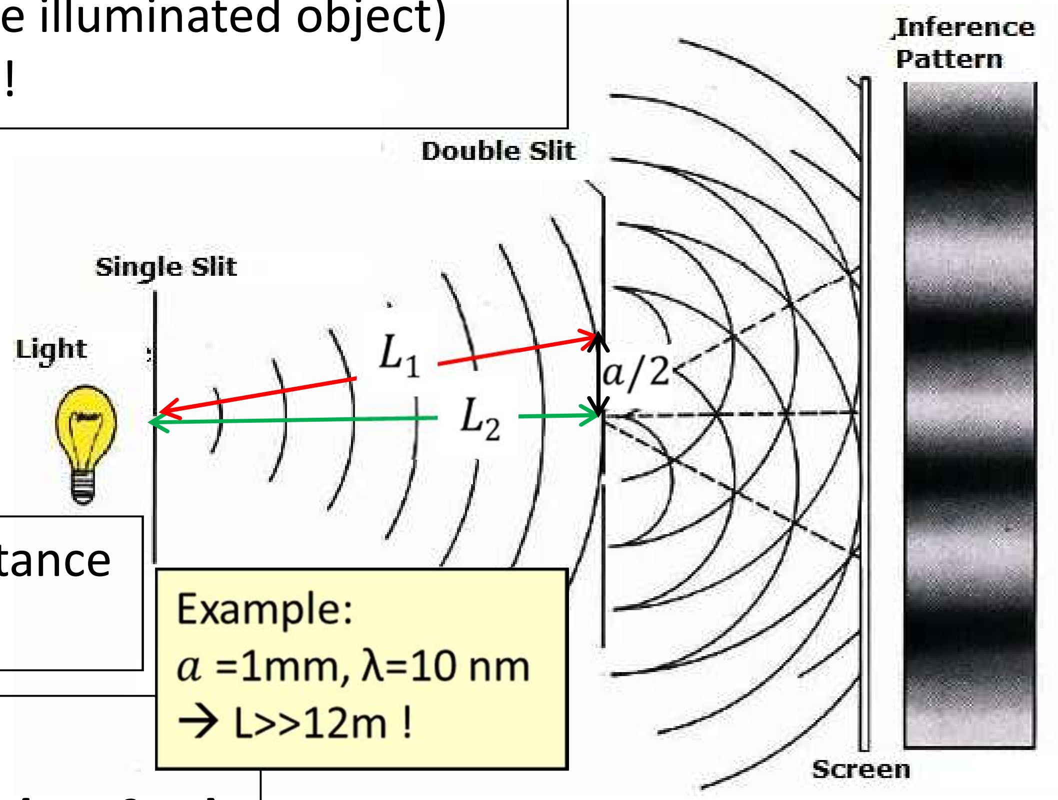
→ There should be no significant phase difference of the wave between the two slits (or the illuminated object)
→ Compare the optical path lengths !

$$L_1 - L_2 = \frac{a^2}{8L} << \lambda$$

→ Point source distance should be $L >> \frac{a^2}{8\lambda}$

The same condition holds for the distance from object to observation screen !

→ Fraunhofer diffraction ☺
Otherwise: Fresnel diffraction (no fun)



Example:
a =1mm, λ=10 nm
→ L>>12m !

transverse coherence

What matters for interference is the superposition of (two) e.m. waves.
So far we assumed they both come from the same source. This is not realistic.
Let's now drop this assumption but let's still consider monochromatic waves.

(complex) electric field at observation point (x), originating from two slit points 1,2, transversely separated by a :

$$E = E_1 + E_2 = A_1 e^{i\delta_1} + A_1 e^{i\delta_2}$$

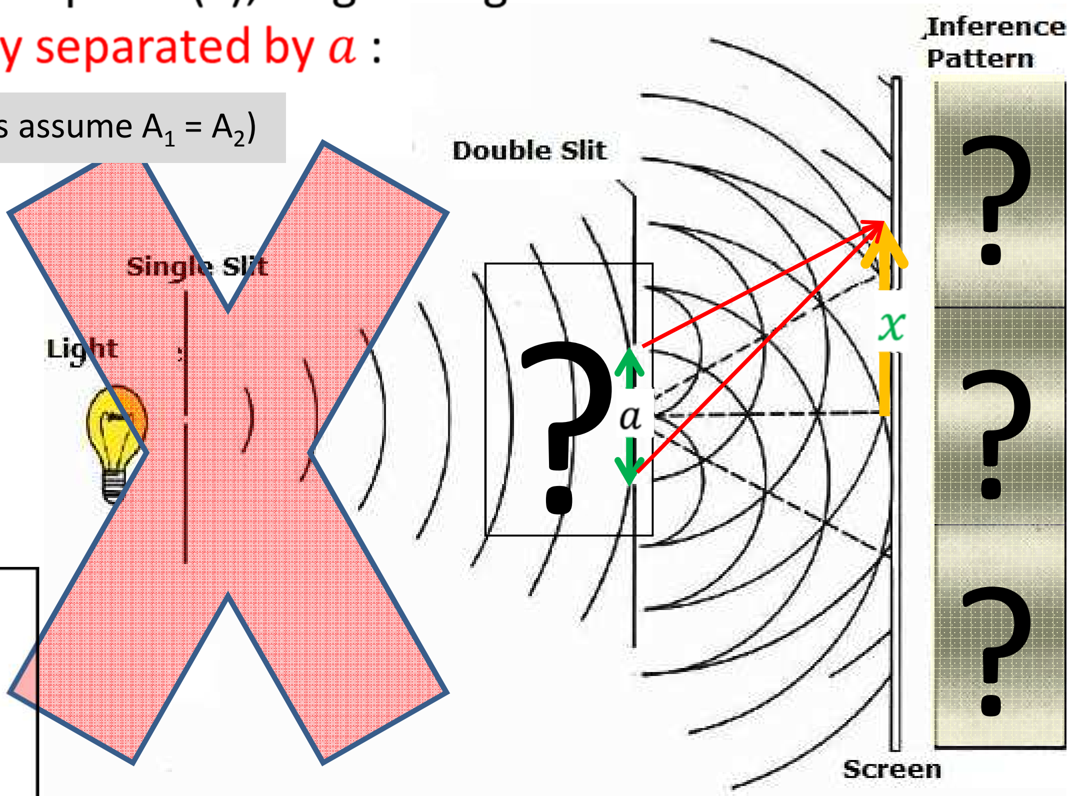
(Let's assume $A_1 = A_2$)

Intensity:

$$I \propto \left\langle |E|^2 \right\rangle_{period} = 2A_1^2 \left(1 + \cos(\delta_2 - \delta_1)\right)$$

$$\Rightarrow I = I_0 \left(1 + \cos\Delta\delta\right)$$

$$\Delta\delta = 2\pi \frac{a \sin\theta}{\lambda} + const \propto a \cdot x + const$$



If the phase difference between the slits is stable, $\Delta\delta$ depends only on a and x .
It only represents geometry of observation.
→ We expect perfect modulation vs. x

→ Such radiation from slits is called transversely **coherent**

transverse coherence

$$I = I_0 \left(1 + \cos \Delta\delta\right)$$

Moving along the screen, the cos$\Delta\delta$ - term oscillates between +1 and -1 (see page 9) → The interference contrast (="visibility") is then V=1:

$$V \equiv \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \rightarrow 1 \text{ (perfect coherence and } A_1 = A_2\text{)}$$

$\boxed{\text{Phase difference } \Delta\delta \text{ can be expressed as time delay, e.g.:}}$ $\Delta\delta = \frac{2\pi a\theta}{\lambda} = \frac{2\pi c}{\lambda}\tau = \omega_0 \tau$

Imagine, the two fields E_1 and E_2 do NOT stem from the same point source. We can still calculate the intensity on the screen according to p.9, but the phase difference on the screen does NOT only stem from the time delay τ (due to the observation angle), but there may be a further time difference τ' between E_1 and E_2 :

$$E_2 = E_1 e^{i\Delta\delta} e^{i\omega_0 \tau'} = E_1 e^{i\omega_0 (\tau + \tau')}$$

τ' is best described by electric field E_O <u>at the object (slits)</u>, when $E_{O,1}$ is observed at t and $E_{O,2}$ at (t- τ):

$$\boxed{\frac{\left\langle E_{O,1}(t) \cdot E_{O,2}^{*}(t-\tau) \right\rangle_T}{\sqrt{\left\langle \left|E_{O,1}\right|^2 \right\rangle \cdot \left\langle \left|E_{O,2}\right|^2 \right\rangle}} \equiv \gamma_{12}^{c}(a,\tau)}$$ **normalized complex correlation function**

We consider intensities → need to average over osc. period

transverse coherence

$$\frac{\left\langle E_{O,1}(t)\cdot E^*_{O,2}(t-\tau)\right\rangle}{\sqrt{\left\langle \left|E_{O,1}\right|^2\right\rangle \cdot \left\langle \left|E_{O2}\right|^2\right\rangle}} \equiv \gamma^c_{12}(a,\tau)$$ **normalized complex correlation function**

- depends on observation angle through $\tau = \frac{2\pi a\theta}{\lambda\omega_0} = \frac{a\theta}{c}$

In our simple case $A_1 = A_2$:

$$I = I_0\left(1+\cos\Delta\delta\right) \Rightarrow I_0\left(1+\Re\gamma^c_{12}\right) = I_0\left(1+\gamma_{12}\right)$$

Indeed: $$\gamma^c_{12}(a,\tau) = \frac{\left\langle A_1 e^{i\omega_0 t} A_1^* e^{-i\omega_0(t+\tau'-\tau)}\right\rangle}{\left|A_1\right|^2} = e^{-i\omega_0(\tau'-\tau)}$$

and if $\tau' = 0$ (perfectly coh. source): $\gamma^c_{12}(a,\tau) \Rightarrow e^{\omega_0\tau} = e^{\Delta\delta}$

$$\gamma_{12} = \Re\gamma^c_{12} = \Re e^{\Delta\delta} = \cos\Delta\delta$$

The visibility of interference is determined by: $\left|\gamma^c_{12}\right|$

In this case ($A_1 = A_2$): $V = \left|\gamma^c_{12}\right| = 1$ q.e.d.

transverse coherence

$$\frac{\left\langle E_{O,1}(t)\cdot E^*_{O,2}(t-\tau)\right\rangle}{\sqrt{\left\langle \left|E_{O,1}\right|^2\right\rangle \cdot \left\langle \left|E_{O2}\right|^2\right\rangle}} \equiv \gamma^c_{12}(a,\tau)$$

normalized complex correlation function

If $I_1 \neq I_2 \rightarrow$

$$I_{total} \propto \left|E_{sum}\right|^2 = \left|A_1 e^{i\omega_0 t} + A_2 e^{i\omega_0 (t+\tau'-\tau)}\right|^2 = A_1^2 + A_2^2 + A_1 A_2 \left(e^{i\omega_0(\tau'-\tau)} + e^{-i\omega_0(\tau'-\tau)}\right)$$

$$\Rightarrow I_{total} = I_1 + I_2 + \sqrt{I_1 I_2}\cos\left(\omega_0(\tau'-\tau)\right) \Rightarrow I_1 + I_2 + \sqrt{I_1 I_2}\,\Re\gamma^c_{12}$$

$$V \equiv \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}\left|\gamma^c_{12}\right| \equiv \gamma^{eff}_{12}$$

transverse coherence

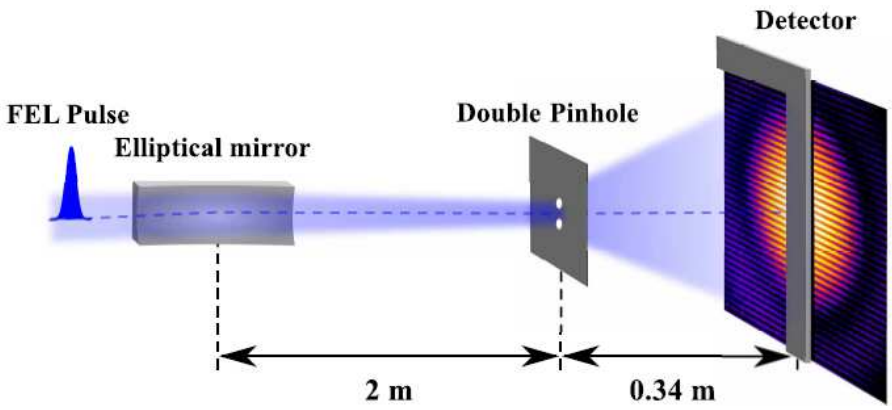
We expect that coherence is the better, the smaller the transverse distance a is.
For the **normalized degree of transverse coherence** we ignore τ :

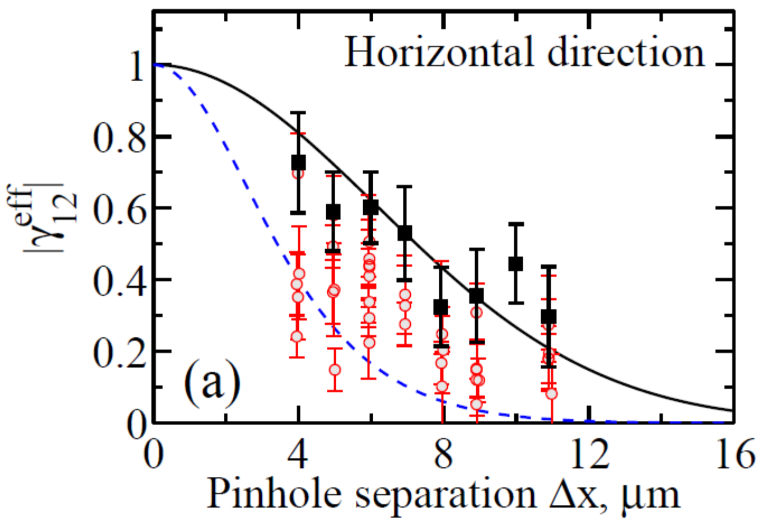
$$\varsigma \equiv \frac{\int \left| E_1(y_1,0)\cdot E_2^*(y_2,0)\right|^2 dy_1 dy_2}{\left(\int \left| E_1(y_1,0)\right| dy_1\right)^2}$$

This is a single number describing the entire beam from slit (object), independent of angle of observation.
$\varsigma \to 1$ for perfectly coherent radiation.

Example from FLASH FEL:

From: A. Singer, et. al:
Opt.Express 20, No.16 (2012)



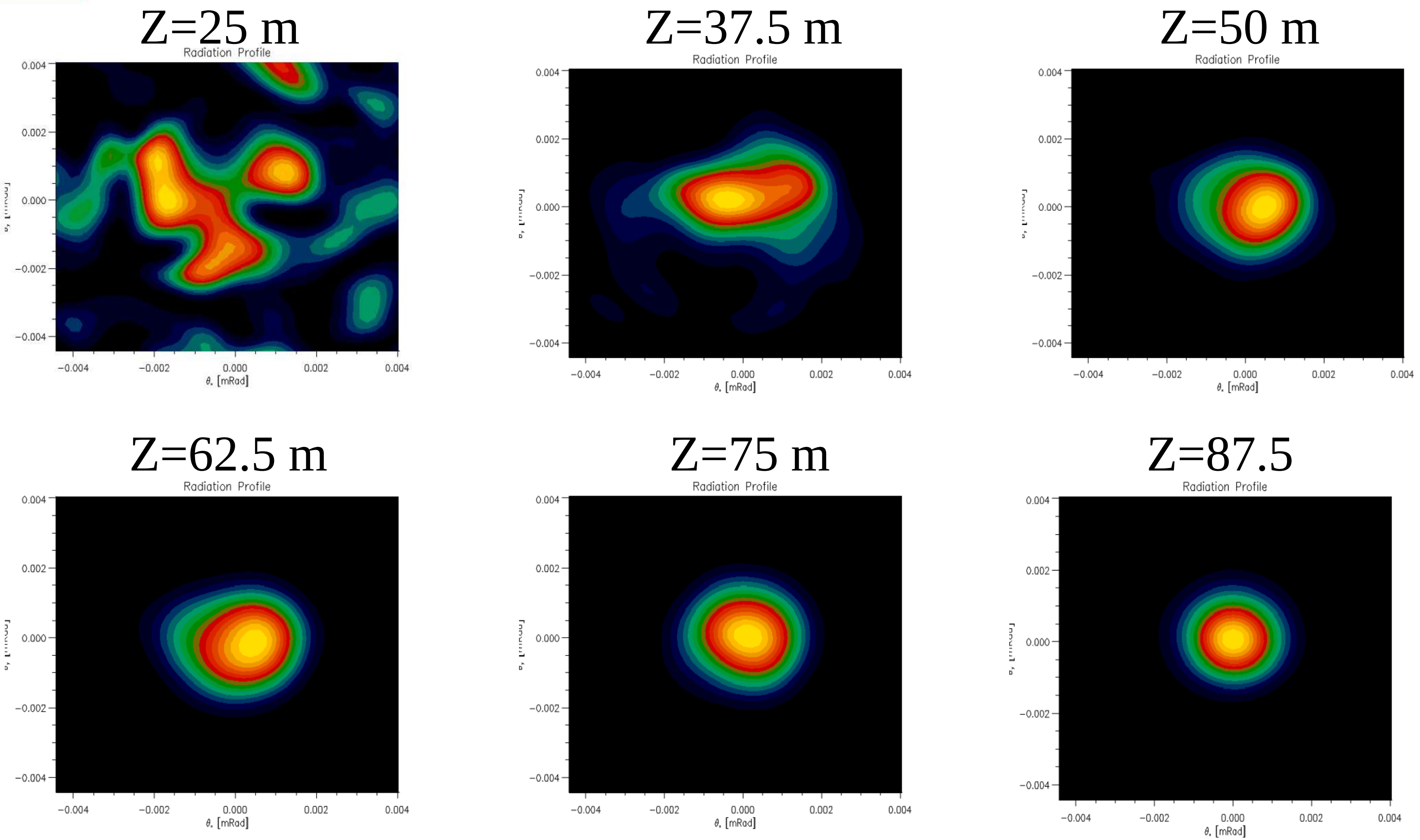


$$\varsigma_x = 0.59 \pm 0.1$$

$$\varsigma_y = 0.72 \pm 0.08$$

Transverse coherence

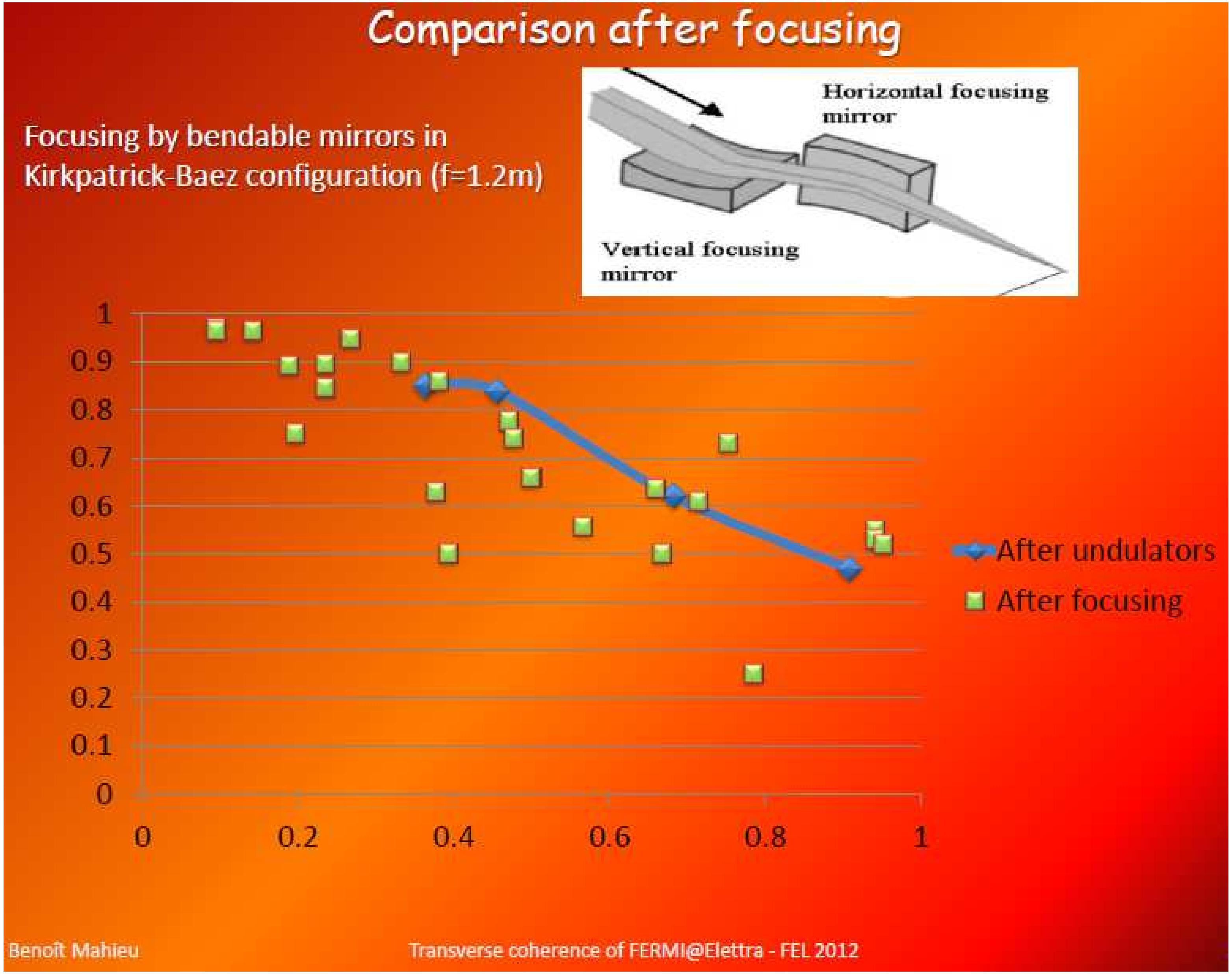
S. Reiche



Single mode dominates ➔ close to 100% transverse coherence

impact of focusing

(needed for most experiments)

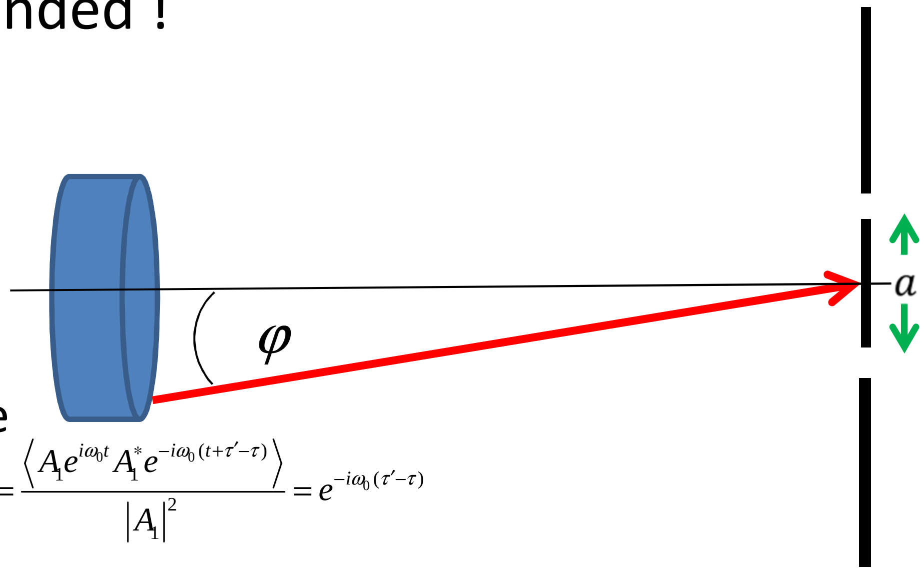


transverse coherence

Why are the phases in 1 and 2 uncorrelated?
Probably because the source is extended !

Consider a ray arriving at double slit at an angle φ and with intensity $I(\varphi)$
It contributes a phase difference to double slit of: $\Delta\delta = \dfrac{2\pi a\varphi}{\lambda}$

The averaged phase difference between the extended source to the double slit is: $\gamma_{12}^{c}(a,\tau) = \dfrac{\left\langle A_1 e^{i\omega_0 t} A_1^{*} e^{-i\omega_0(t+\tau'-\tau)} \right\rangle}{|A_1|^2} = e^{-i\omega_0(\tau'-\tau)}$



$$\frac{1}{I_{total}} \int_{-\infty}^{+\infty} I(\varphi) e^{-i\frac{2\pi a\varphi}{\lambda}} d\varphi = FT\left(\frac{I(\varphi)}{I_{total}}\right) \equiv I\left(\frac{a}{\lambda}\right) \Rightarrow V = \left| I\left(\frac{a}{\lambda}\right) \right|$$

Van Zitter-Zernicke-Theorem:
Visibility of interference pattern corresponds to Fourier Transform of source distribution function

Example:

As long as $\dfrac{a\varphi}{\lambda} << 1$, the source acts like a fully coherent point source !

→ Observe a far distant star ($\varphi \ll \frac{\lambda}{a}$) vs. a double star system with $\varphi \approx \frac{\lambda}{a}$ with a telescope with large a

temporal coherence

So far we have discussed perfectly monochromatic waves.
Such waves have constant phases forever.
Because: Any phase change requires a certain bandwidth $\Delta\nu$.
→ Monochromatic waves, even originating from completely independent sources will be mutually coherent, because the relative phases cannot change.

But this is not realistic !

e.g. Gaussian wave packet with duration $\Delta\tau$:

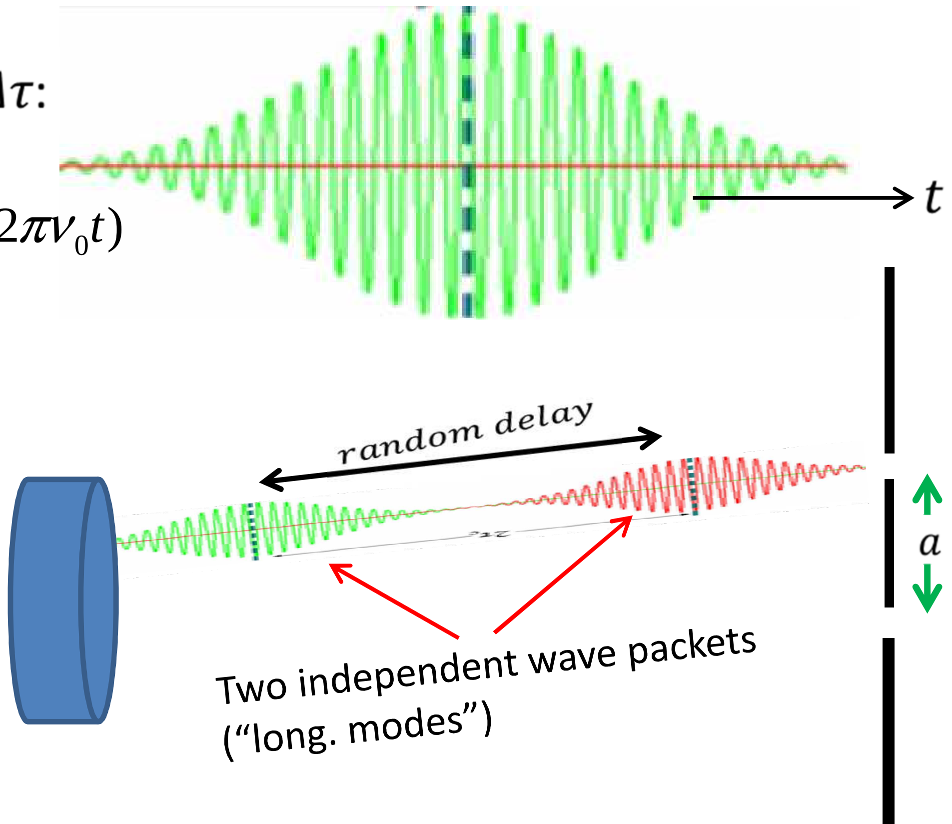
$$E(t) = E_0 e^{-\pi\frac{t^2}{(\Delta\tau)^2}} e^{i2\pi\nu_0 t};\ \Re E(t) = E_0 e^{-\pi\frac{t^2}{(\Delta\tau)^2}} \cos(2\pi\nu_0 t)$$

$$FT(E(t)) \propto e^{-\pi(\Delta\tau)^2(\nu-\nu_0)^2} = e^{-\pi\frac{(\nu-\nu_0)^2}{(\Delta\nu)^2}}$$

→ Frequency width: $\Delta\nu = 1/\Delta\tau$

Phase information will be lost after
longitudinal coherence time:

$$\tau_c \approx \frac{1}{\Delta\nu}$$



temporal coherence

Again: Calculate evolution of phase,
but now for the radiation originating from the same source, observed at two different times → replace E_2 by E_1 and average over all times:

$$\frac{\left\langle E_1(t)\cdot E_2^*(t-\tau)\right\rangle}{\sqrt{\left\langle |E_1(t)|^2\right\rangle\cdot\left\langle |E_2(t-\tau)|^2\right\rangle}} \equiv \gamma_{12}^c(a,\tau) \quad\rightarrow\quad \frac{\left\langle E_1(t)\cdot E_1^*(t-\tau)\right\rangle}{\sqrt{\left\langle |E_1(t)|^2\right\rangle\cdot\left\langle |E_1(t-\tau)|^2\right\rangle}} \equiv \gamma^c(\tau)$$

normalized complex autocorrelation function

And again: $$I(\tau)=I_0\left(1+\Re\gamma^c(\tau)\right)=I_0\left(1+\gamma(\tau)\right)$$

Can also relate $\gamma^c(\tau)$ to normalized spectrum of field *FT(E)*:
(Wiener-Khintchine-theorem):

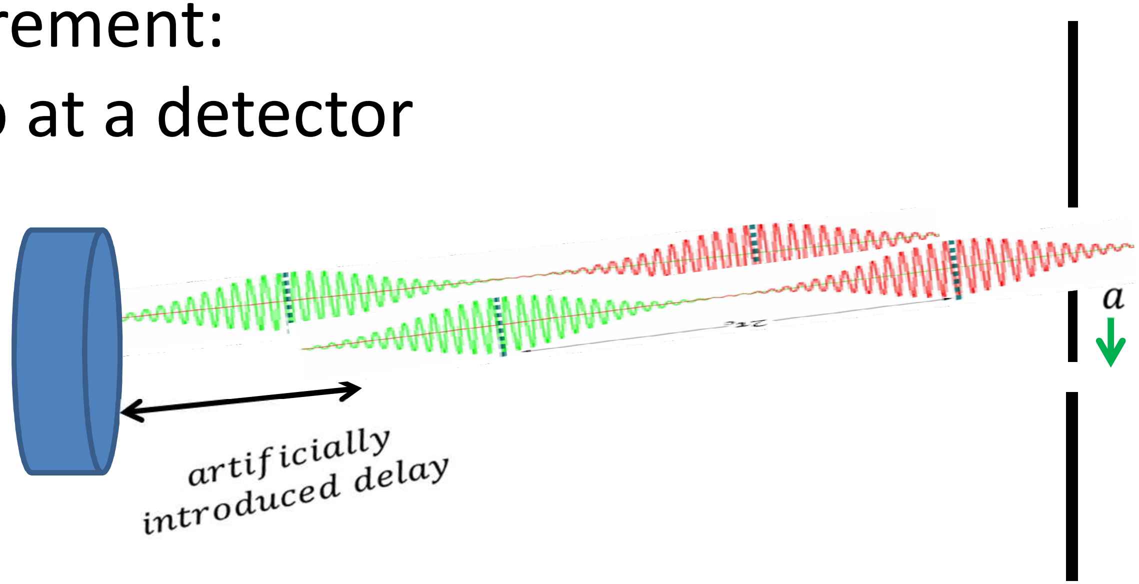
$$FT\left(\gamma^c(\tau)\right)=\left(FT(E)\right)^2$$ = Power spectral density

e.g. our single Gaussian wave packet:

$$\gamma^c(\tau)=e^{-\frac{\pi}{2}\frac{\tau^2}{(\Delta\tau)^2}}e^{i2\pi\nu_0\tau}\Rightarrow\Re\gamma^c(\tau)=\gamma(\tau)=e^{-\frac{\pi}{2}\frac{\tau^2}{(\Delta\tau)^2}}\cos(2\pi\nu_0\tau)$$

temporal coherence

Possible way of measurement:
Split, delay and overlap at a detector



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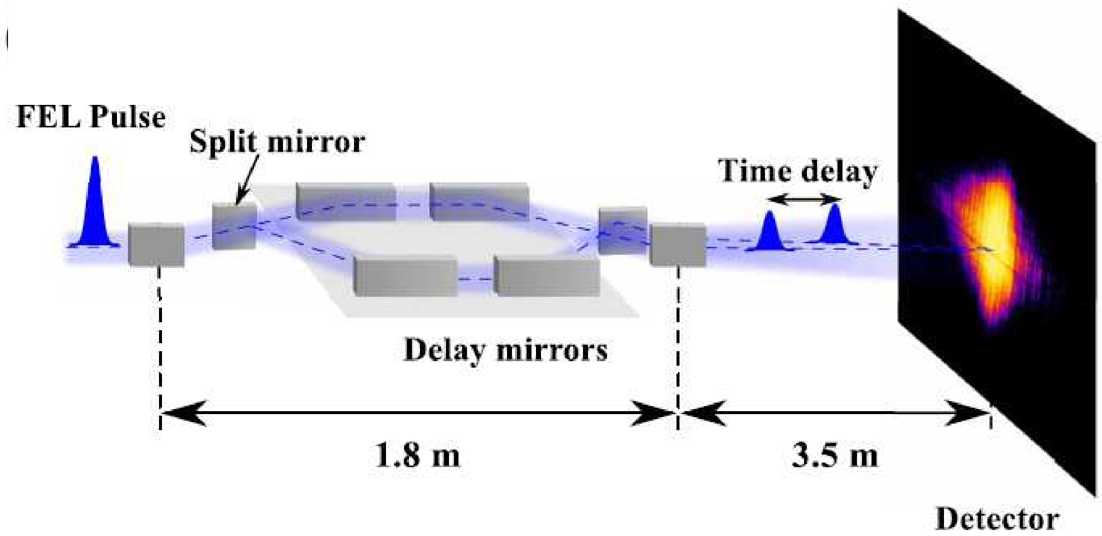
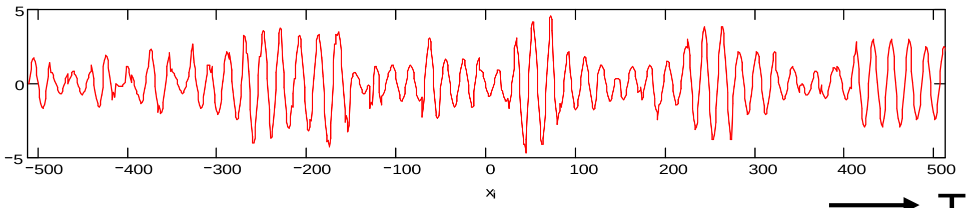


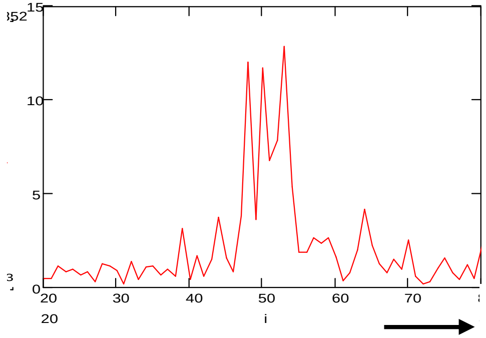
temporal coherence

Add up many wave packets at arbitrary time (phase) differences:



→ T

Fourier



→ ω

If we want a single characteristic number for the average duration of coherence in the beam, we can calculate the **coherence time**:

$$\tau_c = \int_{-\infty}^{+\infty} \left|\gamma^c(\tau)\right|^2 d\tau$$

<u>Typical values for FELs:</u>

FLASH in SASE mode @ 8 nm: τ_c = 6 fs

FERMI in seeded mode @ ca. 20 nm: τ_c = 100 fs

Full optical phase control

If the transverse correlation is perfect, one may think of manipulating the optical phase in a perfectly controlled way.

→ Use a micro-electrical mechanical system (MEMS) filter

→ See http://www.physik.uni-wuerzburg.de/femto-welt/formerframe.html

General case coherence

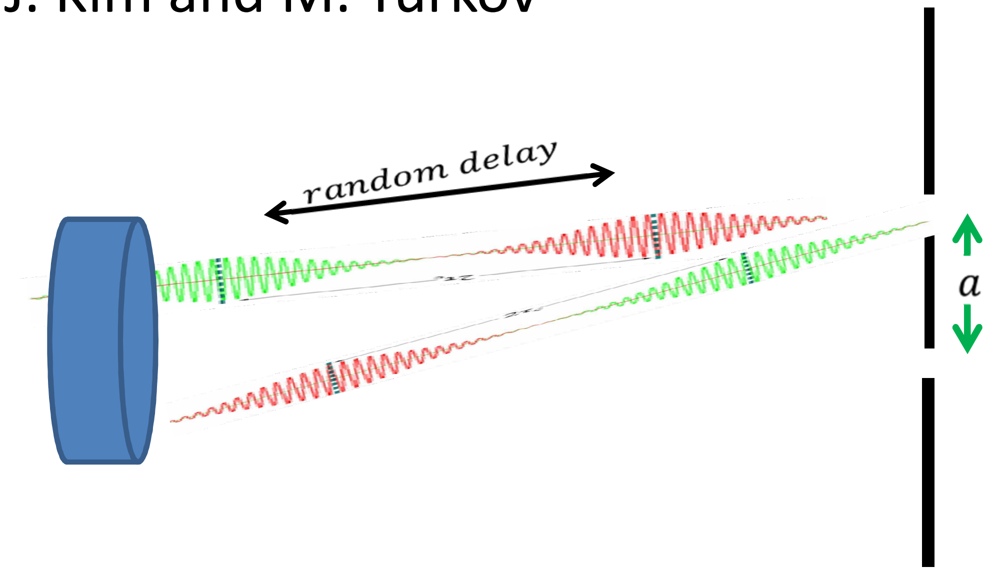
$$\frac{\left\langle E_{O,1}(t)\cdot E^*_{O,2}(t-\tau)\right\rangle}{\sqrt{\left\langle \left|E_{O,1}\right|^2\right\rangle\cdot\left\langle \left|E_{O2}\right|^2\right\rangle}} \equiv \gamma^c_{12}(a,\tau) \qquad \frac{\left\langle E_1(t)\cdot E^*_1(t-\tau)\right\rangle}{\sqrt{\left\langle \left|E_1(t)\right|^2\right\rangle\cdot\left\langle \left|E_1(t-\tau)\right|^2\right\rangle}} \equiv \gamma^c(\tau)$$

Maybe, the autocorrelation function depends on the source point?

Maybe the transverse correlation function depends on wavelength?

→ All this will happen and is characteristic for an FEL

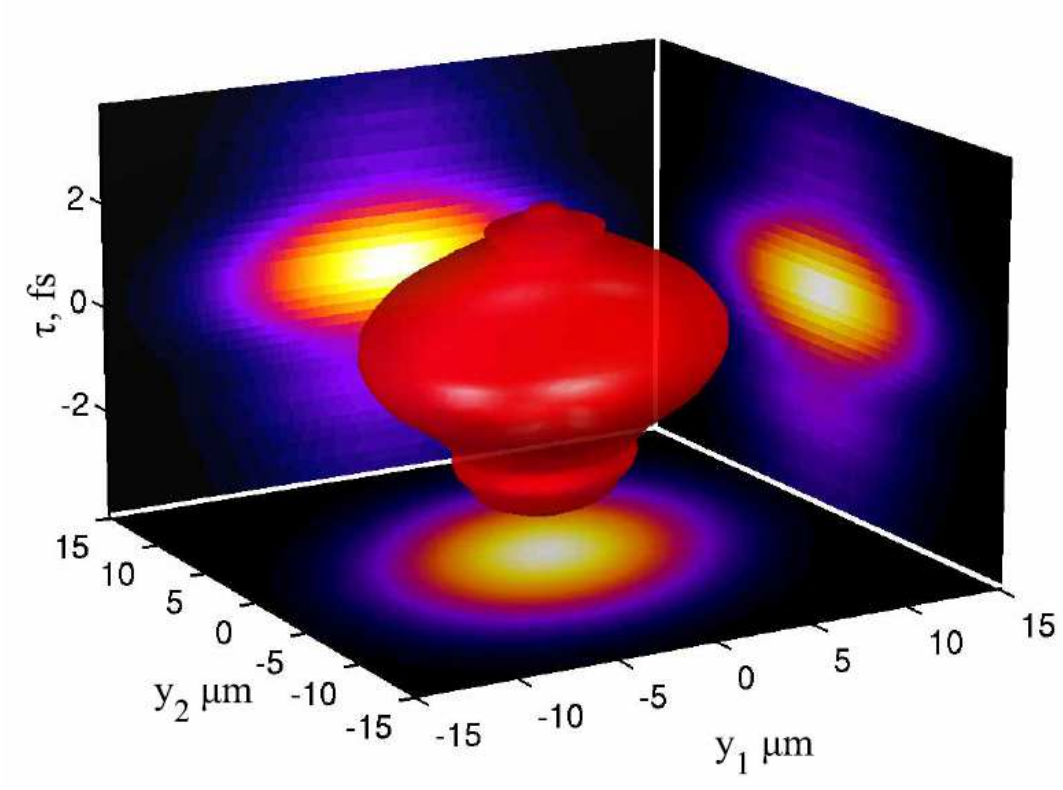
→ See talks by K.-J. Kim and M. Yurkov



Coherence combined

For a full description of the coherence properties, we need to combine transverse and longitudinal coherence properties.

Example from A. Singer, et. al:
Opt.Express 20, No.16 (2012)
Note:
This is only vertical/longitudinal,
the horizontal/longitudinal similar



Electron diffraction

According to quantum mechanics, each electron carries wave properties at the

de Broglie wavelength: $$\lambda_e = \frac{h}{p} = \frac{h}{m_0 \gamma\beta c}$$

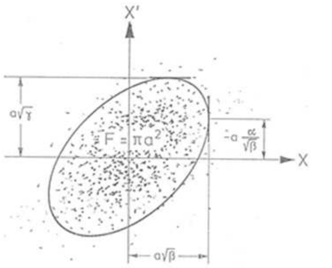
with h the Planck constant,
p the electron‘s momentum,
m_0 the electron‘s rest mass,
$\gamma\beta$ the relativistic parameters

e.g. 1 MeV kinetic energy
→ $\gamma\beta$ =2.7, λ_e= 0.9 ·10^{-12} m

According to p. 19 each (incoherently - remember the star!) radiating source with opening angle Δφ will act like a point source and generate perfect interference as long as the slit separation a is

$$a < \frac{\lambda}{\Delta\varphi}$$ → transverse coherence length of electron beam: $$L_{coh} \approx \frac{\lambda_e}{2\pi\sigma_\varphi}$$

Opening angle σ_φ of electron beam is determined by beam emittance: $$\sigma_x \sigma_\varphi = \varepsilon_x = \frac{\varepsilon_x^n}{\gamma\beta}$$



Electron diffraction

Combine $\lambda_e = \frac{h}{p} = \frac{h}{m_0 \gamma \beta c}$, $L_{coh} \approx \frac{\lambda_e}{2\pi\sigma_\varphi}$, and $\sigma_x \sigma_\varphi = \varepsilon_x = \frac{\varepsilon_x^n}{\gamma\beta}$ →

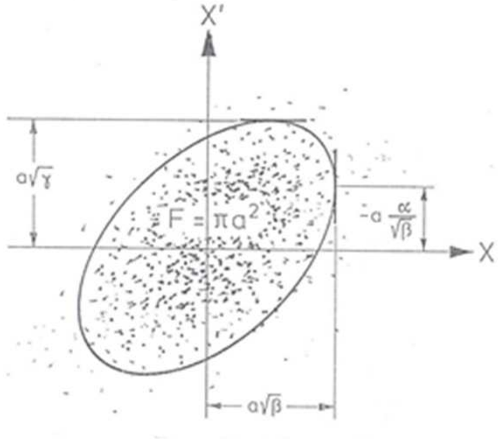
Transv. coherence length of electron beam: $\boxed{L_{coh} \approx \frac{\hbar \sigma_x}{m_0 c \varepsilon_x^n} = \frac{\hbar}{m_0 c}\sqrt{\frac{\beta_x}{\gamma\beta\varepsilon^n}}}$

a) Prefer a large β_x-function → put the crystal far away from waist

b) what matters then is the slice angular spread

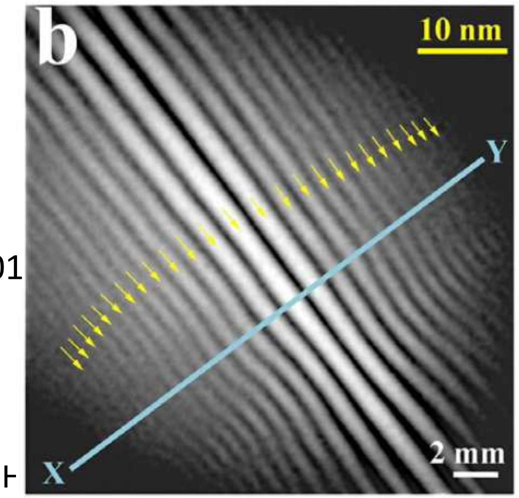
-- but: this reduces the # electrons hitting the crystal

e.g. 1 MeV kinetic energy,
horizontal β- function = 100m,
normalized emittance = 10^{-8} m
$L_{coh} \approx 2.4 \cdot 10^{-8}$ m



→ can perform interference experiment on nanocrystals of 24nm diameter

From: C. C. Chang, et. al:
Nanotechnology **20** (2009) 115401
(89 eV, carbon nanotubes)



Left open….

1. I restricted myself to 1D; extension to 2D needed.
2. I restricted myself to linear polarization case.
3. What happens with coherence when focusing?

   (Nothing in certain cases, Fermat's principle helps.)
4. I discussed only 1st order correlations.
5. Most things can be done also in frequency domain.

Further reading

Text books:

1. Many illustrations are from:
   D. C. Giancoli: Physics (Pearson, 2006)
2. A. Singer, et. al: Spatial and temporal coherence properties of single free-electron laser pulses, Optics Express 20, No.16 (2012)
2. M.V. Klein, T.E. Furtak: Optics (Wiley, New York, 1986)
3. J. W. Goodman: Statistical Optics (Wiley, New York, 2000)
4. L. Mandel and E. Wolf: Optical Coherence and Quantum Optics (Cambridge University Press, 1995)
5. M. Born and E. Wolf: Principles of Optics(Cambridge University Press, 2002)
6. E.L. Saldin, E.A. Schneidmiller, M.V. Yurkov: The Physics of Free Electron Lasers (Springer 1999)