# The Quantum Free Electron Laser

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# Outline

1. Introduction
2. When should quantum effects be significant?
3. A 1D Quantum Model of the High-Gain FEL
4. Quantum FEL simulations
5. Conclusions & Outlook

# 1. Introduction

- Many of the early theoretical studies of free electron lasers (low gain) were quantum mechanical (see e.g. [1,2]).

- It was realised, however, that the behaviour of low gain FELs were described by expressions which were independent of h
i.e. they were essentially classical .

- All FEL experiments **to date** (from mm-wave $\rightarrow$X-rays ) are well described by classical models where the electron beam is a collection of particles interacting with a classical electromagnetic field.

- As FEL operation moves to generation of shorter wavelengths (emission of photons with larger momenta), eventually classical models will break down.

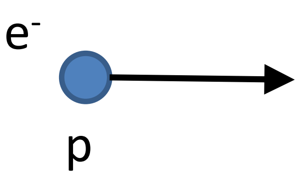
# 2. When could quantum effects become significant?

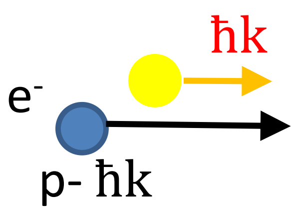
## 2.1 Energy / momentum considerations

The FEL process involves electrons emitting photons.

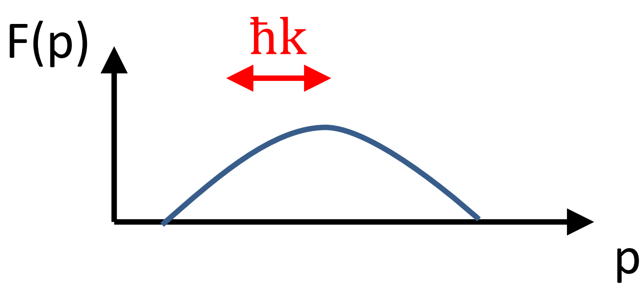
Each photon has a finite amount of momentum = $\hbar k$.

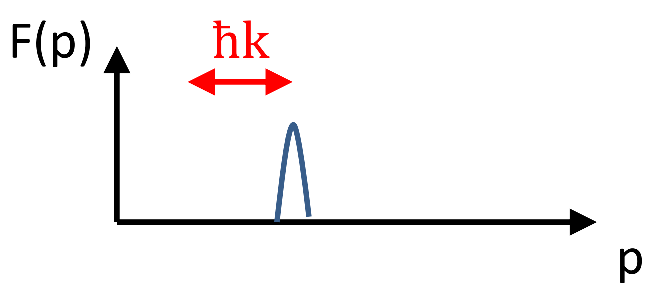
Each photon emission event will therefore result in the electron recoiling, reducing its momentum by $\hbar k$.



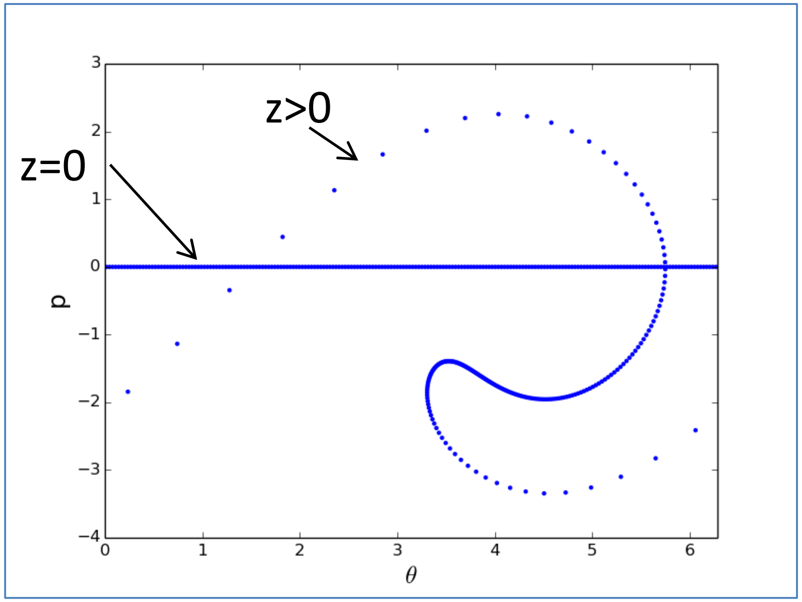

In an electron beam with a finite momentum spread, $\Delta p$, it will only be possible to resolve the electron recoil if $\Delta p < \hbar k$ .



We know from classical FEL physics (see e.g. K-J Kim lectures) that the FEL process induces an energy/momentum spread in the electron beam [3,4].

This can be visualised as electrons moving along continuous trajectories in phase space :




Induced spread : $\dfrac{\Delta\gamma}{\gamma} \sim \rho$

$$\rho = \frac{1}{\lambda_r}\left(\frac{a_w \omega_p}{4ck_w}\right)^{2/3}$$ is the FEL parameter

This validity of the classical model will depend on the ratio : $\dfrac{\Delta p}{\hbar k}$

i.e. $\dfrac{mc\Delta\gamma}{\hbar k}$ or $\boxed{\dfrac{mc\gamma}{\hbar k}\rho \equiv \bar{\rho}}$

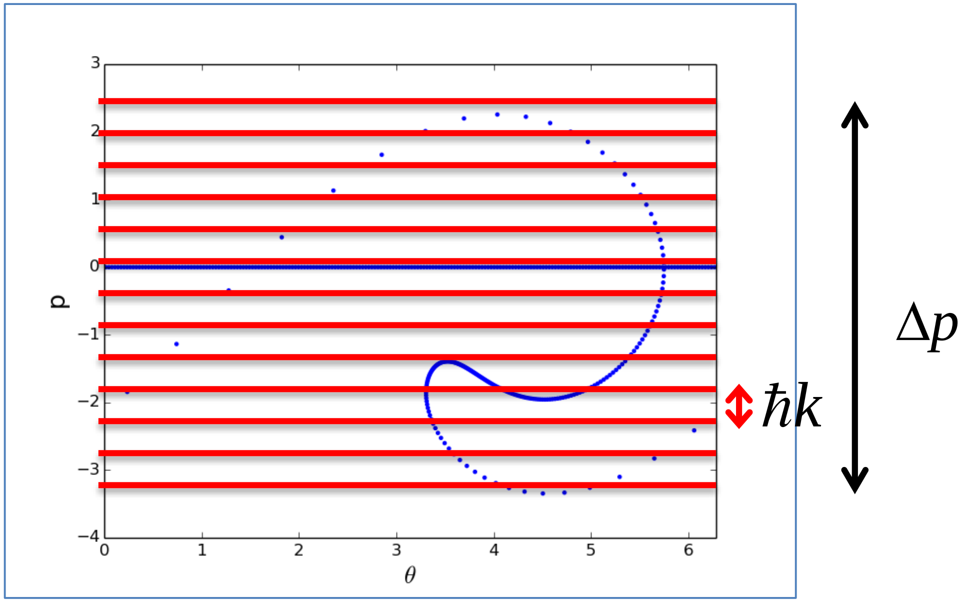
$\bar{\rho}$ is the quantum FEL parameter

## Classical FEL limit

Classical description holds well if

$$\hbar k << \Delta p$$

i.e. $\bar{\rho} >> 1$

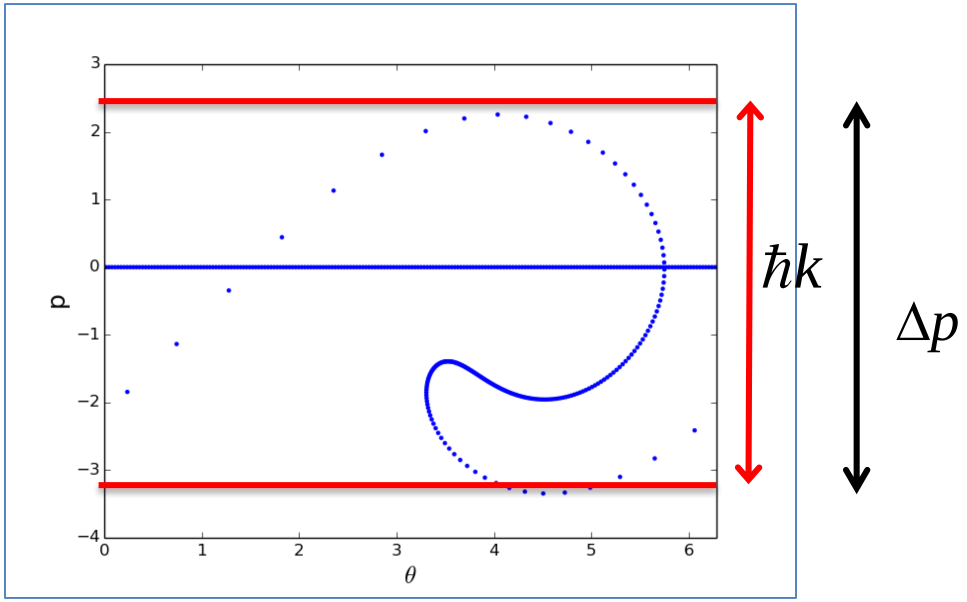



## Quantum FEL limit

Classical description will break down if

$$\hbar k \geq \Delta p$$

i.e. $\bar{\rho} \leq 1$



## How to realise the quantum FEL limit ?

We need $\boxed{\dfrac{mc\gamma}{\hbar k}\rho \equiv \bar{\rho} < 1}$

This can be rewritten as $\boxed{\dfrac{\gamma\lambda}{\lambda_c}\rho < 1}$ where $\boxed{\lambda_c = \dfrac{h}{mc} \approx 2.4\times 10^{-12}m}$

For a magnetostatic X-ray FEL e.g. LCLS

$$\gamma \sim 3\times 10^4,\ \rho \sim 5\times 10^{-4},\ \frac{\lambda}{\lambda_c} \sim 40, \quad \text{so} \quad \boxed{\bar{\rho} >> 1} \quad \text{(classical)}$$

Another option is to use a laser undulator :

**Advantage** : allows use of much smaller $\gamma$ – suggests $\bar{\rho} < 1$ possible when $\dfrac{\lambda}{\lambda_c} \rightarrow 1.$

i.e. approaching $\gamma$–rays.

**Challenge** : shorter interaction lengths/times (see [5] for full details)

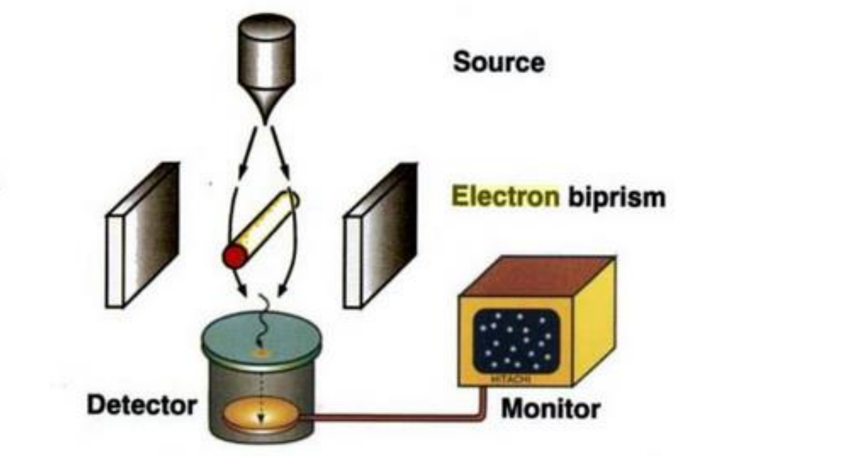
# 2.2 Alternative argument – electron beam coherence

**Q**. Electrons are particles, right ?

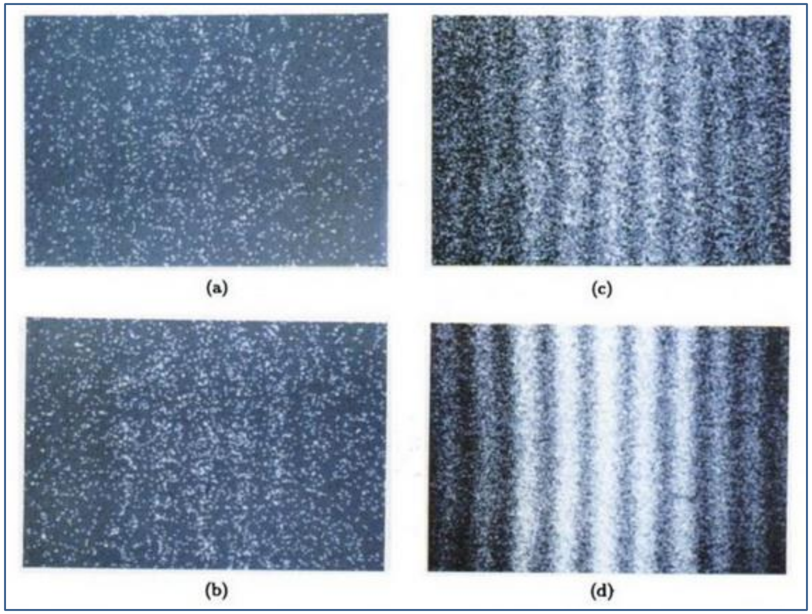
**A**. Sometimes…

Electron beams can demonstrate wave phenomena i.e. interference [6].

The subject of e.g. electron holography, is based on this.




Experimental arrangement of a two-slit experiment for electrons (A. Tonomura *et al.*: Amer. J. Phys. **57** (1989) 117).




Interference pattern visible if path difference < e-beam coherence length.

To describe this, a wavefunction description of the electron beam is required.

$$I \propto |\Psi|^2 = |\Psi_1 + \Psi_2|^2 = |\Psi_1|^2 + |\Psi_2|^2 + 2|\Psi_1||\Psi_2|\cos(\phi_1 - \phi_2)$$

E-beam (longitudinal) coherence length is defined as :

$$L_c = \frac{\lambda_e^{\,2}}{\Delta\lambda_e}$$ where $$\lambda_e = \frac{h}{p}$$ is the de Broglie wavelength of the electrons.

In terms of FEL, wave-like nature of electrons should be significant if

$$L_c > \lambda$$

Rewriting $L_c$ in terms of electron momentum, p : $L_c = \dfrac{h^2}{p^2}\dfrac{p^2}{h\Delta p} = \dfrac{h}{\Delta p}$

so $L_c > \lambda$ implies $\dfrac{h}{\Delta p} > \lambda$ i.e. $\boxed{\hbar k > \Delta p}$

This is the same condition as derived previously for observation of quantum effects.

This suggests that, in this regime, a wavefunction description (or equivalent) of the FEL interaction is required.

# 3. A 1D Model of the Quantum High-Gain FEL

Here I present an outline derivation of a 1D high-gain quantum FEL model.
More rigorous treatments can be found in [7,8].

First, let us look at the classical, 1D high-gain FEL equations i.e. the pendulum-like electrons coupled to the EM field (see e.g. K-J Kim lectures – different notation).

$$\frac{d\theta_j}{d\bar{z}} = \frac{p_j}{\bar{\rho}} \qquad (1)$$

$$\frac{dp_j}{d\bar{z}} = -\bar{\rho}\left(Ae^{i\theta_j} + c.c.\right) \qquad (2)$$

$$\frac{dA}{d\bar{z}} = \left\langle e^{-i\theta} \right\rangle + i\delta A \qquad (3)$$

where

$$\theta = \left(k + k_w\right)z - \omega t$$

$$p = \frac{mc}{\hbar\left(k + k_w\right)}\left(\gamma - \gamma_0\right)$$

$$\rho|A|^2 = \frac{\varepsilon_0|E|^2}{\hbar\omega n_e} = \frac{n_p}{n_e}, \qquad \delta = \frac{\gamma_0 - \gamma_r}{\rho\gamma_0}$$

$$\bar{z} = \frac{z}{L_g}, \quad L_g = \frac{\lambda_w}{4\pi\rho}$$

# (i) Electrons

Consider the equations of motion for electron j :

$$\frac{d\theta_j}{d\bar{z}} = \frac{p_j}{\bar{\rho}}$$

$$\frac{dp_j}{d\bar{z}} = -\bar{\rho}\left(Ae^{i\theta_j} + c.c.\right)$$

These equations can be derived from the single electron Hamiltonian :

$$H_j = \frac{p_j^{\,2}}{2\bar{\rho}} - i\bar{\rho}\left(Ae^{i\theta_j} - c.c.\right)$$

This Hamiltonian can be used to write a Schrodinger equation for the single-electron wavefunction :

$$i\frac{\partial\Psi(\theta,\bar{z})}{\partial\bar{z}} = H_j\Psi(\theta,\bar{z})$$ ,where p is the momentum operator $p = -i\dfrac{\partial}{\partial\theta}$

$$\boxed{i\frac{\partial\Psi(\theta,\bar{z})}{\partial\bar{z}} = -\frac{1}{2\bar{\rho}}\frac{\partial^2\Psi}{\partial\theta^2} - i\bar{\rho}\left(Ae^{i\theta} - c.c.\right)\Psi}$$

# (ii) EM Field

Consider the EM field equation $$\frac{dA}{d\bar{z}} = \left\langle e^{-i\theta} \right\rangle + i\delta A$$

The classical average : $$\left\langle e^{-i\theta} \right\rangle = \frac{1}{N}\sum_{j=1}^{N} e^{-i\theta_j}$$

is replaced by a quantum average defined in terms of $\Psi(\theta, \bar{z})$ i.e.

$$\left\langle e^{-i\theta} \right\rangle \rightarrow \int_0^{2\pi} |\Psi|^2 e^{-i\theta}\, d\theta$$

Consequently, the EM field evolution is described by :

$$\boxed{\frac{dA(\bar{z})}{d\bar{z}} = \int_0^{2\pi} \left|\Psi(\theta, \bar{z})\right|^2 e^{-i\theta}\, d\theta + i\delta A}$$

The equations which describe the quantum FEL interaction are therefore :

$$i\frac{\partial\Psi\left(\theta,\bar{z}\right)}{\partial\bar{z}}=-\frac{1}{2\bar{\rho}}\frac{\partial^{2}\Psi}{\partial\theta^{2}}-i\bar{\rho}\left(Ae^{i\theta_{j}}-c.c.\right)\Psi$$

$$\frac{dA\left(\bar{z}\right)}{d\bar{z}}=\int_{0}^{2\pi}\left|\Psi\left(\theta,\bar{z}\right)\right|^{2}e^{-i\theta}\,d\theta+i\delta A$$

It is possible to solve this coupled set of PDEs/ODEs directly using a number of numerical methods e.g.

- finite difference (e.g. Crank-Nicholson)
- finite element
- splitstep FFT

However it is easier to gain some insight if we rewrite them in terms of <u>momentum states</u>.

# Quantum FEL model : Momentum state representation

The states $|n\rangle = \exp(in\theta)$ are momentum eigenstates because they satisfy the eigenvalue equation

$$\hat{p}|n\rangle = n|n\rangle$$

where $\hat{p} = -i\frac{\partial}{\partial\theta}$ is the momentum operator and n is an integer.

We can expand the electron wavefunction in terms of these momentum eigenstates i.e.

$$\Psi(\theta,\bar{z}) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} c_n(\bar{z})\exp(in\theta)$$

where $|c_n|^2$ is the probability of an electron having momentum

$$(\gamma-\gamma_0)mc = n\hbar k$$

Substituting for $\Psi(\theta,\bar{z})$ , the quantum FEL equations become :

$$i\frac{\partial\Psi(\theta,\bar{z})}{\partial\bar{z}} = -\frac{1}{2\bar{\rho}}\frac{\partial^2\Psi}{\partial\theta^2} - i\bar{\rho}\left(Ae^{i\theta_j} - c.c.\right)\Psi \quad \Longrightarrow \quad \frac{dc_n}{d\bar{z}} = -i\frac{n^2}{2\bar{\rho}}c_n - \bar{\rho}\left(Ac_{n-1} - A^*c_{n+1}\right)$$

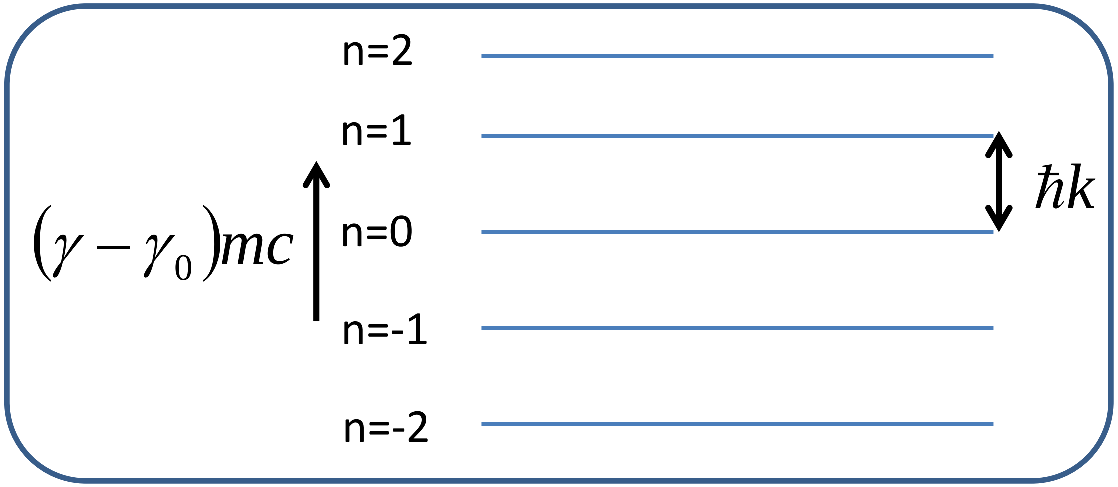
$$\frac{dA(\bar{z})}{d\bar{z}} = \int_0^{2\pi}\left|\Psi(\theta,\bar{z})\right|^2 e^{-i\theta}\,d\theta + i\delta A \quad \Longrightarrow \quad \frac{dA}{d\bar{z}} = \sum_{n=-\infty}^{\infty} c_n c^*{}_{n-1} + i\delta A$$

## Quantum FEL model : Momentum state representation

In the momentum representation the interaction is described as exchange of population between different electron momentum states via the electromagnetic field in **discrete** amounts $\hbar k$ .

$$\frac{dc_n}{d\bar{z}} = -i\frac{n^2}{2\bar{\rho}}c_n - \bar{\rho}\left(Ac_{n-1} - A^* c_{n+1}\right)$$

$$\frac{dA}{d\bar{z}} = \sum_{n=-\infty}^{\infty} c_n c^*{}_{n-1} + i\delta A$$




The EM field is driven by bunching of electrons.

In the position representation , bunching is described by $\int_0^{2\pi} \left|\Psi\left(\theta, \bar{z}\right)\right|^2 e^{-i\theta}\, d\theta$

In the momentum representation , bunching is described by $\sum_{n=-\infty}^{\infty} c_n c^*{}_{n-1}$
i.e. a coherent superposition of momentum states.

## Quantum FEL model : Linear Stability Analysis

$$\frac{dc_n}{d\bar{z}} = -i\frac{n^2}{2\bar{\rho}}c_n - \bar{\rho}\left(Ac_{n-1} - A^* c_{n+1}\right)$$

$$\frac{dA}{d\bar{z}} = \sum_{n=-\infty}^{\infty} c_n c^*_{n-1} + i\delta A$$

A stationary solution to these equations is:

- A=0 (no EM field)
- $c_0=1$, $c_k=0$ for all $k\neq 0$
  (all resonant electrons/
  spatially uniform electron distribution)

Considering small fluctuations in $c_n$ and A about these stationary values i.e.

$$A = 0 + A^{(1)}$$

$$c_0 = 1 + c_0^{(1)}$$

$$c_k = 0 + c_k^{(1)} \quad \text{for all } k\neq 0$$

then retaining only terms linear in the fluctuation variables we obtain :

$$\frac{dc_1}{d\bar{z}} = -\frac{i}{2\bar{\rho}}c_1 - \bar{\rho}A$$

$$\frac{dc_{-1}}{d\bar{z}} = -\frac{i}{2\bar{\rho}}c_{-1} + \bar{\rho}A^*$$

$$\frac{dA}{d\bar{z}} = c^*_{-1} + c_1 + i\delta A$$

## Quantum FEL model : Linear Stability Analysis



$$\frac{dc_1}{d\bar{z}} = -\frac{i}{2\bar{\rho}}c_1 - \bar{\rho}A \qquad (1)$$

$$\frac{dc_{-1}}{d\bar{z}} = -\frac{i}{2\bar{\rho}}c_{-1} + \bar{\rho}A^* \qquad (2)$$

$$\frac{dA}{d\bar{z}} = c^*_{-1} + c_1 + i\delta A \qquad (3)$$

Differentiating eq.(3) twice and substituting eq.(1) and (2) allows us to write an equation in A alone :

$$\frac{d^3A}{d\bar{z}^3} = i\delta\frac{d^2A}{d\bar{z}^2} - \frac{1}{4\bar{\rho}^2}\left(\frac{dA}{d\bar{z}} - i\delta A\right) + iA$$

Looking for solutions of the form

$$A \propto \exp(i\lambda\bar{z})$$

we find the dispersion relation :

$$(\lambda-\delta)\left(\lambda^2 - \frac{1}{4\bar{\rho}^2}\right) + 1 = 0$$

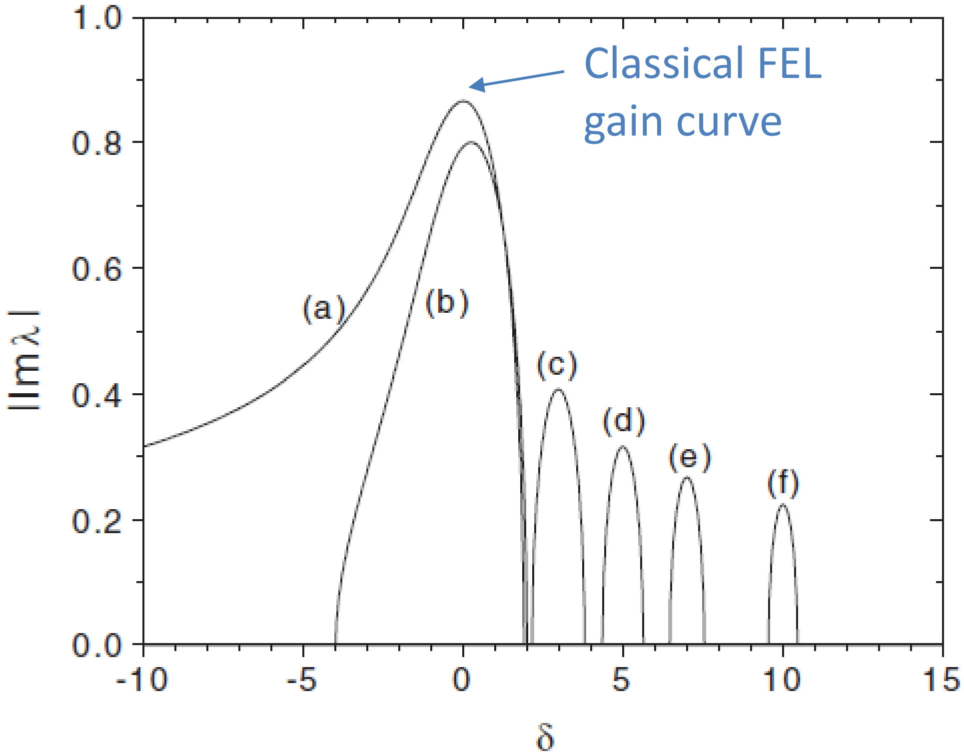
Quantum term

As $\bar{\rho} \to \infty$ , this reduces to the dispersion relation of the classical high-gain FEL :

$$\lambda^2(\lambda-\delta)+1=0$$

# Quantum FEL model : Linear Stability Analysis

Quantum FEL dispersion relation : $(\lambda-\delta)\left(\lambda^2-\frac{1}{4\bar{\rho}^2}\right)+1=0$




Growth rate ,Im(λ), vs δ when

(a) $\bar{\rho}=10$
(b) $\bar{\rho}=1$
(c) $\bar{\rho}=0.167$
(d) $\bar{\rho}=0.1$
(e) $\bar{\rho}=0.071$
(f) $\bar{\rho}=0.05$

Graph from [8]

As $\bar{\rho}$ decreases, gain curve narrows and shifts to increasing $\delta$ (=1/2$\bar{\rho}$)

i.e. $\gamma_0-\gamma_r=\frac{\hbar k}{2mc}$

# 4. Quantum FEL Simulations

Solving the momentum representation equations numerically :

$$\frac{dc_n}{d\bar{z}} = -i\frac{n^2}{2\bar{\rho}} - \bar{\rho}\left(Ac_{n-1} - A^*c_{n+1}\right)$$

$$\frac{dA}{d\bar{z}} = \sum_{n=-\infty}^{\infty} c_n c^*_{n-1} + i\delta A$$

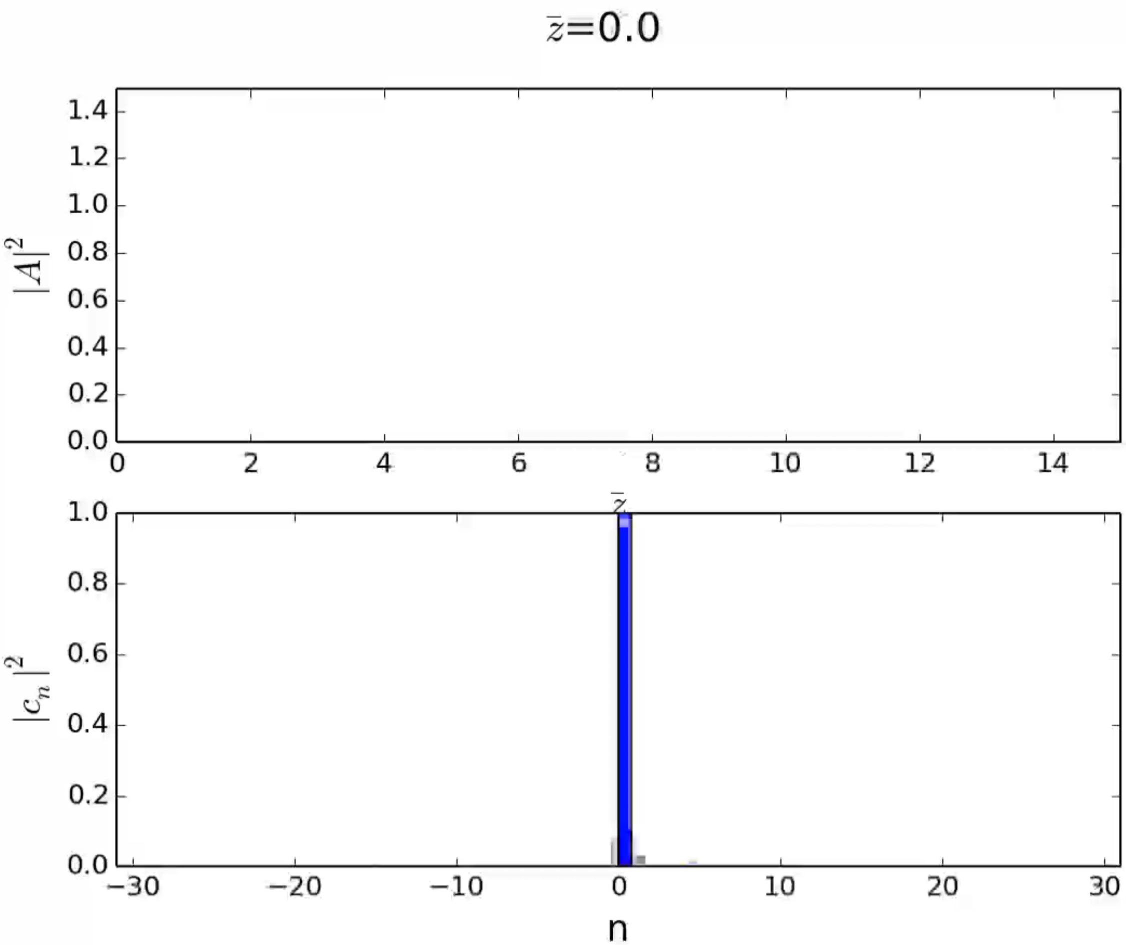
Initial conditions :- A=0 (no EM field)

- $c_0=1$, $c_k=0$ for all $k\neq0$

**Classical limit :**

$$\boxed{\bar{\rho} = 10}$$

- Many momentum states are populated
- Field evolution is identical to that in classical , particle FEL models.

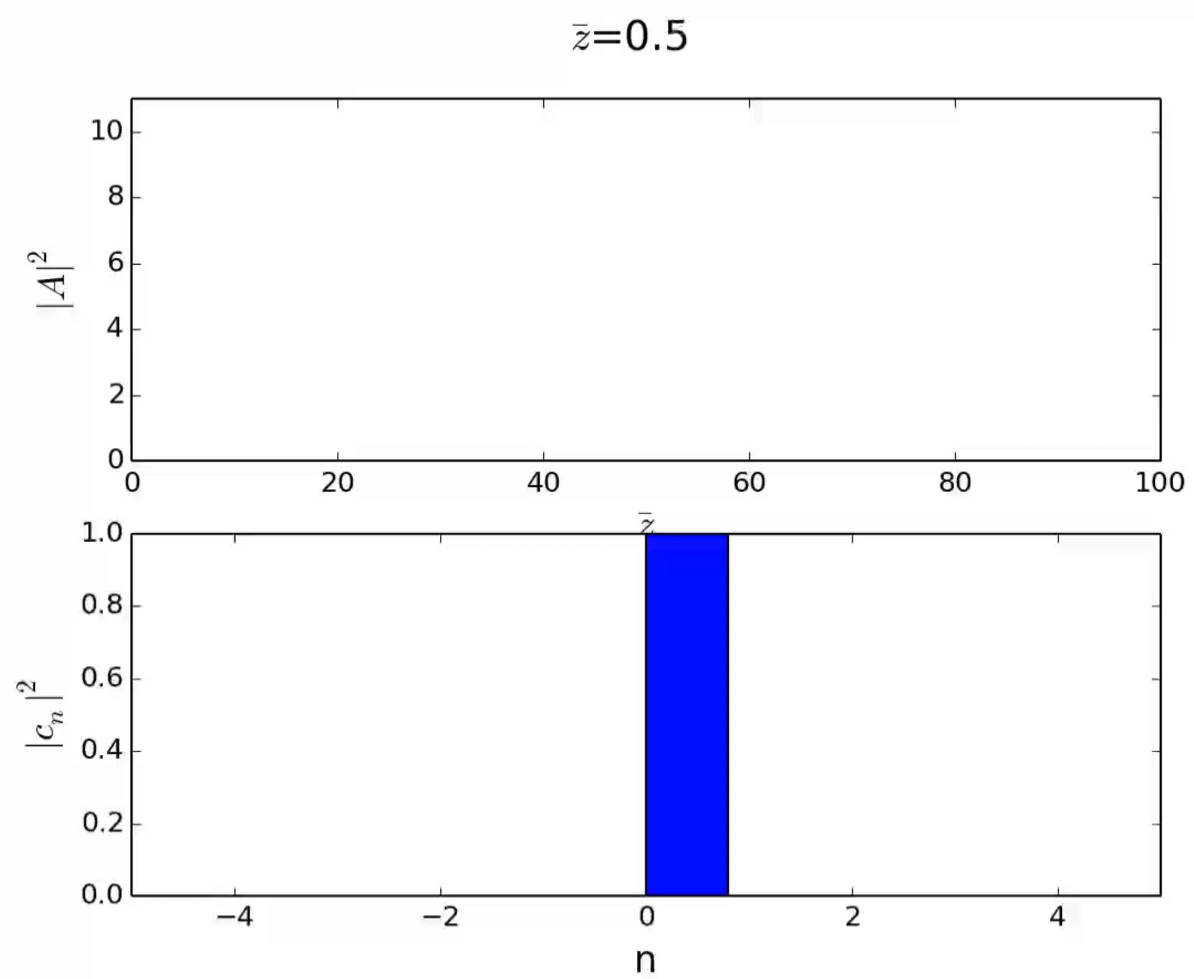


# 4. Quantum FEL Simulations

## Quantum limit :

$$\boxed{\bar{\rho} = 0.1}$$

- Very different evolutio to classical case
- At most 2 momentum states are populated
- FEL behaves as 2-level system



# 4. Quantum FEL Simulations – Including Slippage

So far we have assumed steady-state / single frequency FEL operation :

- Relative slippage between light and electrons is neglected
- E-beam described using a single ponderomotive potential with periodic boundary conditions.
- Every ponderomotive potential in the e-beam behaves the same.

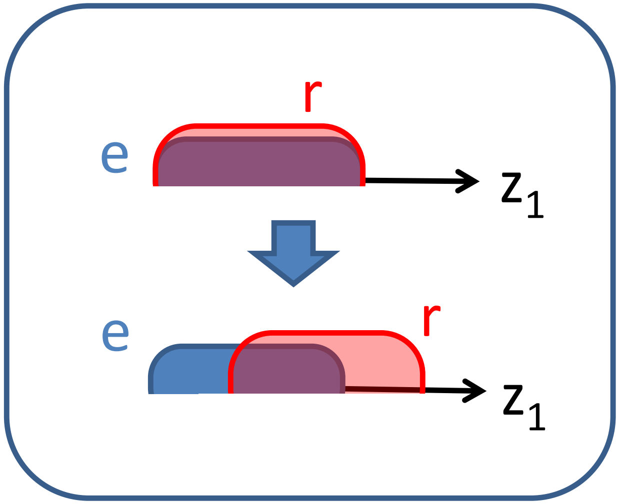
To model Self-Amplified Spontaneous Emission (SASE) this is insufficient

- The FEL interaction starts from random shot noise
- Different parts of the e-beam → different noise

To include slippage we introduce an additional length scale which represents the position along the electron bunch i.e.

$$z_1 = \frac{z - \mathrm{v}_z t}{\mathrm{l}_c}$$

where $l_c = \dfrac{\lambda}{4\pi\rho}$ is the cooperation length




See [9] for full details.

# 4. Quantum FEL Simulations – Including Slippage

In our model , this means that the EM field and momentum state amplitudes must be defined at each position along the electron bunch i.e.

$$A(\bar{z}) \rightarrow A(\bar{z}, z_1)$$
$$c_n(\bar{z}) \rightarrow c_n(\bar{z}, z_1)$$

so the quantum FEL model including slippage is the set of coupled **PDEs**

$$\frac{\partial c_n(\bar{z}, z_1)}{\partial \bar{z}} = -i\frac{n^2}{2\bar{\rho}} - \bar{\rho}\left(Ac_{n-1} - A^* c_{n+1}\right)$$
$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \sum_{n=-\infty}^{\infty} c_n c^*{}_{n-1} + i\delta A$$

As time-dependence is now included, we can look at the frequency spectrum of the emitted radiation.

# 4. Quantum FEL Simulations – Including Slippage

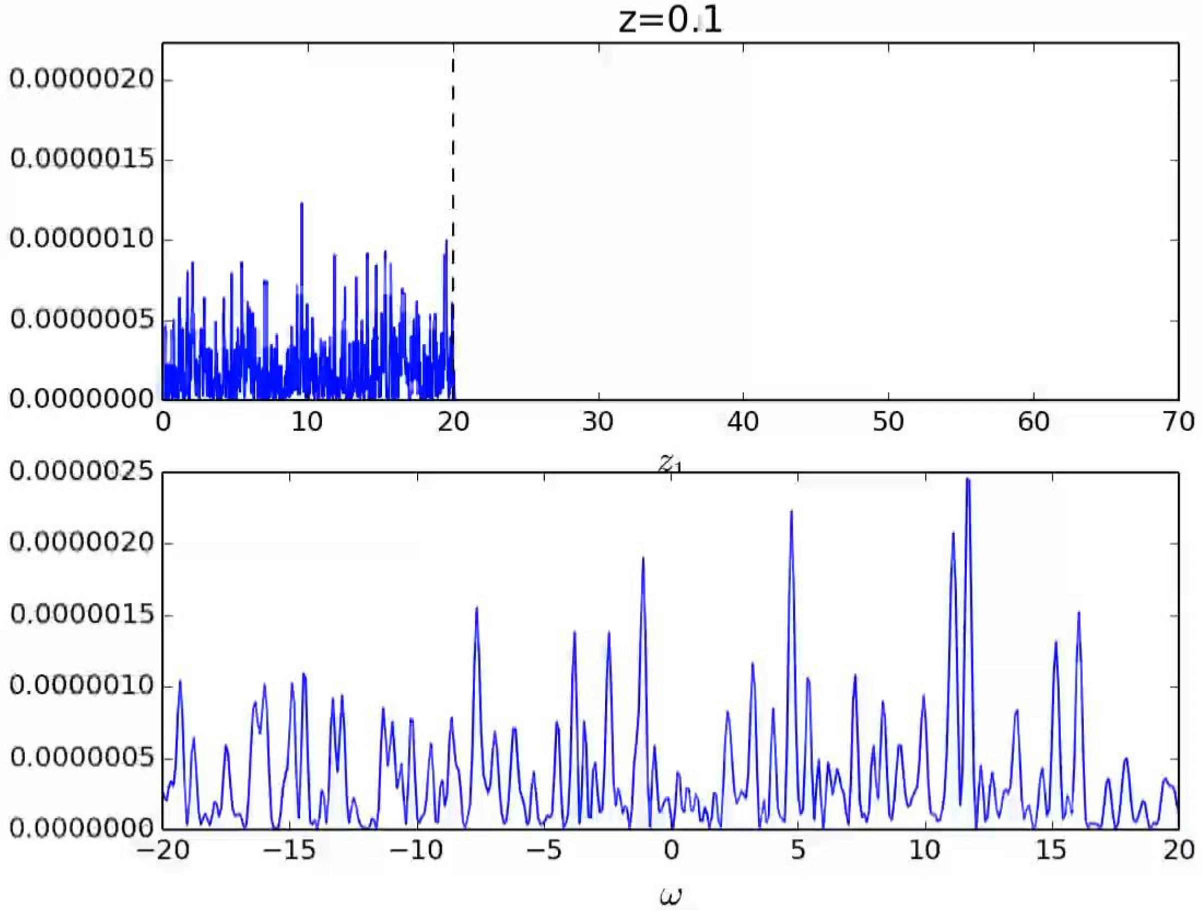
Example : e-beam length = 20 $l_c$
Phases of $c_n$ are random to simulate shot noise.

**Classical limit :**

$$\bar{\rho} = 5$$

- Broad, noisy SASE spectrum as produced from classical particle models

  (see e.g. lecture by M. Yurkov on coherence of SASE FEL)



# 4. Quantum FEL Simulations – Including Slippage

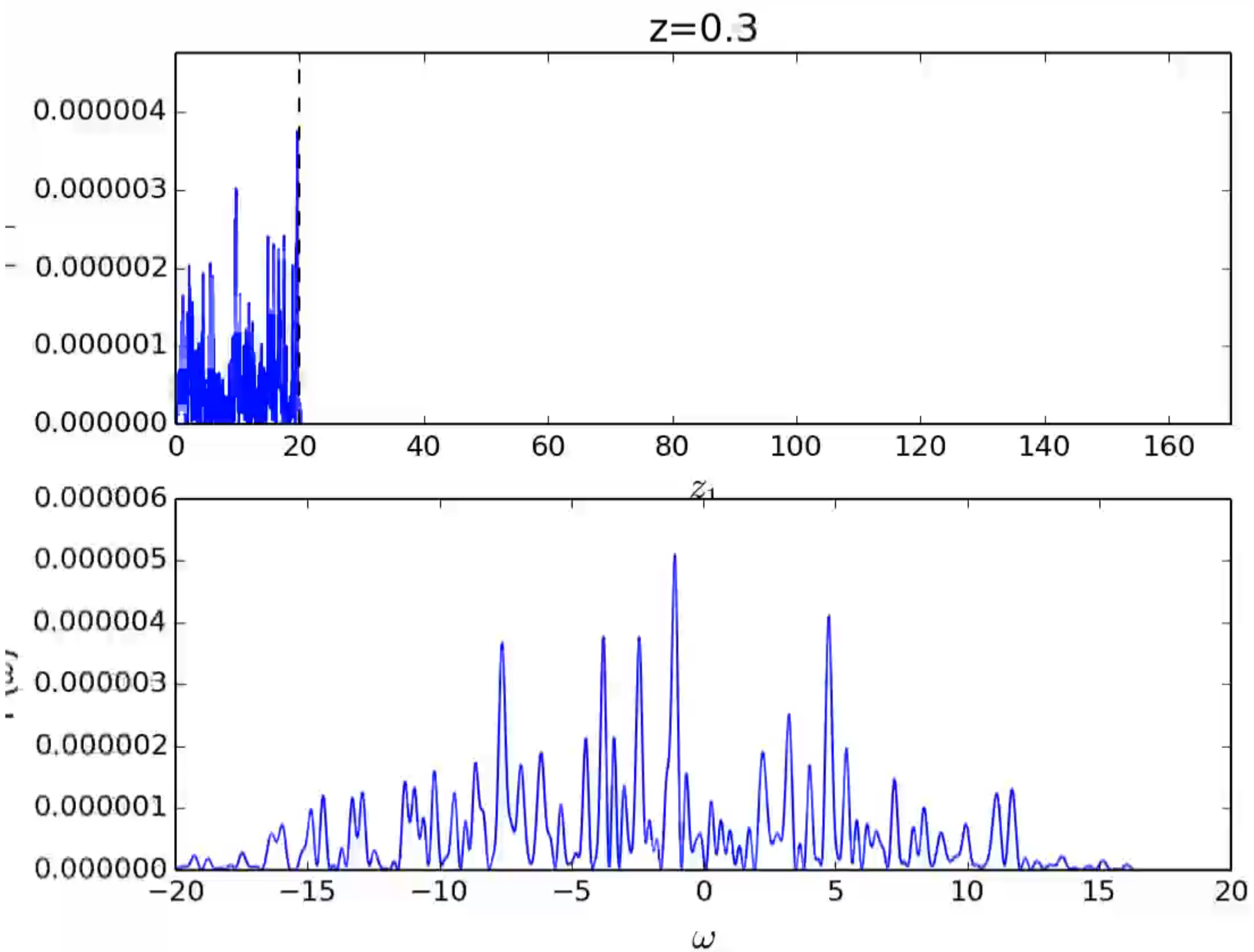
**Quantum limit :**

$$\bar{\rho} = 0.2$$

- Discrete line spectrum
- separation of spectral lines is

$$\Delta\omega = \frac{\hbar k^2}{\gamma m}$$

i.e. relativistic recoil frequency




Hight degree of temporal coherence of quantum FEL is potentially attractive.

# 4. Conclusions

**Covered :**

- When quantum effects may be significant
- Features of quantum FEL operation
- Classical and quantum limits of the quantum FEL model
- Possibility of using quantum regime to produce highly coherent, X-ray/$\gamma$-ray sources.

Quantum FEL regime not realised ….yet.

**Not covered :**

- 3D models and effects (see e.g [10, 11])
- Spontaneous emission (see e.g. [12] and references therein)
- Effects associated with a quantized EM field e.g. entanglement , photon statistics (see e.g. [13,14])

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**+ many others…**



## Acknowledgements

Many of the results presented here were produced in collaboration with R. Bonifacio & N. Piovella.